

Mateo ARMED
MATH 310

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P.S. 1

Problem 1: If F is a finite set and $\mathcal{K}: F \rightarrow F$ is a self map then (\mathcal{K} is injective $\Leftrightarrow \mathcal{K}$ is surjective),

Pf: (\mathcal{K} is injective $\Rightarrow \mathcal{K}$ is surjective)

Given \mathcal{K} is injective by theorem \mathcal{K} is left invertible thus, there exists an f s.t.

$$\text{(*) } f \circ \mathcal{K} = \text{id}_F$$

Consider $\text{id}_F \circ f = f$,

$$\text{(*)} \Rightarrow (f \circ \mathcal{K}) \circ f = f \text{ by associativity}$$

$$\Rightarrow f \circ (\mathcal{K} \circ f) = f$$

$$\Rightarrow (\mathcal{K} \circ f) = \text{id}_F$$

$\Rightarrow \mathcal{K}$ is right invertible by theorem, \mathcal{K} is surjective

• (\mathcal{K} is surjective \Rightarrow \mathcal{K} is injective)

Given \mathcal{K} is surjective
by Theorem, \mathcal{K} is right invertible
by definition, for some \mathcal{H} ,
 $(*) \mathcal{K} \circ \mathcal{H} = id_F$.

Consider,

$$id_F \circ \mathcal{K} = \mathcal{K}$$
$$(*) (\mathcal{K} \circ \mathcal{H}) \circ \mathcal{K} = \mathcal{K} \quad \text{by associativity}$$

$$\Rightarrow \mathcal{K} \circ (\mathcal{H} \circ \mathcal{K}) = \mathcal{K}$$

$$\Rightarrow \mathcal{H} \circ \mathcal{K} = id_F$$

By definition, \mathcal{K} is Left
invertible
By Theorem, \mathcal{K} is injective.



(*) Given finite sets A, B ,
s.t. $B \subseteq A$ and $\text{card}(A) = \text{card}(B) = n \in \mathbb{N}$
then $A = B$

Pf:

Suppose for contradiction

$A \neq B$.

$\Rightarrow \exists x \in A$ s.t. $x \notin B$,

$\Rightarrow \text{card}(A) = \text{card}(B) + 1$

$\Rightarrow n = n + 1$ ~~✓~~

problem 2! prove a set A is infinite
 $\Leftrightarrow \exists f: A \rightarrow A$ not surjective
Pf: injection.

(F) $\exists f: A \rightarrow A$ not surjective
but injective. $\Rightarrow A$ is infinite

Suppose for contradiction
 A is finite.
 $\Rightarrow \text{card}(A) = n \in \mathbb{N}$
Let $f: A \rightarrow A$ be an injection
Since f is injective notice,
• $\text{card}(\text{Range}(f)) = \text{card}(A) = n$
• $f(A) \subseteq A$.

by

$\Rightarrow \text{Range}(f) = A$
 $\Rightarrow f$ is surjective.

contradiction because we assumed f not surjective
Hence, A must be infinite

 $A \cup$ infinite $\Rightarrow f:A \rightarrow A$ onto injective

Using contrapositive

$\neg(A \text{ is infinite}) \equiv A \text{ finite}$.

$\neg(\exists f:A \rightarrow A \text{ not surjective injective})$

$\equiv (\text{all injective function } f:A \rightarrow A \text{ must be surjective})$

$A \text{ is finite} \Rightarrow \text{all } f:A \rightarrow A \text{ must be surjective}$

If immediately follows from

problem $\exists f:A \rightarrow A \text{ must be Surjective}$.

(Since we already showed for
a self map onto a finite set
 f is surjective $\Leftrightarrow f$ is injective.)



Problem 3: Let A, B, C be sets s.t. $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$
 PROVE $\text{card}(A) < \text{card}(C)$

Pf: Given $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$

- $\Rightarrow \exists f: A \hookrightarrow B$ an injection
- $\exists g: B \hookrightarrow C$ an injection
- $\Rightarrow f \circ g: A \hookrightarrow C$ by theorem is an injection
- $\Rightarrow \text{card}(A) \leq \text{card}(C)$ by definition.
- Suppose for contradiction $\exists h: A \rightarrow C$ a bijection $\Rightarrow \text{card}(A) = \text{card}(C)$
- $\Rightarrow \text{card}(C) < \text{card}(B) \leq \text{card}(C)$ (assumption)
- $\Rightarrow \text{card}(C) \neq \text{card}(B)$ (definition)
- $\Rightarrow \text{card}(C) < \text{card}(B) < \text{card}(C)$
- $\Rightarrow \text{card}(C) < \text{card}(B)$ injective (defn)
- $\Rightarrow \exists j: C \hookrightarrow B, k: B \hookrightarrow C$ injective (Cantor-Schöder)
- $\Rightarrow \exists \varphi: C \rightarrow B$ a bijection (Cantor-Schöder)
- $\Rightarrow \text{card}(B) = \text{card}(C)$ ~~#~~
- $\therefore \text{card}(A) < \text{card}(C)$

Problem 4 If $A \subseteq B$ is an inclusion
of sets with A countable
and B uncountable, show $B \setminus A$ is
uncountable.

PF?

Suppose for contradiction

$B \setminus A$ is countable.

Since A & $B \setminus A$ is
countable. By theorem

$A \cup (B \setminus A)$ is countable

$\Rightarrow B$ is countable ~~#~~

contradiction since B is

assumed not countable.

Problem 5: Let $A := \{x \mid x > 0, x^2 \in \mathbb{Q}\}$
prove A is countable.

Pf:

Let $x \in A \Rightarrow x^2 \in \mathbb{Q}, x > 0$.

Consider $\kappa: A \hookrightarrow B \quad \kappa(x) = x^2$

$B := \{\kappa(x) \mid x \in A\}$

κ is trivially injective since $x > 0$.

consider $y \in B$,

$\Rightarrow y = \kappa(x)$ for some $x \in A$

$\Rightarrow y = x^2 \in \mathbb{Q}$ (definition)

Hence $B \subseteq \mathbb{Q}$

since $B \subseteq \mathbb{Q}$ we have

$\zeta: B \hookrightarrow \mathbb{Q}$ an injection from

$B \neq \emptyset$.

$\kappa \circ \zeta: A \hookrightarrow \mathbb{Q}$ is injective (Thm)

$\kappa \circ \zeta: A \hookrightarrow \mathbb{Q}$ is injective $\text{card}(A) \leq \text{card}(\mathbb{Q})$

By definition $\text{card}(\mathbb{Q}) \leq \text{card}(\mathbb{N})$. Hence

Recall, $\text{card}(\mathbb{N}) = \text{card}(\mathbb{W})$. Hence

$\exists f: A \hookrightarrow \mathbb{N}$ an injection
 $\therefore A$ is countable

□.

Problem 6: Consider $\mathcal{F}(\mathbb{N}) := \{A \subseteq \mathbb{N} \mid \text{card}(A) = n \in \mathbb{N}\}$

Show $\mathcal{F}(\mathbb{N})$ is countable.

Pf: Let $A \in \mathcal{F}(\mathbb{N}) \Rightarrow \{a_1, a_2, a_3, \dots, a_n\}$
Consider the function

$p: \mathcal{F}(\mathbb{N}) \rightarrow \mathbb{N}$

$$p(A) = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot \dots \cdot p_n^{a_n}$$

where p_n is the p_n prime

Suppose $p(A) = p(B)$

$$\Rightarrow 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot \dots \cdot p_n^{a_n} = 2^{b_1} \cdot 3^{b_2} \cdot 5^{b_3} \cdot \dots \cdot p_n^{b_n}$$

by fundamental theorem of arithmetic

$a_i = b_i \quad i \in \mathbb{N}$ since each number in \mathbb{N}

has a unique prime factorization

thus $\{p(A) \mid A \in \mathcal{F}(\mathbb{N})\}$ is injective.

$\therefore \mathcal{F}(\mathbb{N})$ is countable \square

Problem 7:

(i) prove $\mathbb{N}^k := \underbrace{\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N}}_{k \text{ times}}$ is countable

(ii) Show $\mathbb{N}^\infty := \left\{ (n_k)_{k \geq 1} \mid n_k \in \mathbb{N} \right\}$
consisting of all natural sequences is
uncountable.

(iii) Prove the set of finite initial segments
natural sequences

$$C_n(\mathbb{N}) := \left\{ (n_k)_{k \geq 1} \mid n_k \in \mathbb{N}, h_k = 0 \quad k \in \mathbb{N}_n \right\}$$

] countable

(iv) Is the set of decreasing
natural sequences

$$D := \left\{ (n_k)_{k \geq 1} \mid n_k \in \mathbb{N}, n_{k+1} \leq n_k \quad k \geq 1 \right\}$$

countable or uncountable.

(i)

Pf. Consider the function $P: \mathbb{N}^n \rightarrow \mathbb{N}$.

$$P(n_1, n_2, n_3, \dots, n_n) = 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdots p^{n_n}$$
$$= \prod_{k=1}^n p_k^{n_k} \text{ where } p_k \text{ is the } k\text{-th prime}$$

Suppose $P(n_1, n_2, n_3, \dots, n_n) = P(m_1, m_2, \dots, m_n)$

$$\Leftrightarrow 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdots p^{n_n} = 2^{m_1} \cdot 3^{m_2} \cdot 5^{m_3} \cdots p^{m_n}$$

Suppose for contradiction $n_i \neq m_i$ for some $i \in \mathbb{N}$

w.l.o.g $n_i < m_i$

$$\Rightarrow l = 2^{m_1-n_1} \cdot 3^{m_2-n_2} \cdots \cdot p_i^{m_i-n_i} \cdots p_n^{m_n-n_n}$$

If $m_k = n_k \Rightarrow p^{m_k-n_k} = 1$.

$$\Rightarrow l = (1 \cdot 1 \cdot 1 \cdots \cdot p_i^{m_i-n_i} \cdots)$$

$$\underset{\text{since } m_i - n_i > 0}{=} p_i^{m_i-n_i} \neq 1 \quad \cancel{\cancel{\cancel{\quad}}}$$

Hence, for all i , $n_i = m_i$

$\Rightarrow P$ is an injection

by definition \mathbb{N}^n is countable.



Lemma 1: If $f: A \hookrightarrow B$ and $g: C \hookrightarrow D$ are injective, $f \times g: A \times C \hookrightarrow B \times D$ is injective.

Pf:

$$\text{Suppose } f \times g(x_1, y_1) = f \times g(x_2, y_2)$$

$$\Rightarrow (f(x_1), g(y_1)) \in (f(x_2), g(y_2))$$

$$\Rightarrow f(x_1) = f(x_2) \text{ and } g(y_1) = g(y_2)$$

Since f and g are injective

$$\Rightarrow x_1 = x_2 \text{ and } y_1 = y_2 \quad \square$$

Lemma 2: If A, B are countable
 $A \times B$ is countable

Pf: Let A, B be countable sets.
By definition

$$\exists \kappa, j \text{ s.t. } \begin{array}{c} \kappa: A \hookrightarrow \mathbb{N} \\ j: B \hookrightarrow \mathbb{N} \end{array}$$

$$Im(A) = \{\kappa(1), \kappa(2), \kappa(3), \dots\}$$

$$Im(B) = \{j(1), j(2), j(3), \dots\}$$

$$\text{consider } \kappa \times j: A \times B \hookrightarrow \mathbb{N}^2$$

$$\text{s.t. } \kappa \times j(m, n), m \in A, n \in B$$

since κ, j are injective $\kappa \times j$ is
injective (Lemma 1)

so $A \times B$ is countable,

since \mathbb{N}^2 is countable,

(ii) P.F. Suppose for contradiction

- \mathbb{N}^∞ is countable.

$\Rightarrow \exists \psi: \mathbb{N} \rightarrow \mathbb{N}^\infty$ a surjection

$$\psi(1) = (n(1))_k = (n(1)_1, n(1)_2, n(1)_3, n(1)_4, \dots)$$

$$\psi(2) = (n(2))_k = (n(2)_1, n(2)_2, n(2)_3, n(2)_4, \dots)$$

$$\vdots$$

Let $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ be a map

$$\text{S.t., } \varphi(n_k) = \begin{cases} 2 & n_k \notin \mathbb{Z}, \\ 3 & n_k \in \mathbb{Z} \end{cases}$$

Consider a sequence

$$S = (\varphi(1), \varphi(2), \dots, \dots) \in \mathbb{N}^\infty$$

Since ψ is a surjection $\exists m \in \mathbb{N}$ s.t

$$\psi(m) = S$$

$$\Rightarrow \psi(m) = (n(m)) = (n(m)_1, n(m)_2, \dots)$$
$$= (\varphi(1), \varphi(2), \dots)$$

$$\Rightarrow n(m_1) = \varphi(1)$$

$$n(m_2) = \varphi(2)$$

Notice, if $n(m_m) = 2$ then $\varphi(m) = 3$
and if $n(m_n) \neq 2$ then $\varphi(m) \neq 3$ ~~X~~

Hence, not countable. \square

(iii) pf:

$$c_c(\mathbb{N}) = \left\{ \left(n_k \right)_{k=1}^{\infty} \mid n_k \in \mathbb{N}, \text{ only } n_k \neq 0 \text{ for } n_k \right\}$$

fix a $m \in \mathbb{N}$ s.t after m
it terminates in 0.

$$\text{Consider } c_c(\mathbb{N})_m = \left\{ \left(n_k \right)_{k=1}^{\infty} \mid n_k \in \mathbb{N}, n_k = 0 \forall k > m \right\}$$

Consider $\varphi: \mathbb{N}^m \rightarrow c_c(\mathbb{N})_m$

$$\varphi(m_1, m_2, \dots, m_m) = m_1, m_2, \dots, m_m, 0, 0, \dots$$

Let $(m) \in c_c(\mathbb{N})_m$

$$(m) = m_1, m_2, \dots, m_m, 0, 0, \dots$$
$$= \varphi(m_1, m_2, \dots, m_m)$$

Hence subject to,

by Thrm, $\exists \varphi: c_c(\mathbb{N}_m) \hookrightarrow \mathbb{N}^m$

since \mathbb{N}^m is countable $c_c(\mathbb{N})_m$

is countable.

$$c_c(\mathbb{N}) \cong \bigcup_{m \in \mathbb{N}} c_c(\mathbb{N})_m$$

by thrm $c_c(\mathbb{N})$ is countable.



(iv) $D = \{(n_k)^\infty \mid n_k \in \mathbb{N}, n_{k+1} \leq n_k \forall k\}$
 is countable

PF: Since $(n_k)^\infty \in D$ is decreasing
 at some point it terminates in $n_k \in \mathbb{N}$

Fix $i \in \mathbb{N}$
 $D_i = \{(n_k)^\infty \mid (n_k)^\infty \in D, n_k = n_i \forall k > i\}$

Consider $\gamma: D_i \rightarrow C_c(\mathbb{N})_i$ ($C_c(\mathbb{N})_i$ from iii)
 $\gamma(n_1, n_2, \dots, n_i, n_{i+1}, \dots) = n_1, n_2, \dots, n_i, 0_{i+1}, 0_{i+2}, \dots$

γ is injective since

$$\text{If } \gamma(n_1, n_2, \dots, n_i, n_{i+1}, \dots) = \gamma(m_1, m_2, \dots, m_i, \dots)$$

$$\Rightarrow n_1, n_2, \dots, n_i, 0_{i+1}, \dots = m_1, m_2, \dots, m_i, 0_{i+1}, \dots$$

2 sequences are equal if and only if all their terms are equal

so they are equal

$\Rightarrow \gamma$ is injective.

Since $C_c(\mathbb{N})_i$ is countable

$\Rightarrow D_i$ is countable

$D = \bigcup_{i \in \mathbb{N}} D_i$ is countable by theorem



Lemma: (*)

$\mathbb{R} \setminus \mathbb{Q}$ is not countable.

Pf?

Suppose for contradiction

$\mathbb{R} \setminus \mathbb{Q}$ is countable,

consider $\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$.

By theorem the

union of 2 countable sets is countable

However $\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$ ~~is not countable~~

contradiction since \mathbb{R} not countable
by Cantor. \square

(*) If A is a countable set and $B \subseteq A$

B is countable

Pf? Given A is countable and $B \subseteq A$

$\exists f: B \hookrightarrow A$ (f) = \mathbb{N} the injection inclusion

map. Since A is countable $\exists f: A \hookrightarrow \mathbb{N}$

By Theorem $\text{cof}: B \hookrightarrow \mathbb{N}$ is an injection

Hence B is countable

~~(*)~~ Lemma: Given $f: X \rightarrow Y$
 $\text{card}(\text{im}(f)) \leq \text{card}(X)$

Pf:

$$X \xrightarrow{f} \text{Im}(f)$$

Let $x \in \text{Im}(f) \Rightarrow x = f(\alpha)$

for some $\alpha \in X$
Hence f is surjective.

Since f is surjective

$\exists g: \text{Im}(f) \rightarrow X$ an injective

function
by defn $\text{card}(\text{Im}(f)) \leq \text{card}(X)$



Problem 8: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ s.t. $f(x) = \begin{cases} \mathbb{Q} & x \in \mathbb{R} \setminus \mathbb{Q}, \\ \mathbb{R} \setminus \mathbb{Q} & x \in \mathbb{Q}, \end{cases}$

Show $\text{Range}(f) \notin \mathcal{I}$ where \mathcal{I} is any interval.

Pf¹

First, $f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$ by (*) $f(\mathbb{Q})$ is countable.
 Second, $f(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{Q}$ b.y. ~~ok~~ $f(\mathbb{R} \setminus \mathbb{Q})$ is countable.

$$\text{Range}(f) = f(\mathbb{Q}) \cup f(\mathbb{R} \setminus \mathbb{Q})$$

b.y. thrm, the union of 2 countable sets is countable.

$$\text{card}(\text{Range}(f)) \leq \text{card}(\mathbb{N})$$

so suppose for contradiction $I \subseteq \text{card}(\text{Range}(f))$
 b.y. ~~ok~~ $\text{card}(I) \leq \text{card}(\mathbb{N})$

b.y. Cantor, $\text{card}(\mathbb{N}) < \text{card}(\mathbb{R})$

and $\text{card}(I) = \text{card}(\mathbb{R})$, $\text{card}(I) > \text{card}(\mathbb{N})$

∴ $I \notin \text{Range}(f)$.

□

Problem 9: prove $\mathcal{P} = \left\{ \sum_{n=0}^{\infty} a_n x^n \mid n \in \mathbb{N}, a_n \in \mathbb{Q} \right\}$

Lemma 3: \mathbb{Q}^n is countable?

PF1, Base case: \mathbb{Q}

\mathbb{Q} is countable by theorem

Suppose for some $n \in \mathbb{N}$,

\mathbb{Q}^n is countable for some n

Check \mathbb{Q}^{n+1}

$\mathbb{Q}^{n+1} = \mathbb{Q}^n \times \mathbb{Q}$
by Lemma 2, $A \times B$ is countable

If A and B are countable

Hence \mathbb{Q}^{n+1} is countable

∴ \mathbb{Q}^n is countable for all $n \in \mathbb{N}$.

□.

Pf: Notice any $p \in \mathcal{P}$ $p = \sum_{k=0}^n a_k x^k$

fix a $m \in \mathbb{N}_0$

$$\mathcal{P}_m = \left\{ p \in \mathcal{P} \mid \deg(p) = m \right\}$$

Let $\vec{a} \in \mathbb{Q}^{m+1}$ s.t. $\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix}$ where $a_i \in \mathbb{Q}$.

$$\text{consider } (\vec{x})^T = (1 \times x^1, \dots, x^m)$$

$$p = (\vec{x})^T (\vec{a}) \\ = \vec{x} \cdot \vec{a}$$

Consider $f: \mathbb{Q}^{m+1} \rightarrow \mathcal{P}_m$, $f(\vec{a}) = \vec{x} \cdot \vec{a}$

$$\text{Let } p \in \mathcal{P}_m \Rightarrow p = \sum_{k=1}^n b_k x^k = b_1 x^1 + b_2 x^2 + \dots + b_n x^n \\ \Rightarrow \vec{x} \cdot \vec{b} \\ = f(\vec{b})$$

so $f: \mathbb{Q}^{m+1} \rightarrow \mathcal{P}_m$ is a surjection.

By theorem, $\exists g: \mathcal{P}_m \hookrightarrow \mathbb{Q}^{m+1}$ a injection
since \mathbb{Q}^{m+1} is countable by lemma 3.

$\Rightarrow \mathcal{P}_m$ is countable.

$$\mathcal{P} = \bigcup_{m \in \mathbb{N}} \mathcal{P}_m, \text{ by theorem } \mathcal{P} \text{ is}$$

countable.



Problem 10. $t \in \mathbb{R}$ is algebraic if

$\exists P$, a polynomial with rational

coefficients s.t. $P(t) = 0$. If t is not algebraic it is transcendental

Show the set of algebraic numbers is countable and conclude there are uncountably many transcendental numbers.

Pf.: Let A be the set of algebraic numbers,

$$A = \{t \in \mathbb{R} \mid P(t) = 0, P \in \mathcal{P}\} \quad (\text{from q})$$

By definition every $t \in A$ is the root of some $P \in \mathcal{P}$

$$\text{Let } B_p = \{t \in A \mid P(t) = 0\}$$

By the fundamental theorem of Algebra

$P(t)$ can have at most n -roots.

So $\text{card}(B_p) \leq n$. More over, from problem 9 we know their are only countably many $P \in \mathcal{P}$, so we

$$\text{can express } A = \bigcup_{P \in \mathcal{P}} B_p$$

By Theorem, A is countable.