Math 310–Occidental College

Problem Set 1–Cardinality and Countability

Problem 1. If F is a finite set and $k : F \to F$ is a self map, prove that k is injective if and only if k is surjective.

Problem 2. Prove that a set A is infinite if and only if there is a non-surjective injection $f : A \to A$.

Problem 3. Let A, B, and C be sets and suppose $\operatorname{card}(A) < \operatorname{card}(B) \leq \operatorname{card}(C)$. Prove that $\operatorname{card}(A) < \operatorname{card}(C)$.

Problem 4. If $A \subseteq B$ is an inclusion of sets with A countable and B uncountable, show that $B \setminus A$ is uncountable.

Problem 5. Is the set $\{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 \in \mathbb{Q}\}$ countable?

Problem 6. Consider the set $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$. Is $\mathcal{F}(\mathbb{N})$ countable?

Problem 7. Let $k \in \mathbb{N}$.

(i) Prove that $\mathbb{N}^k := \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$ is countable.

(ii) Show that the set
$$\mathbb{N}^\infty := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N}\}$$
consisting of all sequences of natural numbers is uncountable.

(iii) Prove that the set of **finitely-supported** natural sequences
$$c_c(\mathbb{N}) := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N},\ n_k = 0 \text{ for all but finitely many } k\}$$
is countable.

(iv) Is the set of decreasing natural sequences
$$D := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N},\ n_{k+1} \leq n_k\ \forall k \geq 1\}$$
countable or uncountable?

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that sends rational numbers to irrational numbers and irrational numbers to rational numbers. Prove that the range $\operatorname{ran}(f)$ can't contain any interval.

Problem 9. Prove that the set
$$\mathcal{P} := \left\{ \sum_{k=0}^{n} a_k x^k \mid n \in \mathbb{N}_0,\ a_k \in \mathbb{Q} \right\},$$
consisting of all polynomials with rational coefficients, is countable.

Problem 10. A real number t is called **algebraic** if there is a nonzero polynomial p with rational coefficients such that $p(t) = 0$. If $t \in \mathbb{R}$ is not algebraic, it is called **transcendental**. For example, $\sqrt{2}$ is algebraic, but π is transcendental. Show that the set of algebraic numbers is countable, and conclude that there are uncountably many transcendental numbers.

Mateo ARMINO
MATH 310
8/27/25
P.S. 1

<u>Problem 1:</u> If F is a finite set and $\mathcal{K}: F \to F$ is a self map then ($\mathcal{K}$ is injective $\iff$ $\mathcal{K}$ is surjective).

<u>Pf:</u>
- ($\mathcal{K}$ is injective $\Rightarrow$ $\mathcal{K}$ is surjective)

Given $\mathcal{K}$ is injective by theorem $\mathcal{K}$ is left invertable
Thus, there exists an f s.t.

(*) $f \circ \mathcal{K} = id_F$.

consider $id_F \circ f = f$,

$$\overset{(*)}{\Rightarrow} (f \circ \mathcal{K}) \circ f = f$$ by associativity

$$\Rightarrow f \circ (\mathcal{K} \circ f) = f$$

$$\Rightarrow (\mathcal{K} \circ f) = id_F$$

$\Rightarrow$ $\mathcal{K}$ is right invertable
by theorem, $\mathcal{K}$ is surjective

• ($\mathcal{K}$ is surjective $\Rightarrow$ $\mathcal{K}$ is injective)

Given $\mathcal{K}$ is surjective
by Theorem, $\mathcal{K}$ is Right invertible
by defintion, for some $\mathcal{H}$,

(*) $\mathcal{K} \circ \mathcal{H} = id_F$.

consider,

$$id_F \circ \mathcal{K} = \mathcal{K}$$

(*) $(\mathcal{K} \circ \mathcal{H}) \circ \mathcal{K} = \mathcal{K}$ by associtivity

$\Rightarrow \mathcal{K} \circ (\mathcal{H} \circ \mathcal{K}) = \mathcal{K}$

$\Rightarrow \mathcal{H} \circ \mathcal{K} = id_F$

By defintion, $\mathcal{K}$ is Left invertable
By Theorem, $\mathcal{K}$ is injective.

$\square$

(*): Given finite sets A, B, s.t. $B \subseteq A$ and $card(A) = card(B) = n \in \mathbb{N}$

then $A = B$

Pf:

Suppose for contradiction

$A \nsubseteq B$.

$\Rightarrow \exists x \in A$ s.t. $x \notin B$,

$\Rightarrow card(A) = card(B) + 1$

$\Rightarrow n = n+1$ #

Problem 2: Prove a set A is infinite $\Leftrightarrow \exists f: A \to A$ not surjective injection.

Pf:

($\Leftarrow$): $\exists f: A \to A$ not surjective but injective. $\Rightarrow A$ is infinite

Suppose for contradiction A is finite.

$\Rightarrow \operatorname{card}(A) = n \in \mathbb{N}$

Let $f: A \hookrightarrow A$ be an injection

Since f is injective notice,

- $\operatorname{card}(\operatorname{Range}(f)) = \operatorname{card}(A) = n$
- $f(A) \subseteq A$.

by ($*$)

$\Rightarrow \operatorname{Range}(f) = A$

$\Rightarrow f$ is surjective. #

contradiction because we assumed f not surjective

Hence, A must be infinite

$(\Rightarrow)$: A is infinite $\Rightarrow \exists f: A \to A$ only injective

$\boxed{\text{using contrapositive}}$

$\neg$(A is infinite) $\equiv$ A is finite.

$\neg$($\exists f: A \to A$ not surjective injective)

$\equiv$ (all injective function $f: A \to A$ must be surjective)

<u>A is finite $\Rightarrow$ all $f: A \hookrightarrow A$ must be surjective</u>

It immmediately follows from <u>Problem 1</u> $f: A \hookrightarrow A$ must be Surjective.

(Since we already showed for a self map onto a finite set f is surjective $\Leftrightarrow$ f is injective.)

$\square$

Problem 3: Let A, B, C be sets s.t. $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$ prove $\text{card}(A) < \text{card}(C)$

Pf: Given $\text{card}(A) < \text{card}(B) \leq \text{card}(C)$

$\Rightarrow \exists f: A \hookrightarrow B$ an injection

$g: B \hookrightarrow C$ an injection

$\Rightarrow f \circ g: A \hookrightarrow C$ by theorem is an injection

$\Rightarrow \text{card}(A) \leq \text{card}(C)$ by definition.

Suppose for contradiction $\exists h: A \to C$ a bijection $\Rightarrow \text{card}(A) = \text{card}(C)$

$\Rightarrow \text{card}(C) < \text{card}(B) \leq \text{card}(C)$ (assumption)

$\Rightarrow \text{card}(C) \neq \text{card}(B)$ (defintion)

$\Rightarrow \text{card}(C) < \text{card}(B) < \text{card}(C)$

$\Rightarrow \exists j: C \hookrightarrow B,\ K: B \hookrightarrow C$ injective (defn)

$\Rightarrow \exists \varphi: C \to B$ a bijection (Cantor-Schröder)

$\Rightarrow \text{card}(B) = \text{card}(C)$ #

$\therefore \text{card}(A) < \text{card}(C)$

<u>problem 4</u>: If $A \subseteq B$ is an inclusion of sets with A countable and B is uncountable. Show $B \setminus A$ is countable

<u>Pf:</u>

Suppose for contradiction $B \setminus A$ is countable.

Since A & $B \setminus A$ is countable. By Theorem

$A \cup (B \setminus A)$ is countable

$\Rightarrow$ B is countable #

Contradiction since B is assumed not countablel-

Problem 5: Let $A := \{x \mid x > 0, x^2 \in \mathbb{Q}\}$

Prove A is countable.

Pf:

Let $x \in A \Rightarrow x^2 \in \mathbb{Q}$ $x > 0$.

consider $\mathcal{K} : A \hookrightarrow B$ $\mathcal{K}(x) = x^2$

$B := \{K(x) \mid x \in A\}$

$\mathcal{K}$ is trivially injective since $x > 0$.

consider $y \in B$,

$\Rightarrow y = \mathcal{K}(x)$ for some $x \in A$

$\Rightarrow y = x^2 \in \mathbb{Q}$ (definition)

Hence $B \subseteq \mathbb{Q}$

Since $B \subseteq \mathbb{Q}$ we have

$\iota : B \hookrightarrow \mathbb{Q}$ an injection from

B to $\mathbb{Q}$.

$\mathcal{K} \circ \iota : A \hookrightarrow \mathbb{Q}$ is injective (THrm)

By definition $\operatorname{card}(A) \leq \operatorname{card}(\mathbb{Q})$

Recall, $\operatorname{card}(\mathbb{Q}) = \operatorname{card}(\mathbb{N})$. Hence

$\exists f : A \hookrightarrow \mathbb{N}$ an injection

$\therefore$ A is countable $\square$.

Problem 6: Consider $\mathcal{F}(\mathbb{N}) := \{A \subseteq \mathbb{N} \mid card(A) = n \in \mathbb{N}\}$

Show $\mathcal{F}(\mathbb{N})$ is countable.

Pf: Let $A \in \mathcal{F}(\mathbb{N}) = \{a_1, a_2, a_3, \ldots, a_n\}$

Consider the function

$\mathbb{P} : \mathcal{F}(\mathbb{N}) \longrightarrow \mathbb{N}$

$\mathbb{P}(A) = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot \ldots p_n^{a_n}$

where P_n is the P_nth prime

Suppose $\mathbb{P}(A) = \mathbb{P}(B)$

$\Rightarrow 2^{a_1} 3^{a_2} 5^{a_3} \ldots p_n^{a_n} = 2^{b_1} \cdot 3^{b_2} \cdot 5^{b_3} \ldots p_n^{b_n}$

By fundemental theorem of aritmatic

$a_i = b_i$ $i \in \mathbb{N}$ since each number in $\mathbb{N}$

Has a unique prime factorization

thus $\mathbb{P}(A)$ is injective.

$\therefore \mathcal{F}(\mathbb{N})$ is countable $\square$

Problem 7:

(i) Prove $\mathbb{N}^k := \underbrace{\mathbb{N}\times\mathbb{N}\times\mathbb{N}\ldots\times\mathbb{N}}_{k \text{ times}}$ is countable

(ii) Show $\mathbb{N}^\infty := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N}\}$

consisting of all natural sequences is uncountable.

(iii) Prove the set of finitely supported natural sequences

$$c_c(\mathbb{N}) := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N}, n_k = 0\ k \in \mathbb{N}_n\}$$

is countable

(iv) Is the set of decreasing natural sequences

$$D := \{(n_k)_{k\geq 1} \mid n_k \in \mathbb{N},\ n_{k+1} \leq n_k\ \forall k \geq 1\}$$

countable or uncountable.

(i)

Pf: Consider the function $\mathbb{P}: \mathbb{N}^n \to \mathbb{N}$.

$$\mathbb{P}(n_1, n_2, n_3, \ldots, n_n) := 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdots p^{n_n} = \prod_{k=1}^{n} p_k^{n_k}$$ where p_k is the kth prime.

Suppose $\mathbb{P}(n_1, n_2, \ldots, n_n) = \mathbb{P}(m_1, m_2, \ldots, m_n)$

$\iff 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdots p^{n_n} = 2^{m_1} \cdot 3^{m_2} \cdot 5^{m_3} \cdots p^{m_n}$

Suppose for contradiction $n_i \neq m_i$ some $i \in \mathbb{N}$

w.l.o.g $n_i < m_i$

$\Rightarrow 1 = 2^{m_1 - n_1} \cdot 3^{m_2 - n_1} \cdots p_{i_0}^{m_{i_0} - n_{i_0}} \cdots p_n^{m_n - n_n}$

If $m_k = n_k \Rightarrow p^{m_k - n_k} = 1$.

$\Rightarrow 1 = 1 \cdot 1 \cdot 1 \cdots p_{i_0}^{m_i - n_i} \cdots 1$

$= p_i^{m_i - n_i}$

since $m_i - n_i > 0$ $p_i^{m_i - n_i} \neq 1$ #

Hence, for all i, $n_i = m_i$

$\Rightarrow \mathbb{P}$ is an injection

by definition $\mathbb{N}^n$ is countable.

$\square$

Lemma 1: If $f: A \hookrightarrow B$ and $g: C \hookrightarrow D$ are injective $f \times g: A \times C \hookrightarrow B \times C$ is injective

pf:

suppose $f \times g(x_1, y_1) = f \times g(x_2, y_2)$

$\Rightarrow (f(x_1), g(y_1)) = (f(x_2), g(y_2))$

$\Rightarrow f(x_1) = f(x_2)$ and $g(y_1) = g(y_2)$

since f and g are injective

$\Rightarrow x_1 = x_2$ and $y_1 = y_2$ $\square$

Lemma 2: If A, B are countable $A \times B$ is countable

Pf: Let A, B be countable sets.

By definition

$\exists \kappa, j$ s.t. $\kappa: A \hookrightarrow \mathbb{N}$

$j: B \hookrightarrow \mathbb{N}$

$Im(A) = \{\kappa(1), \kappa(2), \kappa(3), \ldots\}$

$Im(B) = \{j(1), j(2), j(3), \ldots\}$

consider $\kappa \times j: A \times B \hookrightarrow \mathbb{N}^2$

s.t. $\kappa \times j(m, n)$, $m \in A$, $B \in n$

since κ, j are injective $\kappa \times j$ is injective (Lemma 1)

So $A \times B$ is countable

since $\mathbb{N}^2$ is countable.

(ii) Pf: suppose for contradiction

$\mathbb{N}^{\infty}$ is countable.

$\Rightarrow \exists\, \psi: \mathbb{N} \twoheadrightarrow \mathbb{N}^{\infty}$ a surjection

$\psi(1) = (n(1))_k = (n(1)_1, n(1)_2, n(1)_3, n(1)_4, \ldots$

$\psi(2) = (n(2))_k = (n(2)_1, n(2)_2, n(2)_3, n(2)_4, \ldots$

$\vdots$

Let $\varphi: \mathbb{N} \to \mathbb{N}$ be a map

s.t. $\varphi(n_k) = \begin{cases} 2 & n_k \neq 2, \\ 3 & n_k = 2. \end{cases}$

Consider a sequence

$S = (\varphi(1), \varphi(2), \ldots \ldots) \in \mathbb{N}^{\infty}$

Since ψ is a surjection $\exists\, m \in \mathbb{N}$ s.t.

$\psi(m) = S$

$\Rightarrow \psi(m) = (n(m)) = (n(m)_1, n(m)_2, \ldots)$

$= (\varphi(1), \varphi(2), \ldots)$

$\Rightarrow \quad n(m_1) = \varphi(1)$

$n(m_2) = \varphi(2)$

$\vdots$

Notice, if $n(m)_m = 2$ then $\varphi(m) = 3$ ※

and if $n(m)_m \neq 2$ then $\varphi(m) = 2$

Hence, not countable. $\square$

(iii) Pf:

$C_c(\mathbb{N}) = \{(n_k)_{k=1}^{\infty} \mid n_k \in \mathbb{N}$, only finitely many $n_k \neq 0\}$

fix a $m \in \mathbb{N}$ s.t after m
it terminates in 0.

Consider $C_c(\mathbb{N})_m = \{(n_k)_{k=1}^{\infty} \mid n_k \in \mathbb{N}, n_k = 0 \ \forall k > m\}$

Consider $\varphi : \mathbb{N}^m \to C_c(\mathbb{N})_m$

$\varphi(m_1, m_2, \dots, m_m) = m_1, m_2, \dots, m_m, 0, 0, 0, \dots$

Let $(m) \in C_c(\mathbb{N})_m$

$(m) = m_1, m_2, \dots, m_m, 0, 0, \dots$
$= \varphi(m_1, m_2, \dots, m_m)$

Hence surjective,

by Thrm, $\exists \Phi : C_c(\mathbb{N}_m) \hookrightarrow \mathbb{N}^m$

since $\mathbb{N}^m$ is countable $C_c(\mathbb{N})_m$
is countable.

$C_c(\mathbb{N}) = \bigcup_{m \in \mathbb{N}} C_c(\mathbb{N})_m$

by thrm $C_c(\mathbb{N})$ is countable

$\square$

(iv) $D = \{(n_k)^\infty \mid n_k \in \mathbb{N},\ n_{k+1} \leq n_k\ \forall k\}$

is countable

Pf:

since $(n_k)^\infty \in D$ is decreasing

at some point it terminates in $n_k \in \mathbb{N}$

fix $i \in \mathbb{N}$

$D_i = \{(n_k)^\infty \mid (n_k)^\infty \in D,\ n_k = n_i\ \forall k > i\}$

consider $\gamma : D_i \longrightarrow C_c(\mathbb{N})_i$ ($C_c(\mathbb{N})_i$ from iii)

$\gamma(n_1, n_2, \ldots, n_i, n_i, \ldots) = n_1, n_2, \ldots, n_i, 0_{i+1}, 0 \ldots$

γ is injective since

if $\gamma(n_1, n_2, \ldots, n_i, n_i, n_i, \ldots) = \gamma(m_1, m_2, \ldots, m_i \ldots)$

$\Rightarrow n_1, n_2, \ldots, n_i, 0 \ldots = m_1, m_2, \ldots, m_i, 0 \ldots$

2 sequences are equal if and only if all their term are equal

so they are equal

$\Rightarrow \gamma$ is injective.

since $C_c(\mathbb{N})_i$ is countable

$\Rightarrow D_i$ is countable

$D = \bigcup_{i \in \mathbb{N}} D_i$ is countable by theorem

$\square$

Lemma: (*)

$\mathbb{R}\setminus\mathbb{Q}$ is not countable.

Pf: Suppose for contradiction $\mathbb{R}\setminus\mathbb{Q}$ is countable,

consider $\mathbb{Q}\cup(\mathbb{R}\setminus\mathbb{Q})$.

By theorem the union of 2 countable sets is countable.

However $\mathbb{Q}\cup(\mathbb{R}\setminus\mathbb{Q})=\mathbb{R}$

contradiction since $\mathbb{R}$ not countable by Cantor. $\square$

(*): If A is a countable set and $B\subseteq A$, B is countable

Pf: Given A is countable and $B\subseteq A$

$\exists\, \iota: B\hookrightarrow A$, $\iota(x)=x$ the injective inclusion map. Since A is countable $\exists f: A\hookrightarrow\mathbb{N}$

By Theorem $\iota\circ f: B\hookrightarrow\mathbb{N}$ is an injection

Hence B is countable

(*) **Lemma:** Given $f: X \to Y$

$\text{card}(\text{Im}(f)) \le \text{card}(X)$

Pf:

$$X \overset{f}{\twoheadrightarrow} \text{Im}(f)$$

Let $x \in \text{Im}(f) \Rightarrow x = f(a)$

for some $a \in X$

Hence f is surjective.

Since f is surjection

$\exists g: \text{Im}(f) \to X$ an injective

function

by defn $\text{card}(\text{Im}(f)) \le \text{card}(X)$

$\square$

Problem 8: Let $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) := \begin{cases} \mathbb{Q} & x \in \mathbb{R}\setminus\mathbb{Q}, \\ \mathbb{R}\setminus\mathbb{Q} & x \in \mathbb{Q}. \end{cases}$

Show $\text{Range}(f) \not\subseteq I$ where I is any interval.

Pf:

First, $f(\mathbb{Q}) \subseteq \mathbb{R}\setminus\mathbb{Q}$ by (*) $f(\mathbb{Q})$ is countable.

Second, $f(\mathbb{R}\setminus\mathbb{Q}) \subseteq \mathbb{Q}$ by (*) $f(\mathbb{R}\setminus\mathbb{Q})$ is countable.

$\text{Range}(f) = f(\mathbb{Q}) \cup f(\mathbb{R}\setminus\mathbb{Q})$

by thm, the union of 2 countable sets is countable

So $\text{card}(\text{Ran}(f)) \leq \text{card}(\mathbb{N})$

Suppose for contradiction $I \subseteq \text{card}(\text{Ran}(f))$

by (*) $\text{card}(I) \leq \text{card}(\mathbb{N})$ ※

By Cantor, $\text{card}(\mathbb{N}) < \text{card}(\mathbb{R})$

and $\text{card}(I) = \text{card}(\mathbb{R})$, $\text{card}(I) > \text{card}(\mathbb{N})$

$\therefore I \not\subseteq \text{Ran}(f)$. $\square$

Problem 9: prove $P = \left\{ \sum_{x=0}^{n} a_x x^x \middle| n \in \mathbb{N}_0 \; a_x \in \mathbb{Q} \right\}$

Lemmma 3: $\mathbb{Q}^n$ is Countable:

Pf: Base case: $\mathbb{Q}$

$\mathbb{Q}$ is Countable by theorem

Suppose for some $n \in \mathbb{N}$,

$\mathbb{Q}^n$ is countable for some n

Check $\mathbb{Q}^{n+1}$

$\mathbb{Q}^{n+1} = \mathbb{Q}^n \times \mathbb{Q}^1$

by Lemma 2, $A \times B$ is countable

if A and B are countable

Hence $\mathbb{Q}^{n+1}$ is countable

$\therefore \mathbb{Q}^n$ is Countable for all $n \in \mathbb{N}$. $\square$

Pf: Notice any $p \in \mathcal{P}$ $p = \sum_{k=0}^{n} a_k x^k$

fix a $m \in \mathbb{N}_0$

$\mathcal{P}_m = \{ p \in \mathcal{P} \mid \text{Deg}(p) = m \}$

Let $\vec{a} \in \mathbb{Q}^{m+1}$ s.t. $\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix}$ where $a_i \in \mathbb{Q}$.

consider $(\vec{x})^T = (1 \; x \; x^2 \; \dots \; x^m)$

$p = (\vec{x})^T (\vec{a})$
$= \vec{x} \cdot \vec{a}$

consider $f: \mathbb{Q}^{m+1} \to \mathcal{P}_m$, $f(\vec{a}) = \vec{x} \cdot \vec{a}$

Let $p \in \mathcal{P}_m \Rightarrow p = \sum_{k=1}^{n} b_k x^k = b_1 x^1 + b_2 x^2 + \dots + b_n x^n$
$= \vec{x} \cdot \vec{b}$
$= f(\vec{b})$

So $f: \mathbb{Q}^{m+1} \twoheadrightarrow \mathcal{P}_m$ is a surjection.
By theorem, $\exists g: \mathcal{P}_m \hookrightarrow \mathbb{Q}^{m+1}$ a injection
since $\mathbb{Q}^{m+1}$ is countable by lemma 3.
$\Rightarrow \mathcal{P}_m$ is countable.
$\mathcal{P} = \bigcup_{m \in \mathbb{N}} \mathcal{P}_m$, by theorem $\mathcal{P}$ is countable.

$\square$

Problem 10. $t \in \mathbb{R}$ is algebraic if $\exists P$, a polynomial with rational coefficients s.t. $P(t)=0$. If t is not algebraic it is transcendental

Show the set of algebraic numbers is countable and conclude there are uncountably many transcendental numbers.

Pf: Let A be the set of algebraic numbers.

$$A = \{t \in \mathbb{R} \mid P(t)=0, P \in \mathcal{P}\} \quad (\mathcal{P} \text{ from } 9)$$

By definition every $t \in A$ is the root of some $P \in \mathcal{P}$

Let $B_P = \{t \in A \mid P(t)=0\}$

By the fundamental theorem of Algebra $P(t)$ can have at most n-roots.

So $\operatorname{card}(B_P) \leq n$. Moreover, from problem 9 we know their are only countably many $P \in \mathcal{P}$. So we can express $A = \bigcup_{P \in \mathcal{P}} B_P$.

By Theorem, A is countable.