MATH310:Real Analysis

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1 Sequences

Definition: Let $(x_n)_n$ be a sequence in a metric space (X,d) and $L \in X$
$(x_n)_n$ converges to L or $(x_n)_n \to L$ if
$(\forall \varepsilon > 0)(\exists N_\varepsilon \in \mathbb{N}) : (\forall n \geq N, x_n \in V_\varepsilon(L))$
Remark: N_ε depends only on ε
Example:
Show $(\frac{1}{n})_n \to 0$
Proof:
Given $\varepsilon > 0$,
By the Archimedian property $\exists N_\varepsilon \in \mathbb{N} s.t. \frac{1}{N_\varepsilon} < \varepsilon$
Hence if $n \geq N_\varepsilon \implies \frac{1}{n} \leq \frac{1}{N_\varepsilon}$
So, $|\frac{1}{n} - 0| = |\frac{1}{n}| \leq \frac{1}{N_\varepsilon} \leq \varepsilon$ Example: In $(\mathbb{Z}, d), d = |m - n|$, If $(x_n)_n \to X$ then $(x_n)_n$ is eventually constant
Example: Given $(x_n)_n$ converges to X we have $N \in \mathbb{N}$ such that $\forall n \geq N, |x_n - x| < \varepsilon$
Lets choose $\varepsilon = \frac{1}{2}$
$|x_n - x| < \frac{1}{2} \implies x_n - x = 0 \iff x_n = x$ for $n \geq N$
Fundamental Convergence Lemma:
Let $(x_n)_n$ be a real sequence and $x \in \mathbb{R}$
Suppose $\exists c > 0$ and a sequence $(\epsilon_n)_n$ satisfying:
(i)$(\epsilon_n)_n \to 0 (\epsilon_n > 0)$
(ii) $|x_n - x| \leq c \cdot \epsilon_n$ eventually, then $(x_n)_n \to x$
Proof:
Let $\varepsilon > 0$ be given, Let $\varepsilon' = \frac{\varepsilon}{c}$
Since $(\epsilon_n)_n$ converges to 0, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon, |\epsilon_n - 0| \leq \varepsilon'$
Moreover, $\exists m \in \mathbb{N} s.t. |x_n - x| \leq c\epsilon_n$ when $n \geq m$
Let $N := max\{m, N_\varepsilon\}$ if $n \geq N$ then $|x_n - x| \leq c\epsilon_n$ (since $n \geq m$) and because $n \geq N_\varepsilon, c\epsilon_n < c\varepsilon' = \frac{\varepsilon}{c} c = \varepsilon$ $\square$
Example: Show $(\frac{1}{2^n})_n \to 0$
$|\frac{1}{2^n} - 0| = \frac{1}{2^n} < \frac{1}{n} (Bernouli)$ in the fundamental convergence lemma, $(\epsilon_n)_n = (\frac{1}{n})_n \to 0, c = 1$ hence, it converges.
Proposition: A sequence can have at most 1 limit.
Proof: Suppose for contradiction $(x_n) \to x$ and $(x_n) \to y$, $x \neq y$

x_n are in $V_\delta(x), V_\delta(y)$, By the Hausdorff property ($\delta = |x-y|\frac{1}{2}$) we find that $V_\delta(x) \cap V_\delta(y) = \emptyset$
since the sequence converges we know $\exists n_1 \in \mathbb{N} s.t. n > n_1, |x_n - x| < \delta$ similarly we have $|x_n - y| < \delta$ If we consider $n \geq max\{n_1, n_2\}$ then we have $x \in V_\delta(x), y \in V_\delta(y)$ since we know these are disjoint sets, this is a contradiction.

1.1 Divergence of sequences

Using logical negation we can find, $(x_n)_n \not\to \iff (\exists \varepsilon_0 > 0)(\forall n \in \mathbb{N})(\exists N > n : |x_n - x| \geq \varepsilon_0)$
Example: Show that $((-1)^n)_n$ does not converge
Proof:
Case 1: $((-1)^n) \not\to 1$
Given $n \in \mathbb{N}$, Let $\varepsilon_0 = 2$
find $N > n$ such that N is odd.(N=(2n+1))
Then $|(-1)^N - 1| = |-2| = 2$
Case 2: $((-1)^n) \not\to 1$
Given $n \in \mathbb{N}$, Let $\varepsilon_0 = 2$
find $N > n$ such that N is even.(N=(2n))
Then $|(-1)^N - 1| = |2| = 2$
Case 3: If $L \in \mathbb{R}, L \neq \pm 1$
Given $n \in \mathbb{N}$, Let $\varepsilon_0 = min\{|L-1|, |L+1|\}$
if $\varepsilon_0 = |L-1|$ find $N > n$ such that n is even
then $|(-1)^N - L| = |1 - L| = |L-1| = \varepsilon_0$
if $\varepsilon_0 = |L+1|$ find $N > n$ such that n is even
then $|(-1)^N - L| = |-1 - L| = |-(L+1)| = \varepsilon_0$
Lemma: If $(x_n)_n \to x$ then $(|x_n|)_n \to |x|$
Pf:
Using the reverse triangle inequality we can find that $||x_n| - |x|| \leq |x_n - x|$ so given $\varepsilon > 0$ use the same N where $(x_n)_n$ converges so then $\forall n \geq N, ||x_n| - |x|| \leq |x_n - x| < \varepsilon$
Example: Show that $(sin(n))_n$ diverges
Solution:
if L=1 then Let $\varepsilon_0 = 2$ then we have $|sin(n) - 1| \leq |sin(n)| + 1 \leq 2 = \varepsilon_0$
if L=-1 then Let $\varepsilon_0 = 2$ then we have $|sin(n) + 1| \leq |sin(n)| + 1 \leq 2 = \varepsilon_0$
if $L \neq \pm 1$ Let $\varepsilon_0 = max\{|L+1|, |L-1|\}$
So $|sin(n) - L| \leq |sin(n)| + |L| \leq |1| + |L| \geq \varepsilon_0$
Hence divergent.
Example: $(x_n)_n \to x \iff (|x_n - x|)_n \to 0$
Proof:
$\implies$
Given that $(x_n)_n \to 0$ by definition we can find an N_ε where $\forall n \geq N_\varepsilon$ $|x - x_n| < \varepsilon$ for all $\varepsilon > 0$

so we can the following equality

$$||x_n - x| - 0| = |x_n - x| < \varepsilon$$

Hence we have $(|x_n - x|)_n \to 0$
$\Longleftarrow$
Given $(|x_n - x|)_n \to 0$ by definition we have an N_ε where $\forall n \geq N_\varepsilon, ||x_n - x| - 0| < \varepsilon$
Hence,

$$\varepsilon > ||x_n - x| - 0| = ||x_n - x|| = |x_n - x|$$

Thus $(x_n)_n \to x$

1.2 Important Sequences

Recall the following lemma, Lemma: A sequence $(x_n) \to x \implies (|x_n|)_n \to |x|$

Example: **Geometric Sequence** fix $b \in \mathbb{R}$ we will show $(b^n)_{n=0}^{\infty} \to \begin{cases} 1 & b = 1, \\ 0 & |b| < 1, \\ div & otherwise. \end{cases}$

Proof: First lets get ride of some easy cases,
if b=1 we have a constant sequence converging to 1
if b=0 we have a constant sequence converging 0
if b=-1 we showed this sequence diverges
Suppose $0 < b < 1$
$\implies 1 < \frac{1}{b}$ so we can write $\frac{1}{b} = 1 + a$ for some $a > 0$
$\implies (\frac{1}{b})^n = (1 + a)^n \geq 1 + an(bernouli)$
so we have that $b^n = \frac{1}{1+na} \leq \frac{1}{na} = (\frac{1}{a})(\frac{1}{n})$
So in the fundamental convergence lemma take $c = \frac{1}{a}, \varepsilon_n = \frac{1}{n}$
hence $(b^n)_n \to 0$
if we have that $-1 < b < 0$ we just use the lemma above to find that $(b_n)_n \to 0$
Now suppose $b > 1$ so we write $b = 1 + a, a > 0$
Using Bernoulli we find that $b^n \geq na$
Now lets suppose for contradiction $(b^n)_n \to L$
Let $\varepsilon = 1$ so for all $n \geq N, |b^n - L| < 1$
In particular we have that $b^n < L + 1$
By above we have the chain

$$na \leq b^n < L + 1$$

By the Archimedian property we can find an $M \in \mathbb{N} s.t. M > \frac{L+1}{a}$
Let $N = max\{M, N_\varepsilon\} + 1$
so we have

$$Na < b^n < L + 1 \implies b^N < L + 1\#$$

Example: fix $c > 0, (c^{\frac{1}{n}})_n \to 1$
Proof:

if c=1, $1^{\frac{1}{n}} = 1 \to 1$
if c¿1, then $c^{\frac{1}{n}} > 1$
so we write $c^{\frac{1}{n}} = 1 + a_n, a_n > 0$
so $c = (1 + a_n)^n \geq 1 + na_n \geq na_n$
$\implies a^n \leq \frac{c}{n}$
$\implies |c^{\frac{1}{n}} - 1| = |a_n| \leq \frac{c}{n}$
so by the fundamental convergence lemma $(c^{\frac{1}{n}})_n \to 1$
Now if $0 < c < 1$ we have $c < 1 \implies c^{\frac{1}{n}} < 1^{\frac{1}{n}} = 1 \iff \frac{1}{c^{\frac{1}{n}}} > 1$
$\implies \frac{1}{c^{\frac{1}{n}}} = 1 + a_n, a_n > 0$
$\implies (\frac{1}{c}) = (1 + a_n)^n \geq 1 + a_n n \geq a_n n \implies a_n \leq \frac{1}{c}\frac{1}{n}$
Example: $(n^{\frac{1}{n}})_n \to 1$
Pf:
$n > 1 \iff n^{\frac{1}{n}} > 1$ so we write $n = (1 + a_n)^n = \sum_{k=0}^{n} \binom{n}{k} a_n^k (Binomial) = \binom{n}{0} + \binom{n}{1} a_n + ... + \binom{n}{n} a_n^n \geq \binom{n}{0} + \binom{n}{2} a_n^2 = 1 + \frac{n(n-1)}{2} a_n^2 \implies n-1 \geq \frac{n(n-1)}{2} a_n^2 \implies \frac{2}{n} \geq a_n^2 \implies \frac{\sqrt{2}}{\sqrt{n}} \geq a \implies |n^{\frac{1}{n}} - 1| = a_n \leq \sqrt{2}\frac{1}{\sqrt{n}}$
So by the fundamental convergence lemma we are done.
Example Let $0 \leq b < 1$
Show $(nb^n)_n \to 0$
Proof: If b=0 then we are done
Suppose $0 < b < 1 \implies \frac{1}{b} > 1$, we write $\frac{1}{b} = 1 + a$
so we have $(\frac{1}{b})^n = (1 + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k = \binom{n}{0} + \binom{n}{1} a + ... + \binom{n}{n} a^n \geq \binom{n}{2} a^2 = \frac{n(n-1)}{2} a^2$
$\implies b^n \leq \frac{2}{a^2 n(n-1)} \iff nb^n \leq \frac{2}{a^2}\frac{1}{n-1}$
by the fundamental convergence lemma we are done.

1.3 Convergence Theorems

Proposition: If a sequence $(x_n)_n$ converges then it is bounded
Proof: Since $(x_n)_n \to x$
Let $\varepsilon = 1$ when $n \geq N$
$\implies |x_n - x| < \varepsilon = 1$
$\implies = -1 + x < x_n < 1 + x$ so x_n is bounded when $n \geq N$, Lets call this bound M
When $n \geq N$ we have a finite collection of terms of the sequence so
Let $c = max\{\{x_n \mid n \leq N\}\}$
So we have $-c \leq x_n \leq c$ when $n \leq N$
So for any n let $K = max\{c, M\}$
then $|x_n| \leq K$
Proposition:
The Algebra of Sequences
Let $(x_n)_n \to x, (y_n)_n \to y, (z_n)_n \to z, t \in \mathbb{R}$
(i) $(x_n \pm y_n)_n \to x \pm y$
(ii) $(ty_n) \to ty$

(iii) $(x_n y_n)_n \to xy$
Assume $z_n \neq 0, z \neq 0$
(iv) $(\frac{1}{z_n}) \to \frac{1}{z}$
This is called an algebra because it is an algebra (an algebra is a vector space and a ring)
Proof: (iii)
$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \leq |x_n y_n - x_n y| + |x_n y - xy| = |x_n||y_n - y| + |y||x_n - x|$
Looking at this we see that $|x_n|$ is bounded, $|y_n - y|$ is going to 0 and $|x_n - x|$ is going to 0.
So now we can say $\exists K s.t. x_n \leq K \forall n$
$\implies |x_n y_n - xy| = |x_n||y_n - y| + |y||x_n - x| \leq K|y_n - y| + |y||x_n - x|$
Using (i) we can then apply the fundamental convergence lemma to see this sequence converges
Proof: (iv)
We will try to "bound away from 0"
$|\frac{1}{z_n} - \frac{1}{z}| = |\frac{z - z_n}{z z_n}| = \frac{|z_n - z|}{|z||z_n|}$
Here everything is ok besides $|z_n|$
Since $(z_n)_n$ does not converge to 0 we can find an N where we are are enough away for this sequence to be bounded
Let $\varepsilon = \frac{|z|}{2}$
for $n \geq N, |z_n - z| < \varepsilon = \frac{|z|}{2} \implies \frac{|z|}{2} > z_n - z > \frac{-|z|}{2} \iff \frac{|z|}{2} + z > z_n > \frac{-|z|}{2} + z \implies |z_n| > \frac{|z|}{2} \iff \frac{1}{|z_n|} \leq \frac{2}{|z|}$
So we have $|\frac{1}{z_n} - \frac{1}{z}| \leq \frac{2}{|z|}|z_n - z|$ using the fundamental convergence lemma we are done
Corollary: If $(x_n)_n \to x$ then $(x_n^k)_n \to x^k$
Proof: We apply (3) inductively on k
Theorem: Let $(x_n)_n \to x, (y_n)_n \to y$
If $x_n \leq y_n \forall n \geq 1$ Then $x \leq y$
Proof: Suppose for contradiction x¿y
Let $\varepsilon = \frac{x-y}{2} > 0$
$\exists N_1 \in \mathbb{N} s.t. n \geq N_!, y_n \in V_\varepsilon(y)$
$\exists N_2 \in \mathbb{N} s.t. n \geq N_!, x_n \in V_\varepsilon(x)$
Let $N = max\{N_1, N_2\}$
Then $x_N \in V_\varepsilon(x), y_N \in V_\varepsilon(y)$
Contradiction, Since we assumed $x > y$ and $x_n < y_n$.
Corollary: If $(x_n) \to x, \exists a, b \in \mathbb{R} s.t. a \leq x_n \leq b$
Then $a \leq x \leq b$ we just apply the previous Theorem using the constant sequences containing a and b
The Squeeze Theorem:
Let $(x_n)_n, (y_n)_n, (z_n)_n$ be sequences where $\forall n \geq 1, x_n \leq y_n \leq z_n$
If $(x_n)_n \to L, (z_n)_n \to L$
Then $(y_n)_n \to L$
Proof:

Given $\varepsilon > 0$,
find $N_1, N_2 \in \mathbb{N} s.t. (n \geq N_1 \implies x_n \in V_\varepsilon(L)) \wedge (n \geq N_2 \implies y_n \in V_\varepsilon(L))$
Set $N = max\{N_1, N_2\}$
Then if $n \geq N$
$x_n, z_n \in V_\varepsilon(L)$
So for all $n \geq N$ we have the following string of inequalities

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$$

Hence $y_n \in V_\varepsilon(L) \therefore (y_n)_n \to L\square$
<u>Example:</u> $((a^n + b^n)^{\frac{1}{n}})_n \to max\{a, b\}$
<u>Proof</u>
without loss of generality, $a < b$
$\implies b^n \leq a^n + b^n \leq b^n + b^n = 2b^n$
$\implies b \leq (a^n + b^n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}} b$
By squeeze theorem this goes to b since $(2^{\frac{1}{n}})_n \to 1$
<u>Proposition:</u> Let $(x_n)_n$ be a sequence of strictly positive terms.
$(\frac{x_{n+1}}{x_n})_n \to r < 1$
Then $(x_n) \to 0$
<u>Proof:</u> Since $r < 1 \iff 1 - r > 0$
Let $p = r + \frac{1-r}{2}$
Let $\varepsilon = p - r = \frac{1-r}{2}$
$\exists N \in \mathbb{N} s.t. \forall n \geq N, |\frac{x_{n+1}}{x_n} - r| < \varepsilon$
So if $n \geq N \implies \frac{x_{n+1}}{x_n} < p \implies x_{n+1} < p x_n$
In particular $x_{N+1} < p x_N$
Inductively we find that $x_{N+k} < p^k x_N$
So by squeeze theorem we have that $(x_n)_n \to 0$ since p^k is the geometric sequence.
<u>The Monotone Convergence Theorem:</u>
Let $(x_n)_n$ be a monotone sequence,

(i) if $(x_n)_n$ is increasing and bounded above then $(x_n)_n \to sup(\{x_n \mid n \geq 1\})$

(ii) if $(x_n)_n$ is decreasing and bounded below then $(x_n)_n \to inf(\{x_n \mid n \geq 1\})$

(iii) $(x_n)_n$ converges $\iff (x_n)_n$ is bounded

<u>Proof:</u> We already showed convergent implies bounded so first we assume that $(x_n)_n$ is bounded and monotone
Case 1: $(x_n)_n$increasing and bounded above,

Let $sup(\{x_n \mid n \geq 1\} = u$ which exists by the completeness axiom
we claim that $(x_n) \to u$
Let $\varepsilon > 0$ be given, by the sup lemma $\exists N \in \mathbb{N} s.t. u - \varepsilon < x_N$ so if $n \geq N$
$u - \varepsilon < x_N \leq x_n$ since the sequence is increasing and $x_n \leq u$ by the upper

bound property of u.
so we can conclude that if $n \geq N$ then $|x_n - u| < \varepsilon$
Case 2: $(x_n)_n$ is decreasing and bounded below
Let $inf(\{x_n \mid n \geq 1\} = l$ which exists by the completeness axiom
we claim that $(x_n) \to l$
By the Inf lemma, $\exists N \in \mathbb{N} s.t. x_N < l + \varepsilon$ Since the sequence is decreasing we have $x_n \leq x_N < l + \varepsilon$ and since l is the lower bound, $l \leq x_n \leq x_N < l + \varepsilon$ hence if $n \geq N, |x_n - l| < \varepsilon$ □
Lemma: If $x = (x_n)_n$ is a convergent sequence and $m \in \mathbb{N}$ then the m-th tail (x_{n+m}) converges to the same limit
Proof: Let $\varepsilon > 0$ find N such that $\forall n \geq N, |x_n - L| < \varepsilon$
then if $n \geq N, m + n \geq N$ so $|x_{n+m} - L| < \varepsilon$
Hence (x_{n+m}) converges.
Example: Consider this sequence given recessively $x_1 = 8, x_{n+1} = \frac{1}{2}x_n + 2$
Claim: $x_n \geq 4 \forall n \geq 1$
Proof: (Induction)
$x_1 = 8 \geq 4 \checkmark$
Suppose for some k $x_k \geq^\dagger 4$
Indeed,
$x_{k+1} = \frac{1}{2}x_k + 2 \geq^\dagger 2 + 2 = 4$
Claim: $(x_n)_n$ is decreasing
Proof:
Directly we see that from claim one $4 \leq x_n \iff 4 + x_n \leq 2x_n \iff 2 + \frac{1}{2}x_n \leq x_n \iff x_{n+1} \leq x_n$. So by the monotone convergence theorem we see that this converges.
Then if we apply the m-th tail lemma we see that $x_{n+1} = L = x_n$
So we should have $L = \frac{1}{2}L + 2 \iff L = 4$
Example:
Consider $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$, show $(x_n)_n$ converges.
Proof:
Clearly we can see that $x_1 \leq x_2 \leq x_3 \leq .... \leq x_n \leq x_{n+1}$
Notice, $k^2 \geq k(k+1), k \geq 2$
Hence $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$
so $\sum_{k=2}^{n} \frac{1}{k^2} \leq \sum_{k=2}^{n} \frac{1}{k-1} - \frac{1}{k} = 1 - \frac{1}{n} \leq 1 + 1 - \frac{1}{n}$
So we find that

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n} \leq 2,$$

So by the monotone convergence theorem we see this series converges.
Recall, Given a sequence of nested intervals $I_n = [a_n, b_n], \bigcap_{n=1}^{\infty} I_n \neq \emptyset$
Using the monotone convergence theorme we can prove this as follows.
Proof:
Notice the nesting of the intervals implies:

$$a_1 \leq a_2 \leq a_3 \leq ... \leq a_n \leq b_n \leq b_{n-1} \leq .. \leq b_1$$

So we get that the sequence $(a_n)_n$ is increasing and bounded above and the

sequence $(b_n)_n$ is decreasing and bounded below
So by the monotone convergence theorem

$$(a_n)_n \to sup_n(a_n) = L, (b_n) \to inf(b_n) = R$$

Now fix $n \in \mathbb{N}$,
then $n \leq m, a_n \leq a_m \leq b_m \leq b_n$ by nesting.
so $sup_m(a_m) = L \leq b_n$ is true for all m
similarly we see that $sup_m(a_m) \leq inf_n(b_n) = R \leq b_m$
So we see that $L \leq R$
any number $\xi \in [L, R]$ will be in $\bigcap_n I_n$
Example: Let $a > 0$. We construct a sequence $(x_n)_n \to \sqrt{a}$ using the superumum property
Let $x_1 = 1, x_{n+1} := \frac{1}{2}(x_n + \frac{a}{x_n})$
Claim: $(x_n)_n \to \sqrt{a}$
we will show first that $\forall n \geq 1, x_n^2 \geq a$
Proof:
$2x_{n+1} := x_n + \frac{a}{x_n} \implies 2x_{n+1}x_n = x_n^2 + a$
$0 = x_n^2 - 2x_{n+1}x_n + a, x_n$ is a real root to $x^2 - 2x_{n+1}x + a$
Notice, this mean that $\Delta \geq 0 \implies 4x_{n+1}^2 - 4a \geq 0 \iff 4_{n+1}^2 \geq 4a \iff x_{n+1}^2 \geq a, \forall n \geq 1$ so it is true for $x_n^2 \geq a$
Claim: $(x_n)_n$ is eventually decreasing
proof:

$$x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + \frac{a}{x_n}) = x_n - \frac{1}{2}(\frac{x_n^2 + a}{x_n}) = \frac{1}{2}(\frac{2x_n^2 - x_n^2 - a}{x_n}) = \frac{1}{2}(\frac{x_n^2 - a}{x_n}) > 0$$

By the first claim. We showed the difference between consecutive terms is positive so we know it is decreasing.
So using the monotone convergence theorem we know this converges to some limit x
Now uisng the m-tail we can find the limit.

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}) \to x = \frac{1}{2}(x + \frac{a}{x}) \iff 2x = x + \frac{a}{x} \iff x = \frac{a}{x} \iff x^2 = a \iff x = \sqrt{a}$$

We can approximate this using the following algorithm:
Since we had $\sqrt{a} \leq x_n \iff \frac{1}{\sqrt{a}} \geq \frac{1}{x_n} \iff \sqrt{a} \geq \frac{a}{x_n} \iff \frac{a}{x_n} \leq \sqrt{a} \leq x_n$
Therefore, $0 \leq x_n - \sqrt{a} \leq x_n - \frac{a}{x_n} = (\frac{x_n^2 - a}{x_n})$
So if we want to approximate $\sqrt{2}, a = 2, x_1 = 1, x_2 = \frac{3}{2}, x_3 = \frac{17}{12}$

$$\frac{17}{12} - \sqrt{2} \leq (\frac{(\frac{17}{12})^2 - 2}{\frac{17}{12}}) \approx 0.0049$$

Example: We will now look at Napier Constant:e
Consider $e_n = (1 + \frac{1}{n})^n$ By the Binomial Theorem
$(1 + \frac{1}{n})^n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k}$

Lemma:

$$\binom{n}{k}\frac{1}{n^k} = \frac{1}{k!}\prod_{j=1}^{k-1}(1-\frac{j}{n})$$

To see this we can just write out the left hand side.
$= \frac{n!}{(n-k)!}\frac{1}{n^k} = \frac{1}{k!}\frac{n(n-1)(n-2)...(n-k+1)}{n^k} = \frac{1}{k!}\frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}...\frac{n-k+1}{n} = \frac{1}{k!}1(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{k-1}{n}) = \frac{1}{k!}\prod_{j=1}^{k-1}(1-\frac{j}{n})$
So we can see that we get

$$e_n = \sum_{k=0}^{n+1}(\frac{1}{k!}\prod_{j=1}^{k-1}(1-\frac{j}{n}))$$

First we notice that this has $n+1$ terms in the product
Similarly we see that ,

$$e_{n+1} = \sum_{k=0}^{n+2}(\frac{1}{k!}\prod_{j=1}^{k-1}(1-\frac{j}{n+1}))$$

Which has n+2 terms in the product
So we can see this $1-\frac{j}{n+1} \geq 1-\frac{j}{n} \iff \frac{j}{n+1} \leq \frac{j}{n}$
So each term of the product is bigger in the sequence e_{n+1} so $e_n \leq e_{n+1}, \forall n$ so e_n is increasing.
<u>Claim:</u> $(e_n)_n$ is bounded.
we claim that $(e_n)_n$ is bounded below by two
$e_1 = 2$
we can also see that it is bounded above by 3

$e_n = \sum_{k=0}^{n}\frac{1}{k!}(\prod_{j=1}^{k-1}(1-\frac{j}{n})) \leq \sum_{k=0}^{n}\frac{1}{k!}1 \leq 1+1+\sum_{k=2}^{n}\frac{1}{2^{k-1}} = 2+\sum_{l=1}^{n-1}\frac{1}{2^l} \leq 1+2 = 3$
So By the monotone convergence theorem it converges.
We defined Napier's constant to $e := sup_{n\geq 1}(1+\frac{1}{n})^n$
Corollary to the Monotone Converges Theorem:: A sequence (x_n) monotone and unbounded diverges to $\pm\infty$
<u>Proof:</u>
Recall the definition of being unbounded
$(\forall M > 0)(\exists N \in \mathbb{N}) : (\forall n \geq N, x_n \geq M)$
So we can find an N where $x_N \geq M$
Because $(x_n)_n$ is increasing, we have $x_n \geq x_N > M$ So this diverges properly
<u>Example:</u> Consider the harmonic series $h_n = \sum_{k=1}^{n}\frac{1}{k} = 1+\frac{1}{2}+...+\frac{1}{n}$
We see this is clearly increasing
We will look at the the powers of 2
$h_1 = 1$
$h_2 = 1+\frac{1}{2}$
$h_4 = 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4}) \geq 1+\frac{1}{2}+(\frac{1}{2})$
$h_8 = 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}) \geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+\frac{3}{2}$

$h_{16} \geq 1 + \frac{4}{2}$
Inductivly we can find that
$h_{2^n} \geq 1 + \frac{n}{2}$
Given $M > 0$, find n such that $n > 2(M) - 1 \implies \frac{n}{2} + 1 > M$ So h_n is unbounded.

1.4 Subsequences:

Definition: A natural sequence is a sequence of natural numbers, $(n_k)_k$ where $n_k < n_{k+1} \in \mathbb{N}$
Lemma: $n_k \geq k$ for any natural sequence
Proof:
Base Case: k=1
$n_1 \geq 1, n_1 \in \mathbb{N}$
Suppose for some m $n_m \geq m$
Indeed, $n_{m+1} \geq n_m + 1 \geq m + 1$
Definition: Given a sequence $(x_n)_n \in (X, d)$ a subsequence of $(x_n)_n$ is another sequence $(x_{n_k})_k$ where n_k is a natural sequence
Example: The m-th tail is a subsequence
fix $m \in \mathbb{N}$ then $(x_{m+k})_k$ is a subsequence.
here out $n_k = m - 1 + k$
Proposition: If a sequence $(x_n)_n \to X$ then $(x_{n_k})_k \to X$ for any subsequence n_k
Proof:
Given $\varepsilon > 0$,
$\exists N \in \mathbb{N} : \forall n \geq N, |x_n - x| < \varepsilon$
Set $K = N, \forall k \geq K = N$
$n_k \geq k \geq K = N \implies |x_{n_k} - x| < \varepsilon$
Corollary: Suppose $(x_n)_n$ is a sequence admitting two subsequences
$(x_{n_k})_k \to x, (x_{m_k})_k \to x'$
Then $(x_n)_n$ is divergent
Proof:
Suppose for contradiction we have $(x_n)_n \to L$ by the above proposition we must have
$(x_{n_l})_l \to L, (x_{n_k})_k \to L$ So $L = x = x'$ # Hence we are done
Example: We will show that $((-1)^n)_n$ diverges.
Solution:
Simply pick the subsequence $n_k = 2k$ and the subsequence $n_k = 2k + 1$
$(-1)^{2k} = 1 \to 1$
$(-1)^{2k+1} = -1 \to -1$
So by above Corollary this sequence diverges.
Example: $(b^n)_{n=0}^{\infty}$
Suppose $0 \leq b \leq 1$
inductively we find that $0 \leq b^{n+1} \leq b^n$ So b^n is decreasing
Moreover, b^n is bounded below by 0
Using the monotone convergence theorem we know that $(b^n)_n \to L$

So using the above proposition we know that the subsequence given by $n_k = 2k$ converges to L
$(b^{2k}) = (b^k b^k)_k = (b^k)_k (b^k)_k \to LL = L^2$ So by the uniqueness of limit we know $L^2 = L$ meaning $L = 0 \vee L = 1$
if $b = 1 \to 1$
if $b < 1$ then L=0 since b^n is decreasing.
Example: Let $c > 1$, consider $x_n = c^{\frac{1}{n}}$ show $(x_n)_n \to 1$
Proof:
Given that $1 < c$, inductively we find that $c^n \leq c^{n+1} \implies {}^{?} c^{\frac{1}{n+1}} \leq c^{\frac{1}{n}}$
moreover $(x_n)_n$ is bounded
below by 1
By the monotone convergence Theorem this converges to L
Now consider the subsequence $n_k = 2k$
by the above proposition,
$(c^{\frac{1}{2k}})_k \to L$
$(c^{\frac{1}{2k}})(c^{\frac{1}{2k}}) = (c^{\frac{1}{2k}+\frac{1}{2k}}) = (c^{\frac{1}{k}}) \implies L = L^2$ So since this is decreasing and bounded below by 1 we can deduce it will be 1
Recall,

$$(x_n) \not\to x \iff (\exists \varepsilon_0 > 0) : (\forall N \in \mathbb{N})(\exists n \geq N)(|x_n - x| \geq \varepsilon_0)$$

Proposition: Let $(x_n)_n$ be a real sequence and let $x \in \mathbb{R}$

$$(x_n)_n \not\to x \iff (\exists \varepsilon_0 > 0) \wedge (\exists (x_{n_k})_k \text{ a subsequence}) : (\forall k \in \mathbb{N}, |x_{n_k} - x| \geq \varepsilon_0)$$

Proof:
$\impliedby$:
Given $\exists \varepsilon_0 > 0$ and a subsequence $(x_{n_k})_k$ where for all k, $x_{n_k} \notin V_{\varepsilon_0}(x)$ then suppose for contradiction $(x_n)_n$ converges
then by the above proposition, any subsequence must converge to x
But this is not possible since all the terms of the subsequence lie outside of the ε_0 neighborhood of x
$\implies$:
Given that $(x_n) \not\to x$ by definition $(\exists \varepsilon_0 > 0) : (\forall n \in \mathbb{N})(\exists n_m \geq n, |x_{n_m} - x| \geq \varepsilon_0)$
Fix $\varepsilon_0 > 0$,
Let $m = 1 \implies \exists n_1 : |x_{n_1} - x| \geq \varepsilon_0$
Next set $m = n_1 + 1$ Notice $(n_1 + 1 > 1) \implies \exists n_2 > n_1 : |x_{n_2} - x| \geq \varepsilon_0$
having chosen x_{n_k},
set $m = n_k + 1 \implies \exists n_{k+1} \geq n_k + 1 : |x_{n_{k+1}} - x| \geq \varepsilon_0$
so now by construction we have a subsequence whose terms lie entirely outside of $V_{\varepsilon_0}(x)$
Lemma:
Let $(x_n)_n$ be any real sequence
It always admits a monotone subsequence
Terminology:
A peak of a sequence is a term x_m such that $x_n \leq x_m, \forall n \geq m$

e.g. a strictly increasing sequence has no peaks
e.g. a strictly decreasing sequence has infinity many peaks
Proof:
Case 1: Infinitinly many peaks
Let x_{n_1}, x_{n_2} be the first and second peaks. In this case we can continue to list the peaks by increasing subscripts

$$x_{m_1} \geq x_{m_2} \geq ..... \geq x_{m_k} \geq ....$$

so $(x_{m_k})_k$ is a decreasing monotone subsequence of $(x_n)_n$
Case 2:(Finite number of peaks)
if $(x_n)_n$ has no peaks then it is increasing so assume it has some
Let $x_{m_1}, x_{m_2}, ..., x_{m_k}$ be our peaks
Let $n_1 = m_k + 1$
x_{n_1} is not a peak since its past the last peak
so $\exists n_2 > n_1 : x_{n_2} > x_{n_1}$ Since x_{n_2} is not a peak, there exists $n_3 > n_2 : x_{n_3} > x_{n_2}$
Continuing this way, we obtain an increasing subsequence of $(x_n)_n$
Theorem:(**Bolzano – Weierstrass**)

If $(x_n)_n$ is a bounded sequence

Then it admits a convergent subsequence
Proof: Given $(x_n)_n$ is bounded
Using the Peaks Lemma there exists a monotone subsequence
Since the subsequence $(x_{n_k})_k$ is monotone and bounded,
By The Monotone Convergence Theorem, $(x_{n_k})_k$ converges. $\square$

Limit Superior and Limit Inferior
Idea: Given a family of sets $\{A_k\}_k$ where A_k is countable,
Let $B_n = \bigcap_{k \geq n} A_k$
Let $L = \bigcup_{n \geq 1} B_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k$
$x \in L \iff \exists n \geq 1 : x \in B_n \iff (\exists n \geq 1)(\forall k \geq n)(x \in A_k)$
Now Set $L' = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k$
$x \in L' \iff (\forall n \geq 1)(\exists k \geq n)(x \in A_k)$
We see that L is the set of elements which are all eventually in $A_k, k \geq n$
Similarly we see that L' is the set of elements which are contained in some $A_k, k \geq n$
Limit Points/Subsequential limits
Let $(x_n)_n = X$ be a bounded sequence. By Bolzano-Weierstrass, there exists a convergent subsequence $(x_{n_k}) \to t$
We define $\bar{X} := \{t \in \mathbb{R} \mid \exists (x_{n_k})_k \to t\}$
We call such a t a subsequential limit or a limit point of X
e.g $X = ((-1)^n)_n, \bar{X} = \{-1, 1\}$
Example:
In this example we consider the set $S = \mathbb{Q} \cap [0, 1]$

this is the a countable set. So we enumerate it into the sequence $(r_n)_n = R$
We will show that $\bar{R} = [0,1]$

<u>Proof:</u>

<u>Claim 1:</u> $[0,1] \subseteq \bar{R}$

<u>Proof:</u>

Let $x \in [0,1]$ We will construct a subsequence converging to x
First consider the set $R_1 := \{n \mid r_n \in (x-1, x+1) \cap [0,1]\}, \varepsilon_1 = 1$
Because $\mathbb{Q} \subseteq \mathbb{R}$ is dense $\exists r_{n_1} \in \mathbb{Q} s.t. r_{n_1} \in (x-1, x+1) \cap [0,1]$
So $R_1 \neq \emptyset$, using the well ordering principle, we choose the least element n_1
Next Consider $R_2 := \{n_2 > n_1 \mid r_n \in (x - \frac{1}{2}, x + \frac{1}{2}) \cap [0,1]\}, \varepsilon_2 = \frac{1}{2}$
Because $\mathbb{Q} \subseteq \mathbb{R}$ is dense $\exists r_{n_2} \in \mathbb{Q} s.t. r_{n_2} \in (x-1, x+1) \cap [0,1]$
So $R_1 \neq \emptyset$, using the well ordering principle, we choose the least element n_2
Inductively, let $R_k := \{n_k > n_{k-1} \mid r_n \in (x - \frac{1}{k}, x + \frac{1}{k}) \cap [0,1]\}, \varepsilon_2 = \frac{1}{k}$ and use the well ordering principle to find $n_k > n_{k-1}, r_{n_k} \in (x - \frac{1}{k}, x + \frac{1}{k}) \cap [0,1]$
So by construction, we have $|r_{n_k} - x| < \varepsilon_k, \forall k$
Using the fundamental convergence lemma we are done.
$\therefore (r_{n_k})_k \to x$

<u>Claim 2:</u> $\bar{R} \subseteq [0,1]$

<u>Proof:</u>

Suppose for contradiction, $x \in \bar{R}, x \notin [0,1]$
Because $x \notin [0,1], x < 0 \vee x > 1$
if $x < 0$
Let $\varepsilon = \frac{|x|}{2} = \frac{-x}{2}$
Since $x \in \bar{R} \implies \exists (r_{n_k})_k \to x$

$$\implies \exists N(\varepsilon) \in \mathbb{N} : \forall n \geq N(\varepsilon), |r_{n_k} - x| < \varepsilon$$

$$\implies r_{n_k} < \varepsilon + x = \frac{-x}{2} + x = \frac{x}{2} < 0 \implies r_{n_k} < 0\#$$

if $x > 1$
Let $\varepsilon = \frac{x-1}{2}$
Since $x \in \bar{R} \implies \exists (r_{n_k})_k \to x$

$$\implies \exists N(\varepsilon) \in \mathbb{N} : \forall n \geq N(\varepsilon), |r_{n_k} - x| < \varepsilon$$

$$\implies r_{n_k} > -\varepsilon + x = \frac{-x+1}{2} + x = \frac{x+1}{2} > 0 \implies r_{n_k} > 1\#$$

By Inclusion $\bar{R} = [0,1]$ $\square$ To motive our discussion of the limit superior and limit inferior we consider the bounded sequence $(x_n)_{n=1} = (x_1, x_2, x_3, ..., x_n, ...)$
Let $u_1 = sup_{k \geq 1}(x_k)$
and $l_1 = inf_{k \geq 1}(x_k)$. we can see that $l_1 \leq u_1$
Now consider the same sequence but without the first term
$(x_n)_{n=2} = (x_2, x_3, ..., x_n, ...)$
Let $u_2 = sup_{k \geq 2}(x_k)$
and $l_2 = inf_{k \geq 2}(x_k)$. we can see that $l_2 \leq u_2$
But more importantly we notice since $(x_n)_{n=1}$ has more terms than $(x_n)_{n=2}$
We have $l_1 \leq l_2 \leq u_2 \leq u_1$

Also notice how we are forming a family of bounded nested intervals $I_1 \supseteq I_2$
Inductively we set $u_n = sup_{k \geq n}(x_k)$
and $l_n = inf_{k \geq n}(x_k), I_n = [l_n, u_n]$
By the nested intervals property:

$$\bigcap_{n=1}^{\infty} I_n = [l, u]$$

Where, $l = sup_{n \geq 1}(l_n) = sup_{n \geq 1}(inf_{k \geq n}(x_k) = limit_{n \to \infty}(inf_{k \geq n}(x_k)) := liminf(x_k)$

and

$u = inf_{n \geq 1}(l_n) = inf_{n \geq 1}(sup_{k \geq n}(x_k) = limit_{n \to \infty}(sup_{k \geq n}(x_k)) := limsup(x_k)$

Now we claim that $\bar{X} \subseteq [l, u]$
Proof: Suppose that $t \in \bar{X}$
by definition $\exists (x_{n_k}) \to t$, note that $t \in [l_n, u_n], \forall n \geq 1$
because if we where given n, $K(\varepsilon)$, we have $n_k \geq K(\varepsilon) \implies l_n \leq x_{n_k} \leq u_n$
so if we had $k \to \infty$, $l_n \leq t \leq u_n, \forall n \geq 1$
Since n is arbitrary this is true for all n
so $\bar{X} \subseteq \bigcap_{n \geq 1} [l_n, u_n] = [l, u]$
Example: $x_n = \begin{cases} 2 + \frac{1}{n} & n \text{ even,} \\ -\frac{1}{n} & n \text{ odd.} \end{cases}$
find the limit inferior and limit superior
Solution:
First we will notice that $(x_{2n})_n \to 2, (x_{2n+1})_n \to 0$
and by using the above work we can deduce that $liminf = 0, limsup = 2$
Proposition: Let $(x_n)_n$ be a bounded sequence,

$$(1) : liminf(x_n) \leq limsup(x_n)$$

$$(2) : limsup(x_n) = liminf(x_n) = x \iff (x_n)_n \to x$$

We proved (1) with the nested intervals property above
To prove $\implies$: we know that $l_n \leq x_n \leq u_n$ by the squeeze theorem becuase $l_n \to x, u_n \to x \implies x_n \to x$
Proof: $\impliedby$
Let $u_n = sup_{k \geq n}(x_k), l_n = inf_{k \geq n}(x_k)$
Applying the sup lemma on $\{x_k \mid k \geq n\}$ we know the $\exists k_n \geq n : x_{k_n} \geq u_n - \frac{1}{n}$
Also since u_n is an upper bound we have

$$u_n - \frac{1}{n} \leq x_{k_n} \leq u_n$$

As $n \to \infty$, by the squeeze theorem we see that $(x_{k_n})_n \to limsup(x_k)$
Similarly Applying the inf lemma on $\{x_k \mid k \geq n\}$ we know the $\exists k_n \geq n : x_{m_n} \leq l_n + \frac{1}{n}$
Also since l_n is a lower bound we have

$$l_n + \frac{1}{n} \geq x_{m_n} \geq l_n$$

As $n \to \infty$, by the squeeze theorem we see that $(x_{m_n})_n \to liminf(x_m)$
Since $(x_{m_n})_n, (x_{k_n})_n$ are subsequences of $(x_n)_n$ by uniqueness of limit we must have

$$liminf(x_n) = limsup(x)n) = x$$

$\square$

1.5 Cauchy's Convergence Criterion

Definition:
Let (X, d) be a metric space, $(x_n)_n$ a sequence in X
if $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N,\ d(x_n, x_m) < \varepsilon$ then $(x_n)_n$ is Cauchy
Proposition:
Let $(x_n)_n$ be a sequence in $\mathbb{R}$

$$(x_n)_n \text{ convergent } \implies (x_n)_n \text{ Cauchy}$$

Proof:
Let $(x_n)_n \to x \iff \exists N(\varepsilon) \in \mathbb{N} : \forall n \geq N(\varepsilon), x_n \in V_\varepsilon(x)$
next find $m, n \geq N(\varepsilon) \implies x_n \in V_\varepsilon(x), x_m \in V_\varepsilon(x)$
$\implies |x_m - x| \leq \frac{\varepsilon}{2}, |x_n - x| \leq \frac{\varepsilon}{2}$
adding these inequalities we get

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon > |x_n - x| + |x_m - x| \geq |x_n - x_m|$$

Proposition:
Let $(x_n)_n$ be a sequence in $\mathbb{R}$

$$(x_n)_n \text{ Cauchy } \implies (x_n)_n \text{ Bounded}$$

Proof:
Given that $(x_n)_n$ is Cauchy,
Let $\varepsilon = 1, \exists N_1 \in \mathbb{N} : \forall m, n \geq N_1, |x_m - x_n| < 1$
In particular if $n \geq N$
$|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| \leq 1 + |x_N|$
Set $C = max\{|x_1|, |x_2|, ..., |x_N|, |x_N| + 1\}$
then $\forall n \geq 1, |x_n| \leq C$
$\therefore (x_n)_n$ is bounded $\square$
Lemma: If $(x_n)_n$ is Cauchy and $(x_{n_k})_k \to x$ for some subsequence then $(x_n)_n \to x$
Proof:
Given that $(x_n)_n$ is Cauchy and $\varepsilon > 0$,
$\exists N \in \mathbb{N} : |x_n - x_m| < \frac{\varepsilon}{2}, \forall m, n \geq N$
Similarly since we are given $(x_{n_k})_k \to x$,
$\exists K \in \mathbb{N} : |x_k - K| < \frac{\varepsilon}{2}, \forall k \geq K$
Set $M = max\{N, K\}$ So for $n_M \geq M \geq K$ Then since $(x_{n_k})_k$ converges

$$\implies |x_{n_M} - x| < \frac{\varepsilon}{2}$$

Since $(x_n)_n$ is Cauchy we know:

$$n \geq N \implies |x_n - x_M| < \frac{\varepsilon}{2}$$

Adding these we get

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon > |x_n - x_M| + |x_M - x| \geq |x_n - x|$$

$\therefore (x_n)_n \to x$
Theorem:(**Cauchy′s Convergence Criterion**)

Let $(x_n)_n$ be a real sequence

$$(x_n)_n \text{ Cauchy} \iff (x_n)_n \text{ Convergent}$$

Proof:
We already showed that Convergent $\implies$ Cauchy using a clever choice of $\frac{\varepsilon}{2}$ and the triangle inequality
Now it is simple to show Cauchy $\implies$ Convergent:
We Showed

$$\text{Cauchy} \implies \text{Bounded}$$

$$\text{By Bolzano Weierstrass, } \exists (x_{n_k})_k \text{ Convergent}$$

By the Lemma above we showed if $\exists(x_{n_k})$ Convergent and $(x_n)_n$ Cauchy

Then $(x_n)_n$ is convergent $\square$
Example: $h_n = \sum_{k=1}^{n} \frac{1}{k}$, Show $(h_n)_n$ diverges
Proof:
Let $m > n \implies h_m - h_n = \sum_{k=n+1}^{m} \frac{1}{k}$

$$\sum_{k=n+1}^{m} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{m} \geq \frac{1}{m} + \frac{1}{m} + ... + \frac{1}{m}$$

Because $m > n$

$$\frac{1}{m} + \frac{1}{m} + ... + \frac{1}{m} = \frac{m-n}{m} = 1 - \frac{n}{m}$$

Now Suppose that $m = 2n$
then $|h_{2n} - h_n| \geq 1 - \frac{n}{2n} = 1 - \frac{1}{2} = \frac{1}{2}$
So when $\varepsilon = \frac{1}{2}, |h_{2n} - h_n| \geq \frac{1}{2}$
So by C.C.C this sequence diverges
Example:
$a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$, Show $(a_n)_n$ converges
Aside:
Let $m > n \implies |a_m - a_n| = |\sum_{k=0}^{m} \frac{(-1)^k}{k!} - \sum_{k=0}^{n} \frac{(-1)^k}{k!}| = |\sum_{k=n+1}^{m} \frac{(-1)^k}{k!}| \leq \sum_{k=n+1}^{m} |\frac{(-1)^k}{k!}| = \sum_{k=n+1}^{m} \frac{1}{k!} \leq \sum_{k=n+1}^{n} \frac{1}{2^{k-1}} = \frac{1}{2^n} + \frac{1}{2^n+1} + ... + \frac{1}{2^m-1} =$

$\frac{1}{2^n}(1+\frac{1}{2}+...+\frac{1}{2^{m-n-1}} \leq \frac{1}{2^n}(2) = \frac{1}{2^{n-1}}$
Proof:
Let $\varepsilon > 0$ be given
find $N \in \mathbb{N} : 2^{N-1} > \frac{1}{\varepsilon}$
Then if $m > n \geq N$,
$|a_m - a_n| \leq \frac{1}{2^{n-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$ □
Definition:
Let $(x_n)_n$ be sequence
If $\exists \rho : 0 < \rho < 1, |x_{n+1} - x_n < \rho|x_{n-1} - x|$
Then $(x_n)_n$ is called contractive
Proposition:
Let $(x_n)_n$ be a real sequence
$(x_n)_n$ Contractive $\implies$ $(x_n)_n$ Cauchy
Proof:
Given $(x_n)_n$ contractive, $\implies$ $\exists 0 < \rho < 1 : |x_{n+1} - x_n| < \rho|x_{n-1} - x_n|$, Let $m > n$

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m+1} - .... + x_{n+1} - x_n|$$

By the Triangle Inequality,

$$|x_m - x_{m-1} + x_{m+1} - .... + x_{n+1} - x_n| \leq |x_m - x_{m-1}| + |x_{m+1} - x_{m-2}| + ... + |x_{n+1} - x_n|$$

Inductively using contractivity, we find

$$|x_m - x_{m-1}| + |x_{m+1} - x_{m-2}| + ... + |x_{n+1} - x_n| \leq \rho^{m-2}|x_2 - x_1| + \rho^{m-3}|x_2 - x_1| + ... + \rho^{n-1}|x_2 - x_1|$$

$$\rho^{m-2}|x_2 - x_1| + \rho^{m-3}|x_2 - x_1| + ... + \rho^{n-1}|x_2 - x_1| = \rho^{n-1}|x_2 - x_1|(1 + \rho + \rho^2 + ... + \rho^{m-n-1})$$

$$= \rho^{n-1}|x_2 - x_1|(\frac{1-\rho^{m-n}}{1-\rho}) \leq \frac{|x_2 - x_1|}{1-\rho}\rho^{n-1}$$

So let $\varepsilon > 0$ be given,
find $N \in \mathbb{N} : \frac{|x_2 - x_1|}{1-\rho}\rho^{N-1} < \varepsilon$
then if $m, n > N$
$|x_m - x_n| \leq \frac{|x_2 - x_1|}{1-\rho}\rho^{N-1} < \varepsilon$
Example: We ill consider the following sequence given recursively $x_0 = 0, x_1 = 1, x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$
Show this sequence is convergent Proof:
We will show that this sequence is contractive

$$|x_{n+1} - x_n| = |\frac{1}{2}(x_n + x_{n-1}) - x_n| = |\frac{-1}{2}x_n + \frac{1}{2}x_{n-1}| = \frac{1}{2}|x_{n-1} - x_n|$$

Hence contractive
$\implies$ Cauchy
By C.C.C. $(x_n)_n$ converges
□ Example:
Recall the Fibbonachi Sequence, $f_{n+1} = f_n + f_{n-1}$, We show that $\varphi_n = \frac{f_{n+1}}{f_n}$is

contractive
Solution:
$\varphi_n = \frac{f_n+f_{n-1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} = 1 + \frac{1}{\varphi_{n-1}}$
Then consider:
$|\varphi_{n+1} - \varphi_n| = |1 + \frac{1}{\varphi_n} - 1 - \frac{1}{\varphi_{n-1}}| = |\frac{1}{\varphi_n} - \frac{1}{\varphi_{n-1}}| = \frac{|\varphi_{n-1}-\varphi_n|}{\varphi_n\varphi_{n-1}}$
Now, we claim that $\frac{3}{2} \leq \varphi_n \leq 2, \forall n \geq 3$
Proof:
Base Case n=3
we see that $\varphi_3 = \frac{3}{2}$
Suppose for m $\frac{3}{2} \leq \varphi_n \leq 2$
Indeed,

$$\frac{2}{3} \leq \frac{1}{\varphi_n} \leq \frac{1}{2} \iff \frac{2}{3} + 1 \leq \frac{1}{\varphi_n} + 1 \leq \frac{1}{2} + 1 \iff \frac{5}{3} \geq \varphi_{n+1} \geq \frac{3}{2}$$

So

$$|x_{n+1} - x_n| \leq \frac{4}{9}|x_{n-1} - x_n|$$

Hence Contractive, thus by C.C.C. convergent
so applying the m-th tail lemma we see that

$$\varphi_{n+1} = 1 + \frac{1}{\varphi_n} \to \varphi = 1 + \frac{1}{\varphi}$$

After some algebra we can deduce that
$\varphi = \frac{1+\sqrt{5}}{2}$

1.6 Properly Divergent Sequences

Definition: Let $(x_n)_n$ be a real sequence,
(i) $(x_n)_n$ Diverges Properly to $+\infty$ if $(\forall M > 0)(\exists N \in \mathbb{N}) : (\forall n \geq N, x_n \geq M)$
(ii) $(x_n)_n$ Diverges Properly to $-\infty$ if $(\forall M < 0)(\exists N \in \mathbb{N}) : (\forall n \geq N, x_n \leq M)$
(iii) We says $(x_n)_n$ diverges properly if (i) or (ii)
Example:
Show $x_n = n^2$ diverges properly to $+\infty$
Proof:
Let $M > 0$ be given
By the archimedian property $\exists N \in \mathbb{N} : M < N$
Suppose $n \geq N$
Then $M \leq N \leq n \leq n^2$ $\square$
Example: Let $b > 1$, show $(b^n)_n \to +\infty$
Proof:
Write $b = 1 + a, a > 0$

$$\implies b^n = (1+a)^n \geq 1 + na \geq na$$

So given $M > 0$ find $N > \frac{M}{a}$
Suppose $n \geq N \implies na \geq NA \geq \frac{M}{a}a = M$

Proposition:
Let $(x_n)_n$ be monotone increasing,

$$(x_n)_n \text{ Unbounded above} \iff (x_n)_n \to +\infty$$

Proof:
Let $(x_n)_n$ be unbounded above $\implies (\forall M > 0)(\exists N \in \mathbb{N}) : x_N > M$
Since $(x_n)_n$ is monotone increasing if $n \geq N$ then $x_n \geq x_N > M$

Proposition: Let $(x_n)_n, (y_n)_n$ be strictly positive sequences,
Suppose $(\frac{x_n}{y_n}) \to L > 0$
Then $x_n \to +\infty \iff y_n \to +\infty$
Proof:
Since $\frac{x_n}{y_n} \to L$ Let $\varepsilon = \frac{L}{2}$
$\exists N \in \mathbb{N} : \forall n \geq N, \frac{x_n}{y_n} \in V_\varepsilon(L)$

$$\implies \frac{-L}{2} < |\frac{x_n}{y_n} - L| < \frac{L}{2} \implies \frac{Ly_n}{2} < x_n < \frac{3L}{2} y_n$$

if $x_n \to +\infty$
then given $M > 0$, find $K \in \mathbb{N} : x_n \geq \frac{2L}{3} M$
Then for $n \geq max\{K, M\}$
$\frac{2}{3L} x_n \geq \frac{2}{3L} \frac{3LM}{2} = M$

1.7 Sequences Of Functions

Throughout, fix $\Omega \neq \emptyset$ a subset of $\mathbb{R}$
And recall, $\mathcal{F}(\Omega, \mathbb{R}) = \{f \mid f : \Omega \to \mathbb{R}\}$
Definition:
A sequence $(f_n)_n$ in $\mathcal{F}$ converges to $f \in \mathcal{F}$ **Pointwise** on Ω
if $x \in \Omega, (f_n(x))_n \to f(x)$
equivelently,

$$(\forall x \in \Omega)(\forall \varepsilon > 0)(\exists N(x, \varepsilon) \in \mathbb{N}) : (\forall n \geq N(x, \varepsilon))(|f_n(x) - f(x)| < \varepsilon)$$

Note: $N(x, \varepsilon)$ depends on both x and ε
Example:
Consider $(f_n : \mathbb{R} \to \mathbb{R})_n, f_n(x) = \frac{nx}{1+n^2x^2}$
we will show that $(f_n)_n \to 0$(function)
Proof:
if x=0, $f_n(0) = 0$
so $(f_n(0) = 0)_n \to 0$(constant sequence)
if $x \neq 0$ we claim that $(f_n(x)) \to 0$

$$|f_n(x) - 0| = |\frac{nx}{1 + n^2x^2}| \leq \frac{n|x|}{n^2x^2} = \frac{1}{n|x|} \to 0$$

SO $(f_n)_n \to 0$ point wise on $\mathbb{R}$

Example
$(g_n : [0,1] \to \mathbb{R})_n, g_n(x) = x^n$
Show that $(g_n)_n \to \delta_1 = \begin{cases} 0 & x \neq 1, \\ 1 & x = 1. \end{cases}$ (The Point mass at 1)
Proof:
for $0 \leq x < 1, g_n(x) = x^n \to 0$
for $x = 1, g_n(1) = 1 \to 1$
So $(g_n)_n \to \delta_1$ pointwise on $[0,1]$
Example
$h_n : [0,\infty) \to \mathbb{R}, h_n(x) = x^{\frac{1}{n}}$
Show $(h_n)_n \to \mathbb{1}_{(0,\infty)}$
Proof:
if $x = 0, h_n(0) = 0 \to 0$(constant)
if $0 < x, h_n(x) = x^{\frac{1}{n}} \to 1$
so $h(x) = \mathbb{1}_{(0,\infty)}$
$\implies (h_n)_n \to \mathbb{1}_{(0,\infty)}$ Pointwise on $[0,\infty)$
Example:
$(k_n : [0,\infty) \to \mathbb{R})_n, k_n = e^{-nx}$
show $(k_n)_n \to 0$
Proof:
if $x = 0, k_n(0) = 1 \to 1$(constant)
if $x > 0, k_n(0) = e^{-nx}$
So now consider the following

$$e^{nx} \geq 1 + nx \iff e^{-nx} \leq \frac{1}{1+nx} \leq \frac{1}{nx}$$

So $e^{-nx} \leq \frac{1}{nx} \to 0$
$\implies (k_n)_n \to \delta_0$
Example:
$(f_n : [0,\infty) \to \mathbb{R})_n, f_n(x) = \frac{x^n}{1+x^n}$
Solution:
$x = 0 \implies f_n(0) = \frac{0^n}{1+0^n} = 0$
$0 < x < 1 \implies f_n(x) = \frac{x^n}{1+x^n} \leq x^n \to 0$
$x = 1 \implies f_n(1) = \frac{1^n}{1+1^n} = \frac{1}{2}$
$1 < x \implies f_n(x) = \frac{x^n}{1+x^n} \leq \frac{x^n}{x^n} = 1 \to 1$
So

$$(f_n)_n \to \begin{cases} 0 & 0 \leq x < 1, \\ \frac{1}{2} & x = 1, \\ 1 & x > 1. \end{cases}$$

Point wise on $[0,\infty)$
To motivate the following definition we will look at the following example
Consider $(f_n(x) = n\mathbb{1}_{[0,\frac{1}{n}]})_n$

$$lim_{n\to\infty}(\int_0^1 f_n) = 1, (\int_0^1 lim_{n\to\infty} f_n) = 0,$$

So we see that you cant interchange the two limiting operations. But there are cases where this interchange is justified.

Definition:

If $(f_n)_n$ is a sequence in $\mathcal{F}(\Omega, \mathbb{R}), f \in \mathcal{F}$

$(f_n)_n \to f,$ **Uniformaly on $\boldsymbol{\Omega}$** $\iff (\forall\varepsilon > 0)(\exists N(\varepsilon) \in \mathbb{N})(\forall n \geq N(\varepsilon))(\forall x \in \Omega), (|f_n(x) - f(x)| < \varepsilon)$

Alternatively we can say

$(f_n)_n \to f,$ **Uniformaly on $\boldsymbol{\Omega}$** $\iff (\forall\varepsilon > 0)(\exists N(\varepsilon) \in \mathbb{N})(\forall n \geq N(\varepsilon)), sup_{x\in\Omega}(|f_n - f|) < \varepsilon)$

Example:

$(f_n : [0, 4] \to \mathbb{R})_n, f_n(x) = \frac{x}{x+n}$ Show that $(f_n)_n \to 0$ Uniformally on $[0, 4]$

Proof:

Pointwise we can see that:

$$|f_n(x) - 0| = |\frac{x}{x+n}| \leq \frac{x}{n}$$

However, even stronger we have

$$\frac{x}{n} \leq \frac{4}{n} \implies sup_{x\in[0,4]}(f_n(x) - 0) \leq \frac{4}{n}$$

More formally we can write the following

Given $\varepsilon > 0$

Find $N \in \mathbb{N} : N > \frac{4}{\varepsilon}$

Suppose $n \geq N$

$$sup_{x\in[0,4]}(|f_n(x) - 0|) \leq \frac{4}{n} \leq \frac{4}{N} < \frac{4\varepsilon}{4} = \varepsilon$$

$\therefore (f_n) \to 0$ Uniformly on $[0, 4]$

Example:

$(h_n : (1, \infty) = \Omega \to \mathbb{R})_n, h_n(x) = e^{-nx}$

Show that $(h_n)_n \to 0$ uniformally on $(1, \infty)$

Aside:

$$sup_{x\in\Omega}(|h_n - 0|) = sup_{x\in\Omega}(e^{-nx}) \leq \frac{1}{nx+1} \leq \frac{1}{nx}$$

Since $x > 1 \iff \frac{1}{x} < 1 \iff \frac{1}{nx} < \frac{1}{n}$

So $sup_{x\in\Omega}(|h_n - 0|) \leq \frac{1}{n}$

Proof:

Given $\varepsilon > 0$

Let $N \in \mathbb{N} : N > \frac{1}{\varepsilon}$

Suppose $n \geq N$

$$sup_{x\in\Omega}(|h_n - 0|) \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Observe: $g_n(x) = x^n, g_n : [0,1] \to \mathbb{R}$
we show that $(g_n)_n \to \delta_1$ point wise
Is this uniform?
$|g_n(x) - \delta_1(x)| =^{x\neq 0} |x^n - 0| = x^n \leq sup_x(g_n - \delta_1) = 1$
Lemma: (Non-Uniform Convergence)

$(f_n)_n$ does not converge to f uniformally on $\Omega \iff (\exists \varepsilon_0 > 0)(\exists x_k \in \Omega)(\exists (f_{n_k})_k$ a subsequence$)(|f_{n_k}(x_k) - f($

Example:
$g_n(x) = x^n$ on $[0,1], x_k = (\frac{1}{2})^{\frac{1}{k}}$
Notice $(x_k)_k \to 1$
Let $n_k = k, |g_{n_k}(x_k) - \delta_1(x_k)| \leq |((\frac{1}{2})^k)^{\frac{1}{k}} - 0| = \frac{1}{2}$ So we let $\varepsilon_0 = \frac{1}{2}$
By the Lemma, g_n does not converge uniformally on Ω
Proposition:
(i) if $(f_n)_n \to f$ uniformally on Ω then $(f_n)_n \to f$ pointwise on Ω
(ii) if $(f_n)_n \to f$ and $(g_n)_n \to g$ pointwise on Ω then $f = g$
Proof:(Lemma)
Given $\varepsilon_0 > 0$,
for $N_1 = 1, \exists x_1, \exists n_1 \geq 1 : |f_1(x_1) - f(x_1)| \geq \varepsilon_0$
for $N_2 = N_1 + 1, \exists x_2, \exists n_2 \geq n_1 + 1 : |f_2(x_2) - f(x_2)| \geq \varepsilon_0$
Inductively set $N = n_k + 1, \exists n_{k+1} \geq n_k + 1 \exists x \in \Omega : |f_{k+1}(x) - f(x)| \geq \varepsilon_0$
Remark: $(l_\infty(\Omega), \|\cdot\|)$ forms a normed algebra
If $(f_n)_n$ and f are in $l_\infty(\Omega)$ we see that $(f_n)_n \to f$ uniformally on Ω
$\iff$
$(\|f_n - f\|)_n \to 0,$ In $\mathbb{R}$
So $(f_n)_n \to f$ uniformally on $\Omega \iff (\|f_n\|)_n \to f$
Example:
$(f_n : \mathbb{R} \to \mathbb{R})_n, f_n = \mathbb{1}_{[n,n+1]}$
$(f_n)_n \to 0$ pointwise on $\mathbb{R}$
pf:
Given $x \in \mathbb{R}, \varepsilon > 0$
Using the archimedian property $\exists N \in \mathbb{N} : N > x$
Suppose $n \geq N$

$$|\mathbb{1}_{[n,n+1]}(x) - 0| = |0 - 0| = 0$$

However, this convergence is not uniform since $\|\mathbb{1}_{[n,n+!]} - 0\| = 1$ so take $\varepsilon = 1$
Example:
$f_n = n\mathbb{1}_{(0,\frac{1}{n}]}, (f_n : [0,1] \to \mathbb{R})_n$
Proof:
Given $x \in [0,1]$
if $x = 0, f_n(0) = 0 \to 0$
if $x \neq 0$, by Archemedies $\exists N \in \mathbb{N}, \frac{1}{N} < x$

so if $n \geq N \implies f_n(x) = 0 \to 0$ However, $\|f_n - 0\|_u = n \not\to 0$
So this is not uniform on $[0,1]$
Example: $(g_n : [0,1] \to \mathbb{R})_n, g_n(x) = x^n(1-x)$
Proof:
First we will notice that since g_n is "continuous" on the closed and bounded interval $[0,1]$ it admits a maximum

$$\|g_n - 0\|_u = \|g_n\| = sup_{x\in[0,1]}(g_n(x)) =^{evt} max_{x\in[0,1]}(g_n(x))$$
$$= g_n(\frac{n}{n+1}) = (\frac{n}{n+1})^n(1 - \frac{n}{n+1}) \to \frac{1}{e}0$$

we found the max using Fermat's theorem and we knew it existed using the extreme value theorem.

1.8 Infinite Series

Definition:
Let $(x_k)_k$ be a sequence in $\mathbb{R}$
Let $S_0 = x_0, S_1 = x_1 + S_0, S_n = x_n + S_{n-1}$

$$S_n = \sum_{k=0}^{n} x_k \text{ Is called the sequence of partial sums}$$

if $(S_n)_n \to S$ we say that

$$\sum_{k=0}^{\infty} x_k = S$$

We can also write $\sum_{k=1}^{\infty} x_k < \infty$ if $x_k > 0$ to say it converges without knowing the exact limit
if $(S_n)_n$ diverges then we say $\sum x_k$ diverges
Example:(Geometric)
fix $b \in \mathbb{R}$
we can form the sequence of partial sums $S_n = \sum_{k=0}^{n} b^k = \begin{cases} \frac{1-b^{n+1}}{1-b} & b \neq 1, \\ n+1 & b = 1. \end{cases}$

And $(S_n)_n \to \begin{cases} \frac{1}{1-b} & |b| < 1, \\ diverges & otherwise \end{cases}$ More generally, if $|b| < 1$

$$\sum_{k=k_0}^{\infty} b^k = \frac{b^{k_0}}{1-b}$$

Example: $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+1})$
Recall we where able to calculate the partial sums so we found that

$$S_n = 1 - \frac{1}{n+1}$$

So $(S_n)_n \to 1$
Proposition:

Let $(x_k)_k, (y_k)_k$ be sequences if $\sum x_k, \sum y_k$ converges we have the following
(i) $\sum(x_k \pm y_k) = \sum x_k + \sum y_k$
(ii) $\sum tx_k = t \sum x_k$
Proof:
if we have the sequences of partial sums, $\sum^n x_k, \sum^n y_k$ by sum linearly we have $\sum^n(x_k \pm y_k) = \sum^n x_k \pm \sum^n y_k$ by the algebra of sequences we know that that limits hold this equality
Similarly we for the second case we have $t\sum_k^n x_k = \sum_k^n tx_k$ so we are done.
Proposition: Let $(x_k)_k$ be a sequence and fix $k_0 \in \mathbb{N}$

$$\sum_{k=1} x_k \text{ converges} \iff \sum_{k=k_0+1} x_k \text{ converges}$$

Proof:
$\implies$ Let $S_n = \sum_{k=1}^n x_k = x_1 + x_2 + x_3 + ... + x_n$,
$\sum_{k=k_0+1}^n x_k = S_n - x_{k_0} - x_{k_0-1} - ... - x_1$
Since $(S_n)_n \to S$ we find

$$\sum_{k=k_0+1}^{\infty} x_k = S - \sum_{k=1}^{k_0} x_k$$

So $\sum_{k=k_0+1}^{\infty} x_k$ Converges.
$\impliedby$ Similarly we can write $S_n = \sum_{k=k_0+1}^n x_k$,
$\sum_{k=k1}^n x_k = \sum_{k=k_0+1}^n x_k + x_{k_0} + x_{k_0+1} + ... + x_1$
So since $(S_n)_n \to S$
We have

$$\sum_{k=+1}^{\infty} x_k = S + \sum_{k=1}^{k_0} x_k$$

Proposition: Let $(x_k)_k$ he a real sequence
The Following Statements are Equivelent
(i) $\sum x_k$ converges
(ii) $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m > n \geq N)(|\sum_{k=n+1}^m x_k| < \varepsilon)$
(iii) $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(|\sum_{k>N} x_k| < \varepsilon)$
(iv) $(\sum_{k\geq n} x_k)_n \to 0$
Proof:
(i) $\implies$ (ii)
By C.C.C since $\sum x_k$ converges it is cauchy, so if $m > n$ $|\sum_{k=0}^n x_k - \sum_{k=0}^m x_k| = |\sum_{k=n+1}^m| < \varepsilon$
(ii) $\implies$ (iii)
Using (ii) if in Particular we choose $n = N$, then $\sum_{k=N+1}^m x_k = \sum_{k>N}^m x_k$
so $\forall m \geq k > N, |\sum_{k>N} x_k| < \varepsilon$
(iii) $\implies$ (iv)
Let $t_{N+1} = \sum_{k>N} x_k = \sum_{k=N+1} x_k \to 0$
As t_{N+1} is a subsequence to t_N using the previous proposition, $(t_N)_N \to 0$ by

uniqueness of limit.
(iv) $\implies$ (iv
$(\sum_{k\geq n} x_k)_n \to 0$
take n=1
then $\sum_{k\geq 1} x_k = \sum_{k=1}^{\infty} x_k$ is defined
Proposition:(Divergence Test)

$$\sum x_k \text{ Converges } \implies (x_k) \to 0$$

Proof:
This comes simply from our definition and uniqueness of limit,
Recall,

$$S_n = S_{n-1} + x_n \iff S_n - S_{n-1} = x_n$$

As $n \to \infty$

$$S - S = 0$$

So $(x_n)_n \to 0$
Example:
$\sum_{k=2}^{\infty} \frac{k}{\sqrt{k^2-1}}$ does not converge by above proposition.
Proposition:
Suppose $x_k \geq 0$

$$\sum x_k \text{ Converges } \iff (S_n)_n \text{ Bounded}$$

Proof:
$\implies$: $(S_n)_n$ Converges $\implies (S_n)_n$ Bounded
$\impliedby$ Monotone Convergence Theorem
Example:
$\sum_{k=1}^{n} \frac{1}{n} = S_n$
Solution:
$S_{2^n} \geq 1 + \frac{n}{2}$
Hence not bounded
Example:
$\sum_{k}^{n} \frac{1}{k^2}$ converges since it is bounded
$\sum_1 \frac{1}{k^2} \leq \sum \frac{1}{k(k-1)} + 1 = 1 - \frac{1}{n} + 1 \leq 2$
Important Example:
Claim: $\sum_0^{\infty} \frac{1}{k!} = e$
Proof:
$S_n = \sum_0^n \frac{1}{k!}, e_n = (1+\frac{1}{n})^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n(n-1)(n-2)...(n-k+1)}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} ... \frac{n-k+1}{n} = \sum_{k-0}^{n} \frac{1}{k!} 1(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{k-1}{n}) =$

$$\sum_{k=0}^{n} \frac{1}{k!} \prod_{j=1}^{k-1} (1-\frac{j}{n}) \leq \sum_{k=0}^{n} \frac{1}{k!} \leq 2 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \leq 3$$

So $e_n \leq S_n \to^{n\to\infty} e \leq S$
Next given m find $n \geq m$
$e_n = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \prod_{j=1}^{k-1}(1 - \frac{j}{n})$
$e_m = 1 + 1 + \sum_{k=2}^{m} \frac{1}{k!} \prod_{j=1}^{k-1}(1 - \frac{j}{n})$
as $n \to \infty$
$e_m \to \sum_{k=0}^{m} \frac{1}{k!} \to S$ So $e \geq S \wedge S \geq e \implies e = S$
Fact: $e - S_n < \frac{1}{nn!}$
Proof:

$$e-S_n = \sum_{k\geq n} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ... = \sum_{j=1} \frac{1}{(n+j)!} = \sum_{j=1} (\frac{1}{(n+1)!})(\frac{1}{(n+2)} + \frac{1}{(n+3)} + ... + \frac{1}{(n+j)})$$

$$< \frac{1}{(n+1)!} \sum_{j=1}^{n} (\frac{1}{n+t})^{j-1} = \frac{1}{(n+1)!} \sum_{k=0}^{n} (\frac{1}{n+1}) = \frac{1}{(n+1)!} \frac{1}{1-\frac{1}{n}} = \frac{1}{nn!}$$

Corollary: $e \notin \mathbb{Q}$
Proof:
Suppose for contradiction $e \in \mathbb{Q}$

$$\implies e = \frac{p}{q}, p, q \in \mathbb{N}$$

$$e - S_q < \frac{1}{q}\frac{1}{q!} \iff q!(e - S_q) < \frac{1}{q}$$

$$q!e = q!\frac{e}{q} = (q-1)!p \in \mathbb{Z}$$

$$q!S_q = q! \sum_{k=0}^{q} \frac{1}{k!} = \sum_{k=0}^{q} \frac{q!}{k!} \in \mathbb{Z}$$

So $q!(e - S_q) \in \mathbb{Z}$
and $0 < q!(e - S_q) < \frac{1}{q} \leq 1$ # Proposition: If $(x_k)_k, (y_k)_k$ are sequences such that $0 < x_k < y_k$

$$(i) \sum y_k \text{ Converges } \implies \sum x_k \text{ Converges}$$

$$(ii) \sum x_k \text{ diverges properly } \implies \sum x_k \text{ diverges properly}$$

Example: We say that $h_m = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges
Consider $F = \{n \in \mathbb{N} \mid$ n does not have a 3 in its decimal expansion$\}$
$= \{1, 2, 4, 5, 6, .., 14, ..., 29, 40, ...\}$
$F \subseteq \mathbb{N}$ hence countable
so we can enumerate F as $(n_k)_k$
We will show $\sum_{k=1}^{\infty} \frac{1}{n_k} \leq 80$
Proof: for $j \in \mathbb{N}_0$
$F_j = F \cap [10^j, 10^{j+1} - 1]$
$F_0 = F \cap [1, 9]$ We can just count there are 8

$F_1 = F \cap [10, 99]$ By the multiplication principle we have 8 times 9 Inductively we can find that $|F_j| = 8 \cdot 9^j$
If $n_k \in F_j, 10^j \leq n_k$

$$\sum_{k=1}^{\infty} \frac{1}{n_k} = \sum_{j=0}^{\infty} \sum_{n_k \in F_j} \frac{1}{n_k} =$$

Because $F = \bigsqcup_{i=0} F_i$

$$\sum_{j=0}^{\infty} \sum_{n_k \in F_j} \frac{1}{n_k} \leq \sum_{j=0}^{\infty} \frac{|F_j|}{10^j} = \sum_{j=0}^{\infty} \frac{8 \cdot 9^j}{10^j} = 8 \sum_{j=0}^{\infty} (\frac{9}{10})^j = 8 \frac{1}{1 - \frac{9}{10}} = 80$$

<u>Proposition:</u> (Limit Comperision Test)
<u>$(x_k)_k, (y_k)_k$</u> $: x_k, y_k \geq 0$
if $\sum y_k < \infty$ and $limsup(\frac{x_k}{y_k}) < \infty$
then $\sum x_k < \infty$ if $\sum y_k = \infty$ and $liminf(\frac{x_k}{y_k}) > 0$
then $\sum x_k = \infty$
<u>Proof:</u> Let $limsup(\frac{x_k}{y_k}) = L = inf_{n \geq 1}(sup_{k \geq n}(\frac{x_k}{y_k}))$ By the inf lemma, $\exists N \in \mathbb{N} : \forall k \geq N$

$$\frac{L}{2} \geq Sup(\frac{x_k}{y_k}) \leq \frac{x_k}{y_k}$$

$$\implies \frac{L}{2} y_k \geq x_k$$

$$\implies \sum_{k=0}^{n} \frac{L}{2} y_k \geq \sum_{k=0}^{n} x_k$$

So since $\sum y_k$ converges by the comparison test $\sum x_k$ converges <u>Proof:</u> Let $p = liminf(\frac{x_k}{y_k}) = sup_{n \geq 1}(inf_{k \geq n}(\frac{x_k}{y_k}))$ By the sup Lemma, $\exists N \in \mathbb{N} : \forall k \geq N$

$$\frac{p}{2} \leq inf(\frac{x_k}{y_k}) \leq \frac{x_k}{y_k}$$

$$\implies \frac{p}{2} \leq \frac{x_k}{y_k} \iff \frac{p}{2} y_k \leq x_k \implies \sum_{k=0}^{n} \frac{p}{2} y_k \leq \sum_{k=0}^{n} x_k$$

By the comparison test $\sum x_k$ diverges
<u>Proposition:</u> (Ratio Test)
Let $(x_k)_k$ be a sequence of strictly positive terms
$x_k > 0$
(i) $limsup(\frac{x_{k+1}}{x_k}) < 1 \implies \sum x_k < \infty$
(ii) $liminf(\frac{x_{k+1}}{x_k}) > 1 \implies \sum x_k = \infty$
<u>Proof:</u>
(i)
Let $P = limsup(\frac{x_{k+1}}{x_k}) = inf_{n \geq 1}(sup_{k \geq n}(\frac{x_{k+1}}{x_k}))$

Find $p < p' < 1$ Using the inf lemma, $\exists N \in \mathbb{N} : \forall k \geq N : 1 > p' > sup_{k \geq N}(\frac{x_{k+1}}{x_k}) \geq \frac{x_k+1}{x_k}$ In particular for N

$$\iff x_{N+1} < x_N p'$$

and we have

$$x_{N+3} < p' x_{N+2} < p'^2 x_{N+1} < p'^3 x_N$$

Inductivly,

$$x_{N+k} < p'^k x_N$$

So

$$\sum_{k \geq N} x_k < \sum_j p'^j x_N < \infty$$

Since $\sum p^j$ is geometric
by the comparison tests, $\sum_{k \geq N} x_k$
we showed using C.C.C. that if the tail of the series converges the series converges. Hence, $\sum x_k < \infty$
Proposition: (Root test)
Let $(x_k)_k$ be sequence of positive terms
$limsup((x_k)^{\frac{1}{k}}) = p$
(i) $p < 1 \implies \sum x_k < \infty$
(ii) $p > 1 \implies \sum x_k = \infty$
Proof: (i)
Given that $limsup((x_k)^{\frac{1}{k}}) = p < 1$
by the inf lemma, $\exists r : p < r < 1, \exists N \in \mathbb{N} : \forall k \geq N$

$$r > sup_{k \geq N}(x_k)^{\frac{1}{k}} > x_k^{\frac{1}{k}}$$

$$\iff r^k > x_k$$

$$\implies \sum_{k=0} r^k > \sum_{k=0} x_k$$

So by the comparison test we must have that $\sum x_k$ converges
Proof: (ii)
$\exists((x_{k_j})^{\frac{1}{k_j}})_j \to p$ a subsequence,
Since it converges, $\exists N \in \mathbb{N} : j \geq N$,

$$x_{k_j}^{\frac{1}{k_j}} > \frac{p+1}{2} > 1$$

$$\implies x_{k_j} > 1$$

So by the divergence test $(x_{k_j}) \not\to 0$
Proposition: Let $(x_k)_k$ be a sequence of strictly positive terms,

$$liminf(\frac{x_{k+1}}{x_k}) \leq liminf(x_k^{\frac{1}{k}}) \leq limsup(x_k^{\frac{1}{k}}) \leq limsup(\frac{x_{k+1}}{x_k})$$

This tells us that the root test is more powerful than the ratio test.
e.g. $a_n = \begin{cases} 1/2^n & \text{if } n \text{ is even} \\ 1/3^n & \text{if } n \text{ is odd} \end{cases}$

Proposition: (Cauchy's Condensation Test)
Let $(x_k)_k$ be a decreasing sequence of strictly positive terms,

$$\sum x_k < \infty \iff \sum 2^k x_{2^k} < \infty$$

Proof: Let

$$S_n = \sum_0^n x_k, t_n = \sum_0^n 2^k x_{2^k}$$

Lets write out S_{2^n} with a clever grouping

$$S_{2^n} = x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + ...(x_{2^{n-1}} + ... + x_{2^n - 1}) + x_{2^n}$$

Since x_n is decreasing we have

$$\leq x_1 + 2x_2 + 4x_4 + ... + 2^{n-1} x_{2^{n=1}} + x_{2^n} = \sum_{k=0}^{n-1} 2^n x_{2^n} + x_{2^n} = t_{n-1} + x_{2^n}$$

So $S_{2^n} \leq t_{n-1} + x_{2^n}$ So if $\sum 2^k x_{2^k}$ converges, then $(t_{n-1}) \to t$ so t_n is bounded and $(2^k x_{2^k})_k \to 0$
So $(x_{2^k})_k$ is bounded, So S_{2^n} is bounded-.
$S_n \leq S_{2^n} \implies S_n$ converges by the comparison test.
Example: (P-Series)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ Converges if } p > 1\text{; Diverges if p} < 1$$

Proof: Using the cauchy condensation test,

$$\sum_{k=1}^{n} 2^k \frac{1}{2^{kp}} = \sum_{k=1}^{n} \frac{1}{2^{k(p-1)}} = \sum_{k=1}^{n} (\frac{1}{2^{p-1}})^k$$

This is a geometric series which we know converges if

$$2^{1-p} < 1 \iff p > 1$$

Abel's Lemma
Let $(x_k)_k, (y_k)_k$ be sequence and let $\sum^n y_k = s_n$ where $s_0 = 0$ If $m > n$ then

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$$

Proof:
Notice $y_k = s_k - s_{k-1}$ so we can write

$$\sum_{k=n+1}^{m} x_k(s_k - s_{k-1}) = (x_m s_m - x_{n+1}s_n) + \sum_{k=n+1}^{m-1}(x_k - x_{k+1})s_k$$

By distributing the x_k
Proposition: (Dirichlet's Test)
If $(x_k)_k$ is a decreasing sequence with $(x_k)_k \to 0$ and $\sum_k^n y_k$ is bounded
Then $\sum x_k y_k$ converges
Proof: Let $|s_n = \sum_k^n y_k| \leq B$
Suppose $m > n$
By Abel's Lemma we have

$$|\sum_{k=n+1}^{m} x_k y_k| \leq (x_m + x_{n+1})M + \sum_{k=n+1}^{m-1}(x_k - x_k + 1)M = M[(x_m + x_{n+1}) + (x_{n+1} - x_m)] = 2x_{n+1}M$$

Since $x_{n+1} \to 0$ by the C.C.C, we have the series converges.
Proposition: (Abel's Test)
If $(x_k)_k$ is a convergent monotone sequence and the series $\sum y_n$ converges
then $\sum x_k y_k$ converges
Proof: If $(x_k)_k$ decreases to x, Let $u_k = x_- x$,so that $(u_k)_k$ decreases to 0 , Then
$x_n = x + u_n \implies x_n y_n = x y_n + y_n u_n$

$$\implies \sum x_k y_k = \sum x y_n + \sum u_n y_n$$

It follows from the by the circlet test that $\sum y_k u_k$ converges and $\sum y_k x$ is assumed to converge so $\sum x_k y_k$ converges
If $(x_k)_k$ is increasing, let $v_k = x - x_k$ such that $(v_k) \to 0$ and here we see $x_k = x - v_k \implies x_k y_k = x y_k - v_k y_k$ then continue the above argument.
Definition: A series $\sum x_k$ converges absolutely if the series $\sum |x_k|$ converges
If only $\sum x_k$ converges then it is called conditionally convergent.
Lemma: Let $\sum x_k$ be a series

$$\sum |x_k| < \infty \implies \sum x_k \text{ Converges}$$

Proof: Let $S_n = \sum^n x_k, t_n = \sum^n |x_k|$
Suppose $m > n$

$$|S_n - S_m| = |\sum_{k=n+1}^{m} x_k| \leq \sum_{k=n+1}^{m} |x_k| = |t_m - t_n|$$

Since t_m is convergent, it is Cauchy so S_m is Cauchy hence convergent.
Definition:
An alternating series is a series of the form $\sum(-1)^k b_k, b_k \geq 0$
Proposition:
if b_k decreases to 0, $\sum(-1)^k b_k$ converges
Proof:

Claim: $S_{2n+1} \geq S_{2n+3}$ and $S_{2n} \leq S_{4n}$
Since we know the odd sums are negative adding them gives a even smaller number and with the evens we get postive terms so we get a bigger number
So we have that

$$S_{2n} \leq S_{2n+1} \rightarrow s \leq t$$

Considering $|s-t| = lim|s_{2n} - s_{2n+1}| = lim|b_{2k}| = 0$ So $s = t$

1.9 Infinite Series of Functions

Definition:
Let $(f_k : \Omega \rightarrow \mathbb{R})_k$ be a sequence of functions on $\Omega \subseteq \mathbb{R}$
We then consider the sequence defined as such:

$$S_1(x) = f_k(x)$$

$$S_n = f_n + S_{n-1} = \sum_{k=1}^{n} f_k(x)$$

if $(S_n)_n \rightarrow S(x) \in \mathcal{F}(\Omega, \mathbb{R})$ pointwise then we says

$$\sum_{k=1}^{\infty} f_k(x) = S(x) \text{ pointwise on } \Omega$$

Note: Absolute convergence is still a thing with series of functions
Defintion:
if $(S_n)_n \rightarrow S(x) \in \mathcal{F}(\Omega, \mathbb{R})$ uniformally on Ω then we says

$$\sum_{k=1}^{\infty} f_k(x) = S(x) \text{ uniformally on } \Omega$$

Example: Show $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ Converges uniformally and absolutely on any closed and bounded subset of $\mathbb{R}$
proof:
Consider $[-M, M] \subseteq \mathbb{R}, M > 0$,
Let $S_n = \sum_{k=0}^{n} \frac{x^k}{k!}$,
if $x \in [-M, M]$

$$|f(x) - s_n(x)| = |\sum_{k=n+1}^{\infty} \frac{x^k}{k!}| \leq \sum_{k=n+1}^{\infty} |\frac{x^k}{k!}| \leq \sum_{k=n+1}^{\infty} |\frac{M^k}{k!}|$$

Because this is true for any x,

$$\|f - s_n\| \leq \sum_{k=n+1}^{\infty} \frac{M^k}{k!}$$

Since $\sum_{k=n+1}^{\infty} \frac{M^k}{k!}$ converges (Ratio test),
by Cauchy, its tail goes to 0
$\therefore S_n \to f$ Uniformly on $[-M, M]$
Theorem:(Weierstrass M-Test)
Given a sequence of functions $(f_n : \Omega \to \mathbb{R})_n$ and we have that $\sum_{k=0}^{\infty} \|f_n\|_{\Omega} < \infty$
Then,
$\sum_{k=0}^{\infty} f_k$ converges uniformly and absolutely on Ω to some bounded function f.
Proof: Given a sequence of functions $(f_k : \Omega \to \mathbb{R})_k$ for $x \in \Omega$,
Let $s_n(x) = \sum_{k=0}^{n} f_k(x)$
$\implies \sum_{k=0}^{n} |f_k(x)| \leq \sum_{k=0}^{n} \|f\|_{\Omega}$
By assumption $\sum_{k=0}^{\infty} \|f_k\|_{\Omega} < \infty$
Hence, $\sum_{k=0}^{n} |f_k(x)|$ are convergent.
Consider the following,

$$|f(x) - s_n(x)| \leq |\sum_{k>n} f_k(x)| \leq \sum_{k>n} |f_k(x)| \leq \sum_{k>n} \|f\|_{\Omega}$$

This holds for any x,

$$\implies sup(|f(x) - s_n(x)|) \leq \sum_{k \geq n} \|f\|_{\Omega}$$

$$\implies \|f - s_n\| \leq \sum_{k \geq n} \|f\|_{\Omega}$$

Which by Cauchy goes to zero.
$\therefore (f_n)_n \to f$ Uniformly and Absolutely on Ω

1.10 Power Series

Definition: A series of real functions $\sum f_n$ is a power series centered at $x = c$ if the function $f_n = a_n(x-c)^n$ where $(a_n)_n$ is a real sequence and $c \in \mathbb{R}$ Definition:
Let $\sum a_n x^n$ be a power series,
if $(|a_n|^{\frac{1}{n}})_n$ is a bounded sequence, we set $\rho = limsup(|a_n|^{\frac{1}{n}})$ if not we set $\rho = +\infty$,
The radius of convergence of $\sum a_n x^n$ is given by

$$R := \begin{cases} 0 & \rho = +\infty, \\ \frac{1}{\rho} & 0 < \rho < +\infty. \\ +\infty & \rho = 0. \end{cases}$$

The interval of convergence is the open interval $(c - R, c + R)$
Theorem: (Cauchy- Hadamard)
Given R the radius of convergence for the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$
(i) $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely pointwise on $(c - R, c + R)$
(ii) $\sum_{k=0}^{\infty} a_k(x-c)^k$ diverges on $(-\infty, c - R) \cup (c + R, \infty)$

(iii) $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges uniformally on any any closed and bounded subset of $(c-R, c+R)$

<u>Proof:</u>

(i) and (ii) Applying the root test to $x_k = a_k(x-c)^k$

We get:

$$limsup(|a_k(x-c)|^{\frac{1}{k}}) = limsup(|a_k|^{\frac{1}{k}}|x-c|) = \rho|x-c|$$

if $\rho = 0$, $\implies \sum a_k(x-c)^k$ converges absolutely pointwise by the ratio test
if $\rho = +\infty \implies \sum a_k(x-c)^k$ diverges for all $x \neq c$
if $0 < \rho \sum a_k(x-c)$ converges absolutely $\iff |x-c|\rho < 1 \iff |x-c| < \frac{1}{\rho} = R$
and $\sum a_k(x-c)^k$ diverges $\iff |x-c|\rho > 1 \iff |x-c| > \frac{1}{\rho} = R$
(iii) Let $[a,b] \subseteq (c-R, c+R)$
Observe $(\exists r_1 > 0) : (r_1 < R) \wedge (|x-c| < r_1)$ because there is room between the interval of convergence and our subinterval,
so $\forall x \in [a,b]$

$$|f_k(x)| = |a_k(x-c)^k| \leq |a_k||x-c|^k \leq |a_k|r_1^k$$

find $r_2 : r_1 < r_2 < R$ (Because $r_1 < R$ we can do this)
Hence, $\frac{1}{R} = \rho = limsup(|a_k|^{\frac{1}{k}}) < \frac{1}{r_2}$
Using the inf lemma,
$(\exists K \in \mathbb{N}) : (\forall k \geq K)$

$$|a_k|^{\frac{1}{k}} < \frac{1}{r_2} \implies |a_k| < \frac{1}{r_2^k}$$

Therefore, $f_k(x) \leq (\frac{r_1}{r_2})^k, \forall k \geq K)$ Because r_1, r_2 are independent of our choice of x,

$$\|f_k\|_{[a,b]} \leq (\frac{r_1}{r_2})^k \implies \sum_{k \geq K} \|f\|_{[a,b]} \leq \sum_{k \geq K} (\frac{r_1}{r_2})^k$$

$$\implies \sum_{k \geq K} \|f\| < \infty \implies \sum_{k=0} \|f\| < \infty$$

By Weirstrass M, $\sum a_k(x-c)^k$ converges uniformally and absolutely on $[a,b]$
$\square$ THe reason why you make bring the derivative and the integral in and out of the power series is since the functions are uniformally convergent on that interval. If a functions can be represented as a power series, we call it analytic.