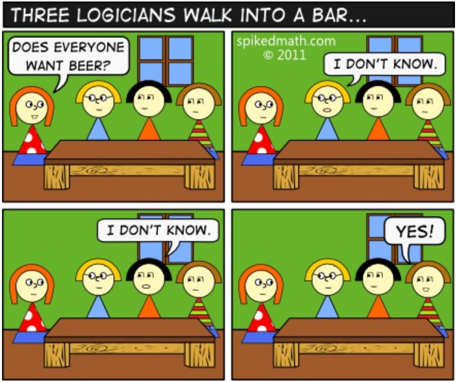
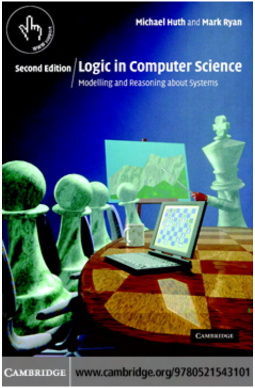
Logic and Sets 1

introduction to propositional logic



Material logic lectures

- M. Huth, M. Ryan, *Logic in Computer Science*, Cambridge University Press, 2nd edition, 2004.
  - we treat only some parts (Chapters 1, 2, 6)
  - some topics in logic lectures are not covered
  - copies of relevant parts of the book on Canvas



- Lecture slides

  available on Canvas

- Exercises for working groups

  also available on Canvas

- Enroll for a Webtutor with interactive logic exercises at

  `https://infinity.few.vu.nl/logic`

Importance of logic

Hardware circuits are based on (Boolean) logic.

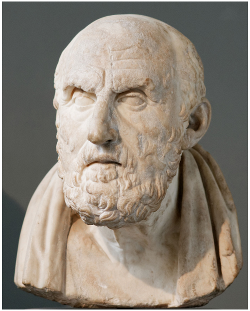
Software is basically a piece of logic.

Artificial intelligence is founded on logic.

Reasoning about an *information system* or *computer program* requires

- a formal logical framework, and
- a trained mind in logical thought.

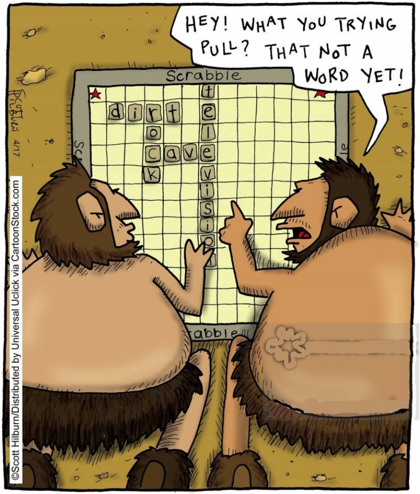
Propositional logic



Chrysippus of Soli *(c. 279 – c. 206 BC)*

introduced a formal system for propositional logic

Propositional logic: Syntax



A line of argument (in natural language)

We want to understand why arguments such as these hold.

Example

If the train arrives late and there are no taxis at the station,
then Jane is late for her meeting.

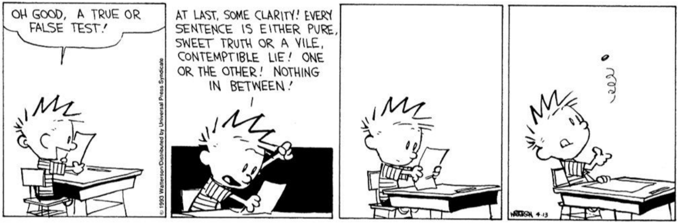
Jane is not late for the meeting.

The train does arrive late.

Therefore, there are taxis at the station.

Declarative sentences

A *declarative sentence* (or *proposition*) is a *true* or *false* statement.



Examples

- $5 > 3$
- $5 < 3$
- grass is green
- grass is green and roses are blue
- grass is green or roses are blue
- if $x > 1$, then $x^2 \neq x$

Declarative sentences

More examples

- All Martians like pepperoni on their pizza.
- Every even number > 2 is the sum of two prime numbers.

  Goldbach conjecture, 1742
- It is possible to derive $0 = 1$ in the natural numbers.

  Gödel's 1st incompleteness theorem, 1931
- You are enjoying this lecture.

Non-examples

- Could you please be quiet during my lecture?
- Prepare for the exercise classes!
- May you do well at the exam.

Argument abstraction

Example (Jane)

If the train arrives late, and there are no taxis at the station, then Jane is late for her meeting.

Jane is not late for her meeting.

The train does arrive late.

Therefore, there are taxis at the station.

Key of translation

p	the train arrives late
q	there are taxis at the station
r	Jane is late for her meeting

Abstraction:

If p and not q, then r. Not r. p. *Therefore,* q.

Argument abstraction (another example)

Example (John)

If it is raining, and John did not take his umbrella with him,
then he will get wet.

John is not getting wet.

It is raining.

Therefore, John took his umbrella with him.

Key of translation

p	it is raining
q	John takes his umbrella with him
r	John is getting wet

Abstraction:

If p and not q, then r. Not r. p. *Therefore,* q.

Argument formalization

	Example (Jane)	Example (John)
p	the train is late	it is raining
q	there are taxis at the station	John has his umbrella with him
r	Jane is late for her meeting	John is getting wet

Both arguments have the same abstraction:

If p and not q, then r. Not r. p. *Therefore,* q.

with as logical formalization:

$$(((p \wedge \neg q) \rightarrow r) \quad \wedge \quad (\neg r \ \wedge \ p)) \quad \rightarrow \quad q$$

That the two arguments hold is due to their *logical form*.

It does not depend on the actual content of propositions p, q and r.

Symbols of propositional logic

We want to study logic without being distracted by the concrete contents of propositions.

Propositional variables:

$p, q, r, \ldots$

Connectives:

$\wedge$	'and'	*conjunction*
$\vee$	'or'	*disjunction*
$\oplus$	'either ... or ...'	*exclusive or*
$\neg$	'not'	*negation*
$\rightarrow$	'if ... then ...'	*implication*
$\leftrightarrow$	'if and only if'	*bi-implication*

Not in the scope of propositional logic are constructs like:

- *for all*, *there exists* (will be treated in lecture 6)
- *eventually* (temporal logic), *I know that* (epistemic logic)

Sentences and formulas

Propositional structure of sentences

- $5 > 3$ — p
- grass is green — p
- grass is green and roses are blue — $p \wedge q$
- grass is green or roses are blue — $p \vee q$
- if $x > 1$, then $x^2 \neq x$ — $p \rightarrow \neg q$

Question

What are p and q in the last example ?

Example

The sentence “I don’t wear glasses” can be expressed by

- $\neg p$ where p represents “I wear glasses”
- q where q represents “I don’t wear glasses”

It's all Greek to me

Following Huth and Ryan, we use *Greek letters* for propositional formulas.

ϕ	phi
ψ	psi
χ	chi

Formulas of propositional logic

Building blocks:

- $p, q, r, \ldots$ are propositional variables
- $\neg$ is a unary connective
- $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$ are binary connectives

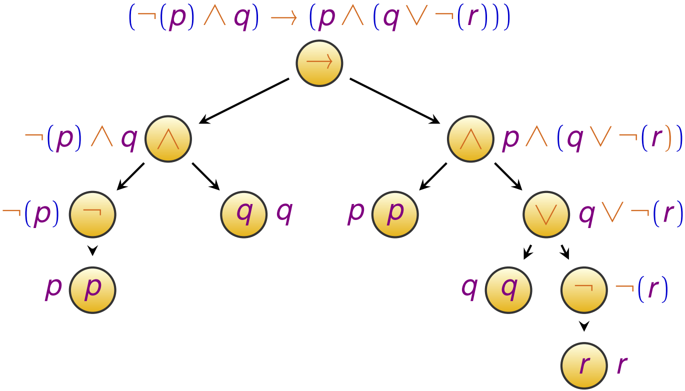
The construction of formulas:

Inductive definition

1. (BASE STEP) every propositional variable is a formula
2. (CONSTRUCTION STEPS)
   - 2.1 if ϕ is a formula, then so is $\neg(\phi)$
   - 2.2 if ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \oplus \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$

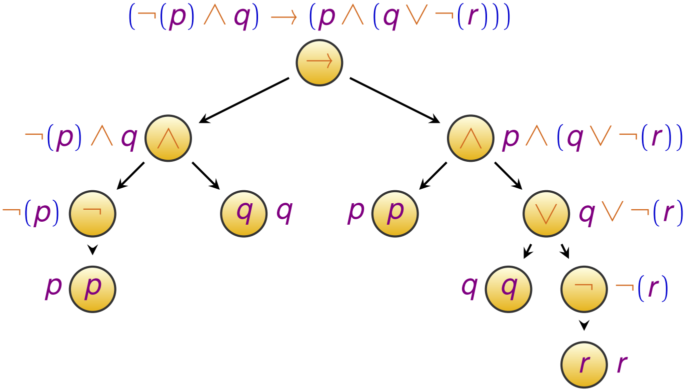
We take the liberty to omit the outermost pair of parentheses.

Parsing a formula



Reconstructing a formula

A formula can be reconstructed from its parse tree:



Omitting parentheses

Question

How do you read the arithmetic expression $3 \times 5 + 2$?

Is it 17 or 21 ?

Omitting parentheses

To omit parentheses from formulas, *without causing ambiguity*, we use the priority schema:

$$\begin{array}{c} \neg \\ \hline \wedge \quad \vee \\ \hline \rightarrow \quad \leftrightarrow \end{array}$$

Examples

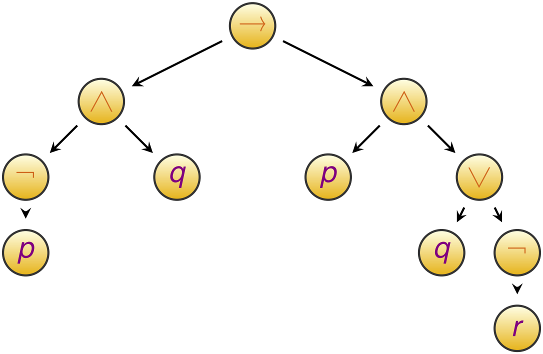
- $\neg(p) \vee q \quad \rightsquigarrow \quad \neg p \vee q$
- $\neg(p \vee q)$ Note: $\not\rightsquigarrow \quad \neg p \vee q$
- $\neg(r) \rightarrow (p \vee q) \quad \rightsquigarrow \quad \neg r \rightarrow p \vee q$
- $r \wedge (p \vee q)$ Note: $\not\rightsquigarrow \quad r \wedge p \vee q$
- $\neg(\neg(p)) \quad \rightsquigarrow \quad \neg\neg p$

Question

Which two pairs of parentheses cannot be omitted from
$(((p \wedge \neg(q)) \rightarrow r) \wedge (\neg(r) \wedge p)) \rightarrow q$?

Parse tree

By the conventions regarding omitted parentheses,
the parse tree of $\neg p \land q \to p \land (q \lor \neg r)$ is:



Remark: Feel free to write all parentheses, if you prefer this.

Syntax and semantics

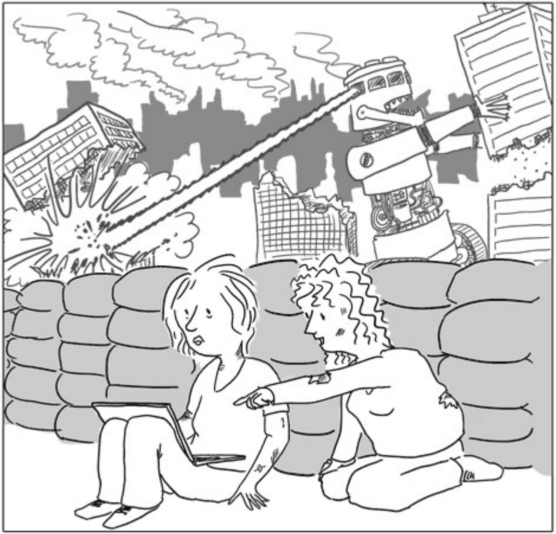
The syntax is how a word (or formula) is *written*.

The semantics is its *meaning*.

For example, the syntax for the animal shown above is *cow*.

The semantics of *cow* is a female mammal, white with black spots, eats grass, gives milk, says moo.

Propositional logic: Semantics



See? You forgot to close a parenthesis in its master control program.

Truth value semantics for propositional logic

Formulas of propositional logic are used to express declarative statements, which are either *true* or *false*.

We introduce the truth values $\mathtt{T}$ and $\mathtt{F}$, corresponding to truth and falsehood, respectively.

The truth value of a formula such as $\phi \wedge \psi$ is determined by the truth values of its components, ϕ and ψ.

For each connective, this is expressed by its truth table.

Negation

A negation $\neg\phi$ ("not ϕ") is

$$\begin{cases} \text{true} & \text{if } \phi \text{ is false} \\ \text{false} & \text{if } \phi \text{ is true} \end{cases}$$

Truth table for $\neg$:

ϕ	$\neg\phi$
T	F
F	T

T = true
F = false

Conjunction

A conjunction $\phi \wedge \psi$ ("ϕ and ψ") is

$$\begin{cases} \text{true} & \text{if } \phi \text{ is true } \textbf{and } \psi \text{ is true} \\ \text{false} & \text{in all other cases} \end{cases}$$

Truth table for $\wedge$:

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

We first list all possible combinations of assignments to ϕ and ψ

Disjunction

A disjunction $\phi \vee \psi$ ("ϕ or ψ") is

$$\begin{cases} \text{true} & \text{if } \phi \text{ is true } \textbf{or } \psi \text{ is true (or both)} \\ \text{false} & \text{otherwise} \end{cases}$$

Truth table for $\vee$:

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

Inclusive versus exclusive or

Examples in natural language:

- Inclusive:

  Do you take sugar or cream in your coffee ?

- Exclusive:

  Do you want a cappuccino or an espresso ?

Exclusive or

An exclusive or $\phi \oplus \psi$ ("either ϕ or ψ") is

$$\begin{cases} \text{true} & \text{if either } \phi \text{ or } \psi \text{ is true (but not both)} \\ \text{false} & \text{otherwise} \end{cases}$$

Truth table for $\oplus$:

ϕ	ψ	$\phi \oplus \psi$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

$\phi \to \psi$ means: *if* ϕ is true, *then* ψ is true.

Question

For which truth values of ϕ and ψ is $\phi \to \psi$ false ?

Implication

An implication $\phi \rightarrow \psi$ ("if ϕ, then ψ") is

$$\begin{cases} \text{false} & \text{if } \phi \text{ is true and } \psi \text{ is false} \\ \text{true} & \text{otherwise} \end{cases}$$

Truth table for $\rightarrow$:

ϕ	ψ	$\phi \rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Implication

Truth table for $\rightarrow$:

ϕ	ψ	$\phi \rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Consider $(n > 2) \rightarrow (n + 1 > 2)$.

Clearly, this implication is *true* for each n.

$n = 1$: both $n > 2$ and $n + 1 > 2$ are *false*.

$n = 2$: $n > 2$ is *false* while $n + 1 > 2$ is *true*.

$n = 3$: both $n > 2$ and $n + 1 > 2$ are *true*.

Finally, consider $(n = 2) \rightarrow (n = 1)$.

This implication is *false* for $n = 2$ (but *true* for any other value of n).

Principle of explosion

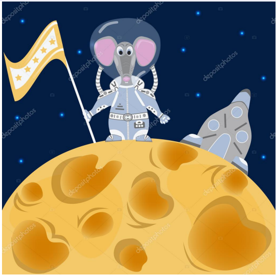
In natural language,

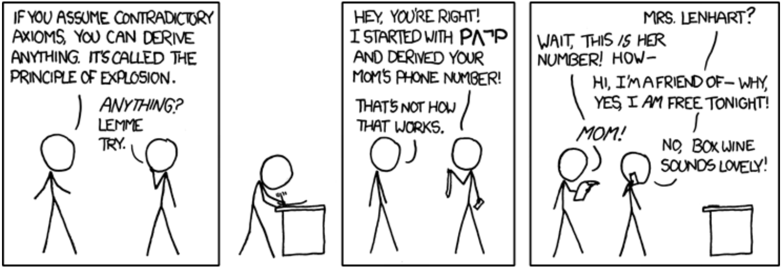
- *if I'm a movie star, then I'm rich*

makes sense, unlike

- *if I'm a movie star, then the moon is made of cheese*.

But as declarative sentences they are both true (because I'm not a movie star).





Bi-implication

A bi-implication $\phi \leftrightarrow \psi$ ("ϕ if and only if ψ") is

$$\begin{cases} \text{true} & \text{if } \phi \text{ and } \psi \text{ have the same truth value} \\ \text{false} & \text{otherwise} \end{cases}$$

Truth table for $\leftrightarrow$:

ϕ	ψ	$\phi \leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Truth values and valuations

The truth value of a formula is determined by
an *assignment of truth values* to its propositional variables.

Such an assignment is called a *valuation*.

Example

Given the valuation: p and q are $\mathtt{T}$, and r is $\mathtt{F}$.

What is the truth value of the formula $p \vee \neg q \to r$?

We reason:

- $p \vee \neg q$ is $\mathtt{T}$ because p is $\mathtt{T}$.
- So, since r is $\mathtt{F}$, it follows that $p \vee \neg q \to r$ is $\mathtt{F}$.

Valuations of a formula in a truth table

A *valuation* corresponds to one row in the truth table of a formula.

Example: Truth table of $p \vee \neg q \rightarrow r$

p	q	r	$\neg q$	$p \vee \neg q$	$p \vee \neg q \rightarrow r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	F

For $p \vee \neg q \rightarrow r$ there are 8 ($= 2^3$) rows in the truth table.

Queston

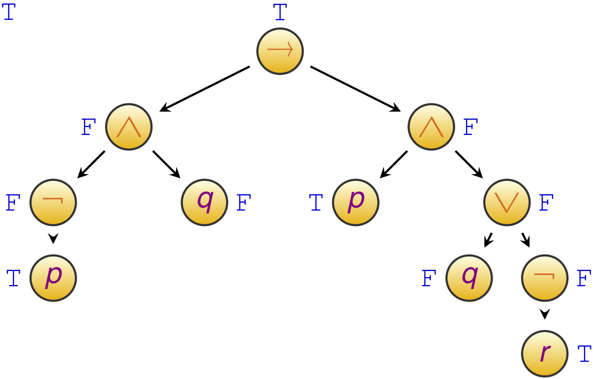
How many rows has a truth table for a formula with n variables ?

Bottom-up truth assignment via a parse tree

The truth value of a formula can be computed via its *parse tree*.

Bottom-up evaluation of $\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$ with:

- $p = \mathtt{T}$
- $q = \mathtt{F}$
- $r = \mathtt{T}$



Semantic equivalence $\equiv$

Definition

Formulas ϕ and ψ are *semantically equivalent*, notation $\phi \equiv \psi$, if they have identical columns in their truth tables.

Example:

Truth tables for $p \rightarrow q$ and $\neg p \vee q$:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The columns for $p \rightarrow q$ and $\neg p \vee q$ are identical.

Hence: $p \rightarrow q \equiv \neg p \vee q$

Expressing connectives

$\leftrightarrow$ can be expressed using $\rightarrow$ and $\wedge$:

$$\phi \leftrightarrow \psi \quad \equiv \quad (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$$

Questions

Express $\phi \oplus \psi$ using $\vee$, $\wedge$ and $\neg$.

Express $\phi \wedge \psi$ using $\vee$ and $\neg$.

Express $\phi \vee \psi$ using $\wedge$ and $\neg$.

Semantic equivalences

$$\neg\neg\phi \quad \equiv \quad \phi$$

$$\phi \wedge \psi \quad \equiv \quad \neg(\neg\phi \vee \neg\psi)$$

So $$\neg(\phi \wedge \psi) \quad \equiv \quad \neg\phi \vee \neg\psi$$

$$\phi \vee \psi \quad \equiv \quad \neg(\neg\phi \wedge \neg\psi)$$

So $$\neg(\phi \vee \psi) \quad \equiv \quad \neg\phi \wedge \neg\psi$$

Important semantic equivalences

$$
\begin{array}{rcll}
\phi \wedge \psi & \equiv & \psi \wedge \phi & \text{commutativity} \\
\phi \vee \psi & \equiv & \psi \vee \phi & \\
\phi \wedge \phi & \equiv & \phi & \text{idempotence} \\
\phi \vee \phi & \equiv & \phi & \\
(\phi \wedge \psi) \wedge \chi & \equiv & \phi \wedge (\psi \wedge \chi) & \text{associativity} \\
(\phi \vee \psi) \vee \chi & \equiv & \phi \vee (\psi \vee \chi) & \\
\neg(\phi \wedge \psi) & \equiv & \neg\phi \vee \neg\psi & \text{De Morgan} \\
\neg(\phi \vee \psi) & \equiv & \neg\phi \wedge \neg\psi & \\
\neg\neg\phi & \equiv & \phi & \text{involution} \\
\phi \rightarrow \psi & \equiv & \neg\phi \vee \psi &
\end{array}
$$

Try to develop intuition for semantic equivalences by practicing (like learning arithmetic as a child).

Associativity of conjunction and disjunction

Conjunction is associative

$$(\phi \wedge \psi) \wedge \chi \quad \equiv \quad \phi \wedge (\psi \wedge \chi)$$

So we can skip parentheses: $\phi \wedge \psi \wedge \chi$

Disjunction is also associative

$$(\phi \vee \psi) \vee \chi \quad \equiv \quad \phi \vee (\psi \vee \chi)$$

So we can skip parentheses: $\phi \vee \psi \vee \chi$

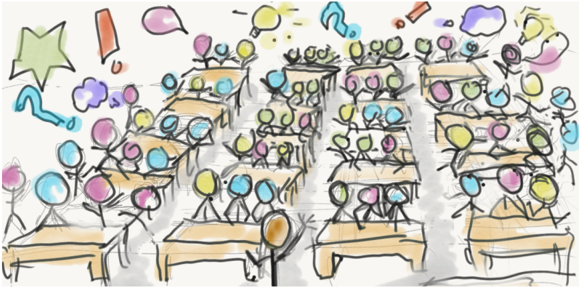
Is the exclusive or associative ?

Question

Does $(\phi \oplus \psi) \oplus \chi \;\equiv\; \phi \oplus (\psi \oplus \chi)$ hold ?

1. Yes
2. No

Feel free to discuss with your neighbors.



Associativity of exclusive or

$\phi \oplus \psi \oplus \chi$ is true *if and only if* an odd number of the arguments ϕ, ψ and χ is true.

ϕ	ψ	χ	$(\phi \oplus \psi) \oplus \chi$	$\phi \oplus (\psi \oplus \chi)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	F	F
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

Is implication associative ?

Question

Does $(\phi \to \psi) \to \chi \quad \equiv \quad \phi \to (\psi \to \chi)$ hold ?

1. Yes
2. No

ϕ	ψ	χ	$(\phi \to \psi) \to \chi$	$\phi \to (\psi \to \chi)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	F	T

Final quiz question

Question

Does $\phi \to (\psi \to \chi) \quad \equiv \quad \phi \wedge \psi \to \chi$ hold ?

1. Yes
2. No

ϕ	ψ	χ	$\psi \to \chi$	$\phi \to (\psi \to \chi)$	$\phi \wedge \psi$	$\phi \wedge \psi \to \chi$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Tautologies, contradictions and contingent formulas

Tautology = always true

$$p \vee \neg p$$

Contradiction = always false

$$p \wedge \neg p$$

Contingent = sometimes true and sometimes false

$$p$$

$$p \wedge \neg q$$

Tautologies

Definition

A formula is a tautology (also called *valid*) if:
its column in a truth table has T on every row.

So: a tautology is true for every valuation.

Examples:

- $p \vee \neg p$
- $p \rightarrow p$
- $p \rightarrow (q \rightarrow p)$
- $(p \wedge \neg q \rightarrow r) \wedge \neg r \wedge p \rightarrow q$

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

(The example of Jane)

Contradictions

Definition

A formula is a contradiction if:
its column in a truth table has $\mathtt{F}$ on every row.

So: a contradiction is false for every valuation.

Examples:

- $p \wedge \neg p$
- $(p \to q) \wedge p \wedge \neg q$

p	q	$p \to q$	$\neg q$	$(p \to q) \wedge p \wedge \neg q$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

More contradictions, and contingent formulas

Questions

Are the following formulas contraditions ?

- $p \rightarrow \neg p$
- $((p \rightarrow q) \rightarrow p) \wedge \neg p$
- $\neg\phi$ with ϕ a tautology

Definition

A propositional formula is contingent if:
it is neither a tautology, nor a contradiction.

Take home

Introduction to propositional logic

- from natural language to formal notation
- syntax of propositional logic
  - from formula to parse tree, and back
- truth value semantics
  - truth tables, valuations
- semantic equivalence
- tautologies and contradictions

Exercise class tomorrow

- *Enroll for working groups on Wednesday and Friday in Canvas.*
- *Try your hand at the exercises !*