Cryptomorphic Descriptions of Matroids

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Outline

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7. Duality

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- These notions share fundamental common properties (e.g. taking an element out of an independent set still yields an independent set)
- Abstracting fundamental properties gives us deeper understanding of the topic (e.g. by developing different ways to visualise the concept of independence)

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Due to all the possible characterizations this structure is useful to relate different areas in mathematics such as linear algebra and graph theory.

Linear Independence, Graph Cycles

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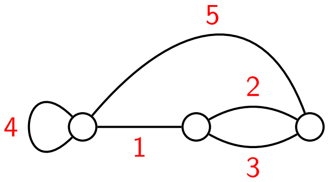

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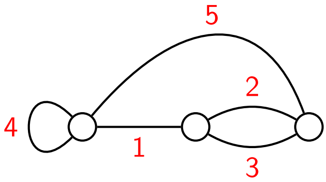

Here, E is the set of edges. The cycles of this graph are $\{1, 2, 5\}, \{1, 3, 5\}, \{2, 3\}, \{4\}$. The independent sets are just the sets of edges that do not contain a cycle.

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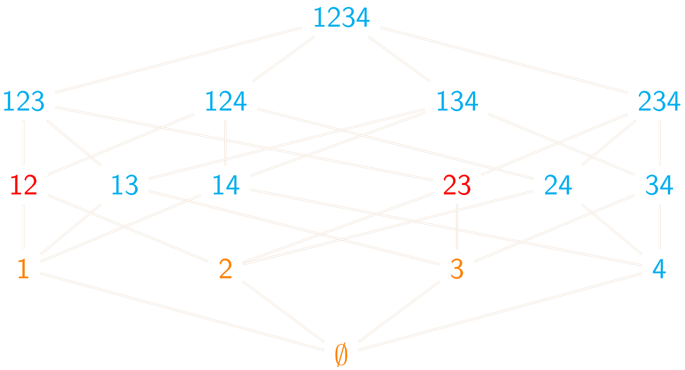

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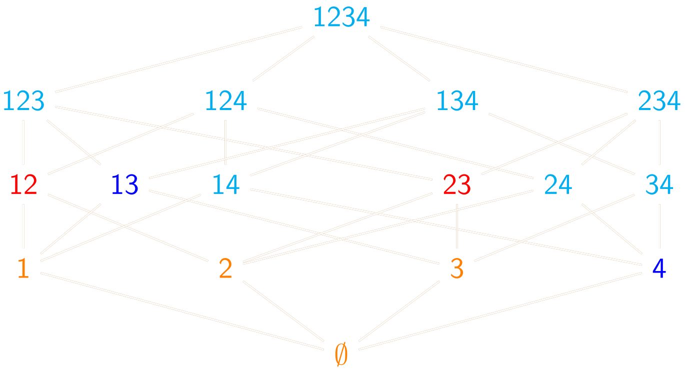

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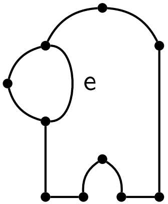

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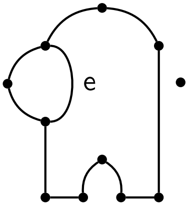

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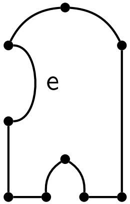

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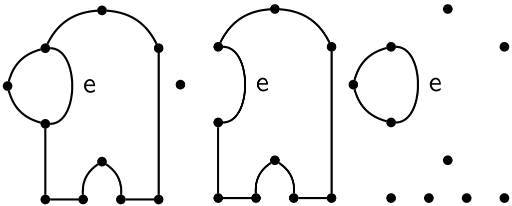

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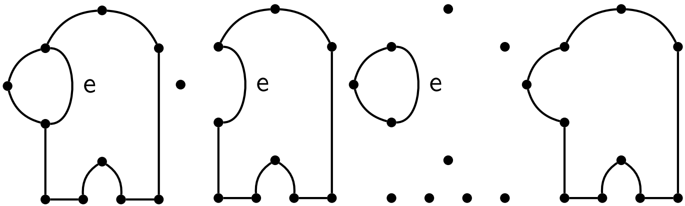

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$$\mathrm{cl}(S) = \{x \in E : \mathrm{r}(S \cup x) = \mathrm{r}(S)\}.$$

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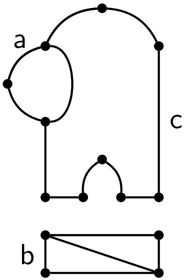
   - the closure operator constructs the span, intersected with E.

Consider the matroid induced by the graph G, and the set S defined below.

(a) G



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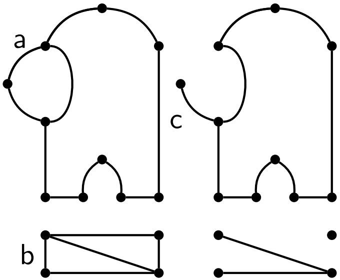

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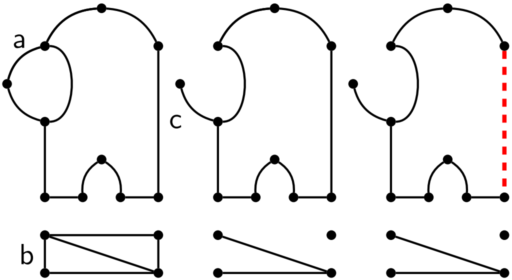

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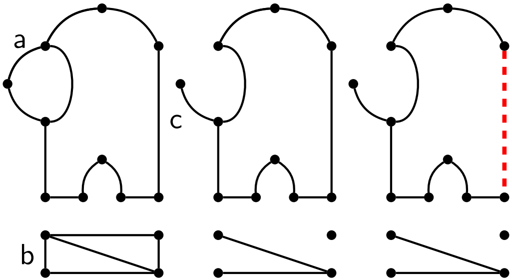

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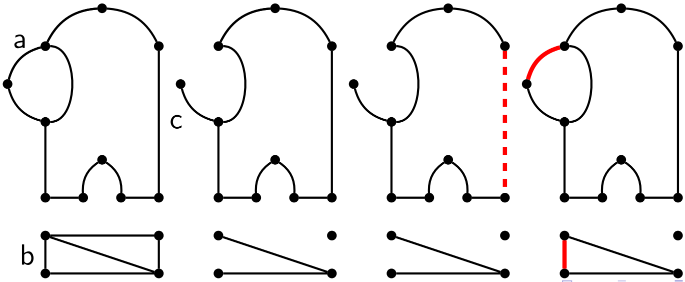
- The only edges which introduce cycles when added to S are a and b, so $\mathrm{cl}(S) = S + a + b$.

(a) G (b) S (c) U (d) $\mathrm{cl}(S)$



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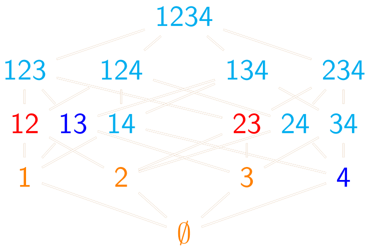

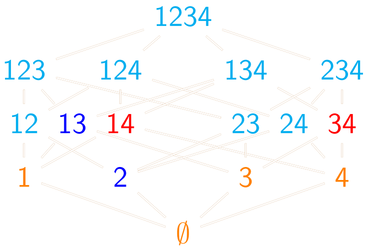

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- $r^*(X) = r(E - X) + |X| - r(M)$

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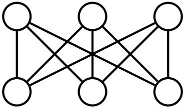

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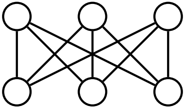

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The matrix associated to the matroid $M(K_{3,3})$ is the following:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

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So, the dual matroid $M^*(K_{3,3})$ is represented by the following matrix:

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Where, $r(M) = 5$ and $r^*(M) = 4$, and $|E(M)| = 9$. Hence, we see that $r(M) + r^*(M) = |E(M)|$ holds.

Duality - Examples

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Also, in this example, both matroids are vector matroids over the field $\mathbb{F}_2$. However, although $M(K_{3,3})$ is a graphic matroid, its dual, $M^*(K_{3,3})$ is not graphic. This results points to the following:

Duality - Examples

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Theorem

The dual of a graphic matroid is itself graphic if and only if the underlying graph is planar.

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To summarize

- We abstracted linear independence with three axioms.
- This abstract independence turns out to have connections to other fields too, like graph theory.
- Each matroid has circuits, bases, rank function and closure operator, each with their unique properties that determine them from the reverse direction.
- Every matroid has an "opposite matroid" called its dual, that is formed using the complements of the bases of the original matroid.

References

- J. G. Oxley. Matroid theory. Oxford Science Publications. Oxford University Press, USA, 1993.

References

- J. G. Oxley. Matroid theory. Oxford Science Publications. Oxford University Press, USA, 1993.
- G. Gordon and J. McNulty. Matroids: A geometric introduction. Cambridge University Press, 2021.

Thank You!

For can now give and example for the concepts given until know.

Example:

Now, we will formulate another important point, the so called, orthogonality, which refers to the link between circuits and cocircuits.

Proposition: For a given matroid M, let C be a circuit and C* be a cocircuit. Then, $|C \cap C*| \neq 1$.

Proof.

We will prove this by contradiction. Suppose $C \cap C* = \{x\}$, for some $x \in E$, this is, there exists and element in the intersection such that the cardinality will be 1. Now consider the hyperplane $H = E - C^*$, and recall that the closure $cl(H) = H$. We can onserve that by the way in which it is defined if $x \in C^*$, then $x \notin H$, but $C - x \subseteq H$. Moreover, we have that $x \in cl(C - x)$ hence $x \in cl(H) = H$. Then $x \notin C^*$, which is a contradiction. So, $C \cap C* \neq \{x\}$, which implies $|C \cap C*| \neq 1$. □