The Optimal Bluffing Strategy In Games With Random Elements – Bluffing in Game Theory

Mathematics Internal Assessment HL

1. Introduction

1.1. Rationale

Game theory always fascinated me. I found games as a form of duel with one or more opponents in which the best player wins. Of course some games require luck, but every single one of them that is respected by the players has skill expression in it and allows for advanced decision making. Those that do not involve such factor are not interesting to strategical players – for example war game or other games in which a player does not make a single decision is played mostly by kids or people that just try to kill time in a boring situation and there is no word about professional play or huge fanbase. On the other hand, there are games like poker, in which people develop many strategies or even unspoken rules. Advanced poker players often hate playing against new players, because of their lack of knowledge and resulting unpredictability and although the strategy of the beginner might work in some cases, causing the annoyance of the professionals, the better player in the end will come on top due to the law of big numbers, which states that after enough events with a probability of p, the result will indeed converge to p.

The analysis of optimal bluffing and optimal gameplay and ran is very important part of game theory. In real life there are many professional gamblers, who manage to beat casinos using optimized strategies. Most of their wins come from games that include skill, such as blackjack, poker or even craps, although the last doesn't offer a strategy that allows the expected outcome to be positive, there are two player strategies that allow to go even with the casino, which in some cases is used to get free drinks, as they are often given away to the guests. This is rather an example of optimal betting than optimal bluffing, but those two concepts are connected. Both of them require immense calculations in more complex games, in fact it was proven in 1950 by John Nash that Nash equilibrium must exist in poker, but this is so complex that contemporary computers are not powerful enough to find it[1]. One explanation why is that

[1] https://www.frontier-economics.com/uk/en/news-and-insights/articles/article-i7113-nashing-in-why-poker-players-are-thinking-like-economists/

if optimal strategy while being used by one player, has to account for other players using it, this creates an almost infinite loop of ifs and whys – based on actions done by the player others will narrow the range of possible hands he has - but what if he is bluffing? But if he is bluffing it must be also the part of the optimal strategy known to those players. So it is basically unknown, if optimal play for one hand allows for only singular play, or maybe there are few equally good options to follow? And there are many more problems with finding the best strategy for poker. But undoubtedly one is certain. If one player had found a winning strategy for such a game and kept it for himself, he would get very rich from winning. Even today advanced mathematical strategies allow the best players to have the expected outcome from blackjack around 1.03 according to Yuchun Lee[2], a professional at the game.

This showcases the importance of the problem. Mathematics can be used to have a winning edge in all bluffing games. And although usually there is psychology involved, the algorithm following the optimal strategy would win anyways (assuming a large number of games played). I find it fascinating that pure analysis is able to beat human beings in psychological endeavours. And the fact that the opponent might know the exact algorithm and still the best move for him would be to also use it, proving the victory of mathematics over humanity, is even more interesting to me.

1.2. Aims of the exploration

Each game has its own winning strategies based on different factors, some harder and easier to find. Because of this the exploration will aim to find general patterns across the games and solutions for some of the simpler games.

1.3. Plan of analysis

The analysis will be based on analysing optimal bluffing strategies for some simpler games and then finding patterns that apply to most of the other games.

[2] https://www.youtube.com/watch?v=3kGlk1E_Cnw

2. Analysis

2.1. First Game Analysis

Two players are playing a card game. Each player draws a card from a pool of six cards numbered from 1 to 6. Then the players are allowed to swap their cards, given that the player with the higher card in the end wins. In which cases should player accept a swap offer? And when should he be the one to offer the swap? Are there any cases where player should accept a swap offer, but should not make a one or vice versa?

In case a player is offered a swap reward there are six possible cards that he has at hand.

Card	Conclusion
6	Player always loses if swaps.
5	If opponent has a six, it is incorrect for him to offer a swap. So player should lose if swaps.
1	Player will always win if swaps.

Table 1 the conclusions for 1, 5 and 6 in hand

It is notable that the opponent has the same knowledge and did not offer a swap that would guarantee a loss for him, which leads to conclusion that he has at most a 4 in hand, so in case the first player has a 4, the swap should be avoided. If the player has a 3, then he wins only if opponent has a 4, but if that is the case, he would not make an exchange, because as I established no offer should be made if a player has a 5 or 6 which the opponent knows and understands swapping with a 4 in hand would be losing. In case the player has a 2, opponent has to have a 1, because as it was deduced previously, he would not offer a swap.

This shows a way of simplification of some games by eliminating so called strictly dominated strategies – ones that can bring no benefit for the player, but can result in loses, making the expected value negative. Such strategies in this case are eliminated from play and by endless repetition we can deduce that the only time when to offer a swap is when the player has a one in hand. But what if this logic is flawed and maybe bluffing sometimes and the assumption that opponent does not offer swap with a 3 on hand and accepting the swap with 2 is profitable?

This would not work against optimal strategy, simply if one player is only swapping with 1, he can only gain from swaps. Indeed, this strategy is always winning against any other strategies and only leads to a draw when also another player is using it. This is the Nash equilibrium for the game.

From the study of this game following general conclusions can be deducted. If there is an optimal bluffing strategy, it has to result in better gains for the player against its best counter-strategy than any other bluffing strategy. Furthermore, the strategy should take the opponents for perfect thinkers and assume neither player will break bounds of logic, as such faults would be ultimately losing. It is proven that every game has at least one Nash equilibrium, it is stated that "In a Nash equilibrium, each player is assumed to know the equilibrium strategies of the other players, and no one has anything to gain by changing only one's own strategy."[3] Which is also confirmed by this example.

2.2. Second Game Analysis

Two players are now playing a game that involves five cards labelled from 1 to 5. Each pulls one card and then the Player 1 places a bet that his card is "higher" or "lower" than the opponents. Then the Player 2 has two options:

1. Check – both players show their cards and the first player wins if his card is higher or lower than the opponents card depending on what he bet earlier.
2. Reverse – the bet is now reversed but his card has its value increased by 3 if the new bet is "lower" and decreases by 1 is the new bet is "higher". If the new values of the cards are the same, the game ends in a draw.

The logical approach to this problem is difficult, only few conclusions are visible without using equations to describe the problem. There are too many possible strategies to solve this problem. For example Player 1 can sometimes bluff and play low with a 5 in hand if opponent reverses too often. However there are some conclusions that can be made, because as it was

[3] https://en.wikipedia.org/wiki/Nash_equilibrium

established earlier, optimal strategies should obey the rules of logic. First conclusion is that the game is symmetrical around the middle value – that is expected values for both players for 1 in hand are the same as for 5 in hand, furthermore the situation is exactly the same for Player 1 if he plays low with 2 or high with 4 in hand. The same logic applies to Player 2, he should have the same probabilities of checking with card n in response to low and checking with card of value 6-n in response to high. Also it would be probably unbeneficial for both players to always play the same in identical setup. This includes a new concept which is mixed strategies – in the same scenario the player has some probability to play option a and some probability to play option b. Of course all probabilities add up to 1, but in some cases there can be more than 2 options.

In most games of such type the optimal strategies should be obtainable through a creation of an equation of the expected value for one player. For the sake of the equation win result in +1 point and lose will result in -1 point, while draw stands at +0 for both players. Firstly, the variables should be defined. Let l_1, l_2, l_3, l_4, l_5 be the probabilities of Player 1 playing low with corresponding cards in hand. Similarly c_1, c_2, c_3, c_4, c_5 are the probabilities of Player 2 checking with corresponding cards in hand in case low is played. And d_1, d_2, d_3, d_4, d_5 are the probabilities of checking in case high is played. Due to previous conclusions $c_n = h_{6-n}$. Also probability of playing high with card n in hand is equal to $1 – l_n$ and $l_n = 1 – l_{6-n}$. Following this comes that $l_3 = 0.5$. If Player 2 has 1 and low was played playing reverse is strictly dominated strategy, so $c_1 = 0$. If player 2 has 5 instead of 1, this time reversing is strictly dominating, as checking is a guaranteed loss, so $c_5 = 1$. The expected values for Player 1 if low is played with certain cards, value is subtracted when the action of the opponent is losing and added if the case is winning, this is divided by 4 as Player 2 can have 4 different cards in hand, are:

$$L_1 = \frac{c_2 + c_3 + c_4 + c_5 - (1 - c_{3)} - (1 - c_4) - (1 - c_5)}{4}$$

$$L_2 = \frac{-c_1 + c_3 + c_4 + c_5 + (1 - c_{1)} - (1 - c_4) - (1 - c_5)}{4}$$

$$L_3 = \frac{-c_1 - c_2 + c_4 + c_5 + (1 - c_{1)} + (1 - c_2) - (1 - c_5)}{4}$$

$$L_4 = \frac{-c_1 - c_2 - c_3 + c_5 + (1 - c_{1)} + (1 - c_2) + (1 - c_3)}{4}$$

$$L_5 = \frac{-c_1 - c_2 - c_3 - c_4 + (1 - c_{1)} + (1 - c_2) + (1 - c_3) + (1 - c_4)}{4}$$

After shortening and substituting c_1 for 1 and c_5 for 0 this becomes:

$$L_1 = \frac{c_2 + 2c_3 + 2c_4 - 3}{4}$$

$$L_2 = \frac{c_3 + 2c_4 - 3}{4}$$

$$L_3 = \frac{-2c_2 + c_4 - 1}{4}$$

$$L_4 = \frac{-2c_2 - 2c_3 + 1}{4}$$

$$L_5 = \frac{-2c_2 - 2c_3 - 2c_4 + 2}{4}$$

Expected values for high played with certain card (n) in hand are corresponding to expected values for low played with card 6-n in hand, so the overall expected value for each card is:

$$E_n = l_n L_n + (1 - l_n) L_{6-n}$$

From this and the fact that $E_n = E_{6-n}$ comes the equation of overall expected value for Player 1, which is:

$$E = \frac{2(l_1 L_1 + (1 - l_1) L_5) + 2(l_2 L_2 + (1 - l_2) L_4) + l_3 L_3 + (1 - l_3) L_3}{5}$$

Which after substitution is equal to:

$$E = \frac{l_1(2c_2 + 4c_3 + 4c_4 - 6) + (1 - l_1)(-4c_2 - 4c_3 - 4c_4 + 4)}{20} + \frac{l_2(2c_3 + 4c_4 - 6) + (1 - l_1)(-4c_2 - 4c_3 + 2)}{20} + \frac{-2c_2 + c_4 - 1}{20}$$

After changing order this becomes:

$$E = \frac{l_1(6c_2 + 8c_3 + 8c_4 - 10) + \ l_2(4c_2 + 6c_3 + 4c_4 - 8) + (-10c_2 - 8c_3 - 3c_4 + 5)}{20}$$

After finding the appropriate equation it is left to find the values of variables such that Player 1 can not change his variables without decreasing the value of the function, while Player 2 can not change his variables without increasing the value of the function. This will create a state where neither player can benefit from any changes in strategy, resulting in achieving the Nash equilibrium. Unfortunately finding such variables in many cases requires immense computing power. Only some games have expected value equations such that variables can be set in such way that partial derivative with respect through each one of them is equal to zero – then a Nash equilibrium. In this case equations resulting equations would be unsolvable for both players – Player 1 would end up with 2 parameters for 3 equations and Player 2 would have 3 parameters for 2 equations, but those would not be solvable in range <0;1>. The way to solve this problem is through converting it to a game matrix. Before doing so it is important to notice two things which simplify the problem:

1. Setting c_2 to 1 is a strictly dominating strategy for Player 2, as $c_2(6l_1 + 4l_2 - 10)$ is never positive.
2. As the expected value function does not contain any polynomial of degree higher than 1, the partial derivatives with respect to certain variables are constant (assuming other variables aren't changed) and so for the matrix there will be assumed scenarios with l_1, l_2, c_3, c_4 equal to 0 or 1.

The matrix itself presents the expected values for Player 1 in each scenario.

	$c_3 = 0, c_4 = 0$	$c_3 = 1, c_4 = 0$	$c_3 = 0, c_4 = 1$	$c_3 = 1, c_4 = 1$
$I_1 = 0, I_2 = 0$	-5	-13	-8	-16
$I_1 = 1, I_2 = 0$	-9	-9	-4	-4
$I_1 = 0, I_2 = 1$	-9	-11	-8	-10
$I_1 = 1, I_2 = 1$	-13	-7	-4	2

Table 2 matrix of the game

Rows of the table are chosen by Player 1 and columns are in Players 2 control. The objective of both players is to find a row/column such that the best choice of opponent in that row/column is worse than other best choices in different rows/columns, in other words both players want to reach a state where change of opponents strategy can not result in benefit for him while also their gain is maximized.[4] For this matrix this is impossible without mixed probability, as if one player plays a certain choice all the time, the opponent always has a counter, that is then countered by a different play by the player. For example, assume Player 1 played row three, then Player 2 plays column two, but in response Payer 1 would want to change to row four. And then Player 2 would change to row one and so on.

Mixed probabilities allow to avoid creation of such cycles. This matrix can be recreated to allow for better demonstration of the concept. But before that it is worth noting that third column is strictly dominated by the second one – why would Player 2 ever play it, when column two always offers a better outcome. Similarly, row three is also strictly dominated by row two. With those eliminated new matrix presents as follows:

	x_1	x_2	x_3
y_1	$-\frac{5}{20}$	$-\frac{13}{20}$	$-\frac{516}{20}$
y_2	$-\frac{9}{20}$	$-\frac{9}{20}$	$-\frac{4}{20}$
y_3	$-\frac{13}{20}$	$-\frac{7}{20}$	$\frac{2}{20}$

[4] Information based on: https://www.matem.unam.mx/~omar/math340/matrix-games.html

Table 3 improved matrix of the game

The new variables x_1, x_2, x_3 have a sum of 1, are greater of equal 0 and represent probabilities of playing respectively ($c_3 = 0$, $c_4 = 0$), ($c_3 = 1$, $c_4 = 0$), ($c_3 = 1$, $c_4 = 1$) by Player 2. Similarly, y_1, y_2, y_3 are probabilities of Player 1 playing ($l_1 = 0$, $l_2 = 0$), ($l_1 = 1$, $l_2 = 0$), ($l_1 = 1$, $l_2 = 1$) and of course they are not negative and sum to 1.

Solving such a matrix is a process of maximizing the expected value which will be marked with z by Player 1 and minimizing it by Player 2. For example in this case Player 2 has a following set of inequalities:

Minimize z for:

$$z \geq \frac{-5x_1 - 13x_2 - 16x_3}{20}$$

$$z \geq \frac{-9x_1 - 9x_2 - 4x_3}{20}$$

$$z \geq \frac{-13x_1 - 7x_2 + 2x_3}{20}$$

This comes from the fact that each inequality represents one row of the matrix and if this is solved then Player 1 can not make a choice that would result in a better outcome than z for him. In many games those equations are very hard to solve without use of technology. Most common way of solving such maximization and minimization problems is Simplex Method which works in a following way "The simplex method uses an approach that is very efficient. It does not compute the value of the objective function at every point; instead, it begins with a corner point of the feasibility region where all the main variables are zero and then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found."[5]

5 https://math.libretexts.org/Bookshelves/Applied_Mathematics/Applied_Finite_Mathematics_(Sekhon_and_Bloom)/04%3A_Linear_Programming_The_Simplex_Method/4.02%3A_Maximization_By_The_Simplex_Method

Computing this problem[6] resulted in following optimal values: $z = -\frac{9}{20}$, $x_1 \in <\frac{1}{3};\frac{1}{2}>$, $x_2 = 1 - x_1$, $x_3 = 0$, $y_1 = 0$, $y_2 = 1$, $y_3 = 0$. This means that probabilities of Player 1 playing low are the weighted averages of those described by those probabilities, so:

$$l_1 \;=\; 0 \times y_1 + 1 \times y_2 + \; 1 \times y_3 = 1$$

$$l_2 \;=\; 0 \times y_1 + 0 \times y_2 + \; 1 \times y_3 = 0$$

Similarly with probabilities of checking (as x_1, x_2 are ranges I will assume that both are equal to half, as such assumption does not impact game output):

$$c_3 \;=\; 0 \times x_1 + 1 \times y_2 + \; 1 \times y_3 = \frac{1}{2}$$

$$c_4 \;=\; 0 \times x_1 + 0 \times y_2 + \; 1 \times y_3 = 0$$

With that the game is solved. The correct strategies for both players are as follows:

Parameter	Value	Parameter	Value
l_1	1	c_1	1
l_2	0	c_2	1
l_3	0.5	c_3	0.5
l_4	1	c_4	0
l_5	0	c_5	0

Table 4 solution to the game

The interesting conclusion is that it is the best for the Player 1 to bluff sometimes and play low with a 4 in hand.

This example shows a very linear approach towards game theory problems, basically showing that to solve the game it is enough to convert it into a equation and then maximize or solve it. Obviously it is easier said than done.

[6] https://www.math.ucla.edu/~tom/gamesolve.html

2.3. The Third Game Analysis

Although the second game was created by me (probably it would be very reasonable to assume that someone else, somewhere else and sometime else had to play it), the third game is very popular, indeed the game known as “Liar’s Dice” was even featured in a movie “Pirates of the Caribbean: Dead Man’s Chest”. I did not even understand the rules of the game while watching the movie, but later I watched a video[7] about how the game is portrayed in the movie, showing how playstyles of the characters are connected to their personalities, which caught my attention.

According to the video the rules are as follows. Firstly, all the players roll five unbiased, six-sided dice each, while keeping them hidden from other players under a special cup. Then they start bidding in a set order how many dice are rolled facing a certain number up. After a bet is made, the next player can either call the bluff or increase the bid. Increasing the bid requires the next player to either increasing the number on the dice, increasing the number of dice with the given value or both. The amount of dice or their value (for simplicity this will be how the number facing up on the dice will be called) can never be decreased. The game ends when a bluff is called and depending if the bid was correct either the accused or accusing player is the winner.

Of course this game with five dice and four players is too complex for analysis in this paper, as it allows for much more options than previous games, in fact I think the current strategy for normal version of the game has not been yet found. Instead I will try to create a model for a simpler version with two players, each with one two sided coin, which can roll either T of F and the former being higher value than the latter.

Even this most simplified version of the game offers a lot of possibilities to the players, in fact so much it is very hard to comprehend it without creating a game tree, which is shown in a figure below. On the branches of the tree there are letters representing chances of making a

[7] https://www.youtube.com/watch?v=T44LuxdH0iw

choice. Vectors a, b and c are controlled by Player 1 and vectors d, e and f depend on Player 2. The letters t and f in the bottom indexes (in Figure 2) add more detail, as the chance of taking a certain branch of the tree by a player is of course affected by the information they have, being in this example a value of the coin under their cup. So for example a_{t3} is the chance of Player 1 calling F as their first move with T under their cup. Also, on the trees FF and TT represent bets that there are respectively two F's and two T's on the top faces of the coins.

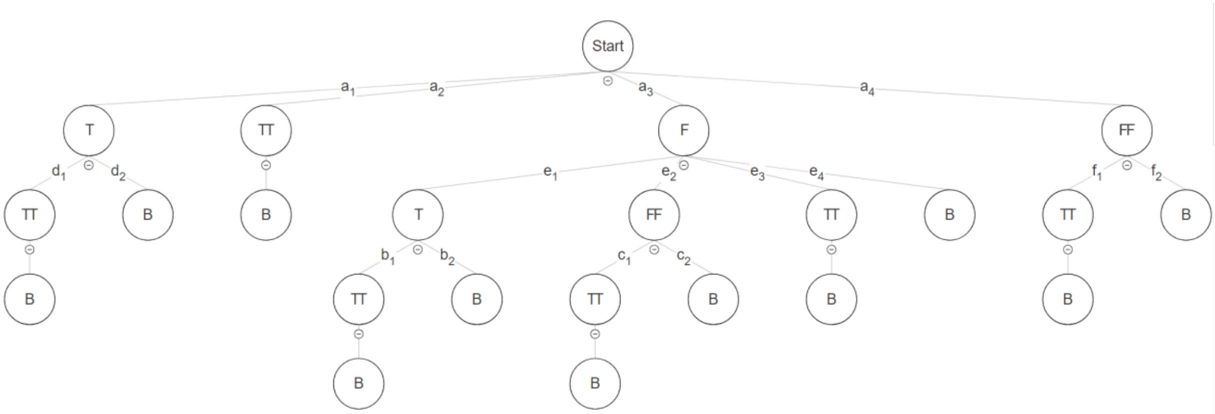

Figure 1 the complete game tree of the simplified liar's dice.
Drawn in: https://app.smartdraw.com

With the main game tree created, the ones for every possible coin rolls can be created. The ones in figures below have most of the strictly dominated strategies removed. For example Player 1 never benefits from calling two coins rolling T with F on his coin, as opponent always has to check such bet and therefore such play is a guaranteed loss.

Many more cases of similar simplification occurred here and all of them are based on pure logic, mostly removing strictly dominated strategies and taking strictly dominated strategies as certain to be played. Of course, as shown earlier, making irrational plays can not be beneficial to the player in most cases, as taking a certain loss will not work against a good opponent. If the opponent is bad, the use of tricks is not necessary to win, but in some cases making the process much easier, although assuming faulty play is far from mathematical models, as it is unknown when the mistakes will be made, and playing optimally against bad strategies is always the safest, as even the mistakes done purposefully can be punished. The topic of purposefully playing sub-optimally in order to maximize the gain is although interesting, as it combines mathematics with psychology, which is always a intriguing combination. On that topic there was an anecdote about Alexander Alekhine, the fourth world chess champion. According to the story, two young players challenged him to two simultaneous games, with Alekhine playing with white in one and with black in the other. The challengers be that one of them would win, or both would draw. To Alekhine's surprise, when he made a move on one board, the move was mirrored on the other board. He was tricked into playing against himself! But he came with the plan, after all he was the world champion. He purposefully made a losing mistake on one board by blundering his queen and the players blinded by the temptation of winning both games broke the symmetry of two boards, making them unable to repeat moves from the one board onto another. As he was the best in the world, he quickly managed to improve his position on the lost board and win both games. This is an example of the case when making a bad play, although in nonstandard conditions, is beneficial for the player.

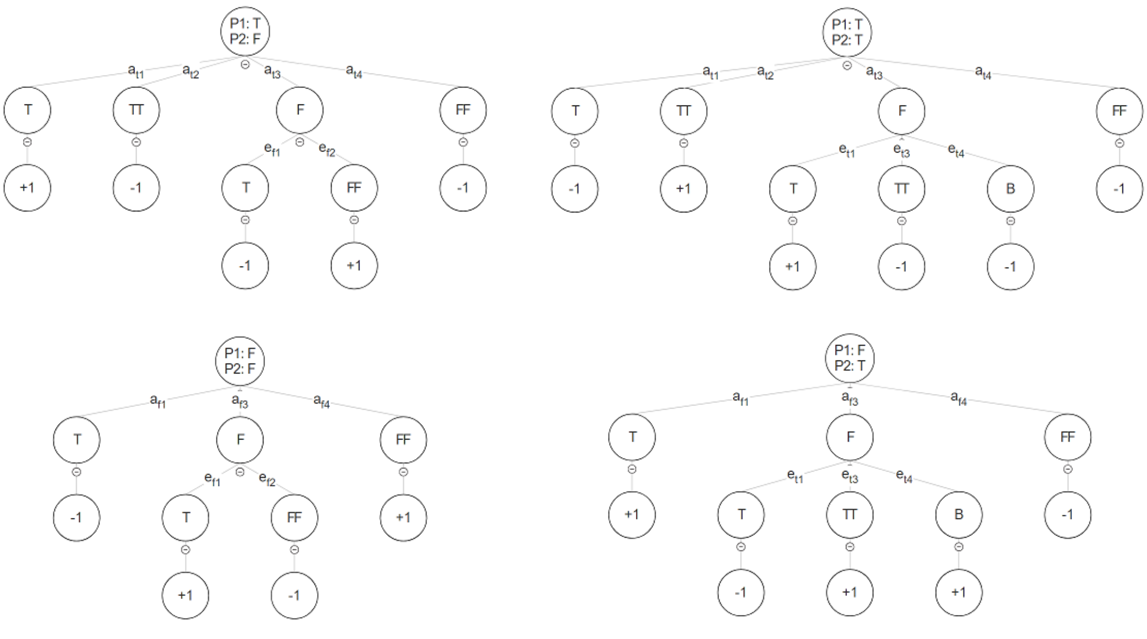

Figure 2 game trees for every possible rolls for both players, simplified and with strictly dominated strategies removed.
Drawn in: https://app.smartdraw.com

From Figure 2 it can be seen that b, c, d and f can be easily simplified using logic with very little effort. With that being said, the equation for expected value of Player 1 is as follows:

$$E = \frac{\left(a_{t1} - a_{t2} + a_{t3}\left(-e_{f1} + e_{f2}\right) - a_{t4}\right) + \left(-a_{t1} + a_{t2} + a_{t3}\left(e_{t1} - e_{t3} - e_{t4}\right) - a_{t4}\right)}{4}$$

$$+\frac{\left(-a_{f1}+\ a_{f3}(e_{f1}-e_{f2})+\ a_{f4}\right)+(a_{f1}+a_{f3}(-e_{t1}+e_{t3}+e_{t4})-a_{f4})}{4}$$

Which after simplification becomes:

$$E=\frac{-2a_{t4}+a_{t3}\left(-e_{f1}+e_{f2}\right)+a_{t3}(e_{t1}-e_{t3}-\ e_{t4})+\ a_{f3}\left(e_{f1}-e_{f2}\right)+a_{f3}(-e_{t1}+e_{t3}+e_{t4})}{4}$$

Keeping a_{t4} above 0 is pointless and also e_{t3} + e_{t4} can be substituted with x, simplifying the equation further:

$$E=\frac{(a_{t3}-a_{f3})\left(-e_{f1}+e_{f2}+\ e_{t1}-x\right)}{4}$$

This equation does not need to be turned into a matrix in order to be solved, as each player has a way of neutralizing the expected value to zero and thus the equilibrium also must result in zero gain to both players, as if it did not, the unbenefited player can always neutralize to game. Thus the only conditions for the equilibrium are, assuming that both players obey the game tree in Figure 2 and do not make losing mistakes in certainly winning game states, that $a_{t3} = a_{f3}$, $e_{f1} + e_{t3} + e_{t4} = e_{t1} + e_{f2}$ and $a_{t4} = 0$.

This result is both disappointing and interesting, as there was no trick to guaranteeing the win. But also it caused my curiosity if the same applies to normal version of the game and each player has a chance to neutralize the unfairness or in more complex states the game favours a player who is x^{th} to move. If the first is true it really shows the magnitude of the psychological warfare performed by the players, as there certainly are some who will win over worse ones. If the latter is true, then it gets really appealing from the mathematical point of view, as there are many possible patterns that could cause such thing. However, the analysis of such complex game would require much more resources, time and experience and is far beyond the level of this paper, but could be a very potent topic for potential research in game theory.

3. Conclusion

The general nature of bluffing is connected to game theory itself, in optimal play lying is dictated just as probability, it is involved in basically any random game and even finds uses in non random games, for which a good example are soccer games studied in “”Testing Mixed-Strategy Equilibria When Players Are Heterogenous: The Case of Penalty Kicks in Soccer”[8] by P.-A. Chiappori, S. Levitt, and T. Groseclose, which shows how bluffing (as long as aiming for a sub-optimal reward can be called a bluff) is applied in sports.

The process of finding the Nash equilibrium is quite simple, excluding the huge amount of computing needed. With powerful enough calculator, simple algorithm, which counts all cases, includes them into equation and then finds optimal parameters, is capable of solving all games. And although humanity lacks such powerful machines, many cases can be simplified, such as in first and second game analysed in this paper, reducing the required amount of sheer calculations. As shown, those simplifications are to be derived then and only then if they come from logical conclusions.

[8] Chiappori, P. -A.; Levitt, S.; Groseclose, T. (2002). "Testing Mixed-Strategy Equilibria when Players Are Heterogeneous: The Case of Penalty Kicks in Soccer" https://pricetheory.uchicago.edu/levitt/Papers/ChiapporiGrosecloseLevitt2002.pdf

4. Evaluation

But what is beyond the conclusions reached in this paper? The game theory is a broad field of mathematics and after all it can not be simply describe by a few step algorithm. Firstly turning a game into an equation can be very difficult, even assuming humanity had sufficient enough calculator, some games would require a mastermind to input them into it. The essence of game theory is finding the tricks to make games look much simpler than they are without changing the outcome. Work of a good mathematician here can make a huge difference, for example, although it is not connected with bluffing, it has a general message, two different chess engines can have an enormous difference in rating although they operate on the same computer.

But non the less the results achieved here are also important, they are coherent with fundamental laws of game theory and although I am not even close to the best game theorists, I can analyse simple games few times faster now than before writing this paper and my understanding of the field increased. In the process of fulfilling the aim of this work I got stuck a few times, due to having wrong approaches. For the second game I tried creating equations from the expected value equation and it took me long before I realized that only inequalities should be applied here. I think I was thinking about the problem for around twelve hours until I realized that matrix should be made, then if I wouldn’t find an article about it, it would be hard to solve. And the pool of the knowledge in this field is much deeper, as what is written here is absolute basics.

Another interesting conclusion I came to about game theory is that every game should be taken with unique approach, as humans have limited mental capability, converting them into equations can be tiering and sometimes very hard. Some games like the second one here can be represented as equations straight up, but for some it is better to draw a tree in order to avoid overly tiring the imagination of the mathematician, and for some constructing a matrix is trivially easy and allows for skipping all the steps.

Finally the game theory is an absolutely beautiful part of mathematics, probably one with the most real life applications and I believe that understanding it better will improve my decision making and allow me to have an edge over most opponents in my endeavours.

Bibliography

- https://www.frontier-economics.com/uk/en/news-and-insights/articles/article-i7113-nashing-in-why-poker-players-are-thinking-like-economists/

- https://www.youtube.com/watch?v=3kGlk1E_Cnw

- https://en.wikipedia.org/wiki/Nash_equilibrium

- https://www.matem.unam.mx/~omar/math340/matrix-games.html

- https://math.libretexts.org/Bookshelves/Applied_Mathematics/Applied_Finite_Mathematics_(Sekhon_and_Bloom)/04%3A_Linear_Programming_The_Simplex_Method/4.02%3A_Maximization_By_The_Simplex_Method

- https://www.math.ucla.edu/~tom/gamesolve.html

- Chiappori, P. -A.; Levitt, S.; Groseclose, T. (2002). "Testing Mixed-Strategy Equilibria when Players Are Heterogeneous: The Case of Penalty Kicks in Soccer" https://pricetheory.uchicago.edu/levitt/Papers/ChiapporiGrosecloseLevitt2002.pdf

- https://app.smartdraw.com

- https://www.youtube.com/watch?v=T44LuxdH0iw