CTMT Axis and Hessian Boundary Constant α: Minimal-Assumption Pipeline on PTB 3D Coil Data

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Abstract

We implement a measurement-first CTMT pipeline to (i) estimate the coil axis and (ii) compute the Hessian boundary constant α from the Fisher axis block, using only admissibility principles (no PDEs or global field models). Synthetic data are generated directly from the traceable parameters reported by PTB for their 3D coil system. We report per-window axis accuracy, α stability, wobble invariance, and unit-invariant conditioning. Results are benchmarked (order-of-magnitude only) against CODATA relative uncertainties.

1 Data and Provenance

PTB 3D coil parameters: coil constants (k_x, k_y, k_z) and inter-axis misalignment angles (millidegrees) are taken from Rott et al. (2022).[1]

Synthetic center-field data at 1 mT are generated using

$$K = R\,\mathrm{diag}(k_x, k_y, k_z),$$

with Gaussian sensor/background noise $\sigma \lesssim 20\,\mathrm{nT}$, consistent with the compensated background reported in the PTB paper. Twelve windows are formed for repeatability and stability checks.

2 CTMT Pipeline (Minimal Assumptions)

2.1 Repair and Admission

- **Unit-invariant conditioning:** conditioning is evaluated on diagonally normalized Fisher, κ_{norm}.
- **Robust covariance:** per-window channel scales estimated via a MAD proxy; Fisher uses $C = \sigma^2 I$.
- **Causality:** static coil case; coherence-time monotonicity not applicable.

2.2 Kernel

We use a Hilbert-space vector kernel with nuisance amplitude and phase. For a unit axis $\mathbf{u}$, the forward map is the complex scalar projection

$$z = (A\,\mathbf{u})\,a\,e^{i\phi},$$

with observables $(\Re z, \Im z)$. Fisher is computed as $F = J^\top C^{-1} J$ using centered finite differences.

[1] N. Rott et al., "Three-dimensional coil system for the generation of traceable magnetic vector fields," *J. Sens. Sens. Syst.* 11 (2022) 211–218.

The 2×2 axis block F_{axis} yields eigenvalues $\lambda_\perp \leq \lambda_\parallel$ and the boundary-constant candidates

$$\alpha_{\text{ratio}} = \frac{\lambda_\perp}{\lambda_\parallel}, \qquad \alpha_{\text{cone}} = \arctan\sqrt{\lambda_\perp/\lambda_\parallel}.$$

2.3 Axis Estimation

For cone excitations around the axis, the CTMT-admitted estimator is the *mean-field direction* per window:

$$\hat{\mathbf{u}} = \frac{\overline{\Re A}}{\|\overline{\Re A}\|}.$$

This requires no PDEs or global models and is sufficient to align the kernel locally.

3 Results

3.1 Axis Accuracy

Mean absolute angular error across 12 windows is **0.57°** (std. 0.35°). Per-window values are shown in Table 1.

Table 1: Per-window axis error (degrees).

Window	Axis Error (deg)
0	0.453 76
1	0.445 65
2	0.547 26
3	0.445 27
4	1.609 28
5	0.311 48
6	0.555 36
7	0.442 51
8	0.662 05
9	0.535 50
10	0.610 74
11	0.226 07

3.2 Hessian Boundary Constant α

Using the admitted axis $\hat{\mathbf{u}}$ in each window, we obtain the statistics:

$$\text{median}\,\alpha_{\text{ratio}} = 0.802, \qquad \text{IQR} = 0.098,$$

$$\text{median}\,\alpha_{\text{cone}} = 41.84^\circ, \qquad \text{IQR} = 1.76^\circ.$$

A wobble test (0.2° perturbation along the near-null direction of F_{axis}) yields a mean absolute drift of 1.4×10^{-3} in α_{ratio}, confirming first-order invariance.

Table 2 shows all per-window values.

Table 2: Per-window α values and wobble invariance.

Window	α_{ratio}	α_{cone} (deg)	α_{ratio} (wobble)	α_{cone} (wobble, deg)	κ_{norm}
0	0.806 87	41.932 10	0.808 27	41.956 75	1.015 85
1	0.861 16	42.860 88	0.862 63	42.885 31	1.098 40
2	0.755 43	40.995 74	0.756 78	41.021 10	1.008 63
3	0.921 66	43.831 87	0.923 04	43.853 14	1.062 51
4	0.835 28	42.425 33	0.836 92	42.453 23	1.197 85
5	0.837 58	42.464 49	0.838 93	42.487 60	1.106 88
6	0.704 93	40.016 90	0.706 17	40.041 68	1.151 59
7	0.796 70	41.751 42	0.797 98	41.774 24	1.233 25
8	0.935 10	44.038 99	0.936 85	44.065 71	1.066 30
9	0.747 51	40.846 20	0.748 71	40.869 05	1.043 95
10	0.737 93	40.663 55	0.739 42	40.692 14	1.013 37
11	0.736 70	40.639 93	0.737 96	40.664 18	1.191 58

3.3 Unit-Invariant Conditioning

Across all windows, κ_{norm} remains bounded (see Table 2), satisfying the CTMT conditioning gate.

4 Benchmark vs. CODATA (Order-of-Magnitude Only)

For context (not equivalence), we compare the *relative uncertainty* of our boundary constant with the CODATA 2022 relative standard uncertainty of the fine-structure constant:

$$\alpha_{\text{FS}} = 7.297\,352\,5643(11) \times 10^{-3}, \qquad u_r(\alpha_{\text{FS}}) = 1.6 \times 10^{-10}.$$

Our present relative spread on α_{ratio} is $\sim 9.3 \times 10^{-2}$, i.e. eight orders of magnitude looser. This is expected prior to applying full CTMT repair (retiming, stacking, axis-focused kernel tuning).

5 Reproducibility

All numerical values used in this report are included directly in Tables 1 and 2. PTB coil parameters are taken from the cited reference; CODATA 2022 values from NIST.

6 Notes on Next Increments

To tighten α constancy without violating CTMT admissibility:

- apply a retiming scan (for live streams) to enforce monotone coherence time τ,
- use robust per-channel covariance C,
- verify stacking admissibility if enriching the kernel,
- evaluate α at the rigidity boundary using an axis-focused excitation cone.

References

- N. Rott et al., *J. Sens. Sens. Syst.* 11 (2022) 211–218.
- NIST CODATA 2022 recommended values.
- CTMT methodology: Fisher rigidity, admissibility gates, coherence diagnostics.