习题 1 设有界数列$\{a_n\}$满足条件$a_{n+1} \geq a_n - \dfrac{1}{2^n}$, $n \in \mathbb{N}$. 请验证数列$\{a_n\}$收敛.

习题 2 请验证以下数列收敛:

$$a_n = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) - 2\sqrt{n};\ b_n = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) - 2\sqrt{n+1}.$$

(提示: 你需要先验证不等式: $2(\sqrt{n+1}-1) < \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} < 2\sqrt{n}$, $n \in \mathbb{N}$.)

习题 3 请验证以下数列收敛:

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}, n \in \mathbb{N}; \quad b_n = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n}, n \in \mathbb{N}.$$

习题 4 请验证以下数列收敛:

$$a_n = \frac{1}{\sqrt{n(n+1)}} + \frac{1}{\sqrt{(n+1)(n+2)}} + \cdots + \frac{1}{\sqrt{(2n-1)2n}}, \quad n \in \mathbb{N}.$$

习题 5 设数列$\{a_n\}$满足$0 < a_n < 1$且$a_n(1 - a_{n+1}) > \dfrac{1}{4}$, $n \in \mathbb{N}$, 请验证数列$\{a_n\}$极限存在并计算其极限.

习题 6 设数列$\{a_n\}$满足$a_1 = \dfrac{3}{2}$, $a_n = \sqrt{3a_{n-1} - 2}$, $n \geq 2$, 请验证数列$\{a_n\}$极限存在并计算其极限.

习题 7 设数列$\{a_n\}$满足$a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$, $n \in \mathbb{N}$, 请验证数列$\{a_n\}$极限存在并计算其极限.

习题 8 计算下列极限: (1) $\lim\limits_{n\to\infty} \dfrac{n^3 \sqrt[n]{2}\left(1-\cos\dfrac{1}{n^2}\right)}{\sqrt{n^2+1}-n}$; (2) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{1}{\sqrt{1^3+2^3+\cdots+k^3}}$.

习题 9 计算下列极限: (1) $\lim\limits_{n\to\infty} \left(\dfrac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n$ $(a \geq 0, b \geq 0)$; (2) $\lim\limits_{n\to\infty} (n!)^{1/n^2}$.

习题 10* 设$a_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}$, $n \in \mathbb{N}$, 请验证$\lim\limits_{n\to\infty} a_n = e$且$0 < e - a_n < \dfrac{1}{nn!}$.

习题 11* 证明$\lim\limits_{n\to\infty} n(\sqrt[n]{n} - 1) = +\infty$.

习题 12* 证明$\lim\limits_{n\to\infty} (n!e - [n!e]) = 0$.