UFCG/CCT/Unidade Acadêmica de Matemática **Revisão da Unidade3**

DISCIPLINA: Álgebra Linear I **PERÍODO: 2021.2e**

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Momento Síncrono **Data: 16/08/2022**

Sejam $\alpha = \{(1,-1),(0,2)\}$ e $\beta = \{(1,0,-1),(0,1,2),(1,2,0)\}$ bases ordenadas de $\mathbb{R}^2$ e $\mathbb{R}^3$respectivamente. Seja $T : \mathbb{R}^2 \to \mathbb{R}^3$ a transformação linear tal que $[T]^{\alpha}_{\beta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$. Determine:

(a) $T(x,y)$.

(b) $[S]^{\beta}_{\beta}$, sabendo-se que $S : \mathbb{R}^3 \to \mathbb{R}^3$ é dada por $S(x,y,z) = (2y, x-y, z)$.

(c) $I_m(T)$ e conclua se T é injetora ou não.

(d) $N(S)$ e justifique porque S é sobrejetora.

(e) A inversa de $S(x,y,z)$, $S^{-1}(x,y,z)$.

(f) $\left(S \circ S^{-1}\right)(x,y,z)$.

(g) O polinômio característico e os autovalores de S.

(h) Uma base para cada autoespaço de S e conclua que S é diagonalizável.

(i) O polinômio minimal de S.

(j) A matriz P^{-1} escrevendo a sua inversa P, tal que as matrizes $[S]$ e $[S]^{\beta'}_{\beta'}$ sejam semelhantes. β' é uma base de $\mathbb{R}^3$ constituída sòmente de autovetores de S determinada no ítem (h).

Solução:

$$(a)\,[T]_{\beta}^{\alpha}=\begin{bmatrix}1&0\\1&1\\0&-1\end{bmatrix},\quad \alpha=\{(1,-1)\,,(0,2)\}\;\; e\;\beta=\{(1,0,-1)\,,(0,1,2)\,,(1,2,0)\}\,.$$

$$T(x,y)=?$$

$$T(1,-1)=1\,(1,0,-1)+1\,(0,1,2)+0\,(1,2,0)=(1,1,1)\,.$$

$$T(0,2)=0\,(1,0,-1)+1\,(0,1,2)-1\,(1,2,0)=(-1,-1,2)\,.$$

$$(x,y)=a\,(1,-1)+b\,(0,2)\Longleftrightarrow a=x\;\;\text{ e }\;\;-a+2b=y\Longrightarrow b=\frac{x+y}{2}.$$

$$(x,y)=x\,(1,-1)+\left(\frac{x+y}{2}\right)(0,2)$$

$$T\,(x,y)=xT\,(1,-1)+\left(\frac{x+y}{2}\right)T\,(0,2)$$

$$T\,(x,y)=x\,(1,1,1)+\left(\frac{x+y}{2}\right)(-1,-1,2)$$

$$\boxed{T\,(x,y)=\left(\frac{x-y}{2},\;\frac{x-y}{2},\;2x+y\right).}$$

(b) $[S]_{\beta}^{\beta} =?$ onde: $\boxed{S(x,y,z) = (2y, x-y, z)}$ e $\beta = \{(1,0,-1)\,,(0,1,2)\,,(1,2,0)\}\,.$

$$S(1,0,-1) = (0,1,-1) = a\,(1,0,-1) + b\,(0,1,2) + c(1,2,0)$$

$$\left\{\begin{array}{rcrcr} a & + & c & = & 0 \\ b & + & 2c & = & 1 \\ -a & + & 2b & = & -1 \end{array}\right. \Longleftrightarrow a = -1;\, b = -1 \;\text{ e }\; c = 1.$$

$$S(0,1,2) = (2,-1,2) = a\,(1,0,-1) + b\,(0,1,2) + c(1,2,0)$$

$$\left\{\begin{array}{rcrcr} a & + & c & = & 2 \\ b & + & 2c & = & -1 \\ -a & + & 2b & = & 2 \end{array}\right. \Longleftrightarrow a = 4;\, b = 3 \;\text{ e }\; c = -2.$$

$$S(1,2,0) = (4,-1,0) = a\,(1,0,-1) + b\,(0,1,2) + c(1,2,0)$$

$$\left\{\begin{array}{rcrcr} a & + & c & = & 4 \\ b & + & 2c & = & -1 \\ -a & + & 2b & = & 0 \end{array}\right. \Longleftrightarrow a = 6;\, b = 12 \;\text{ e }\; c = -2.$$

$$\boxed{[S]_{\beta}^{\beta} = \begin{bmatrix} -1 & 4 & 6 \\ -1 & 3 & 12 \\ 1 & -2 & -2 \end{bmatrix}. \text{ Resolva os 3 sistemas lineares.}}$$

$(c)\ I_m(T) = \left\{ \left(\frac{x-y}{2},\ \frac{x-y}{2},\ 2x+y \right) / x, y \in \mathbb{R} \right\}$

$$I_m(T) = \left\{ \left(\frac{x}{2},\ \frac{x}{2},\ 2x \right) + \left(\frac{-y}{2},\ \frac{-y}{2},\ y \right) / x, y \in \mathbb{R} \right\}$$

$$I_m(T) = \left\{ x \left(\frac{1}{2},\ \frac{1}{2},\ 2 \right) + y \left(\frac{-1}{2},\ \frac{-1}{2},\ 1 \right) / x, y \in \mathbb{R} \right\}$$

$$I_m(T) = \left[\left(\frac{1}{2},\ \frac{1}{2},\ 2 \right), \left(\frac{-1}{2},\ \frac{-1}{2},\ 1 \right) \right]$$

$$\beta_{I_m(T)} = \left\{ \left(\frac{1}{2},\ \frac{1}{2},\ 2 \right), \left(\frac{-1}{2},\ \frac{-1}{2},\ 1 \right) \right\} \text{ ou}$$

$$\beta_{I_m(T)} = \{(1,\ 1,\ 4), (-1,\ -1,\ 2)\} \Longrightarrow \dim I_m(T) = 2.$$

Conclusão: $\boxed{T \textbf{ é injetora, isto é, } N(T) = \{(0,0)\}}$, pois

$$\dim N(T) + \dim I_m(T) = 0 + 2 = \dim \mathbb{R}^2.$$

$(d)\ N(S) = ?$

$$N(S) = \left\{ (x,y,z) \in \mathbb{R}^3 / S(x,y,z) = (0,0,0) \right\}$$

$$N(S) = \left\{ (x,y,z) \in \mathbb{R}^3 / (2y, x-y, z) = (0,0,0) \right\}$$

$N(S) = \{(0,0,0)\}$. S é injetora.

$\boxed{\text{Como } S : \mathbb{R}^3 \to \mathbb{R}^3,\ \dim I_m(T) = 3. \text{ Logo } S \textbf{ é sobrejetora}.}$

Portanto, S é bijetora (um isomorfismo).

(e) $\boxed{S(x,y,z) = (2y, x-y, z)}$.

Determinação de $S^{-1}(x,y,z)$.

$$\left\{\begin{array}{rcr} S(1,0,0) &=& (0,1,0) \\ S(0,1,0) &=& (2,-1,0) \\ S(0,0,1) &=& (0,0,1) \end{array}\right. \Longrightarrow [S] = A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\det A = \begin{vmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = 0 - 2 = -2 \neq 0. \text{ Logo } A^{-1} \text{ existe.}$$

$$\left[\begin{array}{rrr|rrr} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \approx \left[\begin{array}{rrr|rrr} 2 & 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1/2 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]. \quad A^{-1} = \begin{bmatrix} 1/2 & 1 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boxed{S^{-1}(x,y,z) = \begin{bmatrix} 1/2 & 1 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\frac{1}{2}x + y, \frac{1}{2}x, z\right).}$$

(f) Composição de $\boxed{S(x,y,z) = (2y, x-y, z)}$ com $\boxed{S^{-1}(x,y,z) = \left(\frac{1}{2}x + y, \frac{1}{2}x, z\right)}$?

$$
\begin{aligned}
\left(S \circ S^{-1}\right)(x,y,z) &= S\left(S^{-1}(x,y,z)\right) \\
&= S\left(\frac{1}{2}x + y, \frac{1}{2}x, z\right) \\
&= \left(2\left(\frac{1}{2}x\right), \frac{1}{2}x + y - \frac{1}{2}x, z\right) \\
&= (x,y,z)\,. \quad \textit{Coincidência?}
\end{aligned}.
$$

(g) **Polinômio característico e autovalores de** S :

$$
[S] = A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \;\; A - \lambda I = \begin{bmatrix} -\lambda & 2 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}.
$$

$$
\begin{aligned}
p(\lambda) &= \begin{vmatrix} -\lambda & 2 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} && = (1-\lambda)\begin{vmatrix} -\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} \\
&= (1-\lambda)\left(\lambda + \lambda^2 - 2\right) && = (1-\lambda)\left(\lambda^2 + \lambda - 2\right) \\
&= (1-\lambda)(\lambda - 1)(\lambda + 2) && = -(\lambda - 1)^2(\lambda + 2)\,. \\
p(\lambda) &= 0 \qquad \iff && -(\lambda-1)^2 = 0 \text{ ou } (\lambda+2) = 0 \iff
\end{aligned}
$$

$\lambda - 1 = 0$ ou $\lambda + 2 = 0 \iff \lambda = 1$ ou $\lambda = -2$.

$\boxed{\textbf{Autovalores:} \;\; -2 \;\; \textbf{e} \;\; 1.}$

(h) **Base para cada autoespaço de** S. $\left([S]-\lambda I = \begin{bmatrix} -\lambda & 2 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right)$.

$$p/\lambda = -2; \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{cases} 2x+2y &= 0 \\ x+y &= 0 \\ 3z &= 0 \end{cases} \Longleftrightarrow$$

$$\begin{cases} y=-x \quad ; \quad z=0 \\ \qquad x \; - \; livre \end{cases}.$$

$V_{-2} = \{(x,-x,0)\,/x \in \mathbb{R}\} = \{x\,(1,-1,0)\,/x \in \mathbb{R}\} = [(1,-1,0)]$;

$\beta_{V_{-2}} = \{(1,-1,0)\}$. $\dim V_{-2} = 1$.

$$p/\lambda = 1; \begin{bmatrix} -1 & 2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{cases} -x+2y &= 0 \\ x-2y &= 0 \\ 0 &= 0 \end{cases} \Longleftrightarrow$$

$$\begin{cases} x=2y \quad ; \quad y-livre \\ \qquad e \;\; z-livre. \end{cases}.$$

$V_1 = \{(2y,y,z)\,/y,z \in \mathbb{R}\} = \{y\,(2,1,0)+z\,(0,0,1)\,/y,z \in \mathbb{R}\} = [(2,1,0),(0,0,1)]$;

$\beta_{V_1} = \{(2,1,0),(0,0,1)\}$. $\dim V_1 = 2$. $\beta' = \{(1,-1,0),(2,1,0),(0,0,1)\}$ **é uma base de** $\mathbb{R}^3$ **constituida apenas de autovetores de** S**. Logo** S **é diagonalizável.**

$$\boxed{\text{Portanto, } [S]_{\beta'}^{\beta'} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.}$$

(i) **Polinômio minimal de** S. $\left(\boxed{p(\lambda) = -(\lambda-1)^2(\lambda+2)}\right)$.

Candidatos: $(i)\, m_1(x) = -(x-1)(x+2)$ e $(ii)\, m_2(x) = -(x-1)^2(x+2)$.

$$\begin{aligned} m_1(A) &= -(A-I)(A+2I) \\ &= -\begin{bmatrix} -1 & 2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

$\boxed{\textbf{Isto mostra que } m_1(x) = -(x-1)(x+2) \textbf{ é o polinômio minimal de } S.}$

Novamente, constatamos que S **é diagonalizável**.

(j) **Vamos mostrar que as matrizes** $[S]$ **e** $[S]_{\beta'}^{\beta'}$ **são semelhantes.**

Seja $P = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longleftrightarrow P^{-1} = \begin{bmatrix} 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Encontrando P^{-1}, onde $\boxed{P \text{ é a matriz constituída com os autovetores de } S}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & -2 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -2/3 & 0 \\ 0 & 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]. \quad P^{-1} = \begin{bmatrix} 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.

$$
\begin{aligned}
P^{-1}\,[S]\,P &= \begin{bmatrix} 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [S]_{\beta'}^{\beta'}
\end{aligned}
$$

Isto mostra que as matrizes $[S]$ e $[S]_{\beta'}^{\beta'}$ são semelhantes.

U F A !

Boa Prova! Boas Férias!

Boa Sorte nos estudos!