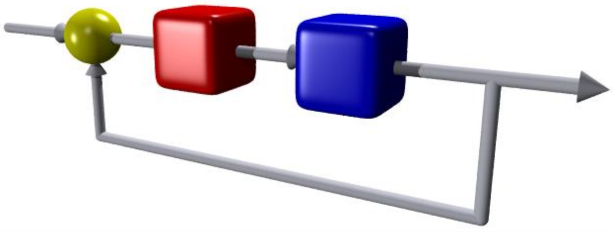

Robust control

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Robust control

Systems
- Linear
- Non-linear

Methods
- Frequency
- State-space
- Polynomial

Control
- Adaptive
- Robust
- Stochastic

Uncertainty
- Non-parametric (non-structured)
- Parametric (structured)

What we call parameters?

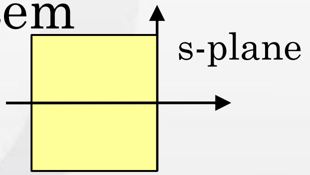
- Time constants
- Time delays
- Gain of a particular element
- Feedback gains
- Coefficients of ODE
- .....
- Stability domain represent domain in the parameter plane where stability is assured
- If the system has μ parameters that can change then μ - dimension stability domain must be investigated

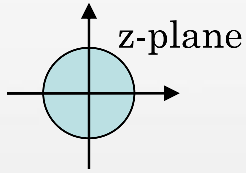
Stability of polynomials

A polynomial *p(•)* is stable if all its roots lie in some given region of the complex plane !

Stability region depends on the nature of the system

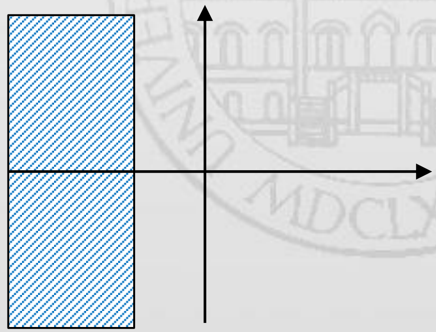
- LHP iff system is continous, *p(s)*
- Unit disk iff system is discrete, *p(z)*

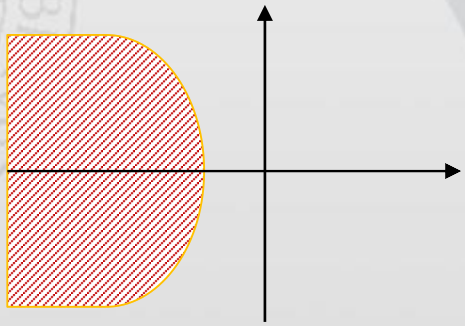


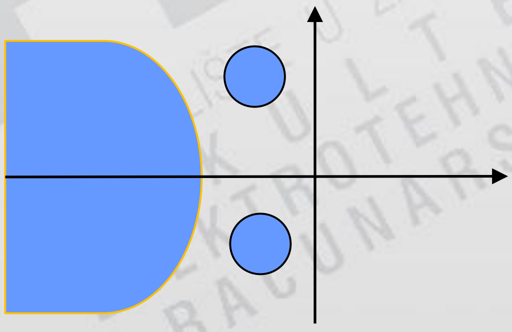


More sophisticated stability regions are required for specific performance specs:

- ➢ Shifted LHP – for desired speed or dominant behaviour or bandwidth
- ➢ Parabola or sector in LHP – for desired damping ζ
- ➢ Some other sector(s) in LHP







Conventional stability analysis methods – changing of parameter(s)

- Root-locus method – by changing one parameter of the closed-loop characteristic equation (Evans, 1948; Teodorčik, 1948)
- Stability analysis in parameter plane (coefficients of the characteristic equation) –
  - D - decomposition (Neimark, 1948), RL by D. Mitrović, 1959
- **Disadvantage:** possibility to analyse stability by changing one parameter (Root-locus by Evans) or two parameters (D - decomposition or Mitrović's Root-locus method)

Modern robust stability methods

- Robust stability by use of the small gain theorem (Doyle et al., 1992)
- Robust stability by polynomial (closed-loop characteristic equation) approach of Kharitonov (Kharitonov, 1979)

By small gain theorem, the stability analysis in case of change of one parameter is possible !

By use of Kharitonov theorems, the stability analysis in case of change of many parameters is possible !

Standard control design problem

Given a linear mathematical model of the system to be controlled (in transfer function or state-space format) and a set of specs. on performance, design a controller such that the closed-loop system is stable and all performance specs. are satisfied.

Implicit in this problem statement is:

1. The given mathematical model is an accurate description of the real system
2. This mathematical model is not subject to perturbations over time

Both of these assumptions are in general false!

Not only is there uncertainty in our knowledge of the system dynamics, but the system will often be subject to perturbations that change dynamics over time!

These perturbations can be due to several factors, including:

- Change in parameter values due to aging or stress or else
- Changes in the operating conditions or working environment
- Failure in a system component (sensor, actuator, ...)
- Coefficient values that depend explicitly on time

The point being made in each of these examples is that response of the real system is governed by mathematical relationships that are different from the linear mathematical model that is usually used to design the control system.

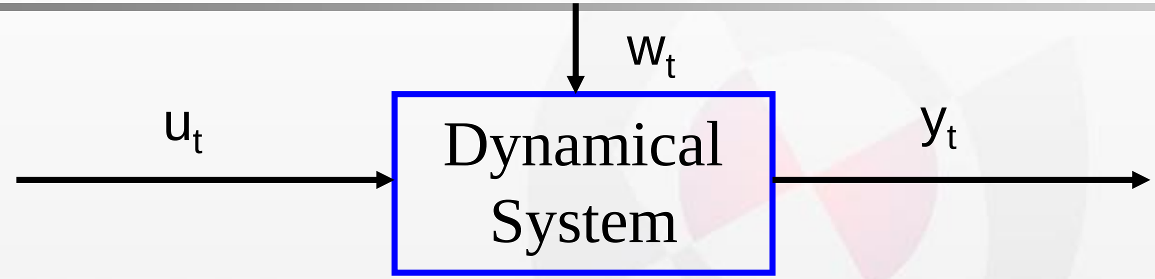
Whatever the reasons for these differences (lack of knowledge about the system dynamics, perturbations to the dynamics, nonlinearities in the system, intentionally neglected dynamics, etc.) the important point to realize is that the control system must stabilize and produce adequate performance from the real system, not just the model used during the design.

This is the task of robust control, namely to design a controller that will satisfy requirements for a family of system models, not just the nominal model!

Sources of model uncertainites

- At HF we know little about the process
  - ✓ Control and model identification concentrate on MF & LF
- Deliberate simplification of the model
  - ✓ It is easier to design controller for simple than for complicated processes
- Uncertainty in the controller is often neglected
  - ✓ Deliberate reduction of controller order for simpler implementation
  - ✓ Implementation issues
    - ➢ Finite floating point precision computers
    - ➢ Error/limit checking in the controller implementation

Information



Information= prior + posterior = $I_0 + I_1$

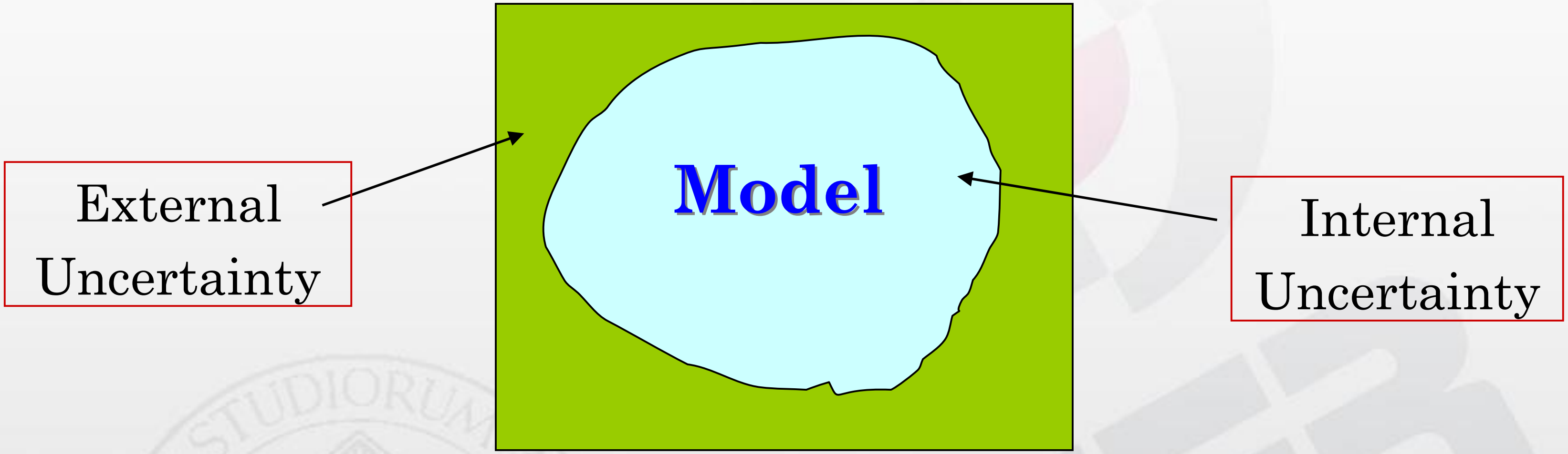
I_0 = prior knowledge about the system

I_1 = posterior knowledge about the system

$=\{u_0,u_1,\ldots u_t,\ y_0,y_1,\ldots,y_t\}$ (Observations)

The posterior information can be used to reduce the uncertainties of the system.

Uncertainties



- External uncertainty:
  - Disturbance
  - Noise
- Internal uncertainty:
  - Parameter uncertainty
  - Signal uncertainty
  - Functional uncertainty

Robust controller

Definition (**Robust controller**):

Given a family of system models, consisting of either

1. A finite number of specified models,
2. A nominal model and a description of uncertainty in the model, or
3. A structural model for the system and a description of the uncertainty in the parameter values in the model,

Then a controller is robust if and only if it internally stabilizes each of the models in the family.

Dynamic uncertainty vs. Parameter uncertainty

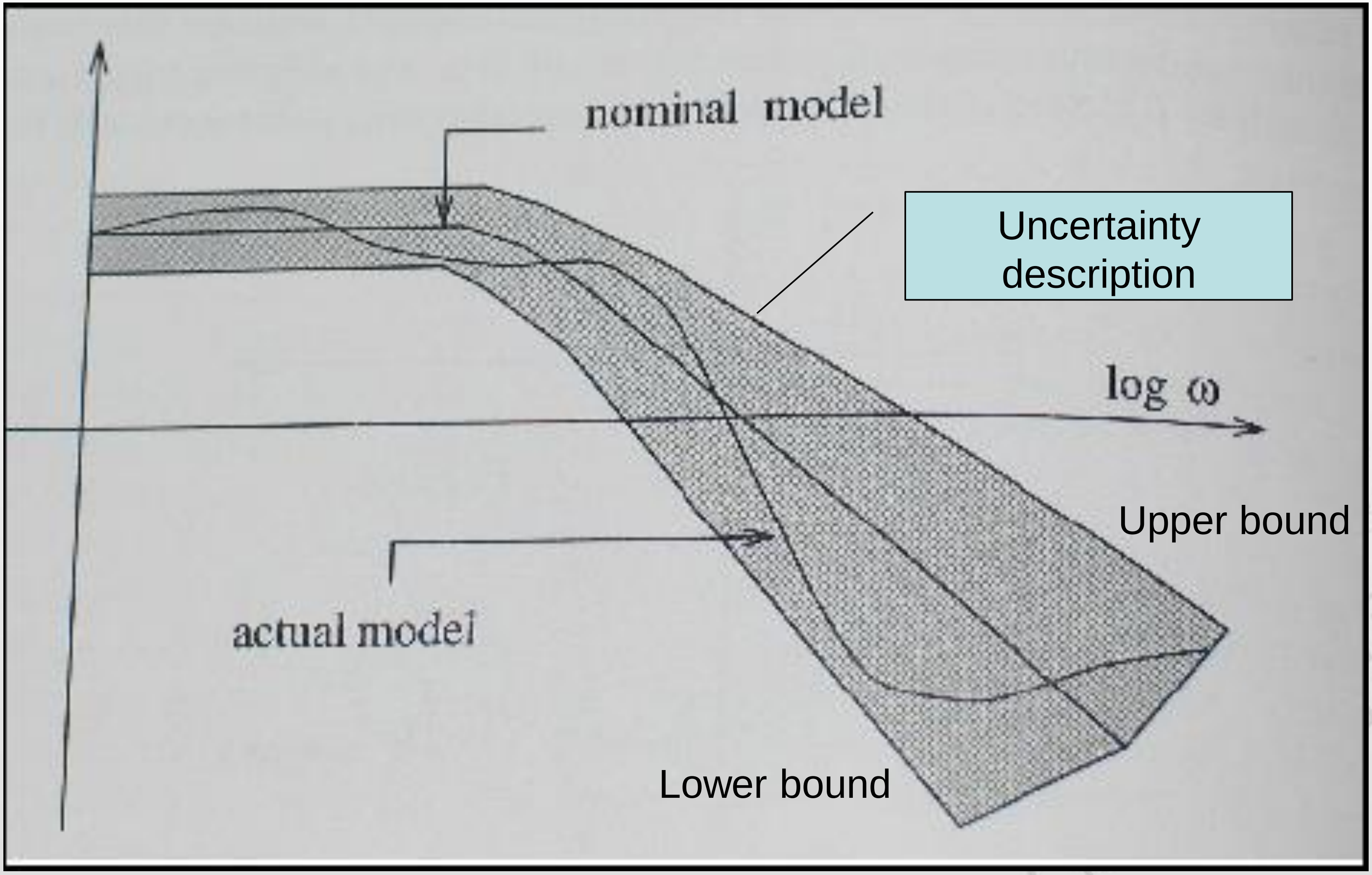
- ✓ Mathematical model of the plant is always obtained under some assumptions and simplifications.
- ✓ It should be recalled that design methods in control engineering rely heavily on the mathematical model of the plant.
- ✓ This nominal model plant model G_n and the uncertainty description Δ_G *associated with the nominal plant model somehow bound the true physical system G, which lies in the set of all plants captured by the pair* (G_n, Δ_G)

Mathematical model uncertainty can be classified to:

- Structured uncertainty – assumes that the uncertainty can be modeled and that parameter constraints are known i.e. Min and Max values for each changing parameter are known!! For instance we can have the transfer function of the process but uncertain positions of poles, zeros, gain etc.

- Unstructured uncertainty – assumes much less knowledge about the process. For instance, we know only that frequency characteristic is upper and lower bounded.

Unstructured uncertainty description

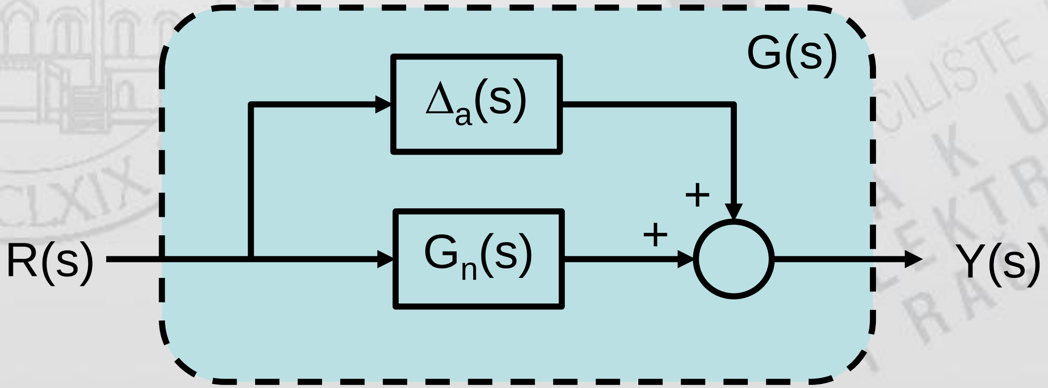


Additive uncertainty or additive perturbation or absolute model error description

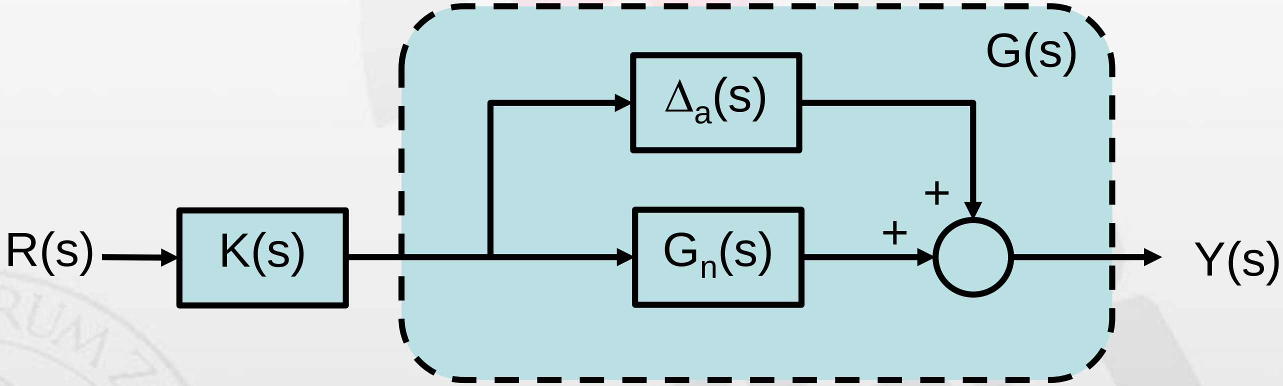
- Assuming that the „true process” is linear and time invariant (LTI) and represented with the model *G(s)*, then the additive uncertainty description is:

$$\Delta_a(s) = G(s) - G_n(s)$$

- This is the simplest aditive description of the difference between the „true process” and its mathematical nominal model $G_n(s)$



- If a series compensator is added we get:



- Open loop transfer function is: $G_o(s) = K(s)G(s)$
- Disadvantage of the additive description of the model error is in that the error in $G_n(s)$ do not represent the error $G_o(s)$, namely

$$\left[G_n(s)+\Delta_a(s)\right]K(s) \neq G_n(s)K(s)+\Delta_a(s)$$

so, it is difficult to determine the effect of $\Delta_a(s)$ to $G_n(s)K(s)$

Multiplicative uncertainty or multiplicative perturbation or relative model error description

- Model error can be described by relative or multiplicative description as:

$$G(s) = \left[1 + \Delta_m(s)\right] G_n(s)$$

- Unstructured uncertainty, described by relative model error in respect to nominal model is given by:

$$\Delta_m(s) = \left[\frac{G(s)}{G_n(s)} - 1\right]$$

- Also, the relative error can be decribed in such a way that the model error is in respect to „real process” i.e.

$$\Delta_M(s) = \left[1 - \frac{G_n(s)}{G(s)}\right]$$

Multiplicative uncertainty or multiplicative perturbation or relative model error description

Relative error in relation to nominal model can be described also by:

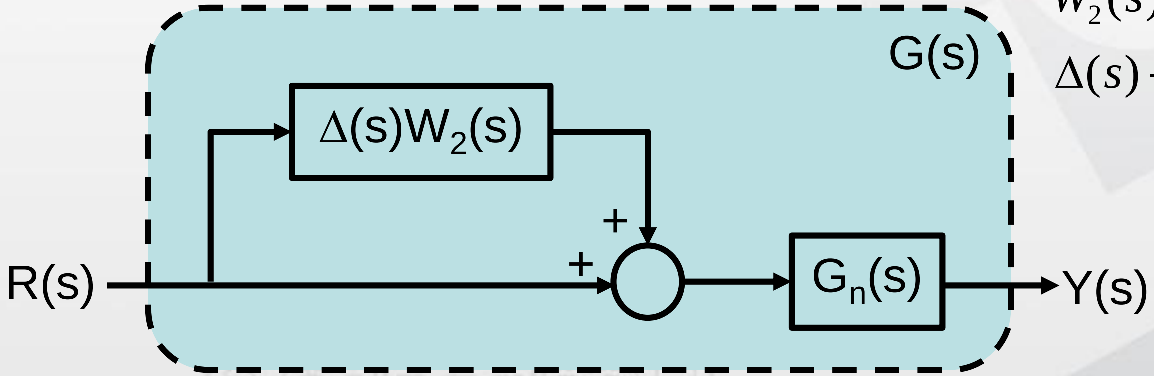
$$\Delta_m(s) = \Delta(s)W_2(s)$$

where:

$W_2(s)$ – uncertainty weight (stable tr.f.)

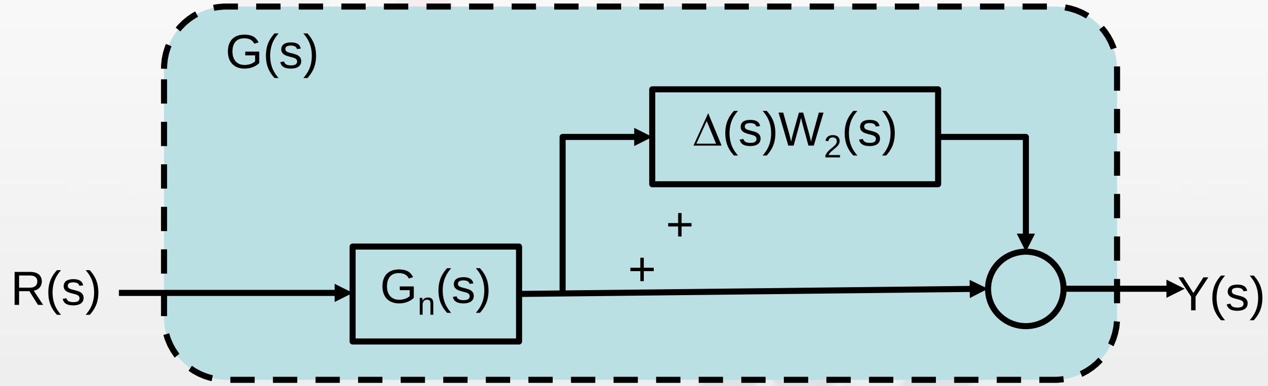
$\Delta(s)$ – stable transfer function with $\|\Delta\|_\infty < 1$

Multiplicative uncertainty at the input

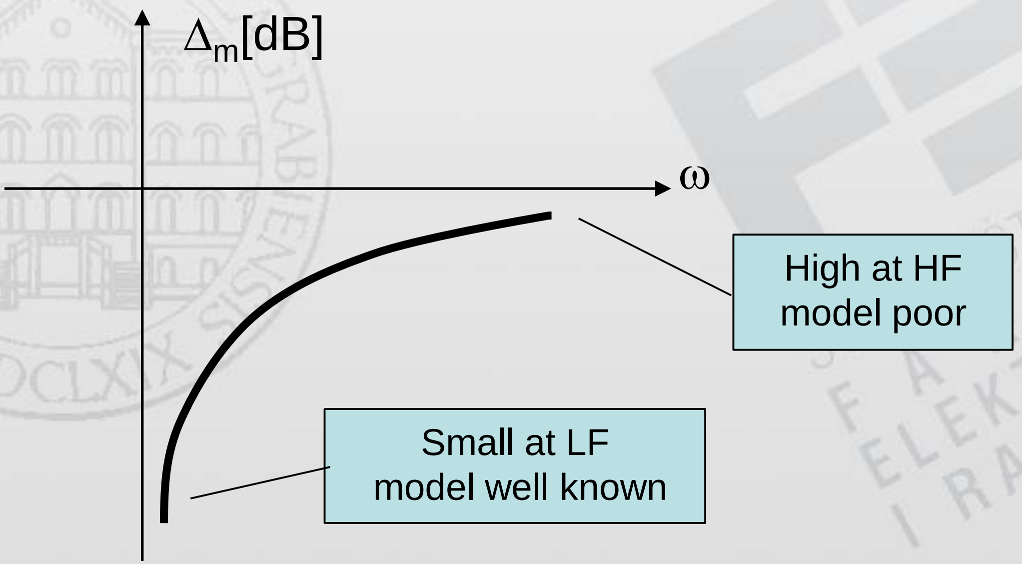


- Multiplicative perturbation do not have the disadvantage as additive one, because the multiplicative perturbation on $G_n(s)$, is also the multiplicative perturbation on $G_n(s)K(s)$
- Moreover, multiplicative model error satisfy intuitive features that it is
  - small at LF (where usually the nominal model is well known and
  - high at HF where nominal model is not able to appropriately describe behaviour of the process

Multiplicative uncertainty at the output



Typical form of the multiplicative model error



Both uncertainty descriptions:

- additive $\Delta_a(s) = G(s) - G_n(s)$

OR

- multiplicative $\Delta_m(s) = \left[\dfrac{G(s)}{G_n(s)} - 1 \right]$

use the nominal model $G_n(s)$ and uncertainty weight W or W_2

- So, if we know the nominal process model $G_n(s)$ and uncertainty weight W or W_2 it is customary to assume that the pair $\{G_n(s), W(s)\}$ or $\{G_n(s), W_2(s)\}$ should be used as the process model for robust stability analysis and robust controller design

- As is often the case, mathematical model of infinite dimension $G(s)$ is aproximated with finite dimensional model $G_n(s)$, and their difference is estimated in order to find uncertainty weight $W(s)$ or $W_2(s)$.

Dynamic (unstructured) uncertainty description

- Assume that the „true plant" is linear and time invariant (LTI), the dynamic modeling uncertainty can be expressed in the Laplace domain as:

$$\boxed{\Delta_a(s) = G(s) - G_n(s)}$$

- One of many possible uncertainty description would be:
  - The number of RHP poles of the „true plant" and of the nominal plant is the same
  - The uncertainty weight W(s) is given :

$$\boxed{|W(j\omega)| > |\Delta_a(j\omega)| = |G(j\omega) - G_n(j\omega)| \quad ; \forall \omega}$$

Plant model used in robust controller design

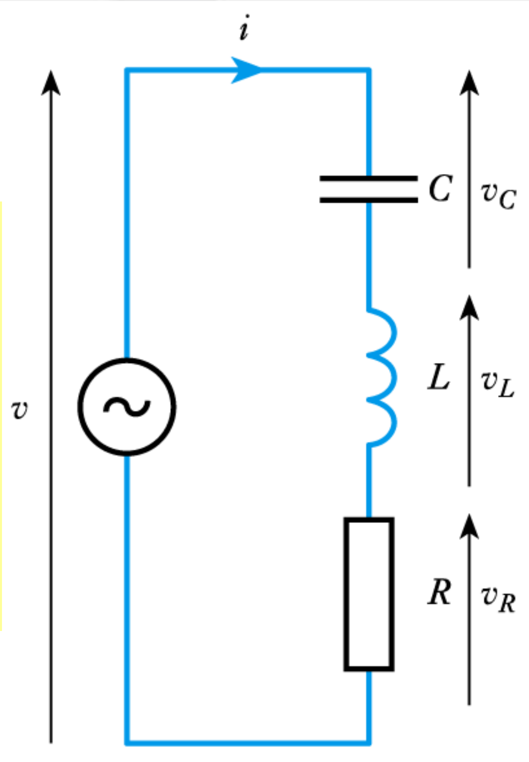
- With the knowledge of the nominal plant model $G_n(s)$ and of the uncertainty weight $W(s)$, it is common to assume that the pair $\{G_n(s), W(s)\}$ should represent the plant model for use in robust controller design.
- NOTE: an infinite dimensional plant model $G(s)$ is aproximated by a finite dimensional model $G_n(s)$ and the difference is estimated to determine W(s).
- Physical system parameters determine coefficients of the nominal plant model $G_n(s)$.
- Uncertain parameters lead to a special type of plant models where the structure of $G(s)$ is fixed, but coefficients are uncertain.

Parametric uncertainty transformed to dynamic uncertainty

That the parametric uncertainty can be transformed to dynamic uncertainty will be shown by the following examples.

- Example 1 (series RLC circuit)

v(t) = *u(t)* – the input (source) voltage
v_c (t) = *y(t)* – output (voltage on capacitor)



The transfer function of this circuit ("true plant")is then:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Nominal plant model: $G_n(s) = \frac{Y(s)}{U(s)} = \frac{1}{L_n C_n s^2 + R_n C_n s + 1}$

R_n, L_n *and* C_n are nominal values of *R, L* and *C* respectively

Example 1 (continued)

- Uncertainities in these parameters appear as uncertainities in the coefficients of the transfer function of the „true plant" G(s).
- We can show that parametric uncertainities can be transformed to dynamic uncertainities if we assume that only the resistance is uncertain.
- With this assumption $\omega_0 = 1/\sqrt{LC}$ remain constant while $2\zeta\omega_0 = R/L$ varies. With this uncertainity in damping the transfer function becomes.

Example (continued)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\omega_0{}^2}{s^2+2\zeta\omega_0 s+\omega_0{}^2}$$

with nominal numerical values

$\omega_n=1\left[s^{-1}\right]$ and $\zeta_n=0.2$ the nominal plant model is:

$$G_n(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+2\zeta_n s+1}$$

and the true plant model assuming uncertainty of the damping factor $\zeta\in[0.1,0.2]$ is:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+2\zeta s+1}\quad;\ \zeta\in[0.1,0.2]$$

For this particular case the uncertainty weight W(s) can be determined from the plot of

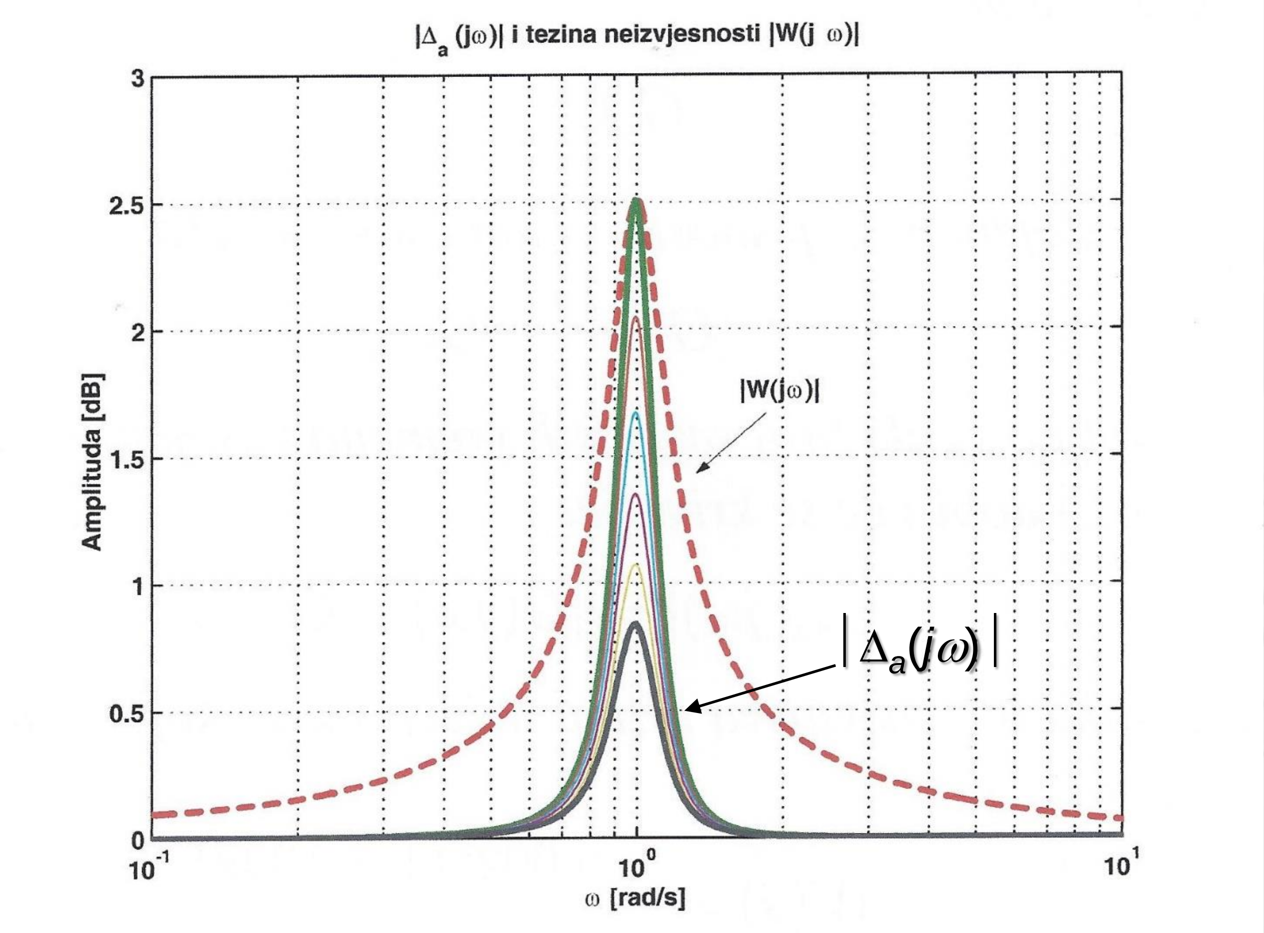
$$\left|\Delta_a(j\omega)\right|=\left|G(j\omega)-G_n(j\omega)\right|$$

for a sufficiently large number of the damping factor in $\zeta\in[0.1, 0.2]$

The weight W(s) should be such that

$$\boxed{|W(j\omega)| > |\Delta_a(j\omega)| = |G(j\omega) - G_n(j\omega)| \quad ; \forall \omega} \quad \textbf{(1)}$$

- Dynamic modeling uncertainty $|\Delta_a(j\omega)|$ for 15 various damping factors between $\zeta \in [0.1, 0.2]$ in the frequency range $\omega \in [10^{-1}, 10^{1}]$ is given in figure.



Example 1 (continued)

- One of many possible weights satisfying (1) is:

$$W(\mathrm{s}) = \frac{0.015(1+80s)}{s^2+0.45s+1}$$

- This uncertainty weight is feasible because it is
  - ✓ of final order,
  - ✓ causal, and
  - ✓ expressed with rational function of complex variable s

Example 2 (Uncertain time delay)

- It is often the case that the time delay of a plant is ignored or aproximated with Padé series (first order approximation)
- Suppose that true plant can be represented with:

$$G(s) = \frac{e^{-\tau s}}{s+1} \quad ; \quad \tau \in [0,\ 0.2]$$

- If the time delay is ignored the nominal plant model becomes:

$$G_n(s) = \frac{1}{s+1}$$

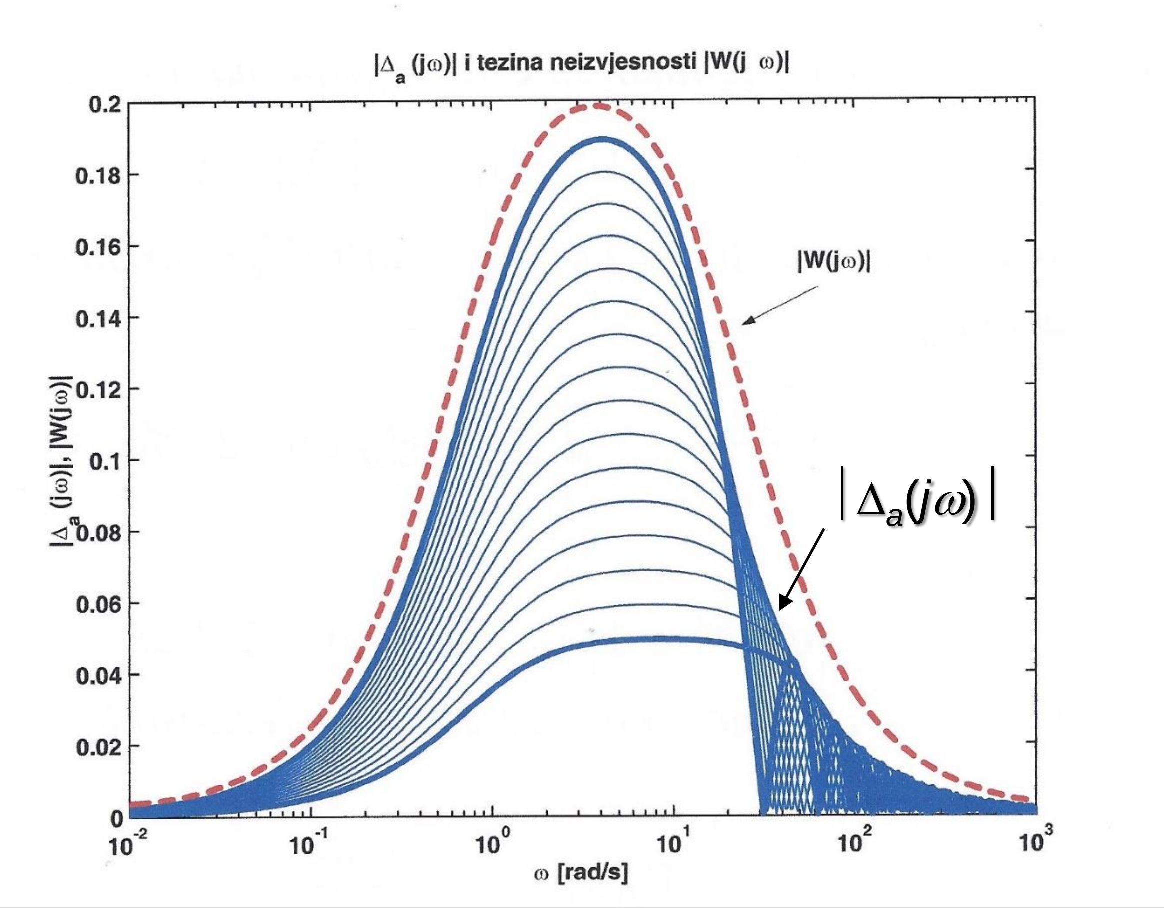
Example 2 (continued)

- Ploting the dynamic modeling uncertainty
  - $|\Delta_a(j\omega)| = |G(j\omega) - G_n(j\omega)|$ for a sufficiently large numbers of possible time delays τ in (0, 0.2] in the frequency range $\omega \in [10^{-2}, 10^{2}]$ we get:

Implementable W(s) could be for instance:

$$W(s) = \frac{0.0025(1+100s)}{(1+0.06s)(1+1.2s)}$$

- finite order
- causal
- rational function of s



Example 3

- If nominal mathematical model of the process is $G_n(s) = 1/s^2$, and we know that the „true process" contains delay that change between $0.03 < \tau < 0.15$ and can be decribed by $G(s) = e^{-\tau s}/s^2$
- This delay can be treated as multiplicative perturbation of the nominal mathematical model, such that $G(s)$ is embedded in the family:

$$\left\{\left(1+\Delta W_2\right)G_n \quad ; \quad \|\Delta\|_\infty \leq 1\right\}$$

- To do this, it is necessary to determine W_2 such that normalized perturbation meet the condition:

Example 3 (contd.)

$$\left|\frac{G(j\omega)}{G_n(j\omega)}-1\right|\leq\left|W_2(j\omega)\right| \quad ; \ \forall\,\omega,\tau$$

or

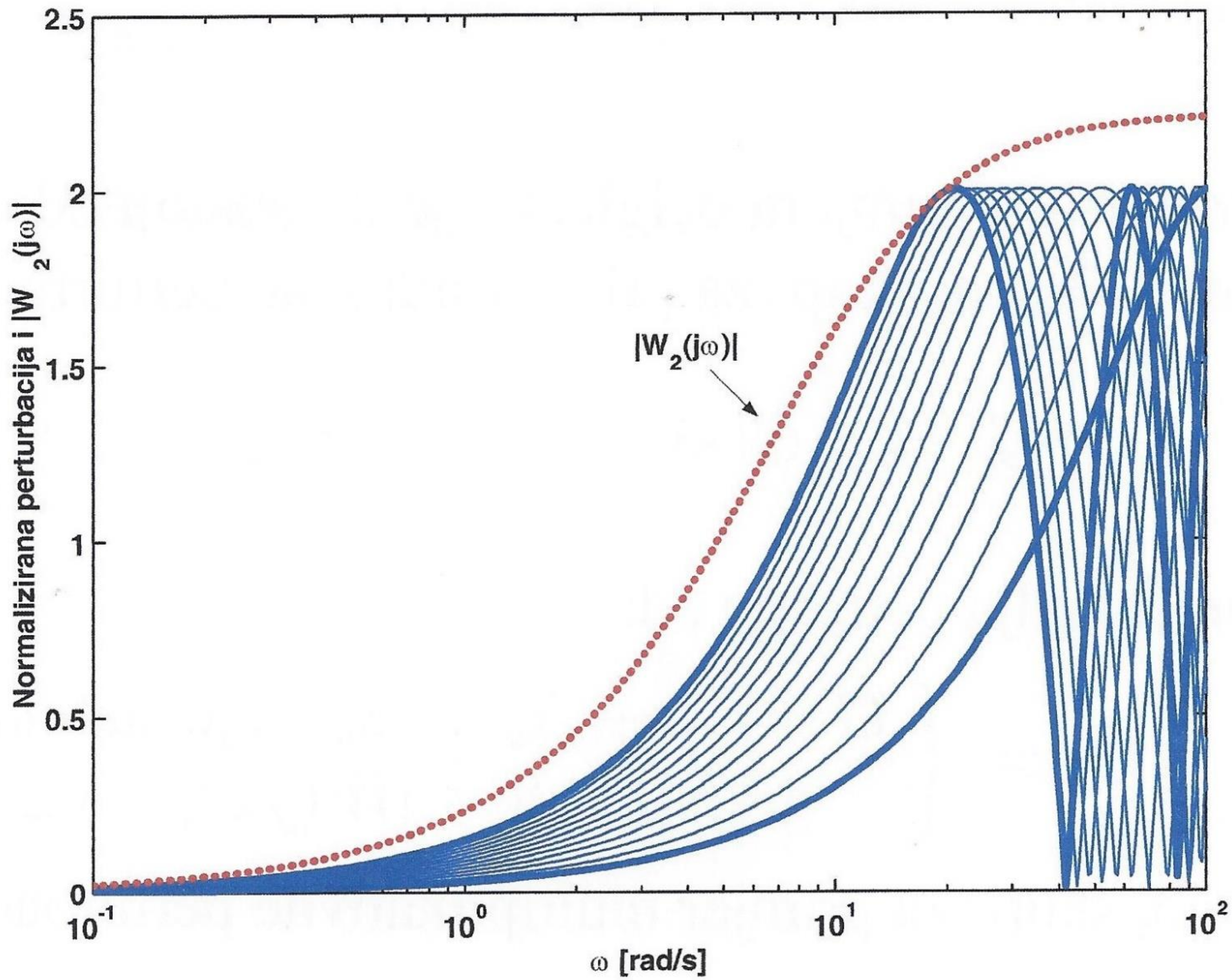
$$\left|e^{-j\omega\tau}-1\right|\leq\left|W_2(j\omega)\right| \quad ; \ \forall\,\omega,\tau$$

- One of infinitely many possible uncertainty weights satisfying above condition is:

$$W_2(s)=\frac{0.21s}{0.095s+0.9}$$

- Figure shows normed perturbation and uncertainty weight

Normed perturbation $\left|\frac{G}{G_n}-1\right|$ and uncertainty weight $\left|W_2(j\omega)\right|$



Discussion

- From example 2 (the case with additive perturbation) it is obvious that there is conservatism here:
  - Namely, the smaller set

$$\mathcal{G}_\tau := \left\{ G(s) = \frac{e^{-\tau s}}{s+1} \quad ; \tau \in [0\ ,\ 0.2] \right\}$$

  - Is a subset of a larger set:

$$\mathcal{G}_\Delta := \left\{ \begin{matrix} G(s) = G_n(s) + \Delta_a(s) \ \ ; \Delta_a(s) \text{ is stable (i.e. LHP)} \\ and \quad |\Delta_a(j\omega)| < |W(j\omega)| \quad ; \forall\, \omega \end{matrix} \right\}$$

  - Or, in example 3 (the case with multiplicative perturbation)

- The smaller set:

$$\mathcal{G}_\tau := \left\{ G(s) = \frac{e^{-\tau s}}{s^2} \quad ; \ \tau \in [0.03 \ , \ 0.15] \right\}$$

- Is a subset of a larger set:

$$\mathcal{G}_\Delta := \left\{ \begin{array}{c} G(s) = [1 + \Delta_m(s)] G_n(s) \ ; \ \Delta_m(s) \text{ is stable (i.e. LHP)} \\ and \quad \left| \dfrac{G(j\omega)}{G_n(j\omega)} - 1 \right| \le |W_2(j\omega)| \quad ; \ \forall \omega \end{array} \right\}$$

Designing controller for $\mathcal{G}_\Delta$

- If controller achieves design objectives $\forall G \in \mathcal{G}_\Delta$, then this controller will be appropriate also for all plants in the set $\mathcal{G}_\tau$
- Since the set $\mathcal{G}_\Delta$ is larger then the actual set of interest $\mathcal{G}_\tau$, design objectives might be more difficult to satisfy in this new setting, where parameter uncertainty is transformed to dynamic uncertainty

We can conclude

- If the regulator achives desired performance specs. for all G belonging to the larger set $\mathcal{G}_\Delta$ then this regulator will be also appropriate for all models in the smaller set $\mathcal{G}_\tau$
- Set $\mathcal{G}_\Delta$ *is larger than the subset* $\mathcal{G}_\tau$ *that we are interested in, it can happen that design goals are more difficult to reach for models in the larger set* $\mathcal{G}_\Delta$*, i.e. in the new ambient in which parameter uncertainty was turned into dynamic uncertainty.*

- *Namely, possible perf. specs. obtained by synthesis based on models in the larger set $\mathcal{G}_{\Delta}$ does not have to be so good as perf. specs. that can be obtained on models from the smaller subset $\mathcal{G}_{\tau}$ i.e. maybe there exist the regulator (even though we can not find it) that gives better perf. specs. on models from the subset $\mathcal{G}_{\tau}$ and which is consequently better from the regulator obtained for models from the larger set $\mathcal{G}_{\Delta}$.*
- *In this respect, regulator obatined for models from the larger set $\mathcal{G}_{\Delta}$ is more conservative for models from the smaller set $\mathcal{G}_{\tau}$.*

What is gained by this transformation from parameter to dynamic uncertainty?

- The answer to this question is related to the difficulty of controller design techniques, because it may be easier to design a controller for the plants in the larger set $\mathcal{G}_{\Delta}$, then for the plants in the subset $\mathcal{G}_{\tau}$

- **NOTE:** In examples 1 & 2 only one parameter changed! What happens if there are more than one parameter which can change? The uncertainty structure becomes more complex and can be quite different. We will also cover this case!

Example 4

- For a family of processes given by:

$$\mathcal{G}_\zeta := \left\{ \frac{1}{s^2 + 2\zeta s + 1} \quad ; \quad \zeta \in [0.1\,,\, 0.2] \right\}$$

- Changing parameter can be expressed as:

$$\zeta = 0.15 + 0.05\Delta \quad ; \text{where } -1 \le \Delta \le 1$$

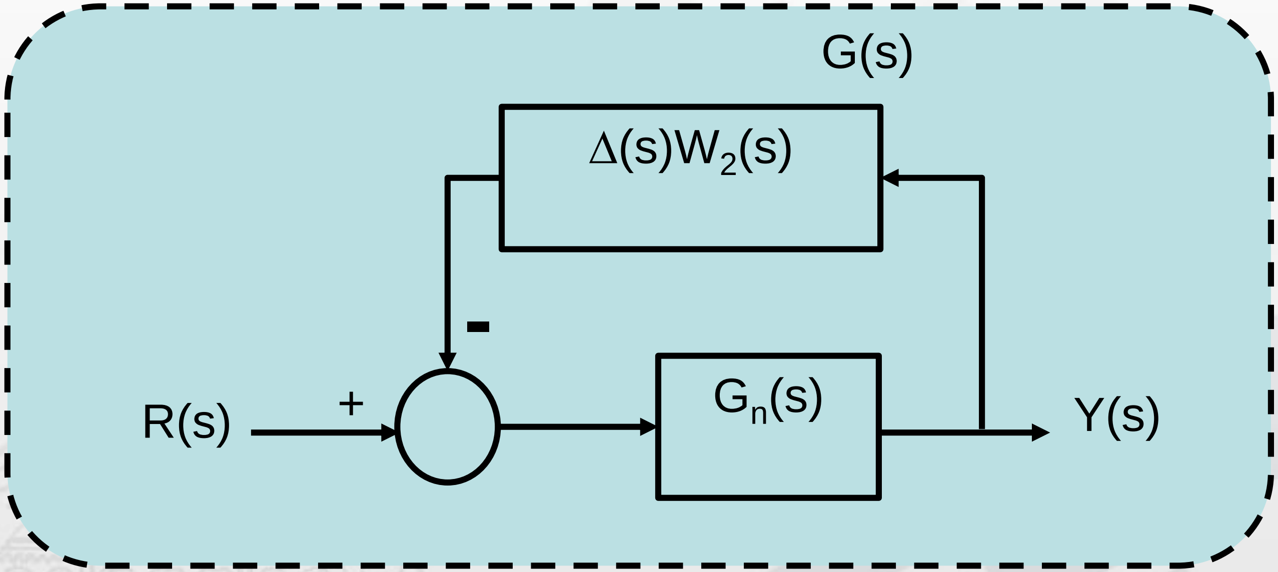
- Then the family of processes can be expressed with:

$$\frac{G_n(s)}{1 + \Delta(s) W_2(s) G_n(s)} \quad ; \quad -1 \le \Delta \le 1$$

$$where$$

$$G_n(s) = \frac{1}{s^2 + 0.3s + 1} \quad , \quad W_2(s) = 0.05s$$

Block scheme (Example 4)



Uncertainty model as a system with the feedback

Most often used uncertainty models

- Multiplicative uncertainty perturbation model

$$G(s) = \left[1 + \Delta_m(s)\right] G_n(s) \quad ; \text{where } \Delta_m = \Delta(s) W_2(s)$$

where: Δ is a stable tr. function

- Additive uncertainty perturbation model

$$\left[G_n(s) + \Delta_a(s)\right] \quad \text{where } \Delta_a(s) = \Delta(\mathrm{s})\,\mathrm{W}(\mathrm{s})$$

- Perturbation model as a system with a feedback

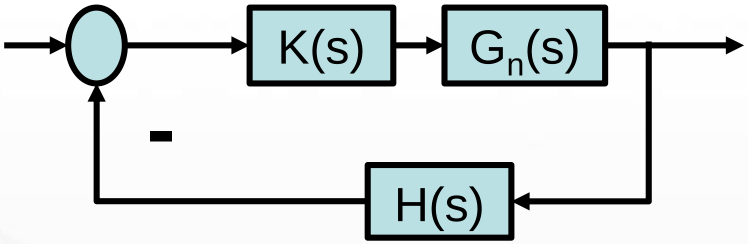
$$\frac{G_n(s)}{1 + \Delta(s) W_2(s) G_n(s)}$$

- Perturbation model $\dfrac{G_n(s)}{1 + \Delta(s) W_2(s)}$

- In each of these perturbation models, a specific assumption on $\Delta(s)$ and $W_2(s)$ should be made. Most often the stability is required for both of them!
- Due to the fact that perturbation models are frequency dependent, sometimes it can be convenient to use different perturbation models for different frequency domains.

- Robust regulator internally stabilizes all models in the family of processes (set) $\mathcal{G}$.
- Nyquist stability and Mikhailov stability methods are very important in this context.
- Small Gain Theorem is very useful for robust stability analysis.
- Small Gain Theorem can be explored for linear and nonlinear systems !!
- The proof can be developed from the Nyquist theorem of stability !

Small Gain Theorem



- Closed loop control system is given in the figure. Assuming stable compensator - *K(s)*, process $G_n(s)$ and feedback element H(s), then the closed loop will remain stable iff

$$\left|G_0(s)\right| = \left|K(s)G_n(s)H(s)\right| < 1$$

- Also,

$$\left|K(s)G_n(s)H(s)\right| \le \left|K(s)G_n(s)\right| \cdot \left|H(s)\right|$$

- So, closed loop system is stable iff

$$\boxed{\left|K(s)G_n(s)\right| \cdot \left|H(s)\right| < 1}$$

- Theorem says that for closed loop stability open loop gain must be small and all components in the loop must be stable ! This guarantees total stability.

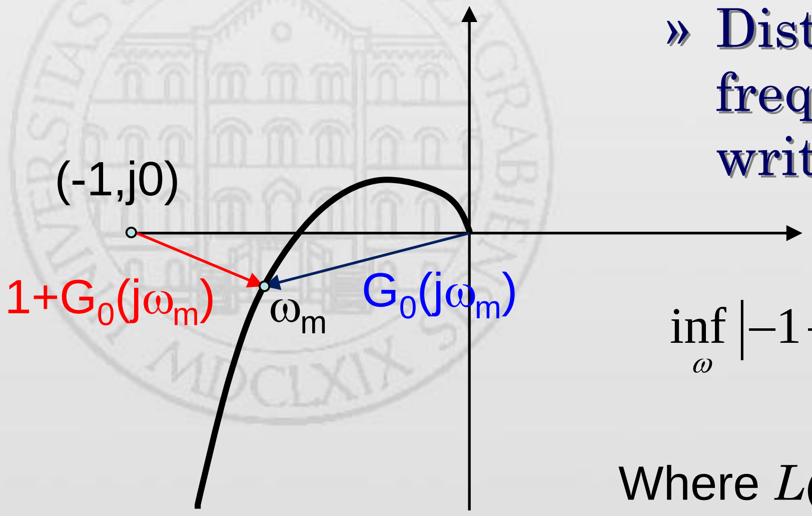
- Assuming H(s) = 1 we get $\left|K(s)G_n(s)\right| < 1$ (*)

- If $G_0(s)$ is stable and if $\left\|G_0(s)\right\|_\infty < 1$ is valid then also $\left(1+G_0\right)^{-1}$ is stable.
- Introduce complementary sensitivity
- $$T(s) \overset{def}{=} 1 - S(s) = 1 - \frac{1}{1+G_0(s)} = \frac{G_0(s)}{1+G_0(s)} = \frac{K(s)G_n(s)}{1+K(s)G_n(s)} \qquad (**)$$
- and introducing the model of multiplicative uncertainty $G(s) = \left[1+\Delta_m(s)\right]G_n(s)$ then the stability condition (*) results with the condition for robust stability $\left\|W_2(s)T(s)\right\|_\infty < 1$ (***)

- Equation (**) gives one very important relation:

$$S(s)+T(s)=1 \quad \text{where } S(s) \overset{def}{=} \frac{1}{1+G_0(s)}$$

- Vector $1+G_0(s)$ is called return difference and is very important in analysis of robustness.



» Distance from point (-1,j0) to frequency characteristic $G_0(j\omega)$ can be written as:

$$\inf_{\omega}\left|-1-L(j\omega)\right| = \inf_{\omega}\left|1+L(j\omega)\right| = \left[\sup_{\omega}\frac{1}{\left|1+L(j\omega)\right|}\right]^{-1}$$

Where $L(s)$ is return ratio or loop gain $L(s) = -G_0(s)$

- Sensitivity is given through return ratio as

$$S(s)=\frac{1}{1-L(s)}=\frac{1}{1+G_0(s)}=1-T(s)$$

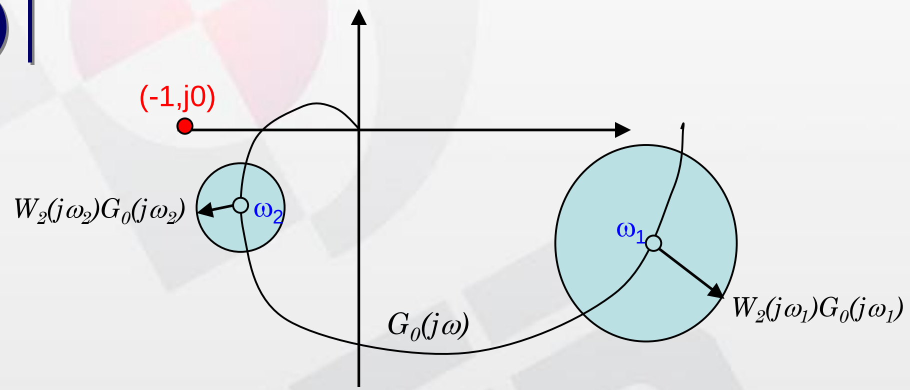
- Robust stability condition $\left\|W_2(s)T(s)\right\|_\infty<1$ can be graphically interpret in the following way.
- Because $\left\|W_2(s)T(s)\right\|_\infty<1 \Leftrightarrow \left|\frac{W_2(j\omega)G_0(j\omega)}{1+G_0(j\omega)}\right|<1 \;,\forall\omega$

or

$$\left\|W_2(s)T(s)\right\|_\infty<1 \Leftrightarrow \left|W_2(j\omega)G_0(j\omega)\right|<\left|1+G_0(j\omega)\right| \;,\forall\omega$$

- This robust stability condition establish that for every ω, the critical point (-1,j0) must be outside the circle!

- The circle has the center at $G_0(j\omega)$ and the radius is $|W_2(j\omega)G_0(j\omega)|$



- So, for robust stability analysis, circles must be drawn with the centre at $G_0(j\omega)$, whose radius will become smaller with higher frequency. If the critical point do not enter in those circles the system is robustly stable.

Robust stability analysis

- From Small Gain Theorem it follows that for robust stability we must have:

$$\left|\Delta_m(s)\right| = \left|\Delta(s)W_2(s)\right| < \frac{1}{\left|T(s)\right|}$$

$$OR$$

$$\left|\Delta_m(j\omega)\right| = \left|\Delta(j\omega)W_2(j\omega)\right| < \frac{1}{\left|T(j\omega)\right|} \quad , \forall \omega$$

- If multiplicative perturbation is bounded and stable then the closed loop control system will be stable iff

$$\left|T(s)\right| < \frac{1}{\left|W_2(s)\right|} \quad \text{OR} \quad \left|W_2(s)T(s)\right| < 1$$

- Importance of the relation $\|W_2(s)T(s)\|_\infty < 1$ can be seen from the block diagram of perturbation uncertainty model (multiplicative uncertainty at output – see Figure 1) in the closed loop control system with serial compensator K(s) – see Figure 2.

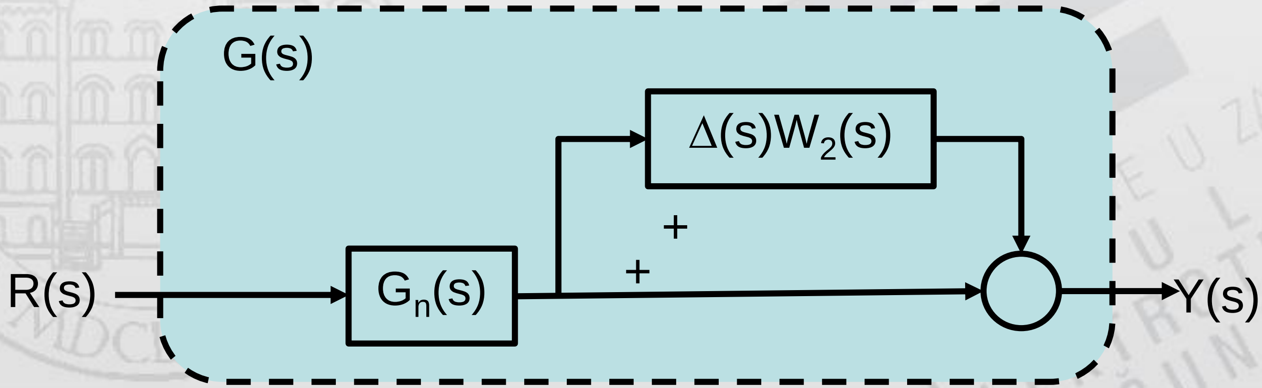


Figure 1. Multiplicative uncertainty at output.

- As we are interested in closed loop control system from the uncertainty standpoint view, we can redraw the multiplicative uncertainty model at output, as seen from input and output terminals

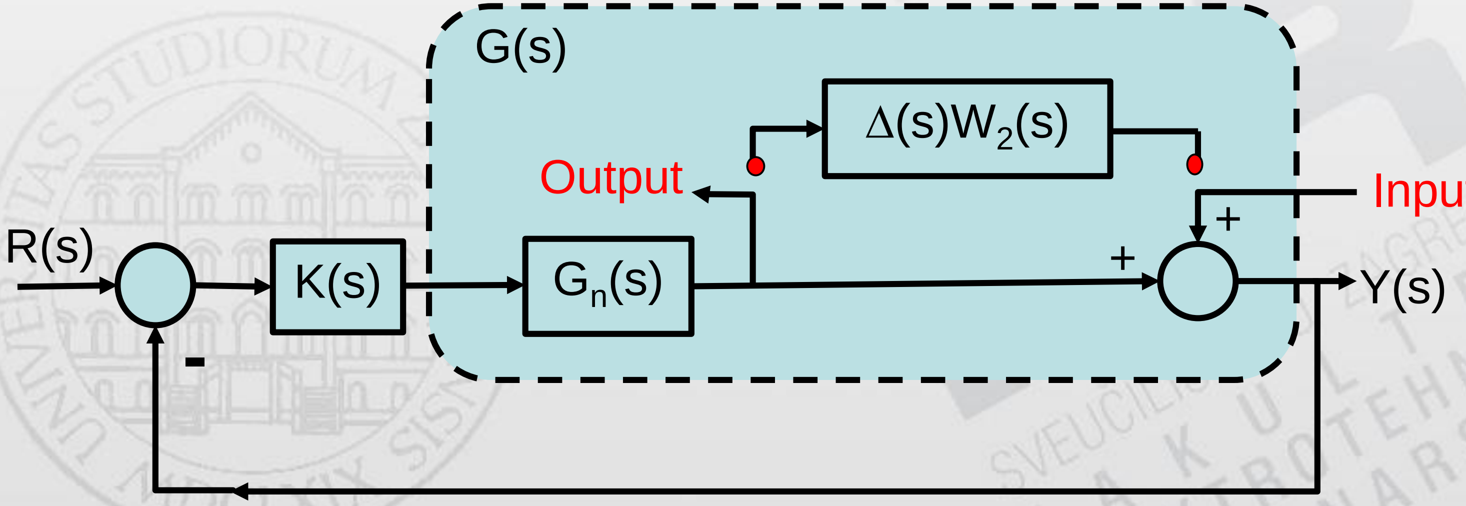
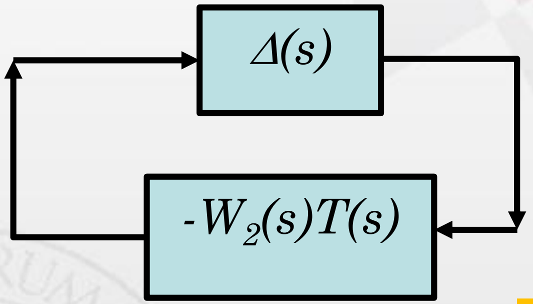


Figure 2. Closed loop control system seen from uncertainty standpoint view.

- Simplified closed loop block diagram from Input/Output terminal (see Figure 2) is given by:



- Maximal open loop gain is: $\left\| -\Delta(s) W_2(s) T(s) \right\|_\infty < 1$
- This gain is valid for all Δ(s) iff condition of the Small Gain Theorem $\left\| W_2(s) T(s) \right\|_\infty < 1$ is valid

Robust stability condition

- Consequently, from the Small Gain Theorem stability condition, it follows that the following must be satisfied for robust stability:

$$\left|\Delta_m(s)\right| = \left|\Delta(s)W_2(s)\right| < \frac{1}{T(s)}$$

$$OR$$

$$\left|\Delta_m(j\omega)\right| = \left|\Delta(j\omega)W_2(j\omega)\right| < \frac{1}{\left|T(j\omega)\right|} \quad , \quad \forall \omega$$

Closed loop system seen from the uncertainty standpoint

- If the multiplicative modeling error is bounded and stable $|\Delta(s)| < W_2(s)$
  - then the closed loop control system will have robust stability iff

$$|T(s)| < \frac{1}{W_2(s)} \quad \text{and respectively } |W_2(s)T(s)| < 1$$

  - If we are interested in smallest uncertainty that could de-stabilize the system, then its value should be determined
  - As multiplicative uncertainty must be smaller than *1/T*, then it must be smaller than minimal amount of *1/T*

- If we want to minimize the right hand side of the robust stability condition

$$|\Delta_m(s)| = |\Delta(s)W_2(s)| < \frac{1}{T(s)}$$

$$OR$$

$$|\Delta_m(j\omega)| = |\Delta(j\omega)W_2(j\omega)| < \frac{1}{|T(j\omega)|} \quad , \ \forall\omega$$

  - then we should maximize $|T(s)|$
- For instance if the system has a resonance (standard 2nd order system with $\zeta < 0.7$) then the maximal amount of $|T(j\omega)|$ for $0 < \omega < \infty$ will be at the resonance frequency ω_m.
- Consequently, the smallest amount of de-stabilizing uncertainty called multiplicative stability margin (MSM) is

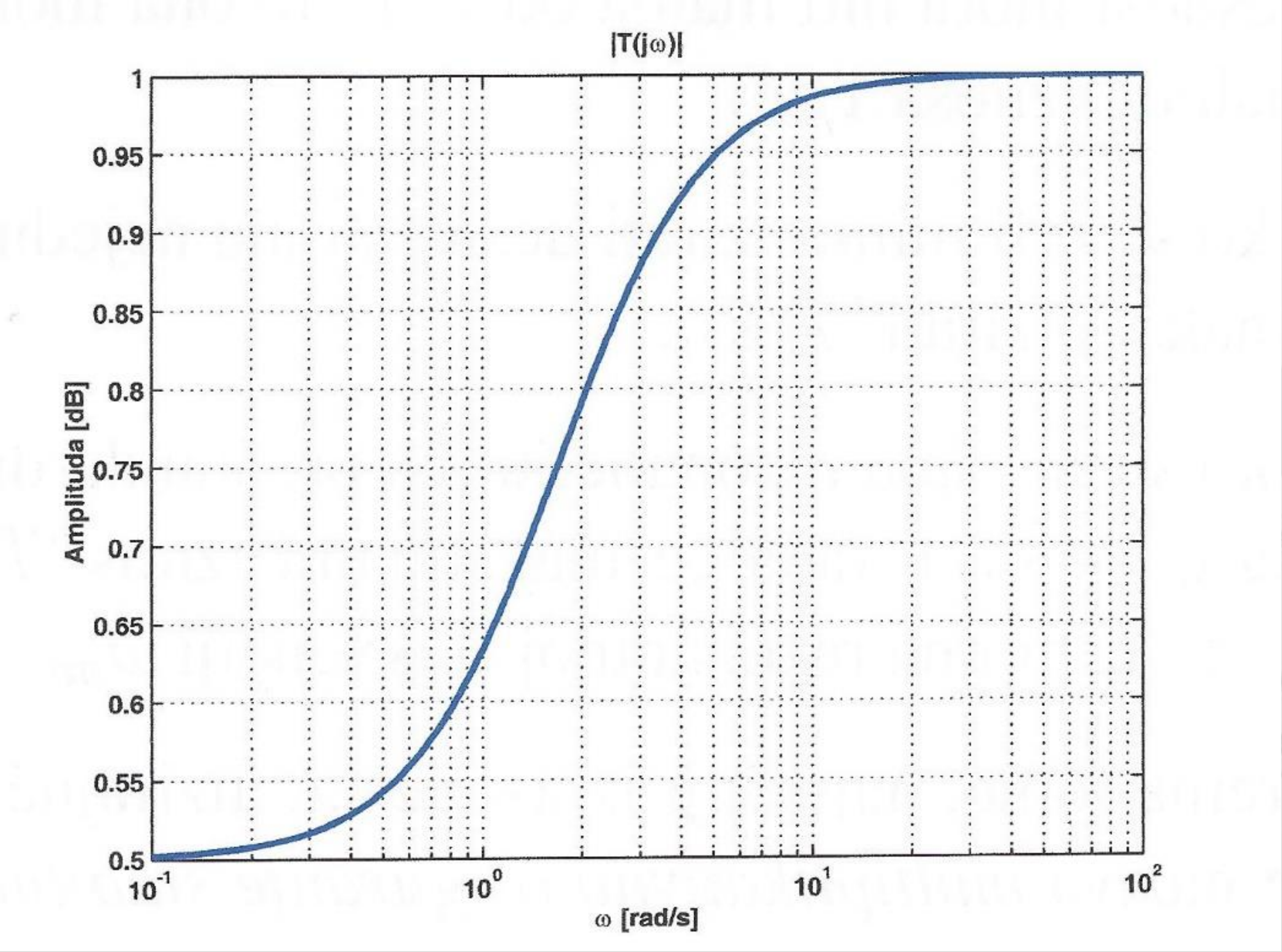
- Multiplicative Stability Margin

$$MSM \overset{def}{=} \frac{1}{M_m} \quad \text{where } M_m = \sup_{\omega} \left| T(j\omega) \right|$$

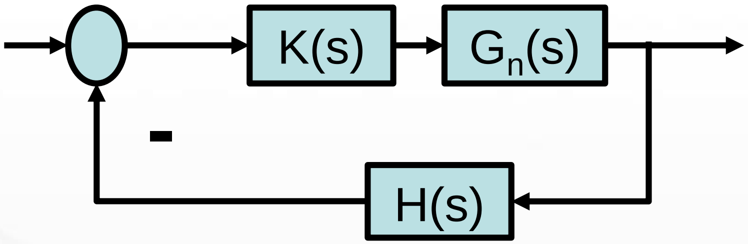
- Symbol *sup* represents the function supremum (or the minimal upper amount) which is its maximal amount, even if the function do not reach this amount.
- For instance $T(s) = \frac{s+1}{s+2}$ do not have a resonance (maximum), but:

$$\sup_{\omega} \frac{|j\omega + 1|}{|j\omega + 2|} = 1$$

- So, the function do not have a maximum, and will never reach 1 (except at ω = ∞) its supremum is 1 (next Fig.)


|T(jω)|
Amplituda [dB]
ω [rad/s]
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
10^-1
10^0
10^1
10^2

Example



- Analyze robust stability of the closed loop with *H(s) = 1*, and

$$G_n(s) = \frac{170000(s+0.1)}{s(s+3)(s^2+10s+10000)}$$

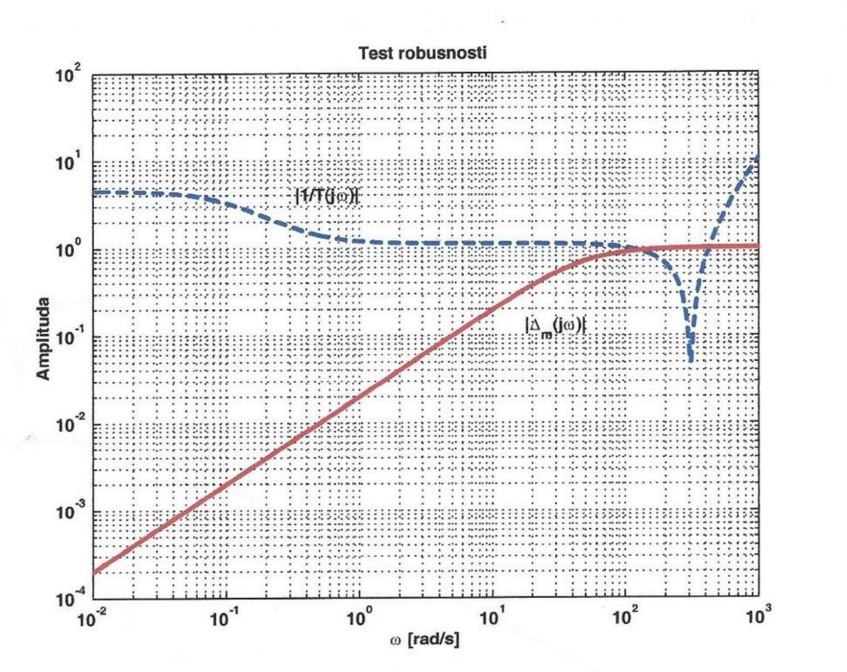
- Series compensator *K(s) = 0.5* stabilizes the closed loop system, but is this robust stability if there is unmodeled pole at 50 [rad/s]?
- If we use multiplicative perturbation:

$$1+\Delta_m(s) = \frac{50}{s+50}$$

and

$$\left|\Delta_m(j\omega)\right| = \left|\frac{-j\omega}{j\omega+50}\right|$$

- Test of robustness consist of drawing amplitude frequency characteristic of multiplicative uncertainty $|\Delta_m(j\omega)|$, and checking if it is on all frequency bellow inverse amplitude frequency characteristic of complementary sensitivity function $|1/T(j\omega)|$

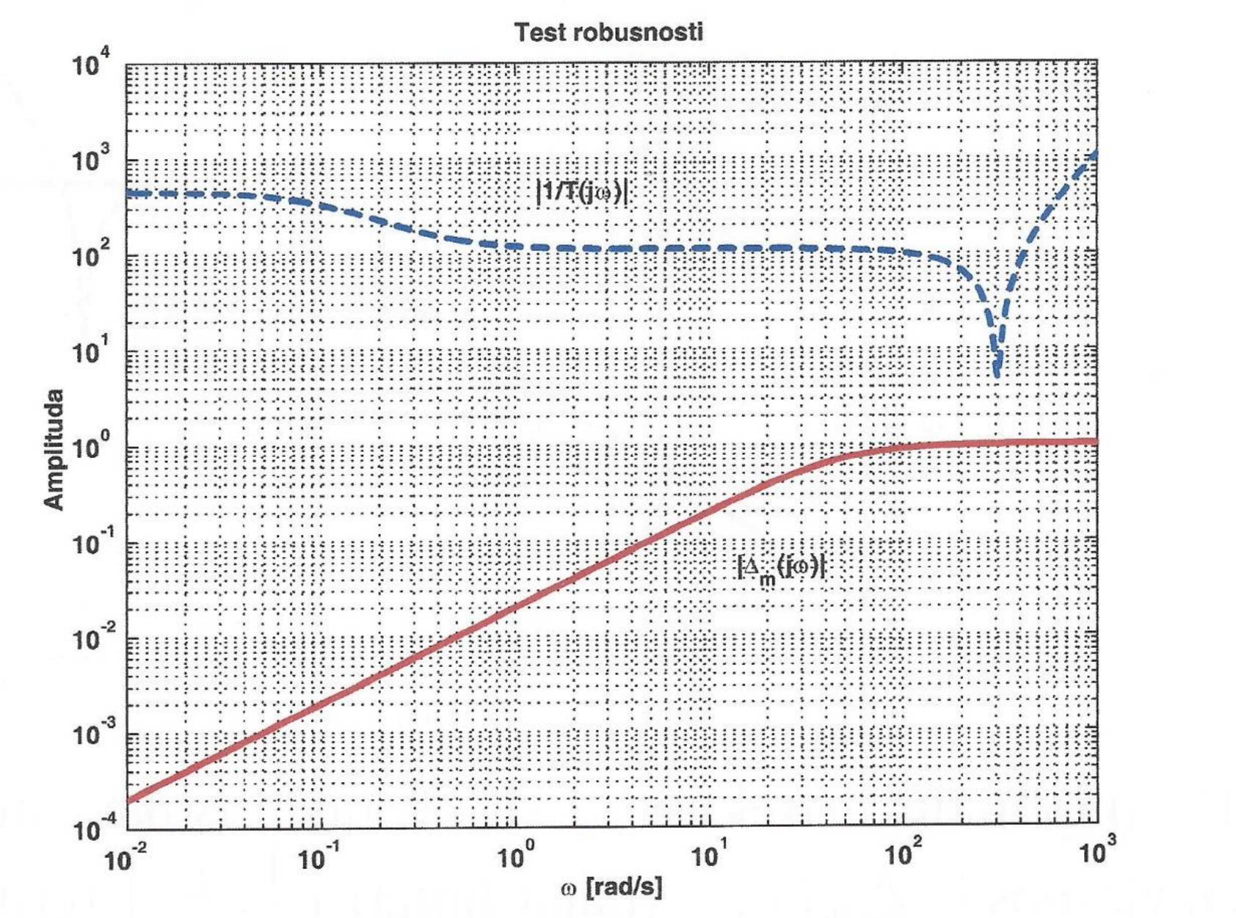


As is seen from the figure $K(s)=0.5$ stabilizes the system but can't assure robust stability !

Namely, robust stability condition is not satisfied

$$\boxed{|\Delta_m(j\omega)| = |\Delta(j\omega)W_2(j\omega)| < \frac{1}{|T(j\omega)|} \quad , \ \forall\omega}$$

- If another compensator is used instead, $K_1(s) = 0.005$ we will get:



Robust stability condition is now valid in the whole frequency range !

- Also,

$$K_2(s) = \frac{0.15(s+25)}{(s+2.5)}$$

will do !!

What if more than one parameter change?

- **NOTE:** In examples so far only one parameter changed! What happens if there are more than one parameter which can change?
- The uncertainty structure becomes more complex and can be quite different. We will also cover this case, because Kharitonov theorem gives us tools for this case!