Supplementary Materials (I):
Conic Sections & Polar Coordinate Parameterizations

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Overview

1. Conic Sections and Quadratic Equations

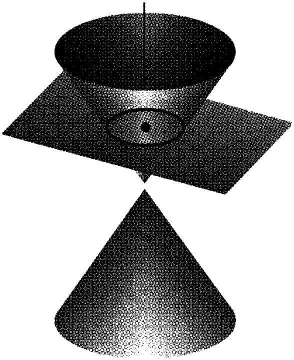
Conic Sections from Greek Geometry

This section shows how the conic sections from Greek geometry are described today as the graphs of quadratic equations in the coordinate plane.

The Greeks of Plato's time described these curves as the curves formed by cutting a double cone with a plane, hence the name conic section.

Conic Sections from Greek Geometry

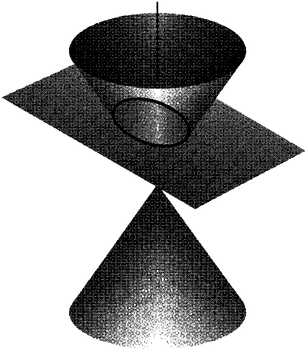
The standard conic sections are the curves in which a plane cuts a double cone.



Circle: plane perpendicular to cone axis

Conic Sections from Greek Geometry

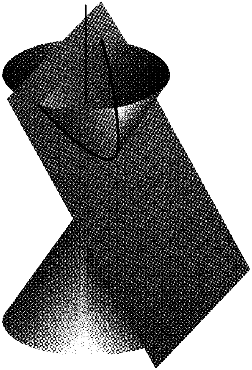
The standard conic sections are the curves in which a plane cuts a double cone.



Ellipse

Conic Sections from Greek Geometry

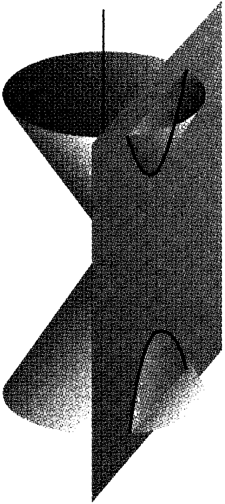
The standard conic sections are the curves in which a plane cuts a double cone.



Parabola: plane parallel to side of cone

Conic Sections from Greek Geometry

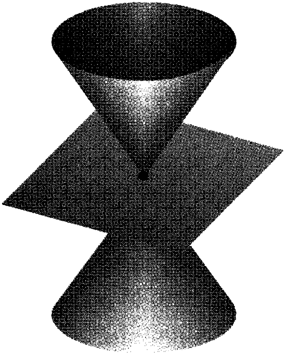
Hyperbolas come in two parts, called branches.



Hyperbola: plane parallel to cone axis

Conic Sections from Greek Geometry

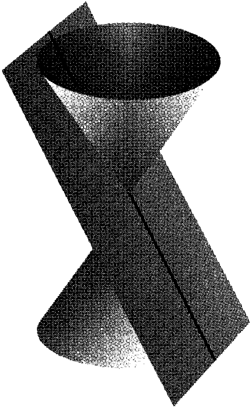
The point and lines obtained by passing the plane through the cone's vertex are degenerate conic sections.



Point: plane through cone vertex only

Conic Sections from Greek Geometry

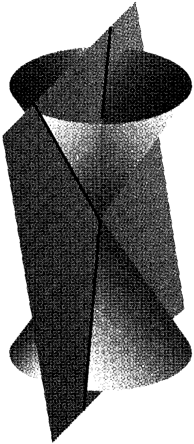
The standard conic sections are the curves in which a plane cuts a double cone.



Single line: plane tangent to cone

Conic Sections from Greek Geometry

The standard conic sections are the curves in which a plane cuts a double cone.



Pair of intersecting lines

Conic Sections from Greek Geometry

Definition

A **circle** is the set of points in a plane whose distance from a given fixed point in the plane is constant. The fixed point is the center of the circle; the constant distance is the radius.

Circle of radius a centered at the origin:

$$x^2 + y^2 = a^2.$$

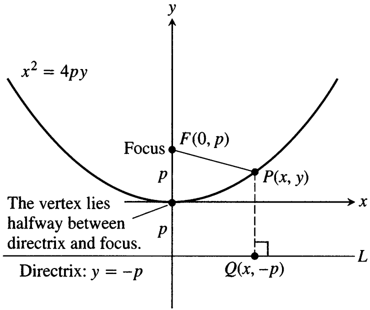
Circle of radius a centered at the point (h, k):

$$(x - h)^2 + (y - k)^2 = a^2.$$

Conic Sections from Greek Geometry

Definition (Parabola)

A set that consists of all the points in a plane equidistant from a given fixed point and a given fixedline in the plane is a **parabola**. The fixed point is the focus of the parabola. The fixed line is the directrix.



The parabola $x^2 = 4py$.

Conic Sections from Greek Geometry

Definition (Parabola)

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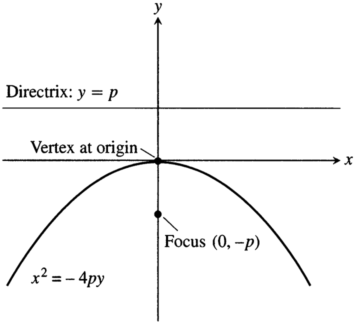
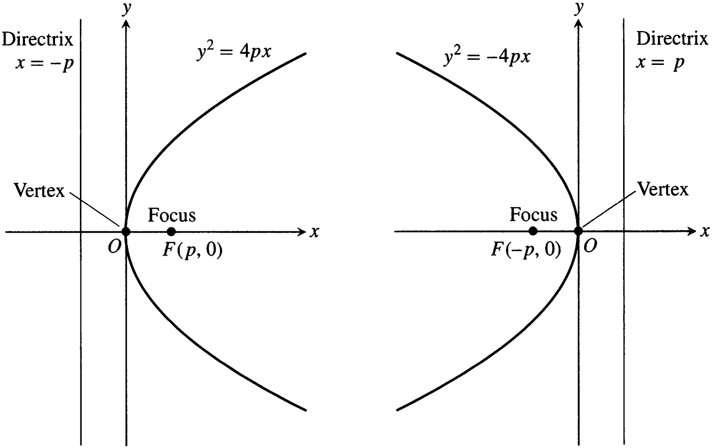


The parabola $x^2 = -4py$.

Conic Sections from Greek Geometry



The parabola $y^2 = 4px$ and $y^2 = -4px$.

Conic Sections from Greek Geometry

Equations for Parabolas

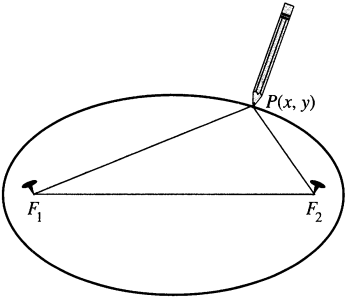
Standard-form equations for parabolas with vertices at the origin ($p > 0$).

Equation	**Focus**	**Directrix**	**Axis**	**Opens**
$x^2 = 4py$	$(0, p)$	$y = -p$	y-axis	Up
$x^2 = -4py$	$(0, -p)$	$y = p$	y-axis	Down
$y^2 = 4px$	$(p, 0)$	$x = -p$	x-axis	To the right
$y^2 = -4px$	$(-p, 0)$	$x = p$	x-axis	To the left

Conic Sections from Greek Geometry

Definition (Ellipse)

An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the foci of the ellipse.

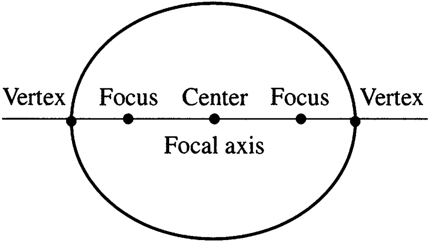


How to draw an ellipse.

Conic Sections from Greek Geometry

Definition (Ellipse)

The line through the foci of an ellispe is the ellipse's focal axis. The point on the axis halfway between the foci is the center. The points where the focal axis and ellipse cross are the ellipse's vertices.



Points on the focal axis of an ellipse.

Conic Sections from Greek Geometry

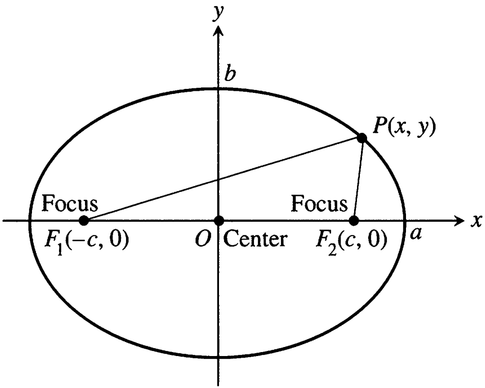
If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, and $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point P on the ellipse satisfy the equation

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b = \sqrt{a^2 - c^2}.$$

Conic Sections from Greek Geometry



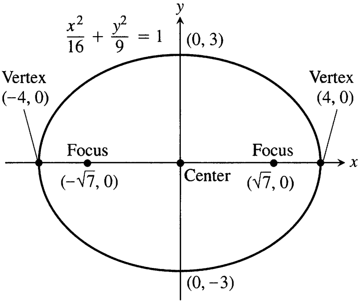
The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Conic Sections from Greek Geometry

Major Axis Horizontal

The ellipse

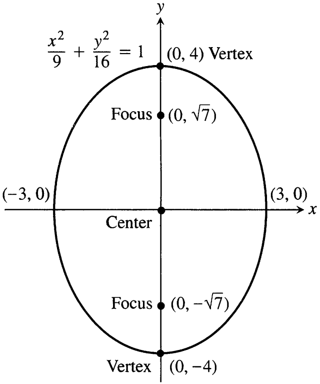
$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$



Conic Sections from Greek Geometry

Major Axis Vertical

The ellipse
$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$



Conic Sections from Greek Geometry

Equations for Ellipses

Standard-form equations for ellipses centered at the origin.

Standard-Form Equations for Ellipses Centered at the Origin

Foci on the x-axis: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$
Foci: $(\pm c, 0)$
Vertices: $(\pm a, 0)$

Foci on the y-axis: $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1 \quad (a > b)$

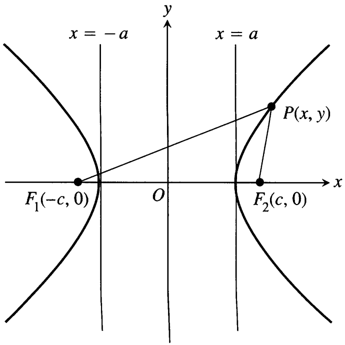
Center-to-focus distance: $c = \sqrt{a^2 - b^2}$
Foci: $(0, \pm c)$
Vertices: $(0, \pm a)$

In each case, a is the semimajor axis and b is the semiminor axis.

Conic Sections from Greek Geometry

Definition (Hyperbola)

A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the foci of the hyperbola.

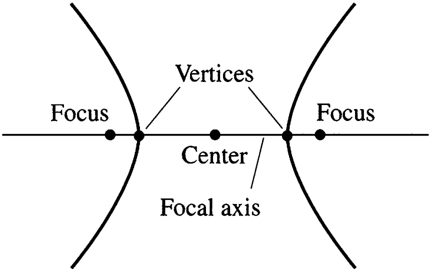


Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here.

Conic Sections from Greek Geometry

Definition (Hyperbola)

The.linethrough.theJociofa.hyperbola is the focal axis. The point on the axis halfway. between the foci is the hyperbola's center. The points where the focal axis and hyperbola cross are the vertices.



Points on the focal axis of a hyperbola.

Conic Sections from Greek Geometry

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, and the constant difference is $2a$, then the coordinates of a point P on the hyperbola satisfy the equation

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we let b denote $b = \sqrt{c^2 - a^2}$, then $a^2 - c^2 = -b^2$. Hence

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Conic Sections from Greek Geometry

Asymptotes of Hyperbolas-Graphing

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

has two asymptotes, the lines

$$y = \pm \frac{b}{a} x.$$

The asymptotes give us the guidance we need to graph hyperbolas quickly. The fastest way to find the equations of the asymptotes is to replace the 1 in hyperbola's equation by 0 and solve the new equation for y:

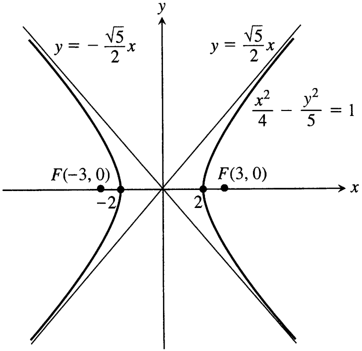
$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbolas}} \Rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{0 for 1}} \Rightarrow \underbrace{y = \pm \frac{b}{a} x.}_{\text{asymptotes}}$$

Conic Sections from Greek Geometry

Foci on the x-axis

The hyperbola

$$\frac{x^2}{4} - \frac{y^2}{5} = 1.$$

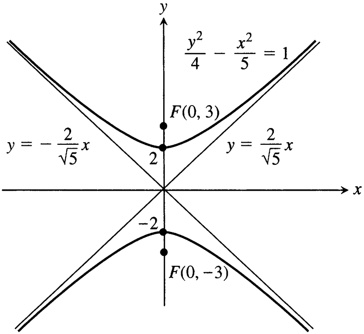


Conic Sections from Greek Geometry

Foci on the y-axis

The ellipse

$$\frac{y^2}{4} + \frac{x^2}{5} = 1.$$



Conic Sections from Greek Geometry

Equations for Hyperbola

Standard-form equations for hyperbolas centered at the origin.

Standard-Form Equations for Hyperbolas Centered at the Origin

Foci on the x-axis: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$
Foci: $(\pm c, 0)$
Vertices: $(\pm a, 0)$
Asymptotes: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$ or $y = \pm \dfrac{b}{a} x$

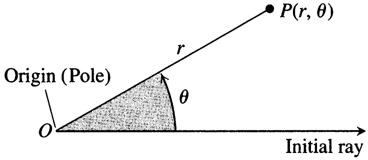
Foci on the y-axis: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$
Foci: $(0, \pm c)$
Vertices: $(0, \pm a)$
Asymptotes: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 0$ or $y = \pm \dfrac{a}{b} x$

Notice the difference in the asymptote equations (b/a in the first, a/b in the second).

2. Polar Coordinate

Definition of Polar Coordinates



To define polar coordinates, we first fix an **origin** O (called the **pole**) and an initial ray from O. Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

Polar Coordinates

$$P(r, \theta)$$

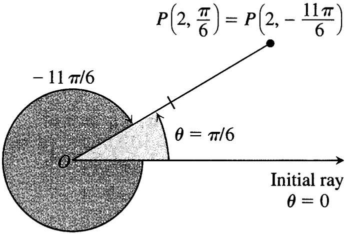
Directed distance from O to P

Directed angle from initial ray to OP

Definition of Polar Coordinates

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique.

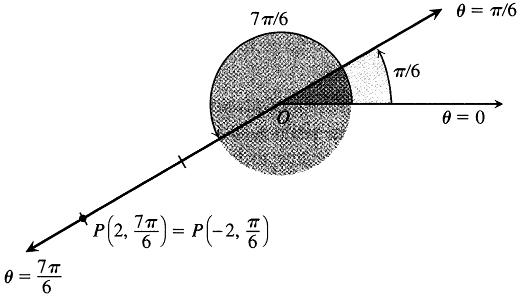
For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2, \theta = \pi/6$. It also has coordinates $r = 2, \theta = -11\pi/6$.



Negative Values of r

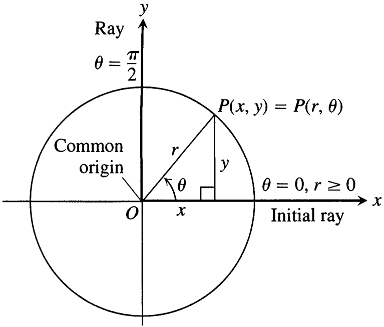
There are occasions when we wish to allow r to be negative.

The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ rad counterclockwise from the initial ray and going forward 2 units. It can also be reached by turning $7\pi/6$ rad counterclockwise from the initial ray and going backward 2 units. So the point also has polar coordinates $r = -2, \theta = \pi/6$.



Cartesian Versus Polar Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive y-axis.



Cartesian Versus Polar Coordinates

The two coordinate systems are related by the following equations:

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2, \qquad \frac{y}{x} = \tan\theta$$

With some curves, we are better off with polar coordinates; with others, we aren't.

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

3. Parametrizations of Plane Curves

Definition of Parametrizations

Definition (Parametrizations)

If x and y are given as continuous functions

$$x = f(t), \quad y = g(t)$$

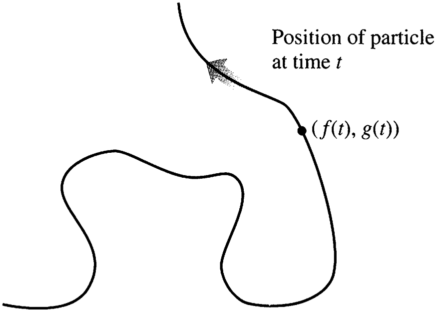
over an interval of t-values, then the set of points

$$(x, y) = (f(t), g(t))$$

defined by these equations is a curve in the coordinate plane. The equations are parametric equations for the curve.

Definition of Parametrizations

When we give parametric equations and a parameter interval for a curve in the plane, we say that we have parametrized the curve. The equations and interval constitute a **parametrization of the curve**.



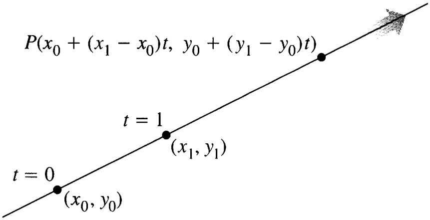
Examples of Parametrizations

A parametrization of lines in the plane

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t, \quad -\infty < t < +\infty$$

can be identified as lines.



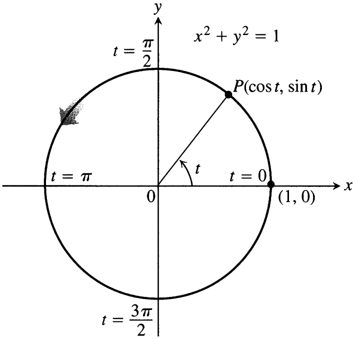
Examples of Parametrizations

The circle $x^2 + y^2 = 1$

The equations and parameter interval

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi,$$

describe the position $P(x, y)$ of a particle that moves counterclockwise around the circle $x^2 + y^2 = 1$ as t increases.



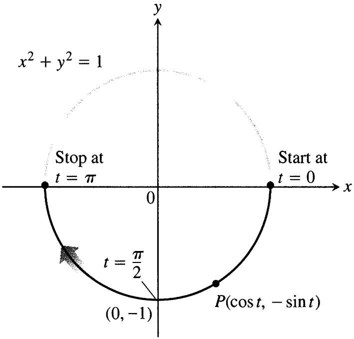
Examples of Parametrizations

A semicircle

The equations and parameter interval

$$x = \cos t, \quad y = -\sin t, \quad 0 \le t \le \pi,$$

describe the position $P(x, y)$ of a particle that moves clockwise around the circle $x^2 + y^2 = 1$ as t increases from 0 to π.



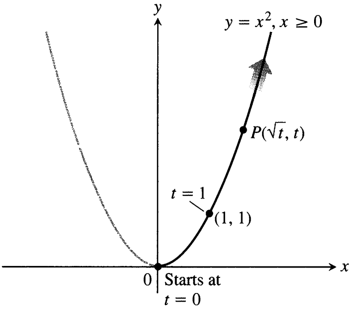
Examples of Parametrizations

Half a parabola

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = \sqrt{t}, \quad y = t, \quad t \geq 0$$

can be identified as half a parabola.



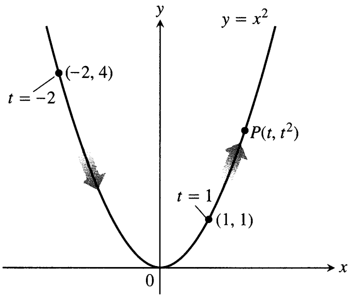
Examples of Parametrizations

An entire parabola

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = t, \quad y = t^2, \quad -\infty < t < +\infty$$

can be identified as a whole parabola.



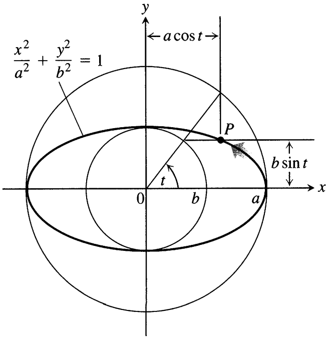
Examples of Parametrizations

A parametrization of the ellipse

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$$

can be identified as an ellipse $x^2/a^2 + y^2/b^2 = 1$.



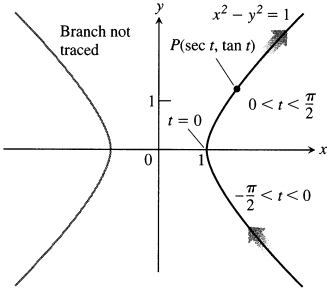
Examples of Parametrizations

A parametrization of the right-hand branch of hyperbola

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

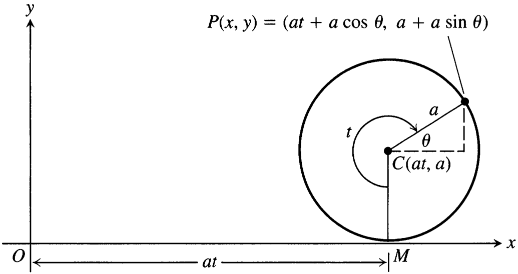
can be identified as the right-hand branch of $x^2 - y^2 = 1$.



Examples of Parametrizations

Definition of cycloids

A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid**.



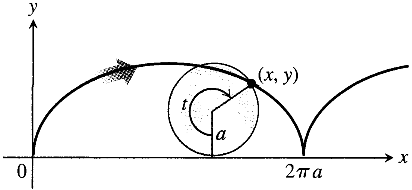
The wheel's center C lies at (at, a) and the coordinates of P are

$$x = at + a\cos\theta, \quad y = a + a\sin\theta.$$

Examples of Parametrizations

To express θ in terms of t, we observe that $t+\theta = 3\pi/2$, so that $\theta = 3\pi/2 - t$. This makes

$$\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t,\ \sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t.$$



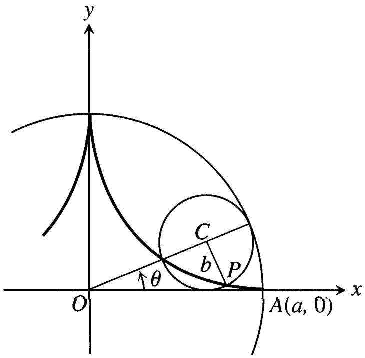
The equations we seek are usually written with the a factored out:

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

Examples of Parametrizations

Definition of hypocycloid

When a circle rolls on the inside of a fixed circle, any point P on the circumference of the rolling circle describes a **hypocycloid**.



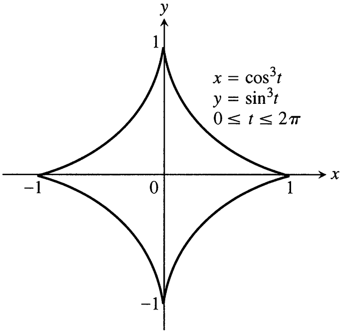
Examples of Parametrizations

A parametrization of hypocycloid

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \le t \le 2\pi$$

can be identified as the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.



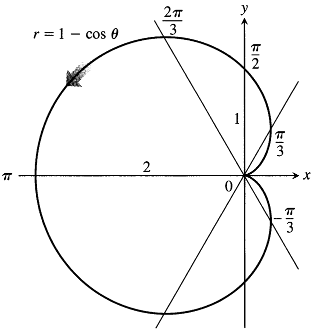
Examples of Parametrizations

A parametrization of cardioid

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$r = 1 - \cos\theta, \quad 0 \le \theta \le 2\pi$$

can be identified as the cardioid.



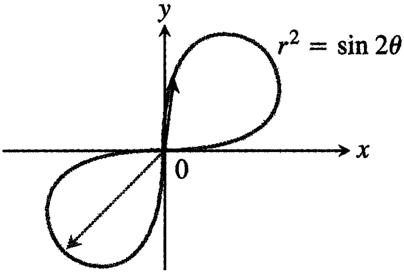
Examples of Parametrizations

A parametrization of lemniscate

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$r^2 = \sin 2\theta, \quad 0 \le \theta \le 2\pi$$

can be identified as the lemniscate.



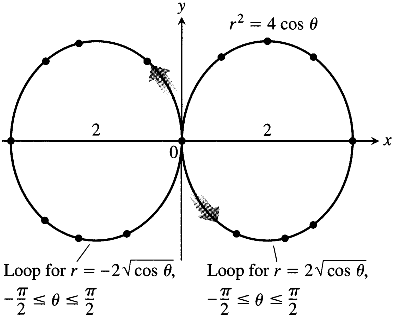
Examples of Parametrizations

Another parametrization of lemniscate

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$r^2 = 4\cos\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

can be identified as the lemniscate.



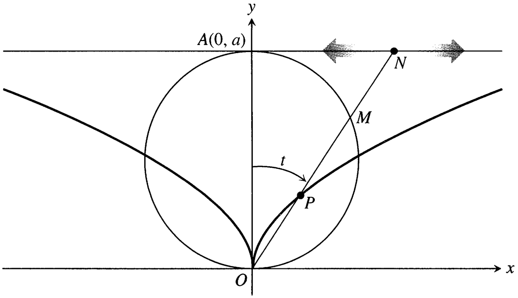
Examples of Parametrizations

Another Examples

As the point N moves along the line $y = a$ in the accompanying figure, P moves in such a way that $OP = MN$. Then the following equation

$$x = a\sin^2 t\tan t, \quad y = a\sin^2 t, \quad -\infty < t < +\infty$$

can be identified as the parametrization of this curve in polar coordinate.



The End