MATH212 Part four: Vector Calculus

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1 Vector Fields

Recall in calculus we saw functions like

- $y = f(x), f : D \to \mathbb{R}$
- $\gamma(t) = (x(t), y(t), z(t)), t \in I$ (Interval)
- $z = f(x, y), (x, y) \in D \subseteq \mathbb{R}^2$
- $w = f(x, y, z), (x, y, z) \in E \subseteq \mathbb{R}^3$

Now we look at Vector Fields, <u>Definition:</u> A vector field in $\mathbb{R}^{\not\models}$ on $U \subseteq \mathbb{R}^{\not\models}(open)$ is a function $\vec{F} : U \to \mathbb{R}^{\not\models}$

$$\vec{F}(x, y, z) =< P(x, y, z), Q(x, y, z), R(x, y, z) >$$

P,Q,R are real valued scalar function called the component functions of the field $\vec{F}$. If P,Q,R have continuous first order partials, then $\vec{F}$ is of class C_1

Remark: Vector Fields exist on $\mathbb{R}^n$ but $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$
<u>Example:</u>
(i) $\vec{F}(x, y) =< -y, x)$
Solution: This is a vector field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$.
$\vec{F}(1, 0) =< 0, 1 >$,
$\vec{F}(0, 1) =< -1, 0 >$,
$\vec{F}(-1, 0) =< 0, -1 >$,
$\vec{F}(0, -1) =< 1, 0 >$

(ii) (Gravity) $\vec{v} =< x, y, z >$ Solution: Newtons Law: There is a force of attraction $\vec{F}$ such that $\|\vec{F}\| = \frac{GMm}{\|\vec{v}^2\|}$ where M and m are mass of two objects.
$\vec{F}(x, y, z) = \frac{GMm}{\|\vec{v}^2\|} - (\frac{\vec{v}}{\|\vec{v}\|})$
$= \frac{-GMm}{\|\vec{v^3}\|}$
$=< \frac{-GMmx}{\sqrt{x^2+y^2+z^2}}, \frac{-GMmy}{\sqrt{x^2+y^2+z^2}}, \frac{-GMmz}{\sqrt{x^2+y^2+z^2}} >$

=Gravitational Field
(iii)(Electric Field) $\vec{E}(x,y,z) = \frac{\varepsilon qQ\vec{Q}}{\|v\|^3}$
if qQ¿0, repulsion
if qQ¡0, attractions
(iv) (Conservative Field, Temperature)
If $T = T(x,y,z)$ is the temperature at any point, the flow of heat is the heat flux vector field.
$\vec{H}(x,y,z) = -\vec{\nabla}T$
$=< -\frac{\partial T}{\partial x}, -\frac{\partial T}{\partial y}, -\frac{\partial T}{\partial z} >$
Notice how our vector field can we written as the gradient vector of a function. If this is the case we call it conservative.
Definition: A vector field $\vec{F} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ is called **Conversative** if there exists a functions $w = f(x,y,z)$ such that $\nabla f \vec{=} \vec{F}$
Notice $< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$
$= \vec{F}(x,y,z)$
$=< P, Q, R >$
Therefore, $\frac{\partial f}{\partial x} = P, \frac{\partial f}{\partial y} = Q, \frac{\partial f}{\partial z} = R$

Example:
(i) $f(x,y,z) = xyz$ find gradient vector field.
Solution:
$\nabla f =< yz, xz, xy >$ is a conservative vector field.
(ii) $\vec{F} =< y^2z^3, 2xyz^3, 3xy^2z^2$
we want a $f(x,y,z)$ with $\frac{\partial d}{\partial x} = y^2z^3, \frac{\partial d}{\partial y} = 2xyz^3, \frac{\partial f}{\partial z} = 3xy^2z^2$
$f(x,y,z) = xy^2z^3 + g(y,z) \implies 2xyz^3 = 2xyz^3 + \frac{\partial g}{\partial y}$
$\iff \frac{\partial g}{\partial y} = 0 \implies g(y,z) = h(z)$
Therefore,
$f(x,y,z) = xy^2z^3 + h(z)$
$\iff \frac{\partial f}{\partial z} = 3xy^2z^2 + h'(z)$
$\iff h'(z) = 0 \iff h(z) = C$
Thus,
$f(x,y,z) = xy^2z^3 + C$
Theorem: Given a vector field in $\mathbb{R}^{\nvDash}, \vec{F} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ with $\vec{F} =< P, Q >$ of class C_2 where U has no holes,
if $\vec{F}$ is conservative,
then, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Proof:
Given $\vec{F} = \vec{\nabla} f$
$=< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$
$=< P, Q >$

$P = \frac{\partial f}{\partial x}, Q = \frac{\partial f}{\partial y}$
Then take the mixed partials of each, $\implies \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$ (clairaut's)

Example: $\vec{F} =< x - y, x - 2 >$
Solution:
$\frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1$ Therefore, Not conservative. (Theorem)

2 Paths

Example: u=$\mathbb{R}^2 \setminus \{0, 0\}, \vec{r}(t) =< cos(t), sin(t) > t \in [0, 2\pi]$
Solution: Notice how the path goes in a circle and the end at t=0 is the same as t=2π
Definition: Let $\vec{r} : [a, b] \to \mathbb{R}^2$ be a path
A path is closed if and only if $\vec{r}(a) = \vec{r}(b)$

Definition: A path is simple if it does not intersect it self any where between its end points.

$$\forall r, s \in (a, b), r(s) \neq r(t)$$

Definition: An open set $U \subseteq \mathbb{R}^n$ is path connected if for any two points $P_1, P_2 \in U$ there is a path $\vec{r} : [a, b] \to U$ in U with $\vec{r}(a) = P_1, \vec{r}(b) = P_2$

Example
(i) $U = \{(x, y) \mid xy > 0\}$, Not connected because U does not include the x and y axis. the path must cross the axes by the intermediate value theorem.
(ii) $U = \{(x, y) \mid x^2 + y^2 \leq 1\}$, is convex and thus connected. (iii) $\dot{U}(0, 1) = \{(x, y) \mid 0 < x^2 + y^2 < 1\}$, connected just go around the origin.
Definition: An open subset of $\mathbb{R}^2$ U is called simply connected if

- U is connected
- every simple closed curve encloses point only in U

Notice, (ii) is simply connected but (iii) is not.
Theorem: If $\vec{F} =< P, Q >$ is of class C_1 defined on an open, simply connected subset of $\mathbb{R}^2$ U and if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on $\mathbb{R}^2$ ONLY,
then $\vec{F}$ is conservative.

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3 Line Integrals

Given a smooth curve C, $C : \vec{r} : [a,b] \rightarrow \mathbb{R}^n (n = 2,3)$ we will define two types of path integrals $\int_C f$ds. and $\int_C \vec{F} \cdot d\vec{r}$.
<u>Definition:</u> Let $U \subseteq \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}^n$ continous on U, C is a smooth curve in U. Let $\vec{r} : [a,b] \rightarrow U$ be a path.

$$\int_C f ds = \int_a^b f(r(t)) \|\vec{r'(t)}\| dt$$

Example: $\int_C x^2 y$ ds
Solution: $\vec{r}(t) =< cos(t), sin(t) > t \in [0,\pi]$
$\vec{r}'(t) =< -sin(t), cos(t) > \iff \|\vec{r}\| = 1$
$f \circ r(\vec{t}) =< cos(t)^2, sin(t) >$
$\int_0^\pi cos(t)^2 sin(t) dt$

u=cos(t) , du=-sin(t)dt

$\int_{-1}^1 u^2 du = [\frac{u^3}{3}]_{-1}^1 = \frac{2}{3}$

Generalization: Consider a piece-wise smooth curve(smooth in pieces) we can integrate it but we integrate it by pieces.
Example: compute $\int_C x ds$ where C is given by $\begin{cases} x^2 & x \leq 2, \\ 4 & x \geq 2. \end{cases}$
Solution: $r_1(t) = (t, t^2), t \in [0,2]$ and $r_2(t) = (t,4), t \in [2,5]$
$r_1'(t)$=$\sqrt{1+4t^2}$, $r_2'(t) = 1$
$\int_C f$ ds =$\int_{C_1} f ds$+$\int_{C_2} f ds$
$\int_{C_1} f ds = \int_0^2 t\sqrt{1+4t^2} dt$

$u = 1 + 4t^2$ du=8t

=$\frac{1}{8} \int_1^{17} \sqrt{u} du$
Example: $C : r(t) =< cos(t), sin(t), t >, t \in [0, 2\pi]$ Evaluate $\int_C y sin(z) ds$
Solution: ... $\frac{\sqrt{2}}{2}\pi$
<u>Definition:</u> Let C be a smooth curve in U and $\vec{r} : [a,b] \rightarrow U \subseteq \mathbb{R}^n$(n= 2 or 3). Moreover, $\vec{F}$ is a continuous vector field in U.

$$\int_C F \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot \vec{\ }(t) dt$$

Example: $\vec{F} =< x, y, z >$, Let C be a turn of the helix.
Solution: .., $2\pi^2$
Geometric Interpretation,

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(\vec{t})) \cdot r'(\vec{t}) dt = \int_a^b \vec{F}(r(\vec{t})) \cdot \frac{r'(\vec{t})}{\|r'(\vec{t})\|} \|r'(\vec{t})\| dt = \int_a^b \vec{F}(r(\vec{t})) \cdot \vec{T} d\vec{r} = \int_C \vec{F} \cdot \vec{T} d\vec{r}$$

We see dr is a small displacement along the curve and the dot product of the vector field of F and T are how much of the vector field of F hits the tangent vector T. Also Work Done=$\int_C F \cdot d\vec{r}$.
Example: Find the work done by the force $\vec{F} =< x^2, y >$ on a particle moving along a line from (1,2) to (3,-4)
Solution: Parametrize the line and then integrate
Exercise: Let C be a smooth path in $\mathbb{R}^3$, suppose $\vec{r}(t) =< x(t), y(t), z(t) > t \in [a, b]$

- Parametrize the opposite path beginning at r(b) and ending at r(a).
- prove $\int_C F \cdot d\vec{r} = -\int_{-C} F \cdot d\vec{r}$

3.1 Fundamental Theorem of Line Integrals

Theorem:(Fundamental Theorem of Line Integrals)
Let C be a piece-wise smooth curve going from a to b, given a differentiable function $f(x, y, z) = w$

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Example: Verify Fundamental Theorem of Line Integrals for $\vec{F} =< y, x >$ C is the line from (0,0) to (1,1)
Solution: ... 1
Definition: A vector field $\vec{F}$ is path independent in U if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any smooth smooth paths C_1, C_2 with equal endpoints.
Corollary: A conservative field is path independent by the Fundamental Theorem of Line Integrals.
Proposition:
Let $\vec{F}$ be a continuous vector field in U, the following are equivalent.

- $\vec{F}$ is path independent
- given any closed curve, $\int_C \vec{F} \cdot d\vec{r} = 0$

From this we have that if $\vec{F}$ is conservative, then $\vec{F}$ is path independent.
Proposition:

Suppose $\vec{F}$ is a vector field continuous on an open and connected set U$\subseteq \mathbb{R}^n$(n=2 or 3),
If $\int_C F \cdot d\vec{r}$ is independent of path in U, then $\vec{F}$ is a conservative vector field on U.
Example: (Conservation of Energy)
Solution:
We fix a continuous force field $\vec{F}$. Consider an object with mass m moving along a path C. With parametrization r(t)=¡x(t),y(t)¿ $t \in [a, b]$
$\vec{F}(\vec{r}(t)) = m\vec{r}''(t)$(Newton)
Work done is equal to $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r'}(t)dt = \int_a^b m\vec{r}''(t)\vec{r}'(t)dt$
$= \frac{m}{2}\int_a^b \frac{d}{dt}\vec{r}'(t)^2 dt = \frac{m}{2}[\|r'(t)\|]_a^b = \frac{m}{2}(\|v(b)\| - \|v(a)\|)$
$= \frac{1}{2}m\|v(t)\|^2(kineticenergy)$

4 Green Theorem

Greens Theorem relates a path integral of a vector field $\int_C F \cdot d\vec{r}$ and a certain double integral where $R \subseteq \mathbb{R}^2$ is a region, and C will be a closed curve.
Example: Find the circulation of the curve C=$C_1 \cup C_2 \cup C_3$ where $C_1 : r_1(t) =< t, 0 >, t \in [0, 1], C_2 : r_2(t) =< 1 - t, t >, t \in [0, 1], C_3 : r_3(t) =< 0, 1 - t > t \in [0, 1]$ on the vector field $\vec{F}(x, y) =< xy, x^2 >$
Solution:
$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 < 0, t^2 > \cdot < 1, 0 > dt = 0$
$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 < t - t^2, (1 - t)^2 > \cdot < -1, 1 > dt = \int_0^1 (t^2 - t) + (1 - t)^2 dt = \frac{1}{6}$
$C_3, \vec{F} \circ \vec{r}_3(t) = 0$
$C_1 + C_2 + C_3 = \frac{1}{6}$
C is a closed curve enclosing a region $R = \{(x, y) \mid x \in [0, 1], 0 \leq y \leq 1 - x\}$
Recall, if $\vec{F}$ is conservative then $\int_C \vec{F} \cdot d\vec{r} = 0$ by the Fundamental Theorem of line integrals. Therefore, $\vec{F}$ is not conservative.
Observe the green function

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

For our case, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - x = x$ tjen $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int_0^1 \int_0^{1-x} xdydx = \int_0^1 x(1 - x)dx = \frac{1}{6}$
<u>Theorem:</u>(Greens Theorem)
Let $U \subseteq \mathbb{R}^2$ be an open set, C a piece-wise smooth, closed, simple and positively oriented (CCW) path which bound a region $D \subseteq U$.Moreover, Let $\vec{F}$ be a C_1 class vector field.
Then, $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$

$(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$ measures how non-conservative $\vec{F}$ is.
Example: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ with $\vec{F}(x, y) =< y^2, 3xy >$ where C goes along the

annulus $1 \leq x^2 + y^2 \leq 4$.
Solution: Notice, it is hard to evaluate all the 4 curves individually. Therefore we can use greens theorem. $D = \{(r, \theta) \mid r \in [1, 2], \theta \in [0, \pi]\}$
$\oint_C F \cdot d\vec{r} = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$(greens)
$= \iint_D (3y - 2y) dA$
$= \int_0^{2\pi} \int_1^2 r^2 sin(\theta) dr d\theta$
$= \frac{14}{3}$
Example: Compute the area enclosed by the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Solution: Let D denote the area enclosed by the ellipse. Notice, $A(D) = \iint_D dA = \int_C \vec{F} \cdot d\vec{r}$
To apply greens Theorem we need to come up with a vector field with $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

$$\vec{F} =< 0, x >, C : \vec{r}(t) =< acos(t), asin(t) >, t \in [0, 2\pi]$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} < 0, acos(t) > \cdot < -asin(t), bcos(t) > dt = \pi ab$$

proof: (Green Theorem)
claim: $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$
$= \int_a^b \vec{F}(r(t)) \cdot r'(t) dt = \int_a^b < P(r(t)), Q(r(t)) > \cdot < x'(t), y'(t) > dt = \int_a^b P((r(t))x'(t) dt + \int_a^b Q(r(t))y'(t) dt$
Claim: $\int_a^b P(r(t))x'(t) dt = -\iint_D \frac{\partial P}{\partial y} dA$
We consider the case where D is a region of the type 1.
$D = \{(x, y) \mid x \in [a, b], f(x) \leq y \leq g(x)\}$
$C_1 : r_1(t) =< t, f(t) >, t \in [\alpha, \beta]$.
$C_2 : r_2(t) =< \beta, t >, t \in [f(b), g(b)]$
$-C_3 : r_3(t) =< t, g(t) >, t \in [\alpha, \beta]$,
$-C_4 : r_4(t) =< \alpha, t >, t \in [f(\alpha), g(\alpha]$
Then do the path integral over the different paths and you prove it.
We can also establish greens theorem for non-simply connected regions. We just create paths to avoid the singularity. And we get

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

To take care of the singularity we make paths we make paths connecting the boarders of the singularity. When we go around the boarders, the times we go around it they cancel out. So we can still apply greens theorem.
Example: Let C be any simply positivity oreinted closed path enclosing O, $\vec{F} =< \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} >$ Show $\int_C \vec{F} \cdot dr = 2\pi, \forall C$
Solution: Notice, we cannot apply green because of the hole, so we put a circle around the origin of raduis a. We now have

$$\int_C \vec{F} \cdot dr + \int_C \vec{F} \cdot dr = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA.$$

Notice the green function is zero. So $\int_C \vec{F}\cdot dr + \int_{C_1} \vec{F}\cdot dr = 0 \iff \int_C \vec{F}\cdot dr = -\int_{C_1} \vec{F}\cdot dr$, Now if we look at only the right hand side of the equality, we can parametrize it easily and then we get that the integral is equal to 2π

5 Divergence and Curl

Recall, If $z = f(x,y)$ or $w = g(x,y,z,)$ the gradient of f is $\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$ is a vector field in $\mathbb{R}^2$ similarly for $\mathbb{R}^3$. We see that the del operator ∇ brings a scalar ed valued function to a vector field. The Divergence goes the other way.
<u>Definition:</u> Let $\vec{F}$ be a class C_1 vector feild in U,
The divergence of $\vec{F}$ is a scalar valued function
$div(\vec{F}) := \frac{\partial P}{\partial x + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}} = \nabla \cdot \vec{F}$
where $\nabla =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$.

Example: Support a field $\vec{F} =< x^2y, z, xyz >$
Solution: $div(\vec{F}) = 2xy + 0 + xy = 3xy$
Geometric Interpretation
In $\mathbb{R}^2$, div(F) can be the rate of expansion of Area. In $\mathbb{R}^3$ if F is the velocity field of a gas or liquid, div(F) is the rate of expansion per unit volume under the flow of the gas. If div(F)¡0 we have compression and if div(F)¿0 we have expansion.
Remark: Supose we begin with $z = f(x,y)$ with $\vec{\nabla} f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$
$div(\vec{F}) = \frac{\partial^2 f}{dx^2} + \frac{\partial^2 f}{dy^2} = 0 \iff f$ is a harmonic function. Then we denote the Laplace Operator as $\nabla \cdot \nabla \vec{f} = \nabla^2 f$.
<u>Defintion:</u>(Curl)
Let F=¡P,Q,R¿ be a C_1 vector field in $\mathbb{R}^3$, the Curl of F is $Curl(\vec{F}) = \nabla \times F$

Note: Curl(F) is again a vector field.
Example: $\vec{F} =< x, xy, 1 >$ find Curl($\vec{F}$).
Solution:
$Curl(\vec{F}) = \nabla \times \vec{F} =< 0, 0, y >$
Example: $\vec{F} =< xy, -sin(z), 1 > find$ Curl($\vec{F}$).
Solution:
$Curl(\vec{F}) = \nabla \times \vec{F} =< cos(z), 0, -x >$
<u>Scalar Curl</u>
If $\vec{F} =< P, Q >$ a vector field in $\mathbb{R}^2$ we can consider it as a field in $\mathbb{R}^3$ we use the field $\vec{F} =< P, Q, 0 >$ We get the scalar curl is greens function.

<u>Proposition:</u> (Gradients are Curl Free)
For any C_2 function, Curl($\vec{\nabla} f$) = $\vec{0}$.

Proof: Use Clariouts Theorem.
Corollary: If $\vec{F}$ is conservative, $curl(\vec{F}) = 0$

Theorem: If $\vec{F}$ is a field defined on all of $\mathbb{R}^3$, of class C_1 and Curl(F)=0, Then F is conservative.

Example: $\vec{F} =< 1, sin(z), ycos(z) >$ is F conservative?
Solution: $Curl(\vec{F}) = \vec{0}$, F is defined everywhere, by theorem, $\vec{F} = \nabla f$ with potential function $f(x, y, z) = ysin(z) + x + C$.
Proposition:
For any Vector Field $\vec{F}$ of class C_2, then div(curl($\vec{F}$)=0. Proof: Use Clairouits.

Example: Show $v =< x, y, z >$ is not the Curl of some vector field $\vec{F}$.
Solution: $div(\vec{v}) = 3 \neq 0$ by theorem, it cannot be the curl.
Note: If Div(G)=0 then G is the curl of some function F if G is on a simply connected domain i.e. all of $\mathbb{R}^3$.

6 Parametric Surface

How can we express path or curves in $\mathbb{R}^2$,$\mathbb{R}^3$?. Recall, In $\mathbb{R}$ we express can it with its graph $y = x^2, r(t) =< x, x^2 >$ or a level curve $x^2 + y^2 = 1$ or a vector valued function $\vec{r}(t) =< x(t), y(t) > t \in [a, b]$. In $\mathbb{R}^3$, we can express a surface by a graph $z = x^2 + y^2$ or a level surface $x^2 + y^2 + z^2 = 1$.
Definition: A parametrization of a surface is a map

$$\vec{r} : D \subseteq \mathbb{R}^{\nvDash} \to \mathbb{R}^3, s.t., \vec{r}(s, t) =< x(s, t,), y(s, t), z(s, t) > (s, t) \in D$$

The image of R is the surface, $\vec{r}$(D)=S. If x(s,t), y(s,t) and z(s,t) are C_1, then S is C_1

Examples:
(i)what does the graph of $\vec{r}(s, t) =< s, cos(t), sin(t) >, s \in [0, 10], t \in [0, \pi]$
(ii) given $z = f(x, y)$ is a surface of a graph, we just parametrize it with $\vec{r}(s, t) =< s, t, f(s, t) >, s, t \in Dom(f)$
(iii) find the parametrization of the plane with points at (0,0,1) , (0,1,0) , (1,0,0).
Solution: we know the function x+y+z=1 and thus can use (ii) to find the parametrization.
(iv) Sphere
Solution: $x^2+y^2+z^2 = a^2 \iff \vec{r}(\phi, \theta) =< asin(\phi)cos(\theta), asin(\phi)cos(\theta), acos(\phi) >$ $\theta \in [0, 2\pi], \phi \in [0, \pi]$(From spherical coordinates).

6.1 Grid Curves

A <u>Grid Curve</u> is a fixed single parameter of a level surface (like a level curve).
Example:
(i) our sphere if we fix $\phi = \frac{\pi}{4}$ then we get a grid curve of circle in space. if we fix $\theta = \frac{\pi}{4}$, we get a semicircle going up and down. (ii) parametrize the cone $z = \sqrt{x^2 + y^2}, z \in [0, 9]$
Solution: we can converge to cylindrical coordinates and use $C(r, \theta) =< rcos(\theta), rsin(\theta), r >$

6.2 Tangent Planes and Normals

Let $\vec{r}(s, t) =< x(s, t), y(s, t), z(s, t) >$ be a C_1 surface.
$\vec{r}_s(s, t) =< \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s} >$ is the tangent in the direction of s of the grid curve.
$\vec{r}_t(s, t) =< \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} >$ is the tangent in the direction of t of the grid curve.
Keeping s_0, t_0 fixed, fixed with $\vec{r}_t(s_0, t_0)$ is the tangent vector to the grid curve. We have a similar result for $\vec{r}_s(s_0, t_0)$. If $r_s \times r_t$ is non 0 across D, the surface is called smooth. (Recall, the cross product is normal to the tangent plane)
<u>Definition:</u> For a smooth parametrized surface $\vec{r} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, the tangent plane to the surface at a point $P_0 = (s_0, t_0)$(In the domain of the curve), is given by its normal $\vec{r}_t \times \vec{r}_s = \vec{n}$. Passing through P_0.
$\vec{n} =< a, b, c >, 0 = a(x - x_0) + b(y - y_0) + c(z - z_0)$

Example:

(i) Cone, find equation of the tangent plane for the cone at point P=$(r(2, \frac{\pi}{4})$.
Solution: $C(r, \theta) =< rcos(\theta), rsin(\theta), r >, \theta \in [0, 2\pi], r \in [0, 9]$ P=¡$\sqrt{2}, \sqrt{2}, 2 >$...
should get 0== $-\sqrt{2}(x - \sqrt{2}) - \sqrt{2}(y - \sqrt{2}) + 2(z - 2)$
An Application of this we can find the surface area given a smooth surface.

$$A(S) = \iint_D \|\vec{r}_s \times \vec{r}_t\| dA$$

Example: Find surface area of the helicoid, $\vec{r}(r, \theta) =< rcos(\theta), rsin(\theta), \theta >, r \in [0, 1], \theta \in [0, 2\pi]$

7 Surface Integrals

We Introduce two new forms of Integration, both over smooth parametrized surfaces.
<u>Definition:</u> Let $\vec{r} : D \to \mathbb{R}^3$ be a smooth parametrized surface. Let $w = f(x, y, z)$ be a continuous function defined on S=$\vec{r}(D)$

$$\iint_S f dS := \iint_D f \circ \vec{r}(s, t) \|\vec{r}_s \times \vec{r}_t\| dA$$

Geometrically, imagine S being a sheet of aluminum and f being the density at each points, then $\iint f dS$ is the mass when f is a constant we get the surface area.
Example: $\iint_S x^2 dS$ where S is the unit sphere
Solution: .... $\frac{4\pi}{3}$. Example: Let S be a triangle with vertices (1,0,0),(0,1,0) and (0,0,1).

7.1 Flux Integrals

Orientation: we start with a smooth surface S, we need to rule put surfaces that don't emit a choice of normal which varies continuously. If f emits a choice normal which varies continuously, S is called an interpreted surface.
Definition: Let S be a smooth parametrized surface oriented with unit normal; $\vec{n}$. Suppose we have a continuous vector field $\vec{F}$ defined on S then

$$\iint_S \vec{F} \cdot d\vec{S} := \iint_S \vec{F} \cdot \vec{N} ds$$

If $\vec{r} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrization, we take the unit normal $\frac{\vec{r}_s \times \vec{r}_t}{\|\vec{r}_s \times \vec{r}_t\|} = \vec{N}$ or the opposite direction. From our definition above,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N} d\vec{S} = \iint_D \vec{F} \circ \vec{r}(s,t) \cdot \vec{N} dA$$

$$= \iint_D \vec{F} \circ \vec{r}(s,t) \cdot \frac{\vec{r}_s \times \vec{r}_t}{\|\vec{r}_s \times \vec{r}_t\|} \|\vec{r}_s \times \vec{r}_t\| dA = \iint_D \vec{F} \circ \vec{r}(s,t) \cdot (\vec{r}_s \times \vec{r}_t) dA$$

Example:
(i) Find flux of $\vec{F} =< z, y, x >$ across the unit sphere. Solution: $\frac{4\pi}{3}$ The sphere is an example of a closed surface, i.e. the boundary of a solid. When we have such a surface, the positive orientation goes outward.
Example: Find the flux over the surface bounded by the curves $z = 1 - x^2 - y^2$ and $z = 0$ for the vector field $\vec{F} =< y, x, z >$
Solution: Break into two surfaces and then sum them. Should get $\frac{\pi}{2}$

8 Stokes Theorem

"Green's Theorem in space", we look at an example to start. Consider the surface S which is part of the paraboloid $z = 5 - x^2 - y^2$ above $z = 1$.We can parametrize the surface with $\vec{S}(x,y) =< x, y, 5 - x^2 - y^2 >$ on the disk $5 - x^2 - y^2 \geq 1 \iff 4 \geq x^2 + y^2$. Notice, this surface has a boundary $C(t) =< 2cos(t), 2sin(t), 1 > t \in [0, 2\pi]$ Given a vector field $\vec{F} =< -2yz, y, 3x >$ be a force field. to find the flux across the sphere we would compute the who surface integral. However, we look at a vector field across S If we want to compute the work done across the curve around the equator. Recall, Greens theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$. Notice Greens function is the scalar curl. So we we compute the surface integral of the curl of F. both methods result into

into $\frac{\pi}{4}$.
<u>Theorem:</u>(Stokes)
Let S be a smooth oriented surface bounded by a simple, closed, piecewise, smooth curve with positive orientation. Let $\vec{F}$ be a vector field on an open set containing S be continuous.
Then

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S curl(\vec{F}) dA$$

Example: Consider the surface part of the sphere $x^2 + y^2 + z^2 = 4, z \geq 0$ inside cylinder $x^2 + y^2 = 1$ find the flux of the curl of a vector field $\vec{F} =< xz, yz, xy >$
Solution: We find that the flux is 0. Example: Find the work done across the tetrahedron with vertices (1,0,0),(0,1,0),(0,0,1).
Solution: We can use stokes to reduce it to a constant times the area of the surface.

9 The Divergence Theorem

Recall, the alternative form of greens theorem, $\oint_C \vec{F} \cdot \vec{n} ds = \iint_S div(\vec{F}) dA$ is how much the vector field is escaping from the surface.
<u>Theorem:</u> Let E be a simple solid and Let S be the boundary enclosing E(Positively Oriented)
Let $\vec{F}$ be a C_1 vector field, Then,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div(\vec{F}) dV$$

Remark, simple means a solid of type 1,2,3, cylindrical or spherical.
Example:
(i) Compute Flux of $\vec{F} =< z, y, x >$ across the unit sphere.
Solution: Use gauss theorem to get $\frac{4\pi}{3}$ (ii) Verify Gauss' theorem for $E = [0,1] \times [0,1] \times [0.1], \vec{F} =< 3x, x, 2xz >$
Solution: you get 0.
Application: Divergence works with solid with a hole (2 boundary surfaces)
Example:
(i) $E = \{(x, y, z) \| 1 \leq x^2 + y^2 + z^2 \leq 4\}$
Solution: Divergence Theorem says $\iiint_E div(\vec{F}) dV = \iint_{S_1 \cup S_2} \vec{F} \cdot dS = \iint_{S_1} \vec{F} \cdot d\vec{S} - \iint_{S_2} \vec{F} \cdot d\vec{S}$
(ii) Coulomb
$\vec{F} = \frac{\varepsilon q Q}{\|\vec{v}\|} \vec{v}$. Notice there is a singularity at 0. So Let S be any closed surface boundary of a solid E. Containing the Origin, now let S_A be the sphere of radius a enclosing E.
$\iiint_E div(\vec{F}) dV = 0 = \iint_{S_a} \vec{F} \cdot d\vec{S} - \iint_S \vec{F} \cdot d\vec{S}$. From this we just compute the easy integral of around the sphere.