# CTMT Axis and Hessian Boundary Constant $\alpha$: Minimal-Assumption Pipeline on PTB 3D Coil Data

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**Abstract**

We implement a measurement-first CTMT pipeline to (i) estimate the coil axis and (ii) compute the Hessian boundary constant $\alpha$ from the Fisher axis block, using only admissibility principles (no PDEs or global field models). Synthetic data are generated directly from traceable PTB coil parameters. Three stages are documented: (I) an early baseline (no retiming), (II) a phase-retimed evaluation, and (III) the final rigidity-boundary evaluation ($5°$ cone). We report per-window axis accuracy, $\alpha$ stability, wobble invariance, unit-invariant conditioning, and Fisher rank in fully inlined tables. We also provide an order-of-magnitude benchmark against CODATA relative uncertainties (context only).

## 1 Data and Provenance

**PTB 3D coil parameters.** Coil constants $(k_x, k_y, k_z)$ and inter-axis misalignment (millidegrees), with compensated local background $\lesssim 20\,\mathrm{nT}$, are taken from Rott et al. (2022). Synthetic center-field data at $1\,\mathrm{mT}$ are generated via

$$K = R\,\mathrm{diag}(k_x, k_y, k_z),$$

and split into 12 windows for stability checks. [1]

## 2 CTMT Pipeline (Minimal Assumptions)

### Repair and Admission

- **Unit-invariant conditioning.** Conditioning is evaluated on diagonally normalized Fisher, $\kappa_{\mathrm{norm}}$.
- **Robust covariance.** Per-window channel scales are estimated by a MAD proxy; Fisher uses $C = \sigma^2 I$ for scalar projection noise.
- **Phase retiming (static surrogate).** Each analytic channel is rotated so its mean is real-positive (removing constant phase retardation; the static analogue of stream retiming).

### Kernel and $\alpha$

Hilbert-space vector kernel with nuisance amplitude and phase:

$$z = (A\,\mathbf{u})\,a\,e^{i\phi}, \qquad \text{observables: } (\Re z, \Im z).$$

Fisher is $F = J^T C^{-1} J$ (centered finite differences). The $2 \times 2$ axis block $F_{\mathrm{axis}}$ yields eigenvalues $\lambda_\perp \le \lambda_\parallel$ and

$$\alpha_{\mathrm{ratio}} = \frac{\lambda_\perp}{\lambda_\parallel}, \qquad \alpha_{\mathrm{cone}} = \arctan\sqrt{\lambda_\perp/\lambda_\parallel}.$$

[1] N. Rott, J. Lüdke, R. Ketzler, M. Albrecht, F. Weickert, *J. Sens. Sens. Syst.* 11 (2022) 211–218.

### Axis Estimation

With axis-focused excitation (cone), the admitted estimator is the mean-field direction,

$$\hat{\mathbf{u}} = \frac{\overline{\Re A}}{\|\overline{\Re A}\|},$$

no PDEs or global model required.

## 3 Stage I — Baseline (No Retiming, Center Evaluation)

### Axis Accuracy

Mean absolute angular error across 12 windows: **0.57°** (std. 0.35°).

Table 1: Stage I: per-window axis error (degrees).

| Window | Axis Error (deg) |
|---|---|
| 0 | 0.453 76 |
| 1 | 0.445 65 |
| 2 | 0.547 26 |
| 3 | 0.445 27 |
| 4 | 1.609 28 |
| 5 | 0.311 48 |
| 6 | 0.555 36 |
| 7 | 0.442 51 |
| 8 | 0.662 05 |
| 9 | 0.535 50 |
| 10 | 0.610 74 |
| 11 | 0.226 07 |

### Hessian Boundary Constant $\alpha$

$$\text{median}\,\alpha_{\text{ratio}} = 0.802, \quad \text{IQR} = 0.098, \qquad \text{median}\,\alpha_{\text{cone}} = 41.84^\circ, \quad \text{IQR} = 1.76^\circ,$$

wobble drift (0.2°): $1.4 \times 10^{-3}$.

Table 2: Stage I: per-window $\alpha$ (with wobble) and $\kappa_{\text{norm}}$.

| Window | $\alpha_{\text{ratio}}$ | $\alpha_{\text{cone}}$ (deg) | $\alpha_{\text{ratio}}$ (wobble) | $\alpha_{\text{cone}}$ (wobble, deg) | $\kappa_{\text{norm}}$ |
|---|---|---|---|---|---|
| 0 | 0.806 87 | 41.932 10 | 0.808 27 | 41.956 75 | 1.015 85 |
| 1 | 0.861 16 | 42.860 88 | 0.862 63 | 42.885 31 | 1.098 40 |
| 2 | 0.755 43 | 40.995 74 | 0.756 78 | 41.021 10 | 1.008 63 |
| 3 | 0.921 66 | 43.831 87 | 0.923 04 | 43.853 14 | 1.062 51 |
| 4 | 0.835 28 | 42.425 33 | 0.836 92 | 42.453 23 | 1.197 85 |
| 5 | 0.837 58 | 42.464 49 | 0.838 93 | 42.487 60 | 1.106 88 |
| 6 | 0.704 93 | 40.016 90 | 0.706 17 | 40.041 68 | 1.151 59 |
| 7 | 0.796 70 | 41.751 42 | 0.797 98 | 41.774 24 | 1.233 25 |
| 8 | 0.935 10 | 44.038 99 | 0.936 85 | 44.065 71 | 1.066 30 |
| 9 | 0.747 51 | 40.846 20 | 0.748 71 | 40.869 05 | 1.043 95 |
| 10 | 0.737 93 | 40.663 55 | 0.739 42 | 40.692 14 | 1.013 37 |
| 11 | 0.736 70 | 40.639 93 | 0.737 96 | 40.664 18 | 1.191 58 |

## 4 Stage II/III — Phase-Retimed, 5° Rigidity-Boundary (Final)

### Axis Accuracy (retimed)

Mean absolute angular error across 12 windows: **0.30°** (std. 0.11°).

Table 3: Stage II/III: per-window axis error (degrees).

| Window | Axis Error (deg) |
|---|---|
| 0 | 0.370 104 |
| 1 | 0.198 866 |
| 2 | 0.268 313 |
| 3 | 0.112 322 |
| 4 | 0.286 401 |
| 5 | 0.233 843 |
| 6 | 0.355 686 |
| 7 | 0.199 983 |
| 8 | 0.327 371 |
| 9 | 0.287 220 |
| 10 | 0.382 768 |
| 11 | 0.552 700 |

### Hessian Boundary Constant $\alpha$ (retimed, boundary)

$$\text{median}\,\alpha_{\text{ratio}} = 0.864, \quad \text{IQR} = 0.096, \qquad \text{median}\,\alpha_{\text{cone}} = 42.91^\circ, \quad \text{IQR} = 1.58^\circ,$$

wobble drift (0.2°): $6.07 \times 10^{-3}$; full Fisher **rank = 4**; $\kappa_{\text{norm}}$ bounded in all windows.

Table 4: Stage II/III: per-window $\alpha$ (with wobble) and $\kappa_{\text{norm}}$.

| Window | $\alpha_{\text{ratio}}$ | $\alpha_{\text{cone}}$ (deg) | $\alpha_{\text{ratio}}$ (wobble) | $\alpha_{\text{cone}}$ (wobble, deg) | $\kappa_{\text{norm}}$ |
|---|---|---|---|---|---|
| 0 | 0.758 584 | 41.054 796 | 0.764 369 | 41.162 612 | 1.240 54 |
| 1 | 0.861 450 | 42.865 737 | 0.867 597 | 42.967 306 | 1.094 40 |
| 2 | 0.922 888 | 43.850 851 | 0.929 121 | 43.947 196 | 1.066 28 |
| 3 | 0.771 890 | 41.301 657 | 0.778 397 | 41.420 923 | 1.295 32 |
| 4 | 0.930 041 | 43.961 360 | 0.935 390 | 44.043 452 | 1.068 10 |
| 5 | 0.879 676 | 43.164 910 | 0.885 375 | 43.257 218 | 1.075 03 |
| 6 | 0.977 785 | 44.678 216 | 0.985 000 | 44.783 513 | 1.003 40 |
| 7 | 0.830 578 | 42.344 811 | 0.836 604 | 42.447 920 | 1.151 96 |
| 8 | 0.911 569 | 43.674 253 | 0.917 798 | 43.771 701 | 1.074 16 |
| 9 | 0.783 482 | 41.513 500 | 0.789 085 | 41.614 841 | 1.167 06 |
| 10 | 0.832 529 | 42.378 279 | 0.838 563 | 42.481 299 | 1.200 40 |
| 11 | 0.867 188 | 42.960 571 | 0.873 202 | 43.059 325 | 1.067 82 |

## 5 Cross-Stage Comparison (Improvement)

Table 5: Summary of improvements across stages.

| Metric | Stage I | Stage II/III | Notes |
|---|---|---|---|
| Axis error (mean) | 0.57° | 0.30° | Phase retiming & boundary excitation |
| $\alpha_{\text{ratio}}$ median | 0.802 | 0.864 | Boundary evaluation (5° cone) |
| $\alpha_{\text{ratio}}$ spread | $\sim$9.3% | $\sim$7.9% | Robust $C$, $\kappa_{\text{norm}}$ bounded |
| Wobble drift (0.2°) | $1.4 \times 10^{-3}$ | $6.1 \times 10^{-3}$ | At boundary (flatter) |
| Fisher rank | 3 | 4 | Enriched (axis+amp+phase) |

## 6 CODATA Yardstick (Order-of-Magnitude Only)

For context (not equivalence), CODATA 2022 reports $\alpha_{\text{FS}} = 7.297\,352\,5643(11) \times 10^{-3}$ with relative standard uncertainty $1.6 \times 10^{-10}$. Our present relative spread on the Hessian boundary constant is $\sim 7.9 \times 10^{-2}$, i.e. $\sim$ 8 orders looser—expected before a full CTMT loop on live streams with causal retiming and fixed-point admission. [2]

## Notes on Next Increments

To tighten $\alpha$ further in live pipelines: (i) enforce monotone coherence time $\tau$ via lag scans (retiming), (ii) maintain robust channel-wise $C$, (iii) verify stacking admissibility on any kernel enrichment, and (iv) evaluate at the rigidity boundary with an axis-focused cone.

## References


- N. Rott et al., *J. Sens. Sens. Syst.* 11 (2022) 211–218.
- NIST CODATA 2022 recommended values.
- CTMT methodology: Fisher rigidity, admissibility gates, coherence diagnostics.


[2] NIST/CODATA (2024): CODATA 2022 recommended values, wall summary and fine-structure $\alpha$ page.