Information–Geometric Trigonometry: Classical Distance as a Limit of Coherence Geometry

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***Abstract*—We derive a notion of distance based solely on information geometry and coherence–limited signal propagation, without assuming Euclidean geometry, rays, or angles. Distance is defined as a geodesic functional on the Fisher information manifold associated with observable signal parameters. Classical trigonometric distance emerges as a flat, stationary, full–rank limit. A closed–form invariant previously observed in empirical transport problems is shown to arise as the boundary solution of this geometry rather than as a primitive postulate.**

I. Motivation

Distance in classical physics is defined geometrically, prior to dynamics or measurement. In contrast, many physical measurements—radar, acoustics, seismology, optics—infer distance indirectly through time delay, phase, and coherence properties of signals. This suggests that distance may be an emergent quantity determined by information and identifiability rather than geometry *a priori*.

Here we construct distance from first principles using only: (i) statistical distinguishability, (ii) coherence–limited signals, and (iii) standard estimation theory.

II. Fisher Information as Physical Metric

Let $Y(t;\theta)$ be an observable signal depending on parameters θ^i (e.g. delay, frequency, phase). The Fisher information matrix is defined as

$$F_{ij}(\theta) = \mathbb{E}[\partial_i \log p(Y|\theta)\, \partial_j \log p(Y|\theta)]\,, \tag{1}$$

which is the unique Riemannian metric invariant under Markov morphisms (Čencov's theorem).

We define the infinitesimal information distance

$$ds^2 = F_{ij}(\theta)\, d\theta^i d\theta^j. \tag{2}$$

No geometric assumptions are made; ds measures distinguishability, not length in space.

III. Information–Geometric Distance

Given a path Γ in parameter space induced by signal transport, the distance between two events is defined as the geodesic length

$$d = \int_\Gamma \sqrt{F_{ij}(\theta)\frac{d\theta^i}{d\lambda}\frac{d\theta^j}{d\lambda}}\, d\lambda. \tag{3}$$

This replaces trigonometric distance with an information–geometric functional.

IV. Delay Estimation and Stationary Phase

Consider a modulated signal with spectrum $S(\omega)$ and phase

$$\Phi(\omega) = \omega t - k(\omega)x. \tag{4}$$

Stationary phase implies

$$\frac{\partial\Phi}{\partial\omega} = t - \frac{x}{v_g(\omega)} = 0, \tag{5}$$

identifying group delay as the observable encoding propagation distance.

For delay estimation, the Fisher information reduces to

$$F_{tt} = \int \frac{1}{S(\omega)}\left(\frac{\partial S(\omega)}{\partial t}\right)^2 d\omega, \tag{6}$$

a standard result from estimation theory.

V. Coherence–Limited Scaling

For narrowband signals with coherence linewidth γ and characteristic frequency Θ, one obtains the scaling

$$F_{tt} \propto \frac{\Theta^2}{\gamma^2}. \tag{7}$$

Thus the information line element becomes

$$ds = \frac{\Theta}{\gamma}\, dt. \tag{8}$$

VI. Emergence of the Distance Invariant

Propagation is weighted by the signal envelope $A(x)$, so that the effective path length is given by its first moment

$$M_1 = \int x\, |A(x)|^2\, dx. \tag{9}$$

Integrating the line element yields

$$d = \int ds = \frac{\Theta}{\gamma}\int d\ell = \frac{M_1\,\Theta}{\gamma}. \tag{10}$$

This expression is not postulated. It is the closed–form solution of the information–geometric distance in the flat, stationary, full–rank limit.

VII. Classical Trigonometry as a Limit Case

When:

- coherence is uniform,
- Fisher curvature vanishes,
- group velocity is constant,

the information manifold is flat and the above expression coincides with classical geometric distance. Classical trigonometry is therefore recovered as a special case of information geometry.

VIII. Scope and Breakdown

In dispersive or scattering media, F_{tt} becomes position–dependent and the distance must be computed via the full integral. The invariant ceases to hold, while the information–geometric formulation remains valid.

Appendix A
Appendix A: Underwater Acoustics

With depth–dependent sound speed $c(z)$ and coherence decay $\gamma(z)$, distance becomes

$$d = \int \frac{\Theta(z)}{\gamma(z)} \frac{dz}{c(z)}. \tag{11}$$

Classical ray trigonometry fails, but the Fisher–geometric distance remains well defined.

Appendix B
Appendix B: Comparison of Distance Definitions

Method	Geometry	Validity	Edge Cases
Euclidean	Assumed	Rigid media	Fails
Ray theory	Kinematic	Weak dispersion	Limited
Invariant form	Flat Fisher	Stationary	Approximate
CTMT trigonometry	Fisher geometry	General	Robust

TABLE I

Comparison of distance notions

Appendix C
Conclusion

Distance can be derived from coherence and statistical distinguishability without assuming geometry. Classical trigonometry emerges as a limit of flat information geometry. The invariant distance expression arises naturally as the boundary solution of this construction, validating its empirical success while clarifying its domain of applicability.