

Kripke Frames

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Abstract

We outline a plan of formalization for Kripke frames, define the modal operators \Box , \Diamond and characterise some of the most relevant modal logic axioms. Time permitting, we will formalize some examples.

This blueprint, and more generally the [kripke-frames](#) Lean project, is part of the assignment for the course [Formalized mathematics and proof assistants](#).

1 Basic definitions

A *Kripke frame* or *modal frame* is a pair $\langle W, R \rangle$ where W is a (possibly empty) set, and R is a **binary relation** on W . Elements of W are called *nodes* or *worlds*, and R is known as the **accessibility relation**.

Definition 1.1. A *Kripke model* is a triple $\langle W, R, \Vdash \rangle$, where $\langle W, R \rangle$ is a Kripke frame, and \Vdash is a relation between nodes of W and modal formulas, such that for all $w \in W$ and modal formulas A and B :

- $w \Vdash \neg A$ if and only if $w \not\Vdash A$,
- $w \Vdash A \rightarrow B$ if and only if $w \not\Vdash A$ or $w \Vdash B$,
- $w \Vdash \Box A$ if and only if $u \Vdash A$ for all u such that $w R u$.

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We read $w \Vdash A$ as “ w satisfies A ”, “ A is satisfied in w ”, or “ w forces A ”. The relation \Vdash is called the *satisfaction relation*, *evaluation*, or *forcing relation*. The satisfaction relation is uniquely determined by its value on propositional variables.

A formula A is *valid* in:

- a model $\langle W, R, \Vdash \rangle$, if $w \Vdash A$ for all $w \in W$,
- a frame $\langle W, R \rangle$, if it is valid in $\langle W, R, \Vdash \rangle$ for all possible choices of \Vdash ,
- a class C of frames or models, if it is valid in every member of C .

Consider the schema $T : \Box A \rightarrow A$. T is valid in any reflexive frame $\langle W, R \rangle$: if $w \Vdash \Box A$, then $w \Vdash A$ since $w R w$. On the other hand, a frame which validates T has to be reflexive: fix $w \in W$, and define satisfaction of a propositional variable p as follows: $u \Vdash p$ if and only if $w R u$. Then $w \Vdash \Box p$, thus $w \Vdash p$ by T , which means $w R w$ using the definition of \Vdash . T corresponds to the class of reflexive Kripke frames.

Common Modal Axiom Schemata

The following table lists common modal axioms together with their corresponding classes.

Name	Axiom	Frame Condition
K	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	Holds true for any frames
T	$\Box A \rightarrow A$	Reflexive: $w R w$
—	$\Box \Box A \rightarrow \Box A$	Dense: $w R u \Rightarrow \exists v (w R v \wedge v R u)$
4	$\Box A \rightarrow \Box \Box A$	Transitive: $w R v \wedge v R u \Rightarrow w R u$
D	$\Box A \rightarrow \Diamond A$ or $\Diamond \top$ or $\neg \Box \perp$	Serial: $\forall w \exists v (w R v)$
B	$A \rightarrow \Box \Diamond A$ or $\Diamond \Box A \rightarrow A$	Symmetric: $w R v \Rightarrow v R w$
5	$\Diamond A \rightarrow \Box \Diamond A$	Euclidean: $w R u \wedge w R v \Rightarrow u R v$

Examples

Time permitting, we will formalize some examples. This section will thus be extended in due time.