1 Naive Set Theory

"Definition:"
A set is a collections of elements considered as a whole.
i.e. all math majors at OXY.
Recall, our sets of numbers.

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

$$\mathbb{Z} = \{0, -1, 1, -2, 2, ...\}$$

$$\mathbb{Q} = \{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\}$$

Example: List F where $F = \{x \mid x \in \mathbb{Z}, x^2 = x\}$ this set is equivalent to asking what are the roots of the polynomial $x^2 - x$. So we see $F = \{0, 1\}$ One important axiom of set theorem is that there exists the empty set denoted $\emptyset$. Also sets can be elements.
i.e. $A = \{1, \{1\}, \{1, 2\}\}$
Definition: Let A and B be sets,
(i) if $a \in A \implies a \in B$ then A $\subseteq$ B
(ii) if $A \subseteq B$ and $B \subseteq A$ then A=B
(ii$_2$) $A = B \iff (x \in A \iff x \in B)$
(iii) if $A \subseteq B$ and $A \neq B$ then $A \subsetneq B$

Lemma: if $x \in \mathbb{Z}$ and x^2 is even then x is even.
Proof: Suppose for contractions x is odd.
$\implies x = 2k + 1$ for some $k \in \mathbb{Z}$ (definition of odd)
$\implies x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
This is a contradiction since we assumed x^2 is even.
$\therefore$ x is even. $\Box$
Claim: $\sqrt{2} \notin \mathbb{Q}$
Proof: Suppose for contradiction $\sqrt{2} \in \mathbb{Q}$,
$\implies \sqrt{2} = \frac{a}{b}$ for $a \in \mathbb{Z}, b \in \mathbb{N}$ (definition of being in Q)
Without loss of generality $\frac{a}{b}$ is in lowest terms
$\implies \sqrt{2}b = a \iff 2b^2 = a^2$ lets call this equation $(\star)$
by Lemma, we since a^2 is even we know a is even.
$\implies a = 2m$ for some $m \in \mathbb{Z}$
$\implies a^2 = 4m^2$ Now lets plug this into $(\star)$
$\implies 2b^2 = 4m^2 \implies b^2 = 2m^2$
by Lemma, we have that b is even.
Hence a and b share a common factor of 2. #
This is a contradiction because we assumed our fraction was in lowest terms.
$\therefore \sqrt{2} \notin \mathbb{Q}$

Remark:
(i) Order does not matter when listing elements of a set. $\{1,2\} = \{2,1\}$
(ii) Repetition does not matter $\{1,2,2\} = \{1,2\}$
(iii) sets can be elements and elements can be sets.

1.1 Set Operations

Definition: Let A,B be sets, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
This is called the union of the two sets.
Definition: Let $\{A_i\}_1^N$ be a finite collection of sets,

$$\bigcup_{i=1}^{N} A_i := \{x \mid x \in A_i \text{ for some } 1 \leq i \leq N\}$$

Definition: Let $A_1, A_2, A_3, ...$ be a collection of sets,

$$\bigcup_{n=1}^{\infty} A_n := \{x \mid x \in A_n \text{ for some } 1 \leq n\}$$

Definition: Let $\{A_i\}_{i \in I}$be a family of sets indexed by an indexing set I,

$$\bigcup_{i \in I} A_i := \{x \mid x \in A_i \text{ for some i} \in I\}$$

Definition:
(i) Let A and B be sets,
$A \cap B := \{x \mid x \in A \text{ and } x \in B\}$
(ii) Let $A_1, A_2, A_3, ...$ be sets,

$$\bigcap_{n=1}^{\infty} A_n := \{x \mid x \in A_n \forall n\}$$

(iii) Let $\{A_i\}_{i \in I}$be a family of sets indexed by an indexing set I,

$$\bigcap_{i \in I} A_i := \{x \mid x \in A_i \forall i \in I\}$$

(iv) Let $\{A_i\}_{i \in I}$be a family of sets indexed by an indexing set I,
If $A_i \cap A_k = \emptyset$ for all $k \neq i$ then we say say $\{A_i\}_{i \in I}$ is mutually disjoint.

Proposition: (Archimedean Property)
(i) $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})$ s.t. $n > x$
(ii)$(\forall x > 0)(\exists n \in \mathbb{N})$ s.t. $x > \frac{1}{n}$

Example: Show $\bigcap_{n \in \mathbb{N}} (\frac{-1}{n}, \frac{1}{n}) = \{0\}$

Proof: Suppose for contractions there is and x$\in \bigcap_{n\in\mathbb{N}}(\frac{-1}{n},\frac{1}{n})$ where $x\neq 0$
$\implies \frac{-1}{n} < x < \frac{1}{n} \forall n \geq 1$
$\implies |x| < 1/n$
by the Archimedean property, $\exists m$ s.t. $\frac{1}{m} < x$
Let n=m
$\implies \frac{1}{m} < |x| < \frac{1}{m}$ # Hence, all $x \neq 0$ are not in $\bigcap_{n\in\mathbb{N}}(\frac{-1}{n},\frac{1}{n})$. It is still let to show that x=0 is in $x \neq 0$ are not in $\bigcap_{n\in\mathbb{N}}(\frac{-1}{n},\frac{1}{n})$.

Definition: Let A, B be sets in a universe U.
(i) $A \setminus B := \{x \mid x \in A, x \notin B\}$
(ii) $A \triangle B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$
(iii) $A^c = U \setminus B$.
Remark: $A \setminus B = A \cap B^C$
Proposition:
(i) $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$
(iii) $A \cup \emptyset = A$
(iv) $A \cap \emptyset = \emptyset$
(v) $A \cup (B \cup C) = (A \cup B) \cup C$
(vi) $A \cap (B \cap C) = (A \cap B) \cap C$
(vii) $(A \cup B)^C = A^C \cap B^C$
(viii) $(A \cap B)^C = A^C \cup B^C$
Proposition: Let $\{A_i\}_{i\in I}$ be a family of sets indexed by I
(i) $A \cap (\bigcup_{i\in I} B_i) = \bigcup_{i\in I}(A \cap B_i)$
(ii) $A \cup (\bigcap_{i\in I} B_i) = \bigcap_{i\in I}(A \cup B_i)$
(iii) $(\bigcup_{i\in I} B_i)^C = \bigcap_{i\in I} B_i^C$
(iv)$(\bigcap_{i\in I} B_i)^C = \bigcup_{i\in I} B_i^C$
Definition: Let A and B be seta,

$$A \times B := \{(a,b) \mid a \in A, b \in B\}$$

Formally, we say the pair $(a,b) := \{\{a\},\{a,b\}\}$
Example: Given $A = (-1,1) \times \mathbb{R}$ write A^C as the disjoint union of sets.
Soln:
$A^C = ((-\infty,-1] \times \mathbb{R}) \sqcup ([1,\infty) \times \mathbb{R})$
We define $A \times B \times C = (A \times B) \times C$
Definiton: Let A be a set,
$\mathcal{P}(A) := \{X \mid X \subseteq A\}$

Techniques of Proof
We have already seen the way to show 2 sets are equal ($A = B \iff (A \subseteq B, B \subseteq A)$
Also we say a proof by contradiction:
(i) If P and Q are statements and we want to show $P \implies Q$. We can prove

by assuming $\neg Q$ and through some steps we can see one of our assumptions are violated.
(ii) Contrapositive: We want to prove $P \implies Q$ it is equivalent to show $\neg Q \implies \neg P$.
(iii) A Direct proof is just using definition, previous finings and theorem to achieve a result.
Definition: Let $m, n \in \mathbb{Z}, m \neq 0$ We say m divides n or $\frac{m}{n}$ if $\exists c \in \mathbb{Z} s.t. m = cn$
Proposition: If a divides b and b divides c then a divides c.
Pf: Given a divides b $\exists a_0 \in \mathbb{Z} s.t. b = a_0 a$ and that b divides c $\exists b_0 \in \mathbb{Z} s.t. c = b_0 b$ $\implies c = a a_0 b_0 \implies$ a divides c
Proposition: If a divides b and a divies c then a divides kb+lc $k, l \in \mathbb{Z}$
Pf: Given a divides b and a divides c
we have $\exists k, l \in \mathbb{Z} s.t. b = ka, c = al$
adding them we get $b + c = ak + al = a(k + l)$ hence a divides b+c
Example: Show $S = \sqrt{\frac{3+\sqrt{2}}{4}} \notin \mathbb{Q}$
Pf: Suppose for contradiction $S \in \mathbb{Q}$ then $S^2 = \frac{3+\sqrt{2}}{4} \iff 4S^2 = 3 + \sqrt{2} \iff 4S^2 - 3 = \sqrt{2}$ hence since the left hand side is in $\mathbb{Q}$ and the right is not this is a contradiction.
Definition: A set X is said to be well ordered if $\forall S \subseteq X$ non empty, $\exists n \in S s.t. n = least(S)$
Proposition: (Well ordering principle) The natural numbers are well ordered.
Principle of Induction
Let $n_0 \in \mathbb{N}$ Suppose $P(n)$ is a statement about natural numbers $n \geq n_0$
If:
(i) $P(n_0)$ is true (base case)
(ii) P_m is true $\implies P_{m+1}$ is true.
Then: P_n is true for all $n \geq n_0$
Proposition: If X is a set with n elements, Then $\mathcal{P}(x)$ has 2^n elements.
Proof: (Induction)
Base Case: X has 1 element. then $\mathcal{P}(X)$ has X and the empty set so it has 2^1 elements.
Suppose for some m we have that $\mathcal{P}(x)$ has 2^m elements,
check m+1:
Let $S = \{A \subseteq X \mid X_{m+1} \in A\}, T = \{A \subseteq X \mid X_{m+1} \notin A\}$
by our induction hypothesis, we know T has 2^m elements. Also since $S = B \cup \{x_{m+1}\}$ where B is an element of T, we get that it has 2^m element. Since these two sets are disjoint we can add them to get $\mathcal{P}(X)$ has $2^m + 2^m = 2(2^m) = 2^{m+1}$
Hence, by the principle of induction it is true for all n.
Proof of Induction:
Let $n \in \mathbb{N}_0$ be fixed,
Suppose $P(n)$ is a statement about natural numbers
Suppose for contradiction $P(n)$ does not hold for at least one natural number $n \geq n_0$,
Let $F = \{n \in \mathbb{N} \mid P(n) \text{ fails }\} \subseteq \mathbb{N}$
By assumption F is non-empty, By the well ordering principle, $\exists l \in F$ s.t. $l =$

$least\{F\}$ Since l is the least element of F that means l-1 is not in F, hence P(l-1) is true. By assumption we have $P(m) \implies P(m+1)$ this would imply $P(l-1)$ holds $\implies P(l)$ holds. # Contradiction since we found P(l) doesnt hold.

Example: Show $e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}, \forall n > 1$.

Proof:

Base Case:(n=1)

we want to show $e^x \geq 1 + x$

consider the function $f(x) = e^x - x - 1$ We will apply Fermat's Theorem for critical points and find x=0 is a critical point. Similarly since $e^x - 1 < 0$ when $x < 0$ and $e^x - 1 > 0$ when $x > 0$ by Darboux's Theorem x=0 is a minimum. So $e^x - x - 1 \geq 0 \iff e^x \geq x + 1, \forall`x$

Suppose for some m we have $e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^m}{m!}$.

Now we check for m+1

Notice since both are positive we have $\int_0^x e^t dt \geq \int_0^x 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^m}{m!}$

Evaluating these integrals we get

$e^x - 1 \geq x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{m+1}}{m+1!} \iff e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^{m+1}}{m+1!}$

Theorem:(Strong Induction)

Fix $n_0 \in \mathbb{N}_0$ Let P(n) be a statement for each natural numbers.

Suppose that $P(n_0)$ hold and that $P(n_0), P(n_0+1), ..P(m) \implies P(m+1)$ then $P(n)$ holds for all n.

Fundamental Theorem of Arithmetic

If $n \in \mathbb{Z} \setminus \{1, -1\}$

then $\exists p_i, i \in \{1, 2, 3, ..l\}$ with $n = (\pm 1)p_1p_2p_3...$

Where p are primes. This theorem is saying that every integer has a unique prime factor decomposition.

Proof of existence

Suppose $n \geq 2$ Check n=2 we see that 2=2 times 1 where 2 and 1 are primes.

Check m+1

If m+1 is prime then $m + 1 = (m + 1)(1)$

If not, then there are at least 2 numbers and at most m a and b whose product is m+1. By our induction hypothesis we have a and b are product of primes.

If A is a "finite" set, we will say n(A)= the number of elements in A

Multiplication Principle:

If A, B are finite $n(A \times B) = n(A)n(B)$

In words we write if there are n ways to do a task and there are m ways to do another task, then there are n times m ways to do the first task then the second task.

Generally, If we have m tasks, $n(A^m) = \prod_{i=1}^m n(A_i)$

Example: There are 12 boxes, there are 2 balls, How many ways are there to place both the 2 balls in any box.

Solution: Our first action is to place the first ball in a box, we have 12 choices, our second action is to place the second ball in a box, we have 12 choices. By the multiplication principle there are 12 times 12 ways to place the 2 balls in

the 12 boxes.
Now Suppose one box can only hold 1 ball.
We will see that there are 12 ways to place the first ball but then only 11 ways to place the second. So by the multiplication principle there are 12 times 11 ways to place the 2 balls in unique boxes.
Example: How many different home addresses are there when we have 4 digits and no repetitions?
Soln: First choice the first digit, we have 10 choices, next we choose the second we have 9 choices, for the third position we have 8 choices then for the last we have 7, By the multiplication principle we have 10 times 9 times 8 times 7 addresses.
Now suppose the addresses cannot start or end with 0.
First we choose the digits which cannot have 0, for those we have 9 then 8 choices then for the two which can have 0 we have 8 then 7. By the multiplication principle we find that there are 9 times 8 times 7 times 8 ways.
Example: A bit string is a string of length n is a sequence $a_1, a_2, ..., a_n : a_i \in \{0, 1\}$ How many bring strings are there of length n.
Solution: We have 2 choices for each n, either 1 or 0 so we will get through the multiplication principle there are 2^n ways.
To motivate permutations lets consider this problem: Given 7 distinct symbols, how many strings of length 4 can we make?
From above result we find there are 7^4 ways.
IF we cant repeat a symbol we find there are $7 \times 6 \times 5 \times 4$ ways by the multiplication principle. If we ask how many words of length 7, no repetitions are there, we fill find there are $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways
Permutations of distinguishable Objects: We have n distinct symbols there are n^r possible arrangements of string of length $r \leq n$ if we allow repetitions. Without repositions we get there are $n \times (n-1) \times (n-2) \times ...(n-r+1) = \frac{n!}{(n-r)!}$ ways to arrange.
Defn: Given $n \in \mathbb{N}_0, r \leq n$

$$P(n, r) = \frac{n!}{(n-r)!}$$

This is the number of arrangements of n distinguishable objects of length r.
Example: A photographer at a wedding arranges 6 people side by side including the bride and groom, how many ways can we do this?
Solution: We are arranging 6 people in a length of 6 so we get 6!
Now Suppose the bridge and groom need to be together,
First we can consider them as just 1 person so we would get 5! from the same reasoning as part one. Then by the multiplication principle we can also we just also multiply it by 2! because there are 2! ways to arrange the bridge-groom unit.
Now Suppose we want to find the ways where the bridge and groom are not together,
We already found all the possible way to take the picture and all the possible ways we can put the bridge and groom together so its natural that the ways

we can have them not together is the total ways without the ways they are together, so we get $6! - (5!)(2)$ ways.

Addition Principle: Let A,B be disjoint lets.

$n(A \cup B) = n(A) + n(B)$

If a task can be done in n ways or in m ways then the total amount of ways to do the task is n+m ways.

Example: Count the number of bit strings or length less than 4.

Solution: From past results we have that each bit string of length n has 2^n possible arrangements. And since each set of bit strings is disjoint, we can apply the addition principle to find there are $2^4 + 2^3 + 2^2 + 2^1$ ways.

Example: A password is 6 to 8 characters consisting of upper case letters and digits.

How many passwords with at least one digit?

Solution: Let $Q_i = \{\text{all passwords of length i}\}$ Since we have 26 letters and 10 digits, we have 36 objects to form a length i arrangements so by permutations we have 36^i ways. Then Let $P_i = \{\text{X has at least 1 digit}\}$ We will find $P_i^C = \{\text{x has only letters}\}$ Using permutations we get there are 26 objects to arrange in length i so we ha we 26^i. Then By the addition principle we can find there are $n(P_i) = n(Q_i) - n(P_i^C) = 36^i - 26^i$ ways. then we will take i=6,7,8 and find there are $36^8 - 26^8 + 36^7 - 26^7 + 36^6 - 26^6$ ways.

Subtraction Principle: If A and B are finite sets, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Example: How many bits strings of length 10 start with 1 or end with 0,0

Solution: Suppose A is set of bit strings starting with 1, $n(A) = 2^9$ and Let B be the set of bit strings ending in 0,0 $n(B) = 2^8$ for $n(A \cap B) = 2^7$ so by the subtraction principle we find that $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 2^9 + 2^8 - 2^7$

Example: Find the numbers of positive integers in $D = \{n \in \mathbb{N} \mid 1 \leq n \leq 100,\ \text{2 does not divide n and 3 does not divide n}\}$

Solution: Let $A = \{n \in \mathbb{N} \mid 1 \leq n \leq 100,\ \text{2 divides n}\}$ we find that $A^C = \{2n \mid 1 \leq n \leq 50\}$ and we can find that $n(A) = 50$, and $n(A^C) = 50$

Similarly we find that $B = \{n \in \mathbb{N} \mid 1 \leq n \leq 100,\ \text{2 does not divde n}\}$ and $B^C = \{3k \mid 1 \leq k \leq 33\}$ have $n(B^C) = 33, n(B) = 100 - n(B^c) = 67$

We want to find $n(A \cap B)$, So we can find that $(A \cup B)^C = \{n \in \mathbb{N} \mid \text{ 2 and 3 divide n}, 1 \leq n \leq 100\}$ This is equivilent to the same $\{6k \mid 1 \leq k \leq 16\}$ hence $n((A \cup B)^C) = 16$ by subtraction principle we find that $n(A \cup B) = 100 - n((A \cup B)^C) = 100 - 16 = 84$

Division Principle: If A is a partitioned set,

$(A = \bigsqcup_{i=1}^{n} A_i$ and $\forall A_i, n(A_i) = d$

then, $\frac{n(A)}{d} = n$

Example: n guests come to dinner,

(i) Suppose we seat them in a row, How many configurations?

We just use the definition of permutations and we get $n!$

(ii) Suppose we have a circular table, How many configurations are there. We say 2 configurations are equal if the left and right neighbors are the same.

Solution: A is the sell of all configurations, $n(A) = n!$ now lets group all config-

urations which are equivalent, we will call these A_i Since our only actions is to rotate, $n(A_i) = n$, by the division principle we find that $m = \frac{n!}{n} = (n-1)!$
Example: Suppose a rook and bishop are placed on a board(Not on the same spot). How many many ways can we do this?
Solution: First we place the rook on the 8 by 8 grid and we have 64 options, for the bishop we just place it on the grid but not where the rook is and we get 64 options.
Now suppose we want to place them in different rows and different columns.
Well for the first option we still have 8 times 8 places. then for the next we cant place it in the same row or column so we have 7 times 7 hence, 64×49.
Now Suppose we have the same set up as 2 but now its 2 indistinguishable bishops.
We have double as many possibilities in part 2 as desired so we take half of 64×49 by the division principle.
Q: Given a set A with n elements, if we have $0 \leq k \leq n$ how many subsets are there of size k?
$A = \{x_1, x_2, x_3, ..., x_n\}$ from last time we know we can form P(n,k) words. In a subset ordered doesn't matter when we take P(n,k) we are over counting. Each choices of k elements gives rise to k! different words. but in a set these are all equivalent. So by the division principle we get that $\frac{P(n,k)}{k!} = \binom{n}{k} = C(n,k)$
Definition: Given $n \in \mathbb{N}_0, 0 \leq k \leq n$,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Given a set A with n elements and we want to choose k where order doesn't matter, we have there are $\binom{n}{k}$ ways.
Proposition: $\sum_{k=1}^{n} \binom{n}{k} = 2^n$Proof: By defintion $\binom{n}{k}$ = number of subsets from a set with n elements of size k.
If we take $\sum_{k=1}^{n} \binom{n}{k} = n(\mathcal{P}(x)) = 2^n$.
Pascals Triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

We first look at just a triangular lattice and we want to find the number of paths there are from the top down to a certain vertex. We can recognize each path leads to a bit string of length n if we go to the point n,k
where 0 is move to the left and 1 is move right. Conversely given a bit string of length n with k ones, we have a unique path. so we have $B(n,k) = n(\{\text{bit string with exactly 1 one}\}) = \binom{n}{k}$. From this we find the following lemma.
Lemma:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

The Binomial Theorem:
If x,y are variables,

$$(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof:(Combinatorial)
$(x+y)^n = (x+y)(x+y)...(x+y), n$ times
$=$ sum of a monomials of the form $bx^m y^l$
To form such a monomial we pick either x or y from each bracket. Note m+l=n. Once we pic the y the other choices must be x so we get $bx^{n-k}y^k$.If we choose which bracket admits a y we get $\binom{n}{k}$. So we get that $(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^{n-k} y^k$
Propn: Given a set N with n elements, there are $\binom{n+r-1}{r}$ ways to choose r objects from the set when repeats are allowed. So we get $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$
Example: A bakery has 4 different types of cookies, we want to choose 6.
Solution: This is a combination problem since order doesnt matter. From the above proposition we get n=4 ,r=6 since we have 4 cookies and we want 6. Hence, $\binom{4+6-1}{6} = \binom{4+6-1}{6-1}$
Example: How many ways can you write eleven as the sum of 3 integers?
Soln: We will use the starts and bars method: Here we have 3 integers so we will have 2 bars. We 11 objects to choose from so we have 11 stars. So we have 13 items and we choose where either the 2 bars or 11 stars go. Hence, $\binom{13}{2}$
Example show $\binom{2n}{2} = 2\binom{n}{2} + n^2$
Pf: First we will count the amount of bits strings of length 2n and this is just $\binom{2n}{2}$
However we can count it as the cases where the ones appear.
If both ones are in the first half
we have there are $\binom{n}{2}$
If both ones are in the second half,
we have the same $\binom{n}{2}$
If 1 one is at the first and the other in the second,
we have there are n^2 ways.
So by the addtional principle we have these are all disjoint so $2\binom{n}{2} + n^2 = \binom{2n}{n}$
Propn: (Vandurmon's Equality)

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Proof: Consider 2 sets A,B such that $n(A) = m, n(B) = n$ chose r objects from $A \cup B$ there are $\binom{m+n}{r}$ possibilities. Another way to count this is through the addition principle,
First we choose only A we get $\binom{m}{r}\binom{n}{0}$

Next 1 from B $\binom{m}{r-1}\binom{n}{1}$
inductively repeat choosing k from B we get $\binom{m}{r-k}\binom{n}{k}$.
By the addition principle the total ways is $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$
Example: Show $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$
Pf: In Vandurmon's choose m=r=n.
$\binom{n+n}{n} = \sum_{k=0}^{r}\binom{n}{n-k}\binom{n}{k}$
$\iff \binom{2n}{n} = \sum_{k=0}^{r}\binom{n}{k}\binom{n}{k}$ By Theorem
$= \sum_{k=0}^{r}\binom{n}{k}^2$
Example: Consider the m by n lattice of $\mathbb{Z}_m \times \mathbb{Z}_n$
How many paths are there from 0,0 to m,n
First notice our movements will add up to m+n and we must go to m so then we can choose m from m+n and the rest of the movements must be n so we get $\binom{m+n}{m}$ ways.

"Defn" A statement is a declarative statement which can be either True or False but not both.
Determine the true values of the following statements:
P: Los Angeles is in California,
Q: 2+2=5
R: Come over

P is true , Q is false and R is not a proposition.
Propositions can be composite, they can be broke down into simple propositions and logical operations.
Ex:R:Roses are red violets are Blue, write R as a competitive proposition.
Solution: We can brake this down into P: Roses are Red , Q: Violets are blue we can write this $R \equiv P \wedge Q$.
Example: R: Occidental College is in LA and Pomona College is in Santa Barbra. Determine truth value of this statement.
Solution:
Let P: Occidental College is in La
Q: Pomona College is in Santa Barbra
$R \equiv P \wedge Q$ Since Q is false and P is true, P and Q is false.
Definition(Conjunction-and-$\wedge$)
Let P,Q be statements.
$P \wedge Q$ is true if and only if P is true and Q is true, otherwise false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition(Disjunction-Or-$\vee$)
Let P,Q be statements.
$P \vee Q$ is false if and only if P is false and Q is false

other it is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition:(Negation-Not-$\neg$
Let P be a statement.
If P is true then $\neg P$ is false

p	$\neg p$
T	F
F	T

Example: Negate the following: (1)P :2+2=5
(2)Q:Everyone is happy
Solution:
$\neg P = 2 + 2 \neq 5$
$\neg Q$: Someone is not happy
Definition: Let P,Q,R be statements An expression $\mathcal{P}(P, Q, R)$ constructed from P,Q,R and logical operators is a compound proposition.
Example: Write the truth table for $\neg(p \wedge \neg q)$

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Example: $(p \wedge \neg p)$

p	$\neg p$	$\neg p$
T	F	T
F	T	T

If a statement has all T in the true table, then it is called a tautology; If a statement has all F in the truth table then it is a contradiction.
Example Verify $p \vee \neg(p \wedge q)$ is a tautology

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \wedge \neg(p \vee q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

When can we say two propositions are the same? Definition:
Given two statements P, Q
$P \equiv Q$ iff their true tables are identical. Example: Prove $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

By definition we have $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Proposition:

Let P,Q,R be propositions.

The Idempotent Laws:

(i) $P \vee P \equiv P$

(ii) $P \wedge P \equiv P$ The Associative Laws:

(iii) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

(iv) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

The Commutative Laws:

(v) $P \wedge Q \equiv Q \wedge P$

(vi) $P \vee Q \equiv Q \vee P$

The Distributive Laws:

(vii) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

(viii) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

The Identity Laws:

(ix) $P \vee FALSE \equiv P$

(x) $P \vee TRUE \equiv TRUE$

(xi) $P \wedge TRUE \equiv P$

(xii) $P \wedge FALSE \equiv FALSE$

Involution:

(xiii) $\neg(\neg P) \equiv P$

Complement Laws:

(xiv) $P \vee \neg P \equiv TRUE$

(xv) $P \wedge \neg P \equiv FALSE$

Demorgan's Laws:

(xvi) $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

(xvii) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Example: Show that $P \vee \neg(P \wedge Q)$ is a tautology

$P \vee \neg(P \wedge Q) \equiv P \vee (\neg P \vee \neg Q)$ (DeMorgan)

$\equiv (P \vee \neg P) \vee \neg Q$(Associativity)

$\equiv TRUE \vee \neg Q$ (Complement Law)

$\equiv TRUE$ (Identity Law)

Definition(Implications- If, then - $\implies$)

Given Statements P and Q,

$P \implies Q$ is False If P is true and Q is false

Otherwise true.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: Show $P \implies Q \equiv \neg P \vee Q$

Solution:

p	q	$\neg p$	$\neg p \vee q$	$p \implies q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Definition: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$ Definition: An Argument is an assertion that given a set of premises $p_1, p_2, p_3, .., p_n$ yields another statement Q. We write
$p_1, ..., p_n \vdash Q$
Example: Law of Detachment, Show the following argument is valid:$P, P \implies Q \vdash Q$
Solution:

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

We see when p and $p \implies q$ are both true we have that q is true. Example: Show $Q, P \implies Q \vdash P$
Solution:

p	q	$q \implies p$
T	T	T
T	F	F
F	T	T
F	F	T

Since in the third row we have both the premises but the conclusion is false we have this is a fallacy.
Theorem: Let $p_1, ..., p_n \vdash Q$ be an argument,
The Following Statements Are Equivalent:
(i) The Argument Is Valid
(ii) $(p_1 \wedge p_2 \wedge .... \wedge p_n) \implies Q$
Example: Show $p, p \implies q \vdash q$ is valid
By Theorem it is equivalent to show $(p \wedge p \implies q) \implies q$ is a tautology.
Solution:

p	q	$p \implies q$	$p \wedge (p \implies q)$	$p \wedge (p \implies q) \implies q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Hence valid.

Example: (Law of Syllogism)
The Following Argument is valid $p \implies q, q \implies r \vdash r$
Proof:
Let $Q \equiv (p \implies q \wedge q \implies r) \implies (p \implies r)$

p	q	r	$p \implies q$	$q \implies r$	$p \implies q \land q \implies r$	$p \implies r$	Q
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

By Theorem this argument is valid.
Example: Is this Valid?
S_1: If a man is a bachelor then he is unhappy.
S_2: If a man is unhappy then he dies young.

—- —— —— —— —- —— —— —— —-

S: Bachelors die young.
Proof:
Let $p \equiv$ A man is a bachelor, $q \equiv$ Is unhappy , $r \equiv$ dies young
We want to show $p \implies q, q \implies r \vdash p \implies r$
By the law of syllogism this argument is valid. Definition: Let A be a set, A propositional Sentence is an expression P(x) s.t. P(a) is a proposition for all $a \in A$. We call A the domain of P
$T_p = \{x \mid x \in A, p(x)\}$ Is the truth set of the propositional sentence P
e.g Let $A = \mathbb{N}, P(x) : x + 2 > 7$, Find T_p, $T_p = \{x \in \mathbb{N} \mid x > 5\} = \{6, 7, 8, 9, ..\}$
Remark: P(X) by it self is an open condition, it has no truth value.
Definition: Let P(x) be a propositional sentence defined on A,
if $T_p = A$ then we write $\forall x \in A, P(x)$
if $T_p \neq \emptyset$ then we write $\exists x \in A, P(x)$
Example: (Negations of quantifiers)
P : all Italians love pasta,
$\neg P$: there exists an Italian who does not love pasta,
Theorem: Let A be a set, P(x) is a propositional sentence defined on A.
(i) $\neg((\forall x \in A)(P(x)) \equiv (\exists x \in A)(P(x))$
(ii) $\neg((\exists x \in A)(P(x))) \equiv (\forall x \in A)(\neg P(x))$
This can be proven using truth tables
Definition: A propositional sentence of n variables over a product set $A_1 \times A_2 \times A_3 \times .. \times A_n$ is an expression $P(x_1, x_2, ..., x_n)$ with the property that $P(a_1, a_2, ..., a_n)$is true or false for any n-tuple in the product set.
Example: $x + 2y + 3z < 18$ on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
define $\exists x \forall y \exists z, P(x, y, z)$
Example: $B := \{1, 2, 3, 4, .., 9\}, P(x, y) : x + y = 10$ On $B \times B$
claim: $\forall x \exists y$ s.t. P(x,y) is true.
Proof: Given $x \in B$ Choose $y = 10 - x \in B$ then $x + y = 10 - x + x = 10$
claim: $\exists x \forall y, P(x, y)$
False, (check all x)

Given a statement with several quantifiers we can apply our theorem successively to negate.
e.g. $\neg((\forall x)(\exists y)(\exists y),(P(x,y,z))) \equiv \exists x \neg((\exists y)(\exists y),(P(x,y,z))) \equiv \exists x \forall y \neg((\exists y),(P(x,y,z)))$
$\equiv \exists x \forall y \forall y \neg P(x,y,z)$
Example: We say $\lim_{x\to c} f(x) = L \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in V_\delta^*(c), |f(x) - L| < \varepsilon)$
What does it mean for f(x) not to converge to L?
Solution:
$\neg(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in V_\delta^*(c), |f(x) - L| < \varepsilon)$
$\equiv (\exists \varepsilon_0 > 0)\neg((\exists \delta > 0)(\forall x \in V_\delta^*(c), |f(x) - L| < \varepsilon))$
$\equiv (\exists \varepsilon_0 > 0)(\forall \delta > 0)\neg(\forall x \in V_\delta^*(c), |f(x) - L| < \varepsilon))$
$\equiv (\exists \varepsilon_0 > 0)(\forall \delta > 0)(\exists x \in V_\delta^*(c), |f(x) - L| \geq \varepsilon_0))$