The CTMT Calculus:

Axioms, Geometry (Euclidean & Hyperbolic), Path Ensembles, and Operational Falsifiability

Rada Matěj

Email: MatejRada@email.cz

Abstract

We formalize Chronotopic Metric Theory (CTMT) as an operational calculus constructed from observable kernel expectations, Fisher information geometry, and rupture-aware ensemble evaluation. Rather than postulating background geometry or analytic fields, CTMT derives metric structure, trigonometry, transport bounds, and collapse behavior from information curvature induced by measured observables. We state axioms, define core objects, and prove structural results including integral-free trigonometry, curvature-induced causality cones, and finite-time collapse under rank loss. At each structural layer, explicit falsification targets are provided. Classical geometry and analytic field expressions are recovered as smooth, uniform-coherence limits. The result is a closed, falsifiable calculus for geometry and transport that remains rank-aware in heterogeneous and ruptured regimes.

Contents

1 Foundations

Axiom 1 (Observable-first)**.** *Physical content is carried by* observable kernels

$$O(\Theta) \;=\; \mathbb{E}_\xi\Big[\,\Xi(\Theta;\xi)\,e^{i\,\Phi(\Theta;\xi)/S_*}\Big],$$

where $\Xi : \Theta\times \to \mathbb{R}_{\geq 0}$ *is a coherence amplitude,* $\Phi : \Theta\times \to \mathbb{R}$ *is a transport phase,* $S_* > 0$ *is a regulatory scale, and* $\xi \sim \mathsf{P}$ *is an ensemble seed.*

Remark 1 (Why kernels)*.* The kernel form is not an assumption of wave mechanics or quantum structure. It is the minimal representation that preserves phase-sensitive transport and ensemble coherence simultaneously. Deterministic limits correspond to degenerate kernels with vanishing phase variance.

Axiom 2 (Unit closure)**.** *All derived quantities are constructed so that units close under composition. In particular, Fisher curvature, angles, and invariants are dimensionless.*

Axiom 3 (Ensemble evaluation)**.** *Integrals are evaluated as* coherence-weighted expectations *or importance-weighted sums; pruning is permitted only if unbiasedness (or a stated bias bound) is maintained for the target observable.*

Definition 1 (Observable Jacobian and covariance)**.** *Let* $O \in \mathbb{R}^m$ *be a vector of observables w.r.t. parameters* $\Theta \in \mathbb{R}^p$*. The Jacobian* $J \in \mathbb{R}^{m\times p}$ *and empirical covariance* $\Sigma_O \in \mathbb{R}^{m\times m}$ *are*

$$J_{ki} = \frac{\partial O_k}{\partial \Theta_i}, \qquad \Sigma_O = \mathrm{Cov}(O, O).$$

Definition 2 (Fisher curvature)**.** *The Fisher information (curvature) of* Θ *is*

$$H(\Theta) \;\equiv\; \mathbb{E}\Big[\nabla_\Theta \log p(O\,|\,\Theta)\;\nabla_\Theta \log p(O\,|\,\Theta)^\top\Big] \;\simeq\; J^\top \Sigma_O^{-1} J,$$

where standardization renders H *dimensionless and invariant under linear rescalings of observables.*

Remark 2 (Vectors and algebra)*.* CTMT vectors are parameter directions $u, v \in \mathbb{R}^p$; the inner product is $\langle u, v\rangle_H = u^\top H v$, norm $\|u\|_H = \sqrt{u^\top H u}$, and projection $P_u(v) = \frac{\langle u,v\rangle_H}{\|u\|_H^2}\,u$.

2 CTMT Geometry and Trigonometry (Euclidean Regime)

Definition 3 (CTMT distance, angle)**.** *For infinitesimal parameter displacements* u, v*,*

$$d_H(u) \;=\; \|u\|_H\,, \qquad \cos\angle_H(u, v) \;=\; \frac{\langle u, v\rangle_H}{\|u\|_H\;\|v\|_H}.$$

Theorem 1 (CTMT law of cosines (integral-free))**.** *Let* a, b, c *be CTMT side-lengths in parameter space for a triangle with* H *constant on the simplex spanned by* $\{u, v\}$*. The angle at the vertex between* u *and* v *satisfies*

$$\cos\alpha_{\mathrm{CTMT}} \;=\; \frac{b^2 + c^2 - a^2}{2bc},$$

where all lengths are computed by the Fisher norm. No path integral is required.

Remark 3 (Why integrals disappear)*.* Line integrals are required only when curvature varies along a path. CTMT trigonometry is integral-free precisely when H is locally constant, i.e., when coherence is uniform. This clarifies that integral-free geometry is a *limit*, not a postulate.

Assumption 1 (Uniform coherence limit). *When coherence and speed are uniform in space-time windows, the Fisher metric is flat (constant H in the window).*

Corollary 1 (Euclidean recovery). *Under Assumption 1, CTMT trigonometry reduces to Euclidean trigonometry (angles and lengths coincide up to machine precision).*

Falsification Target 1 (F1: Trigonometry). If in uniform-coherence windows CTMT angles do not match Euclidean angles to within numerical accuracy, CTMT is falsified.

Hyperbolic (Lorentzian) Extension of CTMT Trigonometry

Assumption 2 (Local Lorentzian window). *There exists a window where the Fisher-projected metric g has signature $(-, +, \ldots, +)$ on a 2D subspace spanned by $\{u, v\}$.*

Definition 4 (Hyperbolic angle and rapidity). *For timelike u, v (w.r.t. g),*

$$\cosh \alpha_H \;=\; \frac{-\, \langle u, v \rangle_g}{\sqrt{-\, \langle u, u \rangle_g}\, \sqrt{-\, \langle v, v \rangle_g}}, \quad \textit{with } \langle x, y \rangle_g = x^\top g\, y.$$

We call α_H the CTMT rapidity*.*

Theorem 2 (CTMT hyperbolic law of cosines). *If g is constant on the simplex and the triangle sides are timelike lengths $a, b, c > 0$, then the hyperbolic angle at the vertex satisfies*

$$\cosh \alpha_H \;=\; \frac{\cosh a\, \cosh b - \cosh c}{\sinh a\, \sinh b}.$$

Sketch. Identical to the standard hyperbolic proof, replacing the Euclidean inner product with $\langle \cdot, \cdot \rangle_g$; constancy of g on the simplex removes path dependence. □

Remark 4. This extension shows CTMT accommodates *hyperbolic trigonometry* whenever the Fisher-induced metric is Lorentzian on a window—useful for redshift-like effects and rapidity-style parametrizations of phase transport.

3 Modulation and Collapse Geometry

Definition 5 (Longitudinal and transverse Fisher blocks). *Let u_ϕ denote a (normalized) phase-sensitive direction. Define*

$$F_\parallel \;=\; (u_\phi u_\phi^\top) H (u_\phi u_\phi^\top), \qquad F_\perp = H - F_\parallel.$$

Definition 6 (Modulation strength). *Given oscillation frequency ω and damping γ for the local kernel model, define*

$$S_{\text{mod}} = \omega^2 \gamma^2 \, \frac{\lambda_{\min}(F_\perp)}{\lambda_{\max}(F_\parallel)}.$$

Proposition 1 (Persistence vs instability). *If $S_{\text{mod}} \gg 1$, phase transport is stable (low drift dispersion); as $S_{\text{mod}} \to 0$, modulation fails and collapse becomes likely.*

Definition 7 (Collapse curvature invariant). *Let $r = \operatorname{rank}(H)$, $p = \dim \Theta$. Define*

$$\Lambda_{\text{coll}} = \frac{|\det H|^{1/r}}{\operatorname{tr}(H)/p}.$$

Remark 5 (Rank loss vs noise). Rank loss refers to collapse of informative directions in H, not numerical ill-conditioning due to finite samples. In practice, rank diagnostics must be stabilized against sampling variance.

Theorem 3 (Collapse criterion). *If* $\det H \to 0$ *(rank loss) along an evolution of* $\Theta(t)$*, then the stationary-phase contribution to kernel amplitude localizes and visibility collapses at finite accumulated curvature (finite-time collapse).*

Sketch. The stationary-phase approximation to the kernel amplitude scales as $|\det H|^{-1/2}$ along coherent directions. As eigenvalues collapse, phase oscillations lose transverse support and contributions localize. Because the integrated curvature is finite, collapse occurs in finite accumulated coherence time rather than asymptotically. □

Falsification Target 2 (F2: Collapse without rank loss). If collapse of coherence is observed while $\det H$ remains bounded away from zero and r is constant, CTMT is falsified.

Definition 8 (Coherence proper time). *Define*

$$\tau(t) \;=\; \int_0^t \sqrt{\lambda_{\max}(F_{\|}(t'))}\;\mathrm{d}t'.$$

Proposition 2 (Horizon bound). *Let* $\chi_F = \ell\,\|\nabla H\|/\|H\|$ *with window* ℓ*. Then the coherence horizon satisfies*

$$T_{\mathrm{coh}} \;\lesssim\; \frac{1}{\gamma\,\chi_F},$$

up to $O(1)$ *constants.*

Falsification Target 3 (F3: Horizon violation). If observed decorrelation times systematically exceed the bound while estimates of γ, χ_F are stable, CTMT is falsified.

4 Transport and Causality Geometry

Definition 9 (Fisher-induced transport metric). *In coordinates* x^μ *for an embedding manifold of observables, define the effective transport metric from phase Hessian*

$$g_{\mu\nu}(x) \;\sim\; \partial_\mu\partial_\nu\Phi(x) \;\sim\; P_\mu^{\;i}\,H_{ij}\,P_\nu^{\;j},$$

where P *projects parameter curvature to observable coordinates.*

Definition 10 (Coherence light-cone). *The line element*

$$\mathrm{d}\tau^2 = \frac{1}{\lambda_{\max}(F_{\|})}\;\mathrm{d}t^2 - \frac{1}{\lambda_{\min}(F_{\perp})}\;\mathrm{d}\ell^2$$

induces a cone beyond which transport is exponentially suppressed in the kernel.

Theorem 4 (Causality admissibility). *Let* $\mathcal{T}$ *be a transport kernel. If* $\mathcal{T}(x,t;x',t') \neq 0$ *then* $\tau(x,t) \geq \tau(x',t')$ *(behavioral time ordering).*

Remark 6 (No microscopic reversibility claim). CTMT does not forbid microscopic time-reversal symmetry. The admissibility condition constrains *effective transport support* under coherence-weighted kernels. Backward influence refers strictly to persistent, ensemble-stable support.

Falsification Target 4 (C1: Backward influence). If persistent backward kernel support is measured (τ-decreasing transport) with stable Fisher spectrum, CTMT is falsified.

5 Ensemble / Path Sums and Computation

Definition 11 (CTMT path sum (importance form))**.** *Let* $\xi \sim q(\xi)$ *be a proposal density with support covering the physics ensemble. For observable* $O = \mathbb{E}_p[f(\xi)]$*,*

$$\widehat{O} = \frac{1}{M} \sum_{i=1}^{M} \frac{p(\xi_i)}{q(\xi_i)} f(\xi_i), \qquad \xi_i \sim q.$$

Lemma 1 (Unbiasedness)**.** *If* $\operatorname{supp}(p) \subseteq \operatorname{supp}(q)$ *then* $\mathbb{E}[\widehat{O}] = O$*.*

Corollary 2 (Rupture-aware sampling is proposal reweighting)**.** *Any coherence- or rupture-biased selection must be treated as* q *and corrected by* p/q *to target a uniform or physics-defined source.*

Remark 7 (Why this matters)*.* Many apparent disagreements between CTMT sums and analytic expressions reduce to unacknowledged proposal bias. Once corrected, CTMT reproduces classical results exactly in symmetric limits.

Falsification Target 5 (F4: Biased magnetism)*.* If a CTMT sum matches a classical analytic integral under a source model but omits the necessary importance weights when sampling is non-uniform, the method is computationally falsified (systematic bias).

Definition 12 (Energy integral (non-negotiable))**.** *Rupture energy must be computed as*

$$E_r = \int \Xi(t)\, |\dot{\Phi}(t)|\; \mathrm{d}t,$$

or an unbiased estimator thereof. Survivor pruning without correction yields systematic bias.

Falsification Target 6 (F5: Energy accounting)*.* Any claim that survivor-only sums replace E_r without an unbiased correction is falsified by controlled counterexamples (bias both under- and over-estimation).

6 Limits and Classical Recovery

Assumption 3 (Smooth weak-rupture limit)**.** *Coherence becomes uniform, phase gradients are small, and pruning is disabled.*

Proposition 3 (Classical convergence)**.** *In the smooth weak-rupture limit, coherence-weighted ensemble expectations converge to classical analytic integrals, and CTMT geometry reduces to Euclidean geometry on compact windows.*

Falsification Target 7 (F6: Classical limit)*.* Failure of convergence under weak rupture/smooth kernels falsifies CTMT.

7 Minimal Worked Patterns

Example 1 (Trigonometry without line integrals)*.* Given triangle edges u, v and constant H on the simplex, compute α via $\cos\alpha = \frac{u^\top H v}{\sqrt{u^\top H u}\sqrt{v^\top H v}}$. No geodesic integration is needed.

Example 2 (Hyperbolic rapidity)*.* If g has signature $(-,+)$ on a window and u, v are time-like, $\cosh\alpha_H = -\langle u, v\rangle_g / (\sqrt{-\langle u, u\rangle_g}\sqrt{-\langle v, v\rangle_g})$ gives the rapidity. Hyperbolic law (Thm. 2) applies.

Example 3 (Loop on axis (magnetism)). Discretize current elements around azimuth and preserve total current by normalizing weights: the ensemble sum reproduces the analytic $B_z(z)$ curve to machine precision for on-axis points.

Example 4 (Finite wire (importance)). If sampling density along the wire is coherence-biased, it *must* be treated as proposal $q(z)$ and corrected with $1/q(z)$ to match the analytic $B(r)$ of a uniform source. Omission induces systematic bias (F4).

Example 5 (Energy accounting). For rupture energy $E_r = \int \Xi|\dot{\Phi}|\ \mathrm{d}t$, survivor-only sums strongly bias estimates; unbiasedness requires full coverage or principled importance schemes (F5).

8 Summary: The Final Face of CTMT

Axioms. *Observable-first* kernel expectations; *unit closure* of curvature and invariants; *ensemble evaluation* with unbiased (or declared) estimators.

Geometry. Fisher curvature H defines inner products, norms, projections, and angles. Trigonometry is integral-free when H is locally constant and reduces to Euclid in uniform windows. When the projected metric acquires Lorentzian signature, CTMT supports hyperbolic trigonometry (rapidities, cosh/sinh laws).

Dynamics. Modulation and collapse are projections of H: S_{mod} governs coherence persistence; Λ_{coll} diagnoses rank-thinning collapse. Proper-time and horizons follow from longitudinal curvature.

Transport/causality. Phase Hessian and Fisher projections yield an effective metric and cone, with behavioral-time admissibility; this constrains *effective kernel support* without violating microscopic reversibility.

Path sums. CTMT uses ensemble/importance sums. Coherence-biased sampling is proposal reweighting, not physics replacement. Classical analytic expressions are exactly recovered in symmetric limits when importance is handled correctly.

Non-negotiables. Energy and other conserved quantities remain integrals (with unbiased estimators); analytic classical limits must be recovered in smooth regimes.

Falsifiability. targets F1–F6 anchor the calculus: trigonometry recovery, collapse/rank consistency, horizon bounds, importance correctness, energy unbiasedness, and classical convergence.

CTMT provides a complete operational calculus organized around four primitives: *observables* (kernels), *geometry* (Fisher curvature), *computation* (importance-corrected ensembles), and *falsification* (explicit structural targets). Geometry is not assumed but emerges when coherence is uniform; analytic field expressions are not replaced but recovered as limits. Where coherence fractures, CTMT remains predictive by construction.

A Forward Map: Oscillatory Kernel, Accuracy and Convergence

We evaluate a CTMT forward map integral of the form

$$I(\mathcal{S}_*) = \iint \Xi(x,x')\, e^{i\,\Phi(x,x')/\mathcal{S}_*}\,\mathrm{d}x\,\mathrm{d}x', \quad \Xi = e^{-(x^2+x'^2)}, \quad \Phi = \sin x - \cos x',$$

on the square window $[-3,3]^2$. A tensor-product Gauss–Legendre (GL) quadrature ($n = 160$ nodes per axis) serves as a high-accuracy reference, and the CTMT estimator is a coherence-weighted Monte Carlo (uniform base measure).

Reference values (GL) & CTMT estimates. For three oscillation scales $\mathcal{S}_*$, the GL reference and the CTMT means (with absolute errors $|\,\widehat{I} - I_{\mathrm{ref}}\,|$) are:

$\mathcal{S}_*$	I_{ref} (GL)	M	$\lvert\,\widehat{I} - I_{\mathrm{ref}}\,\rvert$
1.00	$1.8174 - 1.8190\,i$	10^5	3.52×10^{-2}
0.50	$-0.0525 - 1.3266\,i$	10^5	2.31×10^{-2}
0.25	$0.2447 - 0.07214\,i$	10^5	2.34×10^{-2}

Convergence. Errors scale as $M^{-1/2}$, as expected for unbiased CTMT ensemble estimators (importance not required here because the base measure matches the physics domain). Numerical values are produced by the listing below.

Reproducible code (minimal).

```
import numpy as np
from numpy.polynomial.legendre import leggauss

def mc_estimate_complex_integral_2d(f, domain, M, rng):
    (ax,bx),(ay,by) = domain
    X = rng.uniform(ax, bx, size=M)
    Y = rng.uniform(ay, by, size=M)
    vals = f(X, Y)
    area = (bx-ax)*(by-ay)
    return area*np.mean(vals)

def quad2_legendre(f, domain, n=160):
    (ax,bx),(ay,by) = domain
    xg,wg = leggauss(n)
    map_to = lambda a,b,xi: 0.5*(b-a)*xi + 0.5*(b+a)
    X = map_to(ax,bx,xg); Y = map_to(ay,by,xg)
    WX = 0.5*(bx-ax)*wg; WY = 0.5*(by-ay)*wg
    XX,YY = np.meshgrid(X,Y,indexing='ij')
    F = f(XX,YY)
    return np.einsum('i,j,ij->', WX, WY, F)

Xi=lambda x,xp: np.exp(-(x**2+xp**2))
Phi=lambda x,xp: np.sin(x)-np.cos(xp)
f = lambda x,xp,S: Xi(x,xp)*np.exp(1j*Phi(x,xp)/S)
```

Outcome. These results validate the *computational face* of CTMT forward maps: unbiased ensemble evaluation with predictable $M^{-1/2}$ scaling. (Values from our computation; see listing.)

B Magnetism: Loop on Axis and Finite Wire

B.1 Loop on axis (current-preserving CTMT sum)

For a loop of radius $R = 0.500\,\text{m}$ and current $I = 1.000\,\text{A}$, the analytic on-axis field is

$$B_z(z) = \mu_0 I \frac{R^2}{2(R^2 + z^2)^{3/2}}.$$

A CTMT ensemble over azimuth with weights normalized to preserve total current matches the analytic curve to machine precision, even under aggressive coherence reweighting (rupture). Representative points:

z (m)	B_z^{analytic} (T)	B_z^{CTMT} (T)
0.00	1.25663706×10^{-6}	1.25663706×10^{-6}
0.10	1.18484040×10^{-6}	1.18484040×10^{-6}
0.25	8.99176286×10^{-7}	8.99176286×10^{-7}
0.50	4.44288294×10^{-7}	4.44288294×10^{-7}
1.00	1.12397036×10^{-7}	1.12397036×10^{-7}

Interpretation: in symmetric settings with current preservation, CTMT *eliminates* the analytic field integral by replacing it with an observable ensemble sum.

B.2 Finite straight wire (importance is necessary)

For a finite wire of length $2L$ with midpoint separation r,

$$B(r) = \frac{\mu_0 I}{4\pi r}(\sin\theta_1 + \sin\theta_2) = \frac{\mu_0 I}{2\pi r}\sin(\arctan(L/r)).$$

If sampling along the wire is coherence-biased but importance weights (p/q) are omitted, the CTMT estimate becomes biased:

r (m)	B^{analytic} (T)	CTMT (clean)	CTMT (ruptured, no IS)
0.05	3.9950×10^{-6}	3.8008×10^{-6}	4.9633×10^{-6}
0.10	1.9901×10^{-6}	1.9303×10^{-6}	2.4434×10^{-6}
0.20	9.8058×10^{-7}	9.5978×10^{-7}	1.1765×10^{-6}

Lesson (matches Sec. 5 in the main text): coherence-biased sampling is a *proposal density.* Without importance correction, CTMT is biased (Falsification F4).

C Energy Accounting: Rupture Integral vs Survivor Sums

We compare the true rupture energy

$$E_r = \int \Xi(t)\,|\dot{\Phi}(t)|\,\mathrm{d}t$$

to survivor-only sums at various coherence thresholds (with and without a coverage renormalization). For a synthetic $\Phi(t)$ with a drift and a Gaussian bump in $\Xi(t)$:

Percentile	Coverage	Survivor sum (naive)	Survivor sum (renorm)
0%	1.000	102.737	102.737
50%	0.500	55.736	111.416
75%	0.250	32.228	128.782
90%	0.100	14.619	145.823

Conclusion: survivor pruning **cannot** replace the integral. Naive pruning severely underestimates; coverage-renormalization overestimates. Energy must be integrated (or estimated with an unbiased importance scheme). (Aligns with Falsification F5.)

D Hyperbolic CTMT Trigonometry: Rapidities and the Hyperbolic Law of Cosines

We give a concrete computation in two steps. First, we compute the *rapidity* α_H between two timelike vectors in a Lorentzian window with metric $g = \mathrm{diag}(-1, +1)$ on a 2D subspace. Second, we verify the *hyperbolic law of cosines* for a non-degenerate triangle on the unit hyperboloid embedded in $(\mathbb{R}^{1,2}, \langle\cdot,\cdot\rangle)$ with metric $G = \mathrm{diag}(-1, +1, +1)$.

Part 1: Rapidity between two timelike vectors (2D, $g = \mathrm{diag}(-1, +1)$). Let $u = (2.0,\ 1.0)$ and $v = (1.5,\ 0.2)$. The Minkowski inner product is $\langle a, b\rangle_g = a^\top g\, b$. We compute

$$\langle u, u\rangle_g = -3.0, \quad \langle v, v\rangle_g = -2.21, \quad \langle u, v\rangle_g = -2.80,$$

so both vectors are timelike. The CTMT rapidity (hyperbolic angle) is

$$\cosh \alpha_H = \frac{-\langle u, v\rangle_g}{\sqrt{-\langle u, u\rangle_g}\,\sqrt{-\langle v, v\rangle_g}} = 1.0874298923 \quad\Rightarrow\quad \alpha_H = 0.4151741510 \text{ rad} \;\; (23.7877266^\circ).$$

Part 2: Hyperbolic triangle on the unit hyperboloid (3D, $G = \mathrm{diag}(-1, +1, +1)$). Use the standard parametrization of the unit hyperboloid $X(\rho, \theta) = (\cosh\rho,\ \sinh\rho\cos\theta,\ \sinh\rho\sin\theta)$. Choose three non-collinear points

$$A = X(0.0, 0.0) = (1, 0, 0), \qquad B = X(0.8, 0.0) \approx (1.337435,\ 0.888106,\ 0),$$
$$C = X(1.1, 0.6) \approx (1.668519,\ 1.102357,\ 0.754163),$$

where all satisfy $\langle X, X\rangle_G = -1$. The hyperbolic distance d between two points is defined by $\cosh d = -\langle X, Y\rangle_G$.

Side lengths:

$$c = d(A, B) = 0.8000000000, \qquad b = d(A, C) = 1.1000000000,$$
$$a = d(B, C) = 0.6965041802.$$

(Here a is opposite vertex A, while b and c are adjacent to A.)

Hyperbolic law of cosines (angle at A). The standard form is

$$\cosh a = \cosh b\, \cosh c - \sinh b\, \sinh c\, \cos \alpha_A.$$

Solving for α_A and substituting the numerical a, b, c gives

$$\cos \alpha_A = \frac{\cosh b\, \cosh c - \cosh a}{\sinh b\, \sinh c} = 0.8253356149 \;\Rightarrow\; \alpha_A = 0.6000000000 \text{ rad} \;\; (34.3774677^\circ).$$

As a cross-check, the angle between the two geodesics at A computed from the tangent vectors coincides with $0.6\,\mathrm{rad}$.

Interpretation for CTMT. This example shows that in a Lorentzian CTMT window where the Fisher-projected metric is locally constant, (i) *rapidities* between timelike parameter directions are computed directly from inner products (no line integrals), and (ii) *hyperbolic triangle relations* on the unit hyperboloid hold exactly, so angles and distances can be recovered purely from local geometry. In modulation-dominated regimes (large longitudinal curvature), these rapidities parameterize redshift-like proper-time effects without invoking path integrals.

Minimal Python (reproducible).

```
import numpy as np, math

# --- Part 1: Rapidity in 2D with g = diag(-1, +1)
g = np.diag([-1.0, 1.0])

def mink_inner(u, v):
    return float(u @ g @ v)

u = np.array([2.0, 1.0])
v = np.array([1.5, 0.2])

uu = mink_inner(u, u)
vv = mink_inner(v, v)
uv = mink_inner(u, v)

cosh_alpha = -uv / math.sqrt((-uu) * (-vv))
alpha = math.acosh(max(1.0, cosh_alpha))

print("Rapidity alpha (rad, deg):", alpha, alpha * 180.0 / math.pi)

# --- Part 2: Hyperbolic triangle on unit hyperboloid in R^{1,2}
G = np.diag([-1.0, 1.0, 1.0])

def mink_inner3(a, b):
    return float(a @ G @ b)

def X(rho, theta):
    return np.array([
        math.cosh(rho),
        math.sinh(rho) * math.cos(theta),
        math.sinh(rho) * math.sin(theta)
    ])

def hyp_dist(A, B):
    cosh_d = -mink_inner3(A, B)
    return math.acosh(max(1.0, cosh_d))

A = X(0.0, 0.0)
B = X(0.8, 0.0)
C = X(1.1, 0.6)

c = hyp_dist(A, B)  # AB
b = hyp_dist(A, C)  # AC
a = hyp_dist(B, C)  # BC (opposite A)

cos_alphaA = (math.cosh(b) * math.cosh(c) - math.cosh(a)) / (math.sinh(b) * math.sinh(c))
cos_alphaA = max(-1.0, min(1.0, cos_alphaA))
alphaA = math.acos(cos_alphaA)

print("Sides (a,b,c):", a, b, c)
```

```
print("Angle at A (rad, deg):", alphaA, alphaA * 180.0 / math.pi)
```

Numerical summary (this run).

Rapidity: $\alpha_H = 0.4151741510$ rad (23.7877266°).
Triangle sides: $(a, b, c) = (0.6965041802,\ 1.1000000000,\ 0.8000000000)$.
Angle at A : $\alpha_A = 0.6000000000$ rad (34.3774677°) (law-of-cosines and tangent-space check agree).

E Structured Kernel Source from IGRF–13 (Dipole) & Reproducibility Protocol

E.1 Provenance and Scope

The International Geomagnetic Reference Field (IGRF–13) coefficient set (*Schmidt semi-normalized spherical harmonics*) is used as a structured prior for CTMT kernel prefactors in geospace transport experiments. All values reported in this appendix are taken from the publicly available IGRF–13 coefficient tables provided by the International Association of Geomagnetism and Aeronomy (IAGA): `https://www.ngdc.noaa.gov/IAGA/vmod/igrf.html`.

We focus on the *degree-1 (dipole) terms* at epoch 2020.0: $g_1^0,\ g_1^1,\ h_1^1$, which provide a concise and stable calibration prior for CTMT kernel normalization across global scales.

E.2 Stepwise Computation

1. **Locate the 2020.0 column** in the IGRF–13 coefficient table. The final numeric column corresponds to the 2020–25 secular variation (SV), so the 2020.0 values appear in the *penultimate* column.
2. **Extract the dipole coefficients** ($n = 1$): $g_1^0,\ g_1^1,\ h_1^1$ at epoch 2020.0.
3. **Compute dipole strength**

$$S_1 = \sqrt{(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2},$$

   representing the equatorial surface-field magnitude of the degree-1 component (nT).
4. **Compute dipole-axis orientation** from the geocentric vector $\mathbf{m} = (-g_1^1,\ -h_1^1,\ g_1^0)$, with latitude/longitude obtained via standard spherical mapping (colatitude from the z-component; longitude via $\operatorname{atan2}(y, x)$).
5. **CTMT usage:** treat S_1 as a global scalar prior for kernel amplitude normalization, and the axis (φ, λ) as a directional prior for anisotropy or coherence alignment in the Fisher projection when geospace observables are incorporated.

E.3 Minimal Python Listing (Dipole Extraction & Axis)

The script below reads an IGRF–13 coefficient file in the standard format and returns the dipole coefficients at 2020.0, the dipole strength S_1, and the dipole-axis latitude/longitude.

```
# -*- coding: utf-8 -*-
# Minimal IGRF-13 dipole extractor for epoch 2020.0 (Schmidt semi-normalized)

import math
```

```
path = "igrf13coeffs.txt"
coeffs = []
with open(path, "r", encoding="utf-8", errors="ignore") as f:
    for line in f:
        s = line.strip().split()
        if not s or s[0] not in ("g", "h"):
            continue
        kind, n, m = s[0], int(s[1]), int(s[2])
        vals = [float(x.replace("\\-", "-")) if "\\-" in x else float(x)
                for x in s[3:]]
        if len(vals) < 2:
            continue
        val2020 = vals[-2]      # penultimate column = epoch 2020.0
        coeffs.append((kind, n, m, val2020))

vals2020 = {(k, n, m): v for (k, n, m, v) in coeffs if n == 1}
g10 = vals2020[("g", 1, 0)]
g11 = vals2020[("g", 1, 1)]
h11 = vals2020[("h", 1, 1)]

S1 = math.sqrt(g10*g10 + g11*g11 + h11*h11)
x, y, z = -g11, -h11, g10
r = math.sqrt(x*x + y*y + z*z)
colat = math.degrees(math.acos(z / r))
lat = 90.0 - colat
lon = math.degrees(math.atan2(y, x))

print("g10 =", g10, "nT")
print("g11 =", g11, "nT")
print("h11 =", h11, "nT")
print("S1  =", S1,  "nT")
print("axis latitude  =", lat, "deg")
print("axis longitude =", lon, "deg")
```

E.4 Computed Results (Epoch 2020.0)

Executing the script in §E yields:

$$g_1^0 = -29404.8\ \text{nT}, \quad g_1^1 = -1450.9\ \text{nT}, \quad h_1^1 = +4652.5\ \text{nT},$$
$$S_1 = 2.9806 \times 10^4\ \text{nT}, \quad \text{axis latitude} \approx -80.59^\circ, \quad \text{axis longitude} \approx -72.68^\circ.$$

These values correspond directly to the degree-1 rows of the IGRF–13 table at epoch 2020.0.

CTMT plug-in. Use S_1 as a dimensioned calibration factor C_{phys} for magneto-sensitive observables, and the axis (φ, λ) to initialize a directional prior or orient the phase-Hessian projection P_i^μ when constructing the Fisher blocks $(F_\parallel, F_\perp)$ in coherence-constrained transport scenarios.

E.5 Reproducibility Protocol (Consolidated)

1. **Data source** — record the exact IGRF–13 coefficient file used, obtained from the official IAGA distribution site.

2. **Epoch & subset** — specify the epoch (here: 2020.0) and the degrees/orders used (here: dipole only, $n=1$).

3. **Random seeds** — fix RNG seeds for ensemble or path sampling and report them alongside any stochastic results.

4. **Importance sampling discipline** — when coherence-biased sampling is used, treat it as a proposal density q and correct to the physics density p via p/q; report q and any pruning thresholds.

5. **Windows and regimes** — specify curvature windows where the Fisher metric H (or projected metric g) is approximated as constant; justify the chosen windows.

6. **Energy accounting** — avoid substituting pruned/survivor sums for energy integrals; if pruning is unavoidable, provide an unbiased estimator or a bound on the induced bias.

7. **Version log** — record code versions, parameter files, and a minimal run sheet reproducing the values in §E.4.

E.6 Audit Notes

- *Coefficient provenance:* values in §E.4 were extracted from the IGRF–13 degree-1 rows at epoch 2020.0 (Schmidt semi-normalized).
- *Transform conventions:* the geocentric dipole vector is $\mathbf{m} = (-g_1^1, -h_1^1, g_1^0)$; latitude/longitude follow the usual $(\text{acos}, \text{atan2})$ spherical mapping.