★ Pumping Lemma for RL (RL or NRL?)

For any RL, L $\exists$ an integer n (is dependent on L)
s.t. $\forall\, z \in L$ and $|z| \geq n$

Based on proof by contradiction

i) $z = uvw$

ii) $|uv| \leq |z|$

iii) $|v| \geq 1$ or $|v| \neq 0$

then $uv^iw \in L$ $\forall\, i \geq 0$ u, v, w, z are words built using Σ

Examples

1) $L = \{a^m b^m \mid m \geq 1\}$

Let assume L is a RL for some n

$z \in L$

$z = a^k b^k$; choose k s.t. $|a^k b^k| = 2k \geq n$

Now, we have to divide this ($a^k b^k = z$) (once we get to know value of n we can choose k) as;

$z = a^k b^k$

u (a^{k-1}) v a w b^k

$\Rightarrow$ $z = uvw$

$|uv| \Rightarrow k \leq 2k \Rightarrow |uv| \leq |z|$

&

$|v| = 1 \geq 1$

As the above satisfies all the condition

uv^iw must $\in L$ $\forall\, i \geq 0$

For $i = 2$ $uv^2w \Rightarrow a^{k-1}a^2b^k = a^{k+1}b^k \Rightarrow$ but no. of a's & b's must be $=$

$\hookrightarrow \notin L$

Hence, given L is NRL

2) $L = \{ a^m b^p \mid m > p \}$

Let L is a RL

$z \in L$

As m is alway greater than p, then

Let $z = a^{k+1} b^k$; choose k s.t. $|z| \geq n$

$2k+1 > n$

Now, $z = a^{k+1} b^k$

u v w

a^{k+1} b b^{k-1}

Let us check whether it satisfies all three condiⁿ or not.

$z = uvw$ — (i) satisfied

$|uv| = k+2 \leq 2k+1 \Rightarrow |uv| \leq |z|$ — (ii) satisfied

$|v| = 1 \neq 0 \Rightarrow |v| \geq 1$

As it satisfies all the condiⁿ,

$uv^i w \in L \;\forall\; i \geq 0$, but,

for $i = 2 \Rightarrow uv^2 w = a^{k+1} b^2 b^{k-1}$

$= a^{k+1} b^{k+1} \notin L$ must be

Here $k+1 = k+1$ but # a's > # b's

Hence L is NRL

3) $L = \{ a^{3^n} \mid n \geq 1 \}$

Let $z \in L$

a^{3^n-1} a ϵ

$|uv| \leq |z|$

$|v| \neq 0$

Thus, it must satisfies $uv^i w \in L$ $i \geq 0$

For $i = 2$ $\boxed{}$

$uv^2 w = a^{3^n-1} a^2 \epsilon$

$= a^{3^n+1} \notin L$

Hence L is NRL

* Special Case of pumping lemma when $\Sigma = \{a\}$
L singleton

→ Lengths of the string must follow Arithmetic Progression for a Lang. to be a RL.

Proof: $uv^iw \in L$ let us take $u = a$, $v = aa$, $w = a$

then aa, aaaa, aaaaaa, ...
i=0, 1, 2 ...

len / #a's 2, 4, 6, ...
follows AP

Examples

1) $L = \{a^{3n} \mid n \geq 1\}$

$L = \{a^3, a^6, a^9, a^{12}, \ldots\}$

3, 6, 9, 12, ... ⇒ AP
⇓
L is RL

2) $L = \{a^{2n+1} \mid n \geq 0\}$

$L = \{a, a^3, a^5, a^7, \ldots\}$

Length of string $\in L$ is 1, 3, 5, 7, ... which is an AP
⇓
L is RL

3) $L = \{a^{n^2} \mid n \geq 0\}$

For $n \geq 0$, length of strings;

0, 1, 4, 9, ... which is not an AP
⇓
L is NRL

4) $L = \{a^p \mid p \text{ is prime}\}$

Length of strings corresponding to L;

$2, 3, 5, 7, 11, \ldots \neq AP \Rightarrow L$ is NRL