$$L_{\mathbf{g}}\, z_1 = L_{ad_{\mathbf{f}}\mathbf{g}}\, z_1 = \ldots = L_{ad_{\mathbf{f}}^{\,n-2}\mathbf{g}}\, z_1 = 0 \tag{6.58}$$

From Lemma 6.4, the above equations can be written

$$L_{\mathbf{g}}\, z_1 = L_{\mathbf{g}}\, L_{\mathbf{f}}\, z_1 = \ldots = L_{\mathbf{g}}\, L_{\mathbf{f}}^{\,n-2}\, z_1 = 0 \tag{6.59}$$

This means that if we use $\mathbf{z} = [\, z_1 \;\; L_{\mathbf{f}}\, z_1 \;\; \ldots \;\; L_{\mathbf{f}}^{\,n-1}\, z_1 \,]^T$ as a new set of state variables, the first $n-1$ state equations verify

$$\dot{z}_k = z_{k+1} \qquad k = 1, \ldots, n-1$$

while the last state equation is

$$\dot{z}_n = L_{\mathbf{f}}^{\,n}\, z_1 + L_{\mathbf{g}}\, L_{\mathbf{f}}^{\,n-1}\, z_1\, u \tag{6.60}$$

Now the question is whether $L_{\mathbf{g}} L_{\mathbf{f}}^{\,n-1} z_1$ can be equal to zero. Since the vector fields $\{\, \mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, \ldots, ad_{\mathbf{f}}^{\,n-1}\mathbf{g} \,\}$ are linearly independent in Ω, and noticing, as in the proof of Lemma 6.4, that (6.58) also leads to

$$L_{\mathbf{g}}\, L_{\mathbf{f}}^{\,n-1}\, z_1 = (-1)^{n-1}\, L_{ad_{\mathbf{f}}^{\,n-1}\mathbf{g}}\, z_1$$

we must have

$$L_{ad_{\mathbf{f}}^{\,n-1}\mathbf{g}}\, z_1(\mathbf{x}) \neq 0 \qquad \forall\, \mathbf{x} \in \Omega \tag{6.61}$$

Otherwise, the non-zero vector ∇z_1 would satisfy

$$\nabla z_1\, [\, \mathbf{g} \;\; ad_{\mathbf{f}}\mathbf{g} \;\; \ldots \;\; ad_{\mathbf{f}}^{\,n-1}\mathbf{g} \,] = 0$$

and thus would be orthogonal to n linearly independent vectors, a contradiction.

Therefore, by taking the control law to be

$$u = (-L_{\mathbf{f}}^{\,n}\, z_1 + v)/(L_{\mathbf{g}}\, L_{\mathbf{f}}^{\,n-1}\, z_1)$$

equation (6.60) simply becomes

$$\dot{z}_n = v$$

which shows that the input-output linearization of the nonlinear system has been achieved. □

HOW TO PERFORM INPUT-STATE LINEARIZATION

Based on the previous discussion, the input-state linearization of a nonlinear system can be performed through the following steps:

- Construct the vector fields $\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}, \ldots, ad_{\mathbf{f}}^{\,n-1}\mathbf{g}$ for the given system
- Check whether the controllability and involutivity conditions are satisfied

• If both are satisfied, find the first state z_1 (the output function leading to input-output linearization of relative degree n) from equations (6.58), *i.e.*,

$$\nabla z_1 \, ad_{\mathbf{f}}^{\,i}\mathbf{g} = 0 \qquad i = 0, \ldots, n-2 \tag{6.62a}$$

$$\nabla z_1 \, ad_{\mathbf{f}}^{\,n-1}\mathbf{g} \neq 0 \tag{6.62b}$$

• Compute the state transformation $\mathbf{z}(\mathbf{x}) = [\, z_1 \quad L_{\mathbf{f}}\, z_1 \quad \ldots \quad L_{\mathbf{f}}^{\,n-1} z_1 \,]^T$ and the input transformation (6.53), with

$$\alpha(\mathbf{x}) = -\frac{L_{\mathbf{f}}^{\,n} z_1}{L_{\mathbf{g}} L_{\mathbf{f}}^{\,n-1} z_1} \tag{6.63a}$$

$$\beta(\mathbf{x}) = \frac{1}{L_{\mathbf{g}} L_{\mathbf{f}}^{\,n-1} z_1} \tag{6.63b}$$

Let us now demonstrate the above procedure on a simple physical example [Marino and Spong, 1986; Spong and Vidyasagar, 1989].

Example 6.10: Consider the control of the mechanism in Figure 6.6, which represents a link driven by a motor through a torsional spring (a single-link flexible-joint robot), in the vertical plane. Its equations of motion can be easily derived as

$$I\ddot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) = 0 \tag{6.64a}$$

$$J\ddot{q}_2 - k(q_1 - q_2) = u \tag{6.64b}$$

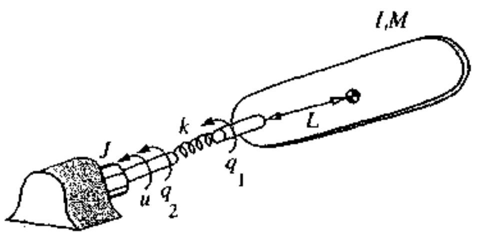

Figure 6.6 : A flexible-joint mechanism

Because nonlinearities (due to gravitational torques) appear in the first equation, while the control input u enters only in the second equation, there is no obvious way of designing a large range controller. Let us now consider whether input-state linearization is possible.

First, let us put the system's dynamics in a state-space representation. Choosing the state vector as

$$\mathbf{x} = [q_1 \quad \dot{q}_1 \quad q_2 \quad \dot{q}_2]^T$$

the corresponding vector fields **f** and **g** can be written

$$\mathbf{f} = [\, x_2 \qquad -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \qquad x_4 \qquad \frac{k}{J}(x_1 - x_3)\,]^T$$

$$\mathbf{g} = [\,0 \quad 0 \quad 0 \quad \frac{1}{J}\,]^T$$

Second, let us check the controllability and involutivity conditions. The controllability matrix is obtained by simple computation

$$[\,\mathbf{g} \quad ad_{\mathbf{f}}\,\mathbf{g} \quad ad^2_{\mathbf{f}}\,\mathbf{g} \quad ad^3_{\mathbf{f}}\,\mathbf{g}\,] = \begin{bmatrix} 0 & 0 & 0 & -\frac{k}{IJ} \\ 0 & 0 & \frac{k}{IJ} & 0 \\ 0 & -\frac{1}{J} & 0 & \frac{k}{J^2} \\ \frac{1}{J} & 0 & -\frac{k}{J^2} & 0 \end{bmatrix}$$

It has rank 4 for $k > 0, IJ < \infty$. Furthermore, since the vector fields $\{\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}, ad_{\mathbf{f}}{}^2\mathbf{g}\}$ are constant, they form an involutive set. Therefore, the system in (6.64) is input-state linearizable.

Third, let us find out the state transformation $\mathbf{z} = \mathbf{z}(\mathbf{x})$ and the input transformation $u = \alpha(\mathbf{x}) + \beta(\mathbf{x})\,v$ so that input-state linearization is achieved. From (6.62), and given the above expression of the controllability matrix, the first component z_1 of the new state vector $\mathbf{z}$ should satisfy

$$\frac{\partial z_1}{\partial x_2} = 0 \qquad \frac{\partial z_1}{\partial x_3} = 0 \qquad \frac{\partial z_1}{\partial x_4} = 0 \qquad \frac{\partial z_1}{\partial x_1} \neq 0$$

Thus, z_1 must be a function of x_1 only. The simplest solution to the above equations is

$$z_1 = x_1 \tag{6.65a}$$

The other states can be obtained from z_1

$$z_2 = \nabla z_1\,\mathbf{f} = x_2 \tag{6.65b}$$

$$z_3 = \nabla z_2\,\mathbf{f} = -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \tag{6.65c}$$

$$z_4 = \nabla z_3\,\mathbf{f} = -\frac{MgL}{I}x_2\cos x_1 - \frac{k}{I}(x_2 - x_4) \tag{6.65d}$$

Accordingly, the input transformation is

$$u = (v - \nabla z_4\,\mathbf{f})/(\nabla z_4\,\mathbf{g})$$

which can be written explicitly as

$$u = \frac{IJ}{k}(v - a(\mathbf{x})) \tag{6.66}$$

where

$$a(\mathbf{x}) = \frac{MgL}{I}\sin x_1\,(x_2^2 + \frac{MgL}{I}\cos x_1 + \frac{k}{I}) + \frac{k}{I}(x_1 - x_3)\,(\frac{k}{I} + \frac{k}{J} + \frac{MgL}{I}\cos x_1)$$

As a result of the above state and input transformations, we end up with the following set of linear equations

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = v$$

thus completing the input-state linearization.

Finally, note that

- The above input-state linearization is actually global, because the diffeomorphism $\mathbf{z}(\mathbf{x})$ and the input transformation are well defined everywhere. Specifically, the inverse of the state transformation (6.65) is

$$x_1 = z_1$$
$$x_2 = z_2$$
$$x_3 = z_1 + \frac{I}{k}\left(z_3 + \frac{MgL}{I}\sin z_1\right)$$
$$x_4 = z_2 + \frac{I}{k}\left(z_4 + \frac{MgL}{I}z_2\cos z_1\right)$$

which is well defined and differentiable everywhere. The input transformation (6.66) is also well defined everywhere, of course.

- In this particular example, the transformed variables have physical meanings. We see that

z_1 is the link position, z_2 the link velocity, z_3 the link acceleration, and z_4 the link jerk. This further illustrates our earlier remark that the complexity of a nonlinear physical model is strongly dependent on the choice of state variables.

• In hindsight, of course, we also see that the same result could have been derived simply by differentiating equation (6.64a) twice, *i.e.*, from the input-output linearization perspective of Lemma 6.3. □

Note that inequality (6.62b) can be replaced by the normalization equation

$$\nabla z_1 \, ad_{\mathbf{f}}^{\,n-1}\mathbf{g} = 1$$

without affecting the input-state linearization. This equation and (6.62a) constitute a total of n linear equations,

$$[\, ad_{\mathbf{f}}^{o}\mathbf{g} \quad ad_{\mathbf{f}}^{1}\mathbf{g} \quad \ldots \quad ad_{\mathbf{f}}^{n-2}\mathbf{g} \quad ad_{\mathbf{f}}^{n-1}\mathbf{g}\,] \begin{bmatrix} \partial z_1/\partial x_1 \\ \partial z_1/\partial x_2 \\ \cdots \\ \partial z_1/\partial x_{n-1} \\ \partial z_1/\partial x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}$$

Given the independence condition on the vector fields, the partial derivatives $\partial z_1/\partial x_1$, ... , $\partial z_1/\partial x_n$ can be computed uniquely from the above equations. The state variable z_1 can then be found, in principle, by sequentially integrating these partial derivatives. Note that analytically solving this set of partial differential equations for z_1 may be a nontrivial step (although numerical solutions may be relatively easy due to the recursive nature of the equations).

CONTROLLER DESIGN BASED ON INPUT-STATE LINEARIZATION

With the state equation transformed into a linear form, one can easily design controllers for either stabilization or tracking purposes. A stabilization example has already been provided in the intuitive section 6.1, where v is designed to place the poles of the equivalent linear dynamics, and the physical input u is then computed using the corresponding input transformation. One can also design tracking controllers based on the equivalent linear system, provided that the desired trajectory can be expressed in terms of the first linearizing state component z_1.

Consider again the flexible link example. Its equivalent linear dynamics can be expressed as

$$z_1^{(4)} = v$$

Assume that it is desired to have the link position z_1 track a prespecified trajectory $z_{d1}(t)$. The control law

$$v = z_{d1}^{(4)} - a_3 \tilde{z}_1^{(3)} - a_2 \tilde{z}_1^{(2)} - a_1 \dot{\tilde{z}}_1 - a_o \tilde{z}_1$$

(where $\tilde{z}_1 = z_1 - z_{d1}$) leads to the tracking error dynamics

$$\tilde{z}_1^{(4)} + a_3 \tilde{z}_1^{(3)} + a_2 \tilde{z}_1^{(2)} + a_1 \dot{\tilde{z}}_1 + a_o \tilde{z}_1 = 0$$

The above dynamics is exponentially stable if the positive constants a_i are chosen properly. To find the physical input u, one then simply uses (6.66).

6.4 Input-Output Linearization of SISO Systems

In this section, we discuss input-output linearization of single-input nonlinear systems described by the state space representation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\,u \tag{6.67a}$$

$$y = h(\mathbf{x}) \tag{6.67b}$$

where y is the system output. By *input-output linearization* we mean the generation of a *linear* differential relation between the output y and a new input v (v here is similar to the equivalent input v in *input-state linearization*). Specifically, we shall discuss the following issues:

- How to generate a linear input-output relation for a nonlinear system?
- What are the internal dynamics and zero-dynamics associated with the input-output linearization?
- How to design stable controllers based on input-output linearizations?

GENERATING A LINEAR INPUT-OUTPUT RELATION

As discussed in section 6.1.3, the basic approach of input-output linearization is simply to differentiate the output function y repeatedly until the input u appears, and then design u to cancel the nonlinearity. However, *in some* cases, the second part of the approach may not be carried out, because the system's relative degree is undefined.

The Case of Well Defined Relative Degree

Let us place ourselves in a region (an open connected set) $\Omega_{\mathbf{x}}$ in the state space. Using