Introduction to Numerical Linear Algebra I

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Overview

We assume basic familiarity with linear algebra and will skip much of the preliminary material in this first set of slides.

1. Notation

   So that we (at least try and) use the same name for the same thing...
   Almost everything is real (at least in my lectures...)

2. Matrix operations

   Matrix multiplication is the hardest

3. Basic Matrix Decompositions

   Main purpose: Linear system solution

4. Determinants

5. Exercises

Notation

Vectors

- $\mathbb{R}$ Set of real numbers (scalars)
- $\mathbb{R}^{17}$ Space of column vectors with 17 real elements

$$\mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{17} \end{pmatrix}$$

- Vectors with all zeros and all ones

$$0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$\mathrm{x} = 0$ *All* elements of x are zero

$\mathrm{x} \neq 0$ *At least* one element of x is non-zero

Matrices

- $\mathbb{R}^{7\times 5}$ Space of 7×5 matrices with real elements

$$A = \begin{pmatrix} a_{11} & \cdots & a_{15} \\ \vdots & & \vdots \\ a_{71} & \cdots & a_{75} \end{pmatrix}$$

- Identity matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}$$

Canonical vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Matrix Operations

Transpose

- Transpose of column vector gives row vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

- Transpose of matrix: Columns turn into rows

$$\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}^T = \begin{pmatrix} c_1^T \\ c_2^T \\ c_3^T \end{pmatrix}$$

- Transposing twice gives back the original $\quad (A^T)^T = A$

- $A \in \mathbb{R}^{n \times n}$ is symmetric if $A^T = A$

Scalar Multiplication

- Every matrix element multiplied by same scalar

$$4\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 4\,a_{11} & 4\,a_{12} & 4\,a_{13} \\ 4\,a_{21} & 4\,a_{22} & 4\,a_{23} \end{pmatrix}$$

- Two different zeros

  If $A \in \mathbb{R}^{7\times 5}$ then $\qquad 0 \cdot A = 0_{7\times 5}$

- Subtraction: $-A \stackrel{\text{def}}{=} (-1)\,A$

Exercises

1. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ show

$$\alpha x = 0 \qquad \iff \qquad \alpha = 0 \quad \text{or} \quad x = 0$$

2. For $A \in \mathbb{R}^{m \times n}$ and $\lambda \in \mathbb{R}$ show $\quad (\lambda A)^T = \lambda A^T$

Matrix Addition

- Elements in corresponding positions are added

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} + \mathsf{I}_3 = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$$

- Adding zero changes nothing

  If $\mathsf{A} \in \mathbb{R}^{7\times 5}$ then $\quad \mathsf{A} + 0_{7\times 5} = \mathsf{A}$

- Distributivity can save work

$$\lambda \mathsf{A} + \lambda \mathsf{B} = \lambda (\mathsf{A} + \mathsf{B})$$

Exercises

1. For $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{m \times n}$ show $\quad (\mathsf{A} + \mathsf{B})^T = \mathsf{A}^T + \mathsf{B}^T$
2. For $\mathsf{A} \in \mathbb{R}^{n \times n}$ show that $\mathsf{A} + \mathsf{A}^T$ is symmetric

Inner Product

- Row times column (with same # elements) gives scalar
  If $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$ then
  $$\mathrm{x}^T \mathrm{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{j=1}^{n} x_j y_j$$
- Commutative (for real vectors)
  $$\mathrm{x}^T \mathrm{y} = \mathrm{y}^T \mathrm{x}$$
- Vectors $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$ are orthogonal if $\mathrm{x}^T \mathrm{y} = 0$

Exercises

1. Sum = inner product with all ones vector

   If $x \in \mathbb{R}^n$ then
   $$\sum_{j=1}^{n} x_j = x^T \mathbb{1} = \mathbb{1}^T x$$

2. Given integer $n \geq 1$, represent $n(n+1)/2$ as an inner product

3. Test for zero vector

   For $x \in \mathbb{R}^n$ show $\quad x^T x = 0 \iff x = 0$

Matrix Vector Product

- Matrix times column vector gives column vector

$$A \in \mathbb{R}^{3\times 4} \qquad A = \begin{pmatrix} r_1^T \\ r_2^T \\ r_3^T \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}$$

$x \in \mathbb{R}^4$ (# vector elements = # matrix columns)

- Column vector of inner products

$$Ax = \begin{pmatrix} r_1^T x \\ r_2^T x \\ r_3^T x \end{pmatrix}$$

- Linear combination of columns

$$Ax = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$

Exercises

1. Show that $\mathrm{A}\mathrm{e}_j$ is column j of matrix A
2. Given $\mathrm{A} \in \mathbb{R}^{m \times n}$ and $\mathbb{1} \in \mathbb{R}^n$, what does $\mathrm{A}\mathbb{1}$ do?
3. For $\mathrm{A}, \mathrm{B} \in \mathbb{R}^{m \times n}$ show

$$\mathrm{A} = \mathrm{B} \qquad \iff \qquad \mathrm{A}\mathrm{x} = \mathrm{B}\mathrm{x} \quad \text{for all } \mathrm{x} \in \mathbb{R}^n$$

Matrix Multiplication

- # columns of A = # rows of B
- A is 2×4 and B is 4×3 $\implies$ AB is 2×3
- Rows of A and columns of B

$$A = \begin{pmatrix} r_1^T \\ r_2^T \end{pmatrix} \qquad B = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$$

- Matrix of inner products

$$A\,B = \begin{pmatrix} r_1^T c_1 & r_1^T c_2 & r_1^T c_3 \\ r_2^T c_1 & r_2^T c_2 & r_2^T c_3 \end{pmatrix}$$

Other Views of Matrix Multiplication

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \qquad B = \begin{pmatrix} b_1^T \\ b_2^T \\ b_3^T \\ b_4^T \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$$

- Row of matrix vector products

$$AB = \begin{pmatrix} Ac_1 & Ac_2 & Ac_3 \end{pmatrix}$$

- Sum of outer products

$$AB = a_1 b_1^T + a_2 b_2^T + a_3 b_3^T + a_4 b_4^T$$

Properties

- Multiplication by the identity changes nothing

$$A \in \mathbb{R}^{8\times 11} \qquad I_8\, A = A = A\, I_{11}$$

- Associativity $\quad A\,(BC) = (AB)\,C$
- Distributivity $\quad A\,(B + D) = AB + AD$
- No commutativity $\quad AB \neq BA$
- Transpose of product $\quad (AB)^T = B^T A^T$

Exercises

1. More distributivity

   For $A, B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{n \times k}$ show $\quad (A + B)\,C = AC + BC$

2. Understanding why matrix multiplication is not commutative

   Find simple examples where $AB \neq BA$

3. Understanding the transposition of a product

   Find simple examples where $(AB)^T \neq A^T B^T$

4. For $A \in \mathbb{R}^{m \times n}$ show that AA^T and $A^T A$ are symmetric

Matrix Powers

For $\mathsf{A} \in \mathbb{R}^{n \times n}$ with $\mathsf{A} \neq 0$

$$\begin{aligned} \mathsf{A}^0 &= \mathsf{I}_n \\ \mathsf{A}^k &= \underbrace{\mathsf{A} \cdots \mathsf{A}}_{k} = \mathsf{A}^{k-1}\,\mathsf{A} = \mathsf{A}\,\mathsf{A}^{k-1} \qquad k \geq 1 \end{aligned}$$

Matrix Powers

For $A \in \mathbb{R}^{n \times n}$ with $A \neq 0$

$$
\begin{aligned}
A^0 &= I_n \\
A^k &= \underbrace{A \cdots A}_{k} = A^{k-1} A = A\, A^{k-1} \qquad k \geq 1
\end{aligned}
$$

$A \in \mathbb{R}^{n \times n}$ is

- idempotent (projector) $\quad A^2 = A$
- nilpotent $\quad A^k = 0$ for some integer $k \geq 1$

For any $\alpha \in \mathbb{R}$

$$
\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} \text{ is idempotent} \qquad \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \text{ is nilpotent}
$$

Exercises

1. Which matrix is idempotent and nilpotent?

2. If $A \in \mathbb{R}^{n\times n}$ is idempotent then $I_n - A$ is idempotent and $A\,(I_n - A) = 0$

3. If A and B are idempotent and $A\,B = B\,A$ then $A\,B$ is idempotent

Outer Product

- Column vector times row vector gives matrix

$$\mathrm{x}\,\mathrm{y}^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \\ x_3 y_1 & x_3 y_2 \\ x_4 y_1 & x_4 y_2 \end{pmatrix}$$

- Columns of $\mathrm{x}\,\mathrm{y}^T$ are multiples of x
- Rows of $\mathrm{x}\,\mathrm{y}^T$ are multiples of y^T

Gram Matrix Multiplication

For $\mathsf{A} = \begin{pmatrix} \mathsf{a}_1 & \cdots & \mathsf{a}_n \end{pmatrix} \in \mathbb{R}^{m \times n}$

- Matrix of inner products

$$\mathsf{A}^T \mathsf{A} = \begin{pmatrix} \mathsf{a}_1^T \mathsf{a}_1 & \dots & \mathsf{a}_1^T \mathsf{a}_n \\ \vdots & & \vdots \\ \mathsf{a}_n^T \mathsf{a}_1 & \cdots & \mathsf{a}_n^T \mathsf{a}_n \end{pmatrix} \quad \in \mathbb{R}^{n \times n}$$

- Sum of outer products

$$\mathsf{A}\,\mathsf{A}^T = \mathsf{a}_1\,\mathsf{a}_1^T + \cdots + \mathsf{a}_n\,\mathsf{a}_n^T \quad \in \mathbb{R}^{m \times m}$$

Exercises

1. Write the matrix as an outer product

$$\begin{pmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{pmatrix}$$

2. Identity matrix is sum of outer products of canonical vectors

   Show $\quad \mathrm{I}_n = \mathrm{e}_1 \mathrm{e}_1^T + \cdots + \mathrm{e}_n \mathrm{e}_n^T$

3. Associativity can save work

   For $\mathrm{x} \in \mathbb{R}^6$ and $\mathrm{y}, \mathrm{z} \in \mathbb{R}^4$ compute both sides of

$$(\mathrm{x}\mathrm{y}^T)\,\mathrm{z} = \mathrm{x}\,(\mathrm{y}^T \mathrm{z})$$

4. For $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$ compute $(\mathrm{x}\mathrm{y}^T)^3\,\mathrm{x}$ with inner products and scalar multiplication only

Inverse

- $A \in \mathbb{R}^{n\times n}$ is nonsingular (invertible), if exists A^{-1} with

$$A\,A^{-1} = I_n = A^{-1}\,A$$

- Inverse and transposition interchangeable

$$A^{-T} \stackrel{\text{def}}{=} (A^T)^{-1} = \left(A^{-1}\right)^T$$

- Inverse of product

  For $A, B \in \mathbb{R}^{n\times n}$ nonsingular $\quad (AB)^{-1} = B^{-1}\,A^{-1}$

- Test

  If $x \in \mathbb{R}^n$ with $x \neq 0$ and $Ax = 0$, then A is singular

Exercises

1. If $A \in \mathbb{R}^{n \times n}$ with $A + A^2 = I_n$ then A is nonsingular

2. The inverse of a symmetric matrix is symmetric

3. Sherman-Morrison-Woodbury Formula

   For $A \in \mathbb{R}^{n \times n}$ nonsingular, and $U, V \in \mathbb{R}^{m \times n}$ show:
   If $I + VA^{-1}U^T$ nonsingular then

$$\left(A + U^T V\right)^{-1} = A^{-1} - A^{-1}U^T \left(I + VA^{-1}U^T\right)^{-1} VA^{-1}$$

Orthogonal Matrices

$A \in \mathbb{R}^{n \times n}$ is orthogonal matrix if $A^{-1} = A^T$

$$A^T A = I_n = AA^T$$

The meaning of orthogonality depends on who you are

- Two vectors $x, y \in \mathbb{R}^n$ are orthogonal if $x^T y = 0$
- Nonsingular matrix A is orthogonal if $A^{-1} = A^T$

Columns/rows of orthogonal matrix are orthogonal vectors

Examples of Orthogonal Matrices

- Identity I_n
- Permutation matrices (Identity with columns or rows permuted)

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- 2×2 rotations

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \quad \text{for} \quad c^2 + s^2 = 1$$

Exercises

1. If $A \in \mathbb{R}^{n \times n}$ is orthogonal then A^T is orthogonal
2. If $A, B \in \mathbb{R}^{n \times n}$ are orthogonal then AB is orthogonal
3. If $x \in \mathbb{R}^n$ with $x^T x = 1$ then $I_n - 2\, xx^T$ is orthogonal
4. For $A \in \mathbb{R}^{m \times n}$ and permutation $P \in \mathbb{R}^{n \times n}$, describe AP.
5. Let $A = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \in \mathbb{R}^{n \times n}$ where $A_1 \in \mathbb{R}^{n \times k}$
   If A orthogonal then
   $$A_1^T A_1 = I_k \qquad A_2^T A_2 = I_{n-k} \qquad A_1^T A_2 = 0$$
6. If $A^T A = B^T B$ for A, B nonsingular then exists orthogonal matrix Q so that $B = QA$

Triangular Matrices

Upper triangular matrix $\lnot$

$$T = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ & \ddots & \vdots \\ & & t_{nn} \end{pmatrix}$$

- $\lnot$ matrix T nonsingular if and only if all $t_{jj} \neq 0$
- Diagonal elements of inverse $(T^{-1})_{jj} = 1/t_{jj}$

Lower triangular matrix $\triangle$

$$L = \begin{pmatrix} l_{11} & & \\ \vdots & \ddots & \\ l_{n1} & \cdots & l_{nn} \end{pmatrix}$$

- Transpose: L^T is $\lnot$

Special Triangular Matrices

- Unit triangular: Ones on the diagonal

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}$$

- Strictly triangular: Zeros on the diagonal

$$\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}$$

- Diagonal: $\triangledown$ and $\triangle$

$$\begin{pmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{pmatrix} = \operatorname{diag}\begin{pmatrix} d_{11} & \cdots & d_{nn} \end{pmatrix}$$

Exercises

1. Sum of ◹ matrices is ◹
   If $A, B \in \mathbb{R}^{n \times n}$ is ◹ then $A + B$ is ◹
2. Product of ◹ matrices is ◹
   If $A, B \in \mathbb{R}^{n \times n}$ is ◹ then AB is ◹
   with diagonal elements $(AB)_{jj} = a_{jj} b_{jj}$ for $1 \leq j \leq n$
3. Diagonal matrices commute
   If $A, B \in \mathbb{R}^{n \times n}$ is diagonal then $AB = BA$ is diagonal
4. Let $A = I_n - \alpha\, e_i e_j$ for some $1 \leq i, j \leq n$ and scalar α.
   When is A ◹? When is A nonsingular?
5. If $D \in \mathbb{R}^{n \times n}$ is diagonal and $D = (I + A)^{-1} A$ for some A
   then A is diagonal

Basic Matrix Decompositions

Gaussian Elimination (LU) with partial pivoting

If $\mathrm{A} \in \mathbb{R}^{n \times n}$ nonsingular then

$$\mathrm{P\,A} = \mathrm{L\,U}$$

P is permutation, L is unit ◺, U is ◹

Linear system solution $\mathrm{A\,x} = \mathrm{b}$

1. Factor $\mathrm{P\,A} = \mathrm{L\,U}$ $\{$Expensive part: $\mathcal{O}(n^3)$ flops$\}$
2. Solve $\mathrm{L\,y} = \mathrm{P\,b}$ $\{$◺ system$\}$
3. Solve $\mathrm{U\,x} = \mathrm{y}$ $\{$◹ system$\}$

QR Decomposition

If $A \in \mathbb{R}^{n \times n}$ nonsingular then

$$A = Q\,R$$

Q is orthogonal matrix, R is ◹

Linear system solution $A\,x = b$

1. Factor $A = Q\,R$ $\quad$ {Expensive part: $\mathcal{O}(n^3)$ flops}
2. Multiply $c = Q^T\,b$ $\quad$ {$\mathcal{O}(n^2)$ flops}
3. Solve $R\,x = c$ $\quad$ {◹ system}

Cholesky Decomposition

Symmetric $A \in \mathbb{R}^{n \times n}$ is positive definite (spd) if

$$x^T A x > 0 \qquad \text{for all } x \neq 0$$

If $A \in \mathbb{R}^{n \times n}$ spd then $A = L L^T$ where L is ◺

Linear system solution $A x = b$

1. Factor $A = L L^T$ {Expensive part: $\mathcal{O}(n^3)$ flops}
2. Solve $L y = b$ {◺ system}
3. Solve $L^T x = y$ {◹ system}

Determinants

A Simple Characterization

1. If $T \in \mathbb{R}^{n \times n}$ is ◹ or ◺ then

$$\det(T) = \prod_{j=1}^{n} t_{jj}$$

2. If $A, B \in \mathbb{R}^{n \times n}$ then

$$\det(AB) = \det(A)\,\det(B)$$

Let $A \in \mathbb{R}^{n \times n}$

- Transpose: $\det(A^T) = \det(A)$
- Singularity: $\det(A) \neq 0 \iff A$ nonsingular

Laplace Expansions

Assume $\mathrm{A} \in \mathbb{R}^{n \times n}$

A_{ij} is $(n-1) \times (n-1)$ submatrix of A
obtained by deleting row i and column j of A

- Expansion is along row i

$$\det(\mathrm{A}) = \sum_{k=1}^{n} (-1)^{i+k}\, a_{ik} \det(\mathrm{A}_{ik}) \qquad 1 \le i \le n$$

- Expansion along column j

$$\det(\mathrm{A}) = \sum_{k=1}^{n} (-1)^{k+j}\, a_{kj} \det(\mathrm{A}_{kj}) \qquad 1 \le j \le n$$

Computation

If $\mathsf{A} \in \mathbb{R}^{n \times n}$ nonsingular

1. Factor $\mathsf{P}\,\mathsf{A} = \mathsf{L}\,\mathsf{U}$
2. $\det(\mathsf{A}) = \pm \det(\mathsf{U}) = \pm\, u_{11} \cdots u_{nn}$

Exercises

Assume: $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{n \times n}$

1. Triangular matrices

   If A is unit ◹ or ◺ then $\det(\mathsf{A}) = 1$

   If A is strictly ◹ or ◺ then $\det(\mathsf{A}) = 0$

2. If $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{n \times n}$ then $\det(\mathsf{AB}) = \det(\mathsf{BA})$

3. Scalar multiplication

   $\det(\alpha\,\mathsf{A}) = \alpha^n \det(\mathsf{A})$ for any $\alpha \in \mathbb{R}$

4. If A nonsingular then $\det(\mathsf{A}^{-1}) = 1/\det(\mathsf{A})$

5. If $\mathsf{A} \in \mathbb{R}^{n \times n}$ is orthogonal then $|\det(\mathsf{A})| = 1$