Math 310–Occidental College

Problem Set 1–Cardinality and Countability

Problem 1. If F is a finite set and $k : F \to F$ is a self map, prove that k is injective if and only if k is surjective.

Problem 2. Prove that a set A is infinite if and only if there is a non-surjective injection $f : A \to A$.

Problem 3. Let A, B, and C be sets and suppose $\operatorname{card}(A) < \operatorname{card}(B) \leq \operatorname{card}(C)$. Prove that $\operatorname{card}(A) < \operatorname{card}(C)$.

Problem 4. If $A \subseteq B$ is an inclusion of sets with A countable and B uncountable, show that $B \setminus A$ is uncountable.

Problem 5. Is the set $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$ countable?

Problem 6. Consider the set $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$. Is $\mathcal{F}(\mathbb{N})$ countable?

Problem 7. Let $k \in \mathbb{N}$.

(i) Prove that $\mathbb{N}^k := \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$ is countable.

(ii) Show that the set

$$\mathbb{N}^\infty := \big\{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N}\big\}$$

consisting of all sequences of natural numbers is uncountable.

(iii) Prove that the set of **finitely-supported** natural sequences

$$c_c(\mathbb{N}) := \big\{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N},\ n_k = 0 \text{ for all but finitely many } k\big\}$$

is countable.

(iv) Is the set of decreasing natural sequences

$$D := \big\{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N},\ n_{k+1} \leq n_k\ \forall k \geq 1\big\}$$

countable or uncountable?

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that sends rational numbers to irrational numbers and irrational numbers to rational numbers. Prove that the range $\operatorname{ran}(f)$ can't contain any interval.

Problem 9. Prove that the set

$$\mathcal{P} := \left\{ \sum_{k=0}^{n} a_k x^k \mid n \in \mathbb{N}_0,\ a_k \in \mathbb{Q} \right\},$$

consisting of all polynomials with rational coefficients, is countable.

Problem 10. A real number t is called **algebraic** if there is a nonzero polynomial p with rational coefficients such that $p(t) = 0$. If $t \in \mathbb{R}$ is not algebraic, it is called **transcendental**. For example, $\sqrt{2}$ is algebraic, but π is transcendental. Show that the set of algebraic numbers is countable, and conclude that there are uncountably many transcendental numbers.