

CTMT Trigonometry: Information–Geometric Distances and the Classical Limit

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Abstract

We derive trigonometric relations on Fisher–information manifolds induced by causal signal transport, without assuming Euclidean geometry *a priori*. Distances arise as geodesic functionals of statistical distinguishability. Under a narrowband stationary–phase regime, the line element reduces to $ds = \Theta\gamma dt$, yielding the invariant $d = \Theta\gamma M_1$. Classical trigonometry is recovered exactly when information curvature vanishes. This establishes Euclidean distance as a coherence–saturated limit of information–geometric transport.

1 Information–Geometric Foundation

Definition 1 (Fisher information metric). *Let $\theta = (\theta^1, \dots, \theta^n)$ be control parameters of an observable process. The Fisher information metric is*

$$F_{ij}(\theta) = \mathbb{E}[\partial_i \log p(x|\theta) \partial_j \log p(x|\theta)], \quad (1)$$

and defines an intrinsic notion of statistical distinguishability [1].

Definition 2 (CTMT distance). *Given a causal path $\Gamma : \lambda \mapsto \theta(\lambda)$ in parameter space, the CTMT distance is defined as the Fisher–geodesic length*

$$d(\Gamma) = \int_{\Gamma} \sqrt{F_{ij}(\theta) \dot{\theta}^i \dot{\theta}^j} d\lambda. \quad (2)$$

Remark 1. No background geometry is assumed. Distance is an emergent property of observable distinguishability, consistent with Čencov invariance [1].

2 Stationary Phase and Coherence Scaling

For wave–like observables admitting a stationary–phase approximation, the phase $\Phi(\omega, x, t)$ satisfies

$$\frac{\partial \Phi}{\partial \omega} = t - \frac{x}{v_g(\omega)} = 0, \quad (3)$$

identifying group delay as the operationally accessible parameter.

For a narrowband signal with carrier frequency Θ and coherence linewidth γ , the Fisher metric collapses to a single scale,

$$ds = \Theta \gamma dt. \quad (4)$$

Lemma 1 (CTMT distance invariant). *Let $A(t)$ be the normalized signal envelope. Then the CTMT distance along a causal trajectory is*

$$d = \Theta \gamma \int A(t)^2 dt \equiv \Theta \gamma M_1, \quad (5)$$

where M_1 is the first temporal moment.

Remark 2. All non-coherent microscopic structure is averaged out; the invariant depends only on operationally accessible coherence.

3 Flat–Coherence Limit and Euclidean Recovery

When coherence is uniform and information curvature vanishes, the Fisher metric reduces to

$$F_{ij} \rightarrow \lambda \delta_{ij}, \quad \lambda = (\Theta \gamma)^2 = \text{const.} \quad (6)$$

Geodesics are straight lines and

$$d^2 = \lambda(\Delta x^2 + \Delta y^2). \quad (7)$$

Theorem 1 (Euclidean recovery). *If $\lambda \rightarrow 1$, CTMT distances coincide exactly with Euclidean distances and classical trigonometry is recovered.*

4 CTMT Trigonometry: Cosine and Sine Laws

Consider a geodesic triangle with CTMT side lengths a, b, c . Angles are defined operationally via the Fisher inner product of geodesic tangents v_b, v_c :

$$\cos_{\text{CTMT}}(\alpha) = \frac{\langle v_b, v_c \rangle_F}{\|v_b\|_F \|v_c\|_F}. \quad (8)$$

Theorem 2 (CTMT law of cosines). *The side lengths satisfy*

$$a^2 = b^2 + c^2 - 2bc \cos_{\text{CTMT}}(\alpha). \quad (9)$$

Corollary 1 (CTMT sine law). *Invariance of geodesic curvature implies*

$$\frac{\sin_{\text{CTMT}}(\alpha)}{a} = \frac{\sin_{\text{CTMT}}(\beta)}{b} = \frac{\sin_{\text{CTMT}}(\gamma)}{c}. \quad (10)$$

Remark 3. In the flat limit $F_{ij} \rightarrow \delta_{ij}$, $\cos_{\text{CTMT}} \rightarrow \cos$ and $\sin_{\text{CTMT}} \rightarrow \sin$, recovering classical trigonometric identities.

5 Numerical Demonstration

Let $\Theta = 10^3$, $\gamma = 10^{-2}$, $\Delta t_x = 2$, $\Delta t_y = 3$. Then

$$d_{12} = 20, \quad d_{13} = 30, \quad d_{23} = \sqrt{20^2 + 30^2} = 36.06.$$

The CTMT cosine evaluates to

$$\cos_{\text{CTMT}}(\alpha) = \frac{20^2 + 30^2 - 36.06^2}{2 \cdot 20 \cdot 30} = 0,$$

corresponding to a right angle, in exact agreement with classical geometry.

Correlation Summary

| γ | \cos_{CTMT} | Classical cos |
|-----------|-------------------------|-------------------------|
| 10^{-2} | 0 | 0 |
| 10^{-3} | 0 | 0 |
| Uniform | \rightarrow classical | \rightarrow classical |

6 Non–Uniform Media

For spatially varying propagation speed $c(z)$ and coherence $\gamma(z)$, the CTMT distance generalizes to

$$d = \int \frac{\Theta(z)\gamma(z)}{c(z)} dz. \quad (11)$$

In such regimes classical ray-based trigonometry fails, while CTMT remains well-defined.

7 Interpretation

CTMT trigonometry makes explicit that:

- distance is emergent,
- angles are operationally defined,
- geometry is coherence-dependent,
- classical trigonometry is a limiting case, not a primitive axiom.

This mirrors analogous reductions observed in CTMT chaos diagnostics and coherence-based magnetostatics.

References

- [1] S. Amari and H. Nagaoka, *Methods of Information Geometry*, AMS / Oxford University Press, 2000.
- [2] “Information geometry,” Wikipedia overview (accessed 2025).
- [3] R. Matěj, “Information–Geometric Trigonometry: Classical Distance as a Limit of Coherence Geometry,” 2 pp. note, 2025.