2. $L \neq 0$. When we do not neglect the length of the bike as compared to the rate of change in the road profile, we obtain the equations

$$\begin{aligned}\frac{M}{2}\frac{d^2}{dt^2}y_1 &= -K(y_1 - h_1) - D(\frac{d}{dt}y_1 - \frac{d}{dt}h_1),\\ \frac{M}{2}\frac{d^2}{dt^2}y_2 &= -K(y_2 - h_2) - D(\frac{d}{dt}y_2 - \frac{d}{dt}h_2),\\ y &= \frac{1}{2}(y_1 + y_2), \quad h_1(t) = H(Vt), \quad h_2(t) = h(Vt - L).\end{aligned}$$

   Explain each of these equations. Explain why it is logical to consider H and y as the manifest variables and h, h_1, h_2, y_1, y_2 as latent variables. The system parameters are M, K, D, L, and V. Take as values for the system parameters $M = 300$ kg, $K = 10,000$ kg/sec^2, $D = 3,000$ kg/sec, $L = 1$ meter, $V = 90$ km/hour. Argue that these figures are in the correct ballpark by reasoning about what sort of value you would expect for the natural frequency, for the steady-state gain obtained by putting a weight on the bike, and for the damping coefficient as observed from the overshoot after taking the weight back off.

3. *Simulation.* Plot the step response in the case $L = 0$. Determine the resonant frequency, the peak gain, and the pass-band. Repeat when L is not neglected. What happens to these plots when the forward velocity V changes? Repeat this for the case that the bike has a defective damper so that its damping coefficient is first reduced to 50%, and subsequently to 10% of its original value. Repeat this again for the case that the bike has a defective spring so that its spring coefficient is first reduced to 50%, and subsequently to 10% of its original value.

A.6 Stabilization of a Double Pendulum

The purpose of this exercise is to illustrate the full extent of the theory developed in Chapters 9 and 10. The exercise uses many of the concepts introduced in this book (modeling, controllability, observability, stability, pole placement, observers, feedback compensation). We recommend that it be assigned after all the theory has been covered, as a challenging illustration of it. This exercise requires extensive use of computer aids: Mathematica© for formula manipulation, and MATLAB© for control system design and numerical simulation.

A.6.1 Modeling

We study the stabilization of a double pendulum mounted on a movable cart. The relevant geometry is shown in Figure A.7. It is assumed that the motion takes place in a vertical plane. The significance of the system parameters is as follows:

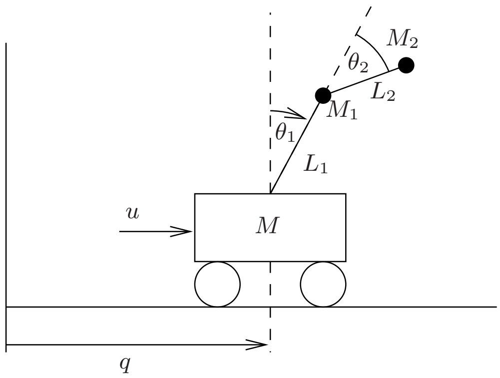

FIGURE A.7. A double pendulum on a cart.

M :	mass of the cart
M_1:	mass of the first pendulum
M_2:	mass of the second pendulum
L_1:	length of the first pendulum
L_2:	length of the second pendulum

The cart and the pendula are all assumed to be point masses, with the masses of the pendula concentrated at the top. It is instructive, however, to consider how the equations would change if the masses of the pendula are uniformly distributed along the bars.

The significance of the system variables is as follows:

u:	the external force on the cart
q:	the position of the cart
θ_1:	the inclination angle of the first pendulum
θ_2:	the inclination angle of the second pendulum

For the output y we take the 3-vector consisting of the horizontal positions of the cart and of the masses at the top of the pendula.

The purpose of this exercise is to develop and test a control law that holds the cart at a particular position with the pendula in upright position. We assume that all three components of the output y are measured and that the force u is the control input. Our first order of business is to find the dynamical relation between u and y. For this, we use Lagrange's equations. In order to express the energy of this system, introduce also the variables

$\dot{q}$:	the velocity of the cart
$\dot{\theta}_1$:	the rate of change of θ_1
$\dot{\theta}_2$:	the rate of change of θ_2

The kinetic energy is given by

$$\begin{aligned} K(q,\theta_1,\theta_2,\dot{q},\dot{\theta}_1,\dot{\theta}_2) \;=\; & \tfrac{1}{2}M\dot{q}^2 + \tfrac{1}{2}M_1[(\dot{q}+L_1\dot{\theta}_1\cos\theta_1)^2 + (L_1\dot{\theta}_1\sin\theta_1)^2] \\ & +\tfrac{1}{2}M_2[(\dot{q}+L_1\dot{\theta}_1\cos\theta_1 + L_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2))^2 \\ & +(L_1\dot{\theta}_1\sin\theta_1 + L_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2))^2]. \end{aligned}$$

The potential energy is given by

$$P(q, \theta_1, \theta_2, \dot{q}, \dot{\theta}_1, \dot{\theta}_2) = M_1 g L_1 \cos\theta_1 + M_2 g[L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)].$$

Lagrange's principle lets us write the equations of motion directly from K and P. In other words, once we have modeled K and P, we have the dynamical equations that we are looking for. Lagrange's principle is a truly amazingly effective modeling tool for mechanical systems. An alternative but much more cumbersome way of obtaining the equations of motion would be to express equality of forces for each of the masses involved. Define the Lagrangian $L := K - P$ and obtain the equations of motion as (please take note of the notation)

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q, \theta_1, \theta_2, \frac{dq}{dt}, \frac{d\theta_1}{dt}, \frac{d\theta_2}{dt}) - \frac{\partial L}{\partial q}(q, \theta_1, \theta_2, \frac{dq}{dt}, \frac{d\theta_1}{dt}, \frac{d\theta_2}{dt}) &= u, \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1}(q, \theta_1, \theta_2, \frac{dq}{dt}, \frac{d\theta_1}{dt}, \frac{d\theta_2}{dt}) - \frac{\partial L}{\partial \theta_1}(q, \theta_1, \theta_2, \frac{dq}{dt}, \frac{d\theta_1}{dt}, \frac{d\theta_2}{dt}) &= 0, \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2}(q, \theta_1, \theta_2, \frac{dq}{dt}, \frac{d\theta_1}{dt}, \frac{d\theta_2}{dt}) - \frac{\partial L}{\partial \theta_2}(q, \theta_1, \theta_2, \frac{dq}{dt}, \frac{d\theta_1}{dt}, \frac{d\theta_2}{dt}) &= 0.
\end{aligned}$$

Note that these equations contain many partial derivatives of the functions K and P, which are rather complex expressions of their arguments. Carrying out such differentiations by hand is not something one looks forward to. However, there are computer tools that do this for us. Use Mathematica© to derive the dynamical equations. You should obtain

$$\begin{aligned}
&-(L_1 M_1 + L_1 M_2)(\tfrac{d\theta_1}{dt})^2 \sin\theta_1 - L_2 M_2 \tfrac{d\theta_1}{dt}\,\tfrac{d\theta_2}{dt} \sin(\theta_1 + \theta_2) \\
&-L_2 M_2 (\tfrac{d\theta_2}{dt})^2 \sin(\theta_1 + \theta_2) + (M + M_1 + M_2)\tfrac{d^2 q}{dt^2} \\
&+(L_1 M_1 + L_1 M_2)\tfrac{d^2\theta_1}{dt^2} \cos\theta_1 + L_2 M_2 \tfrac{d^2\theta_2}{dt^2} \cos(\theta_1 + \theta_2) \qquad = u, \\[6pt]
&-g L_1 (M_1 + M_2) \sin\theta_1 - g L_2 M_2 \sin(\theta_1 + \theta_2) \\
&+L_2 M_2 \tfrac{dq}{dt}\,\tfrac{d\theta_2}{dt} \sin(\theta_1 + \theta_2) - L_1 L_2 M_2 (\tfrac{d\theta_2}{dt})^2 \sin\theta_2 \\
&+(L_1 M_1 + L_1 M_2)\tfrac{d^2 q}{dt^2} \cos\theta_1 + L_1^2 (M_1 + M_2)\tfrac{d^2\theta_1}{dt^2} \\
&+L_1 L_2 M_2 \tfrac{d^2\theta_2}{dt^2} \cos\theta_2 \qquad = 0,
\end{aligned} \tag{A.19a}$$

$$\begin{aligned}
&-g L_2 M_2 \sin(\theta_1 + \theta_2) - L_2 M_2 \tfrac{dq}{dt}\,\tfrac{d\theta_1}{dt} \sin(\theta_1 + \theta_2) \\
&+L_2 M_2 \tfrac{d^2 q}{dt^2} \cos(\theta_1 + \theta_2) + L_1 L_2 M_2 \tfrac{d^2\theta_1}{dt^2} \cos\theta_2 + L_2^2 M_2 \tfrac{d^2\theta_2}{dt^2} \;=\; 0.
\end{aligned} \tag{A.19b}$$

Completed with the output equation

$$y = \begin{bmatrix} q \\ g + L_1 \sin\theta_1 \\ q + L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}, \tag{A.20}$$

we obtain a full system of equations relating the input to the output.

A.6.2 Linearization

Prove that $u^* = 0, q^* = 0, \theta_1^* = 0, \theta_2^* = 0, y^* = 0$ is an equilibrium. Explain physically that this is as expected. Do you see other equilibria?

Introduce as state variables $x_1 = q, x_2 = \theta_1, x_3 = \theta_2, x_4 = \dot{q}, x_5 = \dot{\theta}_1, x_6 = \dot{\theta}_2$. Derive the input/state/output equations; i.e., write the equations in the form

$$\frac{dx}{dt} = f(x,u), y = h(x). \tag{A.21}$$

Note that in order to do this, you have to invert a matrix. It is recommended that you use Mathematica©: who wants to invert matrices by hand if you can let a computer do this for you? Use Mathematica© to linearize the nonlinear input/state/output equations around the equilibrium that you derived. You should obtain the following equations:

$$(M + M_1 + M_2)\frac{d^2\Delta q}{dt^2} + (L_1M_1 + L_1M_2)\frac{d^2\Delta\theta_1}{dt^2} + L_2M_2\frac{d^2\Delta\theta_2}{dt^2} = \Delta u,$$

$$\begin{aligned} &(L_1M_1 + L_1M_2)\tfrac{d^2\Delta q}{dt^2} - g(L_1(M_1 + M_2) + L_2M_2)\Delta\theta_1 \\ &+L_1^2(M_1 + M_2)\tfrac{d^2\Delta\theta_1}{dt^2} - gL_2M_2\Delta\theta_2 + L_1L_2M_2\tfrac{d^2\Delta\theta_2}{dt^2} = 0, \end{aligned} \tag{A.22a}$$

$$\begin{aligned} &L_2M_2\tfrac{d^2\Delta q}{dt^2} - gL_2M_2\Delta\theta_1 + L_1L_2M_2\tfrac{d^2\Delta\theta_1}{dt^2} \\ &-gL_2M_2\Delta\theta_2 + L_2^2M_2\tfrac{d^2\Delta\theta_2}{dt^2} = 0, \end{aligned} \tag{A.22b}$$

$$\begin{bmatrix} \Delta q \\ \Delta q + L_1\Delta\theta_1 \\ \Delta q + (L_1 + L_2)\Delta\theta_1 + L_2\Delta\theta_2 \end{bmatrix} = \Delta y. \tag{A.22c}$$

Or in state space form,

$$\tfrac{d\Delta x}{dt} = A\Delta x + B\Delta u, \qquad \Delta y = C\Delta x,$$

with

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{g(L_1(M_1+M_2)+L_2M_2)}{L_1M} & -\frac{gL_2M_2}{L_1M} & 0 & 0 & 0 \\ 0 & \frac{g(L_1M_1(M+M_1+M_2)+L_2M_2(M+M_1))}{L_1^2MM_1} & \frac{gM_2(-L_1M+L_2(M+M_1))}{L_1^2MM_1} & 0 & 0 & 0 \\ 0 & -\frac{gM_2}{L_1M_1} & \frac{g(L_1(M_1+M_2)-L_2M_2)}{L_1L_2M_1} & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{L_1M} \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & L_1 & 0 & 0 & 0 & 0 \\ 1 & L_1 + L_2 & L_2 & 0 & 0 & 0 \end{bmatrix}.$$

A.6.3 Analysis

For what values of the system parameters M, M_1, M_2, L_1, L_2 (all > 0) is this linearized system stable/asymptotically stable/unstable? Controllable? Observable? Is the equilibrium a stable/asymptotically stable/unstable equilibrium of the nonlinear system?

Assume henceforth the following reasonable choices for the system parameters: $M = 100$ kg, $M_1 = 10$ kg, $M_2 = 10$ kg, $L_1 = 2$ m, $L_2 = 1$ m.

Use MATLAB© to compute the eigenvalues of the resulting system matrix A and plot them in the complex plane. Plot the Bode diagrams, with u as input and y as output. Note that you should have three diagrams, one for each of the output components.

A.6.4 Stabilization

- We first stabilize the system using state feedback. The system is sixth order, the control is a scalar. Thus we have to choose six eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$, in the left half plane and compute the six components of the feedback gain such that the closed loop system matrix has the desired eigenvalues. In order to pick the λs (and from there the feedback gain matrix), you should experiment a bit, using the linearized system. Use the following initial conditions in your experiment: $x_1(0) = -5\text{m}, x_2(0) = 0, x_3(0) = 0, x_4(0) = 0, x_5(0) = 0, x_6(0) = 0$. This corresponds to making a maneuver: the cart is moved from one equilibrium position to the desired one, with the cart at the origin. You should choose the λs such that the transient response does not have excessive overshoot and a reasonable settling time. We suggest that you try the following λs: $-7.5 \pm 0.3i, -6.5 \pm 0.9i, -3.3 \pm 2.3i$. Plot the transient responses x_1, x_2, x_3 for the linearized system, and subsequently for the nonlinear system, with your chosen λs. Explain why you liked your λs better than the others that you tried.

  Note that you obtained a good transient response notwithstanding a rather high initial disturbance. Observe in particular the interesting small time behavior of $x_1 = q$.

- Obtain a state observer based on the measured output y and the input u. Choose the eigenvalues of the error dynamics matrix $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$ by considering the initial estimation error $e_1(0) = 5\text{m}, e_2(0) = 0, e_3(0) = 0, e_4(0) = 0, e_5(0) = 0, e_6(0) = 0$, and tuning the μs so that the resulting error transients e_1, e_2, e_3 show a reasonable settling time without excessive overshoot. Plot these transients for the μs that you selected, for the linearized system, and subsequently for the nonlinear system. Note that the observer gains are not unique in this case, since the observed output is three-dimensional. In this case, MATLAB© optimizes the chosen gains in a judicious way: it minimizes the sensitivity of the error dynamics eigenvalues.

  It appears not easy to obtain a reasonable performance for the observer. The following μs gave us some of the best results: $-10, -10, -5, -3, -1, -1$.

- Combine the state feedback gains and the observer gains obtained before in order to obtain a controller from y to u. Test this controller by plotting the transient responses of x_1, x_2, x_3 for the linearized system, and subsequently for the nonlinear system, with the initial disturbances: $x_1(0) = -5\text{m}, x_2(0) = 0, x_3(0) = 0, x_4(0) = 0, x_5(0) = 0, x_6(0) = 0$. This corresponds to the same maneuver used before. The initial state estimates are $\hat{x}(0) = x(0)$, $\hat{x}(0) = x(0)+$ a small error, and $\hat{x}(0) = [1,\ 0,\ 0,\ 0,\ 0,\ 0]$. The results for the first two initial conditions are good (explain), but not for the third. Conclude that in order to use this controller, one should always reset the observer so that its initial state estimate is accurate.

- Test the robustness of your controller against parameter changes. More concretely, you have obtained a controller that stabilizes the equilibrium for specific values of M, M_1, M_2, L_1, L_2. Now keep the controller fixed, and compute the range of values of M for which this controller remains stabilizing.

A.7 Notes and References

The advent of easy-to-use software packages such as MATLAB© and Mathematica© greatly enhances the applicability of mathematical methods in engineering. There are many recent texts (for example [36]) that aim at familiarizing students with MATLAB©, applied to the analysis of linear systems and the design of control systems. The impossibility of stabilizing a point mass using memoryless position feedback in A.1 is a well-known phenomenon. In [52] it is also used as an example motivating the need for control theory. The occurrence of an adverse response in thermal systems, demonstrated in A.2, is a typical non minimum phase phenomenon. It implies, for example, that high-gain feedback leads to instability and illustrates the need for careful tuning of controller gains. The interesting dynamical response of (weakly) coupled oscillators illustrated in A.3 was already observed by Huygens, and has been the subject of numerous analyses since. The need for control in order to stabilize a geostationary satellite in its station-keeping equilibrium position explained in A.4 is a convincing and very relevant example of a control problem. There is a large literature on this and related topics. See [14] for a recent reference and an entry into the literature. In A.5 we discuss only some very simple aspects of the dynamics of a motorbike. Designing an autonomous device (for example, a robot) that stably rides a bicycle is one of the perennial challenges for control engineering laboratories. Stabilization of a double pendulum in its upright positions (see A.6) is a neat application of the theory of stabilization of a nonlinear system around a very unstable equilibrium. Many control laboratories have an experimental setup in which such a control law is implemented. Note that our results only discuss local stability. Recent papers [6] and experimental setups implement also the swing-up of a double pendulum. Such control laws must, of course, be nonlinear: the double pendulum starts in an initial position in which both pendula hang in a downward position, and by exerting a force on the supporting mass, the pendula swing up to the stabilized upright equilibrium.