Kripke Frames

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Abstract

We outline a plan of formalization for Kripke frames, define the modal operators $\Box$, $\diamond$ and characterise some of the most relevant modal logic axioms. Time permitting, we will formalize some examples.

This blueprint, and more generally the `kripke-frames` Lean project, is part of the assignment for the course Formalized mathematics and proof assistants.

1 Basic definitions

Definition 1.1. A *Kripke frame* or *modal frame* is a pair $\langle W, R\rangle$ where W is a (possibly empty) set, and R is a **binary relation** on W. Elements of W are called *nodes* or *worlds*, and R is known as the **accessibility relation**.

Definition 1.2. A *Kripke model* is a triple $\langle W, R, \Vdash\rangle$, where $\langle W, R\rangle$ is a Kripke frame, and $\Vdash$ is a relation between nodes of W and modal formulas, such that for all $w \in W$ and modal formulas A and B:

- $w \Vdash \neg A$ if and only if $w \nVdash A$,
- $w \Vdash A \to B$ if and only if $w \nVdash A$ or $w \Vdash B$,
- $w \Vdash \Box A$ if and only if $u \Vdash A$ for all u such that $w\ R\ u$.

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We read $w \Vdash A$ as "w satisfies A", "A is satisfied in w", or "w forces A". The relation $\Vdash$ is called the *satisfaction relation*, *evaluation*, or *forcing relation*. The satisfaction relation is uniquely determined by its value on propositional variables.

A formula A is *valid* in:

- a model $\langle W, R, \Vdash\rangle$, if $w \Vdash A$ for all $w \in W$,
- a frame $\langle W, R\rangle$, if it is valid in $\langle W, R, \Vdash\rangle$ for all possible choices of $\Vdash$,
- a class C of frames or models, if it is valid in every member of C.

Consider the schema $T : \Box A \to A$. T is valid in any reflexive frame $\langle W, R\rangle$: if $w \Vdash \Box A$, then $w \Vdash A$ since $w\,R\,w$. On the other hand, a frame which validates T has to be reflexive: fix $w \in W$, and define satisfaction of a propositional variable p as follows: $u \Vdash p$ if and only if $w\,R\,u$. Then $w \Vdash \Box p$, thus $w \Vdash p$ by T, which means $w\,R\,w$ using the definition of $\Vdash$. T corresponds to the class of reflexive Kripke frames.

Common Modal Axiom Schemata

The following table lists common modal axioms together with their corresponding classes.

Name	**Axiom**	**Frame Condition**
K	$\Box(A \to B) \to (\Box A \to \Box B)$	Holds true for any frames
T	$\Box A \to A$	Reflexive: $w\,R\,w$
–	$\Box\Box A \to \Box A$	Dense: $w\,R\,u \Rightarrow \exists v\,(w\,R\,v \wedge v\,R\,u)$
4	$\Box A \to \Box\Box A$	Transitive: $w\,R\,v \wedge v\,R\,u \Rightarrow w\,R\,u$
D	$\Box A \to \Diamond A$ or $\Diamond\top$ or $\neg\Box\bot$	Serial: $\forall w\,\exists v\,(w\,R\,v)$
B	$A \to \Box\Diamond A$ or $\Diamond\Box A \to A$	Symmetric: $w\,R\,v \Rightarrow v\,R\,w$
5	$\Diamond A \to \Box\Diamond A$	Euclidean: $w\,R\,u \wedge w\,R\,v \Rightarrow u\,R\,v$

Examples

Time permitting, we will formalize some examples. This section will thus be extended in due time.