Exercise 2

1. Jacobi - Matrix

$$f(u,v) = R\begin{pmatrix}\cos(u)\\ \sin(u)\\ 0\end{pmatrix} + r\begin{pmatrix}\cos(u)\cos(v)\\ \sin(u)\cos(v)\\ \sin(v)\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} R\cdot\cos(u) + r\cos(u)\cos(v)\\ R\sin(u) + r\sin(u)\cos(v)\\ r\sin(v)\end{pmatrix} \begin{matrix}\to f_1\\ \to f_2\\ \to f_3\end{matrix}$$

Jacobi - Matrix von $f(u,v)$:

$$\Rightarrow \begin{pmatrix} f_1'u & f_1'v\\ f_2'u & f_2'v\\ f_3'u & f_3'v\end{pmatrix} = \begin{pmatrix} -R\sin(u) - r\sin(u)\cos(v) & -r\cos(u)\sin(v)\\ R\cos(u) + r\cos(u)\cos(v) & -r\sin(u)\sin(v)\\ 0 & r\cos(v)\end{pmatrix}$$

2. First Fundamental Form

$$\underline{s}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix} R\cdot\cos(u) + r\cos(u)\cos(v)\\ R\sin(u) + r\sin(u)\cos(v\\ r\sin(v)\end{pmatrix}$$

$$\underline{s}_u\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix} -R\sin(u) - r\sin(u)\cos(v)\\ R\cos(u) + r\cos(u)\cos(v)\\ 0\end{pmatrix} \qquad \underline{s}_v\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix} -r\cos(u)\sin(v)\\ -r\sin(u)\sin(v)\\ r\cos(v)\end{pmatrix}$$

$$I_{\underline{s}}^{\underline{s}} = \begin{pmatrix} E & F\\ F & G\end{pmatrix}$$

$$E = \underline{s}_u^2 = (-R\sin(u) - r\sin(u)\cos(v))^2 + (R\cos(u) + r\cos(u)\cos(v))^2 + 0$$

$$= (\underline{R^2\sin(u)^2} + \underline{2Rr\sin(u)^2\cos(v)} + \underline{r^2\sin(u)^2\cos(v)^2} + \underline{R^2\cos(u)^2} + \underline{2Rr\cos(u)^2\cos(v)} + \underline{r^2\cos(u)^2\cos(v)^2}$$

$$= R^2\underbrace{(\sin(u)^2 + \cos(u)^2)}_{=1} + 2Rr\cos(v)\underbrace{(\sin(u)^2 + \cos(u)^2)}_{=1} + r^2\cos(v)^2\underbrace{(\sin(u)^2 + \cos(u)^2}_{=1}$$

$$= R^2 + 2Rr\cos(v) + r^2\cos(v)^2$$

$$= (R + r\cos(v))^2$$

$$G = \underline{s}_v^2 = r^2\cos(u)^2\sin(v)^2 + r^2\sin(u)^2\sin(v)^2 + r^2\cos(v)^2$$

$$= r^2\underline{\sin(v)^2}(\underline{\cos(u)^2 + \sin(u)^2}) + r^2\underline{\cos(v)^2}$$

$$= r^2$$

$$\underline{S}_u\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}-R\sin(u) - r\sin(u)\cos(v)\\ R\cos(u) + r\cos(u)\cos(v)\\ 0\end{pmatrix} \qquad \underline{S}_v\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}-r\cos(u)\sin(v)\\ -r\sin(u)\sin(v)\\ r\cos(v)\end{pmatrix}$$

$$F = \langle \underline{S}_u, \underline{S}_v \rangle$$

$$= Rr\sin(u)\cos(u)\sin(v) + r^2\sin(u)\cos(v)\cos(u)\sin(v)$$
$$- Rr\cos(u)\sin(u)\sin(v) - r^2\cos(u)\cos(v)\sin(u)\sin(v)$$
$$+ 0$$

$$= 0$$

$$I^{\underline{S}}_{\underline{x}} = \begin{pmatrix}E & F\\ F & G\end{pmatrix} = \begin{pmatrix}(R + r\cos(v))^2 & 0\\ 0 & r^2\end{pmatrix}$$

c) Surface Area

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \sqrt{\det I^{\underline{S}}_{\underline{x}}}\, du\, dv$$

$$\Longrightarrow \sqrt{\det I^{\underline{S}}_{\underline{x}}} : \begin{pmatrix}(R + r\cos(v))^2 & 0\\ 0 & r^2\end{pmatrix}$$

$$= \sqrt{r^2 \cdot (R + r\cos(v))^2}$$

$$= |r(R + r\cos(v))|$$

$$\Rightarrow A = \int_0^{2\pi}\int_0^{2\pi} |Rr + r^2\cos(v)|\, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} Rr\, du\, dv + \int_0^{2\pi}\int_0^{2\pi} r^2\cos(v)\, du\, dv$$

$\sin(2\pi) - \sin(0) = 0 + 0 = 0$

$$= 2\pi Rr \int_0^{2\pi} dv + 2\pi r^2 \int_0^{2\pi} \cos(v)\, dv$$

$$= 4\pi^2 Rr \,//$$