PARUL UNIVERSITY
FACULTY OF ENGINEERING & TECHNOLOGY
Department of Applied Science & Humanities
Third Semester B. Tech (CSE, IT)
Discrete Mathematics (303191202)
UNIT-2 Principles of Mathematical Induction Lecture Notes
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WELL ORDERING PRINCIPLE:
The well-ordering property states that: Every nonempty set of nonnegative integers has a least element.

Question: Write down the well ordering principle. **[Winter 2023 – 24]**

RECURSION:
Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in termsof itself. This process is called **Recursion**.

RECURSIVELY DEFINED FUNCTIONS:
We use two steps to define a function with the set of nonnegative integers as its domain:

Basis Step: Specify the value of the function at zero.

Recursive Step: Give a rule for finding its value at an integer from its values at smaller integers. Such adefinition is called a **recursive** or **inductive.**

Example-1:
Suppose that f is defined recursively by

$$f(0)=3,$$

$$f(n+1)=2f(n)+3$$

Find $f(1), f(2), f(3) \ \& \ f(4)$. **[Winter 2021 – 22]**

Solution:
From the recursive definition it follows that

$$f(1)=2f(0)+3=2*3+3=9$$

$$f(2)=2f(1)+3=2*9+3=21$$

$$f(3)=2f(2)+3=2*21+3=45$$

$$f(4)=2f(3)+3=2*45+3=93$$

Example-2:
Give a recursive definition of a^n, where a is a nonzero real number and n is a non-negativeinteger.
Solution:
The recursive definition contains two parts:

First a^0 is specified, namely, $a^0=1$. Then the rule for finding a^{n+1} from a^n, namely,

$a^{n+1}=a^n \cdot a^1$, for n = 0, 1, 2, 3, . . . , is given. These two equations uniquely define a^n for all nonnegative integers n.

PRINCIPLE OF MATHEMATICAL INDUCTION:

Let S(n) be any statement which is to be proved that it is true for all n∈ Z^+, we must follow three steps forit.

Step 1: Prove for $n = n_0$, the result is true where, n_0 is initial value. this step is called basic/ initial step.Step 2: Assume for $n = k$, the result S(k) is true.

Step 3: Prove for $n = k + 1$ the result is true.

This step is called inductive step. If all steps are valid, then S(n) is true in general, for n∈Z^+.

Example-3:

Prove using mathematical induction,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 +, \ldots\ldots + n\,(n + 1)\,(n + 2) = \frac{n\,(n+1)(n+2)(n+3)}{4}$$

Solution:

Let S(n): $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 +, \ldots\ldots + n\,(n + 1)\,(n + 2) = \frac{n\,(n+1)(n+2)(n+3)}{4}$

Step 1: From n = 1,

S(1): $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 +, \ldots \ldots + 1(1+1)(1+2) = 1(2)(3) = 6 = L.H.S.$

R.H.S. = $\frac{1\,(1+1)(1+2)(1+3)}{4} = \frac{1(2)(3)(4)}{4} = 6$

$\therefore$ L. H. S. = R. H. S.

S(n) is true for n = 1.

Step 2: Assume that S(n) is true for n = k

S(k): $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 +, \ldots\ldots + k\,(k + 1)\,(k + 2) = \frac{k\,(k+1)(k+2)(k+3)}{4}$ …………..(1)

Step 3: We have to Prove that P(k+1) is also whenever P(k) is true.

P(k+1): $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 +, \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \frac{k\,(k+1)(k+2)(k+3)(k+4)}{4}$

From equation (1) we have L.H.S.= $\frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$

$= (k+1)(k+2)(k+3)(\frac{1}{4}k + 1) = (\frac{1}{4}k + 1)(k+2)(k+3)(k+4)$

= R.H.S.

$\therefore$ L. H. S. = R. H. S.

Therefore S(n) is true for all $n = k + 1$, Thus by principle of mathematical induction S(n) is true for all n∈ z^+

Exercise-1

Using mathematical induction method, Prove the following:

1. $1 + 2 + 3 +, \ldots\ldots + n = \frac{n(n+1)}{2}$ **[Winter 2022 - 23]**
2. $1^2 + 2^2 + 3^2 +, \ldots\ldots, +(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ **[Winter 2017 – 18]**
3. $1 + 3 + 3^2 + 3^3 + 3^4 +, \ldots\ldots\ldots, + 3^{n-1} = \frac{(3^n - 1)}{2}$ **[Summer 2017 – 18]**

THE DIVISION ALGORITHM

When an integer is divided by a positive integer, there is a quotient and a remainder, as the division algorithm shows.

Let a be an integer and d a positive integer. Then there are unique integers q and r, with $\mathbf{0 \le r < d}$, such that, $\boldsymbol{a = dq + r}$.
Where, d is called the divisor,
a is called the dividend,
q is called the quotient,
and r is called the remainder.
This notation is used to express the quotient and remainder:

$$\boldsymbol{q = a \text{ div } d,\ r = a \bmod d}$$

Example-4:

What are the quotient and remainder when 101 is divided by 11?
Solution:
We have $101 = 11 * 9 + 2$
Hence, the quotient when 101 is divided by 11 is 9 = 101 div 11, and the remainder is 2 =101 mod 11.

Example-5:
What are the quotient and remainder when - 11 is divided by 3?
Solution:
We have $-11 = 3*(-4) + 1$
Hence, the quotient when - 11 is divided by 3 is -4 = -11 div 3, and the remainder is 1 = -11mod 3.
Note that the remainder cannot be negative. Consequently, the remainder is not -2, even though $-11 = 3(-3) - 2$ because r = -2 does not satisfy $0 \le r < 3$.

PRIMES:
A positive integer p greater than 1 is called prime if the only positive factors of p are 1 and p.

COMPOSITE:
A positive integer that is greater than 1 and is not prime is called composite.

Remark: The integer n is composite if and only if there exists an integer a such that a / n and $1 < a < n$.

For example: The integer 7 is prime because its only positive factors are 1 and 7, whereas the integer 9 is composite because it is divisible by 3.

THE FUNDAMENTAL THEOREM OF ARITHMETIC:

Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of non-decreasing size.

Example-6:
The prime factorizations of 100, 641, 999, and 1024 are given by
Solution:

$100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$

$641 = 641$

$999 = 3 \times 3 \times 3 \times 37 = 3^3 \times 37$

$1024 = 2^{10}$

Remark: If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$.

Example-7:

Show that 101 is prime.

Solution: The only primes not exceeding $\sqrt{101}$ are 2, 3, 5, and 7. Because 101 is not divisible by 2, 3, 5, or 7 (the quotient of 101 and each of these integers is not an integer), it follows that 101 is prime.

GREATEST COMMON DIVISORS

Let a and b be integers, not both zero. The largest integer d such that $d/a \& d/b$ is called the greatest common divisor of a and b. The greatest common divisor of a and b is denoted by $\gcd(a,b)$.

Example-8:

What is the greatest common divisor of 24 and 36?

Solution:

The positive common divisors of 24 and 36 are 1, 2, 3, 4, 6, and 12. Hence, $\gcd(24, 36) = 12$.

Example-9:

What is the greatest common divisor of 17 and 22?

Solution:

The integers 1 7 and 22 have no positive common divisors other than 1. So that $\gcd(17, 22) = 1$.

EUCLIDEAN ALGORITHM:

We will give a more efficient method of finding the greatest common divisor, called the Euclidean algorithm. This algorithm has been known since ancient times.

Divide:

Start with two numbers, let *a* and *b*, where *a* is greater than or equal to *b*

Divide *a* by *b*, to get a quotient *q* and a remainder *r*, $\mathbf{0 \le r < b}$,

$$\boldsymbol{a = bq + r}$$

Swap and Repeat: Replace *a* with *b* and *b* with *r*. This step essentially swaps the numbers and reduces the problem size.

Now, the new equation is: $\boldsymbol{b = rq' + r'}$

So, in essence, the Euclidean algorithm involves repeatedly replacing the larger number by its remainder when divided by the smaller number, until one of the numbers becomes zero.

Example-10:

Find gcd (91,287) using Euclidean algorithm.

Solution:

First, divide 287, the larger of the two integers, by 91, the smaller, 287 = 91 x 3 + 14.

Any divisor of 91 and 287 must also be a divisor of 287 - 91 x 3 = 14.

Also, any divisor of 91 and 14 must also be a divisor of 287 = 91 x 3 + 14.

Hence, the greatest common divisor of 91 and 287 is the same as the greatest common divisor of 91 and 14.

This means that the problem of finding gcd (91, 287) has been reduced to the problem of finding gcd (91, 14).
Next, divide 91 by 14 to obtain 91 = 14 x 6 + 7.
Because any common divisor of 91 and 14 also divides 91 -14 x 6 = 7 and any common divisor of 14 and 7 divides 91.
Therefore gcd(91 ,14) = gcd(14, 7). Continue by dividing 14 by 7, to obtain 14 = 7 x 2.
Because 7 divides 14, it follows that gcd(14, 7) = 7.
Furthermore, because gcd(287, 91) = gcd(91 , 14) = gcd(14, 7) = 7.

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Example-11:
Find the greatest common divisor of 414 and 662 using the Euclidean algorithm. **[Winter 2022-23]**
Solution:
Successive uses of the division algorithm give:662 = 414 x 1+248
414 = 248 x 1 + 166
248 = 166 x 1 + 82
166 = 82 x 2 + 2
82 = 2 x 41.
Hence, gcd(414, 662) = 2, because 2 is the last nonzero remainder.

Exercise-2:
1. Use Euclidean algorithm to find the HCF of 4052 and 12576. **[Winter 2019 – 20]**
2. Using Euclidean algorithm find g.c.d (735,85) **[Winter 2023 – 24]**
3. Using Euclidean algorithm find gcd(252,105) **[Winter 2022 - 23]**

BASIC COUNTING PRINCIPLES:

Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example-12:
A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?
Solution:
The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are 12 x 11 = 132 ways to assign offices to these two employees. ...

Example-13:
The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
Solution:
The procedure of labeling a chair consists of two tasks, namely, assigning one of the 26 letters andthen assigning one of the 100 possible integers to the seat. The product rule shows that there are 26 x 100 = 2600 different ways that a chair can be labeled. Therefore, the largest number of chairs thatcan be labeled differently is 2600.

Example-14:
There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?
Solution:
The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are 32 x 24 = 768 ports.

Example-15:
Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Solution:
There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choosea student who is a mathematics major. Choosing a member of the mathematics faculty is never thesame as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

Example-16:
A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
Solution:
The student can choose a project by selecting a project from the first list, the second list, or the third list. Because n project is on more than one list, by the sum rule there are 23 + 15 + 19 = 57 ways to choose a project.

INCLUSION-EXCLUSION PRINCIPLE:

To correctly count the number of ways to do the two tasks, we add the number of ways to do it in one wayand the number of ways to do it in the other way, and then subtract the number of ways to do the task in a way that is both among the set of n_1 ways and the set of n_2 ways. This technique is called the principle of **inclusion-exclusion**. Sometimes, it is also called the **subtraction principle of counting.**

Example-17:
How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
Solution:
We can construct a bit string of length eight that start with 1 in 2^7 =128

This follows by the product rule, because the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways.

Next, we construct bit string of length eight that end with 00 in 2^6 =64 ways.

This follows by the product rule, because the first 6 bits can be chosen in two ways and the last two bits can be chosen in only one way.

We construct a bit string of length eight that start with 1 and end with 00 in 1 and end with 00 in 2^5 =32 ways. which equals the number of ways to construct a bit string of length eight that begin with a 1 or that ends with 00, equals 128 + 64 - 32 = 160 ways.

Example-18:
A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these people majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business? **[Winter 2019 – 20]**
Solution:
To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants.
Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business.
Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the set of students who majored both in computer science and in business. By the principle of inclusion-exclusion, the number of students who majored either in computer science or in business (or both) equals
$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$= 220 + 147 - 51 = 316.
We conclude that 350 - 316 = 34 of the applicants majored neither in computer science nor in business

THE PIGEONHOLE PRINCIPLE:
If k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects. The pigeonhole principle i s also called the **Dirichlet Drawer Principle**.

Question: State: Pigeonhole Principle. **[Winter 2019 – 20]**

GENERALIZED PIGEONHOLE PRINCIPLE: If n pigeonholes are occupied by kn + 1 or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k + 1 or more pigeons.

Remark: A function *f* from a set with k + 1 or more elements to a set with k elements is not one-to-one.

For example,
- Among any group of 367 people, there must be at least two with the same birthday, because thereare only 366 possible birthdays.
- In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

PERMUTATIONS:
A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an **r-permutation**.

$$P(n,r) = \frac{n!}{(n-r)!}$$

COMBINATIONS:
A **Combination** of these n objects taken r at a time is any selection of r of the objects where order does not count. In other words, **r-combination** of a set of n objects is any subset of r elements.

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Remarks:
1. $P(n\,,n) = n!$
2. $0! = 1$
3. $C(n,n) = 1$

4. $C(n,r) = C(n, n-r)$

5. If n and r are positive integers, where $n \geq r$, then $C(n, r-1) + C(n, r) = C(n+1, r)$

6. Circular Permutation: The number of different circular arrangements of n objects = (n-1) !

In general,

Product rule: If an activity can be performed in r-successive steps and step 1 can be done in n_1 ways, step 2 can be done in n_2 ways,…. Step r can be done in n_r ways then the activity can be done in ($n_1, n_2, n_3, \ldots n_r$) ways.

Sum rule: If r activities can be performed in $n_1, n_2, n_3, \ldots n_r$ ways and if they are disjoint , i.e cannot be performed simultaneously , then any one of the r-activities can be performed in ($n_1+n_2+n_3+\ldots+n_r$) ways.

Example-19:
How many bit strings of length n contain exactly r 1's?

Solution:

The positions of r 1's in a bit string of length n form an r-combination of the set { 1 , 2 , 3 , . . . , n}.

Hence, there are C (n, r) bit strings of length n that contain exactly r 1's.

Example-20:
Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution:
By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem the number of ways to select the committee is

$$C(9,3) * C(11,4) = \frac{9!}{3! * 6!} * \frac{11!}{4! * 7!} = 84 * 330 = 27720$$

Example-21:
Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution:
The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily. Consequently, there are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways for the saleswoman to choose her tour. If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path, she must consider a total of 5040 paths.

Example-22:
How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:
Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is $P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200$.

Example-23:
A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?
Solution:
The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. By Theorem 2, the number of such combinations is $C(30, 6) = 30!\ 6!\ 24! = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25\ 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 593{,}775$.

Example-24:
Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department? **[Winter 2019 – 20]**
Solution:
By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is $C(9, 3) \cdot C(11, 4) = 9!\ 3!6! \cdot 11!\ 4!7! = 84 \cdot 330 = 27{,}720$.