Robust control

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Robust control

Systems
- Linear
- Non-linear

Methods
- Frequency
- State-space
- Polynomial

Control
- Adaptive
- Robust
- Stochastic

Uncertainty
- Non-parametric (non-structured)
- Parametric (structured)

What if more than one parameter change?

- **NOTE:** In examples so far only one parameter changed! What happens if there are more than one parameter which can change?
- The uncertainty structure becomes more complex and can be quite different. We will also cover this case, because Kharitonov theorem gives us tools for this case!

Modern robust stability methods

- Robust stability by use of the small gain theorem (Doyle et al., 1992)
- Robust stability by polynomial (closed-loop characteristic equation) approach of Kharitonov (Kharitonov, 1979)

By small gain theorem, the stability analysis in case of change of one parameter is possible !

By use of Kharitonov theorems, the stability analysis in case of change of many parameters is possible !

Mathematical model uncertainty can be classified to:

- Structured uncertainty – assumes that the uncertainty can be modeled and that parameter constraints are known i.e. Min and Max values for each changing parameter are known!! For instance we can have the transfer function of the process but uncertain positions of poles, zeros, gain etc.

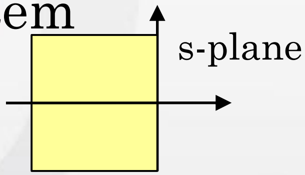
- Unstructured uncertainty – assumes much less knowledge about the process. For instance, we know only that frequency characteristic is upper and lower bounded.

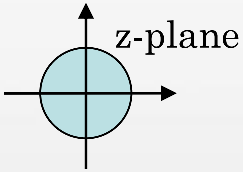
Stability of polynomials

A polynomial *p(•)* is stable if all its roots lie in some given region of the complex plane !

Stability region depends on the nature of the system

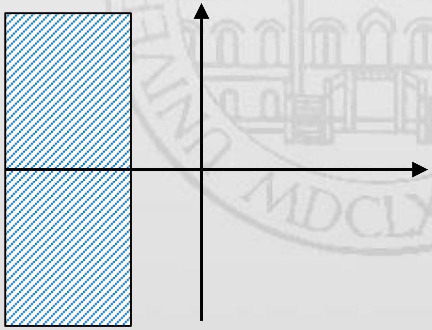
- LHP iff system is continous, *p(s)*
- Unit disk iff system is discrete, *p(z)*

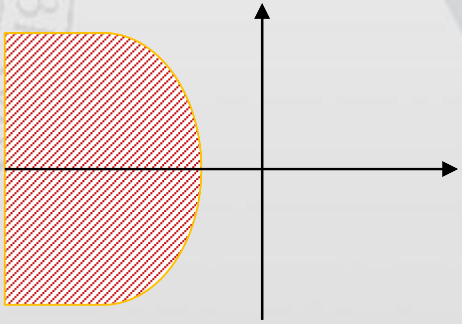


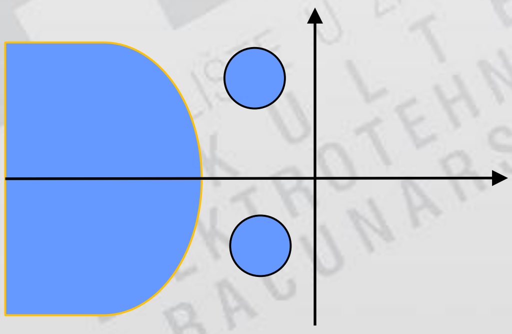


More sophisticated stability regions are required for specific performance specs:

- ➢ Shifted LHP – for desired speed or dominant behaviour or bandwidth
- ➢ Parabola or sector in LHP – for desired damping ζ
- ➢ Some other sector(s) in LHP







Polynomial uncertainty structures
Mathematical preliminaries

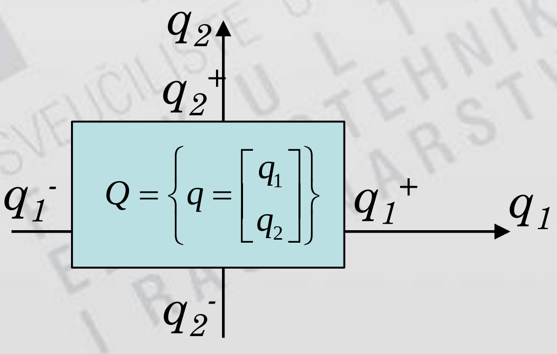
Definition (**Uncertainty Bounding Set**):
The uncertainty bounding set is the set $Q = \left\{ q \in \mathbb{R}^l \,\middle|\, q_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, l \right\}$

NOTE: $\boldsymbol{q}$ is vector and q_i and therefore Q need not be connected!
However, **we will use connected sets** since much of the results in literature apply only to connected sets.

Assumption of connectivity is not restrictive because most of the physical parameters (such as viscous friction coefficients, material properties, lengths, etc.) entering the uncertainty vector vary continously over bounded interval of the real line.

Frequently, each element q_i of vector $\boldsymbol{q}$ is described by its lower q_i^- and upper bound q_i^+.
Then the uncertainty bounding set is the box

$$Q = \left\{ q \in \mathbb{R}^l \,\middle|\, q_i^- < q_i < q_i^+ \in \mathbb{R} \text{ for } i = 1, 2, \ldots, l \right\}$$



Definition (Family)

- An uncertain function together with its uncertainty bounding set is called a family i.e.

$$\mathcal{F}(\bullet,Q)=\left\{ f(\bullet,q) \middle| q\in Q\right\}$$

For example, an uncertain plant G(s,q) and its uncertainty bounding set Q form a family of plants denoted $\mathcal{G}(s,Q) = \{G(s,\boldsymbol{q}) \mid \boldsymbol{q} \in Q\}$.
Similarly, we write $\mathcal{N}(s,Q) = \{N(s,\boldsymbol{q}) \mid \boldsymbol{q} \in Q\}$ for the family of numerators and $\mathcal{D}(s,Q) = \{D(s,\boldsymbol{q}) \mid \boldsymbol{q} \in Q\}$ for the family of denumerators

In general we can deal with the family of polynomials $p(s,q)=\sum_{i=0}^{n} a_i(q)s^i$

Where:

$a_i(\boldsymbol{q})$ are coefficients of family of polynomials $a_i(\boldsymbol{q}) = f_i(\boldsymbol{q})$
$\boldsymbol{q}^T = [q_1\ q_2 \ldots q_l]$
Q – the uncertainty bounding set of all uncertain parameters

Definition (Independent Uncertainty Structure)

An uncertain polynomial

$$p(s,q) = \sum_{i=0}^{n} a_i(q) s^i$$

is said to have *independent uncertainty structure* if each component q_i of $\boldsymbol{q}$ enters into only one coefficient.

Example:

$$p(s,\boldsymbol{q}) = (3q_3 + 2)s^3 + (q_2 + 1)s^2 + (q_1 - 1)s + (2q_0 + 3)$$

NOTE: *An independent uncertainty structure is idealization of a reality where uncertain parameters of a system generally enter into more than one coefficient of* $p(s,\boldsymbol{q})$*. Moreover, in many cases, the* $a_i(\boldsymbol{q})$ *are often nonlinear functions.*

Definition (Interval Polynomial Family)

A family of polynomials

$$\mathcal{P}(s,Q)=\left\{ p(s,q)=\sum_{i=0}^{n} a_i(q)s^i \,\middle|\, q\in Q \right\}$$

Is said to be *interval polynomial family* if ***p(s,q)*** has an independent uncertainty structure, each coefficient depends continously on ***q*** and the uncertainty bounding set ***Q*** is an n-dimensional box.

For brevity, we also refer to $\mathcal{P}(s,Q)$ as an interval polynomial.

Similarly, a family of uncertain plants

$\mathcal{G}(s,Q)=\{G(s,\mathbf{q})=N(s,\mathbf{q})/D(s,\mathbf{q})\ /\ \mathbf{q}\in Q\}$ is said to be an interval plant family if both $N(s,\mathbf{q})$ and $D(s,\mathbf{q})$ are interval polynomials

Definition (Affine Linear Uncertainty Structure)

- An uncertain polynomial $p(s,\mathbf{q})$ is said to have an *affine linear polynomial uncertainty structure* if each coefficient function $a_i(\mathbf{q})$ is of the form:

$$a_i(q) = \beta_i^T q + \gamma_i$$

Where β_i is a column vector and γ_i is a scalar.

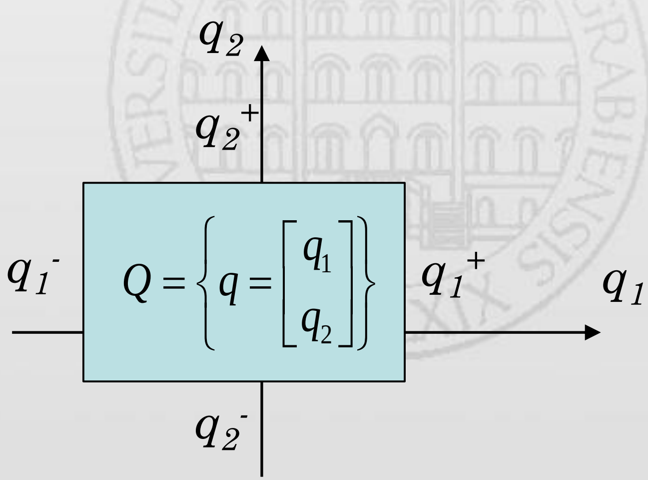
- For affine linear uncertainty structures the uncertainty bounding set is a convex hull of a finite number of points (*polytope*) and not a box!

Example:

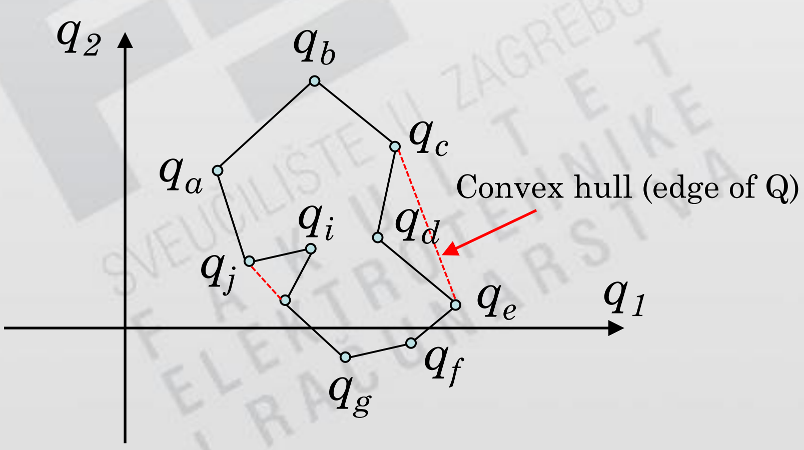
$$p(s,q) = (3q_1 + 4q_2 - 6)s^3 + (2q_1 - 3q_2)s^2 + (q_1 + q_2 s) + (q_2 - 5)$$

- **NOTE: The characteristic polynomial of a closed loop control system with fixed controller parameters and uncertain plant parameter have the affine linear uncertainty structure!**

Interval uncertainty structure
(Q is box)



Affine linear uncertainty structure
(Q is a convex hull of a **finite number of points** – polytope)



Example

- Closed loop characteristic polynomial is given by

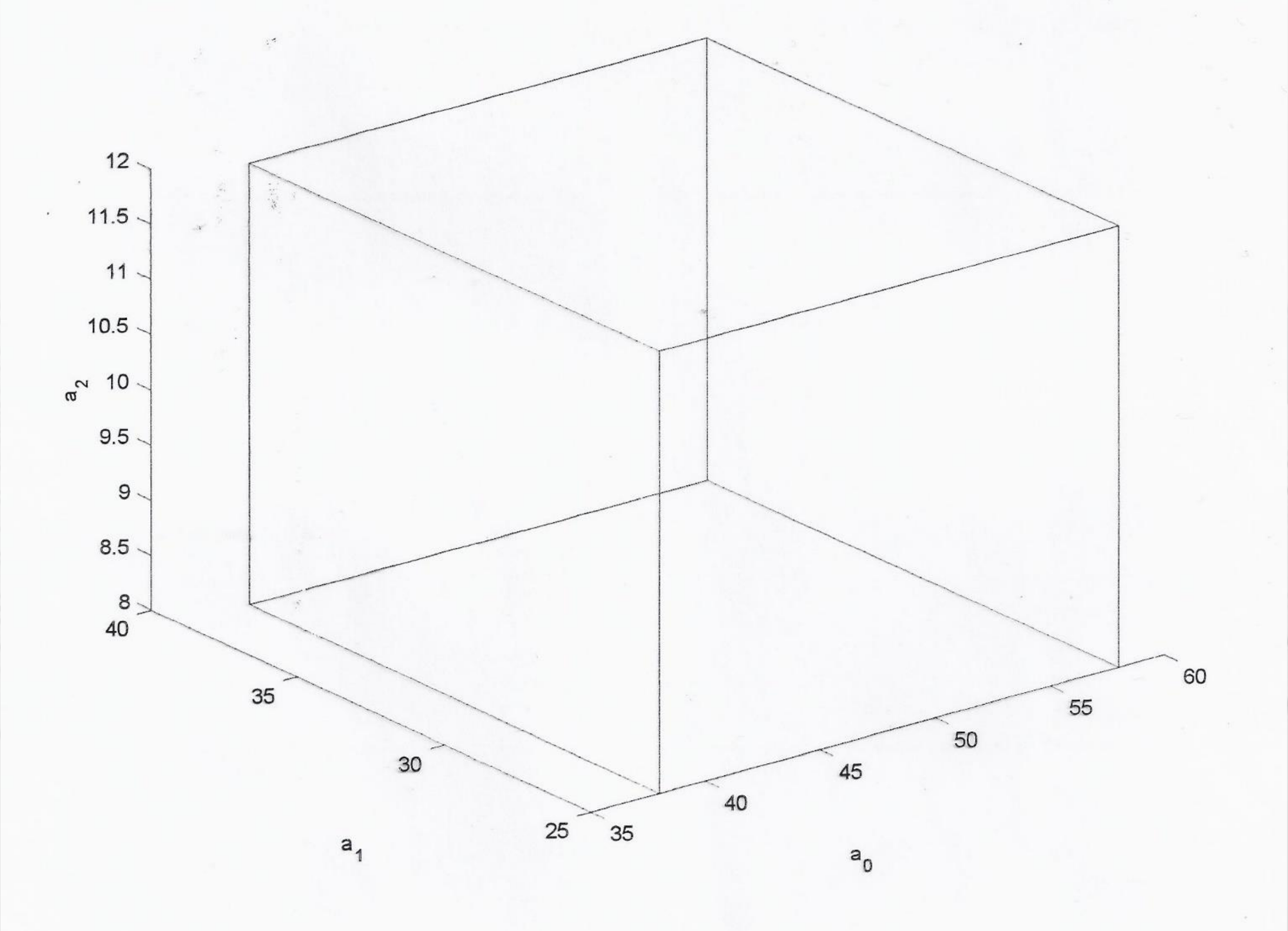
$$\boxed{\alpha_{cl}(s) = \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0}$$

- With the following bounds on the uncertainty:

$$\alpha_0 \in [38, 58] \quad ; \quad \alpha_1 \in [25, 39]$$
$$\alpha_2 \in [8, 12] \quad ; \quad \alpha_3 = 1$$

- The „*box of uncertainty*” i.e. *uncertainty bounding set Q* for the coefficients $\{\alpha_0, \alpha_1, \alpha_2\}$ is shown in the figure

With independent variations in each of the three coefficients, any point inside the box or on its boundary represents a valid set of coefficients for $\alpha_{cl}(s) = \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$
Thus, there is a triply-infinite set of characteristic eqations that must be checked for stability!



Example

Let's suppose that the controller parameters are fixed:

$$N(s) = (s+2)$$

$$D(s) = (s^2 + 2s + 2)$$

while the plant parameters can change

$$A(s,a) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$B(s,b) = b_0$$

$$b_0 \in [3,5]$$
$$a_3 \in [1,1.1]$$
$$a_2 \in [4,4.2]$$
$$a_1 \in [6,8]$$
$$a_0 \in [10,20]$$

The closed-loop characteristic polynomial becomes:

$$\alpha_{cl}(s) = (a_3 s^3 + a_2 s^2 + a_1 s + a_0)(s^2 + 2s + 2) + b_0 (s+2)$$

$$= \alpha_5 s^5 + \alpha_4 s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

where:

$$\alpha_5 = a_3; \qquad \alpha_4 = a_2 + 2a_3$$
$$\alpha_3 = a_1 + 2a_2 + 2a_3; \quad \alpha_2 = a_0 + 2a_1 + 2a_2$$
$$\alpha_1 = 2a_0 + 2a_1 + b_0; \quad \alpha_0 = 2a_0 + 2b_0$$

We end-up with affine linear uncertainty structure

Example cont.

Parameter bounds are:

$$\alpha_5 \in [1, 1.1] \quad ; \quad \alpha_4 \in [6, 6.4] \quad ; \quad \alpha_3 \in [16, 18.6]$$
$$\alpha_2 \in [30, 40.4] \; ; \quad \alpha_1 \in [35, 61] \quad ; \quad \alpha_0 \in [26, 50]$$

We assume that the parameter vector $\boldsymbol{\alpha} = [\alpha_5 \;\; \alpha_4 \;\; \alpha_3 \;\; \alpha_2 \;\; \alpha_1 \;\; \alpha_0]^T$ can take any value in the subset of determined by the above intervals for each parameter. In other words, coefficients vary independent of each other.

However, there are only five truly free parameters (a_3, a_2, a_1, a_0, b_0) which means that there is a dependence between parameter variations.

If robust stability can be shown for all possible values of the parameters α_i in the above intervals, then robust stability of the closed-loop system can be concluded, BUT the converse is not true!!

In that sense, by converting plant (and/or controller) parameter uncertainty into an uncertainty in the coefficients of the α_{cl}(s), some conservativism is introduced.

Degree invariance, boundary crossing, zero exclusion and value sets

- Assume that there is n^{th} – degree polynomial with $a_n = 1$ and independent variations in coefficients $A=\{a_{n-1}, \dots, a_1, a_0\}$. Since the most signifficant coefficient a_n is never equal to zero in this case, all the polynomials in the family are of degree n.

- NOTE: Assumption called *degree invariance* is very important here, because all of analysis methods use this requirement for robust stability analysis.

- Now assume that for at least one set of coefficients $A^a=\{a^a_{n-1}, \dots, a^a_1, a^a_0\}$ *the closed loop system is stable.*

Boundary crossing phenomenon

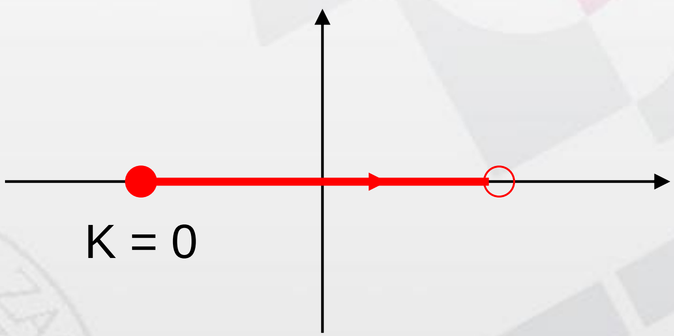
- If it is true that there is another set of coefficients $A^b=\{a^b_{n-1}, \ldots, a^b_1, a^b_0\}$ such that there is *at least one closed-loop pole in the RHP*, then it is also true that there is a set of coeffficients $A^c=\{a^c_{n-1}, \ldots, a^c_1, a^c_0\}$ such that there are no closed-loop poles in the RHP and that there is *at least one pole on the jω axis*.
- What this says is that in order for a system to go from having all LHP roots to having at least one RHP root, there must be a condition where there is one or more roots on the boundary of stability (jω axis), but none in the RHP. This is known as the *boundary crossing phenomenon*

- To go from stable to unstable you have to cross the boundary of stability ($j\omega$ axis) assuming that all polynomials in the family are of degree n (and that the a_i coefficients vary continuously between their upper and lower bounds)

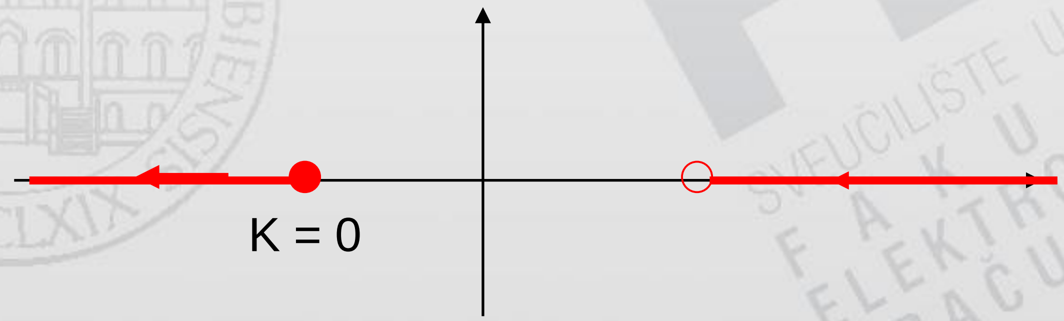
Example $G(s)=\dfrac{K(s-1)}{s+1}$

$$p(s,q)=(1+q)s+(1-q)$$

a) q = K > 0



b) q = K < 0



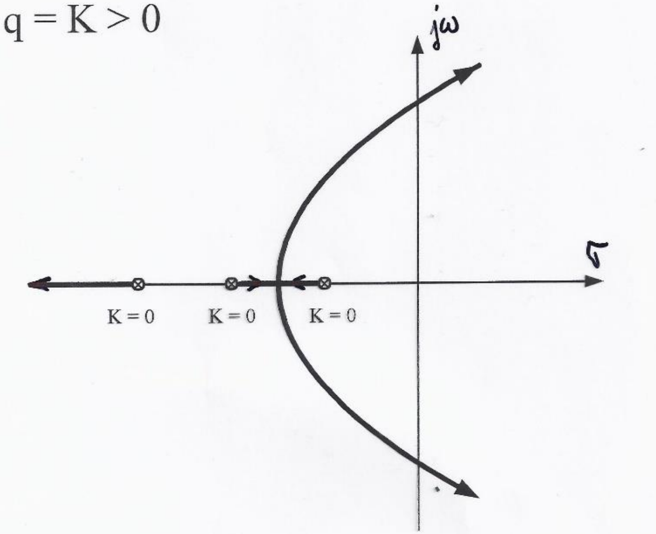
Degree invariance is not satisfied !!

Example

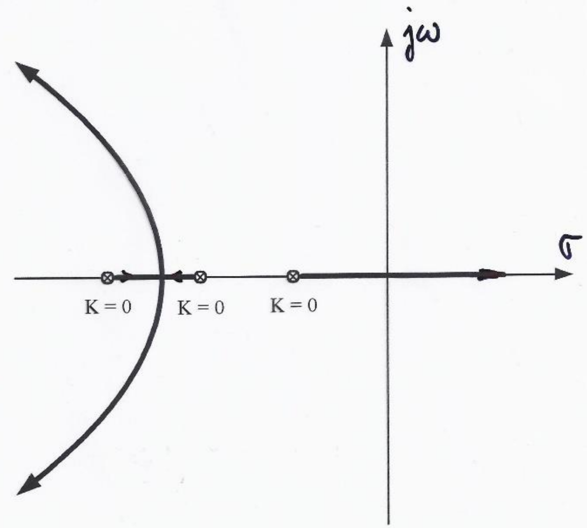
$$G(s) = \frac{K}{(s+1)(s+2)(s+3)}$$

$$p(s,q) = s^3 + 6s^2 + 11s + 6 + q$$

a) $q = K > 0$



b) $q = K < 0$



Degree invariance is satisfied

Value set

- Assume that for our family of n^{th} – degree polynomials we could evaluate each one of them along the boundary of stability. At each point on the boundary, each of the polynomials would evaluate to a complex number.
- For example at $\omega = \omega_1$, the polynomial with coefficients A^a would evaluate to $\alpha^a{}_{CL}(j\omega_1) = x^a{}_1 + jy^a{}_1$.
- As frequency varies and as polynomial coefficients vary, the complex number changes. This set of complex numbers can be plotted in a 2D plane (complex plane, but not s-plane)

- The *value set* is defined to be all the complex numbers generated at a particular frequency $\omega = \omega_1$ by the family of polynomials $\alpha_{CL}(j\omega_1)$ as the coefficients of the polynomial vary over their allowed ranges.
- *Definition (Value set):* The *value set* is the subset of the complex plane consisting of all values which can be assumed by $p(j\omega,\boldsymbol{q})$ as $\boldsymbol{q}$ ranges over Q (ω is a fixed frequency)
- The value set can be represented by a polygon in the complex plane. As frequency varies, the value set moves through the plane.

Zero exclusion

- Therefore, if it is known that there was at least one stable polynomial in the family of $\alpha_{CL}(s)$, the loss of robust stability could be detected by evaluating each polynomial in the family along the $j\omega$ axis and determining if any of the polynomials evaluated to 0 at any frequency $\omega = \omega_1$. If $\alpha_{CL}(j\omega_1) = 0$ for some set of coefficients A^c, then that polynomial has one or more roots on the $j\omega$ axis, and it has been demonstrated that the system is not robustly stable.
- *The test for robust stability then becomes the evaluation (at least conceptually) along the boundary of stability of every polynomial in the family and determining if 0 is included in any of results.*

Zero exclusion principle

- If 0 is not present in any evaluations (zero is exluded from the results), then the family is robustly stable!
- This is known as *the zero exclusion principle*, and serves as the basis for the experimental determination of robust stability !

Theorem (Zero exclusion condition)

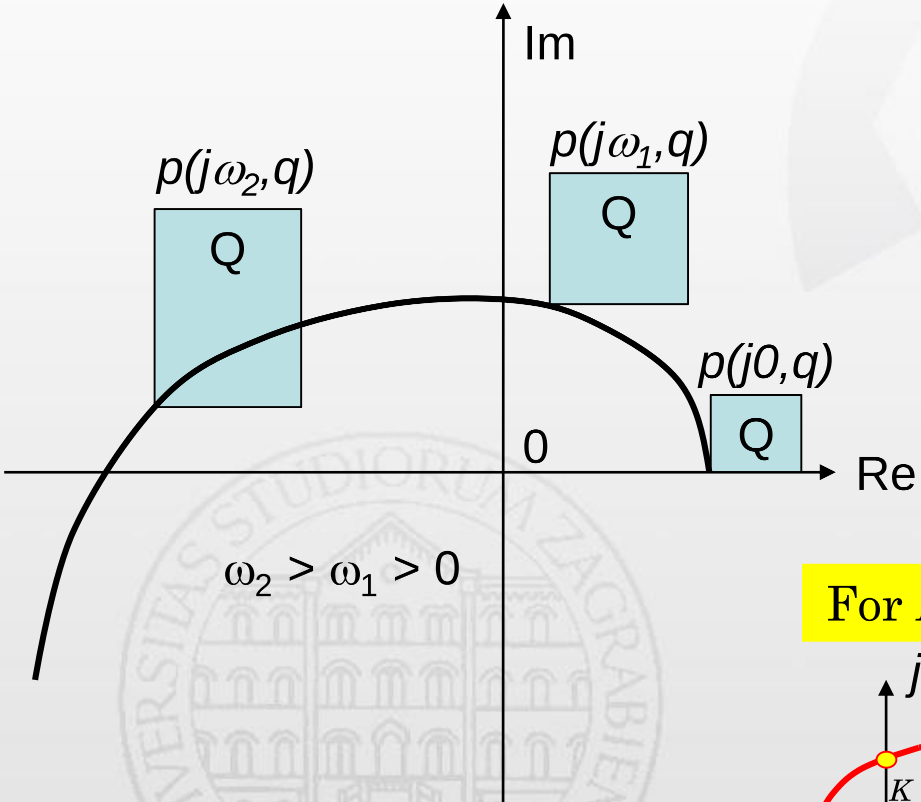
- The value set has mapped the stability analysis of an n^{th} – degree polynomial family into complex plane, which is always *two-dimensional* !!
- *Theorem:* A polynomial family having invariant degree with associated uncertainty bounding set Q which is pathwise connected, continuous coefficient functions $a_i(\mathbf{q})$ for $i = 0, 1, 2, \ldots, n$ and at least one stable member $p(s, \mathbf{q}^*)$ is robustly stable iff the *origin of the complex plane is excluded from the value set at all nonnegative frequencies* i.e.

$$0 \notin p(j\omega, \mathbf{q}) \text{ for all frequencies } \omega \geq 0 \text{ and } \mathbf{q} \in Q.$$

Robust stability

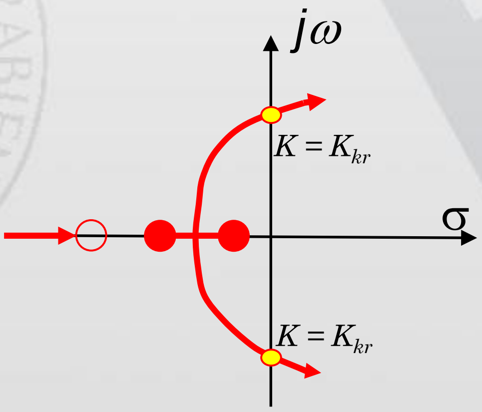
- Theorem (Zero exclusion condition) is useful for checking the stability of uncertain polynomials if we have available and efficient tools to generate the value sets.
- Definition (Robust stability): An uncertain system with the characteristic polynomials $p(s,\boldsymbol{q})$ is robustly stable iff $p(s,\boldsymbol{q})$ is stable for all $\mathbf{q} \in Q$, where Q is the uncertainty bounding set.
- If the family of n^{th} –degree polynomials is robustly stable, the value sets will move in a counter-clockwise direction through n quadrants of the complex plane without passing or touching the origin of the plane !!

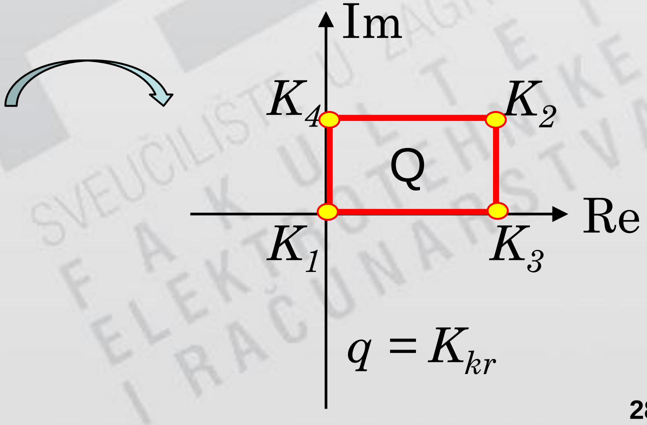
Movement of the value sets through the complex plane



The value sets move in a counter-clockwise direction through n quadrants of the complex plane ! n is the degree of the closed-loop charactersitic polynomial.

For $K = K_{kr} \Rightarrow$ maps to value set as





- The question to be answered is: are all roots of the polynomial $\alpha_{CL}(s,\alpha)$ strictly in the LHP for all values of the α_i coefficients within the bounds?
- *If the answer is yes then the family of systems described by $G_{CL}(s,\alpha)$ is robustly stable !*
- The Routh stability test determines stability of $\alpha_{CL}(s)$ with fixed α_i coefficients.
- How to test stability of this polynomial if α_i coefficients change due to plant uncertainty or due to controller tuning?
- There exist a relatively simple robust stability tests for this particular cases. They are:

Robust stability analysis

- There exist a relatively simple robust stability tests for plants with uncertain parameters – they are:
  - Kharitonov theorem (can be applied only to interval uncertainty type of structures (box) and the stability domain is the LHP – BIBO stability is analyzed)
  - Edge theorem (can be applied to interval uncertainty or affine linear uncertainty type of structures. Contrary to Kharitonov, here the stability domain(s) can be some domain smaller than the whole LHP. Q is a box or a convex hull of a finite number of edges (polytope)
  - Tzypkin-Polyak locus (graphical method for BIBO stability analysis. Enables the robustness stability margin of the polynomial family to be found.)

Kharitonov Theorem

- For interval polynomials, not every polynomial in the family needs to be individually tested for zero exclusion!
- Since each coefficient that varies generates an infinite number of polynomials, this is good news!
- In 1978 the Russian mathematician Vladimir Leonidovich Kharitonov proved that for interval polynomials, there are at most 4 polynomials that have to be tested, regardless of the degree n of the polanomial family.
- An immense savings in computation is thus possible iff the polynomial family has the interval uncertainty structure.

Definition (Kharitonov polynomials)

- Associated with the interval polynomial family

$$\mathcal{P}(s,Q)=\left\{ p(s,\mathrm{q})=\sum_{i=0}^{n} a_i(\mathrm{q})s^i \,\middle|\, \mathrm{q}\in Q \right\}$$

are four fixed Kharitonov polynomials:

$$\mathcal{K}_1(s)=\underbrace{a_0^- + a_1^- s}+\underbrace{a_2^+ s^2 + a_3^+ s^3}+\underbrace{a_4^- s^4 + a_5^- s^5}+\cdots$$

$$\mathcal{K}_2(s)=\underbrace{a_0^+ + a_1^+ s}+\underbrace{a_2^- s^2 + a_3^- s^3}+\underbrace{a_4^+ s^4 + a_5^+ s^5}+\cdots$$

$$\mathcal{K}_3(s)=a_0^+ +\underbrace{a_1^- s + a_2^- s^2}+\underbrace{a_3^+ s^3 + a_4^+ s^4}+a_5^- s^5+\cdots$$

$$\mathcal{K}_4(s)=a_0^- +\underbrace{a_1^+ s + a_2^+ s^2}+\underbrace{a_3^- s^3 + a_4^- s^4}+a_5^+ s^5+\cdots$$

- Coefficients of the four Kharitonov polynomials depend only on the upper and lower bounds of the corresponding coefficients in the polynomial family.
- Therefore, these polynomilas are known in advance !
- The following result applies to interval polynomial families.

- NOTE: These results are necessary and sufficient !

Kharitonov Theorem

- An n^{th} degree interval polynomial family described by:

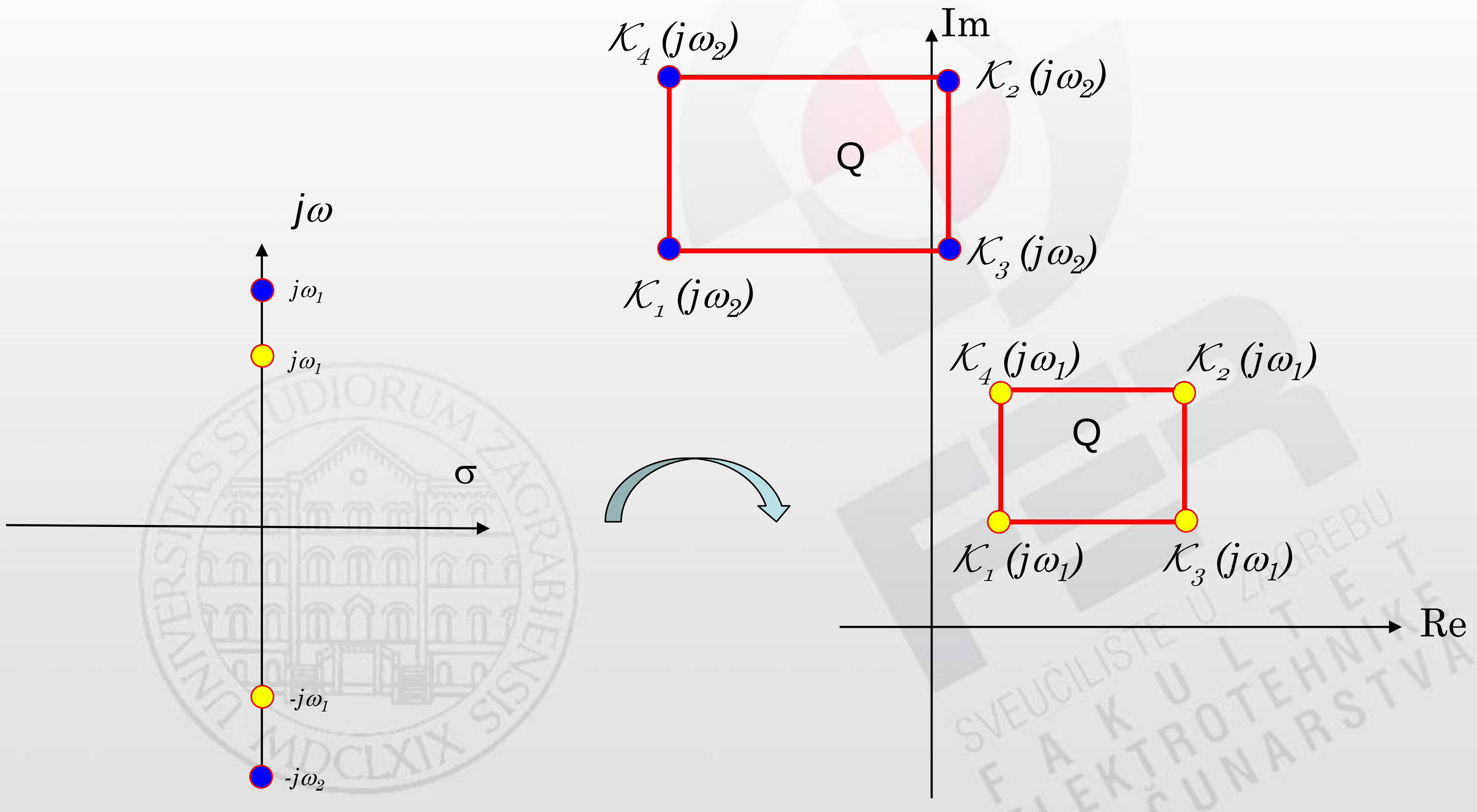
$$\mathcal{P}(s,Q)=\left\{p(s,\mathrm{q})=\sum_{i=0}^{n}a_i(\mathrm{q})s^i \,\middle|\, \mathrm{q}\in Q\right\}$$

is robustly stable if and only if each of four Kharitonov polynomials is stable, that is, all the roots of those polynomials have strictly negative real parts !!

Robust stability - Kharitonov

- Robust stability of an interval polynomial family can be determined by:
  - forming the four Kharitonov polynomials
  - factoring them
  - examining their roots
- If all the roots are in the LHP then the family is robustly stable!
- If any of the roots are on the $j\omega$ axis or in the RHP the family is not robustly stable
- When these Kharitonov polynomials are evaluated at a point on the $j\omega$ axis, they form the four corners of a rectangle whose edges are parallel with the real and imaginary axis.

This rectangle is the value set



Example

- For the *3rd*–order example

$$\alpha_{CL}(s) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

- With the following bounds on the coefficients' uncertainty:

$$\alpha_0 \in [38, 58] \;\; ; \;\; \alpha_1 \in [25, 39] \;\; ; \;\; \alpha_2 \in [8, 12] \;\; ; \;\; \alpha_3 = 1$$

- Kharitonov polynomials are:

Kharitonov polynomials are:

$$\mathcal{K}_1(s) = \underbrace{38 + 25s} + \underbrace{12s^2 + s^3}$$

$$\mathcal{K}_2(s) = \underbrace{58 + 39s} + \underbrace{8s^2 + s^3}$$

$$\mathcal{K}_3(s) = 58 + \underbrace{25s + 8s^2} + s^3$$

$$\mathcal{K}_4(s) = 38 + \underbrace{39s + 12s^2} + s^3$$

- Factoring the polynomials shows that all roots are in the LHP, so the family of polynomials is robustly stable
- If the value sets were plotted for all $\omega > 0$, zero will be excluded from them, providing a graphical conclusion of robust stability !

Example

- Assume that the system to be controlled (plant) and its compensator (controller) are given by the transfer functions: $$G_c(s) = \frac{4(s+3)}{(s+8)} \quad ; \quad G_P(s) = \frac{4}{s(s+2)}$$
- The closed loop characteristic equation for this system is: $$\alpha_{CL}(s) = s^3 + 10s^2 + 32s + 48$$
- Which is a stable polynomial having roots at
$$s_{1,2} = -2 \pm j2,$$
$$s_3 = -6$$

- Assume that parameters change i.e. coefficients in the $\alpha_{CL}(s)$ change except $\alpha_3 = 1$ always.

- Perturbations from nominal values are ±10%, ±20%, ±50% and ±60%. This corersponds to:

$$\text{Case } \pm 10\% : \alpha_0 \in [43.2, 52.8] \; ; \; \alpha_1 \in [28.8, 35.2] \; ; \; \alpha_2 \in [9, 11]$$

$$\text{Case } \pm 20\% : \alpha_0 \in [38.4, 57.6] \; ; \; \alpha_1 \in [25.6, 38.4] \; ; \; \alpha_2 \in [8, 12]$$

$$\text{Case } \pm 50\% : \alpha_0 \in [24, 72] \; ; \; \alpha_1 \in [16, 48] \; ; \; \alpha_2 \in [5, 15]$$

$$\text{Case } \pm 60\% : \alpha_0 \in [19.2, 76.8] \; ; \; \alpha_1 \in [12.8, 51.2] \; ; \; \alpha_2 \in [4, 16]$$

Plotting the value sets for each case we get:

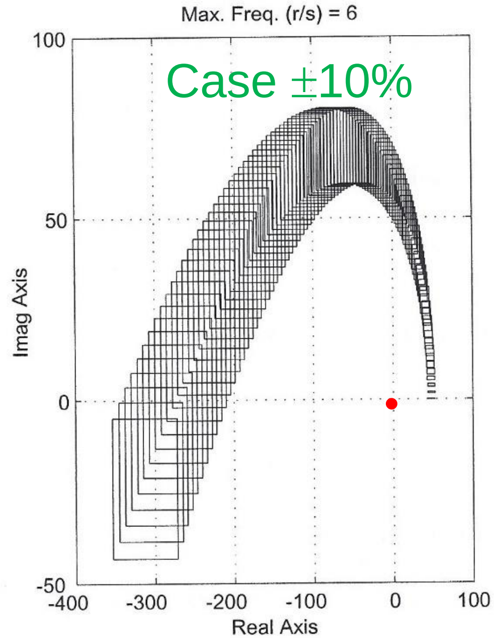
Case ±10% stable

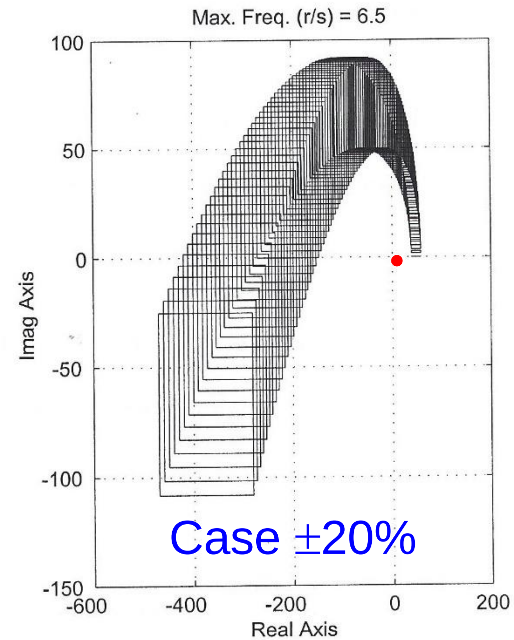
Case ±20% stable

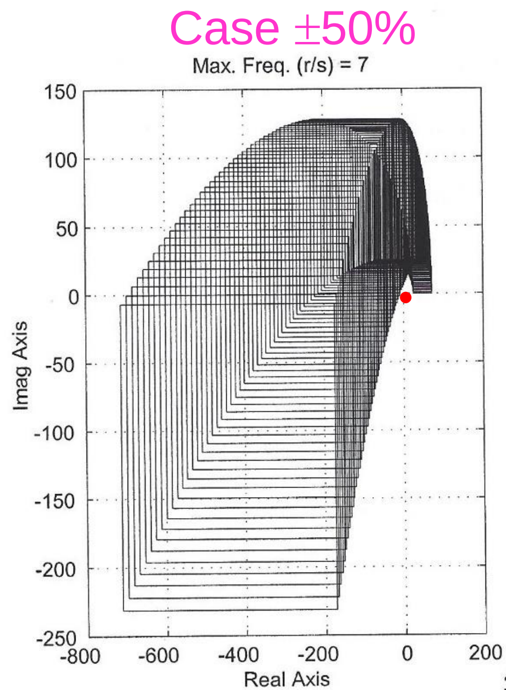
Case ±50% stable

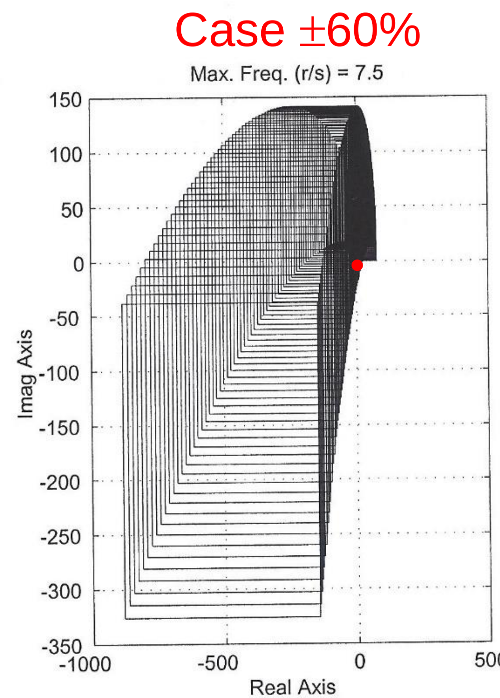
Case ±60% unstable

Value sets for first three cases move from the first quadrant to the second and then to the third quadrant (counter-clockwise) without passing through the origin (zero exclusion).
For the case ±60% perturbation zero exclusion is not satisfied and for 60% change we can't ensure robust stability with this type of controller!









Conclusions

- Since no mathematical model will be a perfect representation of the actual system, a study of stability robustness is an important task in control system design.
- The controller must stabilize and provide good performance for the real system, not just its mathematical model.
- No matter what technique has been used to design a controller, an analysis of robustness should be made when the design is complete.
- This can be done to determine the minimum size of perturbation that will destabilize the system OR to see if stability is maintained for a particular set of assumed perturbations.