Politecnico di Torino

Corso di Laurea Magistrale in Data Science And Engineering

Network Dynamics and Learning

Homework I

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Contents

0.1 Individual contributions

Exercise	Student
1	Alejandra Solarte, Anderson Alvarez
2	Mateo Rivera
3	Alejandro Mesa, Juliana Cortes

0.2 Libraries

[1]:
```
#Libraries required for the notebook
from scipy.io import loadmat
import numpy as np
import networkx as nx
import matplotlib.pyplot as plt
import cvxpy as cp
from collections import Counter
import copy

%matplotlib inline
```

1 Excercise 1

1.1 Point a

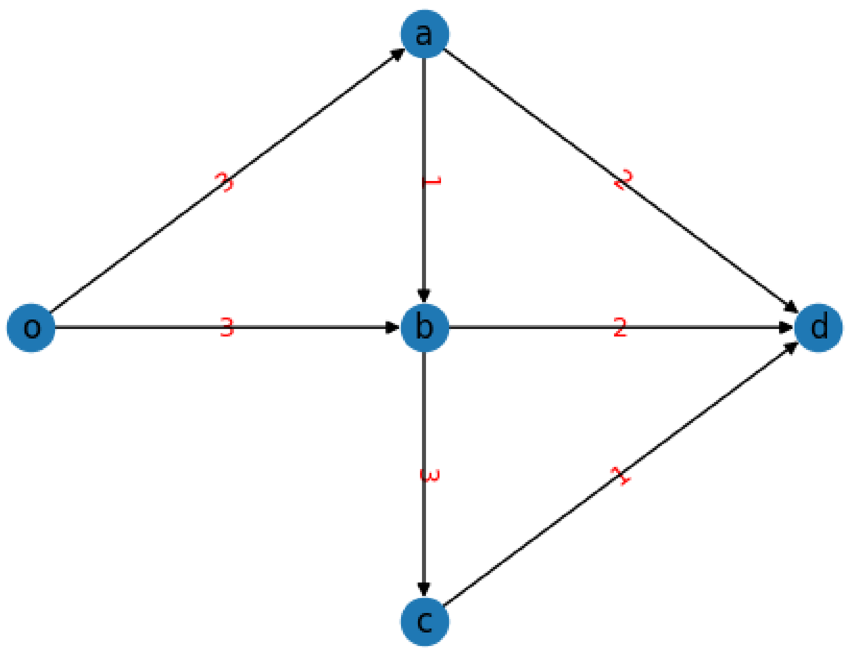
Compute the capacity of all the cuts and find the minimum capacity to be removed for no feasible flow from o to d to exist.

[2]:
```
#Create the graph
def create_graph():
  global G
  G = nx.DiGraph()
  G.add_edge('o', 'a', capacity=3)  # e1
  G.add_edge('o', 'b', capacity=3)  # e3
  G.add_edge('b', 'd', capacity=2)  # e4
  G.add_edge('a', 'd', capacity=2)  # e2
  G.add_edge('a', 'b', capacity=1)  # e7
  G.add_edge('b', 'c', capacity=3)  # e5
  G.add_edge('c', 'd', capacity=1)  # e6

  return G
```

[3]:
```
G= create_graph()
```

[4]:
```
#Define the static positions of the nodes
pos = {"o": (40, 20), "a": (60, 35), "b": (60, 20), "c": (60, 5), "d": (80, 20)}
```

[5]:
```
nx.draw_networkx_edge_labels(G,pos,edge_labels={("o","a"):'3',("o","b"):
↪'3',("a","d"):'2',("b","d"):'2',("b","c"):'3',("c","d"):'1',("a","b"):
↪'1'},font_color='red')
nx.draw(G, pos, with_labels=True)
```



According to the min-cut theorem, when we calculate the min-cut capacity, we obtain the minimum capacity among all cuts between the source and destination nodes, $o - d$.

In the maximum flow problem, a flow is feasible if it satisfies the following constraints:

- It has no negative throughput.
- It respects the capacity limits.
- It maintains mass conservation at each node..

Moreover, by analyzing the min-cuts, we can observe that the minimum capacity belongs to the final cut.

The minimum total capacity that needs to be removed from the network to make d unreachable from o coincides with the min-cut capacity. In this case, it corresponds to the cut between $\{o, a, b, c\}$ and $\{d\}$, with a total capacity of 5.

Cuts - $U = \{o\}, U^C = \{a, b, c, d\}$ -> $C_U = 6$ - $U = \{o, a\}, U^C = \{b, c, d\}$ -> $C_U = 6$ - $U = \{o, a, b\}, U^C = \{c, d\}$ -> $C_U = 7$ - $U = \{o, a, b, c\}, U^C = \{d\}$ -> $C_U = 5$

[6]:
```
#Further-theorem
```

```
nx.algorithms.flow.minimum_cut(G,"o","d") #The minimum capacity to be removed
↪for no feasible flow from o to d to exist is 5.
```

[6]:
```
(5, ({'a', 'b', 'c', 'o'}, {'d'}))
```

[7]:
```
G.edges() #Confirm the existing edges of the graph
```

[7]:
```
OutEdgeView([('o', 'a'), ('o', 'b'), ('a', 'd'), ('a', 'b'), ('b', 'd'), ('b',
'c'), ('c', 'd')])
```

[8]:
```
#Function to calculate the capacity of a given cut

def calculate_cut_capacity(U, U_complement, G):
    capacity = 0
    for u in U: #Iterate over the reachable nodes
        for v in G[u]: #Iterate over the edges of the reachable nodes
            if v in U_complement:#If the edge is shared with the non_reachable
↪section (which in the first case would be a connection with d)
                capacity += G[u][v]['capacity']
    return capacity
```

[9]:
```
# Cuts to do in the graph
cuts = [
    ({'o'}, {'a', 'b', 'c', 'd'}),
    ({'o', 'a'}, {'b', 'c', 'd'}),
    ({'o', 'a', 'b'}, {'c', 'd'}),
    ({'o', 'a', 'b', 'c'}, {'d'})
]
```

[10]:
```
# Calculate the capacities of all the cuts
def get_all_capacities(G):
  capacities=[]
  for U, U_complement in cuts:
      capacity = calculate_cut_capacity(U, U_complement, G)
      capacities.append(capacity)
      print(f"U = {U}, U^C = {U_complement} -> C_U = {capacity}")

  return np.array(capacities)
```

[11]:
```
get_all_capacities(G)
```

```
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 6
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 6
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 7
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 5
```

[11]:
```
array([6, 6, 7, 5])
```

1.2 Point b

You are given $x = 0$ extra units of capacity (x e Z). How should you distribute them in order to maximize the throughput that can be sent from o to d? Plot the maximum throughput from o to d as a function of $x \geq 0$.

Max flow min cut Theorem: the maximal flow that can send from o to d equals the minimal cut capacity among the o-d cuts of the network.

[12]:
```
# Current maxim flow
min_cut= nx.algorithms.flow.maximum_flow(G,"o","d")
# we gey the maximal throughput, plus a dictionary containing the value of the␣
↪flow that goes through each edge.
min_cut
```

[12]:
```
(5,
 {'o': {'a': 3, 'b': 2},
  'a': {'d': 2, 'b': 1},
  'b': {'d': 2, 'c': 1},
  'd': {},
  'c': {'d': 1}})
```

The extra units of capacity should be allocated along the edges of the min-cut to balance capacities between cuts without creating or increasing bottlenecks. This allocation is done incrementally, adding one unit at a time along the min-cut edges. After each allocation, the min-cut is recalculated to check if capacities have changed or if a new min-cut has been created.

[13]:
```
def compute_max_flow_min_cut(G):

    # Compute minimum cut (capacity of the cut)
    _, partition = nx.minimum_cut(G, 'o', 'd')

    # Get the edges in the minimum cut
    reachable, non_reachable = partition #In the first case the reachable␣
↪variable corresponds to o,a,b,c and non_reachable to d.


    cutset={} #Start dictionary to save the edges between the cuts
    esges=[]

    for u in reachable: #Iterate over the reachable nodes
        for v in G[u]: #Iterate over the edges of the reachable nodes
            if v in non_reachable:#If the edge is shared with the non_reachable␣
↪section (which in the first case would be a connection with d)
                #cutset[G[u][v]['capacity']]=(u, v) #Save the edges that connect␣
↪the cuts
                esges.append((u,v))

    return esges, reachable
```

```
    # return cutset
```

[14]:
```
G=create_graph()
borders,reachable = compute_max_flow_min_cut(G) #We get the borders that connect␣
↪U and Uc
```

[15]:
```
borders
```

[15]:
```
[('c', 'd'), ('b', 'd'), ('a', 'd')]
```

[16]:
```
reachable
```

[16]:
```
{'a', 'b', 'c', 'o'}
```

To assign extra capacities effectively, we also need to identify the shared edges between cuts. By increasing the capacity of one of these common edges by one unit, we increase the capacity of two min-cuts simultaneously.

[17]:
```
common_borders={}
common_borders[0]= [("o","b")] #The first cut share the edge e3 with the cut #2
common_borders[1]= [("a","d"), ("o","b")] #The second cut shares the edge e2␣
↪with the cut #3 and #4
common_borders[2]= [("a","d"), ("b","d")] #The third cut shares the edge e2 with␣
↪the cuts #3 and #4 and e4 with the cut#4
common_borders[3]=[("a","d"), ("b","d")] #The forth cut shares the edges e2 and␣
↪e4 with the cut #3
```

[18]:
```
common_borders
```

[18]:
```
{0: [('o', 'b')],
 1: [('a', 'd'), ('o', 'b')],
 2: [('a', 'd'), ('b', 'd')],
 3: [('a', 'd'), ('b', 'd')]}
```

When there is more than one min-cut, we need to ensure that the extra capacity is allocated to the edge shared by the min-cuts or most of them.

[19]:
```
def assign_capacities(x):

  max_flows = {}
  i=0

  while True:
      if i >= x :
          break

      elif i>0:
```

```
        min_cut= nx.algorithms.flow.maximum_flow(G,"o","d")
        #calculate the borders of the min_cut
        borders,reachable = compute_max_flow_min_cut(G)
        print(f"borders{borders}")

        #Get all the mincuts
        capacities= get_all_capacities(G) #get the capacities of the cuts
        mask= capacities==min_cut[0] #Create a mask to know the cuts with the capacity of the min_cut
        positions= np.where(mask)[0] #Position of all the mincuts

        #If just there is a mincut
        if len(positions)==1:
          #Get the minimum value of the borders of the mincut
          #min_capacity_border = min(borders.values())
          min_capacity_border = np.argmin([G[edge[0]][edge[1]]['capacity'] for edge in borders])
          print(f"cut with min capacity {np.array(cuts)[mask]}")

          #Get the border with the minimum value
          u,v =borders[min_capacity_border]
          print(f"u{u},v{v}")

          #Assign the capacity
          G[u][v]['capacity'] += 1
          print(f"The edge between {u} and {v} {G[u][v]['capacity']} has increase 1 unit")

        else:
          #get common borders of the mincuts
          cm = [common_borders[i] for i in positions]
          count=Counter(sum(cm, []))
          duplicados = [{G[elemento[0]][elemento[1]]['capacity']:elemento} for elemento, cuenta in count.items() if cuenta > 1]

          #Assign the extra capacity to the shared border with the minimum capacity
          min_duplicado = min(duplicados, key=lambda d: min(d.keys()))
          u=list(min_duplicado.values())[0][0]
          v=list(min_duplicado.values())[0][1]

          G[u][v]['capacity'] += 1
          print(f"The edge between {u} and {v} {G[u][v]['capacity']} has increase 1 unit")

        max_flows[i],_=nx.algorithms.flow.maximum_flow(G,"o","d")
```

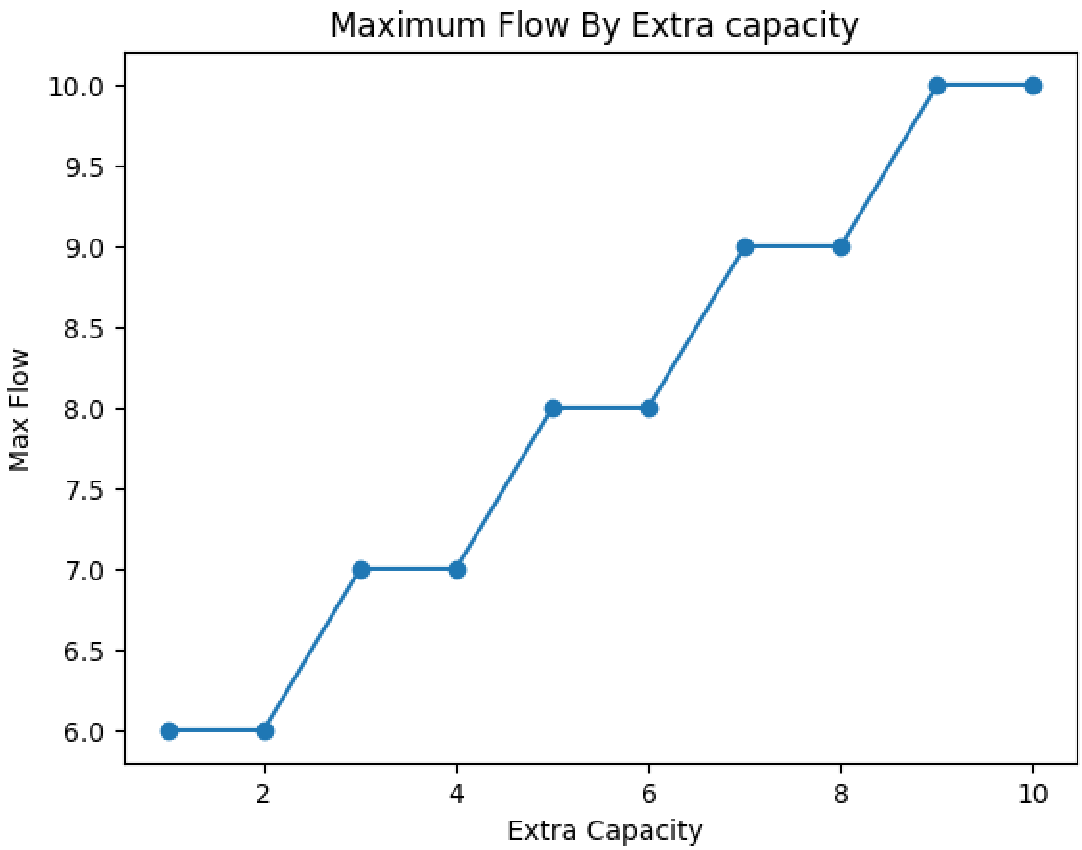
```
            print("*"*50)
        i +=1
        print(i)

    return max_flows
```

[20]:
```
max_flows = assign_capacities(11)

plt.plot(max_flows.keys(), max_flows.values(), marker = 'o')

# Plot formatting
plt.xlabel(' Extra Capacity')
plt.ylabel('Max Flow')
plt.title('Maximum Flow By Extra capacity ')
plt.show()
```

```
1
borders[('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 6
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 6
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 7
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 5
cut with min capacity [[{'o', 'c', 'b', 'a'} {'d'}]]
uc,vd
The edge between c and d 2 has increase 1 unit
**************************************************
2
borders[('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 6
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 6
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 7
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 6
The edge between a and d 3 has increase 1 unit
**************************************************
3
borders[('o', 'a'), ('o', 'b')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 6
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 7
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 8
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 7
cut with min capacity [[{'o'} {'a', 'c', 'b', 'd'}]]
uo,va
The edge between o and a 4 has increase 1 unit
**************************************************
4
borders[('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 7
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 7
```

```
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 8
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 7
The edge between o and b 4 has increase 1 unit
**************************************************
5
borders[('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 8
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 8
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 8
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 7
cut with min capacity [[{'o', 'c', 'b', 'a'} {'d'}]]
uc,vd
The edge between c and d 3 has increase 1 unit
**************************************************
6
borders[('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 8
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 8
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 8
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 8
The edge between b and d 3 has increase 1 unit
**************************************************
7
borders[('o', 'b'), ('a', 'd'), ('a', 'b')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 8
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 8
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 9
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 9
The edge between o and b 5 has increase 1 unit
**************************************************
8
borders[('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 9
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 9
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 9
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 9
The edge between a and d 4 has increase 1 unit
**************************************************
9
borders[('o', 'a'), ('o', 'b')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 9
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 10
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 10
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 10
cut with min capacity [[{'o'} {'a', 'c', 'b', 'd'}]]
uo,va
The edge between o and a 5 has increase 1 unit
**************************************************
```

```
10
borders[('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 10
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 10
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 10
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 10
The edge between b and d 4 has increase 1 unit
**************************************************
11
```



From the image, we can observe that the maximum flow is directly correlated with the extra capacity, increasing proportionally as the capacity increases.

1.3 Point c

You are given the possibility of adding to the network a directed link e8 with capacity c8= 1 and x > O extra units of capacity ($X \in Z$). Where should you add the link and how should you distribute the additional capacity in order to maximize the throughput that can be sent from o to Plot the maximum throughput from o to d as a function of $x \geq 0$.

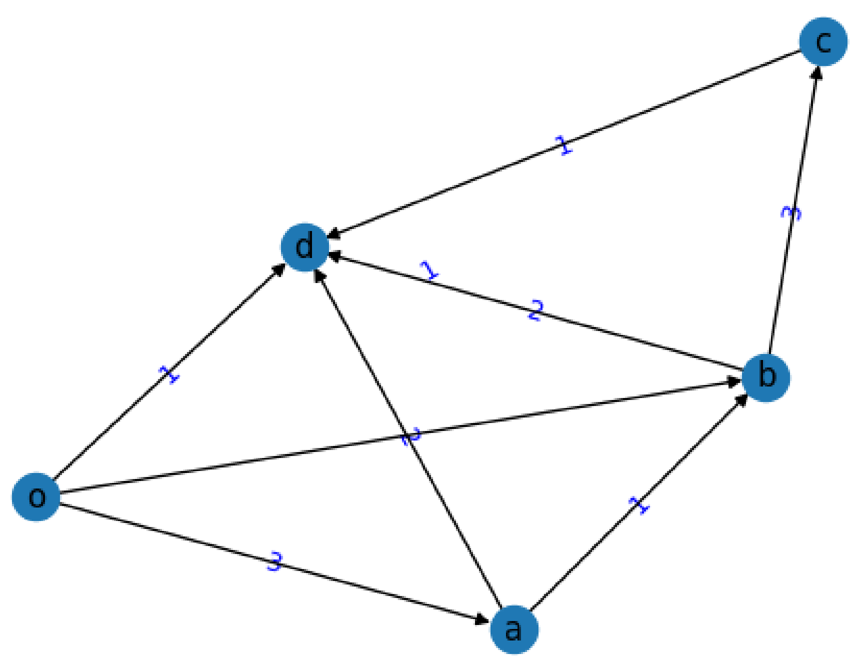
[21]:
```
#Recreate the original graph
G=create_graph()
G.add_edge('o', 'd', capacity=1) #add directed link e8
pos = nx.nx_pydot.graphviz_layout(G)
```

In the original graph, the min-cut is between the sets $\{a, b, c, o\}$ and $\{d\}$. To add the edge e_8 , we should place it in this bottleneck, where there is a missing directed link between nodes o and d. Including this edge in the capacity calculation will directly increase the total capacity. By adding this edge between o and d, we aim to maximize the throughput, as the maximum throughput corresponds to the min-cut capacity.

[22]:
```
G.edges() #Confirming the new edge in the graph
```

[22]:
```
OutEdgeView([('o', 'a'), ('o', 'b'), ('o', 'd'), ('a', 'd'), ('a', 'b'), ('b',
'd'), ('b', 'c'), ('c', 'd')])
```

[23]:
```
#Plot the graph
nx.draw_networkx_edge_labels(G,pos,edge_labels={("o","a"):'3',("o","b"):
 ↪'3',("a","d"):'2',("b","d"):'2',("b","c"):'3',("c","d"):'1',("a","b"):'1',␣
 ↪("o","c"):1, ("o", "d"):1},font_color='blue')
nx.draw(G, pos, with_labels=True) #We have two rows so it is an undirected graph
```



[24]:
```python
#The cuts remains the same
cuts = [
    ({'o'}, {'a', 'b', 'c', 'd'}),
    ({'o', 'a'}, {'b', 'c', 'd'}),
    ({'o', 'a', 'b'}, {'c', 'd'}),
    ({'o', 'a', 'b', 'c'}, {'d'})
]
```

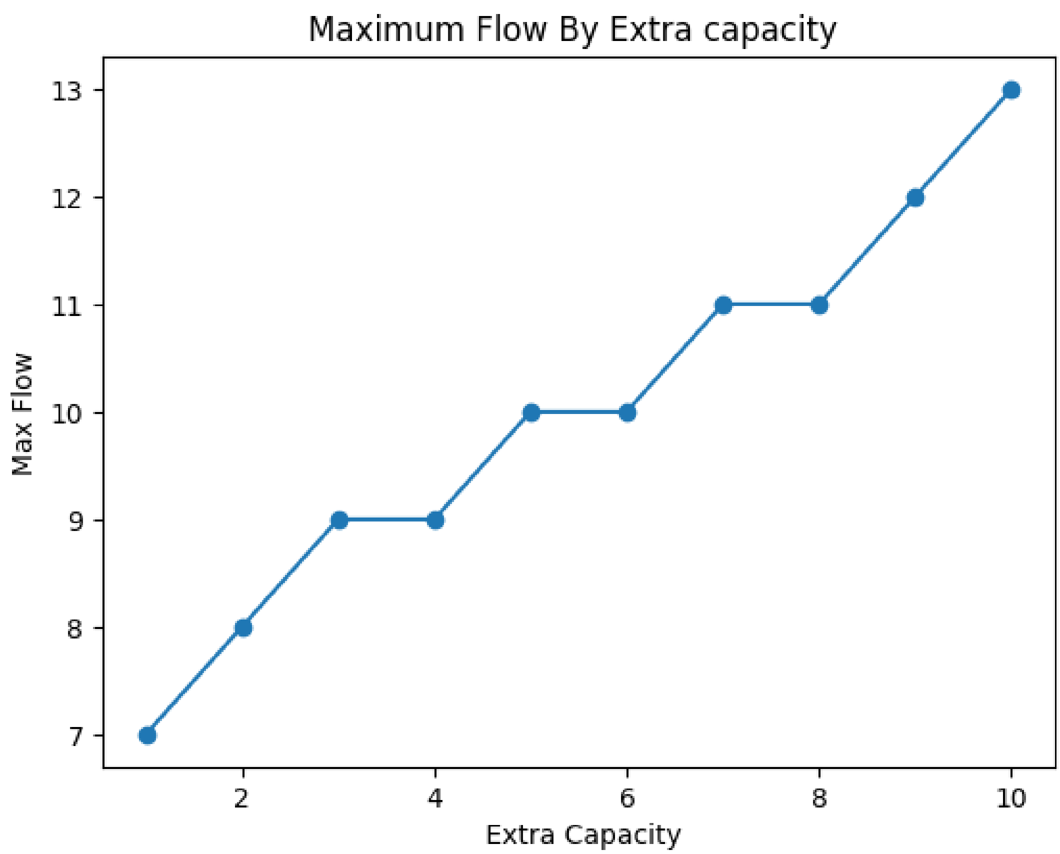
[25]:
```python
#The new edge between o and d is common in all the cuts
common_borders={}
common_borders[0]= [("o","b"),("o", "d")]
common_borders[1]= [("a","d"), ("o","b"),("o", "d")]
common_borders[2]= [("a","d"), ("b","d"),("o", "d")]
common_borders[3]=[("a","d"), ("b","d"),("o", "d")]
```

[26]:
```python
#Use the previous function to know where to allocate the extra capacity
max_flows = assign_capacities(11)



plt.plot(max_flows.keys(), max_flows.values(), marker = 'o')

# Plot formatting
plt.xlabel(' Extra Capacity')
plt.ylabel('Max Flow')
plt.title('Maximum Flow By Extra capacity ')
plt.show()
```

```
1
borders[('o', 'd'), ('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 7
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 7
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 8
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 6
cut with min capacity [[{'o', 'c', 'b', 'a'} {'d'}]]
uo,vd
The edge between o and d 2 has increase 1 unit
**************************************************
2
borders[('o', 'd'), ('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 8
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 8
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 9
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 7
cut with min capacity [[{'o', 'c', 'b', 'a'} {'d'}]]
uc,vd
The edge between c and d 2 has increase 1 unit
**************************************************
3
```

```
borders[('o', 'd'), ('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 8
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 8
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 9
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 8
The edge between o and d 3 has increase 1 unit
************************************************
4
borders[('o', 'd'), ('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 9
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 9
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 10
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 9
The edge between a and d 3 has increase 1 unit
************************************************
5
borders[('o', 'a'), ('o', 'b'), ('o', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 9
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 10
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 11
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 10
cut with min capacity [[{'o'} {'a', 'c', 'b', 'd'}]]
uo,va
The edge between o and a 4 has increase 1 unit
************************************************
6
borders[('o', 'd'), ('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 10
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 10
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 11
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 10
The edge between o and b 4 has increase 1 unit
************************************************
7
borders[('o', 'd'), ('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 11
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 11
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 11
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 10
cut with min capacity [[{'o', 'c', 'b', 'a'} {'d'}]]
uc,vd
The edge between c and d 3 has increase 1 unit
************************************************
8
borders[('o', 'd'), ('c', 'd'), ('b', 'd'), ('a', 'd')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 11
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 11
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 11
```

```
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 11
The edge between b and d 3 has increase 1 unit
**************************************************
9
borders[('o', 'b'), ('o', 'd'), ('a', 'd'), ('a', 'b')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 11
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 11
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 12
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 12
The edge between o and d 4 has increase 1 unit
**************************************************
10
borders[('o', 'b'), ('o', 'd'), ('a', 'd'), ('a', 'b')]
U = {'o'}, U^C = {'a', 'c', 'b', 'd'} -> C_U = 12
U = {'o', 'a'}, U^C = {'b', 'c', 'd'} -> C_U = 12
U = {'o', 'b', 'a'}, U^C = {'c', 'd'} -> C_U = 13
U = {'o', 'c', 'b', 'a'}, U^C = {'d'} -> C_U = 13
The edge between o and b 5 has increase 1 unit
**************************************************
11
```



When we add an extra edge from o to d, the min-cuts improve more quickly due to the presence of a direct path from the start to the end of the graph.

2 Exercise 2

There are a set of people $\{a_1, a_2, a_3, a_4\}$ and a set of foods $\{b_1, b_2, b_3, b_4\}$. Each person is interested in a subset of foods, specifically

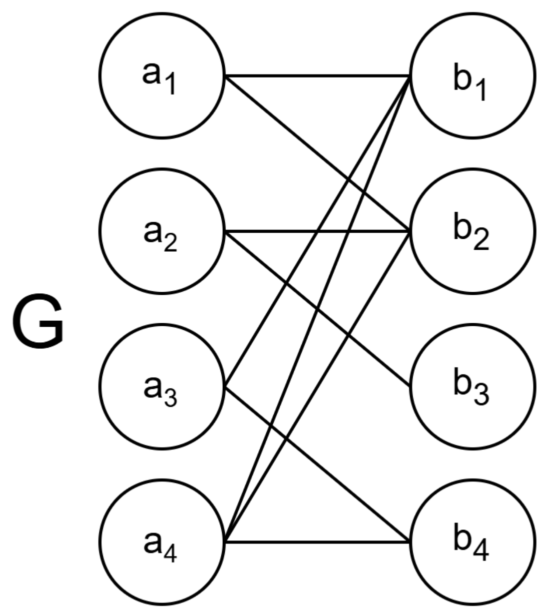
$$a_1 \to \{b_1, b_2\}, a_2 \to \{b_2, b_3\}, a_3 \to \{b_1, b_4\}, a_4 \to \{b_1, b_2, b_4\}$$

2.1 Point a

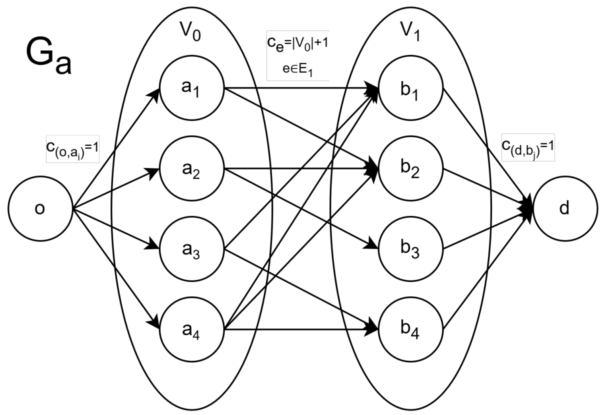
Exploit max-flow problems to find a perfect match (if any).

Solution

In this case, we have a bipartite graph $G = (V = V_0 \bigcup V_1, E)$, where $V_0 = \{a_i\}_{i=1}^4$, $V_1 = \{b_i\}_{i=1}^4$, $E_1 = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_4), (a_4, b_1), (a_4, b_2), (a_4, b_4)\}$, $E_2 = \{(j, i) : (i, j) \in E_1\}$, $E = E_1 \bigcup E_2$

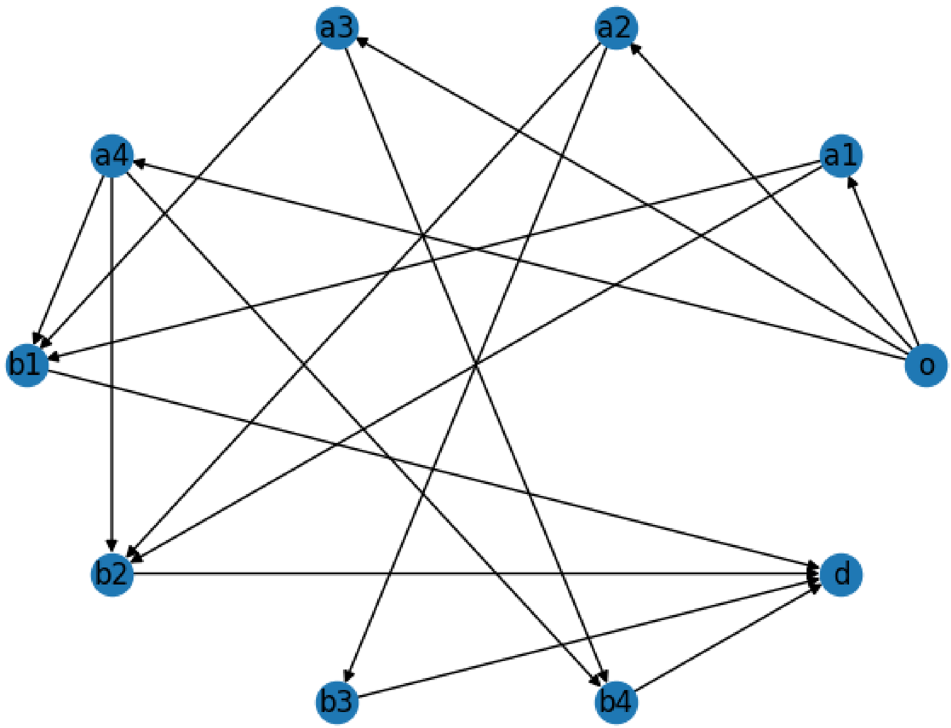


Let us consider the following capacitated directed multigraph $G_a = (V_a = \{o\} \bigcup V \bigcup \{d\}, E_a = \{(o, i) : i \in V_0\} \bigcup E_1 \bigcup \{(i, d) : i \in V_1\}, \vec{c})$ where $\vec{c} \in \mathbb{R}^{E_a}$, $c_e = \begin{cases} 1 & if \quad e \in E_a \setminus E_1 \\ |V_0| + 1 = 5 & if \quad e \in E_1 \end{cases}$



[27]:
```
# G = (V = V0 U V1, E = E1 U E2)
V0 = [f'a{i}' for i in range(1, 4 + 1)]
V1 = [f'b{i}' for i in range(1, 4 + 1)]
V = V0 + V1
E1 = [('a1', 'b1'), ('a1', 'b2'), ('a2', 'b2'), ('a2', 'b3'), ('a3', 'b1'),␣
↪('a3', 'b4'), ('a4', 'b1'), ('a4', 'b2'), ('a4', 'b4')]
E2 = copy.deepcopy(E1)
E2.reverse()
E = E1 + E2

cardinality_V0 = len(V0)
cardinality_V1 = len(V1)

# G_a = (V_a = {O} U V U {d}, E_a = {(o, j)| j \in V0} U E1 U {(i, d)| i \in V1})
V_a = ['o'] + V + ['d']
E_a = [("o", f"a{i}") for i in range(1, 4 + 1)] + E1 + [(f'b{i}', 'd') for i in␣
↪range(1, 4 + 1)]

n_nodes_G_bar = 1 +  cardinality_V0 + cardinality_V1 + 1

G_a = nx.DiGraph()

for e in E_a:
    G_a.add_edge(e[0], e[1], capacity=cardinality_V0 + 1 if e in E1 else 1)
```

[28]:
```
nx.draw_circular(G_a, with_labels=True)
```



“By constructing this network, the goal is to show that there is a maximum flow from o to d of value $|V_0|$; if this flow exists, it implies that each node in V_0 can be matched to a distinct node in V_1, satisfying Hall's condition for a perfect matching”

Let us apply Ford and Fulkerson's algorithm over G_a using the library `NetworkX`, in particular the function `nx.maximum_flow`

[29]:
```
flow_value_a, flow_dict_a = nx.maximum_flow(G_a, "o", "d")
```

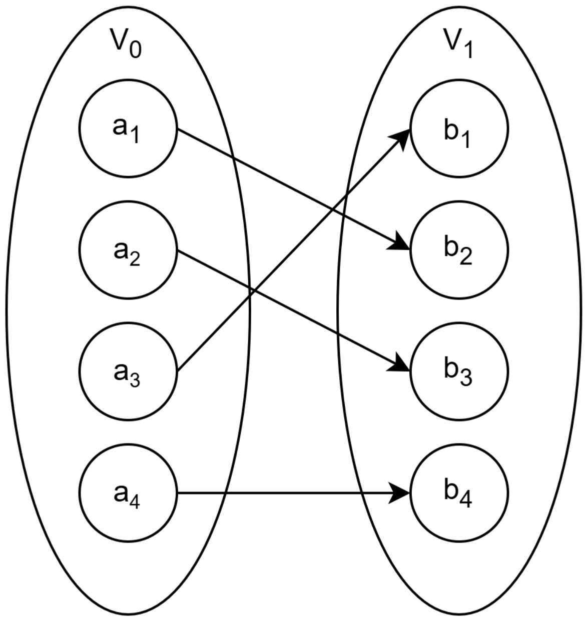
[30]:
```
flow_dict_a
```

[30]:
```
{'o': {'a1': 1, 'a2': 1, 'a3': 1, 'a4': 1},
 'a1': {'b1': 0, 'b2': 1},
 'a2': {'b2': 0, 'b3': 1},
 'a3': {'b1': 1, 'b4': 0},
 'a4': {'b1': 0, 'b2': 0, 'b4': 1},
 'b1': {'d': 1},
 'b2': {'d': 1},
 'b3': {'d': 1},
 'b4': {'d': 1},
 'd': {}}
```

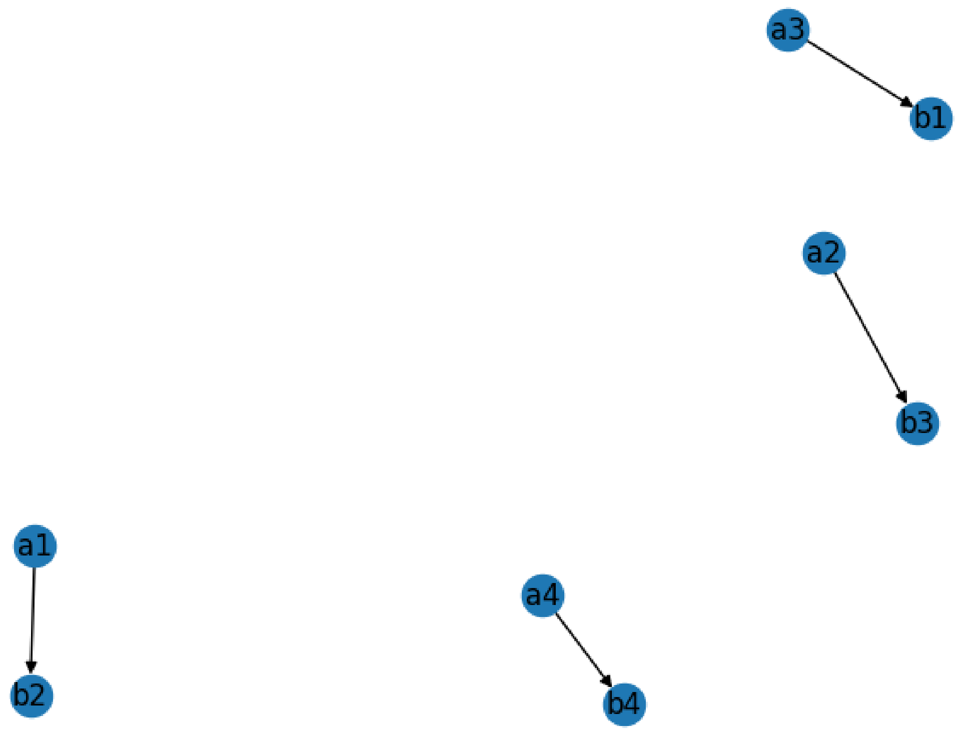
[31]:
```
F = [(i, j) for i in flow_dict_a for j in flow_dict_a[i] if flow_dict_a[i][j] ==␣
↪1 and i != 'o' and j != 'd']
F
```

[31]: [('a1', 'b2'), ('a2', 'b3'), ('a3', 'b1'), ('a4', 'b4')]

And we get a perfect matching because $|F| = |\{(a_1, b_2), (a_2, b_3), (a_3, b_1), (a_4, b_4)\}| = |V_0| = |V_1|$



[32]:
```
nx.draw(nx.DiGraph(F), with_labels = True)
```



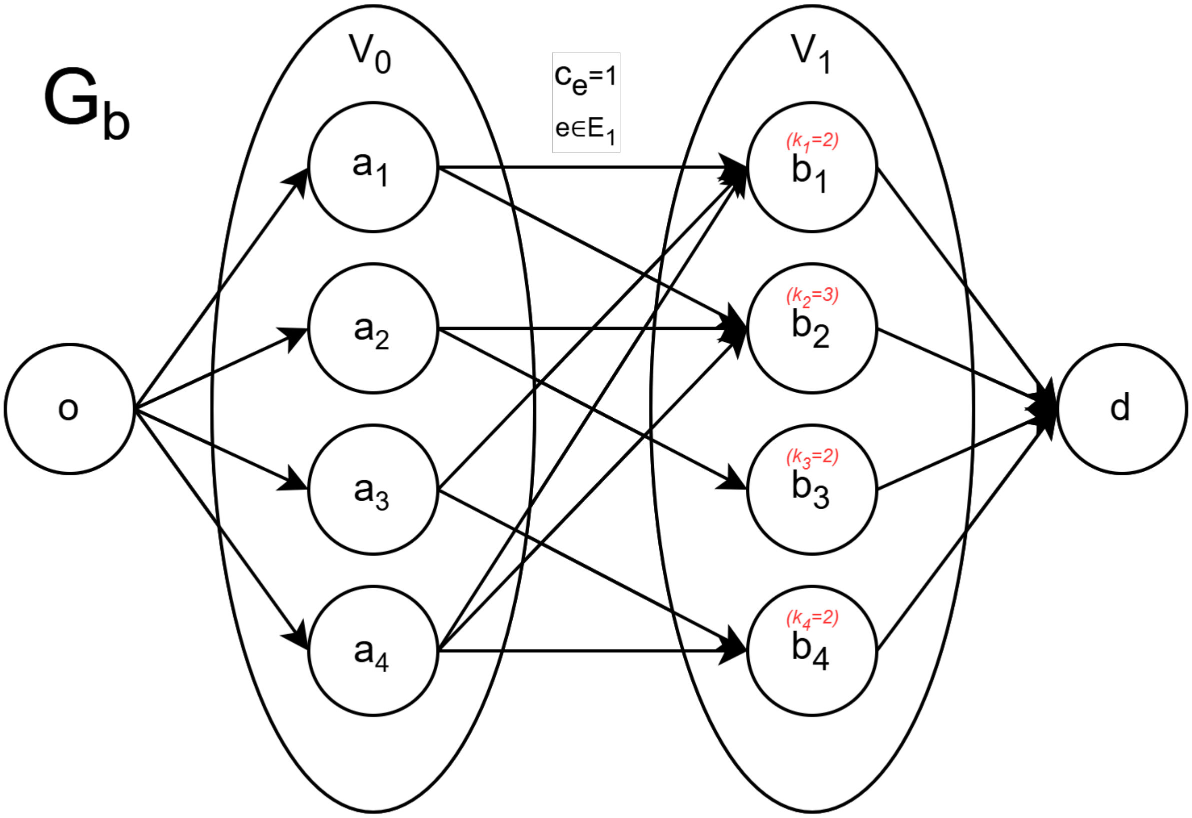
2.2 Point b

Now, assume that there are multiple portions of every food, and the distribution of the portions is $\begin{pmatrix} 2 & 3 & 2 & 2 \end{pmatrix}$. Each person can take an arbitrary number of *different* foods. Exploit the analogy with max-flow problems to establish how many portions of food can be assign in total.

Solution

People can take an arbitrary number of different foods, this means that each person can take one portion from each of the foods they are interested in, but not multiple portions of the same type of food, so it is correct to say that the edges linking people with to their preferences have a capacity of one. We have an additional constraint: The portion distribution of each food is limited, meaning that each food node has a capacity. This concept is called "Maximum flow with vertex capacities", where we aim to maximize the flow under the constraint of vertex capacities.

Let us illustrate this by considering the same graph G_a



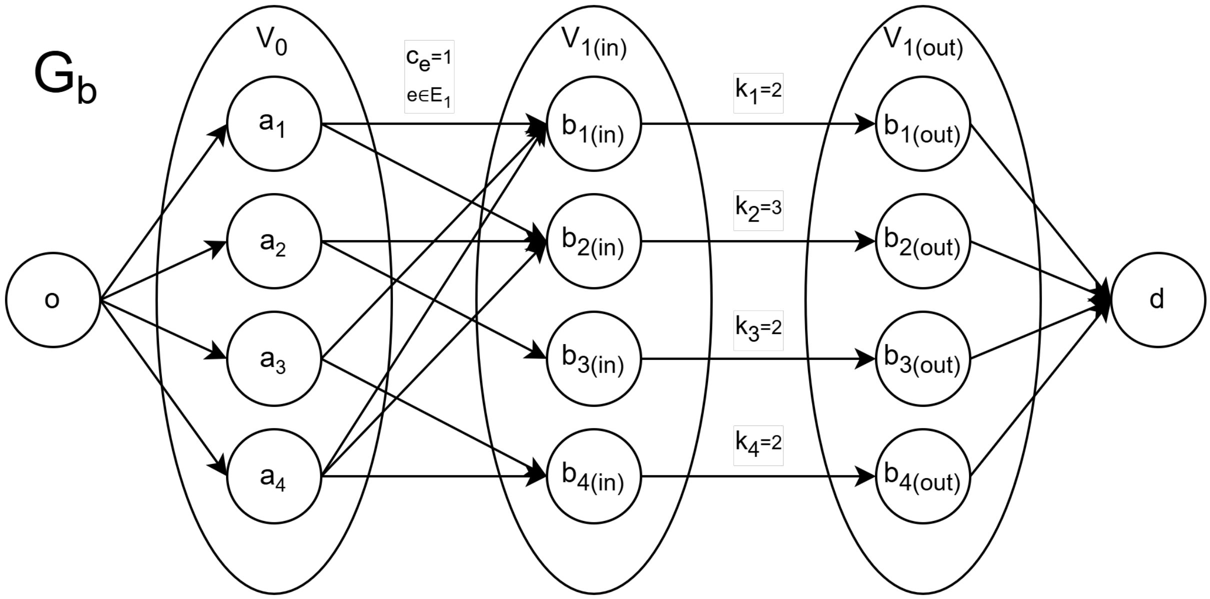
Let us define the portion distribution for each food as $\vec{k} = \begin{pmatrix} 2 & 3 & 2 & 2 \end{pmatrix} \in \mathbb{R}_+^{V_1}$ where k_i represents the capacity of node b_i and $\vec{c} = \vec{1} \in \mathbb{R}_+^{E_1}$ represents the capacities of the edges in E_1

The flow $\vec{f}$ has to satisfy the capacity constraint, conservation flows and vertex capacity constraint: $\sum_{i \in V} f_{ij} \leq k_j \ \forall j \in V_1$

The way to address this problem will be as Trevisan (From Stanford University) [1] suggests:

"... the problem can be reduced to the standard maximum flow problem, by splitting every vertex v into two vertices v_{in} and v_{out}, adding one edge (v_{in}, v_{out}) of capacity c_v, and then converting every edge (u, v) to an edge (u, v_{in}) and every edge (v, w) to an edge (v_{out}, w). It is easy to show that solving the (standard) maximum flow problem on the new network is equivalent to solving the maximum flow with vertex capacity constraints in the original network".

So, our graph becomes the following:



More formally $G_b = \Big(V_b = \{o\} \bigcup V_0 \bigcup V_{1(in)} = \left\{b_{i(in)}\right\}_{i=1}^{4} \bigcup V_{1(out)} = \left\{b_{i(out)}\right\}_{i=1}^{4} \bigcup \{d\}, E_b = \{(o,i) : i \in V_0\} \bigcup \{(i$

where $\vec{c} \in \mathbb{R}^{E_b}$, $c_e = \begin{cases} +\infty & if \quad e \in \{(o,i) : i \in V_0\} \bigcup \left\{(i,d) : i \in V_{1(out)}\right\} \\ 1 & if \quad e \in \left\{\left(i, j_{(in)}\right) : (i,j) \in E_1\right\} \\ 2 & if \quad e = \left(b_{i(in)}, b_{i(out)}\right), i = 1,3,4 \\ 3 & if \quad e = \left(b_{2(in)}, b_{2(out)}\right) \end{cases}$

[33]:
```
V_b = ['o'] + [f'a{i}' for i in range(1, 4 + 1)] + [f'b{i}_in' for i in range(1, 4 + 1)] + [f'b{i}_out' for i in range(1, 4 + 1)] + ['d']
V_b
```

[33]:
```
['o',
 'a1',
 'a2',
 'a3',
 'a4',
 'b1_in',
 'b2_in',
 'b3_in',
 'b4_in',
 'b1_out',
 'b2_out',
 'b3_out',
 'b4_out',
 'd']
```

[34]:
```
E_b = []
for e in E_a:
    if 'b' in e[1]:
```

```
        E_b.append((e[0], e[1] + '_in'))
    elif 'b' in e[0]:
        E_b.append((e[0] + '_out', e[1]))
    else:
        E_b.append(e)
E_b += [(f'b{i}_in', f'b{i}_out') for i in range(1, 4 + 1)]
E_b
```

[34]:
```
[('o', 'a1'),
 ('o', 'a2'),
 ('o', 'a3'),
 ('o', 'a4'),
 ('a1', 'b1_in'),
 ('a1', 'b2_in'),
 ('a2', 'b2_in'),
 ('a2', 'b3_in'),
 ('a3', 'b1_in'),
 ('a3', 'b4_in'),
 ('a4', 'b1_in'),
 ('a4', 'b2_in'),
 ('a4', 'b4_in'),
 ('b1_out', 'd'),
 ('b2_out', 'd'),
 ('b3_out', 'd'),
 ('b4_out', 'd'),
 ('b1_in', 'b1_out'),
 ('b2_in', 'b2_out'),
 ('b3_in', 'b3_out'),
 ('b4_in', 'b4_out')]
```

[35]:
```
n_nodes_G_b = 1 + cardinality_V0 + cardinality_V1 + cardinality_V1 + 1
n_nodes_G_b
```

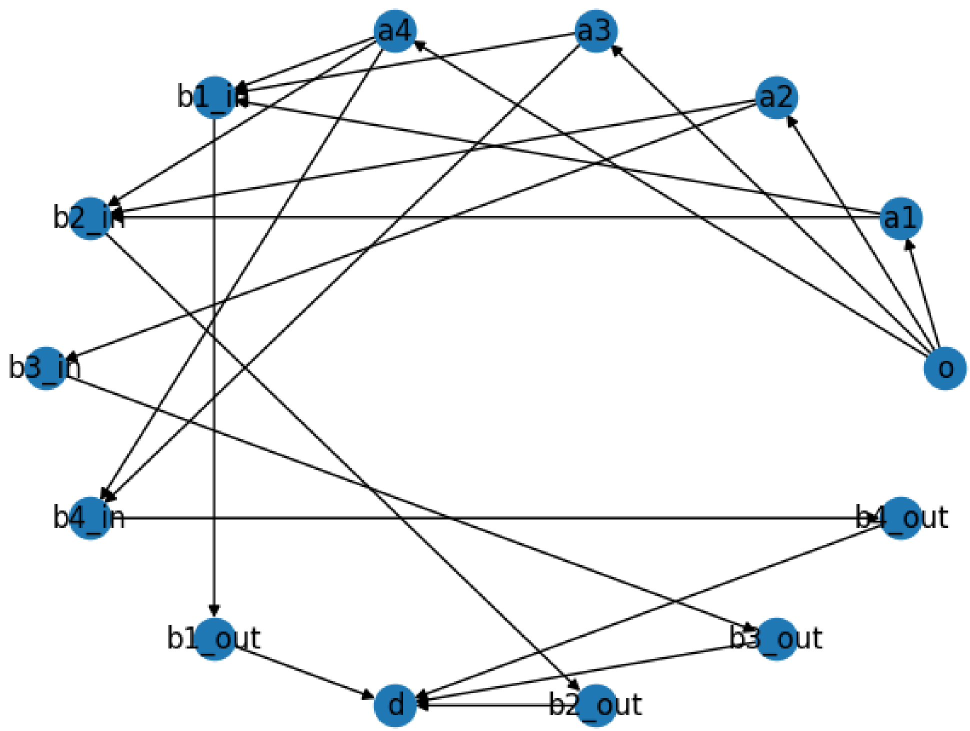
[35]:
```
14
```

[36]:
```
capacities = {
    ('b1_in', 'b1_out'): 2,
    ('b2_in', 'b2_out'): 3,
    ('b3_in', 'b3_out'): 2,
    ('b4_in', 'b4_out'): 2
    }
capacities
```

[36]:
```
{('b1_in', 'b1_out'): 2,
 ('b2_in', 'b2_out'): 3,
 ('b3_in', 'b3_out'): 2,
 ('b4_in', 'b4_out'): 2}
```

[37]:
```
G_b = nx.DiGraph()

for e in E_b:
    if 'b' in e[0] and 'b' in e[1]:
        G_b.add_edge(e[0], e[1], capacity=capacities[e])
    elif 'a' in e[0]:
        G_b.add_edge(e[0], e[1], capacity=1)
    else:
        G_b.add_edge(e[0], e[1])
```

[38]:
```
nx.draw_circular(G_b, with_labels=True)
```



Let us apply Ford and Fulkerson's algorithm over G_b using the same library and function as before, we obtain the following result

[39]:
```
flow_value_b, flow_dict_b = nx.maximum_flow(G_b, "o", "d")
flow_value_b, flow_dict_b
```

[39]:
```
(8,
 {'o': {'a1': 1, 'a2': 2, 'a3': 2, 'a4': 3},
  'a1': {'b1_in': 0, 'b2_in': 1},
```

```
    'a2': {'b2_in': 1, 'b3_in': 1},
    'a3': {'b1_in': 1, 'b4_in': 1},
    'a4': {'b1_in': 1, 'b2_in': 1, 'b4_in': 1},
    'b1_in': {'b1_out': 2},
    'b2_in': {'b2_out': 3},
    'b3_in': {'b3_out': 1},
    'b4_in': {'b4_out': 2},
    'b1_out': {'d': 2},
    'd': {},
    'b2_out': {'d': 3},
    'b3_out': {'d': 1},
    'b4_out': {'d': 2}})
```

We can conclude that the total number of food portions that can be assigned is 8, distributed as follows:

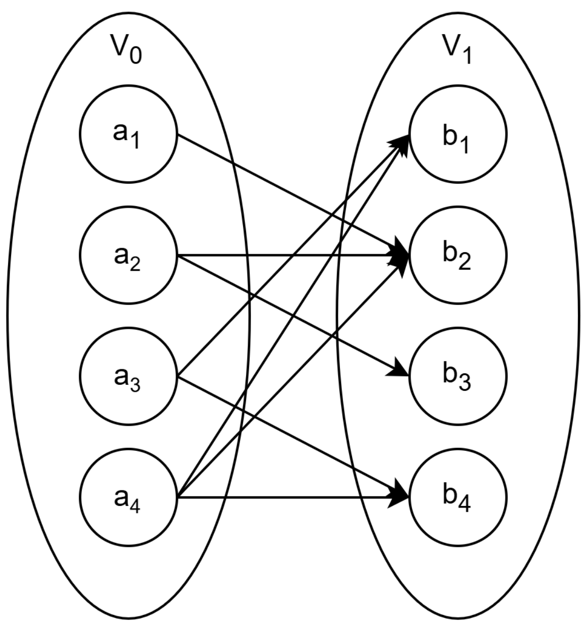
$$a_1 \to \{b_2\}$$

$$a_2 \to \{b_2, b_3\}$$

$$a_3 \to \{b_1, b_4\}$$

$$a_4 \to \{b_1, b_2, b_4\}$$

This means that a_1 receives one portion of b_2, a_2 receives one portion of b_2 and one portion of b_3 and so on. Visually, this shows the distribution clearly:



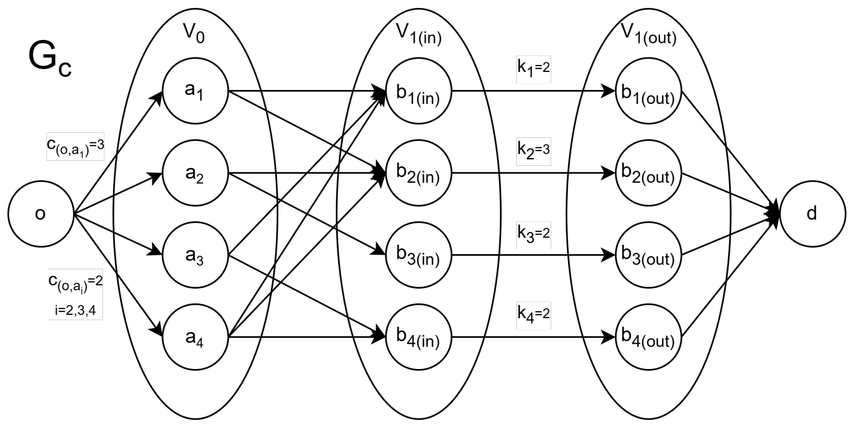
2.3 Point c

Now, assume that a_1 wants 3 portions of food, a_i (For every $i \neq 1$) want 2 portions of food, every person can take multiple portions of the same food, and the distribution of the portions is

$\begin{pmatrix} 2 & 3 & 2 & 2 \end{pmatrix}$. Exploit the analogy with max-flow problems to establish how many portions of food can be assign in total.

Solution

In this part of the exercise, we will consider the graph G_b but with a different distribution of capacities. In this way, we will take into account each person's preferences



More formally $G_c = (V_b, E_b, \vec{c})$ where $\vec{c} \in \mathbb{R}^{E_b}$, $c_e = \begin{cases} 3 & if \quad e \in \left\{(o, a_1), \left(b_{2(in)}, b_{2(out)}\right)\right\} \\ 2 & if \quad e \in \{(o, a_i) : 2 \leq i \leq 4\} \bigcup \left\{\left(b_{i(in)}, b_{i(out)}\right) : i = 1, 3, 4\right\} \\ +\infty & if \quad e \in \left\{\left(i, j_{(in)}\right) : (i, j) \in E_1\right\} \bigcup \left\{\left(b_{i(out)}, d\right) : 1 \leq i \leq 4\right\} \end{cases}$

[40]:
```
capacities |= {('o', 'a1'): 3} | {('o', f'a{i}'): 2 for i in range(2, 4 + 1)}
capacities
```

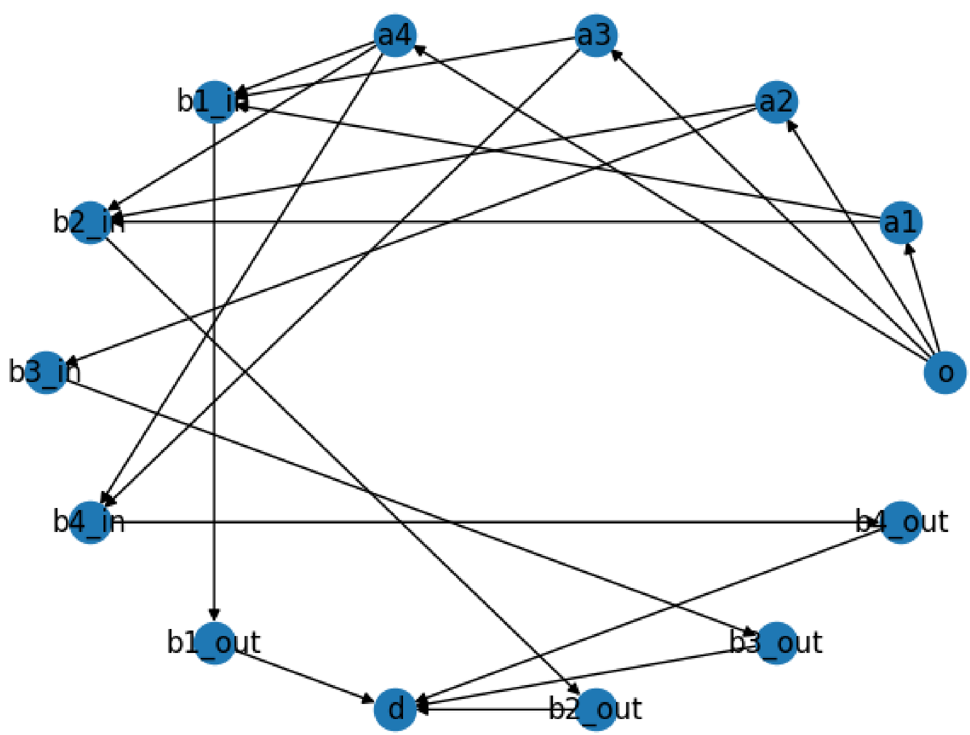
[40]:
```
{('b1_in', 'b1_out'): 2,
 ('b2_in', 'b2_out'): 3,
 ('b3_in', 'b3_out'): 2,
 ('b4_in', 'b4_out'): 2,
 ('o', 'a1'): 3,
 ('o', 'a2'): 2,
 ('o', 'a3'): 2,
 ('o', 'a4'): 2}
```

[41]:
```
G_c = nx.DiGraph()

for e in E_b:
    if e in capacities.keys():
        G_c.add_edge(e[0], e[1], capacity=capacities[e])
    else:
        G_c.add_edge(e[0], e[1])
```

[42]:
```
G_c.edges.data()
```

```
[42]: OutEdgeDataView([('o', 'a1', {'capacity': 3}), ('o', 'a2', {'capacity': 2}),
('o', 'a3', {'capacity': 2}), ('o', 'a4', {'capacity': 2}), ('a1', 'b1_in', {}),
('a1', 'b2_in', {}), ('a2', 'b2_in', {}), ('a2', 'b3_in', {}), ('a3', 'b1_in',
{}), ('a3', 'b4_in', {}), ('a4', 'b1_in', {}), ('a4', 'b2_in', {}), ('a4',
'b4_in', {}), ('b1_in', 'b1_out', {'capacity': 2}), ('b2_in', 'b2_out',
{'capacity': 3}), ('b3_in', 'b3_out', {'capacity': 2}), ('b4_in', 'b4_out',
{'capacity': 2}), ('b1_out', 'd', {}), ('b2_out', 'd', {}), ('b3_out', 'd', {}),
('b4_out', 'd', {})])
```

```
[43]: nx.draw_circular(G_c, with_labels=True)
```



After implementing this graph with `NetworkX` library and using the same function as before, we obtain the following result

```
[44]: flow_value_c, flow_dict_c = nx.maximum_flow(G_c, "o", "d")
flow_value_c, flow_dict_c
```

```
[44]: (9,
 {'o': {'a1': 3, 'a2': 2, 'a3': 2, 'a4': 2},
  'a1': {'b1_in': 0, 'b2_in': 3},
  'a2': {'b2_in': 0, 'b3_in': 2},
```

```
'a3': {'b1_in': 2, 'b4_in': 0},
'a4': {'b1_in': 0, 'b2_in': 0, 'b4_in': 2},
'b1_in': {'b1_out': 2},
'b2_in': {'b2_out': 3},
'b3_in': {'b3_out': 2},
'b4_in': {'b4_out': 2},
'b1_out': {'d': 2},
'd': {},
'b2_out': {'d': 3},
'b3_out': {'d': 2},
'b4_out': {'d': 2}})
```

We can conclude that the total number of food portions that can be assigned is 9 (The whole table) distributed of the following way:

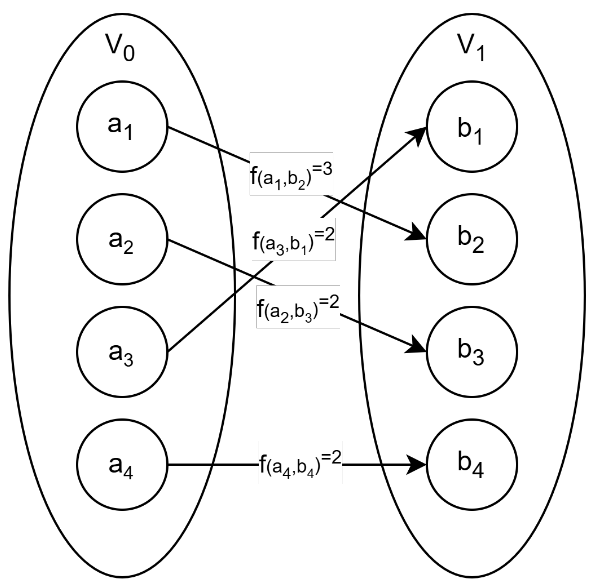
$$a_1 \to \{b_2 : 3\}$$

$$a_2 \to \{b_3 : 2\}$$

$$a_3 \to \{b_1 : 2\}$$

$$a_4 \to \{b_4 : 2\}$$

This means that a_1 receives three (3) portions of b_2, a_2 receives two (2) portions of b_3 and so on. Visually, this shows the distribution clearly:



3 Exercise 3

We are given the highway network in Los Angeles, see Figure 2. To simplify the problem, an approximate highway map is given in Figure 3, covering part of the real highway network. The node-link incidence matrix B, for this traffic network is given in the file `traffic.mat`. The rows of B are associated with the nodes of the network and the columns of B with the links. The i-th column of B has 1 in the row corresponding to the tail node of link e_i and (-1) in the row corresponding to the head node of link e_i . Each node represents an intersection between highways (and some of the area around).

Each link $e_i \in \{e_1, ..., e_{28}\}$, has a maximum flow capacity c_{ei}. The capacities are given as a vector c_e in the file `capacities.mat`. Furthermore, each link has a minimum travelling time lei, which the drivers experience when the road is empty. In the same manner as for the capacities, the minimum travelling times are given as a vector le in the file `traveltime.mat`. These values are simply retrieved by dividing the length of the highway segment with the assumed speed limit 60 miles/hour. For each link, we introduce the delay function

$$\tau_e(f_e) = \frac{l_e}{1 - \frac{f_e}{c_e}}, 0 \leqslant f_e < c_e$$

For $f_e \geq c_e$, the value of $\tau\_e(f\_e)$ is considered as $+\infty$.

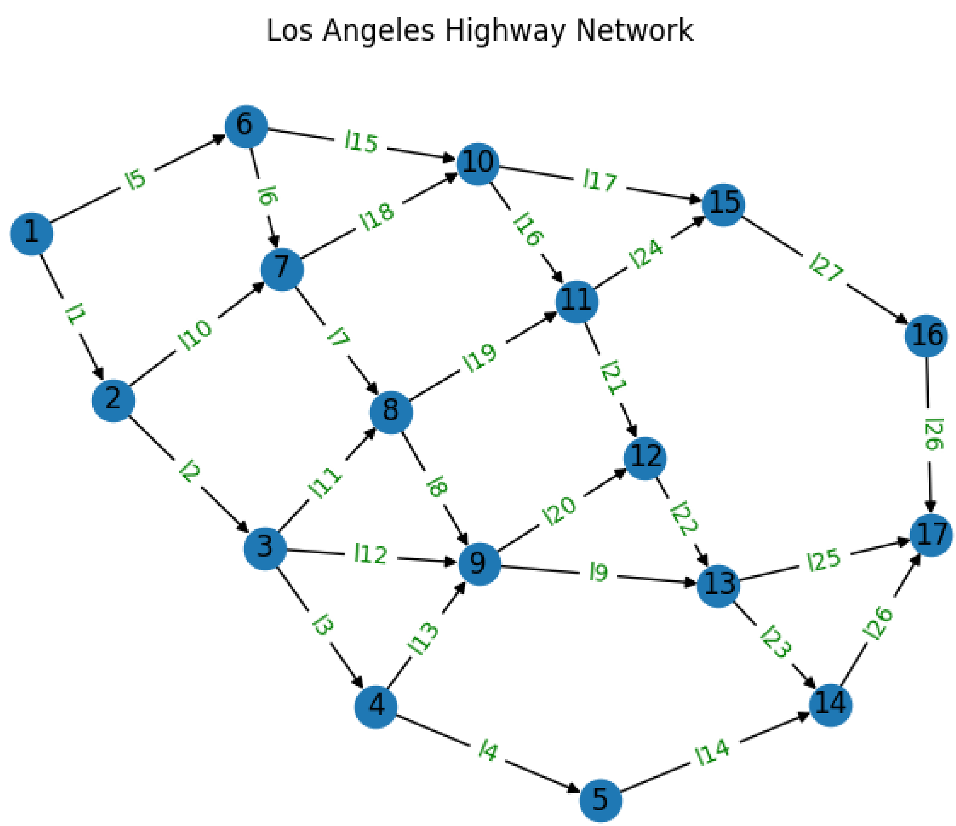
First of all, we need to load the resources given for the excersice and construct the graph.

[46]:
```
f = loadmat('resources/flow.mat')["flow"].reshape(28,)
C = loadmat('resources/capacities.mat')["capacities"].reshape(28,)
B = loadmat('resources/traffic.mat')["traffic"]
l = loadmat('resources/traveltime.mat')["traveltime"].reshape(28,)
```

[47]:
```
labels = {(1,2):'l1',(2,3):'l2',(3,4):'l3',(4,5):'l4',(1,6):'l5',
          (6,7):'l6',(7,8):'l7',(8,9):'l8',(9,13):'l9',(2,7):'l10',
          (3,8):'l11',(3,9):'l12',(4,9):'l13',(5,14):'l14',(6,10):'l15',
          (10,11):'l16',(10,15):'l17',(7,10):'l18',(8,11):'l19',
          (9,12):'l20',(11,12):'l21',(12,13):'l22',(13,14):'l23',
          (11,15):'l24',(13,17):'l25',(14,17):'l26',(15,16):'l27',(16,17):'l26'}
```

[48]:
```
W = [(
    np.argwhere(b==1).reshape(-1)[0]+1,
    np.argwhere(b==-1).reshape(-1)[0]+1,
    {'time': l[i], 'capacity':C[i]}) for i,b in enumerate(B.T)]

G = nx.DiGraph()
for w in W:
    G.add_edge(w[0],w[1], time=w[2]['time'], capacity=w[2]['capacity'])
pos = nx.nx_pydot.graphviz_layout(G)
edges = G.edges
```

[49]:
```
nx.draw(G,pos,with_labels=True)
nx.draw_networkx_edge_labels(
    G, pos,
    edge_labels= labels,
    font_color='green'
)
plt.axis('off')
plt.title("Los Angeles Highway Network")
plt.show()
```



3.1 Point a

Find the shortest path between node 1 and 17. This is equivalent to the fastest path (path with shortest traveling time) in an empty network.

First, as a sanity check, we execute the shortest path algorithm provided by the `networkx`library

[50]:
```
nx.shortest_path(G,1,17)
```

[50]: `[1, 2, 3, 9, 13, 17]`

Here we got that the shortest path in an empty network is $(1) \to (2) \to (3) \to (9) \to (13) \to (17)$.

Now, let's compare it with the separable convex network flow optimization.

Given a multigraph (V, E), and an **exogenous network flow** vector $\nu \in \mathrm{R}^V$ such that

$$\sum_{i \in V} \nu_i = 0. \tag{1}$$

Let $f \in \mathrm{R}^E$ be a network flow vector satisfying the constraints

$$f \geq \mathbf{0}, \quad Bf = \nu. \tag{2}$$

And let $\psi_e(f_e)$ be a separable non-decreasing convex cost function $\psi_e(f_e)$ such that $\psi_e(0) = 0$, this function is applied to every edge.

Therefore we have the following optimization problem, which will be solved using the library `cvxpy`:

$$f^* \in \underset{\substack{f \in \mathbb{R}_+^E \\ Bf = \nu}}{\arg\min} \sum_{e \in E} \psi_e(f_e). \tag{3}$$

[51]:
```
n_edges = B.shape[1]
n_nodes = B.shape[0]

tmp = np.zeros(n_nodes)
tmp[0] = 1
tmp[-1] = -1
# exogenous flow vector: one unit of flow enters the origin and exits the␣
 ↪destination node
# the solution of the shortest path problem is actually independent on the␣
 ↪throughput value because of the linearity
tau = 1
nu = tmp * tau



# Construct the problem.
flow = cp.Variable(n_edges)
objective = cp.Minimize(l.T @ flow)
constraints = [B @ flow == nu, flow >=0]
prob = cp.Problem(objective, constraints)

# The optimal objective value is returned by `prob.solve()`.
result = prob.solve()
# The optimal value for f is stored in `flow.value`.
path = np.around(flow.value, decimals=0, out=None)
print("Optimal flow array:\n",  path)
print("Optimal path:", [f"l{i[0]+1}" for i in list(np.argwhere(path==1))])
```

```
Optimal flow array:
 [1. 1. 0. 0. 0. 0. 0. 0. 1. 0. 0. 1. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 1. 0. 0. 0.]
Optimal path: ['l1', 'l2', 'l9', 'l12', 'l25']
```

Therefore, following the order of edges proposed in the graph, the optiaml path to travel from (1) to (17) is:

$$(1) - l_1 \rightarrow (2) - l_2 \rightarrow (3) - l_{12} \rightarrow (9) - l_9 \rightarrow (13) - l_{25} \rightarrow (17)$$

Which is congruent with the previous result using the shortest path algorithm.

3.2 Point b

Find the maximum flow between node 1 and 17.

For this excercise, lets use the `maximum_flow` algorithm available in the `Networkx` library.

[52]:
```
value, Gflow = nx.algorithms.flow.maximum_flow(G, 1, 17, capacity='capacity')
```

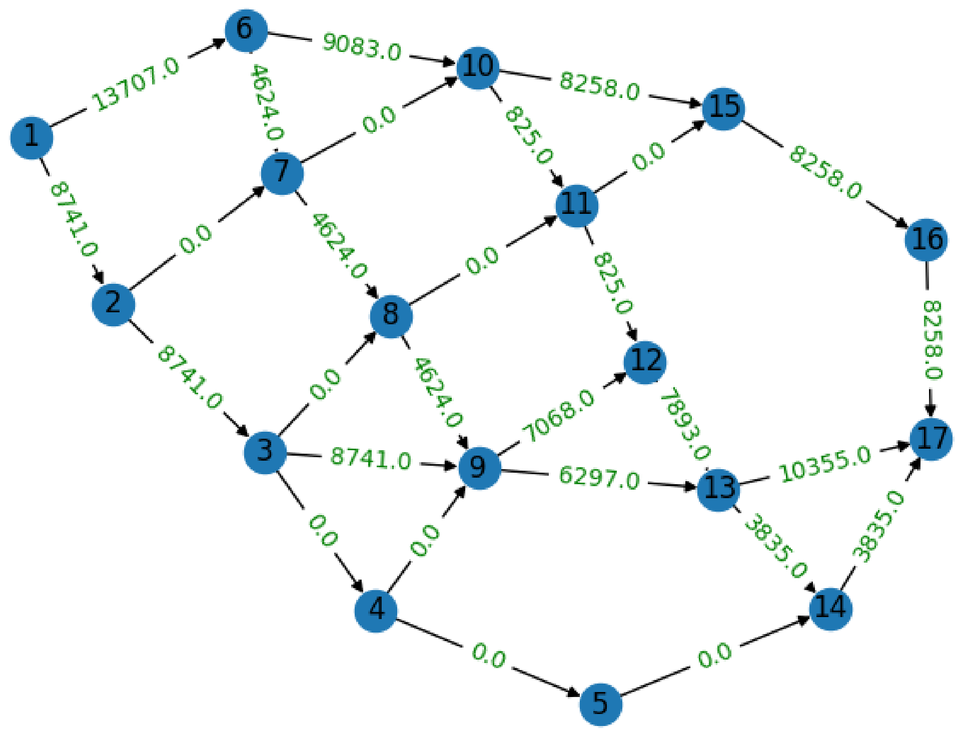
[53]:
```
found_flow = np.zeros(n_edges)
edge_list = [e for e in edges]

for el in Gflow:
  for val in Gflow[el]:
    found_flow[edge_list.index((el,val))] = Gflow[el][val]

print("Max flow is: ", value)
print("With flow vector:",end=" ")
print(found_flow)
```

```
Max flow is:  22448
With flow vector: [ 8741. 13707.  8741.     0.     0.     0.  8741.     0.
0.     0.
  4624.  9083.  4624.     0.  4624.     0.  6297.  7068.  3835. 10355.
  3835.   825.  8258.   825.     0.  8258.  7893.  8258.]
```

As seen in the output of the algorithm, the maximum flow between the nodes 1 and 17 is 22448 and the flow distribution can be observed inside the links of the following graph

[54]:
```
nx.draw(G,pos,with_labels=True)
nx.draw_networkx_edge_labels(
    G, pos,
    edge_labels= {e:i for i,e in zip(found_flow,edges)},
    font_color='green'
)
plt.axis('off')
plt.title("Flow distribution between nodes 1 to 17")
plt.show()
```

Flow distribution between nodes 1 to 17



3.3 Point c

Given the flow vector in flow.mat, compute the vector ν satisfying $Bf = \nu$ We have to verify that

$$f \geq \mathbf{0} \tag{4}$$

[55]:
```
f>= 0
```

[55]:
```
array([ True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True])
```

and also

$$\sum_{i \in V} \nu_i = 0. \tag{5}$$

[56]:
```
# Compute net flow at each node
v = B @ f
```

```
# Compute inflows and outflows.
inflows = [max(flow_value, 0) for flow_value in v]
outflows = [min(flow_value, 0) for flow_value in v]

print("The vector v is:", v)
# Here we are verifying that it satisfies the zero-sum constraint
print("The total net flow =", sum(v))

print("The inflow relative to the given flow vector f is: v_+ =", inflows)␣
↪#positive values
print("The outflow relative to the given flow vector f is: v_- =", outflows)␣
↪#negative values
```

```
The vector v is: [ 16282   9094  19448   4957   -746   4768    413     -2  -5671
1169
     -5  -7131   -380  -7412  -7810  -3430 -23544]
The total net flow = 0
The inflow relative to the given flow vector f is: v_+ = [16282, 9094, 19448,
4957, 0, 4768, 413, 0, 0, 1169, 0, 0, 0, 0, 0, 0, 0]
The outflow relative to the given flow vector f is: v_- = [0, 0, 0, 0, -746, 0,
0, -2, -5671, 0, -5, -7131, -380, -7412, -7810, -3430, -23544]
```

3.4 Point d

Find the social optimum f^* with respect to the delays on the different links $\tau_e(f_e)$. For this, minimize the cost function subject to the flow constraints.

$$\sum_{e\in\mathcal{E}} f_e\tau_e(f_e) = \sum_{e\in\mathcal{E}} \frac{f_e l_e}{1 - f_e/c_e} = \sum_{e\in\mathcal{E}} \left(\frac{l_e c_e}{1 - f_e/c_e} - l_e c_e\right)$$

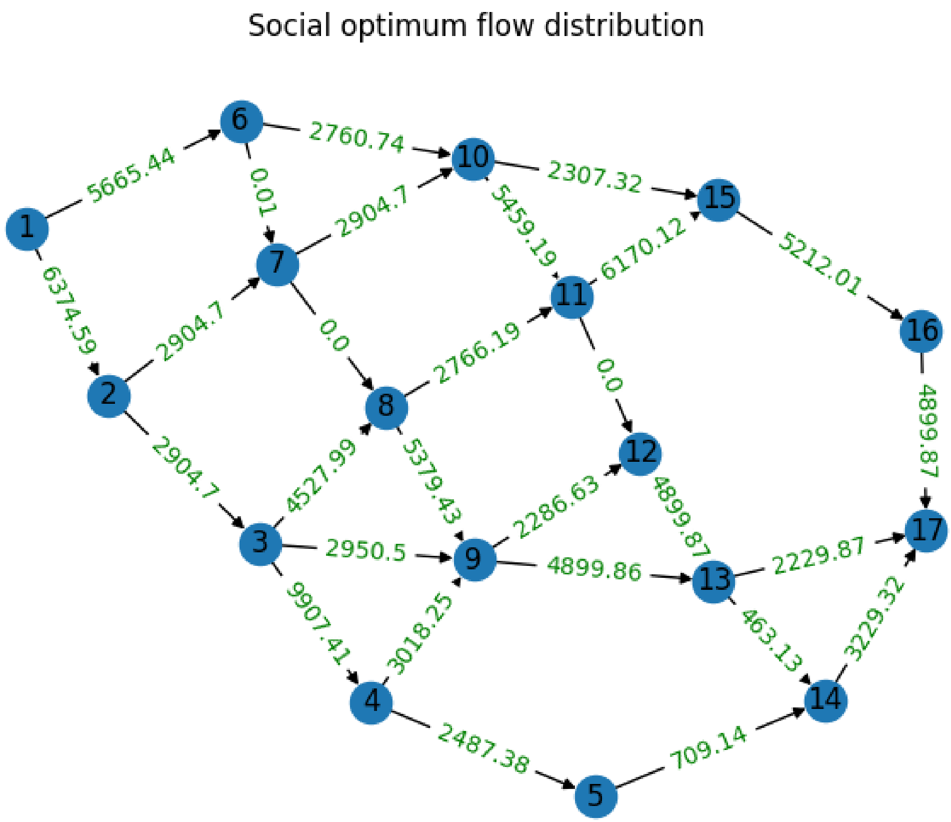
For this excercise we apply again the separable convex network flow optimization, using the ν found in the previous excersise and the proposed cost function, the distribution is presented in the graph below.

[57]:
```
# exogenous flow vector: one unit of flow enters the origin and
# exits the destination node
nu = np.zeros(v.shape[0])
nu[0] = v[0]
nu[-1] = -v[0]





# Construct the problem.
fe = cp.Variable(n_edges)
function = cp.sum(cp.multiply(cp.multiply(l,C),cp.inv_pos(1 - fe/C))- cp.
↪multiply(l,C))
objective = cp.Minimize(function)
```

```
constraints = [B @ fe == nu, fe >=0, fe<=C]
prob = cp.Problem(objective, constraints)

# The optimal objective value is returned by `prob.solve()`.
cost_opt = prob.solve()
# The optimal value for f is stored in `f.value`.
opt_flow = fe.value
print("Social optimal flow:", np.around(opt_flow, decimals=0, out=None))
print("Optimal cost:", np.around(cost_opt, decimals=2, out=None))
social_opt = fe
```

```
Social optimal flow: [6375. 5665. 2905. 2905. 9907. 4528. 2951. 2487. 3018.
709.    0. 2761.
    0. 2905. 5379. 2766. 4900. 2287.  463. 2230. 3229. 5459. 2307.    0.
 6170. 5212. 4900. 4900.]
Optimal cost: 23997.16
```

[58]:
```
nx.draw(G,pos,with_labels=True)
nx.draw_networkx_edge_labels(
    G, pos,
    edge_labels= {e:i for i,e in zip(np.around(opt_flow, decimals=2,␣
 ↪out=None),edges)},
    font_color='green'
)
plt.axis('off')
plt.title("Social optimum flow distribution")
plt.show()
```



3.5 Point e

Find the Wardrop equilibrium $f^{(0)}$. For this, use the cost function

$$min \sum_e \int_0^{f_e} \tau_e(s)\ \mathrm{d}s.$$

The solution of the expression:

$$min \sum_e \int_0^{f_e} \frac{l_e}{1-\frac{s}{c_e}}\ \mathrm{d}s.$$

Is the objective function:

$$min \sum_e -l_e c_e \ln\left(1-\frac{f_e}{c_e}\right)$$

Subject to constraints:

$$Bf = \nu, f \geq 0$$

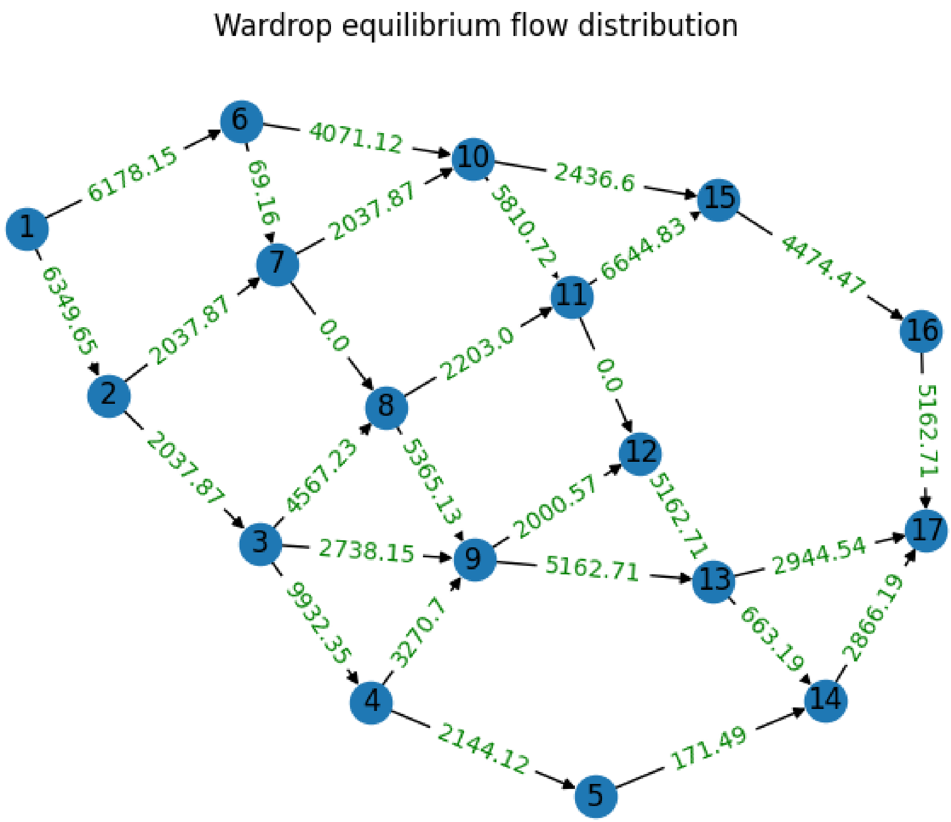
[59]:
```
B # Traffic matrix
n_edges #Number of edges of B
```

```python
# Exogenous flow vector:
# flow entering the origin node is set as v[0], flow exiting the destination␣
↪node is -v[0].
nu = np.zeros(v.shape[0])
nu[0] = v[0]
nu[-1] = -v[0]

# Coefficients for objective function
C # diagonal matrix
l # 1d array
fe = cp.Variable(n_edges) # Variable representing the flow on each edge

# Construct the problem
function = -cp.sum(cp.multiply(cp.multiply(C, l), cp.log(1 - cp.multiply(fe, cp.
↪inv_pos(C)))))
objective = cp.Minimize(function)
constraints = [B @ fe == nu, fe >=0]
prob = cp.Problem(objective, constraints)

# The optimal objective value is returned
result_w = prob.solve()
# The optimal value for f is stored in `fe.value`.
print("Wardrop equilibrium:", np.around(fe.value, decimals=1, out=None))
```

```
Wardrop equilibrium: [6349.6 6178.2 2037.9 2037.9 9932.4 4567.2 2738.1 2144.1
3270.7  171.5
   69.2 4071.1    0.  2037.9 5365.1 2203.  5162.7 2000.6  663.2 2944.5
 2866.2 5810.7 2436.6    0.  6644.8 4474.5 5162.7 5162.7]
```

```
/usr/local/lib/python3.10/dist-packages/cvxpy/problems/problem.py:1407:
UserWarning: Solution may be inaccurate. Try another solver, adjusting the
solver settings, or solve with verbose=True for more information.
  warnings.warn(
```

[60]:
```python
nx.draw(G,pos,with_labels=True)
nx.draw_networkx_edge_labels(
    G, pos,
    edge_labels= {e:i for i,e in zip(np.around(fe.value, decimals=2,␣
↪out=None),edges)},
    font_color='green'
)
plt.title("Wardrop equilibrium flow distribution")
plt.axis('off')
plt.show()
```



The social cost under Wardrop equilibrium is:

$$\sum_e f_e^{(0)} \tau_e(f_e^{(0)}),$$

[61]:
```
wardrop_f = fe
wardrop_cost = sum(wardrop_f.value*l/(1 - wardrop_f.value/C))
print("Wardrop cost: ",  np.around(wardrop_cost, decimals=1, out=None))
```

```
Wardrop cost:  24341.2
```

3.6 Point f

Introduce tolls, such that the toll on link e is $\omega_e = \psi_e'(f_e^*) - \tau_e(f_e^*)$. For the considered $\psi_e(f_e)$, $\omega_e = f_e^* \tau_e'(f_e^*)$, where f_e^* is the flow at the system optimum. Now the delay on link e is given by $\tau_e(f_e) + \omega_e$. compute the new Wardrop equilibrium $f^{(\omega)}$. What do you observe?

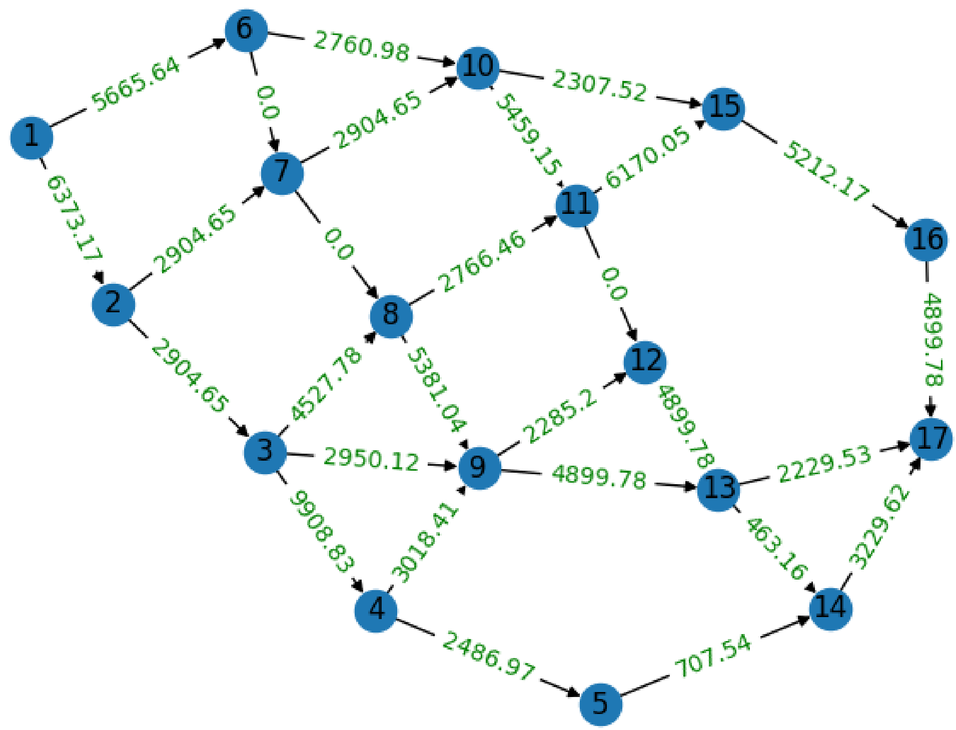
[62]:
```
# exogenous flow vector: one unit of flow enters the origin and
# exits the destination node
nu = np.zeros(v.shape[0])
nu[0] = v[0]
```

```python
nu[-1] = -v[0]
# Q and l store the coefficients of the quadratic
# and linear terms of the objective function.


# Construct the problem.
fe = cp.Variable(n_edges)
t_prime = cp.multiply(cp.multiply(l,C),cp.inv_pos(cp.power((C - social_opt.
 ↪value),2)))
omega = cp.multiply(social_opt.value,t_prime)
function = cp.sum(-cp.multiply(cp.multiply(C,l),cp.log(1-cp.multiply(fe,cp.
 ↪inv_pos(C)))) + cp.multiply(fe,omega))
objective = cp.Minimize(function)
constraints = [B @ fe == nu, fe >=0]
prob = cp.Problem(objective, constraints)

# The optimal objective value is returned by `prob.solve()`.
result_w = prob.solve()
# The optimal value for f is stored in `f.value`.
print("Wardrop equilibrium with tolls:", np.around(fe.value, decimals=2,␣
 ↪out=None))
tolls_f = fe
wardrop_tolls_cost = sum(tolls_f.value*l/(1 - tolls_f.value/C))
print("Wardrop cost with tolls: ", wardrop_tolls_cost)
```

```
Wardrop equilibrium with tolls: [6373.17 5665.64 2904.65 2904.65 9908.83 4527.78
2950.12 2486.97 3018.41
  707.54    0.   2760.98    0.   2904.65 5381.04 2766.46 4899.78 2285.2
  463.16 2229.53 3229.62 5459.15 2307.52    0.   6170.05 5212.17 4899.78
 4899.78]
Wardrop cost with tolls:  23997.162424961676
```

[63]:
```python
nx.draw(G,pos,with_labels=True)
nx.draw_networkx_edge_labels(
    G, pos,
    edge_labels= {e:i for i,e in zip(np.around(tolls_f.value, decimals=2,␣
 ↪out=None),edges)},
    font_color='green'
)
plt.title("Wardrop equilibrium with tolls flow distribution with tolls")
plt.axis('off')
plt.show()
```

Wardrop equilibrium with tolls flow distribution with tolls



[64]:
```python
# Substraction of the social optimum flow and wardrop equilibrium with tolls
np.around(opt_flow-tolls_f.value, decimals=2, out=None)
```

[64]:
```
array([ 1.41, -0.19,  0.04,  0.04, -1.41,  0.2 ,  0.38,  0.42, -0.16,
        1.61,  0.01, -0.24,  0.  ,  0.04, -1.62, -0.27,  0.09,  1.43,
       -0.03,  0.33, -0.3 ,  0.03, -0.2 ,  0.  ,  0.07, -0.16,  0.09,
        0.09])
```

As seen in the graph and the substraction above, the wardrop equilibrium gives a very good aproximation of the social optimum f^* obtained in the numeral **d)**

3.7 Point g

Instead of the total travel time, let the cost for the system be the total additional travel time compared to the total travel time in free flow, given by

$$\psi_e(f_e) = f_e(\tau_e(f_e) - l_e)$$

- Compute the system optimum f^* for the costs above.

From the numeral (d), we know:

$$\sum_e f_e \tau_e(f_e) = \sum_e \left(\frac{l_e c_e}{1 - \frac{f_e}{c_e}} - l_e c_e \right)$$

The objective function will be:

$$\psi_e(f_e) = min \sum_e \frac{l_e c_e}{1 - \frac{f_e}{c_e}} - l_e c_e - f_e l_e$$

Subject to the constraints:

$$Bf = \nu, f \geq 0$$

[65]:
```
B # Traffic matrix
n_edges #Number of edges of B

# Exogenous flow vector:
# flow entering the origin node is set as v[0], flow exiting the destination␣
↪node is -v[0].
nu = np.zeros(v.shape[0])
nu[0] = v[0]
nu[-1] = -v[0]

# Coefficients for objective function
C # diagonal matrix
l # 1d array
fe = cp.Variable(n_edges) # Variable representing the flow on each edge

# Construct the problem, psi function
function = cp.sum(cp.multiply(cp.multiply(l,C),cp.inv_pos(1 - fe/C))- cp.
↪multiply(l,C) - cp.multiply(fe,l))
objective = cp.Minimize(function)
constraints = [B @ fe == nu, fe >=0]
prob = cp.Problem(objective, constraints)

# The optimal objective value is returned
result = prob.solve()

# The optimal value for f is stored in `f.value`.
print("System optimum f*:", np.around(fe.value, decimals=1, out=None))
print("Total cost = ", np.around(result, decimals=1))
new_soc_flow = fe
new_soc_opt_value = fe.value
```

```
System optimum f*: [6393.9 5420.8 3243.7 3243.7 9888.1 4530.6 3051.6 2612.1
2895.4  973.1
    0.  2177.1    0.  3243.7 5357.5 2969.8 4839.8 2452.1  439.5 1893.8
 3409.3 5303.  2140.2    0.  6058.3 5383.9 4839.8 4839.8]
Total cost =  13550.2
```

- Construct a toll vector ω^* such that the Wardrop equilibrium $f^{\{(\omega *)\}}$ coincides with f^*. Compute the new Wardrop equilibrium with the constructed tolls $f^{(w^*)}$to verify your result.

The toll vector is defined as: $\omega_e^* = f_e^* \tau_e'(f_e^*)$

In order to obtain the toll vector, we need to derivate the next expression and multiply it by f_e^*.

$$\psi_e(f_e) = f_e(\tau_e(f_e) - l_e)$$

Since, l_e is a constant, the result of the derivation is:

$$\tau_e'(f_e^*) = \frac{l_e c_e}{(c_e - f_e^*)^2}$$

We also have that the objective function is:

$$f^{(w^*)} = min \sum_e \int_0^{f_e} \psi_e(s)\ \mathrm{d}s + \omega_e^* f_e$$

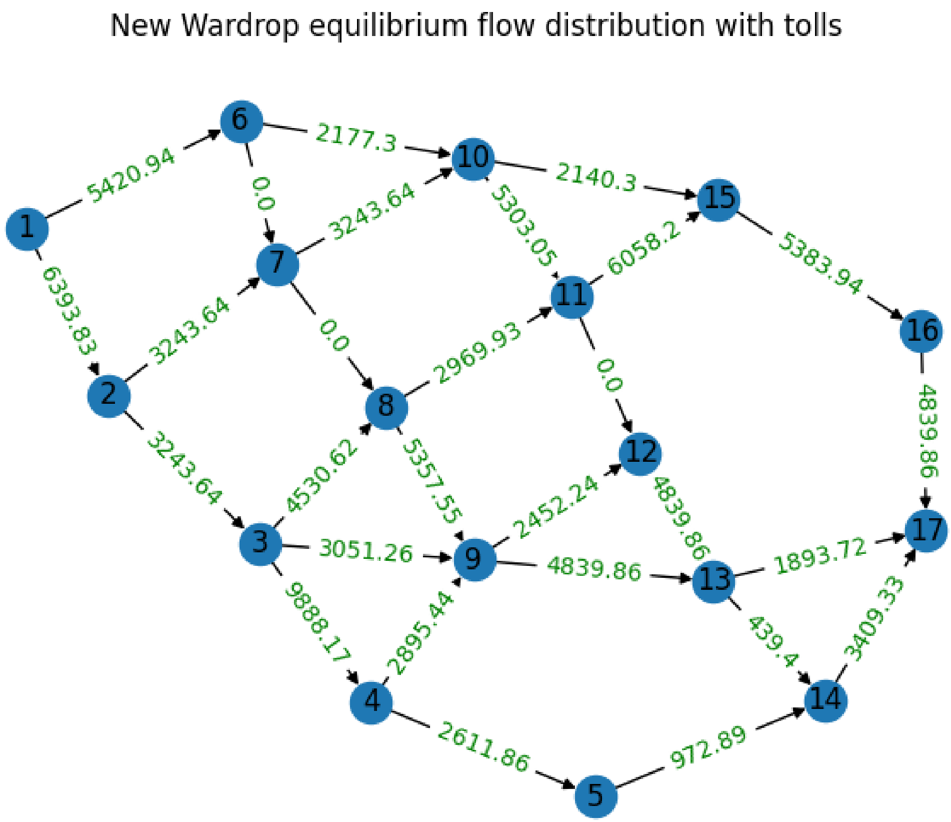
$$f^{(w^*)} = min \sum_e -l_e c_e \ln\left(1 - \frac{f_e}{c_e}\right) - f_e l_e + f_e \omega_e^*$$

[66]:
```
B # Traffic matrix
n_edges #Number of edges of B

# Exogenous flow vector:
# flow entering the origin node is set as v[0], flow exiting the destination␣
↪node is -v[0].
nu = np.zeros(v.shape[0])
nu[0] = v[0]
nu[-1] = -v[0]

# Coefficients for objective function
C # diagonal matrix
l # 1d array
fe = cp.Variable(n_edges) # Variable representing the flow on each edge

t_prime = cp.multiply(cp.multiply(l,C),cp.inv_pos(cp.power((C -␣
↪new_soc_opt_value),2)))
omega = cp.multiply(new_soc_opt_value,t_prime)

function = cp.sum(-cp.multiply(cp.multiply(C,l),cp.log(1-cp.multiply(fe,cp.
↪inv_pos(C)))) - cp.multiply(fe,l) + cp.multiply(fe,omega))
objective = cp.Minimize(function)
constraints = [B @ fe == nu, fe >=0]
prob = cp.Problem(objective, constraints)

# The optimal objective value is returned by `prob.solve()`.
```

```
new_result_w = prob.solve()
print("New Wardrop equilibrium:", np.around(fe.value, decimals=1, out=None))
new_wardrop_f = fe
new_wardrop_cost = sum(new_wardrop_f.value*l/(1 - new_wardrop_f.value/C) -␣
↪new_wardrop_f.value*l)
print("New Wardrop cost: ",np.around (new_wardrop_cost, decimals = 1))
```

```
New Wardrop equilibrium: [6393.8 5420.9 3243.6 3243.6 9888.2 4530.6 3051.3
2611.9 2895.4  972.9
    0.  2177.3    0.  3243.6 5357.6 2969.9 4839.9 2452.2  439.4 1893.7
 3409.3 5303.1 2140.3    0.  6058.2 5383.9 4839.9 4839.9]
New Wardrop cost:  13550.2
```

[67]:
```
nx.draw(G,pos,with_labels=True)
nx.draw_networkx_edge_labels(
    G, pos,
    edge_labels= {e:i for i,e in zip(np.around(new_wardrop_f.value,decimals=2,␣
↪out=None),edges)},
    font_color='green'
)
plt.title("New Wardrop equilibrium flow distribution with tolls")
plt.axis('off')
plt.show()
```



[68]:
```python
# Substraction of the social optimum flow and wardrop equilibrium with tolls
np.around(new_soc_opt_value-new_wardrop_f.value, decimals=2, out=None)
```

[68]:
```
array([ 0.08, -0.14,  0.05,  0.05, -0.08, -0.06,  0.35,  0.24, -0.  ,
        0.22,  0.  , -0.19,  0.  ,  0.05, -0.02, -0.17, -0.04, -0.19,
        0.11,  0.05, -0.06, -0.01, -0.07,  0.  ,  0.06, -0.02, -0.04,
       -0.04])
```

As seen in the graph and the substraction above, the wardrop equilibrium gives a very good aproximation of the new social optimum f^* calculated above.

References

[1] L. Trevisan, *Lecture 16*, Mar. 2011. [Online]. Available: https://theory.stanford.edu/~trevisan/cs261/lecture16.pdf.