PARUL UNIVERSITY
Faculty of Engineering & Technology
Department of Applied Sciences and Humanities
1ST SEMESTER B.Tech PROGRAMME (CSE,IT)
CALCULUS(03019101BS01)
ACADEMIC YEAR – 2025-26

Unit: 2 Multivariate Calculus

Introduction: In calculus, we study how a function changes when its input variables change. For functions of a single variable, this rate of change is measured using the derivative. However, many real-world problems involve functions of several variables—for example, the temperature at a point (x, y, z) in space, the profit of a company depending on price and demand, or the height of a surface given by $z = f(x, y)$, Such functions are called **multivariable (or multivariate) functions**, and to study them we use the concept of **partial derivatives**.

Functions of two variables:

If for every x and y a unique value $f(x, y)$ is associated, then f is said to be a function of the two independent variables x and y and is denoted by

$$z = f(x, y)$$

Geometrically, in three dimensional xyz-coordinate space represents a surface.

Limit:

The function $f(x, y)$ is said to approach to the **limit** L as (x, y) approaches to (x_0, y_0), If for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x,y) - L| < \epsilon \text{ wherever } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

and we write

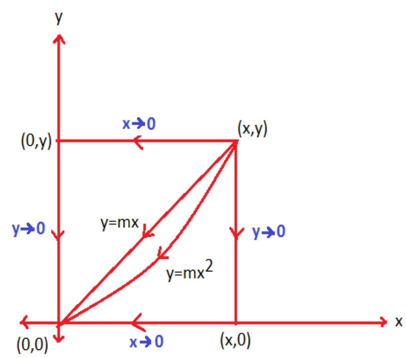
$$\mathbf{lim}_{(x,y)\to((x_0,y_0))} f(x, y) = L.$$

Methods to evaluate limit:

(i) By direct substitution: If example is regarding limit $(x,y) \to (x_0, y_0)$ and on direct substitution we obtain finite value.

(ii) By definition of limit: If in example it is mentioned.

(iii) By path: If example is regarding $(x,y) \to (0,0)$ and on direct substitution we obtain an indeterminate form (mostly $\left(\frac{0}{0}\right)$)



Working rules:

<u>Path 1</u>: Evaluate $\lim\limits_{x\to 0}\left\{\lim\limits_{y\to 0} f(x,y)\right\}$

<u>Path 2</u>: Evaluate $\lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} f(x,y)\right\}$

If limit along path 1 and path 2 are same then proceed further otherwise limit doesn't exist.

<u>Path 3</u>: Evaluate the limit along $y = mx$ as $x \to 0$.

If limit along path 1, path 2 and path 3 are same then proceed further otherwise limit doesn't exist.

<u>Path 4</u>: Evaluate the limit along $y = mx^n$ as $x \to 0$.

If limit along all the paths is same then limit exist.

Examples:

1. Evaluate $\lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5}$ (Example by direct substitution)

Sol:
$$\lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5} = \frac{3(1)^2(2)}{(1)^2+2^2+5}$$
$$= \frac{6}{10} = \frac{3}{5}$$
$$\therefore \lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5} = \frac{3}{5}$$

2. Applying the definition of limit, show that $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$.

Sol. Let $\varepsilon > 0$ be given. We want to find a $\delta > 0$ such that

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$$

Since, $y^2 \leq x^2 + y^2 \Rightarrow \frac{y^2}{x^2+y^2} \leq 1$

$$\Rightarrow \frac{4|x|y^2}{x^2+y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2}$$

Now, $\left|\frac{4xy^2}{x^2+y^2} - 0\right| = \frac{4|x|y^2}{x^2+y^2} \leq 4\sqrt{x^2+y^2} < 4\delta$

By taking, $4\delta = \varepsilon$

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$$

$\therefore \lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$ by definition.

3. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ by Path method. (Summer 2023 – 24)

Sol: By putting $(x,y) \to (0,0)$ we obtain an indeterminate form $\left(\left(\frac{0}{0}\right)\right)$

Therefore, we will apply the Path method to evaluate the limit of a function

Path 1:

$$\lim_{x\to0}\left\{\lim_{y\to0} f(x,y)\right\} = \lim_{x\to0}\left\{\lim_{y\to0}\left(\frac{x-y}{x+y}\right)\right\} = \lim_{x\to0}\left(\frac{x}{x}\right) = 1$$

Path 2:

$$\lim_{y\to0}\left\{\lim_{x\to0} f(x,y)\right\} = \lim_{y\to0}\left\{\lim_{x\to0}\left(\frac{x-y}{x+y}\right)\right\} = \lim_{y\to0}\left(-\frac{y}{y}\right) = (-1) = -1$$

Since both the limits are different, the limit does not exist.

4. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

Sol: By putting $(x, y) \to (0,0)$ we obtain an indeterminate form $\left(\left(\frac{0}{0}\right)\right)$

Therefore, we will apply the Path method to evaluate the limit of a function

<u>Path 1</u>:

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0}\left(\frac{x^3+y^3}{x^2+y^2}\right)\right\} = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right) = \lim_{x\to 0} x \;= 0$$

<u>Path 2</u>:

$$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\left\{\lim_{x\to 0}\left(\frac{x^3+y^3}{x^2+y^2}\right)\right\} = \lim_{y\to 0}\left(\frac{y^3}{y^2}\right) = \lim_{y\to 0} y \;= 0$$

<u>Path 3</u>:

Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^3+(mx)^3}{x^2+(mx)^2}\right) = \lim_{x\to 0}\left(\frac{x^3+m^3x^3}{x^2+m^2x^2}\right) = \lim_{x\to 0} x\left(\frac{1+\mathrm{m}^3}{1+\mathrm{m}^2}\right) = 0$$

<u>Path 4</u>:

Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^3+(mx^2)^3}{x^2+(mx^2)^2}\right) = \lim_{x\to 0}\left(\frac{x^3+m^3x^6}{x^2+m^2x^4}\right) = \lim_{x\to 0} x\left(\frac{1+m^3x^3}{1+m^2x^2}\right) = 0$$

Since, limit along all the paths are same. Hence, the limit exists and its value is 0.

5. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ by Path method.

Sol: By putting $(x, y) \to (0,0)$ we obtain an indeterminate form $\left(\left(\frac{0}{0}\right)\right)$

Therefore, we will apply the Path method to evaluate the limit of a function

<u>Path 1</u>:

$$\lim_{(x,y)\to(0,0)} \left(\frac{2xy}{x^2+y^2}\right) = 0$$

Path 2:

$$\lim_{(x,y)\to(0,0)} \left(\frac{2xy}{x^2+y^2}\right) = 0$$

Path 3:

Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{2x(mx)}{x^2+(mx)^2}\right) = \lim_{x\to 0}\left(\frac{2mx^2}{x^2+m^2x^2}\right) = \frac{2m}{1+m^2}$$

As the limit depends on m and m is not fixed, the limit doesn't exist.

Exercise:

1) Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{3x+y^2}$ [Ans. Limit does not exist

2) Applying the definition of limit, show that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

3) Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ [Ans. Limit does not exist

4) Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ [Ans. 0

5) Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^2+y^2}$ [Ans. 0

Continuity of function of two Variables

A function $f(x, y)$ is continuous at the point (a, b) if it satisfies following properties,

1. $f(x, y)$ is defined at (a, b)
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists
3. $\lim_{(x,y)\to(\mathrm{a,b})} f(x, y) = f(a, b)$.

Examples:

1. Discuss the continuity of

$$\boldsymbol{f(x,y) = \begin{cases} \dfrac{x^2y}{x^3+y^3}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}}$$

At the point $\boldsymbol{(0,0)}$.

Sol: Here, $f(x, y)$ is defined at $(0,0)$ and $f(0,0) = 0$

<u>Path 1</u>:

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2y}{x^3+y^3}\right) = 0$$

<u>Path 2</u>:

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2y}{x^3+y^3}\right) = 0$$

<u>Path 3</u>: Put $y = mx$, as $x \to 0$

$$\lim_{x\to0} \left(\frac{x^2(mx)}{x^3+(mx)^3}\right) = \lim_{x\to0} \left(\frac{mx^3}{x^3+m^3x^3}\right) = \frac{m}{1+m^3}$$

As the limit depends on m and m is not fixed, the limit doesn't exist. Hence, $f(x, y)$ is discontinuous at (0, 0).

2. Discuss the continuity of

$$\boldsymbol{f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}}$$

At the point $(0,0)$. (Summer, Winter 2023 - 24)

Sol: Here, $f(x,y)$ is defined at (0,0) and $f(0,0) = 0$.

Path 1:

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x^2 - y^2}{\sqrt{x^2 + y^2}}\right] = \lim_{x\to 0}\frac{x^2}{x} = x \ = 0$$

Path 2:

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x^2 - y^2}{\sqrt{x^2 + y^2}}\right] = \lim_{y\to 0}\left(-\frac{y^2}{y}\right) = \lim_{y\to 0}(-y) \ = 0$$

Path 3: Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^2 - (mx)^2}{\sqrt{x^2 + (mx)^2}}\right) = \lim_{x\to 0}\left(\frac{x^2(1 - m^2)}{x\sqrt{1 + m^2}}\right) = 0$$

Path 4: Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^2 - (mx^2)^2}{\sqrt{x^2 + (mx^2)^2}}\right) = \lim_{x\to 0}\left(\frac{x^2(1 - m^2x^2)}{x\sqrt{1 + m^2x^2}}\right) = 0$$

Since, limit along all the paths are same.

Therefore, limit exists.

Also, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0 = f(0,0)$

Hence, $f(x,y)$ is continuous at (0, 0).

3. Check whether the given function is continuous at origin or not, if yes then find point of continuity.

$$\boldsymbol{f(x,y) = \begin{cases} \dfrac{x+y}{\sqrt{x}-\sqrt{y}}, & (x,y) \neq (0,0) \\ -1 & (x,y) = (0,0) \end{cases}}$$

Sol: Here, $f(x,y)$ is defined at $(0,0)$ and $f(0,0) = -1$

<u>Path 1</u>:

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x+y}{\sqrt{x}-\sqrt{y}}\right] = \lim_{x\to 0}\frac{x}{\sqrt{x}} = \sqrt{x} = 0$$

<u>Path 2</u>:

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x+y}{\sqrt{x}-\sqrt{y}}\right] = \lim_{y\to 0}\left(\frac{y}{-\sqrt{y}}\right) = \lim_{y\to 0}(-\sqrt{y}) = 0$$

<u>Path 3</u>: Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x+y}{\sqrt{x}-\sqrt{y}}\right) = \lim_{x\to 0}\left(\frac{x+mx}{\sqrt{x}-\sqrt{mx}}\right) = \lim_{x\to 0}\sqrt{x}\left(\frac{1+m}{1-\sqrt{m}}\right) = 0$$

<u>Path 4</u>: Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x+y}{\sqrt{x}-\sqrt{y}}\right) = \lim_{x\to 0}\left(\frac{x+mx^2}{\sqrt{x}-\sqrt{mx^2}}\right) = \lim_{x\to 0}\left(\frac{x+mx^2}{\sqrt{x}-\sqrt{m}x}\right) = \lim_{x\to 0}\frac{x}{\sqrt{x}}\left(\frac{1+mx}{1-\sqrt{mx}}\right) = 0$$

Since, limit along all the paths are same.

Therefore, limit exists.

But, $\lim\limits_{(x,y)\to(0,0)}\frac{x+y}{\sqrt{x}-\sqrt{y}} = 0 \neq -1 = f(0,0)$

Hence, $f(x,y)$ is discontinuous at (0, 0).

Exercise:

1) Discuss the continuity of

$$\boldsymbol{f(x,y)} = \begin{cases} \dfrac{x}{3x+5y}, & \boldsymbol{(x,y) \neq (0,0)} \\ \boldsymbol{1} & \boldsymbol{(x,y) = (0,0)} \end{cases}$$

at the point (0, 0). [Ans. Not continuous

2) Show that

$$\boldsymbol{f(x,y)} = \begin{cases} \dfrac{x^2y}{y^2+x^2}, & \boldsymbol{(x,y) \neq (0,0)} \\ \boldsymbol{1} & \boldsymbol{(x,y) = (0,0)} \end{cases}$$

at the origin. [Ans. Continuous

3) Discuss the continuity of

$$\boldsymbol{f(x,y)} = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & \boldsymbol{(x,y) \neq (0,0)} \\ \boldsymbol{0} & \boldsymbol{(x,y) = (0,0)} \end{cases}$$

at the point (0, 0). [Ans. Continuous

4) **Discuss the continuity of**

$$\boldsymbol{f(x,y)} = \begin{cases} \boldsymbol{\dfrac{x^2-y^2}{x^2+y^2}}, & \boldsymbol{(x,y) \neq (0,0)} \\ \boldsymbol{0} & \boldsymbol{(x,y) = (0,0)} \end{cases}$$

At the point $\boldsymbol{(0,0)}$. [Ans. Not continuous

Partial Derivatives:

The partial derivative of $f(x, y)$ with respect to x at the point (a, b) is

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}, provided\ the\ limit\ exists.$$

The partial derivative of $f(x, y)$ with respect to y at the point (a, b) is

$$\left(\frac{\partial f}{\partial y}\right)_{(a,b)} = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}, provided\ the\ limit\ exists.$$

Second order partial derivative:

Two successive partial differentiations of $z = f(x, y)$ with respect to x (holding y constant) is denoted by $\frac{\partial^2 f}{\partial x^2} or f_{xx}(x, y)$.That is, we define

$$f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

Similarly, two successive partial differentiations of $f(x, y)$ with respect to y (holding x constant) is denoted by $\frac{\partial^2 f}{\partial y^2} or f_{yy}(x, y)$. That is, we define

$$f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

We use $\frac{\partial^2 f}{\partial x \partial y}$ to mean differentiate first with respect to y then with respect to x, and we use $\frac{\partial^2 f}{\partial y \partial x}$ to mean differentiate first with respect to x then with respect to y.

Notation: $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s$ and $\frac{\partial^2 z}{\partial y^2} = t$

Remark: The crossed or mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are in general, equal

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

i.e., the order of differentiation is immaterial if the derivatives involved are continuous.

1. **Find $\frac{\partial f}{\partial z}$ at (1,2,3) for $f(x, y, z) = \boldsymbol{x^2yz^2}$ using the definition. Also find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at same point.**

Sol: Here, $\frac{\partial f}{\partial z} = \lim\limits_{h\to 0} \frac{f(x_0,y_0,z_0+h)-f(x_0,y_0,z_0)}{h}$

$$\left(\frac{\partial f}{\partial z}\right)_{(1,2,3)} = \lim_{h\to 0} \frac{f(1,2,3+h)-f(1,2,3)}{h}$$

$$= \lim_{h\to 0} \frac{2(9+6h+h^2)-18}{h}$$

$$= \lim_{h\to 0}(12+2h)$$

$$= 12$$

2. **If $\boldsymbol{f(x,y) = x^3 + y^3 - 2xy^2}$. Find all second order partial derivatives of $\boldsymbol{f(x,y)}$ at $\boldsymbol{(1,-1)}$**

Sol: Here, $f_x(x,y) = 3x^2 - 2y^2, f_y(x,y) = 3y^2 - 4xy.$

$f_{xx}(x,y) = 6x$, $f_{yy}(x,y) = 6y - 4x$, $f_{xy}(x,y) = -4y$, $f_{yx}(x,y) = -4y$

At the point (1, -1)

$$f_{xx}(1,-1) = 6$$

$$f_{yy}(1,-1) = -10$$

$$f_{xy}(1,-1) = 4$$

$$f_{yx}(1,-1) = 4$$

3. **If $\boldsymbol{u = log(tanx + tany + tanz)}$, then show that**

$$\boldsymbol{sin2x\frac{\partial u}{\partial x} + sin2y\frac{\partial u}{\partial y} + sin2z\frac{\partial u}{\partial z} = 2}$$

Sol: Given that $u = log\ (tan\ x + tan\ y + tan\ z)$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = \frac{1}{tan\ x + tan\ y + tan\ z}\ sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{tan\ x + tan\ y + tan\ z}\ .sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{tan\ x + tan\ y + tan\ z}.sec^2 z$$

Hence,

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{2 sinx cosx sec^2 x + 2 siny cosy sec^2 y + 2 sinz cosz sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z \ \ )}{\tan x + \tan y + \tan z} = 2$$

4. If $u(x,y,z) = e^{3xyz}$ **show that** $\frac{\partial^3 u}{\partial x \partial y \partial z} = (3 + 27xyz + 27x^2y^2z^2)e^{3xyz}$

Sol: Given that $u(x,y,z) = e^{3xyz}$

Differentiating u w.r.t z,

$$\frac{\partial u}{\partial z} = 3xye^{3xyz}$$

Differentiating $\frac{\partial u}{\partial z}$ w.r.t y,

$$\frac{\partial^2 u}{\partial y \partial z} = 3x \frac{\partial}{\partial y}(ye^{3xyz})$$

$$= 3x(e^{3xyz}.1 + ye^{3xyz}.3xz)$$

$$= e^{3xyz}(3x + 9x^2yz)$$

Differentiating $\frac{\partial^2 u}{\partial y \partial z}$ w.r.t x,

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}[[\ e^{3xyz}(3x + 9x^2yz)]$$

$$= e^{3xyz}(3 + 18xyz) + (3x + 9x^2yz).e^{3xyz}.3yz$$

$$= e^{3xyz}(3 + 18xyz + 9xyz + 27x^2y^2z^2)$$

$$= e^{3xyz}(3 + 27xyz + 27x^2y^2z^2)$$

4. If $u(x,y,z) = \ln(x^3 + y^3 + z^3 - 3xyz)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$.

Sol: Given that $u(x,y,z) = \ln(x^3 + y^3 + z^3 - 3xyz)$

Differentiating u with respect to x partially

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{(x^3+y^3+z^3-3xyz)}$$

Similarly, $\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{(x^3+y^3+z^3-3xyz)}$ and $\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{(x^3+y^3+z^3-3xyz)}$

Now

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 - 3yz}{(x^3+y^3+z^3-3xyz)} + \frac{3y^2 - 3xz}{(x^3+y^3+z^3-3xyz)} + \frac{3z^2 - 3xy}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}$$

$$= \frac{3}{x+y+z}.$$

Exercise:

1) Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at point (4, -5) if $f(x,y) = x^2 + 3xy + y - 1$. [Ans. -7, 13

2) If $z = x + y^x$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

3) If $= (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$.

4) If $z = \log(e^x + e^y)$, prove that $rt - s^2 = 0$.

Homogeneous function:

A function $f(x,y)$ is said to be homogeneous in which the power of each term is the same.

A function $f(x,y)$ is a homogeneous function of degree$n$, if degree of each of its term is n.

Thus $f(x,y) = a_0x^n + a_1x^{n-1}y + \cdots\cdots + a_ny^n$ is the homogeneous function of degree n.

Note:

1. If the function $u = f(x,y)$ is a homogeneous function of degree 'n' in x and y then it can be written as $u = x^n\emptyset\left(\frac{y}{x}\right)$ or $u = y^n\emptyset\left(\frac{x}{y}\right)$.

2. For a homogeneous function f of degree n; $f(tx,ty) = t^nf(x,y)$

Euler's theorem for homogeneous functions:

Statement: If u is a homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof: Let $u = f(x,y)$ be a homogeneous function of degree 'n' in x and y, then it can be written as $u = x^n\emptyset\left(\frac{y}{x}\right)$ ___________(1)

Differentiate (1) partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = x^n\frac{\partial}{\partial x}\left[\emptyset\left(\frac{y}{x}\right)\right] + \emptyset\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(x^n)$$

$$= x^n\emptyset'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) + \emptyset\left(\frac{y}{x}\right)(nx^{n-1})$$

$$= x^n\emptyset'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \emptyset\left(\frac{y}{x}\right)(nx^{n-1})$$

$$= -yx^{n-2}\emptyset'\left(\frac{y}{x}\right) + nx^{n-1}\emptyset\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = -yx^{n-1}\emptyset'\left(\frac{y}{x}\right) + nx^n\emptyset\left(\frac{y}{x}\right) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

Differentiate (1) partially w.r.t y, we get

$$\frac{\partial u}{\partial y} = x^n \frac{\partial}{\partial y}\left[\emptyset\left(\frac{y}{x}\right)\right]$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^{n-1}\emptyset'\left(\frac{y}{x}\right)$$

$$\Rightarrow y\frac{\partial u}{\partial y} = yx^n\emptyset'\left(\frac{y}{x}\right) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3)}$$

Adding (2) and (3) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -yx^{n-1}\emptyset'\left(\frac{y}{x}\right) + nx^n\emptyset\left(\frac{y}{x}\right) + yx^n\emptyset'\left(\frac{y}{x}\right) = nx^n\emptyset\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Note: If u is a homogeneous function of three variables x, y and z of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

Cor.1. If u is a homogeneous function of degree n in x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Cor.2. If $\emptyset(u) = f(x, y)$ is a homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{\phi(u)}{\phi'(u)}$$

Cor.3. If $\emptyset(u) = f(x, y)$ is a homogeneous function of degree n in x and $y, then$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1], \text{Where, } g(u) = n\frac{\phi(u)}{\phi'(u)}.$$

Note: Corollary 2 and 3 are also known as Modified Euler's theorem of first and second order respectively.

Examples:

1. If $u = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx - logy}{x^2+y^2}$, prove that (Summer 2023 – 24)

$$(i)\ x\frac{\partial u}{\partial x} + \ y\frac{\partial u}{\partial y} = -\ 2u$$

$$(ii) x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + \ y^2\frac{\partial^2 u}{\partial y^2} = 6u$$

Sol: Let $u = f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{log\frac{x}{y}}{x^2+y^2}$

Replacing x by tx and y by ty,

$$f(tx,ty) = \frac{1}{t^2x^2} + \frac{1}{txty} + \frac{log\frac{tx}{ty}}{t^2x^2+t^2y^2}$$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{log\frac{x}{y}}{x^2+y^2}\right] = t^{-2}f(x,y)$$

Hence, u is a homogeneous function of degree -2

By Euler's Theorem,

(i)
$$x\frac{\partial u}{\partial x} + \ y\frac{\partial u}{\partial y} = \ nu$$

$$\Rightarrow x\frac{\partial u}{\partial x} + \ y\frac{\partial u}{\partial y} = -\ 2u$$

(ii)
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + \ y^2\frac{\partial^2 u}{\partial y^2} = \ n(n-1)u$$

$$i.e\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + \ y^2\frac{\partial^2 u}{\partial y^2} = \ -2(-2-1)u = 6u$$

2. If $u = \ tan^{-1}(\frac{x^2+y^2}{x+y})$, prove that

$$(i) x\frac{\partial u}{\partial x} + \ y\frac{\partial u}{\partial y} = \frac{sin2u}{2}$$

$$(ii) x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + \ y^2\frac{\partial^2 u}{\partial y^2} \ = -2sin^3 ucosu.$$

Sol: Let $u = \ tan^{-1}(\frac{x^2+y^2}{x+y})$, then $\tan u = \ \frac{x^2+y^2}{x+y}$,

Replacing x by tx and y by ty, we get

$$\tan u = t\left(\frac{x^2+y^2}{x+y}\right)$$

Thus, $\tan u = \left(\frac{x^2+y^2}{x+y}\right)$ is a homogeneous function of degree 1.

Let $f(u) = \tan u$, by Modified Euler's Theorem,

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.\frac{tanu}{sec^2u} = sinu\,cosu = \frac{sin2u}{2}$$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where, $g(u) = n\frac{f(u)}{f'(u)} = 1.\frac{tanu}{sec^2u} = \frac{sin2u}{2}$ and $g'(u) = cos2u$

Therefore

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{sin2u}{2}(cos2u - 1)$$

$$= \sin u\cos u\,(\cos 2u - 1)$$

$$= \sin u\cos u\,(-2\sin^2 u)$$

$$= -2sin^3ucosu$$

Exercise:

1) If $u = y^2e^{\frac{y}{x}} + x^2\,log\left(\frac{x}{y}\right)$, show that

$$(i)x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

$$(ii)x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$$

2) If $u = \log\frac{x^2+y^2}{x-y}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

3) If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that **(Winter 2023 - 24)**

$$(i)x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}tanu$$

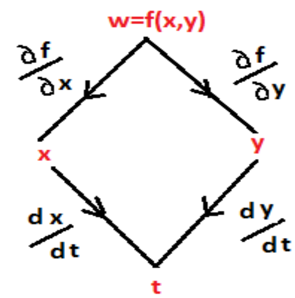
$$(ii)x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}(tan^3u - tanu)$$

4) If $u = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then find **(Summer 2023 – 24)**

$(i) x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ $\quad$ $(ii) x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ $\quad$ [Ans. $\sin 2u$. $2\cos 3u \sin u$

Chain rule for the function of two independent variables:

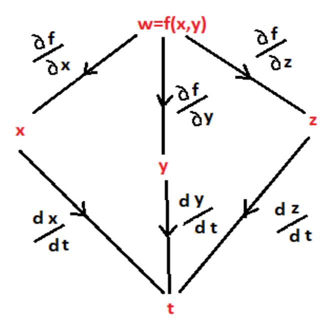
If $w = f(x, y)$ has continuous partial derivative f_x, f_y and if $x = x(t),\ \ y = y(t)$ are differentiable function of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and $\quad \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$.



Chain rule for the function of three independent variables:

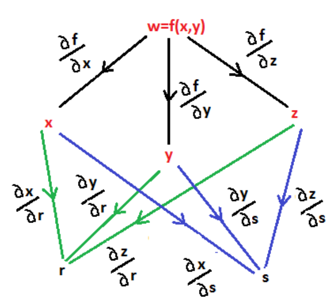
If $w = f(x, y, z)$ is differentiable and x, y *and* z are differentiable function of t then w is a differentiable function of t and

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$



Chain Rule for the function of two independent variables and three intermediate variables

Suppose that $w = f(x, y, z), x = g(r, s), y = h(r, s)$ *and* $z = k(r, s)$. If all four functions are differentiable then w



has partial derivative with respect to r and s given by the formulas

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Examples:

1. Let $z=x^2y^3$, where $x=t^2$ and $y=t$, then verify chain rule by expressing z in terms of t.

(Summer 2023 – 24)

Sol: Here, the chain rule is $\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$

$$=(2xy^3)(2t)+(3x^2y^2)(1)$$

$$=(2t^2t^3)(2t)+(3t^4t^2)$$

$$=4t^6+3t^6$$

$$=7t^6 - - - - - - - -(1)$$

Also

$z=x^2y^3$ then $z=(t^4)(t^3)=t^7$

and $\frac{dz}{dt}=7t^6 - - - - - - - -(2)$

Hence, from (1) and (2), the chain rule is verified.

2. If $u=xy^2+yz^3, x=logt, y=e^t, z=t^2$ find $\frac{du}{dt}$ at $t=1$.

Sol: $u=xy^2+yz^3, x=logt, y=e^t, z=t^2$

By Chain rule

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$=(y^2)\frac{1}{t}+(2xy+z^3)e^t+(3yz^2)2t$$

Substituting x ,y and z,

$$\frac{du}{dt} = 2(e^{2t})\frac{1}{t} + \left(2(logt) \quad e^t + t^6\right)e^t + 3e^t t^4 . 2t$$

Substituting $t = 1$,

$$\frac{du}{dt} = 2e^2 + e^1 + 6e^1$$

$$= 2e^2 + 7e^1$$

3. If $u = f(x - y, y - z, z - x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol: Let $x - y = l, y - z = m, z - x = n$, then $u = f(l, m, n)$ and

$\frac{\partial l}{\partial x} = 1, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -1, \quad \frac{\partial l}{\partial y} = -1, \quad \frac{\partial m}{\partial y} = 1, \quad \frac{\partial n}{\partial y} = 0, \quad \frac{\partial l}{\partial z} = 0,$

$\frac{\partial m}{\partial z} = -1, \quad \frac{\partial n}{\partial z} = 1$

By chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}.1 + \frac{\partial u}{\partial m}.0 + \frac{\partial u}{\partial n}.-1$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n} \quad --------(1)$$

Also

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}.-1 + \frac{\partial u}{\partial m}.1 + \frac{\partial u}{\partial n}.0$$

$$= -\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m} --------(2)$$

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}.0 + \frac{\partial u}{\partial m}.-1 + \frac{\partial u}{\partial n}.1$$

$$= -\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n} -----------(3)$$

Adding Eq.(1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Exercise:

1) If $u=f(x^2+2yz, y^2+2xz)$, prove that $(y^2-xz)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}=0$.

2) If $u=f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

3) If $u=f(r)$, where $r^2=x^2+y^2$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$.

Derivative of implicit functions:

Suppose that $f(x,y)$ is differentiable and that the equation $f(x,y)=0$ defines y as a differentiable function of x. Then at any point where $f_y\neq 0$,

$$\frac{dy}{dx}=-\frac{f_x}{f_y}$$

Examples:

1. If $ysinx=xcosy$, find $\frac{dy}{dx}$. (Summer 2023 – 24)

Sol: Let $f(x,y)=\ ysinx-xcosy$, then

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$=\ -\frac{y\,cosx-cosy}{sinx+x\,siny}$$

$$=\frac{cosy-y\,cosx}{sinx+xsiny}$$

2. If $(cosx)^y = (siny)^x$, find $\frac{dy}{dx}$.

Sol: Given that $(cosx)^y = (siny)^x$

Taking log on both the sides,

$$y \log(cosx) = x \log(siny)$$

Let $f(x, y) = y \log(cosx) - x \log(siny)$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{\frac{y}{cosx}(-sinx) - \log(siny)}{\log(cosx) - \frac{x}{siny}(cosy)}$$

$$= \frac{y\ tanx + \log(siny)}{\log(cosx) - xcoty}$$

Exercise:

1) If $x^3 + y^3 + 3xy = 1$ then, find $\frac{dy}{dx}$. [Ans. $-\frac{x^2+y}{y^2+x}$

2) If $x^y + y^x = c\ then,$ find $\frac{dy}{dx}$. [Ans. $-\frac{x^y\frac{y}{x}+y^x \ln y}{x^y \ln x+y^x\frac{x}{y}}$

Jacobian:

If u and v are continuous and differentiable functions of two independent variables x and y, then the Jacobian of u, v with respect to x, y and is defined by

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarly, if u, v and w are continuous and differentiable functions of three independent variables x, y and z, then the Jacobian of u, v, w with respect to x, y, z and is defined by

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobian:

1. If u and v are functions of x and y, then $J.J' = 1$, where $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$.

2. If u, v are functions of r, s and r, s are in turn functions of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} . \frac{\partial(r,s)}{\partial(x,y)}$.

Examples:

1. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u = x^2 - y^2, v = 2xy$**.**

Sol: Given that $u = x^2 - y^2,\ v = 2xy$

Now, $\frac{\partial u}{\partial x} = 2x,\ \frac{\partial v}{\partial x} = 2y, \frac{\partial u}{\partial y} = -2y$ and $\frac{\partial v}{\partial y} = 2x$. The Jacobian is

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

2. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u = x - y, v = x + y$. Also, verify that $J.J' = 1$.

Sol: Given that $u = x - y,\ v = x + y$, then

$$\frac{\partial u}{\partial x} = 1,\ \frac{\partial v}{\partial x} = 1,\ \frac{\partial u}{\partial y} = -1,\quad \frac{\partial v}{\partial y} = 1$$

Now,
$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

Now, $x = \frac{u+v}{2}, y = \frac{v-u}{2}$, then

$$\frac{\partial x}{\partial u} = \frac{1}{2},\ \frac{\partial y}{\partial u} = -\frac{1}{2},\ \frac{\partial x}{\partial v} = \frac{1}{2} \text{ and } \frac{\partial y}{\partial v} = \frac{1}{2}$$

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Therefore
$$JJ' = 1.$$

Exercise:

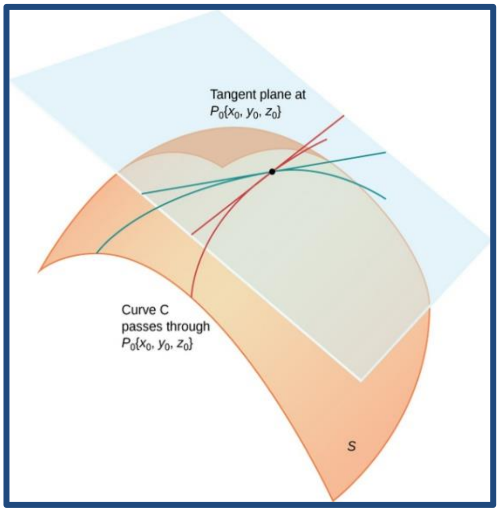
1) Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u = xsiny, v = ysinx$.

2) If $u = 2xy,\ v = x^2 - y^2$ and $x = rcos\theta,\ y = rsin\theta$ then, evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$.

APPLICATIONS OF PARTIAL DERIVATIVES

Tangent Plane and Normal Line

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.



For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on surface $f(x, y, z) = 0$ of the differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$. It is given by

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$. It is given by

$$\frac{x - x_0}{f_x(P_0)} = \frac{y - y_0}{f_y(P_0)} = \frac{z - z_0}{f_z(P_0)}$$

Examples:

1. Find the equation of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 3$ at the point (1,1,1). (Summer 2023-24)

Sol: Let $f(x,y,z) = x^2 + y^2 + z^2 - 3$, then

$f_x(x,y,z) = 2x$, $f_x(1,1,1) = 2$, $f_y(x,y,z) = 2y$, $f_y(1,1,1) = 2$, $f_z(x,y,z) = 2z$ and $f_z(1,1,1) = 2$

Hence, the equation of the tangent plane at (1,1,1) is

$$(x-1)2 + (y-1)2 + (z-1)2 = 0$$

$$\Rightarrow x + y + z = 3$$

The equation of normal line is $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$.

2. Find the equation of the tangent plane and normal line to the surface $z + 8 = xe^y \cos z$ at the point (8,0,0).

Sol: Let $f(x,y,z) = xe^y \cos z - z - 8$, then

$$f_x(x,y,z) = e^y cosz, \qquad f_x(8,0,0) = 1$$

$$f_y(x,y,z) = xe^y cosz, \qquad f_y(8,0,0) = 8$$

$$f_z(x,y,z) = sinz - 1, \qquad f_z(8,0,0) = -1$$

Hence, the equation of the tangent plane at (8,0,0) is

$$1(x-8) + 8(y-0) - 1(z-0) = 0$$

$$\Rightarrow x - 8 + 8y - z = 0$$

$$\Rightarrow x + 8y - z - 8 = 0$$

The equation of normal line is $\frac{x-8}{1} = \frac{y-0}{8} = \frac{z-0}{-1}$

Exercise:

1) Find the equations of tangent plane and normal line to the surface $z = x^2 + 3y^2 - 4$ at $(1,1,0)$.

2) Find the equations of tangent plane and normal line to the surface $2x^2 + y^2 + 2z = 3$ at $(2,1,-3)$. **(Winter 2023-24)**

Local Maximum and Local Minimum

Let $f(x,y)$ be defined on a region R containing the point (a,b), then

1. $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).
2. $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

Second Derivative Test for Local Extreme Values

Suppose that $f(x,y)$ and its first and second partial derivative is continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Then, for

$$r = \frac{\partial^2 f}{\partial x^2},\ s = \frac{\partial^2 f}{\partial x \partial y},\ t = \frac{\partial^2 f}{\partial y^2}$$

(i) f has a local maximum at (a,b) if $r < 0$ and $rt - s^2 > 0$ at (a,b)
(ii) f has a local minimum at (a,b) if $r > 0$ and $rt - s^2 > 0$ at (a,b)
(iii) f has a saddle point at (a,b) if $rt - s^2 < 0$ at (a,b)
(iv) The test has no conclusion at (a,b) if $rt - s^2 = 0$ at (a,b).

Examples:

1. Find the extreme value of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (Summer 2023-24)

Sol: Let $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$, then

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \text{ and } \frac{\partial f}{\partial y} = 6xy - 30y$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, this implies that

$3x^2 + 3y^2 - 30x + 72 = 0$ and $6xy - 30y = 0$

$\Rightarrow x^2 + y^2 - 10x + 24 = 0$ and $6y(x-5) = 0$

$\Rightarrow x^2 + y^2 - 10x + 24 = 0$ and $y = 0, x = 5$

When $y = 0$, we have $x^2 - 10x + 24 = 0$

$$x = 4, 6$$

and when $x = 5$, we have $25 + y^2 - 50 + 24 = 0$

$$y = \pm 1$$

Therefore, the stationary points are $(4,0), (6,0), (5,1), (5,-1)$.

Now, $r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$, $s = \frac{\partial^2 f}{\partial x \partial y} = 6y$, $t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$

(x,y)	r	s	t	$rt - s^2$	Conclusion	$f(x,y)$
$(4,0)$	$-6 < 0$	0	-6	$36 > 0$	Maximum	112
$(6,0)$	$6 > 0$	0	6	$36 > 0$	Minimum	108
$(5,1)$	0	6	0	$-36 < 0$	Saddle Point	--
$(5,-1)$	0	-6	0	$-36 < 0$	Saddle Point	--

2. Find the points on the surface $z^2 = x^2 + y^2$ that are closed to P (1, 1, 0).

Sol: Let $A(x, y, z)$ be any point on the surface, then by distance formula, the distance d

Between A and P is given by $d = \sqrt{(x-1)^2 + (y-1)^2 + z^2}$

$$\Rightarrow d^2 = (x-1)^2 + (y-1)^2 + z^2$$

$$= 2x^2 + 2y^2 - 2x - 2y + 2 = f \text{ (say)}$$

Now, $f_x(x,y) = 4x - 2, \quad f_y(x,y) = 4y - 2$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, this implies that

$$4x - 2 = 0, \quad 4y - 2 = 0$$

$$\Rightarrow x = \frac{1}{2}, \quad y = \frac{1}{2}$$

Thus, $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a stationary point. Now,

$$r = \frac{\partial^2 f}{\partial x^2} = 4, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0, \quad t = \frac{\partial^2 f}{\partial y^2} = 4$$

So, $\quad rt - s^2 = (4)(4) - 0 = 16 >$

Hence, function is minimum at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the minimum value of given surface

$z^2 = x^2 + y^2$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$$z^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So, $z = \pm\frac{1}{2}$.

<u>Exercise:</u>

1) Discuss the maxima and minima of the function $f(x,y) = x^2 + y^2 + 6x + 12$. **(Winter 2023-24)** [Ans. $f_{min} = 3\ at\ (-3,0)$

2) Find the extreme values of the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$.

[Ans. $f_{max} = 7$ at (0,0) $f_{min} = 3$ at (2,0)

3) Discuss the maxima and minima of the function $f(x,y) = x^3 + y^3 - 3axy$.

[Ans. Min if $a > 0$ and Max if $a < 0$.

4) Discuss the maxima and minima of the function $f(x, y) = x^3y^2(1 - x - y)$

[Ans. Max at $\left(\frac{1}{2}, \frac{1}{3}\right)$

MAXIMA AND MINIMA WITH CONSTRAINED VARIABLES

The Method of Lagrange Multipliers:

The method of Lagrange multipliers allows us to maximize or minimize function with a constraint.

Let $f(x, y, z)$ be the given function subject to the constraint $\emptyset(x, y, z) = 0$__________(1)

Solve by following Steps:

1. Construct an equation $f(x, y, z) + \lambda\emptyset(x, y, z) = 0$_________(2)

where, λ is a variable called Lagrange multiplier.

2. Differentiate Eq. (2) partially w.r.t x, y, z to obtain

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 0 \text{\_\_\_\_\_\_\_\_\_\_\_\_\_(3)}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 0 \text{\_\_\_\_\_\_\_\_\_\_\_\_\_(4)}$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z} = 0 \text{\_\_\_\_\_\_\_\_\_\_\_\_\_(5)}$$

3. Solve and eliminate λ from Eqs. $(1), (3), (4), (5)$ to obtain the stationary points (x, y, z).

4. Substitute the stationary points (x, y, z) into f to see where, f attains its maximum and mimum values.

<u>Note:</u> For function of two independent variables, the condition is similar, but without the variable z.

Examples:

1. Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse

$\frac{x^2}{8} + \frac{y^2}{2} = 1.$

Sol: Given that $f(x,y) = xy$.

Let $\emptyset(x,y,z) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

Let the equation be $xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right) = 0$__________(2)

Differentiating Eq. (2) partially w. r. t. x, we get

$$y + \frac{\lambda x}{4} = 0 \text{ \_\_\_\_\_\_\_\_\_\_(3)}$$

Differentiating Eq. (2) partially w. r. t. y, we get

$$x + \lambda y = 0 \text{ \_\_\_\_\_\_\_\_\_\_(4)}$$

From eqns. (3) and (4),

$$\frac{-4y}{x} = \frac{-x}{y}$$

$$\Rightarrow 4y^2 = x^2$$

Substituting x^2 in eqn. (1),

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1$$

$$\Rightarrow y^2 = 1$$

$$\Rightarrow y = \pm 1$$

$$\Rightarrow x = \pm 2$$

Therefore, the function $f(x,y) = xy$ takes extreme values on the ellipse at four points $(2,1)$, $(-2,1), (-2,-1), (2,-1)$.

The maximum value is $xy = 2$ and minimum value is $xy = -2$.

2. Find the maximum value of $x^2y^3z^4$, subject to the condition $x+y+z=5$. (Winter 2023-24)

Sol: Let $f(x,y,z)=x^2y^3z^4$ and $\emptyset(x,y,z)=x+y+z-5=0$__________(1)

Let the equation be $x^2y^3z^4+\lambda(x+y+z-5)=0$__________(2)

Differentiating eqn. (2) partially w. r. t. x, we get

$$2xy^3z^4+\lambda=0$$

$$\Rightarrow \lambda=-2xy^3z^4 \quad \text{\_\_\_\_\_\_\_\_\_\_(3)}$$

Differentiating eqn. (2) partially w.r.t y,

$$3x^2y^2z^4+\lambda=0$$

$$\Rightarrow \lambda=-3x^2y^2z^4 \quad \text{\_\_\_\_\_\_\_\_\_\_(4)}$$

Differentiating eqn. (2) partially w.r.t z,

$$4x^2y^3z^3+\lambda=0$$

$$\Rightarrow \lambda=-4x^2y^3z^3 \quad \text{\_\_\_\_\_\_\_\_\_\_(5)}$$

From eqns. (3), (4) and (5),

$$-2xy^3z^4=-3x^2y^2z^4=-4x^2y^3z^3$$

$$\Rightarrow 2yz=3xz=4xy$$

$$\Rightarrow y=\tfrac{3}{2}x \text{ and } z=2x$$

Substituting y and z in eqn. (1), we get

$$x+\frac{3}{2}x+2x=5$$

$$\Rightarrow 9x=10$$

$$\Rightarrow x=\frac{10}{9}$$

$$y=\frac{3}{2}\left(\frac{10}{9}\right)=\frac{5}{3}$$

$$z=2\left(\frac{10}{9}\right)=\frac{20}{9}$$

Maximum value of $x^2y^3z^4$ is $\left(\frac{10}{9}\right)^2\left(\frac{5}{3}\right)^3\left(\frac{20}{9}\right)^4=\frac{(2^{10})(5^9)}{3^{15}}$.

Exercise:

1) Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange's multipliers.

2) Find the minimum value of $x^2 + y^2 + z^2$, subject to the condition $xyz = a^3$. [Ans. $3a^2$

Taylor's series for $f(x, y)$

If $f(x, y)$ and all its derivatives are finite and continuous at all points of (x, y), then the taylor series near the point (a, b) is

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

At the point $(0, 0)$

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right)$$
$$+\frac{1}{3!}\left(x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right) + \cdots$$

This equation is known as Maclaurin's series.

Note: For quadratic expansion, find the Taylor's series up to second degree terms and for cubic expansion, find the Taylor's series up to third degree terms.

Examples:

1. Expand $x^2y + 3y - 2$ in powers of $(x - 1)$and $(y + 2)$ up to second degree terms.

Sol: Let $f(x, y) = x^2y + 3y - 2$. By Taylor's Expansion,

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + \quad 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \cdots$$

Here, $a = 1, b = -2$

$f(x,y) = x^2y + 3y - 2,$ $\qquad f(1,-2) = (1)^2(-2) + 3(-2) - 2 = -10$

$f_x(x,y) = 2xy,$ $\qquad f_x(1,-2) = 2(1)(-2) = -4$

$f_y(x,y) = x^2 + 3,$ $\quad$ $f_y(1,-2) = (1)^2 + 3 = 4$

$f_{xx}(x,y) = 2y,$ $\quad$ $f_{xx}(1,-2) = 2(-2) = -4$

$f_{xy}(x,y) = 2x,$ $\quad$ $f_{xy}(1,-2) = 2(1) = 2$

$f_{yy}(x,y) = 0,$ $\quad$ $f_{yy}(1,-2) = 0$

Substituting these values in Taylor's Expansion,

$$f(x,y) = -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \ldots$$

$$\Rightarrow x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + \cdots$$

2. Expand $e^x log(1+y)$in powers of x and y upto third degree.

Sol: Let $f(x,y) = e^x log(1+y)$. By Maclaurin's series,

$$f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[(x)^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + (y)^2 f_{yy}(0,0)]$$

$$+ \frac{1}{3!}[(x)^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) 3xy^2 f_{xyy}(0,0) + (y)^3 f_{yyy}(0,0)] + \ldots$$

$f(x,y) = e^x \, log\, log\,(1+y)\,,$ $\quad$ $f(0,0) = 0$

$f_x(x,y) = e^x \, log\, log\,(1+y)\,,$ $\quad$ $f_x(0,0) = 0$

$f_y(x,y) = \dfrac{e^x}{1+y},$ $\quad$ $f_y(0,0) = 1$

$f_{xx}(x,y) = e^x \, log\, log\,(1+y)\,,$ $\quad$ $f_{xx}(0,0) = 0$

$f_{xy}(x,y) = \dfrac{e^x}{1+y},$ $\quad$ $f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -\dfrac{e^x}{(1+y)^2},$ $\quad$ $f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = e^x \, log\, log\,(1+y)\,,$ $\quad$ $f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = \dfrac{e^x}{1+y},$ $\quad$ $f_{xxy}(0,0) = 1$

$$f_{xyy}(x,y) = -\frac{e^x}{(1+y)^2}, \qquad f_{xxy}(0,0) = -1$$

$$f_{yyy}(x,y) = \frac{2e^x}{(1+y)^3}, \qquad f_{yyy}(0,0) = 2$$

Substituting these values in Taylor's series,

$$f(x,y) = 0 + [x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(-1)]$$

$$+\frac{1}{3!}[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)] + \cdots$$

$$e^x\, log\, log\,(1+y) \ = y + \frac{1}{2!}(2xy - y^2) + \frac{1}{3!}(3x^2y - 3xy^2 + 2y^3) + \cdots$$

$$= y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} - \cdots$$

Exercise:

1) Expand $x^2 + xy + y^2$in powers of $(x-1)$and $(y-2)$ upto second- degree terms.

2) Expand $e^x cosy$ in powers of x and y upto third degree.