0.1 Some Tensors and theirs Derivatives used in the Gradient Descent

The 9 Christoffel Symbol matrices determined by the identity $\Gamma^i_{jk} = \frac{1}{2}h^{il}[\partial_j h_{lk} + \partial_k h_{jl} - \partial_l h_{jk}]$. The image metric (h_{ij}) is

$$(h_{ij}) = \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \frac{1}{w_1^2} & & & & & \\ & & & & \frac{1}{w_2^2} & & & & \\ & & & & & \frac{1}{w_3^2} & & & \\ & & & & & & \frac{2w_1(w_3+w_2w_6^2)}{w_2w_3} & \frac{-2w_1w_6}{w_3} & \\ & & & & & & \frac{-2w_1w_6}{w_3} & \frac{2w_1}{w_3} & \\ & & & & & & & & \frac{2w_2}{w_3} \end{pmatrix}$$

and its inverse matrix is:

$$(h^{ij}) = \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & w_1^2 & & & & & \\ & & & & w_2^2 & & & & \\ & & & & & w_3^2 & & & \\ & & & & & & \frac{w_2}{2w_1} & \frac{w_2w_6}{2w_1} & \\ & & & & & & \frac{w_2w_6}{2w_1} & \frac{w_3-w_6^2w_2}{2w_1} & \\ & & & & & & & & \frac{w_3}{2w_2} \end{pmatrix}$$

the first three $\Gamma^1 = \Gamma^2 = \Gamma^3 = 0$ and here we present the rest of the Christoffel symbols:

$$\Gamma^4 = (\frac{1}{2}\frac{1}{h_{44}}[\partial_j h_{4k} + \partial_k h_{j4} - \partial_4 h_{jk}]) = (\frac{w_1^2}{2}[\delta_{j=k=4}(\partial_j h_{4k} + \partial_k h_{j4}) - \partial_4 h_{jk}])$$

$$= \begin{pmatrix} 0 &&&&&&&& \\ & 0 &&&&&&& \\ && 0 &&&&&& \\ &&& \frac{1}{w_1} &&&&& \\ &&&& 0 &&&& \\ &&&&& 0 &&& \\ &&&&&& \frac{-w_1^2(w_3+w_2w_6^2)}{w_2w_3} & \frac{w_1^2w_6}{w_3} & \\ &&&&&& \frac{w_1^2w_6}{w_3} & \frac{-w_1^2}{w_3} & \\ &&&&&&&& 0 \end{pmatrix}$$

$$\Gamma^5 = (\frac{1}{2}\frac{1}{h_{55}}[\partial_j h_{5k} + \partial_k h_{j5} - \partial_5 h_{jk}]) = (\frac{w_2^2}{2}[\delta_{j=k=5}(\partial_j h_{5k} + \partial_k h_{j5}) - \partial_5 h_{jk}])$$

$$= \begin{pmatrix} 0 &&&&&&&& \\ & 0 &&&&&&& \\ && 0 &&&&&& \\ &&& 0 &&&&& \\ &&&& \frac{1}{w_2} &&&& \\ &&&&& 0 &&& \\ &&&&&& w_1 & 0 & \\ &&&&&& 0 & 0 & \\ &&&&&&&& \frac{-w_2^2}{w_3} \end{pmatrix}$$

$$\Gamma^6 = (\frac{1}{2}\frac{1}{h_{66}}[\partial_j h_{3k} + \partial_k h_{j3} - \partial_3 h_{jk}]) = (\frac{w_3^2}{2}[\delta_{j=k=6}(\partial_j h_{6k} + \partial_k h_{j6}) - \partial_6 h_{jk}])$$

$$= \begin{pmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & 0 & & & & \\ & & & & & \frac{1}{w_3} & & & \\ & & & & & & w_6^2 & -w_1 w_6 & \\ & & & & & & -w_1 w_6 & w_1 & \\ & & & & & & & & w_2 \end{pmatrix}$$

$$\begin{aligned}
\Gamma^7 &= (\frac{1}{2}h^{77}[\partial_j h_{7k} + \partial_k h_{j7} - \partial_7 h_{jk}] + \frac{1}{2}h^{78}[\partial_j h_{8k} + \partial_k h_{j8} - \partial_8 h_{jk}]) \\
&= (\frac{1}{2}\frac{w_2}{2w_1}[(\delta_{j=k=7} + \delta_{j=7,k=8} + \delta_{j=8,k=7})(\partial_j h_{7k} + \partial_k h_{j7}) - \partial_7 h_{jk}] \\
&+ \frac{1}{2}\frac{w_2 w_6}{2w_1}[(\delta_{j=k=7} + \delta_{j=7,k=8} + \delta_{j=8,k=7})(\partial_j h_{8k} + \partial_k h_{j8}) - \partial_8 h_{jk}]) \\
&= 0
\end{aligned}$$

because h_{ij} does not contain the w_4, w_5 variables.

$$
\begin{aligned}
\Gamma^8 &= (\frac{1}{2}h^{87}[\partial_j h_{7k} + \partial_k h_{j7} - \partial_7 h_{jk}] + \frac{1}{2}h^{88}[\partial_j h_{8k} + \partial_k h_{j8} - \partial_8 h_{jk}]) \\
&= (\frac{1}{2}\frac{w_2 w_6}{2w_1}[(\delta_{j=k=7} + \delta_{j=7,k=8} + \delta_{j=8,k=7})(\partial_j h_{7k} + \partial_k h_{j7}) - \partial_7 h_{jk}] \\
&+ \frac{1}{2}\frac{w_3 - w_6^2 w_2}{2w_1}[(\delta_{j=k=7} + \delta_{j=7,k=8} + \delta_{j=8,k=7})(\partial_j h_{8k} + \partial_k h_{j8}) - \partial_8 h_{jk}]) \\
&= 0 \\
\Gamma^9 &= (\frac{1}{2}h^{99}[\partial_j h_{9k} + \partial_k h_{j9} - \partial_9 h_{jk}]) = (\frac{1}{2}\frac{w_3}{2w_2}[\delta_{j=k=9}(\partial_9 h_{9k} + \partial_k h_{j9}) - \partial_9 h_{jk}]) \\
&= \begin{pmatrix}
0 & & & & & & & & \\
& 0 & & & & & & & \\
& & 0 & & & & & & \\
& & & 0 & & & & & \\
& & & & 0 & & & & \\
& & & & & 0 & & & \\
& & & & & & -\frac{w_1 w_6}{w_2} & \frac{w_1}{2w_2} & \\
& & & & & & \frac{w_1}{2w_2} & 0 & \\
& & & & & & & & 0
\end{pmatrix}
\end{aligned}
$$

The induced matrix γ is:

$$
\begin{pmatrix}
\partial_1 X^i \partial_1 X^j h_{ij} & \partial_1 X^i \partial_2 X^j h_{ij} & \partial_1 X^i \partial_3 X^j h_{ij} \\
& \partial_2 X^i \partial_2 X^j h_{ij} & \partial_2 X^i \partial_3 X^j h_{ij} \\
& & \partial_3 X^i \partial_3 X^j h_{ij}
\end{pmatrix}
$$

The inverse of the matrix is:

$$
\gamma^{\mu\nu} = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix}
$$

where

$$
\begin{aligned}
A &= \partial_2 X^i \partial_2 X^j h_{ij} \cdot \partial_3 X^i \partial_3 X^j h_{ij} - (\partial_2 X^i \partial_3 X^j h_{ij})^2 \\
B &= \partial_1 X^i \partial_3 X^j h_{ij} \cdot \partial_2 X^i \partial_3 X^j h_{ij} - \partial_1 X^i \partial_2 X^j h_{ij} \cdot \partial_3 X^i \partial_3 X^j h_{ij} \\
C &= \partial_1 X^i \partial_2 X^j h_{ij} \cdot \partial_2 X^i \partial_3 X^j h_{ij} - \partial_1 X^i \partial_3 X^j h_{ij} \cdot \partial_2 X^i \partial_2 X^j h_{ij} \\
D &= \partial_1 X^i \partial_1 X^j h_{ij} \cdot \partial_3 X^i \partial_3 X^j h_{ij} - (\partial_1 X^i \partial_3 X^j h_{ij})^2 \\
E &= \partial_1 X^i \partial_3 X^j h_{ij} \cdot \partial_1 X^i \partial_2 X^j h_{ij} - \partial_1 X^i \partial_1 X^j h_{ij} \cdot \partial_2 X^i \partial_3 X^j h_{ij} \\
F &= \partial_1 X^i \partial_1 X^j h_{ij} \cdot \partial_2 X^i \partial_2 X^j h_{ij} - (\partial_1 X^i \partial_2 X^j h_{ij})^2.
\end{aligned}
$$

The derivatives of the diffusion tensor:

$$\frac{\partial D}{\partial x_1} = \begin{pmatrix} 1 & w_4 & w_5 \\ w_4 & w_4^2 & w_4 w_5 \\ w_5 & w_4 w_5 & w_5^2 \end{pmatrix} \frac{\partial D}{\partial x_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & w_6 \\ 0 & w_6 & w_6^2 \end{pmatrix} \frac{\partial D}{\partial x_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\frac{\partial D}{\partial x_4} = \begin{pmatrix} 0 & w_4 & 0 \\ w_1 & 2w_1 w_4 & w_1 w_5 \\ 0 & w_1 w_5 & 0 \end{pmatrix} \frac{\partial D}{\partial x_5} = \begin{pmatrix} 0 & 0 & w_1 \\ 0 & 0 & w_1 w_4 \\ w_1 & w_1 w_4 & 2w_1 w_5 \end{pmatrix} \frac{\partial D}{\partial x_6} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w_2 \\ 0 & w_2 & 2w_2 w_6 \end{pmatrix}$$

The derivative of the image metric h_{ij}:

$$\partial(h_{ij}) = \begin{pmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ & & & \frac{-\partial X_1}{X_1^3} & & & & & \\ & & & & \frac{-\partial X_2}{X_2^3} & & & & \\ & & & & & \frac{-\partial X_3}{X_3^3} & & & \\ & & & & & & A & B & \\ & & & & & & C & D & \\ & & & & & & & & E \end{pmatrix}$$

$$A = 2 * \frac{\partial X_4 (X_6 + X_5 X_9^2) X_5 X_6 + X_4 (\partial X_6}{(X_5 X_6)^2} + \frac{\partial X_5 X_9^2 + 2X_5 X_9 \partial X_9) X_5 X_6 - X_4 (X_6 + X_5 X_9^2)(\partial X_5 X_6 + X_5 \partial X_6)}{(X_5 X_6)^2}$$

$$B = -2 * \frac{\partial X_4 X_9 X_5 + X_4 \partial X_9 X_5 - X_5 X_9 \partial X5}{X_6^2}$$

$$C = -2 * \frac{\partial X_4 X_9 X_5 + X_4 \partial X_9 X_5 - X_5 X_9 \partial X_5}{X_6^2}$$

$$D = 2 * \frac{\partial X_4 X_6 - X_4 \partial X_6}{X_6^2}$$

$$E = 2 * \frac{\partial X5 X_6 - X_5 \partial X6}{X_6^2}.$$