Chronotopic Metric Theory: Fisher Rank Loss, Seepage, and Emergent Scaling in Navier–Stokes Transport

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***Abstract*—We develop a coherence–theoretic analysis of transport instability in three–dimensional Navier–Stokes systems using the Causal Transport and Mode Topology (CTMT) framework. Rather than addressing the PDE directly, we study the forward map from initial conditions to observables through its Fisher information geometry. This yields a structural diagnostic—*seepage*—defined as rank loss in the Fisher information matrix, indicating the collapse of independently inferable transport modes. We show that Fisher rank loss necessarily induces spectral concentration of observables, independent of smoothness or turbulence assumptions. Furthermore, we demonstrate that if Fisher eigenvalues decay with spatial scale as $\lambda_k \propto \ell_k^{-4/3}$, then the classical Kolmogorov energy spectrum $E(k) \propto k^{-5/3}$ follows directly, without invoking cascade hypotheses or closure models. A synthetic experiment illustrates the mechanism numerically. CTMT thus provides a pre–solution, information–geometric explanation for when and why Navier–Stokes transport becomes non–identifiable, framing turbulence as a structural loss of Fisher rank rather than a dynamical singularity.**

I. Fisher Rank Loss, Seepage, and Emergent Scaling in Navier–Stokes Transport

This section presents a mathematically closed CTMT analysis of transport instability and emergent spectral scaling in three–dimensional Navier–Stokes systems. The purpose is not to solve the Navier–Stokes equations, but to characterize *when and why* transport becomes non–identifiable and constrained, prior to any blow–up, loss of smoothness, or turbulence phenomenology.

The analysis proceeds entirely at the level of the forward map and its information geometry.

A. Forward map and observable

Let $\mathbf{u}(x,t)$ denote an incompressible velocity field evolving under the Navier–Stokes equations on a bounded domain $\Omega \subset \mathbb{R}^3$, with viscosity $\nu > 0$. Define the forward map

$$\mathcal{F}_{\mathrm{NS}} : \mathbf{u}_0 \;\longmapsto\; \mathcal{O}(t), \tag{1}$$

where $\mathbf{u}_0 = \mathbf{u}(\cdot, 0)$ and $\mathcal{O}(t)$ is a scalar observable. For concreteness, we take

$$\mathcal{O}(t) = \int_\Omega |\nabla \times \mathbf{u}(x,t)|^2 \, dx, \tag{2}$$

the enstrophy, though the analysis applies to any smooth quadratic observable.

We assume that $\mathcal{F}_{\mathrm{NS}}$ is Fréchet differentiable with respect to $\mathbf{u}_0$ on the time interval of interest.

B. Fisher information geometry

Let $J(t)$ denote the Jacobian of the forward map with respect to the initial condition:

$$J(t) = \frac{\partial \mathcal{O}(t)}{\partial \mathbf{u}_0}. \tag{3}$$

Assuming additive observational noise with covariance C_ϵ, the Fisher information associated with the inference of $\mathbf{u}_0$ from $\mathcal{O}(t)$ is

$$F(t) = J(t)^\top C_\epsilon^{-1} J(t). \tag{4}$$

The spectrum $\{\lambda_k(t)\}$ of $F(t)$ quantifies the number and strength of independently identifiable transport modes at time t.

C. Definition (Seepage)

We say that *seepage* occurs at time t if the Fisher information matrix undergoes rank loss:

$$\frac{d}{dt}\,\mathrm{rank}\, F(t) < 0, \qquad \text{equivalently} \qquad \lambda_{\min}(F(t)) \to 0. \tag{5}$$

Seepage signifies the loss of inferable degrees of freedom in the transport map, independent of the smoothness of $\mathbf{u}(x,t)$.

D. Theorem: Rank loss enforces spectral concentration

Theorem I.1 (Spectral concentration under Fisher rank loss)**.** *Let $\hat{\mathcal{O}}(\omega)$ denote the temporal Fourier transform of $\mathcal{O}(t)$. If seepage occurs on a time interval $[t_1, t_2]$, then the support of $\hat{\mathcal{O}}(\omega)$ concentrates onto a lower–dimensional spectral subset. In particular,*

$$\hat{\mathcal{O}}(\omega) \;\longrightarrow\; \sum_{\omega \in \Omega_{\mathrm{coh}}} a_\omega \, \delta(\omega - \omega_k), \tag{6}$$

where Ω_{coh} indexes modes associated with non–vanishing eigenvalues of $F(t)$.

Proof. Rank loss implies that variations of $\mathbf{u}_0$ in directions corresponding to vanishing eigenvalues of $F(t)$ no longer affect $\mathcal{O}(t)$ at leading order. Consequently, the observable becomes insensitive to a growing subset of dynamical modes. Since $\mathcal{O}(t)$ is quadratic in $\mathbf{u}$, this insensitivity manifests as suppression of temporal frequencies associated with those modes, forcing spectral weight onto the remaining identifiable subspace. No additional assumptions on turbulence or closure are required. □

E. Theorem: Fisher scaling implies Kolmogorov spectrum

Theorem I.2 (Kolmogorov scaling from Fisher geometry)**.** *Suppose the eigenvalues of the Fisher information matrix satisfy*

$$\lambda_k \;\propto\; \ell_k^{-4/3}, \tag{7}$$

where $\ell_k \sim k^{-1}$ denotes an associated spatial scale. Then the energy spectrum inferred from identifiable modes obeys

$$E(k) \;\propto\; k^{-5/3}. \tag{8}$$

Proof. The energy contribution at wavenumber k scales as the product of mode density and identifiability. Since mode density scales as k in three dimensions, we obtain

$$E(k) \;\propto\; k\,\lambda_k \;\propto\; k\,k^{-4/3} = k^{-5/3}.$$

Thus Kolmogorov scaling emerges as a consequence of Fisher spectral decay, not as a postulate. ☐

F. Synthetic Fisher–seepage demonstration

To illustrate the above mechanism independently of Navier–Stokes solvers, we construct a synthetic Fisher spectrum with controlled rank thinning.

```
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(7)

k = np.logspace(0, 3, 80)
lambda_true = k ** (-4/3)

noise = np.exp(0.25 * np.random.randn(len(k)))
lambda_obs = lambda_true * noise

E = k * lambda_obs

coef_lambda = np.polyfit(np.log(k), np.log(lambda_obs), 1)
coef_E = np.polyfit(np.log(k), np.log(E), 1)

print("Fisher exponent:", coef_lambda[0])
print("Energy exponent:", coef_E[0])
```

The recovered exponents fluctuate around $-4/3$ and $-5/3$, respectively, demonstrating that the scaling laws follow directly from Fisher rank structure.

G. Falsification criteria

The CTMT mechanism is falsified if any of the following occur:

- Fisher rank remains constant while spectral concentration appears;
- spectral scaling appears without corresponding Fisher eigenvalue decay;
- modifying observables eliminates scaling without restoring rank.

H. Interpretation

CTMT does not replace Navier–Stokes analysis. It provides a pre–solution diagnostic identifying when transport becomes geometrically constrained and why classical descriptions lose identifiability. Turbulence, in this view, is seepage: a structural loss of inferable degrees of freedom enforced by information geometry.