Parul® University

Artificial Intelligence

CHAPTER-5

Fuzzy Sets and Fuzzy Logic

Fuzzy Logic

- Fuzzy logic is the super set of Boolean logic.
- It is a form of many valued logic, where the values are real number lies between o and 1.
- It is used to handle the partial truth which varies between completely true and completely false.
- Furthermore , when linguistic language is used (very, some, little) , these degrees may be managed by specific degree membership function.
- These methods are used in control systems like AC, train , washing machine,etc.
- The concept of fuzzy logic is extensively used in the fields of business, defense, finance, aerospace.

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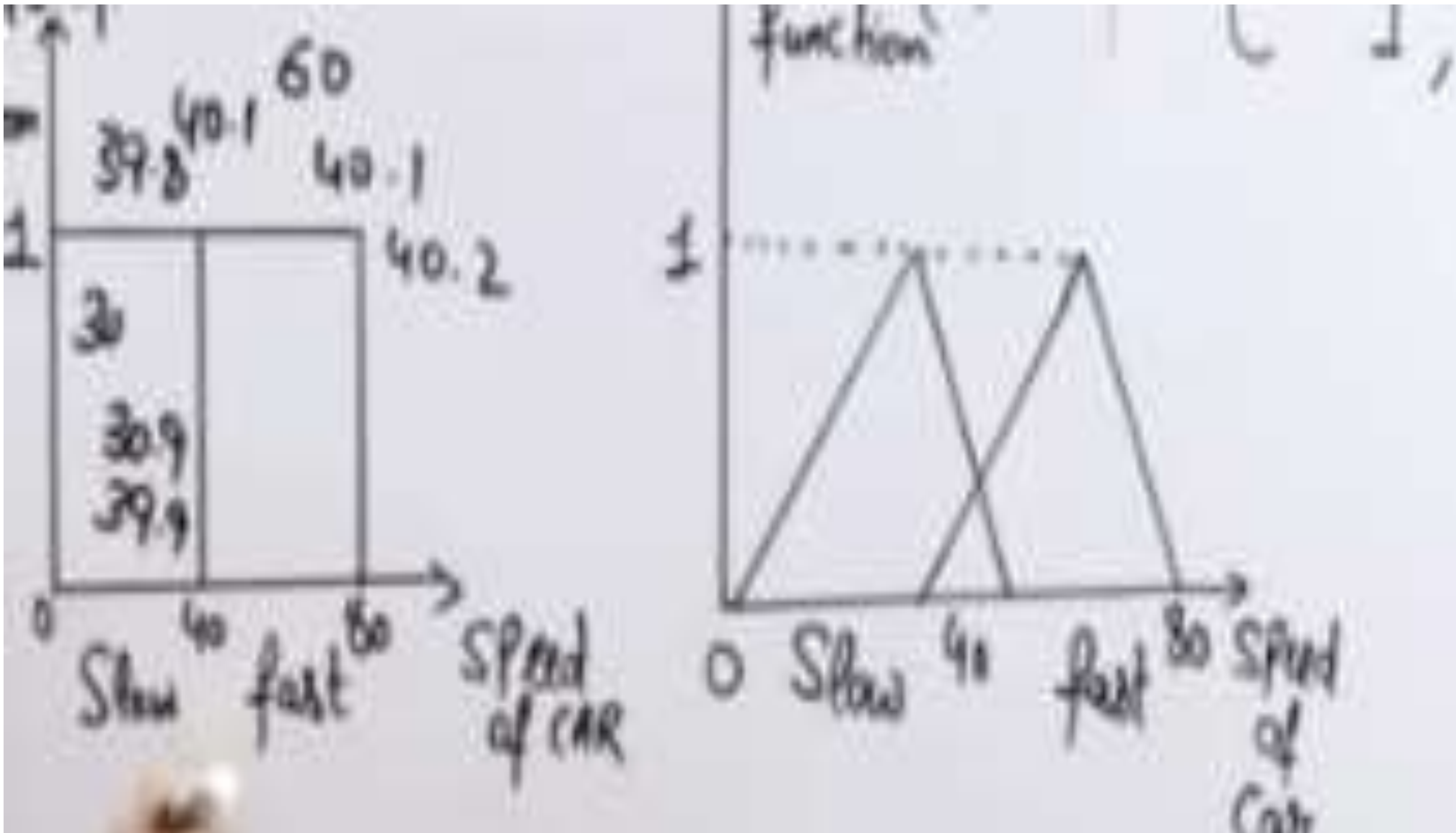
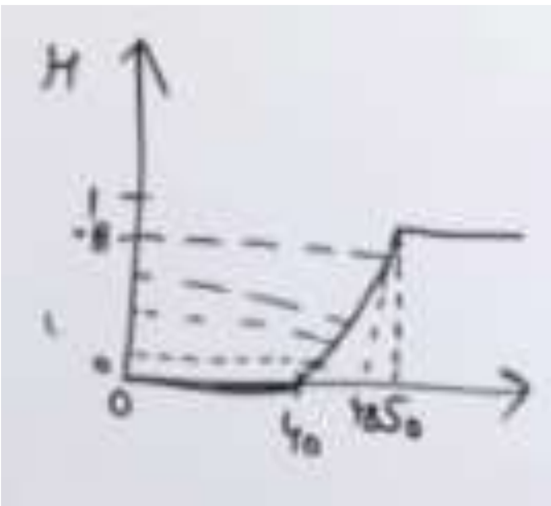




'Checks the degree of fastness'

$$\mu = \begin{cases} 0, & \text{if Speed}(x) \le 40 \\ \dfrac{\text{Speed}(x) - 40}{10}, & \text{if } 40 < \text{Speed}(x) < 50 \\ 1, & \text{if Speed}(x) \ge 50 \end{cases}$$

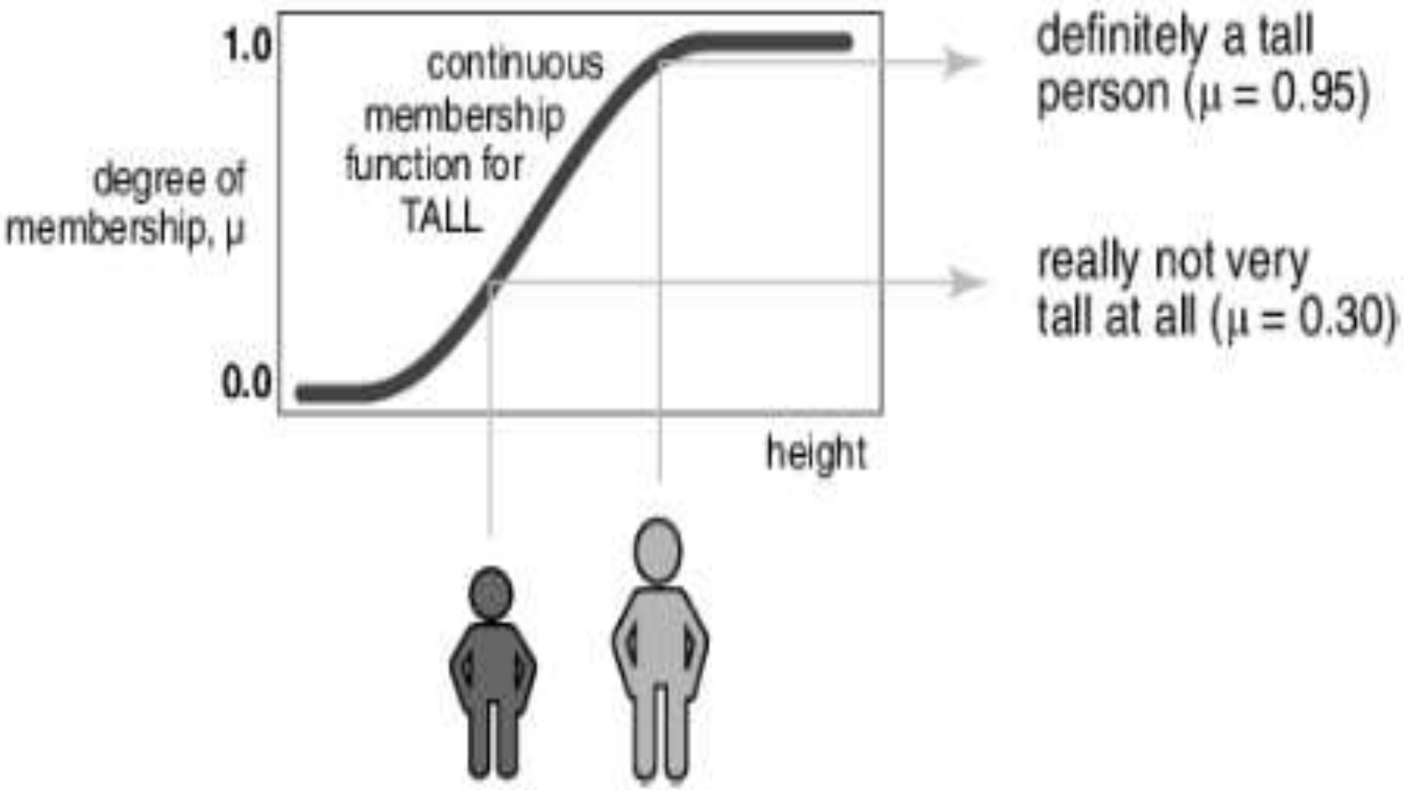
$x = 30 \quad (30, 0)$

$x = 60 \quad (60, 1)$

$x = 42 \quad \frac{42-40}{10} = \frac{2}{10} = \frac{1}{5} = .2$

$x = 45 \quad \frac{45-40}{10} = \frac{5}{10} = \frac{1}{2} = .5$

$x = 48 \quad \frac{48-40}{10} = \frac{8}{10} = .8$


1.0
0.0
degree of
membership, μ
continuous
membership
function for
TALL
height
definitely a tall
person (μ = 0.95)
really not very
tall at all (μ = 0.30)

Basic Concepts: Several fuzzy sets representing linguistic concepts such as low, medium, high, and so one are often employed to define states of a variable. Such a variable is usually called a fuzzy variable.

For example:

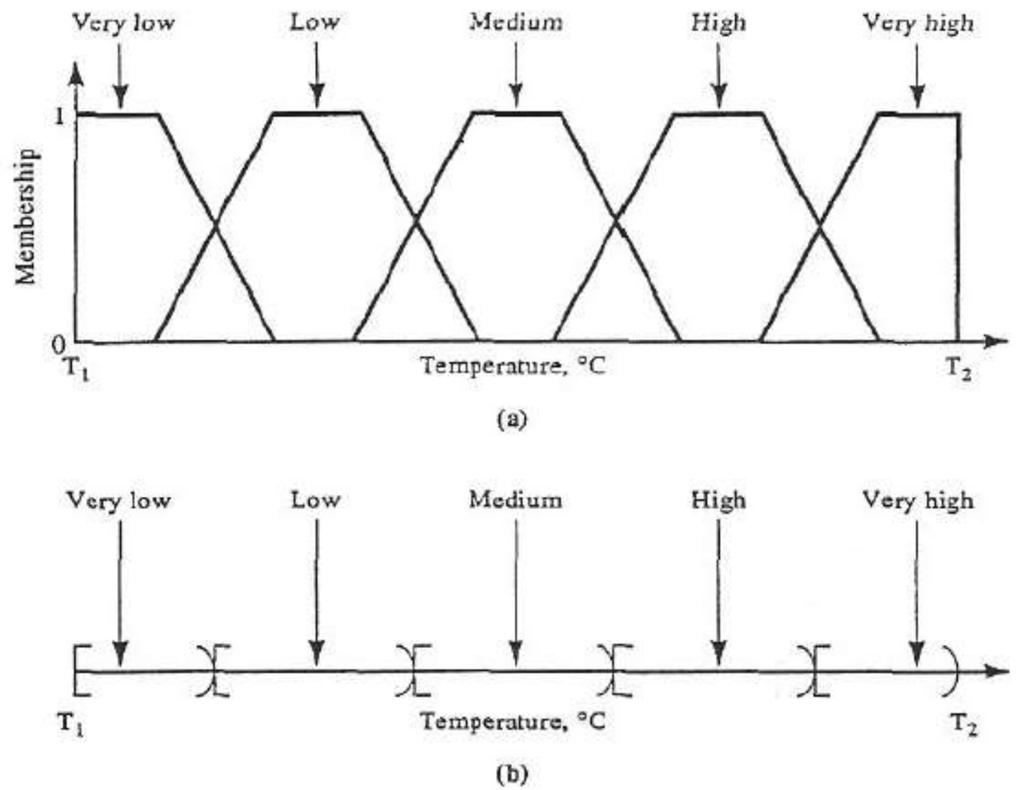


Figure 1.4 Temperature in the range $[T_1, T_2]$ conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.

Hedges

- In **fuzzy logic**, *hedges* are words or modifiers that change the **degree of membership** of a fuzzy set—basically, they make a fuzzy statement stronger or weaker.
- They work a bit like how we use words in everyday language to intensify or soften meaning:

  Very hot
  Somewhat tall
  Extremely cold
  More or less fast

- In fuzzy logic, a hedge is applied as a **mathematical operation** on the fuzzy set's membership function.

Example:

Suppose we have a fuzzy set **Tall** with a membership value μ = 0.7 for a person of height 180 cm.

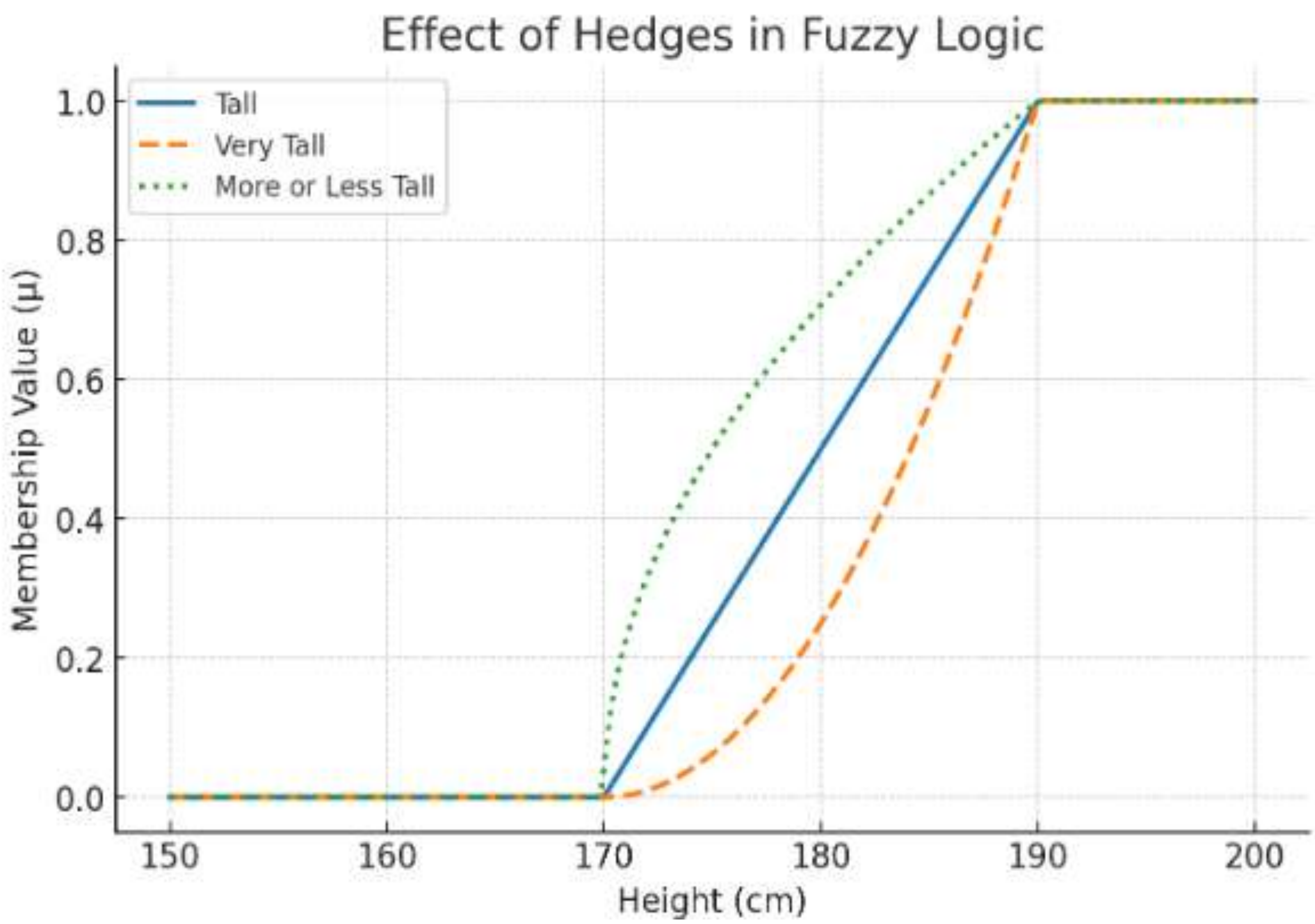
Hedge	Mathematical Effect (example)	New μ value
Very	Square the membership value (μ^2)	0.49
Extremely	Raise to higher power (μ^3)	0.343
Somewhat	Take square root ($\sqrt{\mu}$)	0.836
More or less	Power less than 1 (μ^0.5)	0.836
Not	Complement ($1 - \mu$)	0.3

✅ **Key Idea:**

- *Very* → makes the set **narrower** (more strict).
- *Somewhat / More or less* → makes the set **wider** (more lenient).
- *Not* → flips the membership degree.

📌 **सीधा Logic:**

- Very → सख्ती → membership value घटती है।
- More or Less / Somewhat → ढील → membership value बढ़ती है।



Effect of Hedges in Fuzzy Logic
Tall
Very Tall
More or Less Tall
Membership Value (μ)
1.0
0.8
0.6
0.4
0.2
0.0
150
160
170
180
190
200
Height (cm)

Why does μ decrease with "Very"?

When we say **"Very Tall"**, we are making the definition **stricter**.

Why does μ increase with "More or Less"?

"More or Less Tall" means **"somewhat tall"** or **"almost tall"** — the definition becomes **looser**.

Fuzzy sets and Membership Function

- Fuzzy logic is a set of mathematical principles for knowledge representation based on degrees of membership rather than on crisp membership of classical binary logic.
- Unlike two-valued Boolean logic, fuzzy logic is multi-valued. It deals with degrees of membership.
- The concept of a set is fundamental to mathematics. Crisp set theory is governed by a logic that uses one of only two values: true or false.
- This logic cannot represent vague concepts, and therefore fails to give the answers on the inconsistencies.
- In fuzzy set theory, an element is with a certain degree of membership. Thus, a proposition is not either true or false, but may be partly true (or partly false) to any degree.
- This degree is usually taken as a real number in the interval [0,1].

- The classical example in fuzzy sets is tall men. The elements of the fuzzy set "tall men" are all men, but their degrees of membership depend on their height.

Name	Height in cm	Degree of Membership	
		Crisp	Fuzzy
John	208	1	1
Tom	181	1	0.8
Bob	152	0	0.0
Mike	198	1	0.9
Billy	158	0	0.4

- In the classical logic, we have a crisp set in which if we ask the question: Is the man tall? Then the tall men are above 180, and not tall men are below 180.
  Fuzzy set asks the question: How tall is the man? The tall is partial membership in the fuzzy set, e.g. Mike is 0.9 tall.

Fuzzy Set Operations

1. Union (A∪B)
2. Intersection (A∩B)
3. Complement (A^{-})
4. Bold Union (Bounded Sum) (A⊕B)
5. Bold Intersection (Bounded Difference) (A⊖B)
6. Equality (A=B)

1. Union ($A \cup B$)

•The fuzzy union corresponds to the **OR** operation in classical logic. It is defined by taking the **maximum** of the membership values for each element in the sets.

•**Formula:** $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

2. Intersection ($A \cap B$)

•The fuzzy intersection corresponds to the **AND** operation. It's defined by taking the **minimum** of the membership values for each element.

•**Formula:** $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

3. Complement ($\bar{A}$)

•The complement of a fuzzy set is the opposite of the original set. It's defined by subtracting the membership value of each element from 1.

•**Formula:** $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ This operation represents the degree to which an element does **not** belong to fuzzy set A.

4. Bold Union (Bounded Sum) (A⊕B)

- The bold union, also known as the bounded sum, is a different way to combine fuzzy sets. It's calculated by adding the membership values and "bounding" the result at a maximum of 1.
- **Formula:** $\mu_{A \oplus B}(x) = \min(1, \mu_A(x) + \mu_B(x))$
- This operation can be useful when you want the combined membership to accumulate but not exceed the maximum possible value of 1.

5. Bold Intersection (Bounded Difference) (A⊖B)

- The bold intersection, or bounded difference, is defined by taking the sum of the membership values, subtracting 1, and "bounding" the result at a minimum of 0.
- **Formula:** $\mu_{A \ominus B}(x) = \max(0, \mu_A(x) + \mu_B(x) - 1)$
- This operation is more restrictive than the standard intersection, resulting in a nonzero membership value only when the sum of the individual memberships exceeds 1.

6. Equality (A=B)

•Two fuzzy sets, A and B, are considered equal if and only if every element in their universe has the **exact same membership value** in both sets.

•**Formula:** $A=B \Leftrightarrow \mu A(x)=\mu B(x)$ for all x in the universe of discourse.

•This is a strict condition; even a single differing membership value means the sets are not equal.

Example

Let's use a universe of discourse X=1,2,3,4,5 and two fuzzy sets, A and B.
Set A (e.g., "tall" people): A={1/1,2/0.8,3/0.6,4/0.3,5/0}
Set B (e.g., "heavy" people): B={1/0.2,2/0.7,3/0.9,4/0.5,5/0.1}

1)**Union**={1/1,2/0.8,3/0.9,4/0.5,5/0.1}
2)**Intersection**={1/0.2,2/0.7,3/0.6,4/0.3,5/0}
3)**Complement**(A)={1/0,2/1,3/0.4,4/0.7,5/1}
4)**Bold Union**={1/1,2/1,3/1,4/0.8,5/0.1}
5)**Bold Intersection**={1/0.2,2/0.5,3/0.5,4/0,5/0}
6)**Equality**={}

Universe of Discourse(UoD)

•In **fuzzy logic**, the **Universe of Discourse (UoD)** is simply the **range of all possible values** for the variable we are dealing with in a fuzzy system.

•The set of all possible values that a variable can take in a given problem domain, over which fuzzy sets are defined is called UoD.

Example 1 — Height of a Person

•**Variable:** Height

•**Universe of Discourse:** 100 cm to 250 cm (reasonable human height range)

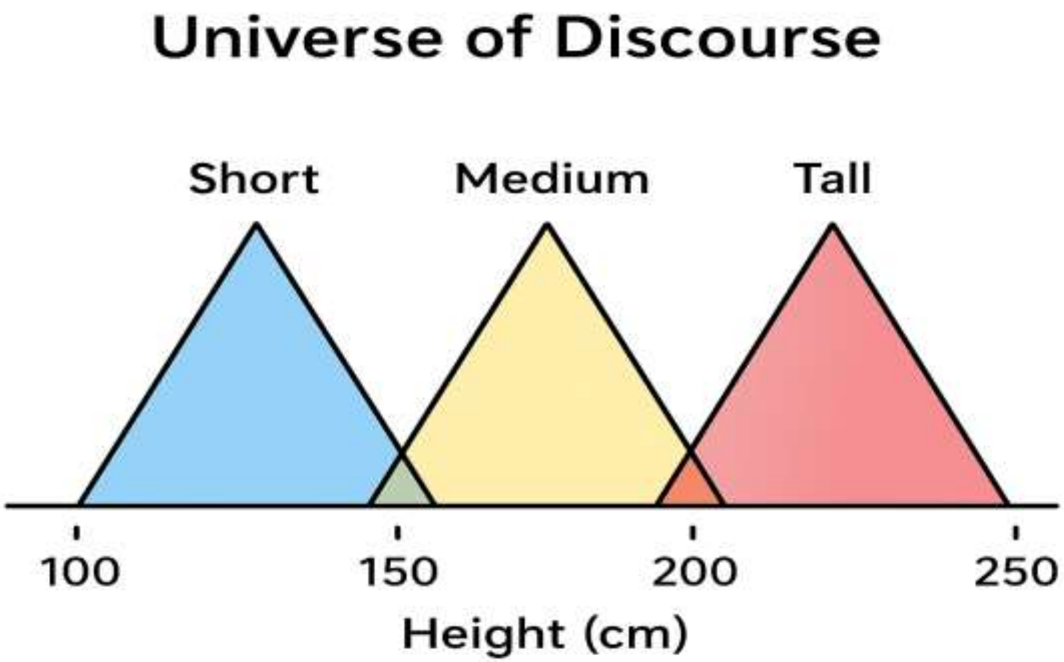
•**Fuzzy sets within this UoD:**

Short

Medium

Tall

•Here, “Short”, “Medium”, and “Tall” are fuzzy sets, each with its own **membership function** over the same Universe of Discourse: **[100, 250]**.



Example 2 — Room Temperature Control

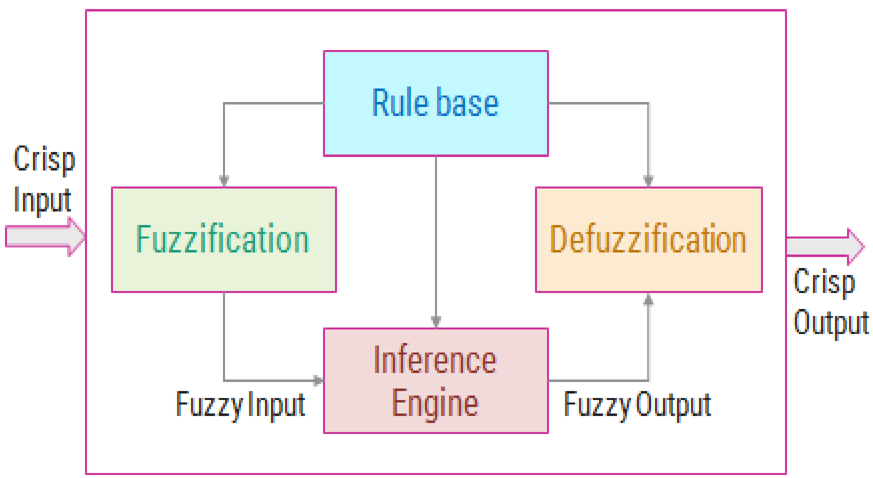
- **Variable:** Temperature (°C)
- **Universe of Discourse:** 0°C to 50°C (possible indoor range for the system)
- **Fuzzy sets:**
    - *Cold*
    - *Warm*
    - *Hot*

If temperature = 28°C, it lies within the UoD, and its membership might be:

- $\mu\_cold(28) = 0.1$
- $\mu\_warm(28) = 0.6$
- $\mu\_hot(28) = 0.3$

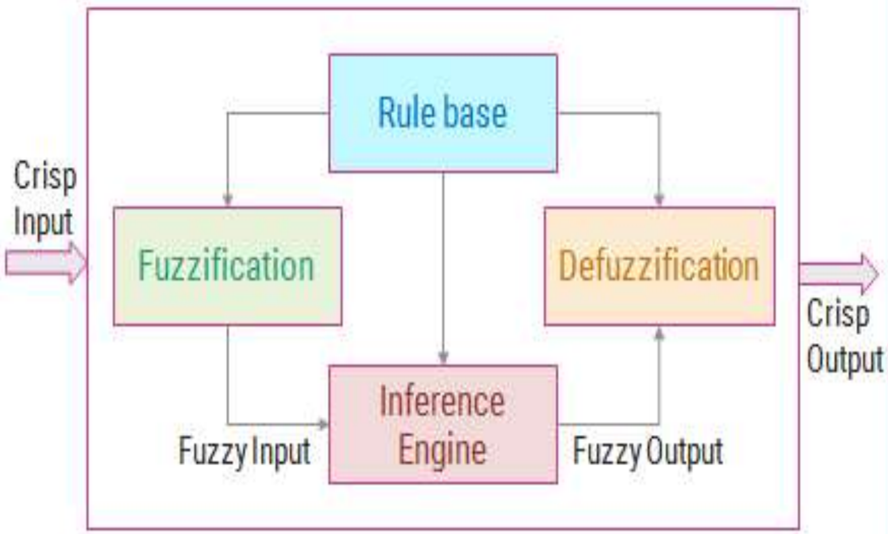
Fuzzy Systems

Architecture of a Fuzzy Logic System



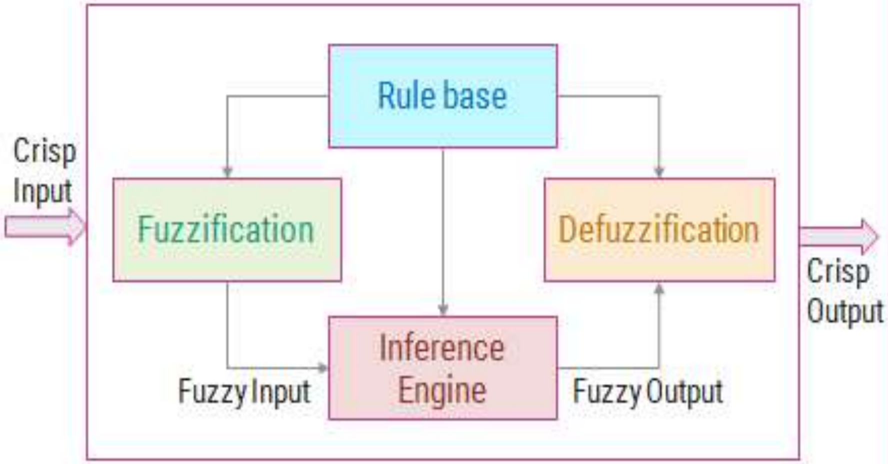
- In the architecture of the Fuzzy Logic system, each component plays an important role. The architecture consists of four different components.
  1. Rule Base
  2. Fuzzification
  3. Inference Engine
  4. Defuzzification

Architecture of a Fuzzy Logic System



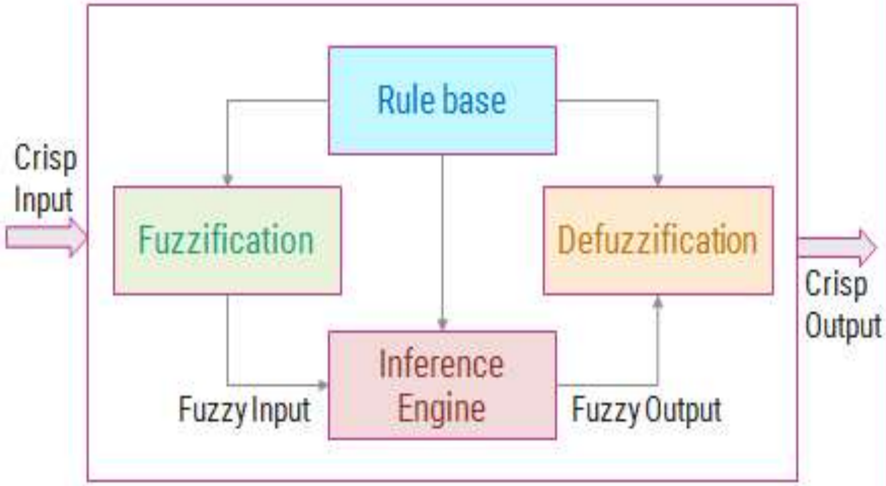
1. Rule Base :
   - Rule Base is a component used for storing the set of rules and the If-Then conditions given by the experts are used for controlling the decision-making systems.
   - There are so many functions which offer effective methods for designing and tuning of fuzzy controllers.
   - These updates or developments decreases the number of fuzzy set of rules.

Architecture of a Fuzzy Logic System



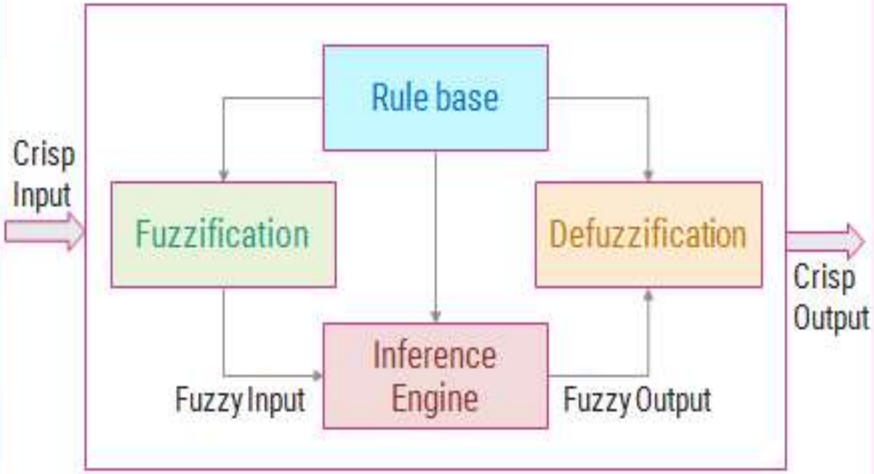
2. Fuzzification :
   - Fuzzification is a module or component for transforming the system inputs, i.e., it converts the crisp number into fuzzy steps.
   - The crisp numbers are those inputs which are measured by the sensors and then fuzzification passed them into the control systems for further processing.
   - This component divides the input signals into following five states in any Fuzzy Logic system:
     i. Large Positive (LP)
     ii. Medium Positive (MP)
     iii. Small (S)
     iv. Medium Negative (MN)
     v. Large negative (LN)

Architecture of a Fuzzy Logic System



3. Inference Engine :
   - This component is a main component in any Fuzzy Logic system (FLS), because all the information is processed in the Inference Engine.
   - It allows users to find the matching degree between the current fuzzy input and the rules.
   - After the matching degree, this system determines which rule is to be added according to the given input field. When all rules are fired, then they are combined for developing the control actions.

Architecture of a Fuzzy Logic System



4. Defuzzification
   - Defuzzification is a module or component, which takes the fuzzy set inputs generated by the Inference Engine, and then transforms them into a crisp value.
   - It is the last step in the process of a fuzzy logic system.
   - The crisp value is a type of value which is acceptable by the user.
   - Various techniques are present to do this, but the user has to select the best one for reducing the errors.

Example of Fuzzy System: Washing Machine

Let's imagine a washing machine that adjusts its wash cycle based on the dirtiness of the clothes and the size of the load.

- **Inputs:** The system has two crisp inputs:
  - **Dirtiness:** A numerical value from 0 to 100.
  - **Load Size:** A numerical value from 0 to 100.
- **Outputs:** The system produces one crisp output:
  - **Wash Time:** A numerical value in minutes.

•**Fuzzification:** The crisp inputs are converted into fuzzy sets with membership functions.
Dirtiness is fuzzified into "**low**," "**medium**," or "**high**."
Load Size is fuzzified into "**small**," "**medium**," or "**large**." For example, a dirtiness level of 65 might be a member of both "**medium**" (e.g., with a membership value of 0.8) and "**high**" (with a membership value of 0.2).

•**Rule Base (Fuzzy Rules):** The system uses simple IF-THEN rules.
Rule 1: IF **dirtiness** is **low** AND **load size** is **small**, THEN **wash time** is **short**.
Rule 2: IF **dirtiness** is **medium** AND **load size** is **medium**, THEN **wash time** is **medium**.
Rule 3: IF **dirtiness** is **high** OR **load size** is **large**, THEN **wash time** is **long**.

Inference Engine: Based on the fuzzified inputs, the inference engine applies the rules. If the dirtiness is 65 (medium dirtiness with a degree of 0.8) and the load size is 80 (large load with a degree of 0.9), the inference engine will combine the rules to calculate a fuzzy output. Rule 3 would be strongly activated because the load size is large.

Defuzzification: The fuzzy output (e.g., a combination of "**medium**" and "**long**" wash times) is converted into a single, precise wash time. The defuzzifier might calculate a weighted average, resulting in a specific number like 55 minutes, which the washing machine's control system can then use.

Fuzzy Proposition

- A fuzzy proposition is a statement that uses **linguistic variables** (like "hot" or "fast") to express a degree of truth, rather than being strictly true or false. Its truth value is not binary (0 or 1) but a number between 0 and 1, representing how much the statement is true.

A fuzzy proposition typically takes the form "P:V is F",
where:
V is a **linguistic variable** (e.g., temperature, speed, height).
F is a **fuzzy set** defined on the universe of discourse of V (e.g., "hot," "fast," "tall").

- The truth value of the proposition, T(P), is determined by the membership grade of a specific value in the fuzzy set.

•**p:V is F**

e.g. Proposition (p): "The temperature is high."
Variable (V): Temperature.
Fuzzy Set (F): High.

•**Unconditional** means the statement is not dependent on any other condition. It doesn't use "if-then" clauses. The truth of the proposition is evaluated on its own, without reference to another proposition.

•**Unqualified** means the proposition doesn't include any linguistic hedges or truth qualifiers. It avoids words like "very," "somewhat," "highly," or "likely," which would modify the proposition's truth value.

Types of Fuzzy Proposition:

1)Unconditional and Unqualified: These are simple, direct propositions.
Form: p:V is F
Example: "The temperature is high."
Truth Value: The truth value T(p) is the membership grade of the variable's value in the fuzzy set, $T(p)=\mu F(v)$.

2)Unconditional and Qualified: These propositions include a "truth qualifier" or linguistic hedge.
Form: p:V is F is S
Example: "The temperature is high is very true."
Truth Value: The truth value is a function of the truth qualifier and the membership grade. For a modifier like "very," this might be $T(p)=(\mu F(v))^2$.

3)Conditional and Unqualified: These propositions express a relationship between two variables.
Form: p: If X is A, then Y is B
Example: "If the temperature is high, then the fan speed is fast."
Truth Value: This is represented as a fuzzy relation between the fuzzy sets A and B.

4)Conditional and Qualified: These are the most complex type, combining a conditional statement with a truth qualifier.
Form: p: If X is A, then Y is B is S
Example: "If the temperature is high, then the fan speed is fast is quite likely."
Truth Value: The truth value is determined by combining the methods for conditional and qualified propositions.

Fuzzy Inference Rules

•Fuzzy inference rules are the core of a fuzzy logic system. They are **IF-THEN statements** that link fuzzy propositions to one another, forming the knowledge base for decision-making. These rules are also known as **fuzzy implications**.

- The general form of a fuzzy inference rule is:

  IF x is A THEN y is B

  where, A and B are fuzzy sets.

Example: A simple air conditioner controller.
Linguistic Variables: "Temperature" and "Fan Speed."
Fuzzy Sets:
For "Temperature": **Cold**, **Warm**, **Hot**.
For "Fan Speed": **Slow**, **Medium**, **Fast**.
Let's define the following fuzzy rules for our controller:
Rule 1: IF Temperature is **Warm** THEN Fan Speed is **Medium**.
Rule 2: IF Temperature is **Hot** THEN Fan Speed is **Fast**.

•Now, consider a scenario where the current temperature is 26°C. The fuzzifier determines that this temperature is **Warm** to a degree of 0.8 and **Hot** to a degree of 0.3.

The inference process works as follows:
Evaluate Rule 1: The truth value of the premise "Temperature is Warm" is 0.8. The rule is partially activated, and the consequence, "Fan Speed is Medium," is applied to the same degree of truth (0.8).
Evaluate Rule 2: The truth value of the premise "Temperature is Hot" is 0.3. The rule is partially activated, and the consequence, "Fan Speed is Fast," is applied to a degree of 0.3.
Combine Outputs: The outputs from all activated rules are combined (aggregated) to form a single fuzzy set for the final output, "Fan Speed." This new fuzzy set represents a combination of "Medium" and "Fast."
Defuzzify: The final fuzzy output is converted into a single, crisp value (e.g., 650 RPM) that the air conditioner can use to set the fan speed.

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