MATH310:Real Analysis

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1 Discrete Math Review

Recall the Canonical sets:

$$\mathbb{N} := \{1, 2, 3, 4, ...\}$$
$$\mathbb{N}_{\nvdash} := \mathbb{N} \cup \{0\}$$
$$\mathbb{Z} := \{0, -1, 1, -2, 2, ..\}$$
$$\mathbb{Q} := \{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}\}$$
$$\mathbb{R} = (-\infty, \infty)$$
$$\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$$

Also recall, if A and B are sets, $A \times B := \{(a, b) \mid a \in A, b \in B\}$.
We say R is a relation from A to B if $R \subseteq A \times B$. A function from A to B is $f \subseteq A \times B$ such that $\forall a \in A \exists! b_a \in B$ and $(a, b_a) \in f$. We write $f(a) = b_a$ and $f : A \to B$. The domain of f or dom(f)=A and the Codomain of f is B. The range of f is $f(A) = Im(A)_f \subseteq B$ The graph of f is $\{(a, f(a)) \mid a \in A\}$
some important functions are:
The characteristic function
$\mathbb{1}_A := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$
The identity function
$id_A : A \to A, id_A(a) = a$

Recall, A function $f : A \to B$ is infective if $x' \neq x \implies f(x') \neq f(x)$. f is surjection if $\forall b \in B \exists a \in A s.t. f(a) = b$
<u>Example:</u> The shift function $S(n) = n + 1, S : \mathbb{N} \to \mathbb{N}$ The range of S is $\{2, 3, 4, ...\}$ hence not surjective. However, S is an injection. $S(n') = S(n) \iff n' + 1 = n + 1 \iff n' = n$.
<u>Lemmma:</u> Given functions $A \to^f B \to^g C$
(i) f, g are surjective $\implies f \circ g$ is surjective.
(ii) f, g are injective $\implies f \circ g$ is injective.

Proof:
(i) Let f and g be surjective
Let $c \in C$ since g is surjective $\exists b \in B$ s.t. $g(b) = c$ moreover, since f is surjective $\exists a \in A$ s.t. $f(a) = g(b)$
Hence, $f \circ g$ is surjective
(ii) Suppose,

$$f \circ g(x') = f \circ g(x) \iff f(g(x') = f(g(x))$$

Because g is injective, $g(x') = g(x) \implies x' = x$

$$f(x') = f(x)$$

similarly we have $x' = x$. Hence, $f \circ g$ is injective.

2 Cardinality

Defintion: Given $f : A \to B$,
(i) if $\exists g : B \to A$ such that $g \circ f = Id_A$ then f is left invertable
(ii) if $\exists h : B \to A$ such that $f \circ h = Id_B$ then f is right invertable
(iii)if $\exists k : B \to A$ such that $k \circ f = Id_A$ and $f \circ k = Id_B$ then f is left invertable

Example: $S : \mathbb{N} \to \mathbb{N}$ S(n)=n+1 find a left inverse,
$g(m) := \begin{cases} m-1 & m \geq 2, \\ 2025 & m = 1. \end{cases}$
Notice we have $g \circ s = (n+1) - 1 = n = Id_A$
However, S(g(1))=2026$\neq$ 1 So g is not a right inverse.
Propn:
Let $f : A \to B$ be a map,
(i) f is left invertable $\iff$ f is injective
(ii) f is right invertable $\iff$ f is surjective
(iii) f is invertable $\iff$ f is bijective
(iv) f is invertable $\iff$ f is left invertable and f is right invertable. Proof:
(i) $\implies$:
let f be an injective function we will construct a left inverse g
If $b \in Range(f)$ by defintion $\exists a \in A$ with f(a)=b
Define $g : B \to A$ such that $g(b) := \begin{cases} a_b & b \in Im(A))_f, \\ a_0 & b \notin Im(A))_f. \end{cases}$ where $a_0 \in A$ is fixed.
$g \circ f = Id_A$ hence, g is a left inverse.
(ii) $\implies$:
Let f be right invertable,
$\implies \exists g : B \to A$ such that $f \circ g = Id_B$
$\implies f(g(b)) = b$
Hence, given $b \in B, \exists g(b) \in A$ such that f(g(b))=b. Therefore, f is surjective.

(iv) $\Longleftarrow$:
Given $g \circ f = Id_A$ and $f \circ h = Id_B$,
Consider,

$$h = Id_A \circ h = (g \circ f) \circ h \text{ (by assumption)}$$
$$= g \circ (f \circ h) = g \circ Id_B \text{ (by assumption)}$$
$$= g$$

Hence h=g so f is invertible.

Defintion: Given Sets A and B
If $\exists f : A \to B$ a bijection then we say $card(A) = card(b)$

e.g. $card(\mathbb{N}) = card(\mathbb{N}_0)$ because of $S : \mathbb{N}_0 \to \mathbb{N}, S(n) = n + 1$ is a bijection.
Example:
(i) $card((a, b)) = card((c, d))$
Solution: Let $f : (a, b) \to (c, d), f(x) := \frac{d-b}{c-a}(x - a) + c$ clearly a bijection.
(ii) $card((\frac{-\pi}{2}, \frac{\pi}{2})) = card(\mathbb{R})$
Solution: Tangent

Lemma: If $card(A) = card(B)$ and $card(B) = card(C)$ then $card(A) = card(C)$
pf: Since $card(A) = card(B) \implies \exists f : A \to B$ a bijection
Similarly, $\exists g : B \to C$ a bijection. So by lemma $f \circ g : A \to C$ is a bijection. Hence, card(A)=card(C).
Corrollary: $card((a, b)) = card(\mathbb{R})$.
Definition: Let A be a set,
(i) A is finite $\iff \exists f : A \to \mathbb{N}_n$ is a bijection or if A is empty
(ii) else, A is called infinite.
Example: Show $card(\mathbb{N}_m) \neq card(\mathbb{N}_n)$ for $n \neq m$
proof: Suppose for contradiction $\exists f : \mathbb{N}_n \to \mathbb{N}_m$ a injection. Without loss of generality, n¿m.
Since f is an injection f(1), f(2), f(3), ..., f(m) are distinct. #
By the pigeonhole principle, $f(x_i) = f(x_j)$ for at least one $i \neq j$. Therefore, f cannot be an injection.
Proposition: $\mathbb{N}$ is infinite.
Proof: Suppose for contradiction $\mathbb{N}$ is finite.
$\implies \exists f : \mathbb{N} \to \mathbb{N}_n abijection.$
Consider the inclusion map $\iota : \mathbb{N}_{n+1} \to \mathbb{N}, \iota(n) = n$ since $\mathbb{N}_{n+1} \subseteq \mathbb{N}$
$\implies f \circ \iota : \mathbb{N}_{n+1} \to \mathbb{N}$ is an injection,#
By lemma, no injection exists between $\mathbb{N}_{n+1}$ and $\mathbb{N}$,
thus, $\mathbb{N}$ is finite.

Lemma: If A is infinite, $\exists f : \mathbb{N} \to A$ an injection.
Proposition: $card(\mathbb{Z}) = card(\mathbb{N})$
Reason: $f : \mathbb{Z} \to \mathbb{N}$

$$f(m) := \begin{cases} 2m+1 & m > 0, \\ -2m & m \leq 0. \end{cases}$$

Definition: Let X be a set,
(i) $\mathcal{P}(x) = \{A \mid A \subseteq X\}$
(ii) $\mathbf{2}^X = \{f \mid f : X \to \{0,1\}\}$
Propn: $card(\mathcal{P}(x)) = card(\mathbf{2}^X)$
pf: Consider $\varphi : \mathcal{P}(x) \to \mathbf{2}^X, \varphi(A) = \mathbb{1}_A$ Where $A \subseteq X$.
This is a bijection.
Theorem:(Cantor's Diagonalization Argument
$\nexists r : \mathbb{N} \to (0,1)$ a surjection.
Lemma: If $0 < \sigma < 1$ then $\sigma = \sum_{k=1}^{\infty} \frac{\sigma_k}{10^k}$ not terminating in "9"'s.
Proof: Suppose for contradiction, $\exists r : \mathbb{N} \to (0,1)$ a surjection.
$r(n) := 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\sigma_4(n)....$ where $\sigma_i(n) \in \{0,1,2,3,4,..9\}$

Consider $\tau : \mathbb{N} \to \{0,1,2,3,4,..9\}$, $\tau(n) := \begin{cases} 3 & \sigma_n = 2, \\ 2 & \sigma_n \neq 2. \end{cases}$

Consider $S = 0.\tau(1)\tau(2)\tau(3)\tau(4).... \in (0,1)$ because r is a surjection,
$\exists m \in \mathbb{N}$ such that r(n)=S.
So $= 0.\sigma_1(m)\sigma_2(m)\sigma_3(m)\sigma_4(m).... = 0.\tau(1)\tau(2)\tau(3)\tau(4)....$
By lemma, we must have
$\sigma_1(m) = \tau(1)$
$\sigma_2(m) = \tau(2)$
$\sigma_3(m) = \tau(3)$
....
$\sigma_m(m) = \tau(m)$
Notice, if $\sigma_m(m) = 2 \implies \tau(m) = 3$
and if $\sigma_m(m) \neq 2 \implies \tau(m) = 2$ #
$\therefore$ r cannot be a surjection.
Corollary: $card(\mathbb{N}) \neq card(\mathbb{R})$
Proof: By theorem since $card(0,1) = card(\mathbb{R})$ by above, $card(\mathbb{N}) \neq card(\mathbb{R})$

Definition: Let A and B be sets,
(i) $card(A) \leq card(B) \iff \exists f : A \to B$ an injection
(ii) $card(A) < card(B) \iff card(A) \leq card(B)$ and $card(A) \neq card(B)$
e.g. $card(\mathbb{N}) \leq card(\mathbb{Z}) \leq card(\mathbb{Q}) \leq card(\mathbb{R})$
Lemma: If $A \subseteq B$ then $card(A) \leq card(B)$ by the inclusion map.
Also we showed $card(\mathbb{N}) < card(\mathbb{R})$
Example: Show $card(A) \leq card(B)$ and $card(B) \leq card(C) \implies card(A) \leq card(C)$

Pf:
Since $card(A) \leq card(B) \exists f : A \to B$ an injection similarly, $\exists g : B \to C$ an injection
By theorem, $f \circ g : A \to C$ is an injection. By definition, $card(A) \leq card(C)$
Cantor's Theorem:
If A is non empty, $card(A) \leq card(\mathcal{P(A)})$
Proof:
First we will show $card(A) < card(\mathcal{P}(A))$. Indeed we have an injection from A to $\mathcal{P}(A)$ where $a \in A \mapsto \{a\} \in \mathcal{P}(A)$.
Now suppose for contradiction, $\exists g : A \to \mathcal{P}(A)$ a surjection,
Let $S := \{a \in A \mid a \notin g(a)\} \subseteq A$
Since g is a surjection, $\exists x \in A$ such that g(x)=S. #
If $x \in S \implies x \notin g(x)$ but $x \in S \iff x \in g(x) = S$
Lemma: Let A and B be sets. The following statements are equivalent
(i) $card(A) \leq card(B)$
(ii) $\exists f : A \to B$ an injection
(iii) $\exists g : B \to A$ a surjection
Proof:
$(i) \iff (ii)$ by definition
$(ii) \iff (iii)$:
$f : A \to B$ an injection by theorem f is left invertible.
$\iff \exists g : B \to A$ such that $g \circ f = Id_A$
$\iff$ g is right invertible
$\iff g : B \to A$ is surjective.
Lemma: Let A,B,C,D be sets. if we have $f : A \to B$ and $g : C \to D$ bijections then $f \times g : A \times C \to B \times D$ is a bijection
Pf: $(f \times g)(a, c) = (f(a), g(c))$
(i) Suppose $(f(a'), g(c')) = (f(a), g(c)) \iff f(a') = f(a) \wedge g(c') = g(c)$ since f and g are injections we have $a' = a \wedge c' = c$
(ii) Let $(b, d) \in B \times D$ since f, g are surjections, $\exists a \in A s.t. f(a) = b \wedge c \in C s.t. g(c) = d$ so (f(a),g(c))=(b,d).
$\therefore f \times g$ is a bijection. Proposition: $card(\mathbb{Q}) \leq card(\mathbb{N})$
Proof:
First we observe that $g : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, g(m, n) = \frac{m}{n}$ is a surjection. By lemma, $\exists f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ a injection. Hence, $card(\mathbb{Q}) \leq card(\mathbb{Z} \times \mathbb{N})$
Now consider $H : \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}, H(m, n) = (h(m), n)$ where h is the bijection from $\mathbb{Z} \to \mathbb{N}$, by lemma, H is a bijection.
$\implies card(\mathbb{Z} \times \mathbb{N}) = card(\mathbb{N} \times \mathbb{N})$ We have $\mathbb{Q}) \leq card(\mathbb{N} \times \mathbb{N})$
Consider $K : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, K(m, n) = 2^m 3^n$
Claim: K is injective
Suppose $K(m', n') = K(m, n)$
$\iff 2^{m'} 3^{n'} = 2^m 3^n$ by the fundamental theorem of algebra all natural numbers have a unique prime factorization hence $m' = m \wedge n' = n$
Hence we have an injection from $\mathbb{Q} \to \mathbb{N} \implies card(\mathbb{Q}) \leq card(\mathbb{N})$

Theorem: (Cantor-Schroeder-Bernstein)
If $\exists f : A \to B$ an injection and $\exists g : B \to A$ an injection
then $\exists h : A \to B$ a bijection
(This implies that cardinality is anti symmetric.)
So we can conclude that $\mathbb{Q}$ is countable because $\mathbb{N} \subseteq \mathbb{Q}$.
Example: Let I be any non degenerate interval, Show $card(I) = card(\mathbb{R})$
Proof:
Notice $I \subseteq \mathbb{R}$ so we have the inclusion map $\iota : I \to \mathbb{R}$ an injection. Moreover, let $a < b \in I, (a,b) \subseteq I$ By theorem $card(\mathbb{R}) = card((a,b))$. Also we have $\iota_{a,b} : (a,b) \to I$ so we have an injection from $\mathbb{R} \to I$ and $I \to \mathbb{R}$ so by Cantor-Schroeder-Bernstein $card(I) = card((a,b))$.
Theorem: Given sets A,B
either $card(A) \leq card(B) \vee card(A) \geq card(B)$
the Proof relies on Zorans lemma.
Definition: A set X is countable If $\exists f : X \to \mathbb{N}$ an injection
$(card(X) \leq card(\mathbb{N})$
If X is infinite and countable then X is denumerable.
Q: Does there exist a infinite set A which is countable and $card(A) < card(\mathbb{N})$
If A is infinite and countable then $card(A) = card(\mathbb{N})$
Proposition: If i have a family of sets $\{A_n\}_{n=1}^{\infty}$ where each A_n is countable then $\bigcup_{n\in\mathbb{N}} A_n$ is countable.
Proof: Each set A_n is countable
$\implies$ for each n, $\exists \pi_n : \mathbb{N} \to A_n$ a surjection
Consider $\pi : \mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n, \pi(m,n) = \pi_m(n)$
Let $x \in \bigcup_{n=1}^{\infty} A_n \implies x \in A_i$ for some $i \in \mathbb{N}$
$\implies$ Since π_i is a surjection $\exists i_0 \in \mathbb{N}$
such that $\pi_i(i_0) = x, \pi(i, i_0) = x$ so we have:
$card(\bigcup_{n=1}^{\infty} A_n) \leq card(\mathbb{N} \times \mathbb{N}) \leq card(\mathbb{N})$
Claim: If $\{A_i\}_{i\in I}$ is family of countable sets. Show $A_1 \times A_2 \times A_3 \times A_4 \times ... \times A_n$ is countable.
Proof: Given $\{A_i\}_{i\in I}$ is family of countable sets,
$\implies \exists f_i : A_i \to \mathbb{N}$ an injection.
Consider $\mathbb{P} : A^n \to \mathbb{N}, \mathbb{P} = 2^{f_1(a)} 3^{f_2(a)} 5^{f_3(a)} 7^{f_4(a)} ... p_n^{f_n(a)}$
where p_n is the n-th prime. By the fundamental theorem of algebra, $\mathbb{P}$ is an injection. Hence A^n is countable.
Claim: If A,B are countable $A \times B$ is countable.
Pf: Given A,B countable $\exists f : A \to \mathbb{N}, g : B \to \mathbb{N}$
consider $f \times g : A \times B \to \mathbb{N} \times \mathbb{N}$ since f,g are injective $\implies f \times g$ are injective.
Since $\mathbb{N} \times \mathbb{N}$ is countable $\exists h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so $(f \circ g) \circ h : A \times B \to \mathbb{N}$ injective.
$\therefore A \times B$ is countable

Proposition $card(\mathbf{2}^{\mathbb{N}})$ is uncountable.
Recall, $\mathbf{2}^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to \{0,1\}\} = \{(a_n)_n^{\infty} \mid a_n \in \{0,1\}\}$
Lemma 1: $card([0,1]) = card(\mathbf{2}^{\mathbb{N}}))$

proof: Consider $\varphi : \mathbf{2}^{\mathbb{N}} \to [0,1], \varphi(f) = \sum_{n=1}^{\infty} \frac{f(k)}{2^k} \leq 1$
Fact: Every $t \in \mathbb{R}$ has $t = \sum_{n=1}^{\infty} \frac{t_k}{2^k}$ so φ is surjective.
Lemma 2: $card([0,1]) = card(\mathbb{R})$ (Proved earlier)
Lemma 3: $card(\mathbf{2}^{\mathbb{N}}) \leq card([0,1])$
Proof: $\Psi : \mathbf{2}^{\mathbb{N}} \to [0,1], \Psi(f) = \sum_{n=1}^{\infty} \frac{f(k)}{3^k}$
Let $f \neq g \in 2^{\mathbb{N}} \implies f(k) \neq g(k)$ for at least 1 k.
Let k_0 be the smallest k for which they differ.
$\Psi(f) = \sum_{n=1}^{\infty} \frac{f(k)}{3^k} = \sum_{n=1}^{k_0-1} \frac{f(k)}{3^k} + f(k) + \sum_{k>k_0} \frac{f(k)}{3^k}$
$\Psi(g) = \sum_{n=1}^{\infty} \frac{g(k)}{3^k} = \sum_{n=1}^{k_0-1} \frac{g(k)}{3^k} + g(k) + \sum_{k>k_0} \frac{g(k)}{3^k}$
Consider $\Psi(f) - \Psi(g) = \sum_{n=1}^{k_0-1} \frac{f(k)}{3^k} + f(k) + \sum_{k>k_0} \frac{f(k)}{3^k} - \sum_{n=1}^{k_0-1} \frac{g(k)}{3^k} - g(k) - \sum_{k>k_0} \frac{g(k)}{3^k}$ Because for for k¿$k_0, f = g$
$f(k) + \sum_{k>k_0} \frac{f(k)}{3^k} + g(k) + \sum_{k>k_0} \frac{g(k)}{3^k}$
Suppose for contradiction $\Psi(f) = \Psi(g)$,
$\implies 0 = f(k) + \sum_{k>k_0} \frac{f(k)}{3^k} + g(k) + \sum_{k>k_0} \frac{g(k)}{3^k}$
$\iff f(k) - \sum_{k>k_0} \frac{f(k)}{3^k} - g(k) - \sum_{k>k_0} \frac{g(k)}{3^k}$
$\iff g(k) - f(k) = \sum_{k>k_0} \frac{f(k)-g(k)}{3^k}$
$\iff 1 =| g(k) - f(k)| = | \sum_{k>k_0} \frac{f(k)-g(k)}{3^k}| \leq \sum_{k>k_0} \frac{|f(k)-g(k)|}{3^k}$
Notice, $f(k) - g(k) \leq 1$ So $\sum_{k>k_0} \frac{|f(k)-g(k)|}{3^k} \leq \sum \frac{1}{3^k} = \frac{1}{2}3^{-k_0} < 1\#$ So, $\Psi(f) \neq \Psi(g)$ hence injective. By Cantor Schroeder Bernstein, $card(\mathbb{R}) = card([0,1]) = card(2^{\mathbb{N}})$

2.1 Ordering of $\mathbb{R}$

Recall, if X is a non-empty set, a relation is $R \subseteq X \times X$.
Definition: Let R be a relation on X
(i) R is reflexive if $(x,x) \in R, \forall x \in X$
(ii) R is transitive if $(x,y) \in R, (y,z) \in R \implies (x,y) \in R$
(iii) R is symmetric if $(x,y) \in R \implies (y,x) \in R$
(iv) R is anti-symmetric if $(x,y) \in R$ and $(y,x) \in R$
Notation: If R is a relation on X, $(x,y) \in R \iff xRy$
Definition: If a relation R is reflexive,transitive and antisymmetric then R is an ordering on X.
Notation: R is an ordering we write $(x,y) \in R \iff xRy \iff x \leq_R y$
Example: Consider $\mathbb{N}$ we define, D is a relation $aDb \iff^{defn} \frac{a}{b}$
IS D an ordering?
D is reflexive since $aDa = a/a$ for all a.
D is transitive since $aDb, bDc \implies a/b$ and d/c so a/c=aDc.
D is anti symmetric aDb,bDa $\iff$ a/b and b/a so a=b. So by definition D is an ordering.
Example Consider $\mathbb{Z}, m \leq_a n \iff \exists k \in \mathbb{N}_0 s.t. m + k = n$
Is A reflexive? mAm take k=0.
Is A transitive? mAn and nAj so $m + k = n, n + k = j$ so take k=k+n for mAj.

Is A anti symmetric ? mAn and nAm $\implies m+k=n, n+k=m \iff k=m-n=n-m \iff m=n$
Example: S is non empty ,$X=\mathcal{P}(S)=\{A \mid A\subseteq S\}$
$A\leq_i B \iff A\subseteq B$ defines an ordering.
We say $\mathbb{Z}$ with $\leq_a$ is an ordering we can restrict to a subset $X=\mathbb{N}, \leq_a$ to get an ordering.
Definition: If $\leq$ is an order on X, the pair $(X,\leq)$ an partially ordered set . What is the difference between $(\mathbb{N},\leq_D)$ and $(\mathbb{N},\leq_a)$ notice $2\nleq_D 3$ but $2\leq_a 3$.
Definition: An ordering on a set X is total if any 2 elements are comparable.
Ex: $(\mathbb{N},\leq_a)$ is total
Ex: $(\mathbb{N},\leq)$ is not total
Ex: $(\mathcal{P}(S),\leq_i)$ is not total
<u>Definition:</u> Suppose we have an ordered set $(X,\leq)$ Let $A\subseteq X$
(i) A is bounded above if $\exists M\in X s.t. a\leq M\forall a\in A$ such an M is called an upper bound for A.
(ii) A is bounded below if $\exists m\in X s.t. m\leq a\forall a\in A$ such an m is called an lower bound for A.
(iii) If $\exists M_A\in A$ and M_A is an upper bound of A then M_A is the maximum element of A
(Note M_A is unique because of anti symmetry)
(iv) If $\exists m_A\in A$ and m_A is an lower bound of A then m_A is the minimum element of A
(Note m_A is unique because of anti symmetry)
(v) Suppose A is bounded above, Let $u\in X$ s.t. u is an upper bound for A and if v is an upper bound for A then $u\leq v$ then u is the supremum of A , u=Sup(A) (Least Upper Bound)
(vi)Suppose A is bounded below Let $u\in X$ s.t. u is an lower bound for A and if v is an lower bound for A then $v\leq u$ then u is the infimum of A , u=Inf(A)(Greatest Lower Bound)
(vii) $M\in A$ is maximal if $a\in A, a\geq M \implies a=M$
(viii) $m\in A$ is minimal if $a\in A, a\leq m \implies a=m$
<u>Example:</u> Consider the po-set $(\mathcal{P}(S),\subseteq)$, Clearly not totally ordered (unless only 1 element). Consider $A\subseteq\mathcal{P}(S)$ Where A is a collection of $\{S_i\}_{i\in I}$ where $S_i\subseteq S\forall i\in I$
(i) Show that Sup(A)=$\bigcup_{i\in I} S_i$
(ii) Show that Inf(A)=$\bigcap_{i\in I} S_i$
<u>Solution:</u>
(i)
By definition of the union, $\forall S_i\in A, S_i\subseteq\bigcup_{i\in I} S_i$ for $\bigcup_{i\in I} S_i$ is an upper bound of A.
Let V be another upper bound, if $V\subseteq\bigcup_{i\in I} S_i$ then V is not an upper bound , and becuase $S_i\subseteq V,\forall i\in I \implies \bigcup_{i\in I} S_i\subseteq V$ Hence $\bigcup_{i\in I} S_i$ is the leadst upper bound.
(ii) By definition of the intersection $\bigcap_{i\in I} S_i\subseteq S_i\forall i\in I$ Hence, $\bigcap_{i\in I} S_i$ is a lower bound.
Let V be another lower bound, if V is a lower bound we have $V\subseteq S_i\forall i\in I$ So

$V \subseteq \bigcap_{i\in I} S_i$
Defintion: An ordered set $(A, \leq)$ is well ordered if for all $a \subseteq A(a \neq \emptyset)$
$\exists a \in A s.t. a = min(A)$
Definition: An ordered set $(A, \leq)$ is complete if $\forall a \subseteq A$ the Sup(a) and Inf(a) exists.
Ordering of $\mathbb{Z}$:
$m \leq_a n \iff m \leq n \iff \exists k \in \mathbb{N}_0 s.t. m + k = n$
$\leq_a$ properties:
(i) $m \leq n \iff n - m \geq 0$
(ii) $m \leq n \wedge p \leq q \iff m + p \leq n + q$
(iii) $m \leq n \iff -n \leq -m$
(iv) $m \leq n, p \geq 0 \implies pm \leq pn$ (v) $\leq_a$ is total
(vi) $m > 0, mn \geq 0 \implies n \geq 0$
(vii) $m > 0, mn \geq mp \implies n \geq p$
Once we can order $\mathbb{Z}$ we can define an ordering on $\mathbb{Q}$. Recall, $Q = \mathbb{Z} \times \mathbb{N}$ then we define an equivalence relation on Q, $(m', n') \sim (m, n) \iff mn' = m'n$. Then we say $\mathbb{Q} = \{[(m, n)] \mid (m, n) \in Q\}$ all equivalence classes. Then we say that $\frac{a}{b} \leq_Q \frac{c}{d} \iff ad \leq_a bc$

2.2 Inequalities

Definition: $\mathbb{Z}^+ = \{m \in \mathbb{Z} \mid m \geq 0\}$ is the cone of positive elements.
Definition: Let S be a set with 2 binary operations,

$$+ : S \times S \to S : (s, t) \mapsto s + t$$

$$\cdot : S \times S \to S : (s, t) \mapsto st$$

If S is an Abelian Group
(i) (s+t)+r=s+(t+r) (Associativity)
(ii) $\exists 0_s \in S s.t. 0_S + s = s, \forall s \in S$
(iii) $\forall s \in S, \exists t \in S : s + t = 0 = t + s$ we call t the addiative inverse of s
(iv) s+t=t+s
and if for $\cdot$
(v) $(s \cdot t) \cdot r = s \cdot (t \cdot r)$(Associativity)
(vi) $(s + t) \cdot r = sr + tr$(right distributivity)
$r \cdot (s + t) = rs + rt$ (left distributivity)
Then S is called a Ring.
Definition: If $\exists 1_S \in S s.t. s \cdot 1_s = s, \forall s \in S$ then we call S a unital ring.
Definition: If $\forall s, t \in S, s \cdot t = t \cdot s$ then S is called a commutative ring.
$\mathbb{Z}$ and $\mathbb{Q}$ are commutative unital rings.
Definition: Let S be a set if S is commutative,unital and $\forall t \in S, t \neq 0 \exists s \in$

$Ss.t.ts = 1_s$ then S is a field.

Definition: Let F be a field when F has a total order satisfying

$$(1) x \leq y, s \leq t \implies x + s \leq y + t$$

$$(2) x \leq y, 0 \leq z \implies zx \leq zy$$

Then F is an ordered field.

Proposition: Let F be an ordered field

(i) $x, y \in F^+ \implies x + y \in F^+$

(ii) $x, y \in F^+ \implies xy \in F^+$

(iii) $\forall x \in F \implies x \in F^+ \vee -x \in F^+$

(iv) $x \in F^+ \wedge -x \in F^+ \implies x = 0$

Proposition: Let F be an ordered field,

(i) $\forall a \in F, a^2 \in F^+$

(ii) $0_F, 1_F \in F^+$

(iii) If $n \in \mathbb{N}$ the element $n \cdot 1_F := 1 + 1 + 1 + 1 + . + 1$ n times is in F^+

(iv) If $x \in F^+, x \neq 0$ then $x^{-1} \in F^+$

(v) If xy¿0 then x¿0,y¿0 $\vee$ x¡0,y¡0

(vi) If $0 < x \leq y \implies 0 < y^{-1} \leq x^{-1}$

(vii) If $x \leq y \implies -y \leq x$

(viii) $x \geq 1 \implies 1 \leq x \leq x^2$

$0 \leq x \leq 1 \implies 0 \leq x^2 \leq x \leq 1$

Order Axiom:

There is an ordered field $\mathbb{R} s.t. \mathbb{Q} \subseteq \mathbb{R}$

Propn: $\mathbb{Q}^+ \subseteq \mathbb{R}^+, q_1, q_2 \in \mathbb{Q}, q_1 \leq_{\mathbb{Q}} q_2 \implies q_1 \leq_{\mathbb{R}} q_2$

Proof:

By Proposition, $1, 0 \in \mathbb{R}$ and $n \in \mathbb{N}, n = 1 + 1 + ... + 1 \in \mathbb{R}$

So $\mathbb{N}_0 = \mathbb{Z}^+ \subseteq \mathbb{R}^+$

let $q_1 \in \mathbb{Q}^+ \implies q = \frac{a}{b}, a \in \mathbb{Z}^+, b \in \mathbb{N} \implies ab^{-1} \in \mathbb{R}^+$

So given $q_1 \leq q_2 \iff q_2 - q_1 \in \mathbb{Q}^+ \iff q_2 - q_1 \in \mathbb{R}^+$

Proposition: Let a,b$\in \mathbb{R}$ if$a \leq b$ then $a \leq \frac{1}{2}(a + b) \leq b$

Proof: Let $a \leq b \in \mathbb{R}$

$\implies a + a \leq a + b \implies 2a \leq (a + b) \leq 2b \iff a \leq \frac{1}{2}(a + b) \leq b$

Corollary: $0 < b \implies \frac{1}{2}b \leq b$

Proposition: Let $a \in \mathbb{R}$ if $\forall \varepsilon > 0, 0 \leq a < \varepsilon$ then a=0

Corollary: If $a, b \in \mathbb{R} s.t. \forall \varepsilon > 0, a \leq b + \varepsilon$ then $a \leq b$ Proposition: Let $a_1, a_2, a_3, ..., a_n \in \mathbb{R}^+$

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$$

Proof:(n=2)

$(a - b)^2 \geq 0 \iff a^2 - 2ab + b^2 \geq 0 \iff a^2 + b^2 \geq 2ab \iff a^2 + 2ab + b^2) \geq$

$4ab \iff (a+b)^2 \geq 4ab \iff (a+b) \geq 2\sqrt{ab} \iff \frac{1}{2}(a+b) \geq \sqrt{ab}$
Bernoulli's inequality: $(1+x)^n \geq 1 + nx \forall n > 0, x > -1$ (Proof: Induction)
Cauchy's inequality
Let $a_1, a_2, a_3, ..., a_n, b_1, b_2, b_3, ..., b_n \in \mathbb{R}$

$$\sum_{i=1}^{n} a_i b_i \leq (\sum_{i=1}^{n} a_i^2)^{1/2} (\sum_{i=1}^{n} b_i^2)^{1/2}$$

Proof: Consider the following quadratic,
$F : \mathbb{R} \to \mathbb{R}, F(x) = \sum_{i=1}^{n} (a_i - b_i x)^2$
Note, $F(x) \geq 0 \forall x \in \mathbb{R}$ If we write $F(x) = Ax^2 + Bx + C$ we know $B^2 - 4AC \leq 0$
$F(x) = \sum_{i=1}^{n} a_i^2 - 2a_i b_i x + b_i^2 x^2 = x^2 \sum_1^n b_i^2 - 2x(\sum_1^n a_i b_i) + \sum_1^n a_i^2$
$\iff 4(\sum_1^n a_i b_i)^2 \leq 4 \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \iff \sum_{i=1}^{n} a_i b_i \leq (\sum_{i=1}^{n} a_i^2)^{1/2} (\sum_{i=1}^{n} b_i^2)^{1/2}$
This is a sharp inequality because when $\vec{a} = c\vec{b}$ we have equality.
Proposition:(Triangle Inequality)
Let $a_1, a_2, a_3, ..., a_n, b_1, b_2, b_3, ..., b_n \in \mathbb{R}$

$$|\sum_{i=1}^{n} a_i + b_i| \leq |\sum_{i=1}^{n} a_i| + |\sum_{i=1}^{n} b_i|$$

Proof: By Cauchy-swartz, $\sum_{i=1}^{n} (a_i + b_i)^2 = \sum_{i=1}^{n} a_i^2 + 2a_i b_i + b_i^2$ By Cauchy-Swartz,
$\leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2(\sum_{i=1}^{n} a_i^2)^{1/2} (\sum_{i=1}^{n} b_i^2)^{1/2} = (\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2)^2 \iff$
$|\sum_{i=1}^{n} a_i + b_i| \leq |\sum_{i=1}^{n} a_i| + |\sum_{i=1}^{n} b_i|$

Definition: $|\cdot| : \mathbb{R} \to \mathbb{R}, |x| = \begin{cases} x & x \in \mathbb{R}^+, \\ x & -x \in \mathbb{R}. \end{cases}$

Proposition:
Let $a, b \in \mathbb{R}, \delta > 0$
(i) $|ab| = |a||b|$
(ii) $|a^2| = |a|^2$
(iii) $|-a| = |a|$
(iv) $|a| \in \mathbb{R}^+$
(v) $|a| \leq \delta \iff -\delta \leq a \leq \delta$
(vi) $|a+b| \leq |a| + |b|$
$|a-b| \leq |a| + |b|$
$||a| - |b|| \leq |a-b|$
Proof: (i) Given $a, b \in \mathbb{R}$
Suppose $a, b \in \mathbb{R}^+ \implies |ab| = ab = |a||b|$
if $-b, a \in \mathbb{R}^+$
$|a| = a, |b| = -b \implies -ab = |ab|$
Proof: (v)Given $|a| \leq \delta$

Suppose $a \in \mathbb{R}^+ \implies a \leq \delta$ since $a \in \mathbb{R}^+ \wedge \delta > 0 \implies a \geq 0 \geq -\delta \implies -\delta \leq a \leq \delta$

Proof:(vi)
$|a| = |a-b+b| \leq |a-b| + |b|$ by the normal triangle inequality
$\iff |a| - |b| \leq |a-b|$ likewise, with b $|b| - |a| \leq |a-b| \implies \pm(|a| - |b|) \leq |a-b| \iff ||a| - |b|| \leq |a-b|$
Definition: Given $y \in \mathbb{R}^+$
$\exists x \in \mathbb{R}^+ s.t. x^2 = y$ we then define $x := \sqrt{y}$
Propn: $\sqrt{x^2} = |x|$
Proof:
If $x \in \mathbb{R}^+ \implies |x| = x \wedge \sqrt{x^2} = z^2 \implies z = x$
If $-x \in \mathbb{R}^+ \implies |x| = -x \wedge \sqrt{(x)^2} = z \iff z = x$

Example:
(i) $\forall x \in \mathbb{R}, -|x| \leq x \leq |x|$
(ii) $A \subseteq \mathbb{R}$ A is bounded $\iff \exists c > 0, s.t. |a| \leq c \ \forall a \in A$
Proof: (i)
Given $x \in \mathbb{R}$,
if $x \in \mathbb{R}^+, x = |x|$ so $x \leq |x| \wedge -|x| \leq x$ since $-x \leq x \iff 2x \geq 0 \iff x \geq 0$
if $-x \in \mathbb{R}^+$, $x = -|x|$ so $x \geq -|x|$ and $x \leq |x| = -x \iff 2x \leq 0$ Hence $-|x| \leq x \leq |x|$ Proof: (ii)
By the completeness axiom, $l = Sup(A), u = Inf(A)$ exists.
Notice $|l| \leq l \leq a \leq u \leq |u|$
Let $c = max\{|l|, |u|\}$
So $c \geq |u|, |l| \iff -c \leq -|l| \iff -c \leq a \leq c \ \forall a \in A \iff |a| \leq c$
Definition: Let X be any non empty set. $f : X \to \mathbb{R}$ is bounded if Im(f) is a bounded subset of $\mathbb{R}$. i.e. $\exists c \in \mathbb{R} s.t. |f(x)| \leq c \forall x \in X$
Example: Show $f : [3, 7] \to \mathbb{R}, f(x) = \frac{x^2+2x+1}{x-1}$ is bounded
Solution:
Given $3 \leq x \leq 7 \iff 2 \leq x - 1 \leq 6 \iff \frac{1}{2} \geq \frac{1}{x-1} \geq \frac{1}{6} \iff |\frac{1}{x=1}| \leq \frac{1}{2}$
Similarly $3 \leq x \leq 7 \iff 9 \leq x^2 \leq 49 \iff 6 \leq 2x \leq 14 \iff 16 \leq x^2 + 2x + 1 \leq 64 \implies |f(x)| \leq 32$
Definition: Given s,t$\in \mathbb{R}$
$d(s, t) := |s - t|$
Propn: If s,t,r$\in \mathbb{R}$
(i) $d(s, t) = d(t, s)$ (Symmetry)
(ii) $d(s, t) \leq d(r, s) + d(r, t)$ (Triangle inequality)
(iii) $d(s, t) = 0 \iff s = t$
(identity of indecernables) (iv) $d(s, t) \geq 0 \forall s \neq t$
(non negativity) Definition: A metric space is a set $X \neq \emptyset$ equipped with a function $d : X \times X \to \mathbb{R}^+$ Satisfying:
(i) Symmetry
(ii) triangle inequality
(iii) non negativity
(iv) identity of indecernables

The function d is called a metric on X and the pair (X,d) is called a metric space.
Examples:On $\mathbb{R}^n$
(i) $d_2(x,y) = \sqrt{(y_1-x_1)^2 + ((y_2-x_2)^2 + .. + (y_n-x_n)^2}$
(ii) $d_1(x,y) = \sum_{i=1}^n |y_i - x_i|$
(iii) $d_\infty(x,y) = max_{i=1}^n \{|y_i - x_i|\}$
are all metrics.
Definition: Let (X,d) be a metric space. Let $x_0 \in X, \delta > 0$
The open ball centered at x_0 with radius δ is $V_\delta(x_0) := \{x \mid d(x,x_0) < \delta\}$
The closed ball is $B_\delta(x_0) := \{x \mid d(x,x_0) \leq \delta\}$
Example: In $(\mathbb{R}, d), d(x,y) = |x-y|$
$V_\delta(x) := \{x \mid |x-y| < \delta\} = (-\delta + x, x + \delta)$
$B_\delta(x) := \{x \mid |x-y| \leq \delta\} = [-\delta + x, x + \delta]$ In $(\mathbb{R}^2, d), B_1(0,0) = \{(x,y) \mid x^2 + y^2 \leq 1\}$
Now look at $(\mathbb{R}^2, d_\infty)$
$B_1(0,0) = \{(x,y) \mid \max\{(|x-0|, |y-0|)\} \leq 1\} = \{(x,y) \mid \max\{(|x|,|y|)\} \leq 1\}$
So we have $|x| \leq 1, |y| \leq 1 \iff -1 \leq x \leq 1, -1 \leq y \leq 1$ Hence $[-1,1] \times [-1,1]$
In a metric set we can talk about open and closed spaces,
Definition: Let X be a metric space, $A \subseteq X$
(i) A is open if $\forall a \in A \exists \delta > 0 : V_\delta(a) \subseteq A$
(ii) A is closed of A^C is open.
Notice being open and closed are not mutually exclusive.
Example:
(i) $(\mathbb{R}, d), d(s,t) = |s-t|$ Let $A = (-\infty, -2) \cup [1,3)$
Is A open? No since $\nexists \delta > 0 s.t. V_\delta(1) \subseteq A$
Is A closed? No since A^C is not open because $V_\delta(3) \subseteq A$
(ii) $A = (1, \infty)$ is open
Proof: Let $a \in A$, choose $\delta = a - 1 > 0 \iff a > 1$
Claim: $V_\delta(a) \subseteq A$

$V_\delta(a) = \{a \in A \mid d(x,a) < a-1\} \iff |x-a| < a-1 \iff -a+1 < x-a < a-1 \iff 1 < x < 2a-1$ So $a \in A$
Claim: Any subset $A \subseteq \mathbb{Z}$ is open
Pf: let $a \in A$ choose $V_{\frac{1}{2}}(a) = \{a\} \subseteq A$

2.3 Applications of the Supremum

Recall, a complete set is if any bounded subset has a sup and inf
Axiom: $(\mathbb{R}, \leq)$ is complete.
Example: Let $A \subseteq B$ be bounded sets. Then $Sup(A) \leq Sup(B)$
Proof: Let $u = Sup(A), Sup(B) = v$ since $A \subseteq B \implies (a \in A \implies a \in B)$ since v is an upper bound of B and all a in A are in B. Since u is less than all upper bounds of A and v is an upper bound of A $Sup(A) \leq Sup(B)$
Example: If A is bounded then Inf(-A)=-Sup(A)
Pf: Since A is bounded Sup(A)=u,

$\implies \forall a \in A, a \leq u, u \leq v$ for all upper bounds v.
by ordered field properties $-a \geq -u$
Hence -u is a lower bound for -A. Let w be a lower bound of -A
Let w be a lower bound of -A.
$\implies \forall a \in A, w \leq -a \implies -w \geq a$ Hence, -w is an upper bound for A. By the least upper bound property of u. we know $-w \geq u \iff w \leq -u \implies Inf(-A) = -u$

Lemma: (Sup Lemma)
Let $\emptyset \neq A \subseteq \mathbb{R}$ Suppose $\forall x \in A, x \leq u$ The following statements are equivalent:
(i) $u = sup(A)$
(ii) $\forall t < u, \exists a_t \in As.t.t < a_t$
(iii) $\forall \varepsilon > 0, \exists a_\varepsilon \in As.t.u - \varepsilon < a_\varepsilon$
<u>Proof:</u>
(i) $\implies$ (ii):
Given $u = sup(A)$ Let, $t < u$
if $\nexists a_t \in As.t.t < a_t$ then by total ordering $a \leq t \forall a \in A$ Hence, t is an upper bound of A. Since $t < u$ then u cannot be sup(A) so there must $\exists a_t \in As.t.t < a_t$
(ii) $\implies$ (iii):
Given $\varepsilon > 0$, let $t = u - \varepsilon$ By (ii) $\exists a_t \in As.t.t < a_t$, Thus we have $u - \varepsilon < a_t = a_\varepsilon$
(iii) $\implies$ (i)
Given $\forall \varepsilon > 0, \exists a_\varepsilon \in As.t.u - \varepsilon < a_\varepsilon \iff u < a_\varepsilon + \varepsilon$. Since u is an upper bound, $a_\varepsilon + \varepsilon$ is an upper bound.
Let V be any upper bound for A
Let $\varepsilon = u - v > 0$ by (iii) $\exists a_\varepsilon \in As.t.u - \varepsilon < a_\varepsilon$
$\implies v < a_\varepsilon$ hence v is not an upper bound. THus $u \leq v$ So u=sup(A)
<u>Example:</u> $Sup([0, 1)) = 1$
<u>Proof:</u> Clearly $\forall t \in [0, 1) \implies t < 1$ so 1 is an upper bound. Given $0 < \varepsilon < 1 \implies 0 < \frac{\varepsilon}{2} < \varepsilon$
Let $a_\varepsilon = 1 - \frac{\varepsilon}{2}$ which satisfies $1 - \varepsilon < a_\varepsilon$
<u>Definition:</u> A real valued function $f : D \to \mathbb{R}$,
(i) $f : D \to \mathbb{R}$ is bounded if $Im(f) \subseteq \mathbb{R}$ is bounded.
(ii) if $f : D \to \mathbb{R}$ is bounded then $\|f\|_u = Sup_{x \in D}(f(x))$ is finite and is called the uniform norm of f
<u>Example:</u> $f : (1, \infty) \to \mathbb{R}, \|f\|_u = 1, Im(f) = (0, 1)$
<u>Example:</u> $f : [0, 1] \to \mathbb{R}, f(t) = \begin{cases} t & 0 \leq t < 1, \\ 0 & t = 1. \end{cases}$
$\|f\|_u = 1, Im(f) = (0, 1)$ Notice $Sup(f) \notin Im(f)$ <u>Definition:</u>
Let $\emptyset \neq D$ be any set,

$$l_\infty(D) := \{f \mid f : D \to \mathbb{R} \text{ bounded}\}$$

Define $d_u(f, g) := \|f - g\|_u, f, g \in l_\infty(D)$ this is called the uniform metric.
Recall the well ordering principle on $\mathbb{N}$ says any non empty subset of $\mathbb{N}$ has a

minimum element.

Archimedean Property:

(i) If $x \in \mathbb{R}$ then $\exists n_x \in \mathbb{N} s.t. x < n_x$

(ii) If $x > 0$ then $\exists n_x \in \mathbb{N} s.t. \frac{1}{n_x} < x$

Pf:

Suppose for contradiction $\mathbb{N}_0$ is bounded, by the completeness axiom $Sup(\mathbb{N}_0)$ exists. Let $\varepsilon = 1$ by the sup lemma we have $m_\varepsilon \in \mathbb{N} s.t. u - 1 < m_\varepsilon \implies u < m_\varepsilon + 1$ However, $m_\varepsilon + 1 \in \mathbb{N}$ hence u cannot be an upper bound.

For (ii) Let $x := t^{-1}$ by AP1, $\exists N \in \mathbb{N} s.t. t^{-1} < N$ by ordered field property we have $\frac{1}{N} < t$

Corollary: Given $t > 0 \exists m \in \mathbb{N} s.t. \frac{1}{2^m} < t$(Bernoulli)

Corollary: Let $x \in \mathbb{R}, \exists n_x \in \mathbb{Z} s.t. n_x - 1 \leq x \leq n_x$

Proof: if $x \geq 0$ Let $S_x = \{n \in \mathbb{N} \mid n > x\}$ By the Archimedean property, $S_x \neq \emptyset$ by the well ordering principle since $S_x \subseteq \mathbb{N}, \exists min\{S_x\} = n_x$ Moreover, $n_x - 1 \in \mathbb{Z}, \notin S_x$ so $n_x - 1 \leq x \leq n_x$.

if $x < 0$,

Let $z \in \mathbb{Z} s.t. z + x > 0$ we apply case 1 ,

$\implies m - 1 \leq z + x \leq m \iff m - z - 1 \leq x \leq m - z$ Let $n_x = m - z \in \mathbb{Z}$

Hence, $n_x - 1 \leq x \leq n_x$

Density Theorem:

Definition: Let (X,d) be a metric space. $A \subseteq X$ is dense in X if $\forall \delta > 0, x_0 \in X : V_\delta(x_0) \cap A \neq \emptyset$

In $\mathbb{R}$ a subset A is sense if $\forall a < b, (a, b) \cap A \neq \emptyset$

This is saying that any open interval contains a point in A. ($\exists t \in A s.t. a < t < b$

Theorem: $\mathbb{Q} \subseteq \mathbb{R}$ is dense.

Proof: Let $I \subseteq \mathbb{R}$ be any open interval. We know $\exists a, b \in \mathbb{R} s.t. (a, b) \in I$ without loss of generality $0 < a < b$ by ordered field properties $b - a > 0$ by AP2, $\exists n \in \mathbb{N} s.t. 0 < \frac{1}{n} < b - a \iff 1 < nb - na \iff na + 1 < nb$ by our above corollary, $\exists m \in \mathbb{N} s.t. m - 1 \leq na \leq m \implies a < \frac{m}{n}$ moreover, $m \leq na + 1 \leq nb$ So we have $a < \frac{m}{n} < b$ clearly $\frac{m}{n} \in \mathbb{Q}$ so Q is dense.

Example: Show the irrationals are dense in $\mathbb{R}$

Pf: Assume $\sqrt{2}$ exists and we know $\sqrt{2} \notin \mathbb{Q}$ Let I be any open interval $\exists a, b \in \mathbb{R} s.t. (a, b) \subseteq I$ so we have $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ because $\mathbb{Q} \subseteq \mathbb{R}$ is dense, $\exists q \in \mathbb{Q} s.t. \frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}} \iff a < \sqrt{2}q < b$ clearly $\sqrt{2}q \notin \mathbb{Q}$

Theorem: $\exists! x \in \mathbb{R} s.t. x > 0, x^2 = 2$

Proof:

(existence)

Define $S = \{t \mid t \geq 0, t^2 < 2\}$ S is clearly not empty since 1 and 0 are in it.

Moreover, S is bounded above if $y > 2 \implies y^2 > 4$ (ordered field property)

hence, $t \leq 2 \forall t \in S$

By the completeness axiom, $x := Sup(S) \in \mathbb{R}$ exists.

Claim 1: $x^2 \geq 2$ suppose for contradiction $x \in S$

$x + \frac{1}{m} \in S \iff (x + \frac{1}{m})^2 < 2 \iff x^2 + \frac{2x}{m} + \frac{1}{m^2} < 2$

Notice we have $x^2+\frac{2x}{m}+\frac{1}{m^2}\le x^2+\frac{2x}{m}+\frac{1}{m}=x^2+\frac{1}{m}(2x+1)\iff\frac{1}{m}(2x+1)<2-x^2\iff\frac{1}{m}<\frac{2-x^2}{2x+1}$ if $x^2<2$ then $0<2-x^2$ so $\frac{2-x^2}{2x+1}>0$ by the Archimedean property $\exists n\in\mathbb{N}s.t.\frac{1}{n}<\frac{2-x^2}{2x+1}\implies x+\frac{1}{n}\in S$ by the sup lemma this is a contradiction.

<u>Claim 2:</u> x=2

Since $x=Sup(S)$ by the sup lemma, $\exists t_m\in Ss.t.x-\frac{1}{m}<t_m\iff(x-\frac{1}{m})^2<t_m^2<2\iff x^2-\frac{2x}{m}+\frac{1}{m^2}<2\iff x^2-\frac{2x}{m}<2\iff x^2-2<\frac{2x}{m}\iff\frac{x^2-2}{2x}<\frac{1}{m}$

By above claim $0\le\frac{x^2-2}{2x}<\frac{1}{m}\forall m\ge1$

$\implies\frac{x^2-2}{2x}=0\iff x^2=2$

(Uniqueness)

Suppose for contradiction $\exists 0<x_1,x_2s.t.x_1^2=2,x_2^2=2,x_1\ne x_2$

then $x_1^2=x_2^2\iff 0=x_1^2-x_2^2\iff 0=(x_1+x_2)(x_1-x_2)$ So either $(x_1+x_2)\vee(x_1-x_2)=0$ not possible since both are positive and not equal.

<u>Definition:</u> The unique positive x s.t. $x^2=2,x:=\sqrt{2}$

<u>Theorem:</u> Given $x\ge0,a\ge0\exists!xs.t.x^2=a$ we define $x:=\sqrt{a}=a^{\frac{1}{2}}$

Remark: We could have defined $T=\{t\in\mathbb{Q}\mid t\ge0,t^2<0\}$ this tells us that because T being bounded above and a subset of Q, $Sup(T)\notin\mathbb{Q}$ hence $(\mathbb{Q},\le)$ is not complete.

<u>Nested Intervals:</u>

Recall,

(1) Open Interval - $(a,b),(-\infty,b),(a,\infty)$

(2) Closed Interval - $[a,b],(-\infty,b],[a,\infty)$

(3) Half open half closed- $(a,b],[a,b)$

Remark Given any interval I and any 2 points in the interval.

If $x_1,x_2\in Is.t.x_1<x_2$ and $x_1\le y\le x_2\implies y\in I$

<u>Theorem:</u> Let $S\subseteq\mathbb{R}$ contain at least 2 points,

If $x,y\in Ss.t.x<y$ and $[x,y]\subseteq S$ then S is an interval.

<u>Proof:</u> Given $S\subseteq\mathbb{R}$ has 2 points,

Case 1: S is bounded. By the completeness axiom, $\exists Sup(S)=b,Inf(S)=a$ by definition $S\subseteq[a,b]$ Because b is an upper bound and a is a lower bound. We will show $(a,b)\subseteq S$ once this is done we have case 1 done.

Let $t\in(a,b)\iff a<t<b$ by the sup-inf lemma. $\exists t_a,t_b\in Ss.t.a\le t_a<t<t_b\le b$ by above remark we have $t\in S$. Case 2: S is bounded above and un bounded below

Since S is bounded above, $\exists b=Sup(S)$ by definition we have that $t\le x\forall t\in S$ by definition $S\subseteq(-\infty,x]$

Let $y\in(-\infty,x)\implies y<x$ by the sup lemma $\exists y_\varepsilon\in Ss.t.y<y_\varepsilon\le x$

Since S is not bounded below $\exists t\in Ss.t.t<y$ so we have $t<y\le y_\varepsilon$ by above remark we have an interval.

<u>Definition:</u> A sequence of Intervals $(I_n)_{n\ge1}$ if

$I_1\supseteq I_2\supseteq I_3\supseteq I_4\supseteq ...$

<u>Defn:</u> The "Measure" of an interval is the difference of the endpoints.

$m([a,b])=b-a$

Theorem: Let $(I_n)_{n\geq 1}$ be a sequence of closed and bounded nested intervals $I_n = [a_n, b_n]$

$$(i) \bigcap_{n\geq 1} I_n \neq \emptyset$$

$$(ii) \text{ If } Inf(m(I_n)) = 0, \text{ then } \exists! \xi \in \bigcap_n I_n$$

Corollary: [0,1] is uncountable

Pf: Suppose for contradiction [0,1] is countable. This means we can list its elements in a sequence. $\{t_1, t_2, t_3, ...\} t_i \in [0, 1]$

Select a closed sub interval $I_1 \subseteq I s.t. t_1 \notin I_1, I_1$ is closed.

Select another closed sub interval $I_2 \subseteq I_1 s.t. t_2 \notin I_2, I_2$ is closed. Repeat Inductively so $I_n \subseteq I_{n-1} s.t. t_n \notin I_n, I_n$ is closed. $(I_n)_n$ is a nested sequence. By the Nested interval property $\bigcap_{n\geq 1} I_n \neq \emptyset$ By definition $\exists t \in \bigcap_{n\geq 1} I_n$ In particular $t \in [0, 1]$ so $\exists m s.t. t = t_m$ but $t_m \notin I_m$ # contradiction because t needs to be in all I_n.

Proof of NIP:

Given $I_n = [a_n, b_n]$ notice:

$$a_1 \leq a_2 \leq a_3 \leq ...$$

$$b_1 \geq b_2 \geq b_3 \geq ...$$

Claim 1: $\forall m, n \geq 1, a_m \leq b_m$

if $m \leq n$,

then we have $a_m \leq a_n \leq b_n$ because $(a_m)_m$ is an increasing sequences and because I_n is an interval.

If $m \geq n$

then $a_m \leq b_m \leq b_n$ because $(b_m)_m$ is decreasing and I_m is an interval.

Fix $n \geq 1$ then $a_m \leq b_n \forall m$

So b_n is an upper bound of the set $\{a_n \mid n \geq 1\}$ Since we have a set which is bounded above we know by the completeness axiom $Sup(a_i)_{i\geq 1}^n = \xi \leq b_n$

This holds $\forall n \geq 1$

hence, ξ is a lower bound of $\{b_n \mid n \geq 1\}$ Since the set is bounded $\eta = Inf(b_n)_{n\geq 1}$ exists with $\xi \leq \eta$

Claim 2: $\bigcap_{n\geq 1} I_n = [\xi, \eta]$

First $\bigcap_{n\geq 1} I_n \supseteq [\xi, \eta]$

Let $x \in [\xi, \eta] \iff \xi \leq x \leq \eta$ Since ξ is the upper bound of a_n and η is lower bound for b_n we have $a_n \leq \xi \leq x \leq \eta \leq b_n \iff a_n \leq x \leq b_n \iff x \in \bigcap_{n\geq 1} I_n$

Next $\bigcap_{n\geq 1} I_n \subseteq [\xi, \eta]$

Let $x \in \bigcap_{n\geq 1} I_n \implies x \in I_n \forall n \geq 1 \iff a_n \leq x \leq b_n$ Since x is an upper bound of a and x is a lower bound of b we must have $\xi \leq x \leq \eta$

(2) Suppose $Inf([b_n - a_n]) = 0$

Suppose for contradiction $\xi < \eta$, Let $\delta = \eta - \xi > 0$
$\forall n \geq 1, a_n \leq \xi < \eta \leq b_n$ hence $b_n - a_n \geq \eta - \xi = \delta > 0$ So δ is a lower bound of the set $b_n - a_n$ and $\delta > 0$ so if $Inf\{[b_n - a_n]\} = 0$ then δ cannot be a lower bound.