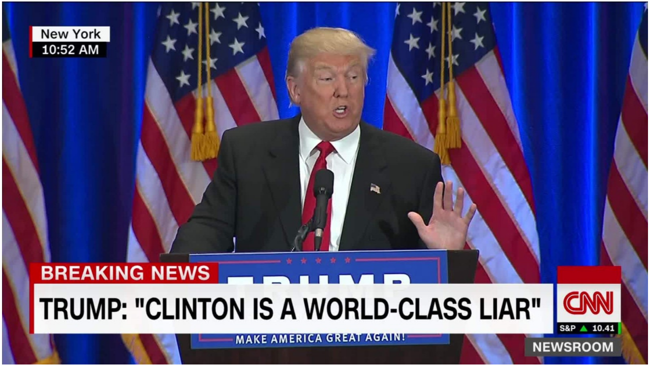
Logic and Sets 2

semantic entailment   metalogic   logic puzzles



Last Tuesday

Introduction to propositional logic

- declarative sentences (propositions)
- syntax of propositional logic
  1. every propositional variable $p, q, r, \ldots$ is a formula
  2. if ϕ and ψ are formulas, then so are $\neg(\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \oplus \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$
- parse trees
- omitting parentheses
- semantics of propositional logic
  - truth tables
  - valuations
  - semantic equivalence $\phi \equiv \psi$
  - tautologies, contradictions

Today

- semantic entailment $\models$
  - $\phi_1, \ldots, \phi_n \models \psi$ (truth of $\phi_1, \ldots, \phi_n$ entails truth of ψ)
  - counterexamples
- metalogic statements:
  - general properties of propositional formulas
  - such as: if $\phi \wedge \psi$ is a tautology, then what about ϕ and ψ ?
- logic puzzles:
  - an island with liars and truth speakers
  - finding one's way on this island
  - a brief encounter with Donald Trump

Semantic entailment

Semantic entailment

Definition

A formula ψ is *semantically entailed* by formulas $\phi_1, \ldots, \phi_n$, denoted by $\phi_1, \ldots, \phi_n \vDash \psi$, if:

> Each valuation that makes $\phi_1, \ldots, \phi_n$ true,
> also makes ψ true.

Each valuation: Each row of the corresponding truth table.

Semantic entailment $\vDash$ is at the core of logical reasoning.

Arguments Jane/John: Formalization

	Example (Jane)	Example (John)
	If the train arrives late, and there are no taxis at the station, then Jane is late for her meeting.	If it is raining, and John did not take his umbrella with him, then John is getting wet.
	Jane is not late for her meeting.	John is not getting wet.
	The train was late.	It is raining.
	Therefore, there were taxis at the station.	*Therefore*, John did take his umbrella with him.
p	the train is late	it is raining
q	there are taxis at the station	John has his umbrella with him
r	Jane is late for her meeting	John is getting wet

$$p \wedge \neg q \rightarrow r, \quad \neg r, \quad p \quad \models \quad q$$

Argumentation Jane/John revisited

We check that

$$p \wedge \neg q \rightarrow r, \quad \neg r, \quad p \quad \models \quad q$$

is a semantic entailment that holds.

p	q	r	$\neg r$	$\neg q$	$p \wedge \neg q$	$p \wedge \neg q \rightarrow r$
T	T	T	F	F	F	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	T	F	T	F	F	T
F	F	T	F	T	F	T
F	F	F	T	T	F	T

- ✓

Counterexamples

Turned around:

$\phi_1, \ldots, \phi_n \models \psi$ does not hold (denoted by: $\phi_1, \ldots, \phi_n \not\models \psi$)
if there exists a valuation v such that:

v makes $\phi_1, \ldots, \phi_n$ true, but *not* ψ.

Such a v is called a *counterexample*.

Question

Give a counterexample if premise $p \wedge \neg q \rightarrow r$ is omitted from

$$p \wedge \neg q \rightarrow r, \qquad \neg r, \qquad p \qquad \models \qquad q$$

Question: Does $p \to q, \neg q \vDash \neg p$ hold ?

The truth table is:

	p	q	$p \to q$	$\neg q$	$\neg p$		
1	T	T	T	F	F		
2	T	F	F	T	F		
3	F	T	T	F	T		
4	F	F	T	T	T	•	✓

- Only in row 4, both $p \to q$ and $\neg q$ are true.
- In that row also $\neg p$ is true.

  So in each row where both *premises* are true, the *conclusion* is also true.

Therefore, the semantic entailment holds.

Question: Does $p \vee q \models p \to q$ hold ?

The truth table is:

	p	q	$p \vee q$	$p \to q$		
1	T	T	T	T	•	✓
2	T	F	T	F	•	×
3	F	T	T	T	•	✓
4	F	F	F	T		

- In rows 1–3, $p \vee q$ is true
- In rows 1 and 3, also $p \to q$ is true
- But in row 2, $p \to q$ is false

This shows that the semantic entailment does not hold:

The valuation $p \mapsto \mathtt{T}$, $q \mapsto \mathtt{F}$ (row 2) is a counterexample.

More questions on semantic entailment

Questions

Do the following semantic entailments hold ?

- $p \wedge q \ \vDash \ p \rightarrow q$
- $p \rightarrow (q \rightarrow r), \ p \ \vDash \ r$
- $p \rightarrow (q \rightarrow r), \ p, \ q \ \vDash \ r$

Truth table for $p \rightarrow (q \rightarrow r)$

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

$p \rightarrow (q \rightarrow r),\ \ p\ \ \not\models\ \ r$

$p \rightarrow (q \rightarrow r),\ \ p,\ \ q\ \ \models\ \ r$

Properties of $\models$

$\models \phi$ denotes that ϕ is a *tautology*.

$\models$ is *reflexive*: $\phi \models \phi$.

$\models$ is *transitive*: If $\phi \models \psi$ and $\psi \models \chi$, then $\phi \models \chi$.

If $\phi_1 \models \psi$, then also $\phi_1, \phi_2 \models \psi$.

Expressing semantic entailment as a formula

Question

Represent $\phi_1, \ldots, \phi_n \models \psi$ as a propositional formula.

Answer: $\models \phi_1 \land \ldots \land \phi_n \to \psi$

Deduction theorem

For all formulas $\phi_1, \ldots, \phi_n$ and ψ:

$$\phi_1, \ldots, \phi_n \models \psi \quad \Longleftrightarrow \quad \phi_1, \ldots, \phi_{n-1} \models \phi_n \to \psi$$

Semantic equivalence via $\vDash$

$\phi \equiv \psi$ holds precisely if both $\phi \vDash \psi$ and $\psi \vDash \phi$.

$$\begin{aligned}
& \phi \equiv \psi \\
\iff \quad & \phi \text{ and } \psi \text{ have the same truth table} \\
\iff \quad & \phi \text{ and } \psi \text{ are } \mathtt{T} \text{ on exactly the same rows} \\
\iff \quad & \phi \vDash \psi \ \text{ and } \ \psi \vDash \phi
\end{aligned}$$

Distributivity law

Distributivity

$$(\phi \wedge \psi) \vee \chi \quad \equiv \quad (\phi \vee \chi) \wedge (\psi \vee \chi)$$

$\models$: $(\phi \wedge \psi) \vee \chi$ means **(1)** $\phi \wedge \psi$ is true or **(2)** χ is true.

(1) If $\phi \wedge \psi$ is true, then ϕ and ψ are true.

This implies that $\phi \vee \chi$ and $\psi \vee \chi$ are true.

So then $(\phi \vee \chi) \wedge (\psi \vee \chi)$ is true.

(2) If χ is true, then $\phi \vee \chi$ and $\psi \vee \chi$ are true.

So then again $(\phi \vee \chi) \wedge (\psi \vee \chi)$ is true.

Hence $(\phi \wedge \psi) \vee \chi \models (\phi \vee \chi) \wedge (\psi \vee \chi)$.

Distributivity law

$\Dashv$: $(\phi \vee \chi) \wedge (\psi \vee \chi)$ means $\phi \vee \chi$ and $\psi \vee \chi$ are both true.

So if χ is false, then ϕ and ψ are both true.

So χ is true, or ϕ and ψ are true.

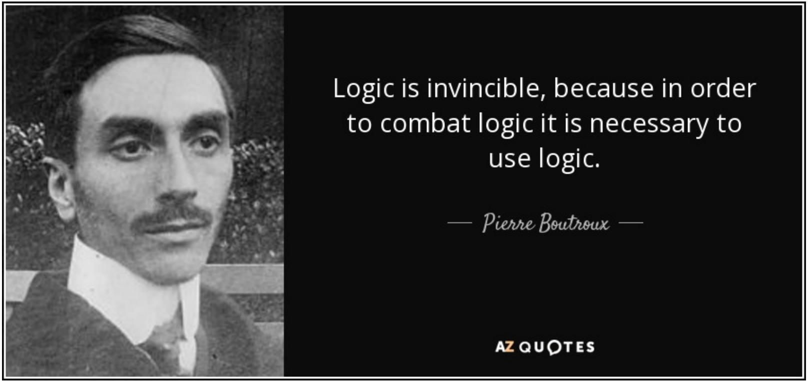
That is, ϕ and ψ are true, or χ is true.

Hence $(\phi \vee \chi) \wedge (\psi \vee \chi) \models (\phi \wedge \psi) \vee \chi$.

Distributivity

$$(\phi \wedge \psi) \vee \chi \equiv (\phi \vee \chi) \wedge (\psi \vee \chi)$$

Metalogic


Logic is invincible, because in order to combat logic it is necessary to use logic.
— Pierre Boutroux —
AZ QUOTES

Metalogic

Now we know how to answer for *specific* formulas ϕ, ψ :

- Does the semantic entailment $\phi \models \psi$ hold ?
- Does $\phi \equiv \psi$ hold ?
- Is ϕ a tautology ?

One can also ask questions of a more general kind, concerning <u>all</u> propositional formulas (irrespective of their syntactic structure).

What do you think ?

1. If $\phi \wedge \psi$ is a tautology, must ϕ and ψ then be tautologies ?
2. If $\phi \vee \psi$ is a tautology, must ϕ or ψ then be a tautology ?

We need to reason at a higher abstraction level: *Metalogic*.

Metalogic answers (1)

If $\phi \wedge \psi$ is a tautology, then what about ϕ and ψ ?

Suppose $\phi \wedge \psi$ is a tautology. Consider its truth table.

- The column of $\phi \wedge \psi$ contains only $\mathtt{T}$'s.
- *Assume* that the column of ϕ or ψ contains a $\mathtt{F}$.
- Then the column of $\phi \wedge \psi$ contains $\mathtt{F}$ in the same row.
- But this is impossible by what we know from above !
- So the assumption that the column of ϕ or ψ contains a $\mathtt{F}$ cannot hold.
- Therefore the columns of ϕ and ψ contain only $\mathtt{T}$'s.
- Hence ϕ and ψ must be tautologies.

Metalogic answers (2)

If $\phi \vee \psi$ is a tautology, then what about ϕ and ψ?

$\phi \vee \psi$ can be a tautology, while ϕ and ψ are not.

For example, take $\phi = p$ and $\psi = \neg p$.

- $p \vee \neg p$ is a tautology.
- But p and $\neg p$ are both contingent.

Semantic entailment and implication (1)

Question

Suppose $\psi_1 \vDash \psi_2$.

Can we conclude $\phi \rightarrow \psi_1 \ \vDash \ \phi \rightarrow \psi_2$?

Yes !

Suppose $\phi \rightarrow \psi_2$ is $\mathtt{F}$.

Then ϕ is $\mathtt{T}$ and ψ_2 is $\mathtt{F}$.

Since $\psi_1 \vDash \psi_2$ and ψ_2 is $\mathtt{F}$, it follows that ψ_1 is $\mathtt{F}$.

Hence $\phi \rightarrow \psi_1$ is $\mathtt{F}$.

Semantic entailment and implication (2)

Question

Suppose $\phi_1 \vDash \phi_2$.

Can we conclude $\phi_1 \to \psi \;\vDash\; \phi_2 \to \psi$?

No !

For example, ϕ_1 is $n = 1$, ϕ_2 is $n > 0$, and ψ is $n = 1$.

$n = 1 \;\vDash\; n > 0$ and $n = 1 \;\to\; n = 1$ both hold.

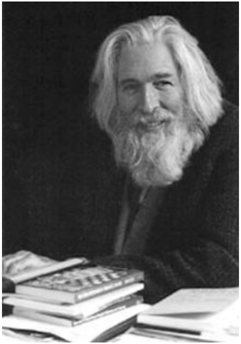
But $n > 0 \;\to\; n = 1$ does *not* hold.

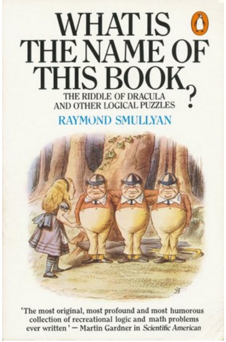
A general counterexample: Let ϕ_1 and ψ be $\mathtt{F}$, while ϕ_2 is $\mathtt{T}$.

The island of liars and truth speakers

We present some puzzles from the book:

"What is the name of this book ?"





Raymond Smullyan (1919 – 2017)

A question that is its own answer

Question

Formulate a (non-declarative) question that is its own answer.

Such a question typically refers to itself.

Possible answer: “Which question do I pose right now ?”

Island of liars and truth speakers

On the *Island of Liars and Truth Speakers*, every inhabitant has the peculiar property of either:

- always lying, or
- always speaking the truth

They only utter declarative sentences, and answer ‘yes’ or ‘no’.

For each islander x, the propositional variable t_x expresses:

x is a truth speaker

Question

Give a formula expressing that islander x is a liar.

Answer: $\neg t_x$

An important bi-implication

If an islander x makes an assertion ϕ, then:

- ϕ is true if x is a truth speaker
- ϕ is false if x is a liar

Hence, irrespective of whether or not x is a truth speaker, the following bi-implication is true:

$$t_x \leftrightarrow \phi$$

What are these two islanders ?

Puzzle

You meet two islanders a and b.

a says: "*We are both liars.*"

What are a and b?

a says, in propositional logic: $\neg t_a \wedge \neg t_b$.

So $t_a \leftrightarrow \neg t_a \wedge \neg t_b$ is true.

Given this fact, we determine the truth values of t_a and t_b by two different approaches:

- *logical reasoning*
- *a truth table*

Solution method 1: Logical reasoning

We know: $t_a \leftrightarrow \neg t_a \wedge \neg t_b$ is true.

- *Assumption:* t_a is true.
  - Then, because of the bi-implication, also $\neg t_a \wedge \neg t_b$ is true.
  - Then $\neg t_a$ is true as well, and therefore t_a is false.
  - This *contradicts* our assumption that t_a is true.

    So this assumption cannot hold.
- Hence t_a must be false.
  - Then, because of the bi-implication, also $\neg t_a \wedge \neg t_b$ is false.
  - Since t_a is false, $\neg t_a$ is true.
  - It follows that $\neg t_b$ is false, so t_b is true.
- *Conclusion:* a must be a liar and b a truth speaker.

  Indeed this is a (or better, the) correct solution.

Solution method 1: Logical reasoning

Question

Why should logical reasoning start with a *wrong* assumption (in the example: t_a is true) instead of a *correct* assumption ?

Answer: This way we can argue there is a single correct solution.

A wrong assumption restricts the space of possible solutions.

A starting assumption "t_a is false" would have led to the conclusion that then t_b must be true.

But this had not excluded the possibility that a is a truth speaker.

Question

Why must it be checked at the end whether a being a liar and b being a truth speaker is a correct solution ?

Solution method 2: Truth table

- We know: $t_a \leftrightarrow \neg t_a \wedge \neg t_b$ is true.

- In order to find out what this fact tells us about t_a and t_b, we make a truth table:

t_a	t_b	$\neg t_a$	$\neg t_b$	$\neg t_a \wedge \neg t_b$	$t_a \leftrightarrow \neg t_a \wedge \neg t_b$
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	T
F	F	T	T	T	F

  We find that the bi-implication is T only if t_a is F and t_b is T.

- So again: a is a liar and b a truth speaker.

We have proved the semantic entailment

$$t_a \leftrightarrow \neg t_a \wedge \neg t_b \quad \models \quad \neg t_a \wedge t_b$$

Solution method 2: Truth table

Question

Why is it not sufficient to give a single row from the truth table where all formulas $t_a \leftrightarrow \phi_a$, for each islander a and his/her statement ϕ_a, have the value $\mathtt{T}$?

Answer: This single row does not exclude the possibility that there may be multiple solutions, i.e., multiple rows where all formulas $t_a \leftrightarrow \phi_a$ have the value $\mathtt{T}$.

Logical reasoning versus a truth table

Logical reasoning

- works by excluding possibilities,
- and makes the logical chain of arguments explicit,
- but requires intuition, in part to make wrong assumptions,
- and the logical argumentation can be flawed.

A *truth table*

- is exhaustive and mechanical,
- but also laborious and error-prone,
- and buries intuition under a pile of `T`'s and `F`'s.

Quiz time

Question

- Suppose an islander says: "*I am a truth speaker.*"
- Suppose an islander says: "*I am a liar.*"

What can you conclude about this islander ?

1. Truth speaker 2. Liar 3. Both possible 4. Impossible

t_a	$\neg t_a$	$t_a \leftrightarrow t_a$	$t_a \leftrightarrow \neg t_a$
T	F	T	F
F	T	T	F

Love on the island

Puzzle: *Who does a love ?*

Islander a stares into the void and sighs:

"I love Martha."

After some thought, he adds:

"If I love Martha, then I love Kathy."

Formalization

Let us use the following translation key

m: *a* loves Martha
k: *a* loves Kathy

and, as always,

t_a: *a* is a truth speaker

Love on the island: Logical reasoning

The assertions of a in formulas: m and $m \rightarrow k$.

We cannot reason on the basis of these formulas.

Because we do not yet know whether a speaks the truth.

- We do know that $t_a \leftrightarrow m$ and $t_a \leftrightarrow (m \rightarrow k)$ are true.
- By these bi-implications, t_a and m, and also t_a and $m \rightarrow k$, must have the same truth value.
- Therefore m and $m \rightarrow k$ have the same truth value.
- *Assumption*: m is false. Then $m \rightarrow k$ is true.

  But then m and $m \rightarrow k$ have different truth values.
- So the assumption cannot hold: m must be true.
- Then $m \rightarrow k$ is true. Hence t_a and k are also true.

Conclusion: a must be a truth speaker who loves Martha and Kathy.

Indeed this is a (or better, the) correct solution.

Love on the island: Truth table method

We know: $t_a \leftrightarrow m$ and $t_a \leftrightarrow (m \rightarrow k)$ are true.

t_a	m	k	$t_a \leftrightarrow m$	$m \rightarrow k$	$t_a \leftrightarrow (m \rightarrow k)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	T	T	F
F	F	F	T	T	F

So again: a is a truth speaker and loves Martha and Kathy.

We have proved the semantic entailment:

$$t_a \leftrightarrow m, \quad t_a \leftrightarrow (m \rightarrow k) \quad \models \quad t_a \wedge m \wedge k$$

Quiz time

Question

Suppose an islander says: "*If I am a truth speaker, then* ..."

The rest of the sentence is lost, due to a passing lorry.

What can you conclude about this islander ?

1. Truth speaker 2. Liar 3. Both possible 4. Impossible

Finding one's way on the island

Puzzle

On the island you arrive at a T-crossing.

You wonder whether the harbour is to the left or to the right.

An islander hurries by.

You only have time to ask one question.

The only possible answers are ‘yes’ and ‘no’.

Which question can you ask to determine the right way ?

Hint 1: Recall the question that is its own answer.

Hint 2: Lying about a (‘yes’ or ‘no’) lie yields a true statement.

Just one question to find one's way

If I asked you the question:

"Does the way to the right lead to the harbour ?"

would your answer be "yes" ?

The islander is a truth speaker or a liar, and the harbour is to the right or to the left, leading to 4 cases:

- The islander is a *truth speaker*.
  - The harbour is to the right: the answer is *yes*.
  - The harbour is to the left: the answer is *no*.
- The islander is a *liar*.
  - The harbour is to the right: the answer is *yes*.
  - The harbour is to the left: the answer is *no*.

Two islanders pointing at each other - part I

Who is lying ? - part I

Islander a says: "*b is a truth speaker.*"

Islander b says: "*a is a truth speaker.*"

What can you conclude ?

Answer: a and b are either both truth speakers or both liars.

t_a	t_b	$t_a \leftrightarrow t_b$	$t_b \leftrightarrow t_a$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Two islanders pointing at each other - part II

Who is lying ? - part II

Islander *a* says: “*b is a liar.*”

Islander *b* says: “*a is a liar.*”

What can you conclude ?

Answer: Either *a* or *b* is a liar, but not both.

t_a	t_b	$\neg t_a$	$\neg t_b$	$t_a \leftrightarrow \neg t_b$	$t_b \leftrightarrow \neg t_a$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

Two islanders pointing at each other - part III

Who is lying ? - part III

Islander *a* says: "*b is a liar.*"

Islander *b* says: "*a is a truth speaker.*"

What can you conclude ?

Answer: You're on the wrong island...

t_a	t_b	$\neg t_b$	$t_a \leftrightarrow \neg t_b$	$t_b \leftrightarrow t_a$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	F	T

Take home

- semantic entailment $\models$
  - $\phi_1, \ldots, \phi_n \models \psi$
  - $\phi \equiv \psi \iff \phi \models \psi$ and $\psi \models \phi$

- metalogic statements, such as:

  $\phi \wedge \psi$ is a tautology $\iff$ ϕ and ψ are tautologies.

- logic puzzles on liars and truth speakers

- try the exercises for tomorrow