Notes on an Introduction to Complex Analysis

Based on the Introduction to Complex Analysis course on Coursera

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Contents

1 Complex Numbers Basics

1.1 Algebra and Geometry in the Complex Plane

1.1.1 The Complex Plane

- Complex numbers are numbers of the form: $z = x + iy$.
- Set of complex numbers is denoted: $\mathbb{C}$.
- Real numbers are a subset of the complex numbers.

1.1.2 Adding Complex Numbers

Addition of complex numbers is defined as: $(x+iy)+(u+iv) = (x+u)+i(y+v)$
Thus $\Re(z + w) = \Re(z) + \Re(w)$ and $\Im(z + w) = \Im(z) + \Im(w)$

1.1.3 The Modulus of a Complex Number

The modulus of the complex number $z = x + iy$ is the length of the vector z:

$$|z| = \sqrt{x^2 + y^2}$$

1.1.4 Multiplication of Complex Numbers

Motivation: $(x + iy)(u + iv) = xu + ixv + iyu + i^2yv$
So we define:

Definition 1.1.

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu)$$

The usual properties hold:

- $(z_1z_2)z_3 = z_1(z_2z_3)$ (associative property)
- $z_1z_2 = z_2z_1$ (commutative property)
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (distributive property)

1.1.5 How Do You Divide Complex Numbers?

Suppose that $z = x + iy$ and $w = u + iv$. What is $\frac{z}{w}$(for $w \neq 0$)?

$$\frac{z}{w} = \frac{x + iy}{u + iv} = \frac{(x + iy)(u - iv)}{(u + iv)(u - iv)} = \frac{xu + yv}{u^2 + v^2} + i\frac{yu - xv}{u^2 + v^2}$$

1.1.6 The Complex Conjugate

If $z = x + iy$ then $\overline{z} = x - iy$ is the complex conjugate of z.
Properties:

- $\overline{\overline{z}} = z$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $|z| = |\overline{z}|$
- $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$
- $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}$

1.1.7 More Properties of the Complex Conjugate

- When is $z = \overline{z}$?
- $z + \overline{z} = (x + iy) + (x - iy) = 2x$ so
$$\Re(z) = \frac{z + \overline{z}}{2}, similarly\ \Im(z) = \frac{z - \overline{z}}{2i}$$
- $|z \cdot w| = |z| \cdot |w|$
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}, (w \neq 0)$
- $|z| = 0$ if and only if $z = 0$

1.1.8 Some Inequalities

- $-|z| \leq \Re(z) \leq |z|$
- $-|z| \leq \Im(z) \leq |z|$
- $|z + w| \leq |z| + |w|$ (triangle inequality)
- $|z - w| \geq |z| - |w|$ (reverse triangle inequality)

1.1.9 The Fundamental Theorem of Algebra

Theorem 1.1. *If $a_0, a_1, ..., a_n$ are complex numbers with $a_n \neq 0$, then the polynomial*
$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$
has n roots $z_1, z_2, ..., z_n$ in $\mathbb{C}$. It can be factored as
$$p(z) = a_n(z - z_1)(z - z_2)...(z - z_n)$$

1.2 Polar Representation of Complex Numbers

1.2.1 Polar Coordinates

Consider a complex number $z = x + iy$. Z can also be described by the distance r from the origin and the angle θ between the positive x-axis and the line segment from 0 to z.
(r, θ) are the polar coordinates of z.

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

1.2.2 The Argument of a Complex Number

Definition 1.2. *The principal argument of z, called $Argz$, is the value of θ for which $-\pi < \theta \leq \pi$.*

Remark. *θ is not unique!*
$argz = (Argz + 2\pi k : k = 0, \pm 1, \pm 2, ...)\,, z \neq 0$

1.2.3 Exponential Notation

Convenient notation: $e^{i\theta} = \cos\theta + i\sin\theta$
So $z = r(\cos\theta + i\sin\theta)$ becomes $z = re^{i\theta}$

1.2.4 Properties of the Exponential Notation

- $\left|e^{i\theta}\right| = 1$
- $\overline{e^{i\theta}} = e^{-i\theta}$
- $\frac{1}{e^{i\theta}} = e^{-i\theta}$
- $e^{i(\theta+\varphi)} = e^{i\theta} \cdot e^{i\varphi}$

1.2.5 De Moivre's Formula

- $(e^{i\theta})^n = e^{in\theta}$
- Since $e^{i\theta} = \cos\theta + i\sin\theta$ means that:
  $(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

1.3 Roots of Complex Numbers

1.3.1 The nth Root

Definition 1.3. *Let w be a complex number. An nth root of w is a complex number z such that $z^n = w$*

If $w \neq 0$ there are exactly n distinct nth roots. Use the polar form for w and z: $w = \rho e^{i\varphi} and z = re^{i\theta}$
The equation $z^n = w$ then becomes

$$r^n e^{in\theta} = \rho e^{i\varphi}, so\ r^n = \rho\ and\ e^{in\theta} = e^{i\varphi}$$

Thus

$$r = \sqrt[n]{\rho}\ and\ n\theta = \varphi + 2k\pi, k \in \mathbb{Z}$$

So

$$\theta = \frac{\varphi}{n} + \frac{2k\pi}{n}, k = 0, 1, ..., n-1$$

1.3.2 Roots of Unity

Definition 1.4. *The nth roots of 1 are called the nth roots of unity.*

1.4 Topology in the Plane

1.4.1 Sets in the Complex Plane

Circles and disks: center $z_0 = x_0 + iy_0$, radius r.

- $B_r(z_0) = \{z \in \mathbb{C} :\ z$ has distance less than r from $z_0\}$ disk of radius r, centered at z_0
- $K_r(z_0) = \{z \in \mathbb{C} :\ z$ has distance r from $z_0\}$ circle of radius r, centered at z_0

How to measure distance?

- $$\begin{aligned} d &= \sqrt{(x - x_0)^2 + (y - y_0)^2} \\ &= |(x - x_0) + i(y - y_0)| \\ &= |z - z_0| \end{aligned}$$
- So $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$

1.4.2 Interior Points and Boundary Points

Definition 1.5. *Let $E \subset \mathbb{C}$. A point z_0 is an interior point of E if there is some $r > 0$ such that $B_r(z_0) \subset E$.*

Definition 1.6. *Let $E \subset \mathbb{C}$. A point b is a boundary point of E if every disk around b contains a point in E and a point not in E. The boundary of the set $E \subset \mathbb{C}, \partial E$, is the set of all boundary points of E.*

1.4.3 Open and Closed Sets

Definition 1.7. *A set $U \subset \mathbb{C}$ is open if every one of its points is an interior point.*
A set $A \subset \mathbb{C}$ is closed if it contains all of its boundary points.

Examples:

- $\{z \in \mathbb{C} : |z - z_0| < r\}$ and $\{z \in \mathbb{C} : |z - z_0| > r\}$ are open.
- $\mathbb{C}$ and $\emptyset$ are open.
- $\{z \in \mathbb{C} : |z - z_0| \leq r\}$ and $\{z \in \mathbb{C} : |z - z_0| = r\}$ are closed.
- $\mathbb{C}$ and $\emptyset$ are closed.
- $\{z \in \mathbb{C} : |z - z_0| < r\} \cup \{z \in \mathbb{C} : |z - z_0| = r$ and $\Im(z - z_0) > 0\}$ is neither open nor closed.

1.4.4 Closure and Interior of a Set

Definition 1.8. *Let E be a set in $\mathbb{C}$.*
The closure of E is the set E together with all of its boundary points: $\overline{E} = E \cup \partial E$.
The interior of E, $\mathring{E}$ is the set of all interior points of E.

Examples:

- $\overline{B_r(z_0)} = B_r(z_0) \cup K_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$
- $\overline{B_r(z_0) \setminus \{z_0\}} = \{z \in \mathbb{C} : |z - z_0| \leq r\}$
- $\overline{K_r(z_0)} = K_r(z_0)$
- With $E = \{z \in \mathbb{C} : |z - z_0| \leq r\}, \mathring{E} = B_r(z_0)$
- With $E = K_r(z_0), \mathring{E} = \emptyset$

1.4.5 Connectedness

Definition 1.9. *Two sets, X, Y in $\mathbb{C}$ are separated if there are disjoin open set U, V so that $X \subset U$ and $Y \subset V$. A set W in $\mathbb{C}$ is connected if it is impossible to find two separated non-empty sets whose union equals W.*

Example:

$$X = [0, 1) \textit{ and } Y = (1, 2]$$

are separated: choose $U = B_1(0)$ and $V = B_1(2)$. Thus

$$X \cup Y = [0, 2] \setminus \{1\}$$

is not connected. It is hard to chek whether a set is connected.

1.4.6 Connectedness for Open Sets in $\mathbb{C}$

For open sets the is a much easier criterion to check wheter or not a set is connected:

Theorem 1.2. *Let G be an open set in $\mathbb{C}$. Then G is connected if and only if any two points in G can be joined in G by successive line segments.*

1.4.7 Bounded Sets

Definition 1.10. *A set A in $\mathbb{C}$ is bounded if there exists a number $R > 0$ such that $A \subset B_R(0)$. If no such R exists then A is called unbounded.*

2 Complex Functions, Julia Sets, Mandelbrot Set

2.1 Complex Functions

2.1.1 Functions

- Recall: a function $f : A \to B$ is a rule that assigns to each element of A exactly one element of B.
- Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$
- Graphs help us understand the function.

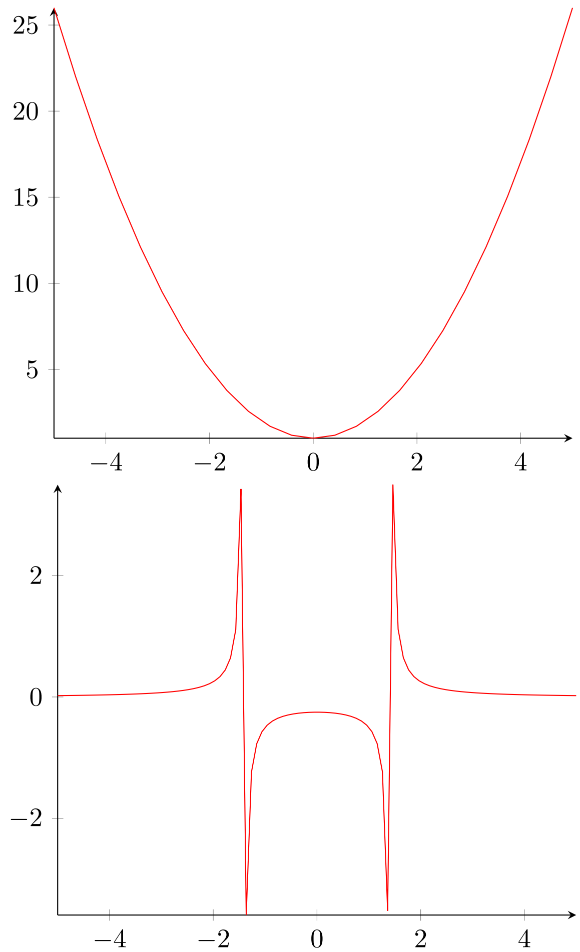


Figure 1: $f(x) = x^2 + 1$ and $g(x) = \frac{1}{2x^2-4}$

2.1.2 Complex Functions

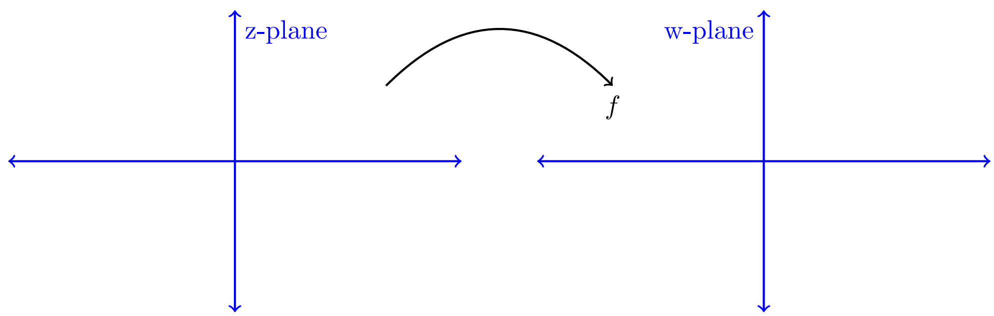
Changing the domain and the codomain results in: $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2 + 1$
How to graph this? 4 dimensions would be necessary, which is a bit impractical. Writing $z = x + iy$ we see:

$$\begin{aligned} w = f(z) &= (x + iy)^2 + 1 \\ &= (x^2 - y^2 + 1) + i \cdot 2xy \\ &= u(x, y) + iv(x, y) \end{aligned}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$. This is one way to plot complex functions, by using two two-dimensional graphs. There is another option that is much more practical though!

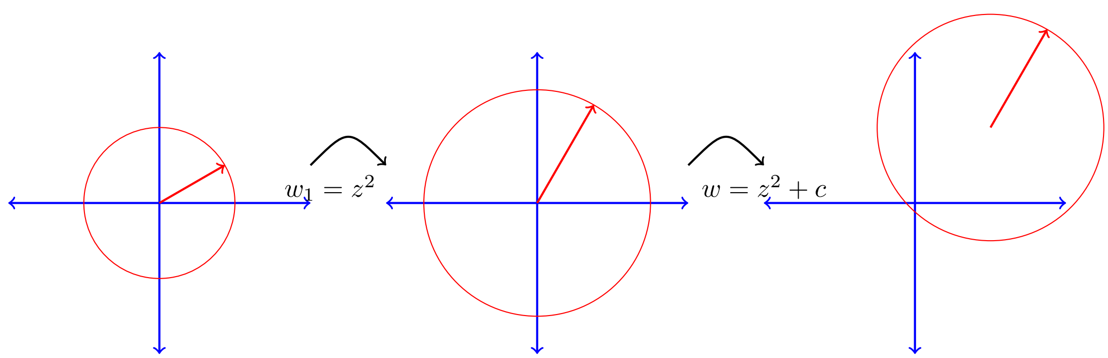
2.1.3 Graphing Complex Functions

The idea is to consider 2 complex planes: one for the domain, and one for the range of the function. To analyze how a function behaves, we analyze how geometric configurations in the z-plane are mapped under f to the w-plane.



2.1.4 More Complicated Functions

How to understand more complicated functions, such as $f(z) = z^2 + c, c \in \mathbb{C}$? Same idea!



2.1.5 Iteration of Functions

Let $f(z) = z + 1$.
Then $f^2(z) = f(f(z)) = f(z+1) = (z+1) + 1 = z + 2$.
$f^3(z) = z + 3$
... $f^n(z) = z + n$
f^n is called the nth iterate of f

2.2 Sequences and Limits of Complex Numbers

2.2.1 Limits

Definition 2.1. *A sequence $\{s_n\}$ of complex numbers converges to $s \in \mathbb{C}$ if for every $\epsilon > 0$ there exists an index $N \geq 1$ such that*

$$|s_n - s| < \epsilon \; for \; all \; n \geq N$$

In this case we write

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

2.2.2 Rules for Limits

1. Convergent sequences are bounded
2. If $\{s_n\}$ converges to s and $\{t_n\}$ converges to t, then
   - $s_n + t_n \to s + t$
   - $s_n \cdot t_n \to s \cdot t$(in particular: $a \cdot s_n \to a \cdot s$ for any $a \in \mathbb{C}$)
   - $\frac{s_n}{t_n} \to \frac{s}{t}$, provided $t \neq 0$
3. A sequence of complex numvers, $\{s_n\}$, converges to 0 if and only if the sequence $\{|s_n|\}$ of absolute values converges to 0
4. A sequence of complex numbers, $\{s_n\}$, with $s_n = x_n + iy_n$, converges to $s = x + iy$ if and only if $x_n \to x$ and $y_n \to y$ as $n \to \infty$

2.2.3 Squeeze Theorem

Theorem 2.1. *Suppose that $\{r_n\}, \{s_n\}$ and $\{t_n\}$ are sequences of real numbers such that $r_n \leq s_n \leq t_n$ for all n. If both sequences $\{r_n\}$ and $\}t_n\}$ converge to the same limit, L, then the sequence $\{s_n\}$ has no choice but to converge to the limit L as well.*

Next theorem is an equivalent of a sequence running against a wall:

Theorem 2.2. *A bounded, monotone sequence of real numbers converges.*

2.2.4 Limits of Complex Functions

Definition 2.2. $\lim_{z \to z_0} f(z) = L$ *if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$*

2.2.5 Continuity

Definition 2.3. *The function f is continuous at z_0 if $f(z) \to f(z_0)$ as $z \to z_0$*

2.3 Iteration of Quadratic Polynomials, Julia Sets

2.3.1 Quadratic Polynomials

Polynomials of interest are of the form $f(z) = z^2 + c$, where $c \in \mathbb{C}$ is a constant. Depending on c, iterates of f will behave differently.
What about other polynomials? What about more general polynomials, $p(z) = az^2 + bz + d$, for constants $a, b, d \in \mathbb{C}$
As it turns out, for each triple of constants (a, b, d) there is exactly one constant c such that $p(z)$ and $f(z)$ behave the same under iteration
Why? Given a, b and d define $c = ad + \frac{b}{2} - (\frac{b}{2})^2$. Then letting $\phi(z) = az + b/2$ one can check that $p(z) = \phi^{-1}(f(\phi(z)))$ for all z.
This is usually written as $p = \phi^{-1} \circ f \circ \phi$
Under iteration:

$$
\begin{aligned}
p \circ p &= (\phi^{-1} \circ f \circ \phi) \circ (\phi^{-1} \circ f \circ \phi) = \phi^{-1} \circ f \circ f \circ \phi \\
p^2 &= \phi^{-1} \circ f^2 \circ \phi \\
p^3 &= \phi^{-1} \circ f^3 \circ \phi \\
p^n &= \phi^{-1} \circ f^n \circ \phi
\end{aligned}
$$

2.3.2 The Julia Set

The Julia set (named after the French mathematician Gaston Julia) of $f(z) = z^2 + c$ is the set of all $z \in \mathbb{C}$ for which the behaviour of the iterates is 'chaotic' in a neighbourhood.
The Fatou set (named after the French mathematician Pierre Fatou) is the set of all $z \in \mathbb{C}$ for thich the iterates behave 'normally' in a neighbourhood.
The iterates of f behave normally near z if nearby points remain nearby under iteration.
The iterates of f behave chaotically at z if in any small neighbourhood of z the behaviour of the iterates depends sensitively on the initial point.
Example for the function $f(z) = z^2$
The unit circle $\{z : |z| = 1\}$ is the locus of chaotic behaviour, whereas $\{z : |z| > 1\}$ (iterates attracted to ∞) and $\{z : |z| < 1\}$ (iterates attracted to 0) form the locus of normal behaviour.
We write $J(f) = \{z : |z| = 1\}$ (Julia set) and $F(t) = \{z : |z| > 1\} \cup \{z : |z| < 1\}$ (Fatou set)

2.3.3 The Basin of Attraction to ∞

More generally, let's look at $f(z) = z^2 + c$. Let

$$A(\infty) = \{z : f^n(z) \to \infty\}$$

'basin of attraction to ∞'

Theorem 2.3. *The set $A(\infty)$ is open, connected and unbounded. It is contained in the Fatou set of f. The Julia set of f coincides with the boundary of $A(\infty)$, which is a closed and bounded subset of $\mathbb{C}$.*

Recap:

- The Julia set is a closed and bounded set.
- The Fatou set is open and unbounded and contains $A(\infty)$.
- Also: $J(f) \cap F(t) = \emptyset$ and both sets are 'completely invariant' under f, meaning that $f(J) = J$ and $f(F) = F$

2.3.4 Wrap-up

Two examples of Julia sets:

- $f(z) = z^2$. $J(f) = \{z : |z| = 1\}$, the unit circle.
- $f(z) = z^2 - 2$. $J(f) = [-2, 2]$ the closed interval from -2 to 2 on the real axis.

These two examples are only two smooth Julia sets.

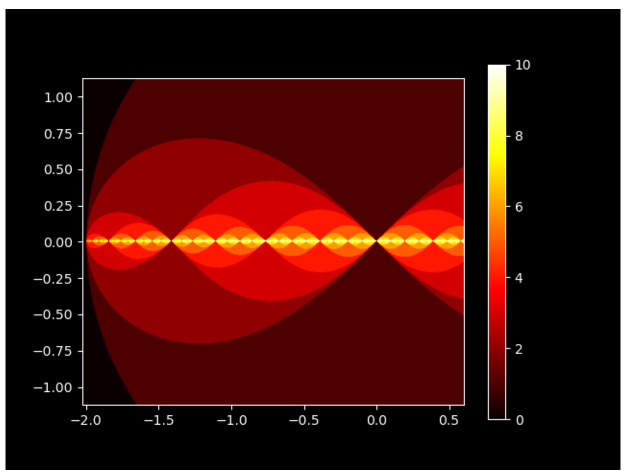


Figure 2: $f(z) = z^2 - 2$

2.3.5 Types of Orbits on the Julia Set of $f(z) = z^2 - 1$

What is $J(f)$? Let's check some orbits.

- $z = 0 : f(0) = -1, f^2(0) = f(-1) = 0, f^3(0) = f(0) = -1, f^4(0) = 0, ...$
  The orbit is thus 0, -1, 0, -1, 0, ...
  This is called a periodic orbit, and 0 is a periodic point of period 2. Clearly, $0 \in J(f)$.

- $z = 1 : 1, 0, -1, 0, -1, 0, -1, ...$
  1 itself isn't a periodic point, but it runs into a periodic orbit. This is called a pre-periodic point. Again, $1 \in J(f)$

- $z = \frac{1+\sqrt{5}}{2}$
  $f(z) = (\frac{1+\sqrt{5}}{2})^2 - 1 = \frac{1+2\sqrt{5}+5}{4} - 1 = \frac{2+2\sqrt{5}}{4} = \frac{1+\sqrt{5}}{2} = z$
  The point $\frac{1+\sqrt{5}}{2}$ is a fixed point of f and thus belongs to $J(f)$ as well.

2.3.6 What Orbits Go Off to ∞?

Theorem 2.4. *Let $f(z) = z^2 + c$, and let $R = \frac{1+\sqrt{1+4|c|}}{2}$. Let $z_0 \in \mathbb{C}$. If for some $n > 0$ we have that $|f^n(z_0)| > R$, then $f^n(z_0) \to \infty$ as $n \to \infty$, i.e. $z_0 \in A(\infty)$, so $z_0 \notin J(f)$*

2.3.7 The Mandelbrot Set

Definition 2.4. *The Mandelbrot set M is the set of all parameters $c \in \mathbb{C}$ for which the Julia set $J(f)$ of $f(z) = z^2 + c$ is connected.*

Remark. *The Mandelbrot set is a subset of the parameter space (the space of all possible c-values), whereas Julia sets are sets of z-values.*

2.4 The Mandelbrot Set

2.4.1 Finding the Mandelbrot Set

Theorem 2.5. *Let $f(z) = z^2 + c$. Then $J(f)$ is connected if and only if 0 does not belong to $A(\infty)$, that is if and only if the orbit $\{f^n(0)\}$ remains bounded under iteration.*

In fact, it is possible to show the following:

Theorem 2.6. *A complex number c belongs to M if and only if $|f^n(0)| \leq 2$ for all $n \geq 1$ (where $f(z) = z^2 + c$)*

2.4.2 Properties of the Mandelbrot Set

- M is a connected set.
- M is contained in the disk of radius 2, centered at zero.
- The boundary of M is very intricate - this is where you will find the most beautiful zooms.
- Moreover, for c-values near the boundary of M, their Julia sets have many different patterns.

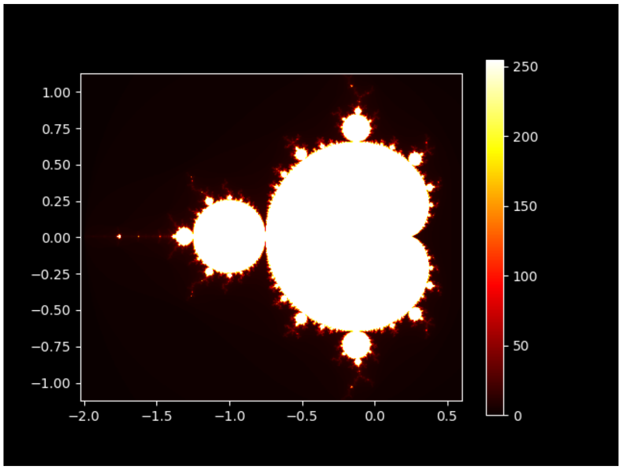


Figure 3: The Mandelbrot Set

- The boundary of the main cardioid is given by $c = \frac{1}{2}e^{i\phi} - \frac{1}{4}e^{2i\phi}, 0 \leq \phi < 2\pi$.
  Writing $\phi = 2\pi\alpha, 0 \leq \alpha < 1$ we can distinguish whether α is a rational or an irrational number.
- Rational α: of the form $\frac{p}{q}$
  - Example: $\alpha = \frac{1}{2}$. Then $c = \frac{1}{2}e^{\pi i} - \frac{1}{4}e^{2\pi i} = -0.75$. Here is a picture for $J(z^2 - 0.75)$:

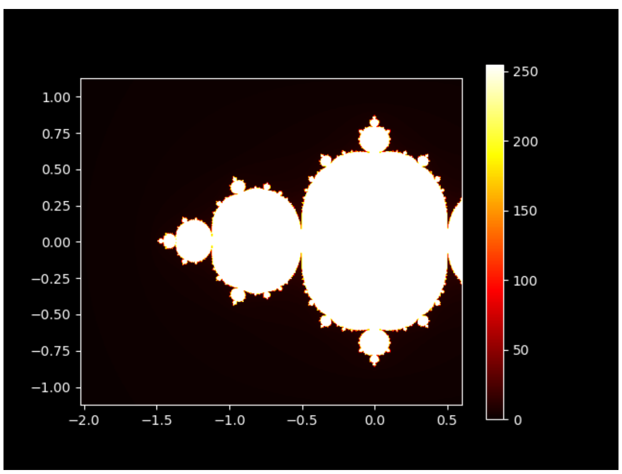


Figure 4: $f(z) = z^2 - 0.75$

- Irrational α: no values p and q such that $\alpha = \frac{p}{q}$.
  - Example: $\alpha = \frac{1+\sqrt{5}}{2}$. Then $c = -0.3905... - 0.5868...i$.

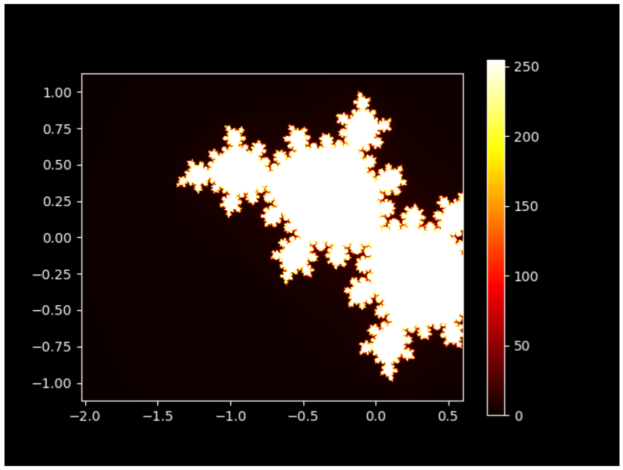


Figure 5: Interior has a so-called 'Siegel disk'

2.4.3 Misiurewicz Points

A point $c \in \mathbb{C}$ is called a Misiurewicz point if the orbit of 0 under $f(z) = z^2 + c$ is pre-periodic, but not periodic.
Example is $c = i$.

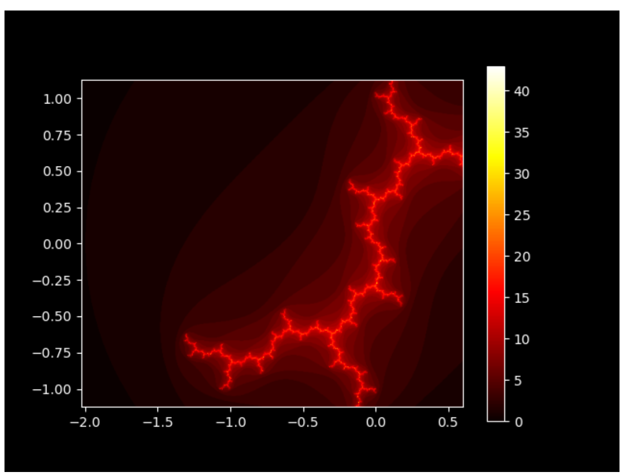


Figure 6: $f(z) = z^2 + i$

2.4.4 Big Open Conjecture

Conjecture 2.1. *The Mandelbrot set is locally connected, that is, for every $c \in M$ and every open set V with $c \in V$, there exists an open set U such that $c \in U \subset V$ and $U \cap M$ is connected*

3 Analytic Functions

3.1 The Complex Derivative

3.1.1 The Complex Derivative

Definition 3.1. *A complex-valued function f of a complex variable is (complex) differentiable at $z_0 \in domain(f)$ if $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.*
If this limit exists, it is denoted $f'(z_0)$ or $\frac{df}{dz}(z_0)$ or $\frac{d}{dz}f(z)|_{z=z_0}$

Example:

- $f(z) = c$ (a constant function, $c \in \mathbb{C}$). Let $z_0 \in \mathbb{C}$ be arbitrary. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{c - c}{z - z_0} = 0 \to 0 \ as \ z \to z_0.$$

  Thus $f'(z) = 0$ for all $z \in \mathbb{C}$.

3.1.2 Other Forms of the Difference Quotient

Instead of

$$\frac{f(z) - f(z_0)}{z - z_0}$$

we often write $z = z_0 + h$ (where $h \in \mathbb{C}$), and the difference quotient becomes

$$\frac{f(z_0 + h) - f(z_0)}{h} \ or \ simply \ \frac{f(z + h) - f(z)}{h}$$

where the limit is take as $h \to 0$.
Another example:

- $f(z) = z^2$. Then

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{(z_0 + h)^2 - z_0^2}{h} = \frac{2z_0h + h^2}{h} = 2z_0 + h \to 2z_0 \ as \ h \to 0.$$

  Thus $f'(z) = 2z$ for all $z \in \mathbb{C}$.

3.1.3 Differentiation Rules

Theorem 3.1. *Suppose f and g are differentiable at z, and h is differentiable at $f(z)$. Let $c \in \mathbb{C}$.*
Then

- $(cf)'(z) = cf'(z)$.
- $(f+g)'(z) = f'(z) + g'(z)$.
- $(f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$.
- $\frac{f}{g}'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}, for\ g(z) \neq 0$.
- $(h \circ f)'(z) = h'(f(z))f'(z)$.

3.1.4 Differentiability Implies Continuity

Theorem 3.2. *If f is differentiable at z_0 then f is continuous at z_0.*

Proof.

$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} (\frac{f(z) - f(z_0)}{z - z_0} \cdot (z - z_0)) = f'(z_0) \cdot 0 = 0$$

□

Remark. *Note that a function can be continuous without being differentiable.*

3.1.5 Analytic Functions

Definition 3.2. *A function f is analytic in an open set $U \subset \mathbb{C}$ if f is (complex) differentiable at each point $z \in U$.*
A function which is analytic in all of $\mathbb{C}$ is called an entire function.

Examples:

- polynomials are analytic in $\mathbb{C}$ (hence entire)
- rational functions $\frac{p(z)}{q(z)}$ are analytic wherever $q(z) \neq 0$
- $f(z) = \overline{z}$ is not analytic
- $f(z) = \Re z$ is not analytic

3.2 The Cauchy-Riemann Equations

3.2.1 The Equations

Recall: a complex function f can be written as

$$f(z) = u(x, y) + iv(x, y)$$

where $z = x + iy$ and u, v are real-valued functions that depend on the two real variables x and y.

Theorem 3.3. *Suppose that $f(z) = u(x, y)+iv(x, y)$ is differentiable at a point z_0. Then the partial derivatives u_x, u_y, v_x, v_y exist at z_0 and satisfy:*

$$u_x = v_y \text{ and } u_y = -v_x$$

These are called the Cauchy-Riemann Equations. Also

$$\begin{aligned} f'(z_0) &= u_x(x_0, y_0) + iv_x(x_0, y_0) = f_x(z_0) \\ &= -i(u_y(x_0, y_0) + iv_y(x_0, y_0)) = -if_y(z_0). \end{aligned}$$

Proof. Suppose that $f(z) = u(x, y)+iv(x, y)$ is differentiable at a point z_0. Let's look at the limit definition of the derivative and take the limit as h approaches 0 along the real axis, i.e. $h = h_x + i \cdot 0$:

$$\begin{aligned} f'(z_0) &= \lim_{h_x \to 0} \frac{f(z_0 + h_x) - f(z_0)}{h_x} \\ &= \lim_{h_x \to 0} \frac{u(x_0 + h_x, y_0) + iv(x_0 + h_x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h_x} \\ &= \lim_{h_x \to 0} \frac{u(x_0 + h_x, y_0) - u(x_0, y_0)}{h_x} + \lim_{h_x \to 0} i \frac{v(x_0 + h_x, y_0) - v(x_0, y_0)}{h_x} \\ &= u_x + iv_x \end{aligned}$$

Now let's take the limit as h approaches 0 along the imaginary axis, i.e. $h = 0 + i \cdot h_y$.

$$\begin{aligned} f'(z_0) &= \lim_{h_y \to 0} \frac{f(z_0 + ih_y) - f(z_0)}{ih_y} \\ &= \lim_{h_y \to 0} \frac{u(x_0, y_0 + h_y) + iv(x_0, y_0 + h_y) - u(x_0, y_0) - iv(x_0, y_0)}{ih_y} \\ &= \lim_{h_y \to 0} \frac{u(x_0, y_0 + h_y) - u(x_0, y_0)}{ih_y} + \lim_{h_y \to 0} \frac{iv(x_0, y_0 + h_y) - iv(x_0, y_0)}{ih_y} \\ &= \frac{1}{i} u_y + v_y = -iu_y + v_y \end{aligned}$$

For the derivative to exist, both of these limits must exist and both must be the same. Thus we can equate the two:

$$-iu_y + v_y = u_x + iv_x$$

Equating the real and imaginary parts we get:

$$u_x = v_y$$
$$u_y = -v_x$$

$\square$

3.2.2 Sufficient Conditions for Differentiability

Theorem 3.4. *Let $f = u + iv$ be defined on a domain $D \subset \mathbb{C}$. Then f is analytic in D if and only if $u(x, y)$ and $v(x, y)$ have continuous first partial derivatives on D that satisgy the Cauchy-Riemann equations.*

3.3 The Complex Exponential Function

3.3.1 Definition of the Complex Exponential Function

Definition 3.3. *The complex exponential function, e^z, sometimes also denoted $exp(z)$, is defined by*

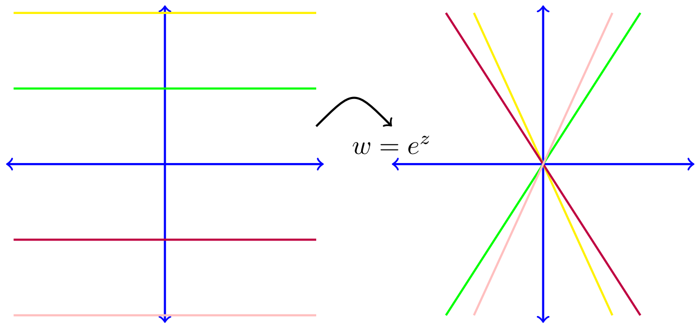
$$e^z = e^x \cdot e^{iy}, \text{ where } z = x + iy$$

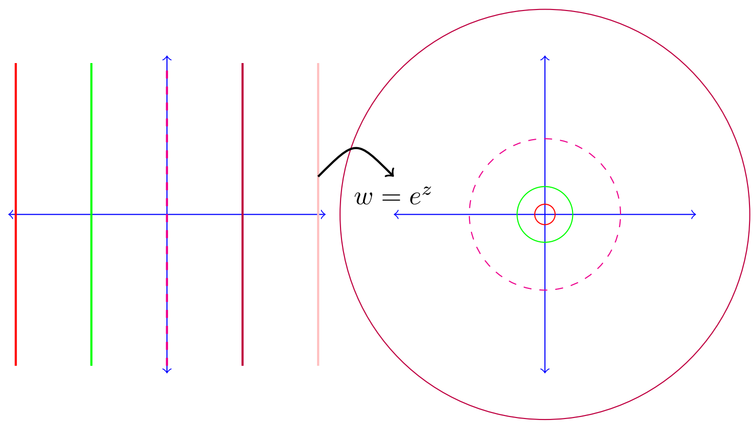
3.3.2 Properties of the Complex Exponential Function

- $|e^z| = |e^x| \left|e^{iy}\right| = e^x$
- $arge^z = arg(e^x e^{iy}) = y$
- $e^{z+2\pi i} = e^x e^{i(y+2\pi)} = e^x e^{iy} = e^z$
- $e^{z+w} = e^z e^w$
- $\frac{1}{e^z} = e^{-z}$
- e^z is an entire function
- $\frac{d}{dz} e^z = e^z$
- $\frac{d}{dz} e^{az} = a \cdot e^{az} (a \in \mathbb{C})$ by the chain rule
- $e^{\overline{z}} = e^{x-iy} = e^x e^{-iy} = e^x \overline{e^{iy}} = \overline{e^x e^{iy}} = \overline{e^z}$
- $e^z = 1 \iff z = 2\pi ik, k \in \mathbb{Z}$
- $e^z = e^w \iff e^{z-w} = 1 \iff z - w = 2\pi ik \iff z = w + w\pi ik$

3.3.3 Understanding the Mapping $w = e^z$

The function $w = e^z$ is a mapping from $\mathbb{C} \to \mathbb{C}$.
Let's map horizontal lines of the form $L = \{x + iy_0 \| x \in \mathbb{R}\}$ for fixed $y_0 \in \mathbb{R}$.
Let's now map vertical lines of the form $L = \{x_0 + iy \| y \in \mathbb{R}\}$ for fixed $x_0 \in \mathbb{R}$





3.4 Complex Trigonometric Functions

3.4.1 Definition

Having extended the exponential function to the complex plane, can the same be done for the trigonometric functions?

Definition 3.4. *The complex cosine and sine dunctions are defined vie*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

3.4.2 Properties

- $\sin z$ and $\cos z$ are analytic functions (in fact, entire).
- For real-valued z (i.e. $z = x + i \cdot 0$) the complex sine and cosine agree with the real-valued sine and cosine functions.
- $\cos(-z) = \frac{e^{-iz}+e^{iz}}{2} = \cos z$.
- $\sin(-z) = \frac{e^{-iz}-e^{iz}}{2i} = -\sin z$

- $\cos(z+w) = \cos z \cos w - \sin z \sin w$,
  $\sin(z+w) = \sin z \cos w + \cos z \sin w$.
- $\cos(z+2\pi) = \cos z$
  $\sin(z+2\pi) = \sin z$
- $\sin^2 z + \cos^2 z = 1$
- $\sin(z + \frac{\pi}{2} = \cos z$

3.4.3 The Zeros of Sine and Cosine

$\sin z = 0 \iff z = k\pi, k \in \mathbb{Z}$
$\cos z = 0 \iff z = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

3.4.4 Relation to Hyperbolic Functions

$\sin z = \sin x \cosh y + i \cos x \sinh y$
$\cos z = \cos x \cosh y - i \sin x \sinh y$

3.5 First Properties of Analytic Functions

3.5.1 Analytic Functions with Zero Derivative

Theorem 3.5. *If f is analytic on a domain D, and if $f'(z) = 0$ for all $z \in D$, then f is constant in D.*

Remark. *Recall the 1-dimension analog:*
If $f : (a, b) \to \mathbb{R}$ is differentiable and satisfies $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Proof. First step of the proof:

- Let $B_r(a)$ be a disk contained in D, and let $c \in B_r(a)$. Pick the point $b \in B_r(a)$ as in the figure. Since $f'(z) = 0$ in D we have $u_x = u_y = v_x = v_y = 0$ in D. In particular, look at u on the horizontal line segment from a to b. It depends only on one parameter (namely x) there, and $u_x = 0$. By the 1-dimensional fact, we find that u is constant on the line segment, in particular, $u(a) = u(b)$. Similarly, $u(b) = u(c)$, thus $u(a) = u(c)$. Since c was an arbitrary point in $b_r(a)$, u is thus constant in $B_r(a)$. Similarly, v, is constant in $B_r(a)$, thus f is constant in $B_r(a)$.

Here is the second step of the proof:

- Let a and b be two arbitrary points in D. Since D is connected, there exists a nice curve in D, joining a and b. By the previous step, f is constant in the disk around the point a (see picture). Furthermore, f is also constant in the neighbouring disk. Since these two disks overlap, the two constants must agree. Continue on in this matter until you reach b. Therefore, $f(a) = f(b)$. Thus f is constant in D.

$\square$

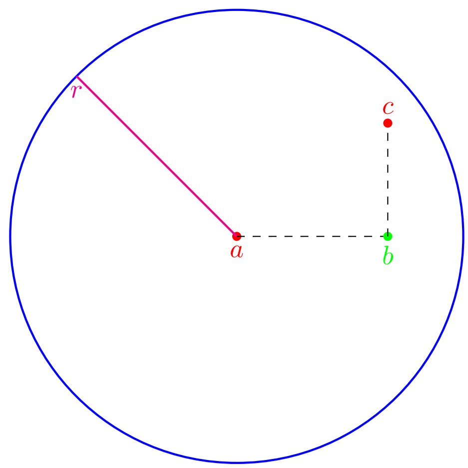


Figure 7: First part of the proof

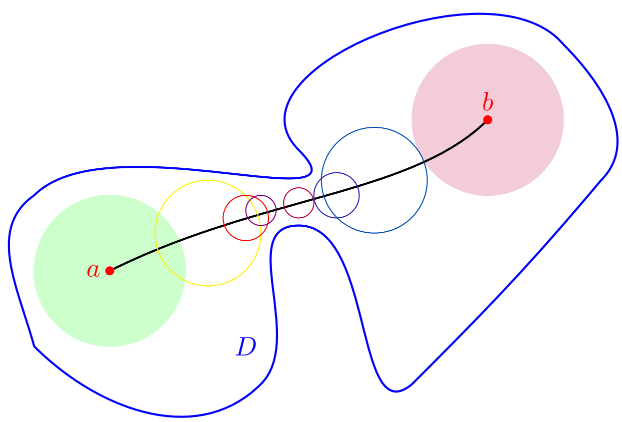


Figure 8: Second Part of the proof

3.5.2 Consequences

The previous theorem, together with the Cauchy-Riemann equations, has strong consequences.

- Suppose that $f = u + iv$ is analytic in a domain D. Suppose furthermore, that u is constant in D. Then f must be constant in D.

  Proof. u constant in D implies that $u_x = u_y = 0$ in D. Since f is analytic, the Cauchy-Riemann equations now imply that $v_x = v_y = 0$ as well. Thus

$f' = u_x + iv_x = 0$ in D. By previous theorem, f is constant in D. □

- Similarly, if $f = u + iv$ is analytic in a domain D with v being constant, then f must be constant in D.
- Suppose next that $f = u + iv$ is analytic in a domain D with $|f|$ being constant in D. This too implies that f itself must be constant

  Proof. Let $f = u + iv$ be analytic in a domain D, and suppose that $|f|$ is constant in D. Then $|f|^2$ is also constant, i.e. there exists $c \in \mathbb{C}$ such that

  $$|f(z)|^2 = u^2(z) + v^2(z) = c \; for \; all \; z \in D.$$

  - If $c = 0$ then u and v must be equal to zero everywhere, and so f is equal to zero in D.
  - If $c \neq 0$ then in fact $c > 0$. Taking the partial derivative with respect to x (and similarly with respect to y) of the above equation yields:

    $$2uu_x + 2vv_x = 0 \; and \; 2uu_y = 2vv_y = 0$$

    Substituting $v_x = -u_y$ in the first and $v_y = u_x$ in the second equation gives

    $$uu_x - vu_y = 0 \; and \; uu_y + vu_x = 0.$$

    Multiplying the first equation by u and the second by v we find

    $$u^2u_x - uvu_y = 0 \; and \; uvu_y + v^2u_x = 0.$$

    Add the two equations.

    $$u^2u_x + v^2u_x = 0$$

    Since $u^2 + v^2 = c$, this last equation becomes

    $$cu_x = 0$$

    But $c > 0$, so it must be the case that $u_x = 0$ in D. We can similarly find that $u_y = 0$ in D, and using the Cauchy-Riemann equations we also obtain that $v_x = v_y = 0$ in D. Hence $f'(z) = 0$ in D, and the theorem yields that f is constant in D.

    Remark. *The assumption of D being connected is important!*

  □

3.6 Inverse Functions of Analytic Functions

3.6.1 The Logarithm Function

Definition 3.5. *For $z \neq 0$ we define*

$$Logz = ln\,|z| + iArgz,\; theprincipalbranchoflogarithm,$$

and

$$\begin{aligned} logz &= ln\,|z| + iargz,\; a\ multi-valued\ function \\ &= Logz + 2k\pi i, k \in \mathbb{Z}. \end{aligned}$$

3.6.2 Continuity of the Logarithm Function

Notice:

- $z \mapsto |z|$ is continuous in $\mathbb{C}$.
- $z \mapsto ln\,|z|$ is continuous in $\mathbb{C}\backslash\{0\}$.
- $z \mapsto Argz$ is continous in $\mathbb{C}\backslash(-\infty, 0]$
- Thus, $Logz$ is continuous in $\mathbb{C}\backslash(-\infty, 0]$.
- However, as $z \to -x \in (-\infty, 0)$ from above, $Logz \to lnx + i\pi$, and as $z \to -x$ from below, $Logz \to lnx - i\pi$, so $Logz$ is not continuous on $(-\infty, 0)$ (and not defined at 0).

Fact: The principal branch of logarithm, $Logz$, is analytic in $\mathbb{C}\backslash(-\infty, 0]$.

3.6.3 Theorem about Finding Derivatives of Functions

Theorem 3.6. *Suppose that $f : U \to \mathbb{C}$ is an analytic function and there exists a continuous function $g : D \to U$ from some domain $D \in \mathbb{C}$ into U such that $f(g(z)) = z$ for all $z \in D$. Then g is analytic in D, and*

$$g'(z) = \frac{1}{f'(g(z))}\; for\ z \in D.$$

3.6.4 Application 1

Let $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2$. Then $f'(z) = 2z$.
Let $g : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}, g(z) = \sqrt{z}$ be the principal branch of the square root. Then

- $f(g(z)) = z$ for all $z \in D = \mathbb{C}\backslash(-\infty, 0]$
- g is continuous in D, thus g is analytic in D, and

$$\begin{aligned} g'(z) &= \frac{1}{f'(g(z))} \\ &= \frac{1}{2\sqrt{z}} \end{aligned}$$

3.6.5 Some Terminology

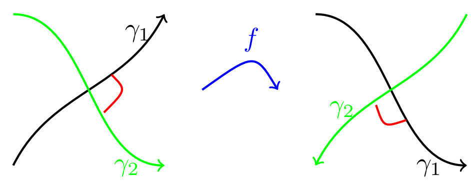
Let $f : U \to V$ be a function.

- f is injective (also called 1-1) provided that $f(a) \neq f(b)$ whenever $a, b \in U$ with $a \neq b$.
- f is surjective (also called onto) provided that for every $y \in V$ there exists and $x \in U$ such that $f(x) = y$.
- f is a bijection (also called 1-1 and onto) if f is both injective and surjective.

3.7 Conformal Mappings

3.7.1 What is a Conformal Mapping

Inuitively, a conformal mapping is a mapping that preserves the angles between curves.



3.7.2 Paths

Definition 3.6. *A path in the complex plane from a point A to a point B is a continuous function $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(a) = A$ and $\gamma(b) = B$.*

3.7.3 Curves

Definition 3.7. *A path $\gamma : [a, b] \to \mathbb{C}$ is smooth if the functions $x(t)$ and $y(t)$ in the representation $\gamma(t) = x(t) + iy(t)$ are smooth, that is, have as many derivatives as desired.*

3.7.4 The Angle Between Curves

Definition 3.8. *Let γ_1 and γ_2 be two smooth curves, intersecting at a point z_0. The angle between the two curves at z_0 is defined as the angle between the two tangent vectors at z_0.*

3.7.5 Conformality

Definition 3.9. *A function is conformal if it preserves angles between curves. More precisely, a smooth complex-valued function g is conformal at z_0 if whenever γ_1 and γ_2 are two curves that intersect at z_0 with non-zero tangents, then $g \circ \gamma_1$ and $g \circ \gamma_2$ have non-zero tangents at $g(z_0)$ that intersect at the same angle.*
A conformal mapping of a domain D onto V is a continuously differentiable mapping that is conformal at each point in D and maps D one-to-one onto V.

3.7.6 Analytic Functions

Theorem 3.7. *If $f : U \to \mathbb{C}$ is analytic and if $z_0 \in U$ such that $f'(z_0) \neq 0$, then f is conformal at z_0.*

3.8 Mobius Transformations

3.8.1 Mobius Transformations

Definition 3.10. *A Mobius transformation (also called fractional linear transformation) is a function of the form*

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Notes:

- As $z \to \infty$, $f(z) \to \frac{a}{c}$ if $c \neq 0$ and $f(z) \to \infty$ if $c = 0$.
  That's why $z = \infty$ is allowed and $f(\infty) = \frac{a}{c}$ is defined if $c \neq 0$ and $f(\infty) = \infty$ if $c = 0$.
- Similarly, $f(-\frac{d}{c}) = \infty$, if $c \neq 0$.
- We thus regard f as a mapping from the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to the extended plane $\hat{\mathbb{C}}$.

3.8.2 Properties of $f(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc \neq 0$

- $f'(z) = \frac{(cz+d)a-(az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$. The condition $ad - bc \neq 0$ thus simply guarantees that f is non-constant.
- If we multiply each parameter of a, b, c, d by a constant $k \neq 0$, we obtain the same mapping, so a given mapping does not uniquely determine a, b, c, d.
- A Mobius transformation is one-to-one and onto from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

Proof. Pick $w \in \hat{\mathbb{C}}$ and observe

$$\begin{aligned} f(z) = w &\iff az + b = w(cz + d) \\ &\iff z(a - wc) = wd - b \\ &\iff z = \frac{wd - b}{-wc + a} \end{aligned}$$

For each $w \in \hat{\mathbb{C}}$ there is one and only one $z \in \hat{\mathbb{C}}$ such that $f(z) = w$. $\square$

3.8.3 Conformality of $f(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc \neq 0$

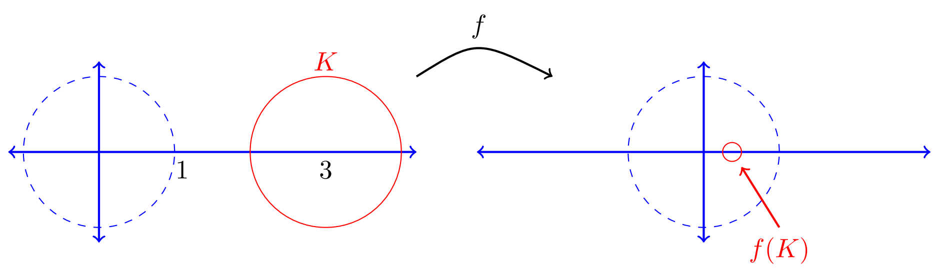
Mobius transformations are thus conformal mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. In fact, Mobius transformations are the only conformal mappings from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

3.8.4 The Inversion $f(z) = \frac{1}{z}$

Let $K = \{z : |z - 3| = 1\}$ be the circle of radius 1, centered at 3.

$$\begin{aligned} w \in f(K) \iff \frac{1}{w} \in K &\iff \left|\frac{1}{w} - 3\right| = 1 \\ &\iff |1 - 3w|^2 = |w|^2 \\ &\iff \left|w - \frac{3}{8}\right| = \frac{1}{8} \end{aligned}$$

Thus the image of the circle $K = \{z : |z - 3| = 1\}$ under $f(z) = \frac{1}{z}$ is another circle, namely the circle of radius $\frac{1}{8}$, centered at $\frac{3}{8}$.



3.8.5 Facts About Mobius Transformations

Every Mobius transformation maps circles and lines to circles or lines.
Given three distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, there exists a unique Mobius transformation f such that $f(z_1) = 0$, $f(z_2) = 1$, and $f(z_3) = \infty$.
This transformation can be written down:

$$f(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

3.8.6 Further Facts

- The composition of two Mobius transformations is a Mobius transformation, and so is the inverse.

- Given three distinct points z_1, z_2, z_3 and three distinct points w_1, w_2, w_3, there exists a unique Mobius transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that maps z_j to $w_j, j = 1, 2, 3$.

  Proof. Let f_1 be the Mobius transformation that maps z_1, z_2, z_3 to $0, 1, \infty$. Let f_2 be the Mobius transformation that maps w_1, w_2, w_3 to $0, 1, \infty$ Then $f_2^{-1} \circ f_1$ maps z_1, z_2, z_3 to w_1, w_2, w_3, respectively. $\square$

3.8.7 Compositions of Mobius transformations

Every Mobius transformation is the composition of maps of the type

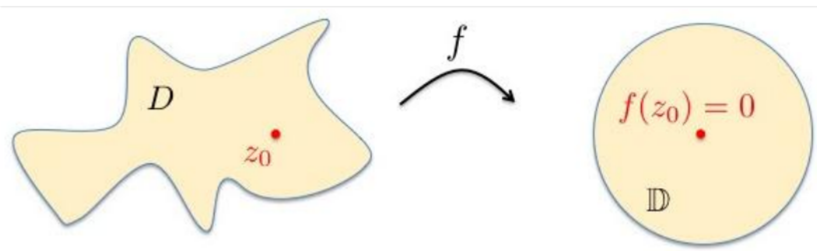
1. $z \longmapsto az$ (rotation and dilation)
2. $z \longmapsto z + b$ (translation)
3. $z \longmapsto \frac{1}{z}$ (inversion)

3.9 The Riemann Mapping Theorem

3.9.1 The Theorem

What conformal mappings are there of the form $f : \mathbb{D} \to D$, where $\mathbb{D} = B_1(0)$ is the unit disk and $D \subset \mathbb{C}$?

Theorem 3.8. *If D is a simply connected domain (open, connected, no holes) in the complex plane, but not the entire complex plane, then there is a conformal map (analytic, one-to-one, onto) of D onto the open unit disk $\mathbb{D}$.*



4 Complex Integration

4.1 Complex Integration

4.1.1 The Fundamental Theorem of Calculus

Theorem 4.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous, and define $F(x) = \int_a^x f(t)dt$. Then F is differentiable and $F'(x) = f(x)$ for $x \in [a, b]$.*

4.1.2 Generalization to $\mathbb{C}$

Integration in $\mathbb{C}$ is happening on curves.

$$\gamma : [a, b] \to \mathbb{C}, \gamma(t) = x(t) + iy(t).$$

If f is complex valued on γ, we define

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j)$$

where $z_j = \gamma(t_j)$ and $a = t_0 < t_1 < ... < t_n = b$.
It can be shown that if $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve and f is continuous on γ, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Proof.

$$\begin{aligned}\sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j) &= \sum_{j=0}^{n-1} f(\gamma(t_j))\frac{\gamma(t_{j+1}) - \gamma(t_j)}{t_{j+1} - t_j}(t_{j+1} - t_j) \\ &\to \int_a^b f(\gamma(t))\gamma'(t)dt \text{ as } n \to \infty\end{aligned}$$

$\square$

4.1.3 Independence of Parametrization

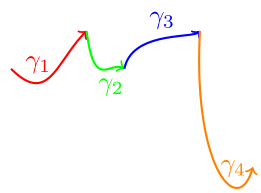
Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve, and let $\beta : [c, d] \to \mathbb{C}$ be another smooth parametrization of the same curve, given by $\beta(s) = \gamma(h(s))$, where $h : [c, d] \to [a, b]$ is a smooth bijection.
Let f be a complex-valued function, defined on γ. Then

$$\begin{aligned}\int_\beta f(z)dz &= \int_c^d f(\beta(s))\beta'(s)ds \\ &= \int_c^d f(\gamma(h(s)))\gamma'(h(s))h'(s)ds \\ &= \int_a^b f(\gamma(t))\gamma'(t)dt = \int_\gamma f(z)dz.\end{aligned}$$

4.1.4 Piecewise Smooth Curves

Let $\gamma = \gamma_1 + \gamma_2 + ... + \gamma_n$ be a piecewise smooth curve.
Then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + ... + \int_{\gamma_n} f(z)dz.$$



4.1.5 Reverse Paths

If $\gamma : [a, b] \to \mathbb{C}$ is a curve, then a curve $(-\gamma) : [a, b] \to \mathbb{C}$ is defined by

$$(-\gamma)(t) = \gamma(a + b - t).$$

If f is continuous and complex-valued on γ, then

$$\int_{(-\gamma)} f(z)dz = -\int_\gamma f(z)dz$$

4.1.6 Linearity of the Path Integral

If γ is a curve, c a complex constant and f, g are continuous and complex-valued on γ, then

- $\int_\gamma (f(z) + g(z))dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz.$
- $\int_\gamma (cf)(z)dz = c\int_\gamma f(z)dz.$
- $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$

4.1.7 Arc Length

Given a curve $\gamma : [a, b] \to \mathbb{C}$, how can we find its length?
Let $a = t_0 < t_1 < ... < t_n = b$. Then

$$length(\gamma) = \lim_{n\to\infty} \sum_{j=0}^{n} \left|\gamma(t_{j+1} - \gamma(t_j)\right|.$$

How to actually calculate this?

$$\sum_{j=0}^{n} \left|\gamma(t_{j+1} - \gamma(t_j)\right| = \sum_{j=0}^{n} \frac{|\gamma(t_{j+1}) - \gamma(t_j)|}{t_{j+1} - t_j}(t_{j+1} - t_j) \to \int_a^b |\gamma'(t)|\, dt$$

Thus:

$$length(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

4.1.8 Integration With Respect To Arc Length

Definition 4.1. *Let γ be a smooth curve, and let f be a complex-valued and continuous function on γ. Then*

$$\int_\gamma f(z)\, |dz| = \int_a^b f(\gamma(t))\, |\gamma'(t)|\, dt$$

is the integral of f over γ with respect to arc length.

4.1.9 The ML-Estimate

Theorem 4.2. *If γ is a curve and f is continuous on γ then*

$$\left| \int_\gamma f(z) dz \right| \leq \int_\gamma |f(z)|\, |dz|\,.$$

In particular, if $|f(z)| \leq M$ on γ, then

$$\left| \int_\gamma f(z) dz \right| \leq M \cdot length(\gamma).$$

4.2 The Fundamental Theorem of Calculus for Analytic Functions

4.2.1 Antiderivatives and Primitives

Definition 4.2. *If $f : [a, b] \to \mathbb{R}$ is continuous and has an antiderivative $F : [a, b] \to \mathbb{R}$, then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

Is there a complex equivalent?

Definition 4.3. *Let $D \subset \mathbb{C}$ be a domain, and let $f : D \to \mathbb{C}$ be a continuous function. A primitive of f on D is an analytic function $F : D \to \mathbb{C}$ such that $F' = f$ on D.*

4.2.2 Functions with Primitives

Theorem 4.3. *If f is continuous on a domain D and if f has a primitive F in D, then for any curve $\gamma : [a, b] \to D$ we have that*

$$\int_\gamma f(z) dz = F(\gamma(b)) - F(\gamma(a)).$$

Remark.

- *The integral only depends on the initial point and the terminal point of γ!*
- *Big 'hidden' assumption: f needs to have a primitive in D!*

4.2.3 When Does f Have a Primitive?

Theorem 4.4 (Goursat)**.** *Let D be a simply connected domain in $\mathbb{C}$, and let f be analytic in D. Then f has a primitive in D. Moreover, a primitive is given explicitly by picking $z_0 \in D$ and letting*

$$F(z) = \int_{z_0}^{z} f(w)dw.$$

where the integral is taken over an arbitrary curve in D from z_0 to z.

One way to prove this theorem is as follows:

1. First, show Morera's Theorem: if f is continuous on a simply connected domain D, and if $\int_\gamma f(z)dz = 0$ for any triangular curve γ in D, then f has a primitive in D.
2. Next, show the Cauchy Theorem for Triangles: for any triangle T that fits into D (including its boundary), $\int_{\partial T} f(z)dz = 0$.

4.2.4 The Cauchy Theorem for Triangles

Theorem 4.5 (Cauchy's Theorem for Triangles)**.** *Let D be an open set in $\mathbb{C}$, and let f be analytic in D. Let T be a triangle that fits into D (including its boundary), and let ∂T be its boundary, oriented positively. Then*

$$\int_{\partial T} f(z)dz = 0.$$

Proof. Assume

$$\left|\int_{\partial T} f\right| = c \geq 0.$$

It will be shown that $c = 0$.
First, subdivide T into four triangles, marked T^1, T^2, T^3, T^4 by joining the midpoints on the sides. Then it is true that

$$\int_{\partial T} f = \sum_{r=1}^{4} \left|\int_{T^r} f\right|$$

Choose r such that

$$\left|\int_{\partial T^r} f\right| \geq \frac{1}{4}c.$$

Defining T^r as T_1, then

$$\left|\int_{\partial T_1} f\right| \geq \frac{1}{4}c \ and\ L(\partial T_1) = \frac{1}{2}L(\partial T)$$

(where $L(\gamma)$ describes length of the curve).
Repeat this process of subdivision to get a sequence of triangles

$$T \supset T_1 \supset T_2 \supset ... \supset ...T_n \supset ...$$

satisfying that

$$\left|\int_{\partial T_n} f\right| \geq (\frac{1}{4})^n c \ and\ L(\partial T_n) = (\frac{1}{2})^n L(\partial T).$$

Claim: The nested sequence $\overline{T} \supset \overline{T_1} \supset \overline{T_2} \supset ... \supset \overline{T_n} \supset ...$ contains a point $z_0 \in \cap_{n=1}^{\infty}\overline{T_n}$. On each step choose a point $z_n \in T_n$. Then it is easy to show that (z_n) is a Cauchy sequence. Then (z_n) converges to a point $z_0 \in \cap_{n=1}^{\infty}\overline{T_n}$ since each of the $\overline{T_n}$ are closed, hence, proving the claim.
We can generate another estimate of c using the fact that f is differentiable. Since f is differentiable at z_0, for a given $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \ implies\ \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

which can be rewritten as

$$0 < |z - z_0| < \delta \ implies\ |f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon\,|z - z_0|$$

For $z \in T_n$ we have $|z - z_0| < L(\partial)$, and so by the Estimation Lemma we have that

$$\left|\int_{\partial T_n} \{f(z) - [f(z_0) + f'(z_0)(z - z_0)]\}dz\right| \leq \epsilon L^2(\partial)$$

As $f(z_0) + f'(z_0)(z - z_0)$ is of the form $\alpha z + \beta$ it has an antiderivative in D, and so $\int_{\partial T_n} f(z_0) + f'(z_0)(z - z_0) = 0$, and the above is then just

$$\left|\int_{\partial T_n} f(z)dz\right| \leq \epsilon L^2(\partial)$$

Notice that

$$(\frac{1}{4})^n c \leq \left|\int_{\partial T_n} f(z)dz\right| \leq \epsilon L^2(\partial) = (\frac{1}{4})^n \epsilon L^2(\partial)$$

Giving

$$c \leq \epsilon L^2(\partial).$$

Since $\epsilon > 0$ can be chosen arbitrary small, then $c = 0$. $\square$

4.2.5 Morera's Theorem

Theorem 4.6 (Morera's Theorem)**.** *If f is contiunous on a simply connected domain D, and if $\int_\gamma f(z)dz = 0$ for any triangular curve in D, then f has a primitive in D.*

Proof. Without loss of generality, it can be assumed that D is connected. Fix a point z_0 in D, and for any $z \in D$, let $\gamma : [0,1] \to D$ be a piecewise C^1 curve such that $\gamma(0) = z_0$ and $\gamma(1) = z$. Then define the function F to be

$$F(z) = \int_\gamma f(\zeta)d\zeta.$$

To see that the function is well-defined, suppose $\tau : [0,1] \to D$ is another piecewise C^1 curve such that $\tau(0) = z_0$ and $\tau(1) = z$. The curve $\gamma\tau^{-1}$ (i.e. the curve combining γ with τ in reverse) is a closed piecewise C^1 curve in D. Then,

$$\int_\gamma f(\zeta)d\zeta + \int_{\tau^{-1}} f(\zeta)d\zeta = \oint_{\gamma\tau^{-1}} f(\zeta)d\zeta = 0$$

And it follows that

$$\int_\gamma f(\zeta)d\zeta = \int_\tau f(\zeta)d\zeta.$$

Then using the continuity of f to estimate difference quotients, we get that $F'(z) = f(z)$. Had we chosen a different z_0 in D, F would change by a constant: namely, the result of integrating f along any piecewise regular curve between the new z_0 and the old, and this does not change the derivative.
Since f is the derivative of the holomorphic function F, it is holomorphic. The fact that derivatives of holomorphic functions are holomorphic can be proved by using the fact that holomorphic functions are analytic, i.e. can be represented by a convergent power series, and the fact that power series may be differentiated term by term. □

4.3 Cauchy's Theorem and Integral Formula

4.3.1 Cauchy's Theorem

Theorem 4.7 (Cauchy's Theorem for Simply Connected Domains)**.** *Let D be a simply connected domain in $\mathbb{C}$, and let f be analytic in D. Let $\gamma : [a,b] \to D$ be apiecewise smooth, closed curve in D (i.e. $\gamma(b) = \gamma(a)$). Then*

$$\int_\gamma f(z)dz = 0.$$

Proof. If one assumes that the partial derivatives of a holomorphic function are continuous, the Cauchy integral theorem can be proved as a direct consequence of Green's theorem and the fact that the real and imaginary parts of $f = u + iv$ must satisfy the Cauchy-Riemann equations in the region bounded by γ, and

moreover in the open neighbourhood U of this region. Cauchy provided this proof, but it was later proved by Goursat without requiring techniques from vector calculus, or the continuity of partial derivatives.
We can break the integrand f, as well as the differential dz into their real and imaginary components:

$$f = u + iv$$
$$dz = dx + idy$$

In this case we have

$$\oint_\gamma f(z)dz = \oint_\gamma (u+iv)(dx+idy) = \oint_\gamma (udx - vdy) + i\oint_\gamma (vdx + udy)$$

By Green's theorem, we may then replace the integrals around the closed contour γ with an area integral throughout the domain D that is enclosed by γ as follows:

$$\oint_\gamma (udx - vdy) = \iint_D (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy$$
$$\oint_\gamma (vdx + udy) = \iint_D (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy$$

However, being the real and imaginary parts of a functionholomorphic in the domain D, u and v must satisfy the Cauchy-Riemann equations there:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We therefore find that both integrands (and hence their integrals) are zero

$$\iint_D (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy = \iint_D (\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y})dxdy = 0$$
$$\iint_D (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy = \iint_D (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x})dxdy = 0$$

This gives the desired result

$$\oint_\gamma f(z)dz = 0$$

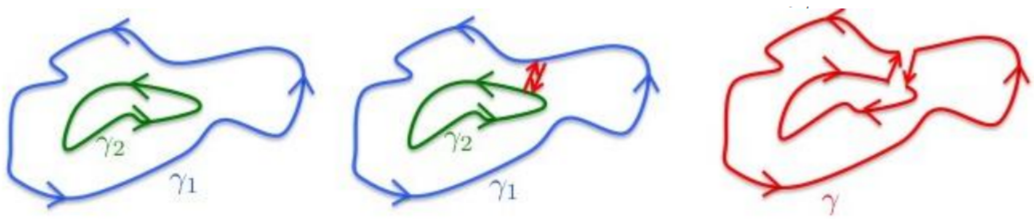
□

4.3.2 A first Conclusion of the Theorem

Corollary 4.7.1. *Let γ_1 and γ_2 be two simple closed curves (i.e. neither of the curves intersects itself), oriented counterclockwise, where γ_2 is inside γ_1. If f is analytic in a domain D that contains both curves as well as the region between them, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof. Take the two curves and form a 'joint curve' γ as in the picture below. As f is analytic in a simply connected region, containing γ, we have $\int_\gamma f(z)dz = 0$. ∎



4.3.3 The Cauchy Integral Formula

Theorem 4.8 (Cauchy Integral Formula)**.** *Let D be a simply connected domain, bounded by a piecewise smooth curve γ, and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then*

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz$$

for all $w \in D$.

Proof. By using the Cauchy integral theorem, one can show that the integral over C (or the closed rectifiable curve) is equal to the same integral taken over an arbitrarily small circle around a. Since $f(z)$ is continuous, we can choose a circle small enough on which $f(z)$ is arbitrarily close to $f(a)$. On the other hand, the integral

$$\oint_C \frac{1}{z-a}dz = 2\pi i$$

over any circle centered at a. This can be calculated directly via a parametrization (integration by substitution) $z(t) = a + \epsilon e^{it}$ where $0 \le t \le 2\pi$ and ϵ is the radius of the circle.
Letting $\epsilon \to 0$ gives the desired estimate

$$\begin{aligned}
\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz - f(a)\right| &= \left|\frac{1}{2\pi i}\oint_C \frac{f(z)-f(a)}{z-a}dz\right| \\
&= \left|\frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{f(z(t))-f(a)}{\epsilon e^{it}}\cdot \epsilon e^{it} i\right)dt\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(z(t))-f(a)|}{\epsilon}\epsilon dt \\
&\le \max_{|z-a|=\epsilon}|f(z)-f(a)| \to 0 \; as \; \epsilon \to 0
\end{aligned}$$

∎

4.3.4 Analyticity of the Derivative

Theorem 4.9. *If f is analytic in an open set U, then f' is also analytic in U.*

Idea of the proof:

- Use the Cauchy Integral Formula to show that for any $w \in U$, the derivative $f'(w)$ can be found via

  $$f'(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2} dz,$$

  where γ is the boundary of a small disk, centered at w; small enough so that it fits into U.

- Next, show that the right-hand side defines an analytic function in w, and therefor f' must be analytic.

4.3.5 The Cauchy Integral Formula for Derivatives

Repeated application of the previous theorem shows that an analytic function has infinitely many derivatives.

Theorem 4.10 (Cauchy Integral Formula for Derivatives)**.** *Let D be a simply connected domain, bounded by a piecewise smooth curve γ, and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then*

$$f^{(k)}(w) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{k+1}} dz \text{ for all } w \in D, k \geq 0.$$

4.4 Consequences of Cauchy's Theorem and Integral Formula

4.4.1 Cauchy's Estimate

Theorem 4.11. *Suppose that f is analytic in an open set that contains $\overline{B_r(z_0)}$, and that $|f(z)| \leq m$ holds on $\partial B_r(z_0)$ for some constant m. Then for all $k \geq 0$,*

$$\left| f^{(k)}(z_0) \right| \leq \frac{k!m}{r^k}.$$

Proof. By the Cauchy Integral Formula we have that

$$\begin{aligned} \left| f^{(k)}(z_0) \right| &= \frac{k!}{2\pi} \left| \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{(k+1)}} dz \right| \leq \frac{k!}{2\pi} \int_{|z-z_0|=r} \frac{|f(z)|}{|z-z_0|^{k+1}} |fz| \\ &\leq \frac{k!m}{2\pi r^{k+1}} \cdot 2\pi r = \frac{k!m}{r^k}. \end{aligned}$$

□

4.4.2 Liouville's Theorem

Theorem 4.12 (Liouville)**.** *Let f be analytic in the complex plane (thus f is an entire function). If f is bounded then f must be constant.*

Proof. Suppose that $|f(z)| \leq m$ for all $z \in \mathbb{C}$. Pick $z_0 \in \mathbb{C}$. Since $\mathbb{C}$ contains $\overline{B_r(z_0)}$ for any $r > 0$, we obtain from Cauchy's estimate:

$$|f'(z_0)| \leq \frac{m}{r}$$

for any $r > 0$. Letting $r \to \infty$ we find that $f'(z_0) = 0$. Since z_0 was arbitrary, $f'(z) = 0$ for all z, hence f is constant. □

4.4.3 Use of Lioville's Theorem to Prove Fundamental Theorem of Algebra

Theorem 4.13 (Fundamental Theorem of Algebra)**.** *Any polynomial $p(z) = a_0 + a_1 z + ... + a_n z^n$ (with $a_0, ..., a_n \in \mathbb{C}, n \geq 1$ and $a_n \neq 0$) has a zero in $\mathbb{C}$, i.e. there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. Suppose to the contrary that there exists a polynomial p as in the theorem that has no zeros. Then $f(z) = \frac{1}{p(z)}$ is an entire function! Apply Liouville's theorem to f!

$$p(z) = z^n(a_n + \frac{a_{n-1}}{z} + ... + \frac{a_0}{z^n})$$

, so

$$|p(z)| \geq |z|^n (|a_n| - \frac{|a_{n-1}|}{|z|} - ... - \frac{|a_0|}{|z|^n}) \to \infty \ as \ |z| \to \infty.$$

Thus $|f(z)| \to 0$ as $|z| \to \infty$, and so f is bounded in $\mathbb{C}$. By Liouville, f is constant, and so p is constant. This is a contradiction. □

4.4.4 The Maximum Principle

Another consequence of the Cauchy Integral Formula is

Theorem 4.14 (Maximum Principle)**.** *Let f be analytic in a domain D and suppose there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Then f is constant in D.*

Remark. *If $D \subset \mathbb{C}$ is a bounded domain, and if $f : \overline{D} \to \mathbb{C}$ is continuous in $\overline{D}$ and analytic in D, then $|f|$ reaches its maximum on ∂D.*

5 Power Series

5.1 Infinite Series of Complex Numbers

5.1.1 Infinite Series

Definition 5.1. *An infinite series*

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + ... + a_n + a_{n+1} + ...$$

(with $a_k \in \mathbb{C}$) converges to S if the sequence of partial sums $\{S_n\}$, given by

$$S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + ... + a_n$$

converges to S.

Theorem 5.1. *If a series $\sum_{k=0}^{\infty} a_k$ converges then $a_k \to 0$ as $k \to \infty$.*

5.1.2 Absolute Convergence

Definition 5.2. *A series $\sum_{k=0}^{\infty} a_k$ converges absolutely if the series $\sum_{k=0}^{\infty} |a_k|$ converges.*

Examples:

- $\sum_{k=0}^{\infty} z^k$ converges and converges absolutely for $|z| < 1$.
- $\sum_{k=1}^{\infty} \frac{i^k}{k}$ converges, but not absolutely.

Theorem 5.2. *If $\sum_{k=0}^{\infty} a_k$ converges absolutely, then it also converges, and $|\sum_{k=0}^{\infty} a_k| \leq \sum_{k=0}^{\infty} |a_k|$.*

5.2 Power Series

5.2.1 Taylor Series

Definition 5.3. *A power series (also called Taylor series), centered at $z_0 \in \mathbb{C}$, is a series of the form*

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

5.2.2 The Radius of Convergence

Theorem 5.3. *Let $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ be a power series. Then there exists a number R, with $0 \leq R \leq \infty$, such that the series converges absolutely in $\{|z - z_0| < R\}$ and diverges in $\{|z - z_0| > R\}$. Furthermore, the convergence is uniform in $\{|z - z_0| \leq r\}$ for each $r < R$.*

5.2.3 Analyticity of Power Series

Theorem 5.4. *Suppose that* $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ *is a power series of radius of convergence* $R > 0$. *Then*

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k \text{ is analytic in } \{|z-z_0| < R\}$$

Furthermore, the series can be differentiated term by term, i.e.

$$f'(z) = \sum_{k=1}^{\infty} a_k \cdot k(z-z_0)^{k-1}, f''(z) = \sum_{k=2}^{\infty} a_k \cdot k(k-1)(z-z_0)^{k-2}, \ldots$$

In particular, $f^{(k)}(z_0) = a_k \cdot k!$, *i.e.* $a_k = \frac{f^{(k)}(z_0)}{k!}$ *for* $k \geq 0$.

5.2.4 Integration of Power Series

If $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ has radius of convergence R, then for any w with $|w-z_0| < R$ we have that

$$\int_{z_0}^{w} \sum_{k=0}^{\infty} a_k(z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int_{z_0}^{w} (z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \frac{1}{k+1}(w-z_0)^{k+1}.$$

Here, the integral is taken over any curve in the disk $\{|z-z_0| < R\}$ from z_0 to w.

5.3 The Radius of Convergence of a Power Series

5.3.1 The Ratio Test

Theorem 5.5 (Ratio Test)**.** *If the sequence* $\{\left|\frac{a_k}{a_{k+1}}\right|\}$ *has a limit as* $k \to \infty$ *then this limit is the radius of convergence,* R, *of the power series* $\sum_{k=0}^{\infty} a_k(z-z_0)^k$.

5.3.2 The Root Test

Theorem 5.6 (Root Test)**.** *If the sequence* $\{\sqrt[k]{|a_k|}\}$ *has a limit as* $k \to \infty$ *then* $R = \frac{1}{\lim_{k\to\infty}\{\sqrt[k]{|a_k|}\}}$.

Remark.

- *If* $\lim_{k\to\infty}\{\sqrt[k]{|a_k|}\} = 0$ *then* $R = \infty$.
- *If* $\lim_{k\to\infty}\{\sqrt[k]{|a_k|}\} = \infty$ *then* $R = 0$.

5.3.3 The Cauchy Hadamard Criterion

Theorem 5.7. *The radius of convergence of the power series* $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ *equals*

$$R = \frac{1}{\lim_{k\to\infty} \sup \sqrt[k]{|a_k|}}.$$

5.3.4 Analytic Functions And Power Series

Theorem 5.8. *Let $f : U \to \mathbb{C}$ be analytic and let $\{|z - z_0| < r\} \subset U$. Then in this disk, f has a power series representation*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, |z - z_0| < r, where\ a_k = \frac{f^{(k)}(z_0)}{k!}, k \geq 0.$$

The radius of convergence of this power series is $R \geq r$.

Corollary 5.8.1. *If f and g are analytic in $\{|z - z_0| < r\}$ and if $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all k, then $f(z) = g(z)$ for all z in $\{|z - z_0| < r\}$.*

5.4 The Riemann Zeta Function And The Riemann Hypothesis

5.4.1 Introduction to the Zeta Function

Recall:

$$\sum_{n=1}^{\infty} \frac{1}{n}\ diverges\ (harmonic\ series),$$

but

$$\sum_{n=1}^{\infty} \frac{1}{n^s}\ converges\ for\ all\ s > 1.$$

This can be seen as follows:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^s} \leq 1 + \int_1^{\infty} \frac{1}{x^s} dx &= 1 + \left. \frac{1}{1-s} \frac{1}{x^{s-1}} \right|_1^{\infty} \\ &= 1 - \frac{1}{1-s} \\ &= \frac{s}{s-1}\ (s > 1) \end{aligned}$$

5.4.2 The Zeta Function

Now consider $s \in \mathbb{C}$ instead of $s \in \mathbb{R}$.

Definition 5.4. *For $s \in \mathbb{C}$ with $\Re(s) > 1$, the zeta function is defined as*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Note that for real s, we have that $n^s = e^{\ln n^s} = e^{s \ln n}$, so we define

$$n^s = e^{s \log n} = e^{s \ln n}\ for\ s \in \mathbb{C}$$

5.4.3 Convergence of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

Since $n^s = e^{s \ln n}$, we have that $|n^s| = \left|e^{s \ln n}\right| = e^{\Re(s) \ln n} = n^{\Re(s)}$. Thus

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{\Re(s)(}},$$

and since $\Re(s) > 1$, the series on the right converges. Thus $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely in $\{\Re(s) > 1\}$.
In fact, the convergence is uniform in $\{\Re(s) \geq r\}$ for any $r > 1$, and this can be used to show that $\zeta(s)$ is analytic in $\{\Re(s) > 1\}$.

5.4.4 Analytic Continuation of the Zeta Function

Theorem 5.9. *The zeta function has an analytic continuation into $\mathbb{C} \setminus \{1\}$, and this continuation satisfies that $\zeta(s) \to \infty$ as $s \to 1$.*

Slightly easier to construct is an extension to the right half plane $\{\Re(s) > 0\}$, minus the point 1, here is the outline:

$$\sum_{n=1}^{N} \frac{1}{n^s} = \int_1^{N+1} \frac{1}{x^s} dx + \sum_{n=1}^{N} \delta_n(s),$$

where

$$\delta_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s} dx.$$

Observe that $\sum_{n=1}^{N} \delta_n(s)$ is analytic in $\{\Re(s) > 0\}$. One can show that $\sum_{n=1}^{N} \delta_n(s)$ converges, as $N \to \infty$, to an analytic function $H(s)$ in $\{\Re(s) > 0\}$. Thus

$$\zeta(s) = \frac{1}{s-1} + H(s) \; holds \; for \; \Re(s) > 1,$$

where $H(s)$ is analytic in $\{\Re(s) > 0\}$. We can therefore use this to define the zeta function in all of $\{\Re(s) > 0\} \setminus \{1\}$. This definition agrees with the original definition in $\{\Re(s) > 1\}$.

5.4.5 The Zeros of the Zeta Function

Of much interest are the zeros of the zeta function, i.e. those $s \in \mathbb{C}$, for which $\zeta(s) = 0$.

Theorem 5.10. *The only zeros of the zeta funciton outside of the strip $\{0 \leq \Re(s) \leq 1\}$ are at the negative even integers, $-2, -4, -6, ...$*

- The zeros $-2, -4, -6, ...$ are often called the 'trivial zeros', and the region to be studied remains the strip $\{\leq \Re(s) \leq 1\}$.
- A key result is that zeta has no zeros on the line $\{\Re(s) = 1\}$, this is an essential fact in the proof of the prime number theorem.
- From the fact that zeta has no zeros on $\{\Re(s) = 1\}$ it can easily be deduced that it has no zeros on $\{\Re(s) = 0\}$ either, via a functional equation.

5.4.6 The Riemann Hypothesis

In his seminal paper in which he proved the analytic continuation of the zeta function to $\mathbb{C} \setminus \{1\}$, Riemann initiated important insights into the distribution of prime numbers. In this paper, he expressed his belief in the veracity of the following:

Conjecture 5.1 (Riemann Hypothesis). *In the strip* $\{0 < \Re(s)\ 1\}$*, all zeros of* ζ *are on the line* $\{\Re(s) = \frac{1}{2}\}$*.*

Much research has been done in attempts to prove this conjecture:

- $\zeta(s)$ has infinitely many zeros in $\{0 < \Re(s) < 1\}$.
- The asymptotic distribution of the zeros of ζ in $\{0 < \Re(s) < 1\}$ is known.
- At least on third of the zeros in $\{0 < \Re(s) < 1\}$ lie on the critical line $\{\Re(s) = \frac{1}{2}\}$.
- Trillions of zeros of zeta have been calculated - so far all of them lie on the critical line.
- Numerical evidence and much research point to the validity of this conjecture, but it is to this day unproved and remains one of the most famous unsolved problems in mathematics.
- The Riemann Hypothesis is on the list of seven 'Millennium Prize Problems' (declared by the Clay Mathematics Institute in 2000). Only one of these has been solved so far (as of summer 2013) - the so-called Poincare Conjecture (by Grigory Perelman).
- The Riemann Hypothesis has strong implications on the distribution of prime numbers and on the growth of many other important arithmetic functions. It would greatly sharpen many number-theoretic results.

5.5 The Prime Number Theorem

5.5.1 The Prime Counting Function

Definition 5.5. *Let* $\pi(x)$ = *number of primes less than or equal to* x*. This funciton is called the prime counting function.*

It is impossible to find an explicit formula fo $\pi(x)$, that's why we study the asymptotic behaviour of $\pi(x)$ as x becomes very large.

5.5.2 The Prime Number Theorem

Theorem 5.11 (Prime Number Theorem).

$$\pi(x) \sim \frac{x}{\ln x} \text{ as } x \to \infty.$$

5.5.3 How is $\zeta(s)$ Related to Prime Numbers?

$$\begin{aligned}
\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + ... \\
&= (1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + ...)(1 + \frac{1}{3^s} + \frac{1}{9^s} + ...)(1 + \frac{1}{5^s} + \frac{1}{25^s} + ...)... \\
&= \prod_p \sum_{k=0}^{\infty} \frac{1}{p^{ks}} \\
&= \prod_p \frac{1}{1 - \frac{1}{p^s}}.
\end{aligned}$$

Note:

- This product formula shows that $\zeta(s) \neq 0$ for $\Re(s) > 1$.
- The key step in the proof of the prime number theorem is that ζ has no zeros on $\{\Re(s) = 1\}$.
- The prime number theorem says that $\pi(x) \sim \frac{x}{\ln x}$, but it doesn't have any information about the difference $\pi(x) - \frac{x}{\ln x}$.
- However, the prime number theorem can also be written as $\pi(x) \sim Li(x)$, where $Li(x) = \int_2^x \frac{1}{\ln t} dt$ is the (offset) logarithmic integral function.
- The proofs of the prime number theorem by Hadamard and de la Vallee Poussin actually show that $\pi(x) = Li(x)+$error term, where the error term grows to infinity at a controlled rate.
- Van Koch (in 1901) was able to give the best possible bounds on the error term, assuming the Riemann hypothesis is true. Schoenfeld (in1976) made this precise and proved that the Riemann hypothesis is equivalent to

  $$|\pi(x) - li(x)| < \frac{\sqrt{x} \ln x}{8\pi},$$

  where $li(x) = \int_0^x \frac{1}{\ln t} dt$ is the (un-offset) logarithmic integral function, related to $Li(x)$ via $Li(x) = li(x) - li(2)$.

The veracity of the Riemann Hypothesis would therefore imply further results about the distribution of prime numbers, in particular, they'd be distributed beautifully regularly about 'expected' locations.

6 Laurent Series and the Residue Theorem

6.1 Laurent Series

6.1.1 Review of Taylor Series

If $f : U \to \mathbb{C}$ is analytic and $\{|z - z_0| < R\} \subset U$ then f has a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \ where \ a_k = \frac{f^{(k)}(z_0)}{k!}, k \geq 0.$$

6.1.2 Laurent Series Expansion

Theorem 6.1. *If $f : U \to \mathbb{C}$ is analytic and $\{r < |z - z_0| < R\} \subset U$ then f has a Laurent series expansion*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \ldots \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$

that converges at each point of the annulus and coverges absolutely and uniformly in each sub annulus $\{s \leq |z - z_0| \leq t\}$, where $r < s < t < R$.

6.1.3 The Coefficients a_k

For a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, r < |z - z_0| < R$$

f may not be defined at z_0, so we need a new approach.

$$a_k = \frac{f^{(k)}(z_0)}{k!} =^{Cauchy} \frac{1}{2\pi i} \int_{|z - z_0| = s} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

Theorem 6.2. *If f is analytic in $\{r < |z - z_0| < R\}$, then*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i} \int_{|z - z_0| = s} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

for any s between r and R and all $k \in \mathbb{Z}$.

6.2 Isolated Singularities of Analytic Functions

6.2.1 Isolated Singularities

Definition 6.1. *A point z_0 is an isolated singularity of f if f is analytic in a punctured disk $\{0 < |z - z_0| < r\}$ centered at z_0.*

- $f(z) = \frac{1}{z}$ has an isolated singularity at $z_0 = 0$.
- $f(z) = \frac{1}{\sin z}$ has isolated singularities at $z_0 = 0, \pm\pi, \pm 2\pi, ...$
- $f(z) = \sqrt{z}$ and $f(z) = \log z$ do not have isolated singularities at $z_0 = 0$ since these functions cannot be defined to be analytic in any punctured disk around 0.
- $f(z) = \frac{1}{z-2}$ has an isolated singularity at $z_0 = 2$.

6.2.2 Classification of Isolated Singularities

Definition 6.2. *Suppose z_0 is an isolated singularity of an analytic function f with Laurent series $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k, 0 < |z - z_0| < r$. Then the singularity z_0 is*

- *removable if $a_k = 0$ for all $k < 0$.*
- *a pole if there exists $N > 0$ so that $a_N \neq 0$ but $a_k = 0$ for all $k < -N$. The index N is the order of the pole.*
- *essential if $a_k \neq 0$ for infinitely many $k < 0$.*

6.2.3 Removable Singularities

Example: $f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ....$ The Laurent series looks like a Taylor series. Taylor series are analytic within their region of convergence. Thus, if we define $f(z)$ to have the value 1 at $z_0 = 0$, then f becomes analytic in $\mathbb{C}$:

$$f(z) = \begin{cases} \frac{\sin z}{z}, & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

The singularity has been removed.

Theorem 6.3 (Riemann's Theorem)**.** *Let z_0 be an isolated singularity of f. Then z_0 is a removable singularity if and only if f is bounded near z_0.*

6.2.4 Poles

Theorem 6.4. *Let z_0 be an isolated singularity of f. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Remark. *If $f(z)$ has a pole at z_0 then $\frac{1}{f(z)}$ has a removable singularity at z_0 (and vice versa).*

6.2.5 Essential Singularities

Example: $f(z) = e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{z^k} = 1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+...$ has an essential singularity at $z_0 = 0$.

Theorem 6.5 (Casorati-Weierstraß). *Suppose that z_0 is an essential singularity of f. Then for every $w_0 \in \mathbb{C}$ there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $f(z_n) \to w_0$ as $n \to \infty$.*

6.2.6 Picard's Theorem

Theorem 6.6 (Picard). *Suppose that z_0 is an esential singularity of f. Then for every $w_0 \to \mathbb{C}$ with at most one exception there exists a sequence $\{z_n\}$ with $z_n \to z_0$ such that $f(z_n) = w_0$.*

6.3 The Residue Theorem

6.3.1 Motivation

Recall: f has an isolated singularity at z_0 if f is analytic in $\{0 < |z - z_0| < r\}$ for some $r > 0$. In that case, f has a Laurent series expansion

$$f(z) = \sum_{-\infty}^{\infty} a_k(z - z_0)^k, 0 < |z - z_0| < r.$$

Observe: if $0 < \rho < r$ then

$$\int_{|z-z_0|=\rho} f(z)dz = \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=\rho} (z - z_0)^k dz.$$

What is $\int_{|z-z_0|=\rho}(z - z_0)^k dz$?

- For $k \neq -1$, the function $h(z) = (z - z_0)^k$ has a primitive, namely $H(z) = \frac{1}{k+1}(z - z_0)^{k+1}$. Therefore, $\int_{|z-z_0|=\rho}(z - z_0)^k dz = 0$ for $k \neq -1$.
- For $k = -1$, the integral is $\int_{|z-z_0|=\rho} \frac{1}{z-z_0}dz$. We can use the Cauchy Integral Formula (or compute this directly) and find $\int_{|z-z_0|=\rho}(z-z_0)^k dz = 2\pi i$ for $k = -1$.

Hence

$$\int_{|z-z_0|=\rho} f(z)dz = \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=\rho} (z - z_0)^k dz = 2\pi i a_{-1}.$$

That is why a_{-1} gets special attention.

6.3.2 The Residue

Definition 6.3. *If f has an isolated singularity at z_0 with Laurent series expansion*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, 0 < |z - z_0| < r,$$

then the residue of f at z_0 is $Res(f, z_0) = a_{-1}$.

6.3.3 The Residue Theorem

Theorem 6.7 (Residue Theorem)**.** *Let D be a simply connected domain, and let f be analytic in D, except for isolated singularities. Let C be a simple closed curve in D (oriented counterclockwise), and let $z_1, ..., z_n$ be those isolated singularities of f that lie inside of C. Then*

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k).$$

6.4 Finding Residues

6.4.1 Residues at Removable Singularities

z_0 is a removable singularity if $a_k = 0$ for all $k < 0$. In particular $a_{-1} = 0$ in that case, so that $Res(f, z_0) = 0$.
Example:

$$\begin{aligned} f(z) = \frac{\sin z}{z} &= \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= 1 - \frac{1}{3!} z^2 + \frac{1}{5!} z^4 - ... \end{aligned}$$

Thus $Res(f, 0) = 0$.

6.4.2 Residues at Simple Poles

z_0 is a simple pole if $a_{-1} \neq 0$ and $a_k = 0$ for all $k \leq -1$.

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

How to isolate a_{-1}?

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + ...,$$

so that

$$Res(f, z_0) = a_{-1} = \lim_{z \to z_0} ((z - z_0)f(z))$$

Example: $f(z) = \frac{1}{z^2+1}$ has a simple pole at $z_0 = i$ (and another one at $-i$).

$$\begin{aligned} Res(\frac{1}{z^2+1}, i) &= \lim_{z\to i}((z-i)\frac{1}{z^2+1}) \\ &= \lim_{z\to i}((z-i)\frac{1}{(z-i)(z+i)}) \\ &= \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i} = -\frac{i}{2}. \end{aligned}$$

6.4.3 Residues at Double Poles

z_0 is a double pole if $a_{-2} \neq 0$ and $a_k = 0$ for all $k \leq -3$.

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + ...$$

How to isolate a_{-1}?

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + ...$$

so that

$$\frac{d}{dz}((z-z_0)^2 f(z)) = a_{-1} + 2a_0(z-z_0) + ...$$

Hence

$$Res(f, z_0) = a_{-1} = \lim_{z\to z_0}\frac{d}{dz}((z-z_0)^2 f(z)).$$

Example: $f(z) = \frac{1}{(z-1)^2(z-3)}$ has a double pole at $z_0 = 1$ (and a simple one at 3).

$$\begin{aligned} Res(\frac{1}{(z-1)^2(z-3)}, 1) &= \lim_{z\to 1}\frac{d}{dz}((z-1)^2\frac{1}{(z-1)^2(z.3)}) \\ &= \lim_{z\to 1}\frac{-1}{(z-3)^2} = -\frac{1}{4}. \end{aligned}$$

6.4.4 Residues at Poles of Order n

z_0 is a pole of order n if $a_{-n} \neq 0$ and $a_k = 0$ for all $k \leq -(n+1)$.

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + ... + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + ...$$

Then

$$Res(f, z_0) = a_{-1} = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}((z-z_0)^n f(z)).$$

6.4.5 More on Residues

Remark. *If $f(z) = \frac{g(z)}{h(z)}$, where g and h are analytic near z_0, and h has a simple zero at z_0, then*

$$Res(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

6.5 Evaluating Integrals via the Residue Theorem

Example:
$\int_{|z|=1} e^{\frac{3}{z}} dz = 2\pi i Res(f,0)$, where $f(z) = e^{\frac{3}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!}(\frac{3}{z})^k$. Thus $Res(f,0) = 3$, so that

$$\int_{|z|=1} e^{\frac{3}{z}} dz = 6\pi i.$$

6.5.1 More Examples

The Residue Theorem can also be used to evaluate real integrals, for example of the following forms:

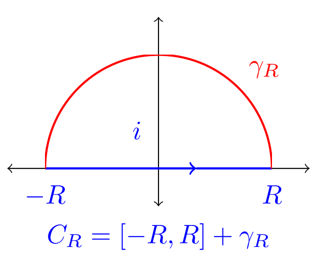
- $\int_0^{2\pi} R(\cos t, \sin t)dt$, where $R(x,y)$ is a rational function of the real variables x and y.
- $\int_{-\infty}^{\infty} f(x)dx$, where f is a rational function of x.
- $\int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx$, where f is a rational function of x.
- $\int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx$, where f is a rational function of x.

6.6 Evaluating an Improper Integral via the Residue Theorem

6.6.1 An Improper Integral

Evaluate $\int_0^{\infty} \frac{\cos x}{1+x^2} dx$.
Idea:

$$\begin{aligned}
\int_0^R \frac{\cos x}{1+x^2} dx &= \frac{1}{2}\int_{-R}^R \frac{\cos x}{1+x^2} dx \\
&= \frac{1}{2}\int_{-R}^R \frac{\cos x + i\sin x}{1+x^2} dx \\
&= \frac{1}{2}\int_{-R}^R \frac{e^{ix}}{1+x^2} dx.
\end{aligned}$$



$$\begin{aligned}
\frac{1}{2}\int_{-R}^{R}\frac{e^{ix}}{1+x^2}dx &= \frac{1}{2}\int_{[-R,R]}\frac{e^{iz}}{1+z^2}dz \\
&= \frac{1}{2}\int_{C_R}\frac{e^{iz}}{1+z^2}dz - \frac{1}{2}\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz \\
&= \frac{1}{2}\cdot 2\pi i Res(\frac{e^{iz}}{1+z^2}, i) - \frac{1}{2}\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz.
\end{aligned}$$

We thus need to find the residue of $f(z) = \frac{e^{iz}}{1+z^2}$ at $z_0 = i$ and estimate the integral over γ_R.

1. Finding the residue of $f(z) = \frac{e^{iz}}{1+z^2}$ at $z_0 = i$:
   - f has a simple pole at $z = 0$.
   - Thus $Res(f, i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{(z-i)e^{iz}}{1+z^2} = \lim_{z\to i}\frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}$.
   - Hence $\frac{1}{2}\int_{C_R}\frac{e^{iz}}{1+z^2}dz = \frac{1}{2}2\pi i\frac{1}{2ie} = \frac{\pi}{2e}$.
2. Estimating $\frac{1}{2}\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz$:
   - We are only interested in what happens as $R\to\infty$.
   - Want to show: $\frac{1}{2}\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz \to 0$ as $R\to\infty$.
   - Therefore, it suffices to show that $\left|\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz\right| \le const(R)$, where the constant goes to zero as $R\to\infty$.
   - $\left|\int_{\gamma_R} f(z)dz\right| \le length(\gamma_R)\cdot \max_{z\in\gamma_R}|f(z)|$.
   - $\left|\frac{e^{iz}}{1+z^2}\right| = \frac{e^{\Re(iz)}}{|1+z^2|} = \frac{e^{-y}}{|1+z^2|} \le \frac{e^{-y}}{R^2-1} \le \frac{1}{R^2-1}$ for $z\in\gamma_R$, since $|z| = R$ and $y\ge 0$ on γ_R.
   - $\left|\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz\right| \le \pi R\cdot\frac{1}{R^2-1} \to 0$ as $R\to\infty$.
   - Thus $\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz \to 0$ as $R\to\infty$.

Conclusion
To find $\int_0^\infty \frac{\cos x}{1+x^2}dx$ here is what we have:

1. $$\int_0^\infty \frac{\cos x}{1+x^2}dx = \lim_{R\to\infty}\int_0^R\frac{\cos x}{1+x^2}dx$$
2. $$\int_0^R \frac{\cos x}{1+x^2}dx = \frac{1}{2}\int_{C_R}\frac{e^{iz}}{1+z^2}dz - \frac{1}{2}\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz$$

3.

$$\frac{1}{2}\int_{C_R}\frac{e^{iz}}{1+z^2}dz = \frac{1}{2}\cdot 2\pi i Res(\frac{e^{iz}}{1+z^2}, i) = \frac{\pi}{2e}$$

4.

$$\int_{\gamma_R}\frac{e^{iz}}{1+z^2}dz \to 0 \; as \; R \to \infty.$$

Hence $\int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}$.