

## Chapter 1

# Hyperbolic Partial Differential Equations

We begin our study of finite difference methods for partial differential equations by considering the important class of partial differential equations called hyperbolic equations. In later chapters we consider other classes of partial differential equations, especially parabolic and elliptic equations. For each of these classes of equations we consider prototypical equations, with which we illustrate the important concepts and distinguishing features associated with each class. The reader is referred to other textbooks on partial differential equations for alternate approaches, e.g., Folland [18], Garabedian [22], and Weinberger [68]. After introducing each class of differential equations we consider finite difference methods for the numerical solution of equations in the class.

We begin this chapter by considering the simplest hyperbolic equation and then extend our discussion to include hyperbolic systems of equations and equations with variable coefficients. After the basic concepts have been introduced, we begin our discussion of finite difference schemes. The important concepts of convergence, consistency, and stability are presented and shown to be related by the Lax–Richtmyer equivalence theorem. The chapter concludes with a discussion of the Courant–Friedrichs–Lewy condition and related topics.

## 1.1 Overview of Hyperbolic Partial Differential Equations

### The One-Way Wave Equation

The prototype for all hyperbolic partial differential equations is the one-way wave equation:

$$u_t + au_x = 0, \quad (1.1.1)$$

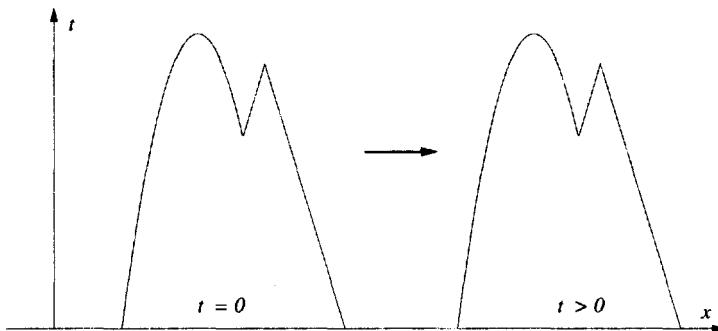
where  $a$  is a constant,  $t$  represents time, and  $x$  represents the spatial variable. The subscript denotes differentiation, i.e.,  $u_t = \partial u / \partial t$ . We give  $u(t, x)$  at the initial time, which we always take to be 0—i.e.,  $u(0, x)$  is required to be equal to a given function  $u_0(x)$  for all real numbers  $x$ —and we wish to determine the values of  $u(t, x)$  for positive values of  $t$ . This is called an *initial value problem*.

By inspection we observe that the solution of (1.1.1) is

$$u(t, x) = u_0(x - at). \quad (1.1.2)$$

(Actually, we know only that this is a solution; we prove later that this is the unique solution.)

The formula (1.1.2) tells us several things. First, the solution at any time  $t_0$  is a copy of the original function, but shifted to the right, if  $a$  is positive, or to the left, if  $a$  is negative, by an amount  $|a|t_0$ . Another way to say this is that the solution at  $(t, x)$  depends only on the value of  $\xi = x - at$ . The lines in the  $(t, x)$  plane on which  $x - at$  is constant are called *characteristics*. The parameter  $a$  has dimensions of distance divided by time and is called the speed of propagation along the characteristic. Thus the solution of the one-way wave equation (1.1.1) can be regarded as a wave that propagates with speed  $a$  without change of shape, as illustrated in Figure 1.1.



**Figure 1.1.** The solution of the one-way wave equation is a shift.

Second, whereas equation (1.1.1) appears to make sense only if  $u$  is differentiable, the solution formula (1.1.2) requires no differentiability of  $u_0$ . In general, we allow for discontinuous solutions for hyperbolic problems. An example of a discontinuous solution is a shock wave, which is a feature of solutions of nonlinear hyperbolic equations.

To illustrate further the concept of characteristics, consider the more general hyperbolic equation

$$\begin{aligned} u_t + au_x + bu &= f(t, x), \\ u(0, x) &= u_0(x), \end{aligned} \tag{1.1.3}$$

where  $a$  and  $b$  are constants. Based on our preceding observations we change variables from  $(t, x)$  to  $(\tau, \xi)$ , where  $\tau$  and  $\xi$  are defined by

$$\tau = t, \quad \xi = x - at.$$

The inverse transformation is then

$$t = \tau, \quad x = \xi + a\tau,$$

and we define  $\tilde{u}(\tau, \xi) = u(t, x)$ , where  $(\tau, \xi)$  and  $(t, x)$  are related by the preceding relations. (Both  $u$  and  $\tilde{u}$  represent the same function, but the tilde is needed to distinguish

between the two coordinate systems for the independent variables.) Equation (1.1.3) then becomes

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial \tau} &= \frac{\partial t}{\partial \tau} u_t + \frac{\partial x}{\partial \tau} u_x \\ &= u_t + au_x = -bu + f(\tau, \xi + a\tau).\end{aligned}$$

So we have

$$\frac{\partial \tilde{u}}{\partial \tau} = -bu + f(\tau, \xi + a\tau).$$

This is an ordinary differential equation in  $\tau$  and the solution is

$$\tilde{u}(\tau, \xi) = u_0(\xi)e^{-b\tau} + \int_0^\tau f(\sigma, \xi + a\sigma)e^{-b(\tau-\sigma)} d\sigma.$$

Returning to the original variables, we obtain the representation for the solution of equation (1.1.3) as

$$u(t, x) = u_0(x - at)e^{-bt} + \int_0^t f(s, x - a(t-s))e^{-b(t-s)} ds. \quad (1.1.4)$$

We see from (1.1.4) that  $u(t, x)$  depends only on values of  $(t', x')$  such that  $x' - at' = x - at$ , i.e., only on the values of  $u$  and  $f$  on the characteristic through  $(t, x)$  for  $0 \leq t' \leq t$ .

This method of solution of (1.1.3) is easily extended to nonlinear equations of the form

$$u_t + au_x = f(t, x, u). \quad (1.1.5)$$

See Exercises 1.1.5, 1.1.4, and 1.1.6 for more on nonlinear equations of this form.

## Systems of Hyperbolic Equations

We now examine systems of hyperbolic equations with constant coefficients in one space dimension. The variable  $u$  is now a vector of dimension  $d$ .

**Definition 1.1.1.** *A system of the form*

$$u_t + Au_x + Bu = F(t, x) \quad (1.1.6)$$

*is hyperbolic if the matrix  $A$  is diagonalizable with real eigenvalues.*

By saying that the matrix  $A$  is *diagonalizable*, we mean that there is a nonsingular matrix  $P$  such that  $PAP^{-1}$  is a diagonal matrix, that is,

$$PAP^{-1} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} = \Lambda.$$

The eigenvalues  $a_i$  of  $A$  are the characteristic speeds of the system. Under the change of variables  $w = Pu$  we have, in the case  $B = 0$ ,

$$w_t + \Lambda w_x = PF(t, x) = \tilde{F}(t, x)$$

or

$$w_t^i + a_i w_x^i = \tilde{f}^i(t, x),$$

which is the form of equation (1.1.3). Thus, when matrix  $B$  is zero, the one-dimensional hyperbolic system (1.1.6) reduces to a set of independent scalar hyperbolic equations. If  $B$  is not zero, then in general the resulting system of equations is coupled together, but only in the undifferentiated terms. The effect of the lower order term,  $Bu$ , is to cause growth, decay, or oscillations in the solution, but it does not alter the primary feature of the propagation of the solution along the characteristics. The definition of hyperbolic systems in more than one space dimension is given in Chapter 9.

**Example 1.1.1.** As an example of a hyperbolic system, we consider the system

$$\begin{aligned} u_t + 2u_x + v_x &= 0, \\ v_t + u_x + 2v_x &= 0, \end{aligned}$$

which can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0.$$

As initial data we take

$$\begin{aligned} u(0, x) = u_0(x) &= \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1, \end{cases} \\ v(0, x) &= 0. \end{aligned}$$

By adding and subtracting the two equations, the system can be rewritten as

$$(u + v)_t + 3(u + v)_x = 0,$$

$$(u - v)_t + (u - v)_x = 0$$

or

$$w_t^1 + 3w_x^1 = 0, \quad w^1(0, x) = u_0(x),$$

$$w_t^2 + w_x^2 = 0, \quad w^2(0, x) = u_0(x).$$

The matrix  $P$  is  $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  for this transformation. The solution is, therefore,

$$w^1(t, x) = w_0^1(x - 3t),$$

$$w^2(t, x) = w_0^2(x - t)$$

or

$$u(t, x) = \frac{1}{2}(w^1 + w^2) = \frac{1}{2}[u_0(x - 3t) + u_0(x - t)],$$

$$v(t, x) = \frac{1}{2}(w^1 - w^2) = \frac{1}{2}[u_0(x - 3t) - u_0(x - t)].$$

These formulas show that the solution consists of two independent parts, one propagating with speed 3 and one with speed 1.  $\square$

## Equations with Variable Coefficients

We now examine equations for which the characteristic speed is a function of  $t$  and  $x$ . Consider the equation

$$u_t + a(t, x)u_x = 0 \quad (1.1.7)$$

with initial condition  $u(0, x) = u_0(x)$ , which has the variable speed of propagation  $a(t, x)$ . If, as we did after equation (1.1.3), we change variables to  $\tau$  and  $\xi$ , where  $\tau = t$  and  $\xi$  is as yet undetermined, we have

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial \tau} &= \frac{\partial t}{\partial \tau}u_t + \frac{\partial x}{\partial \tau}u_x \\ &= u_t + \frac{\partial x}{\partial \tau}u_x. \end{aligned}$$

In analogy with the constant coefficient case, we set

$$\frac{dx}{d\tau} = a(t, x) = a(\tau, x).$$

This is an ordinary differential equation for  $x$  giving the speed along the characteristic through the point  $(\tau, x)$  as  $a(\tau, x)$ . We set the initial value for the characteristic curve through  $(\tau, x)$  to be  $\xi$ . Thus the equation (1.1.7) is equivalent to the system of ordinary differential equations

$$\begin{aligned} \frac{d\tilde{u}}{d\tau} &= 0, & \tilde{u}(0, \xi) &= u_0(\xi), \\ \frac{dx}{d\tau} &= a(\tau, x), & x(0) &= \xi. \end{aligned} \quad (1.1.8)$$

As we see from the first equation in (1.1.8),  $u$  is constant along each characteristic curve, but the characteristic determined by the second equation need not be a straight line. We now present an example to illustrate these ideas.

**Example 1.1.2.** Consider the equation

$$u_t + x u_x = 0,$$

$$u(0, x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Corresponding to the system (1.1.8) we have the equations

$$\frac{d\tilde{u}}{d\tau} = 0, \quad \frac{dx}{d\tau} = x, \quad x(0) = \xi.$$

The general solution of the differential equation for  $x(\tau)$  is  $x(\tau) = ce^\tau$ . Because we specify that  $\xi$  is defined by  $x(0) = \xi$ , we have  $x(\tau) = \xi e^\tau$ , or  $\xi = xe^{-t}$ . The equation for  $\tilde{u}$  shows that  $\tilde{u}$  is independent of  $\tau$ , so by the condition at  $\tau$  equal to zero we have that

$$\tilde{u}(\tau, \xi) = u_0(\xi).$$

Thus

$$u(t, x) = \tilde{u}(\tau, \xi) = u_0(\xi) = u_0(xe^{-t}).$$

So we have, for  $t > 0$ ,

$$u(t, x) = \begin{cases} 1 & \text{if } 0 \leq x \leq e^t, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

As for equations with constant coefficients, these methods apply to nonlinear equations of the form

$$u_t + a(t, x)u_x = f(t, x, u), \quad (1.1.9)$$

as shown in Exercise 1.1.9. Equations for which the characteristic speeds depend on  $u$ , i.e., with characteristic speed  $a(t, x, u)$ , require special care, since the characteristic curves may intersect.

## Systems with Variable Coefficients

For systems of hyperbolic equations in one space variable with variable coefficients, we require uniform diagonalizability. (See Appendix A for a discussion of matrix norms.)

**Definition 1.1.2.** *The system*

$$u_t + A(t, x)u_x + B(t, x)u = F(t, x) \quad (1.1.10)$$

with

$$u(0, x) = u_0(x)$$

is hyperbolic if there is a matrix function  $P(t, x)$  such that

$$P(t, x)A(t, x)P^{-1}(t, x) = \Lambda(t, x) = \begin{pmatrix} a_1(t, x) & & 0 \\ & \ddots & \\ 0 & & a_d(t, x) \end{pmatrix}$$

is diagonal with real eigenvalues and the matrix norms of  $P(t, x)$  and  $P^{-1}(t, x)$  are bounded in  $x$  and  $t$  for  $x \in R$ ,  $t \geq 0$ .

The characteristic curves for system (1.1.10) are the solutions to the differential equations

$$\frac{dx^i}{dt} = a_i(t, x), \quad x^i(0) = \xi^i.$$

Setting  $v = P(t, x)u$ , we obtain the system for  $v$ :

$$v_t + \Lambda v_x = P(t, x) F(t, x) + G(t, x)v,$$

where

$$G = (P_t + \Lambda P_x - PB)P^{-1}.$$

In terms of directional derivatives this system is equivalent to

$$\frac{dv^i}{dt} \Big|_{\text{along } x^i} = \tilde{f}^i(t, x) + \sum_{j=1}^d g_j^i(t, x)v^j.$$

This formula is not a practical method of solution for most problems because the ordinary differential equations are often quite difficult to solve, but the formula does show the importance of characteristics for these systems.

## Exercises

**1.1.1.** Consider the initial value problem for the equation

$$u_t + au_x = f(t, x)$$

with  $u(0, x) = 0$  and

$$f(t, x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that  $a$  is positive. Show that the solution is given by

$$u(t, x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x/a & \text{if } x \geq 0 \text{ and } x - at \leq 0, \\ t & \text{if } x \geq 0 \text{ and } x - at \geq 0. \end{cases}$$

**1.1.2.** Consider the initial value problem for the equation

$$u_t + au_x = f(t, x)$$

with  $u(0, x) = 0$  and

$$f(t, x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that  $a$  is positive. Show that the solution is given by

$$u(t, x) = \begin{cases} (x+1)/a & \text{if } -1 \leq x \leq 1 \text{ and } x - at \leq -1, \\ t & \text{if } -1 \leq x \leq 1 \text{ and } -1 \leq x - at, \\ 2/a & \text{if } x \geq 1 \text{ and } x - at \leq -1, \\ (1-x+at)/a & \text{if } x \geq 1 \text{ and } -1 \leq x - at \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

**1.1.3.** Solve the initial value problem for

$$u_t + \frac{1}{1 + \frac{1}{2} \cos x} u_x = 0.$$

Show that the solution is given by  $u(t, x) = u_0(\xi)$ , where  $\xi$  is the unique solution of

$$\xi + \frac{1}{2} \sin \xi = x + \frac{1}{2} \sin x - t.$$

**1.1.4.** Show that the initial value problem for (1.1.5) is equivalent to the family of initial value problems for the ordinary differential equations

$$\frac{d\tilde{u}}{d\tau} = f(\tau, \xi + a\tau, \tilde{u})$$

with  $\tilde{u}(0, \xi) = u_0(\xi)$ . Show that the solution of (1.1.5),  $u(t, x)$ , is given by  $u(t, x) = \tilde{u}(t, x - at)$ .

**1.1.5.** Use the results of Exercise 1.1.4 to show that the solution of the initial value problem for

$$u_t + u_x = -\sin^2 u$$

is given by

$$u(t, x) = \tan^{-1} \left( \frac{\tan[u_0(x-t)]}{1 + t \tan[u_0(x-t)]} \right).$$

An equivalent formula for the solution is

$$u(t, x) = \cot^{-1} (\cot[u_0(x-t)] + t).$$

**1.1.6.** Show that all solutions to

$$u_t + a u_x = 1 + u^2$$

become unbounded in finite time. That is,  $u(t, x)$  tends to infinity for some  $x$  as  $t$  approaches some value  $t^*$ , where  $t^*$  is finite.

**1.1.7.** Show that the initial value problem for the equation

$$u_t + (1 + x^2) u_x = 0$$

is not well defined. *Hint:* Consider the region covered by the characteristics originating on the  $x$ -axis.

**1.1.8.** Obtain the solution of the system

$$\begin{aligned} u_t + u_x + v_x &= 0, & u(x, 0) &= u_0(x), \\ v_t + u_x - v_x &= 0, & v(x, 0) &= v_0(x). \end{aligned}$$

- 1.1.9.** Show that the initial value problem for (1.1.9) is equivalent to the family of initial value problems for the system of ordinary differential equations

$$\begin{aligned}\frac{d\tilde{u}}{d\tau} &= f(\tau, x(\tau), \tilde{u}), & \tilde{u}(0, \xi) &= u_0(\xi), \\ \frac{dx}{d\tau} &= a(\tau, x(\tau)), & x(0) &= \xi.\end{aligned}$$

The solution to (1.1.9) is given by  $u(t, x(\xi)) = \tilde{u}(t, \xi)$ .

## 1.2 Boundary Conditions

We now consider hyperbolic partial differential equations on a finite interval rather than on the whole real line. Most applications of partial differential equations involve domains with boundaries, and it is important to specify data correctly at these locations. The conditions relating the solution of the differential equation to data at a boundary are called boundary conditions. A more complete discussion of the theory of boundary conditions for time-dependent partial differential equations is given in Chapter 11. The problem of determining a solution to a differential equation when both initial data and boundary data are present is called an *initial-boundary value problem*. In this section we restrict the discussion to initial-boundary value problems for hyperbolic equations in one space variable.

The discussion of initial-boundary value problems serves to illustrate again the importance of the concept of characteristics. Consider the simple equation

$$u_t + au_x = 0 \quad \text{with } 0 \leq x \leq 1, t \geq 0. \quad (1.2.1)$$

If  $a$  is positive the characteristics in this region propagate from the left to the right, as shown in Figure 1.2. By examining the characteristics in Figure 1.2, we see that the solution must be specified on the boundary at  $x$  equal to 0, in addition to the initial data, in order to be defined for all time. Moreover, no data can be supplied at the other boundary or the solution will be overdetermined.

If we specify initial data  $u(0, x) = u_0(x)$  and boundary data  $u(t, 0) = g(t)$ , then the solution is given by

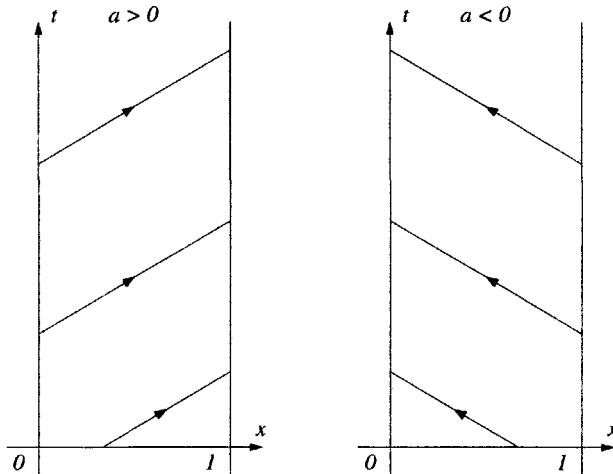
$$u(t, x) = \begin{cases} u_0(x - at) & \text{if } x - at > 0, \\ g(t - a^{-1}x) & \text{if } x - at < 0. \end{cases}$$

Along the characteristic given by  $x - at = 0$ , there will be a jump discontinuity in  $u$  if  $u_0(0)$  is not equal to  $g(0)$ . If  $a$  is negative, the roles of the two boundaries are reversed.

Now consider the hyperbolic system

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t + \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x = 0 \quad (1.2.2)$$

on the interval  $0 \leq x \leq 1$ . The eigenvalues, or characteristic speeds, of the system are easily seen to be  $a + b$  and  $a - b$ . We consider only the cases where  $a$  and  $b$  are



**Figure 1.2.** Characteristics for equation (1.2.1).

positive. If we have  $0 < b < a$ , then both characteristic families propagate to the right, as shown in Figure 1.3. This means that the entire solution, both components  $u^1$  and  $u^2$ , must be specified at  $x$  equal to 0, and no data should be specified at  $x$  equal to 1. Notice that the slope of the characteristic in these figures is the inverse of the speed. Thus the characteristics with the slower speed have the greater slope.

The most interesting case is where  $0 < a < b$ , since then the characteristic families propagate in opposite directions (see the right-hand side in Figure 1.3). If system (1.2.2) is put into the form (1.1.6), it is

$$\begin{pmatrix} u^1 + u^2 \\ u^1 - u^2 \end{pmatrix}_t + \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \begin{pmatrix} u^1 + u^2 \\ u^1 - u^2 \end{pmatrix}_x = 0. \quad (1.2.3)$$

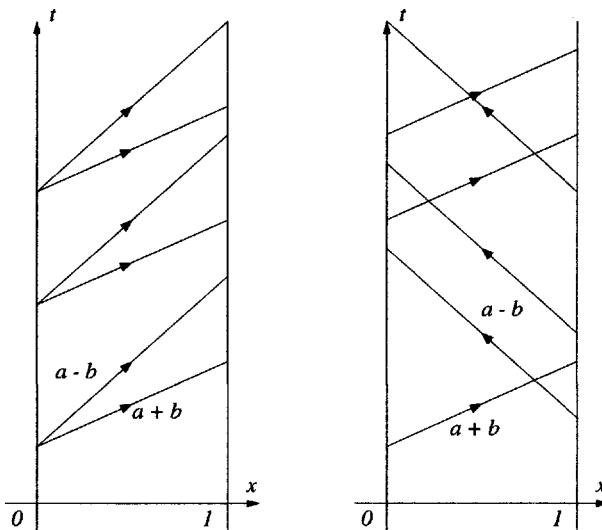
Certainly one way to determine the solution uniquely is to specify  $u^1 + u^2$  at  $x$  equal to 0 and specify  $u^1 - u^2$  at  $x$  equal to 1. However, there are other possible boundary conditions; for example, any of the form

$$\begin{aligned} u^1 + u^2 &= \alpha_0(u^1 - u^2) + \beta_0(t) && \text{at } x = 0, \\ u^1 - u^2 &= \alpha_1(u^1 + u^2) + \beta_1(t) && \text{at } x = 1, \end{aligned} \quad (1.2.4)$$

will determine the solution. The coefficients  $\alpha_0$  and  $\alpha_1$  may be functions of  $t$  or constants.

As examples, we have that the boundary conditions

$$\begin{aligned} u^1(t, 0) &= \beta_0(t), \\ u^2(t, 1) &= \beta_1(t) \end{aligned}$$



**Figure 1.3.** Characteristics for system (1.2.3).

can be put in the form

$$\begin{aligned} u^1(t, 0) + u^2(t, 0) &= -(u^1(t, 0) - u^2(t, 0)) + 2\beta_0(t), \\ u^1(t, 1) + u^2(t, 1) &= \quad u^1(t, 1) - u^2(t, 1) + 2\beta_1(t), \end{aligned}$$

which are equivalent to the conditions in (1.2.4) with  $\alpha_0$  and  $\alpha_1$  equal to  $-1$  and  $1$ , respectively.

Boundary conditions that determine a unique solution are said to be *well-posed*. For the system (1.2.2) the boundary conditions are well-posed if and only if they are equivalent to (1.2.4). The boundary conditions (1.2.4) express the value of the characteristic variable on the *incoming* characteristic in terms of the *outgoing* characteristic variable and the data. By incoming characteristic we mean a characteristic that enters the domain at the boundary under consideration; an outgoing characteristic is one that leaves the domain. We see then that specifying  $u^1$  or  $u^2$  at  $x$  equal to  $0$  is well-posed, and specifying  $u^1$  or  $u^2$  at  $x$  equal to  $1$  is also well-posed. However, specifying  $u^1 - u^2$  at  $x$  equal to  $0$  is ill-posed, as is specifying  $u^1 + u^2$  at  $x$  equal to  $1$ .

For a hyperbolic initial-boundary value problem to be well-posed, the number of boundary conditions must be equal to the number of incoming characteristics. The procedure for determining whether or not an initial-boundary value problem is well-posed is given in Chapter 11.

**Example 1.2.1.** To illustrate how the solution to a hyperbolic system is determined by both the initial and boundary conditions, we consider as an example the system

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t + \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x = 0 \quad (1.2.5)$$

on the interval  $[0,1]$  with the initial conditions

$$u^1(0, x) = 0 \quad \text{and} \quad u^2(0, x) = x.$$

The eigenvalues of the matrix in (1.2.5) are 2 and -1, so this system requires one boundary condition on each boundary. We take boundary conditions

$$u^1(t, 0) = t \quad \text{and} \quad u^1(t, 1) = 0.$$

The two families of characteristic curves are given by

$$t - 2x = \xi_1 \quad \text{and} \quad t + x = \xi_2,$$

where different values of  $\xi_1$  and  $\xi_2$  give the different characteristic curves. The characteristics are displayed in Figure 1.4.

The system (1.2.5) can be rewritten as

$$\begin{pmatrix} u^1 + u^2 \\ u^1 - u^2 \end{pmatrix}_t + \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u^1 + u^2 \\ u^1 - u^2 \end{pmatrix}_x = 0 \quad (1.2.6)$$

and this shows that the characteristic variables  $w^1$  and  $w^2$  are

$$w^1 = u^1 + u^2 \quad \text{and} \quad w^2 = u^1 - u^2.$$

The inverse relations are

$$u^1 = \frac{w^1 + w^2}{2} \quad \text{and} \quad u^2 = \frac{w^1 - w^2}{2}.$$

The equations satisfied by  $w^1$  and  $w^2$  are

$$w_t^1 + 2w_x^1 = 0 \quad \text{and} \quad w_t^2 - w_x^2 = 0.$$

The initial conditions for  $w^1$  and  $w^2$  are

$$w^1(0, x) = x \quad \text{and} \quad w^2(0, x) = -x.$$

In the characteristic variables the boundary conditions are

$$w^1(t, 0) = -w^2(t, 0) + 2t \quad \text{and} \quad w^2(t, 1) = w^1(t, 1). \quad (1.2.7)$$

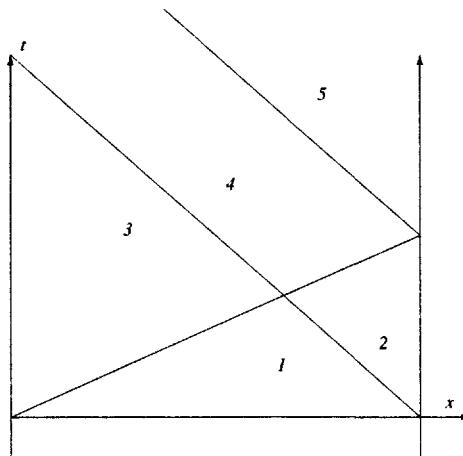


Figure 1.4. Characteristics for Example 1.2.1.

We now use this data to determine the solution in the interior. In region 1 of Figure 1.4, the solution is determined by the initial conditions. Thus, using the characteristics and the initial data we obtain

$$\begin{aligned} w^1(t, x) &= w^1(0, x - 2t) = x - 2t, \\ w^2(t, x) &= w^2(0, x + t) = -(x + t) = -x - t. \end{aligned}$$

Using the inverse relations, we have

$$\begin{aligned} u^1(t, x) &= \frac{w^1(t, x) + w^2(t, x)}{2} = -\frac{3}{2}t, \\ u^2(t, x) &= \frac{w^1(t, x) - w^2(t, x)}{2} = x - \frac{1}{2}t. \end{aligned}$$

In region 2, the values of  $w^1$  are determined since the characteristics for  $w^1$  enter from region 1. Thus, the formula for  $w^1$  is the same for regions 1 and 2:

$$w^1(t, x) = x - 2t.$$

The values of  $w^2$  in region 2 are determined by the values from the characteristics emanating from the boundary at  $x = 1$ . The boundary condition there is (from (1.2.7))

$$w^2(t, 1) = -w^1(t, 1) = -(1 - 2t) = -1 + 2t,$$

and extending to the interior we have

$$w^2(t, x) = w^2(x + t - 1, 1) = -1 + 2(x + t - 1) = -3 + 2x + 2t.$$

Thus in region 2

$$\begin{aligned} u^1(t, x) &= \frac{w^1(t, x) + w^2(t, x)}{2} = \frac{(x - 2t) + (-3 + 2x + 2t)}{2} = -\frac{3}{2} + \frac{3}{2}x, \\ u^2(t, x) &= \frac{w^1(t, x) - w^2(t, x)}{2} = \frac{(x - 2t) - (-3 + 2x + 2t)}{2} = \frac{3}{2} - \frac{1}{2}x - 2t. \end{aligned}$$

Notice that both  $u^1$  and  $u^2$  are continuous along the line  $x + t = 1$  between regions 1 and 2.

In region 3, the values of  $w^2$  are the same as in region 1:

$$w^2(t, x) = -x - t.$$

The boundary condition at  $x = 0$  from (1.2.7) is

$$w^1(t, 0) = -w^2(t, 0) + 2t.$$

Thus at  $x = 0$ ,

$$w^1(t, 0) = -w^2(t, 0) + 2t = 3t.$$

Extending this into the interior along the characteristics gives

$$w^1(t, x) = w^1\left(t - \frac{1}{2}x, 0\right) = 3\left(t - \frac{1}{2}x\right) = -\frac{3}{2}x + 3t.$$

Thus, from the inverse equations, in region 3

$$u^1(t, x) = \frac{-\frac{3}{2}x + 3t + (-x - t)}{2} = -\frac{5}{4}x + t,$$

$$u^2(t, x) = \frac{-\frac{3}{2}x + 3t - (-x - t)}{2} = -\frac{1}{4}x + 2t.$$

In region 4, the values of  $w^1$  are determined by the characteristics from region 3, and the values of  $w^2$  are determined by the characteristics from region 2. Thus

$$w^1(t, x) = -\frac{3}{2}x + 3t,$$

$$w^2(t, x) = -3 + 2x + 2t,$$

and so

$$u^1(t, x) = -\frac{3}{2} + \frac{1}{4}x + \frac{5}{2}t,$$

$$u^2(t, x) = \frac{3}{2} - \frac{7}{4}x + \frac{1}{2}t.$$

Similar analysis can determine the solution in all the regions for all  $t$ .  $\square$

## Periodic Problems

Besides the initial value problem on the whole real line  $R$ , we can also consider periodic problems on an interval. For example, consider the one-way wave equation (1.1.1) on the interval  $[0, 1]$ , where the solution satisfies

$$u(t, 0) = u(t, 1) \tag{1.2.8}$$

for all nonnegative values of  $t$ . Condition (1.2.8) is sometimes called the periodic boundary condition, but strictly speaking it is not a boundary condition, since for periodic problems there are no boundaries.

A periodic problem for a function  $u(t, x)$  with  $x$  in the interval  $[0, 1]$  is equivalent to one on the real line satisfying  $u(t, x) = u(t, x + \ell)$  for every integer  $\ell$ . Thus, the function  $u(t, x)$  is determined by its values of  $x$  in any interval of length 1, such as  $[-\frac{1}{2}, \frac{1}{2}]$ .

A periodic problem may also be regarded as being defined on a circle that is coordinatized by an interval with endpoints being identified. In this view, there is a boundary in the coordinate system but not in the problem itself.

## Exercises

- 1.2.1.** Consider system (1.2.2) on the interval  $[0, 1]$ , with  $a$  equal to 0 and  $b$  equal to 1 and with the boundary conditions  $u^1$  equal to 0 at the left and  $u^1$  equal to 1 at the right boundary. Show that if the initial data are given by  $u^1(0, x) = x$  and  $u^2(0, x) = 1$ , then the solution is  $u^1(t, x) = x$  and  $u^2(t, x) = 1 - t$  for all  $(t, x)$  with  $0 \leq x \leq 1$  and  $0 \leq t$ .

- 1.2.2.** Consider system (1.2.2) on the interval  $[0, 1]$ , with  $a$  equal to 0 and  $b$  equal to 1 and with the boundary conditions  $u^1$  equal to 0 at the left and  $u^1$  equal to  $1 + t$  at the right boundary. Show that if the initial data are given by  $u^1(0, x) = x$  and  $u^2(0, x) = 1$ , then for  $0 \leq x + t \leq 3$  the solution is given by

$$\begin{pmatrix} u^1(t, x) \\ u^2(t, x) \end{pmatrix} = \begin{cases} \begin{pmatrix} x \\ 1-t \end{pmatrix} & \text{if } 0 \leq t < 1-x, \\ \begin{pmatrix} 2x+t-1 \\ 2-x-2t \end{pmatrix} & \text{if } 1-x \leq t < 1+x, \\ \begin{pmatrix} 3x \\ 3(1-t) \end{pmatrix} & \text{if } 1+x \leq t < 3-x. \end{cases}$$

- 1.2.3.** Consider system (1.2.2) on the interval  $[0, 1]$ , with  $a$  equal to 0 and  $b$  equal to 1 and with the boundary conditions  $u^1$  equal to 0 at both the left and the right boundaries. Show that if the initial data are given by  $u^1(0, x) = x$  and  $u^2(0, x) = 1$ , then for  $0 \leq t \leq 1$  the solution is given by

$$\begin{pmatrix} u^1(t, x) \\ u^2(t, x) \end{pmatrix} = \begin{cases} \begin{pmatrix} x \\ 1-t \end{pmatrix} & \text{if } 0 \leq x < 1-t, \\ \begin{pmatrix} x-1 \\ 2-t \end{pmatrix} & \text{if } 1-t \leq x < 1. \end{cases}$$

- 1.2.4.** Show that the initial-boundary value problem of Exercise 1.2.3 has the solution for  $1 \leq t \leq 2$  given by

$$\begin{pmatrix} u^1(t, x) \\ u^2(t, x) \end{pmatrix} = \begin{cases} \begin{pmatrix} x \\ 3-t \end{pmatrix} & \text{if } 0 \leq x < t-1, \\ \begin{pmatrix} x-1 \\ 2-t \end{pmatrix} & \text{if } t-1 < x < 1. \end{cases}$$

- 1.2.5.** Consider system (1.2.2) on the interval  $[0, 1]$ , with  $a$  equal to 1 and  $b$  equal to 2 and with the boundary conditions  $u^1$  equal to 0 at the left and  $u^1$  equal to 1 at the right boundary. Show that if the initial data are given by  $u^1(0, x) = x$  and

$u^2(0, x) = 1$ , then for  $0 \leq t \leq 1 + \frac{1}{3}x$  the solution is given by

$$\begin{pmatrix} u^1(t, x) \\ u^2(t, x) \end{pmatrix} = \begin{cases} \begin{pmatrix} x - t \\ 1 - 2t \end{pmatrix} & \text{if } 0 \leq t \leq \min(\frac{1}{3}x, 1 - x), \\ \begin{pmatrix} \frac{2}{3}x \\ 1 - \frac{1}{3}x - t \end{pmatrix} & \text{if } \frac{1}{3}x \leq t \leq 1 - x, \\ \begin{pmatrix} 2x - 1 \\ 2 - x - 3t \end{pmatrix} & \text{if } 1 - x \leq t \leq \frac{1}{3}x, \\ \begin{pmatrix} t + \frac{5}{3}x - 1 \\ 2 - 2t - \frac{4}{3}x \end{pmatrix} & \text{if } \max(\frac{1}{3}x, 1 - x) \leq t \leq \min(\frac{4}{3} - x, 1 + \frac{1}{3}x), \\ \begin{pmatrix} \frac{2}{3}x + \frac{1}{3} \\ \frac{2}{3} - \frac{1}{3}x - t \end{pmatrix} & \text{if } \frac{4}{3} - x \leq t \leq 1 + \frac{1}{3}x. \end{cases}$$

### 1.3 Introduction to Finite Difference Schemes

We begin our discussion of finite difference schemes by defining a grid of points in the  $(t, x)$  plane. Let  $h$  and  $k$  be positive numbers; then the grid will be the points  $(t_n, x_m) = (nk, mh)$  for arbitrary integers  $n$  and  $m$  as displayed in Figure 1.5. For a function  $v$  defined on the grid we write  $v_m^n$  for the value of  $v$  at the grid point  $(t_n, x_m)$ . We also use the notation  $u_m^n$  for  $u(t_n, x_m)$  when  $u$  is defined for continuously varying  $(t, x)$ . The set of points  $(t_n, x_m)$  for a fixed value of  $n$  is called *grid level*  $n$ . We are interested in grids with small values of  $h$  and  $k$ . In many texts the quantities that we call  $h$  and  $k$  are represented by  $\Delta x$  and  $\Delta t$ , respectively.

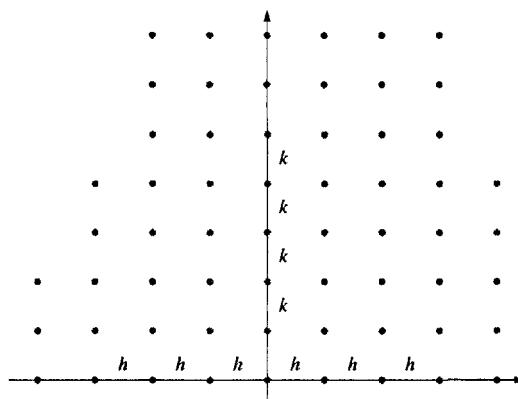


Figure 1.5. The finite difference grid.

The basic idea of finite difference schemes is to replace derivatives by finite differences. This can be done in many ways; as two examples we have

$$\begin{aligned}\frac{\partial u}{\partial t}(t_n, x_m) &\simeq \frac{u(t_n + k, x_m) - u(t_n, x_m)}{k} \\ &\simeq \frac{u(t_n + k, x_m) - u(t_n - k, x_m)}{2k}.\end{aligned}$$

That these are valid approximations is seen from the formulas

$$\begin{aligned}\frac{\partial u}{\partial t}(t, x) &= \lim_{\varepsilon \rightarrow 0} \frac{u(t + \varepsilon, x) - u(t, x)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{u(t + \varepsilon, x) - u(t - \varepsilon, x)}{2\varepsilon},\end{aligned}$$

relating the derivative to the values of  $u$ . Similar formulas approximate derivatives with respect to  $x$ .

Using these approximations we obtain the following five finite difference schemes for equation (1.1.1). Many other schemes are presented later.

$$\frac{v_m^{n+1} - v_m^n}{k} + a \frac{v_{m+1}^n - v_m^n}{h} = 0, \quad (1.3.1)$$

$$\frac{v_m^{n+1} - v_m^n}{k} + a \frac{v_m^n - v_{m-1}^n}{h} = 0, \quad (1.3.2)$$

$$\frac{v_m^{n+1} - v_m^n}{k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0, \quad (1.3.3)$$

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0, \quad (1.3.4)$$

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0. \quad (1.3.5)$$

We refer to scheme (1.3.1) as the forward-time forward-space scheme because forward difference approximations are used for both the time and space derivatives. Similarly, (1.3.2) and (1.3.3) are referred to as the forward-time backward-space scheme and forward-time central-space scheme, respectively. The scheme (1.3.4) is called the leapfrog scheme and (1.3.5) is called the Lax–Friedrichs scheme.

The method of deriving these five schemes is very simple. This is one of the significant features of the general method of finite differences, namely, that it is very easy to derive finite difference schemes for partial differential equations. However, the analysis of

finite difference schemes to determine if they are useful approximations to the differential equation requires some powerful mathematical tools. Moreover, to develop very efficient and accurate schemes requires more work than went into obtaining the schemes (1.3.1)–(1.3.5). Nonetheless, the finite difference method is notable for the great variety of schemes that can be used to approximate a given partial differential equation.

Given this short list of schemes, we are naturally led to the question of which of them are useful and which are not, as indeed some are not. This is a basic question, and we spend some time and care in answering it. In fact, the question can be answered on several levels. We first answer it on the most primitive level, determining which schemes have solutions that approximate solutions of the differential equation at all. Later, we determine which schemes are more accurate than others and also investigate the efficiency of the various schemes.

Each of the schemes (1.3.1)–(1.3.5) can be written expressing  $v_m^{n+1}$  as a linear combination of values of  $v$  at levels  $n$  and  $n - 1$ . For example, scheme (1.3.1) can be written as

$$v_m^{n+1} = (1 + a\lambda) v_m^n - a\lambda v_{m+1}^n,$$

where  $\lambda = k/h$ . The quantity  $\lambda$  will appear often in the study of schemes for hyperbolic equations and will always be equal to  $k/h$ . Those schemes that involve  $v$  at only two levels, e.g.,  $n + 1$  and  $n$ , are called one-step schemes. Of the schemes just listed all except the leapfrog scheme (1.3.4) are one-step schemes. Given the initial data  $v_m^0$ , a one-step scheme can be used to evaluate  $v_m^n$  for all positive values of  $n$ .

The leapfrog scheme (1.3.4) is an example of a multistep scheme. For a multistep scheme it is not sufficient to specify the values of  $v_m^0$  in order to determine  $v_m^n$  for all positive values of  $n$ . To specify completely the means of computing a solution to a multistep scheme, either we must specify  $v$  on enough time levels so that the scheme can be employed or we must specify a procedure for computing the values of  $v$  on these initial time levels. For example, to use the leapfrog scheme we could specify the values of  $v_m^0$  and  $v_m^1$  for all  $m$ , or we could specify that scheme (1.3.1) would be used to compute the values of  $v_m^1$  from the values  $v_m^0$ . In either case the leapfrog scheme (1.3.4) would be used to compute  $v_m^n$  for  $n$  greater than 1.

When we refer to the leapfrog scheme we do not always distinguish between these two ways of initializing the computation. As we show in Section 4.1, many of the properties of the leapfrog scheme are independent of the method used to initialize the solution. Since the usual practice is to use a one-step scheme to initialize the first time level, we usually assume that the initialization is done in this way. This is illustrated in Example 1.3.2. The subject of how to initialize multistep schemes in general is considered in more detail in Section 4.1.

**Example 1.3.1.** Before we proceed with the analysis of finite difference schemes, we present the results of some computations using two of the schemes just presented. We use the initial-boundary value problem

$$u_t + u_x = 0 \quad \text{on} \quad -2 \leq x \leq 3, 0 \leq t$$

with initial data

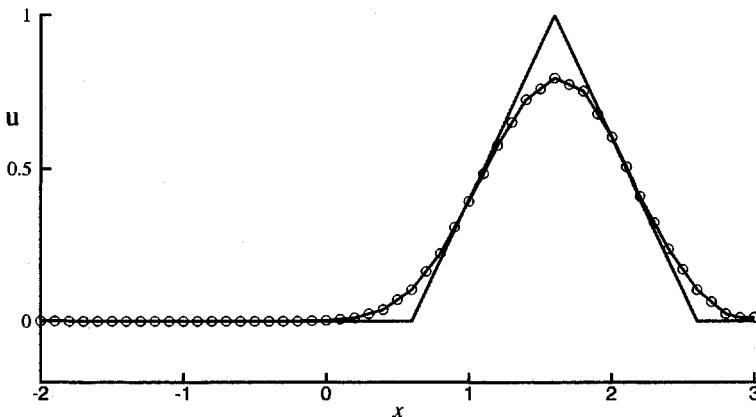
$$u_0(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

On the boundary at  $x$  equal to  $-2$ , we specify that  $u$  is zero.

The first computation uses the Lax–Friedrichs scheme (1.3.5) with  $\lambda = 0.8$  and  $h$  equal to  $0.1$ . At the right-hand boundary we use the condition  $v_M^{n+1} = v_{M-1}^{n+1}$ , where  $x_M = 3$ . For our initial data we take  $v_m^0 = u_0(x_m)$ . The computation proceeds using the formula

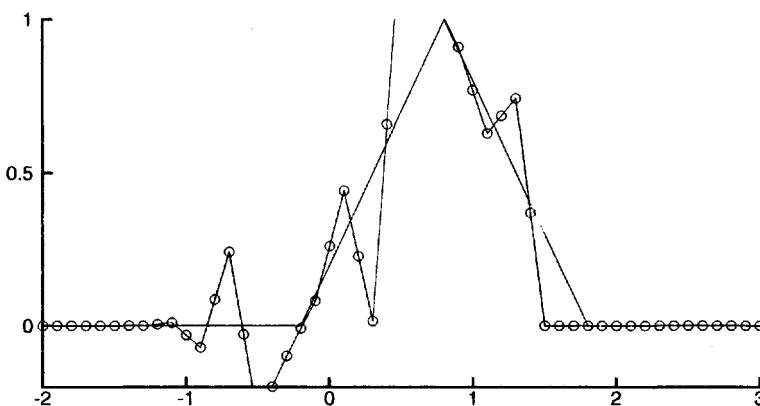
$$v_m^{n+1} = \frac{1}{2}(v_{m+1}^n + v_{m-1}^n) - \frac{1}{2}\lambda(v_{m+1}^n - v_{m-1}^n)$$

to find the values of  $v_m^{n+1}$  for all values except those at the endpoints of the interval. A graph of the solution at  $t = 1.6$  is shown in Figure 1.6. In the figure the exact solution to the differential equation is given by the solid line and the solution of the scheme is shown as the curve with the circles. The figure shows that the finite difference scheme computes a reasonable solution, except that the computed solution does not maintain the sharp corners of the exact solution. A smaller value of  $h$ , with the same value of  $\lambda$ , improves the shape of the computed solution.

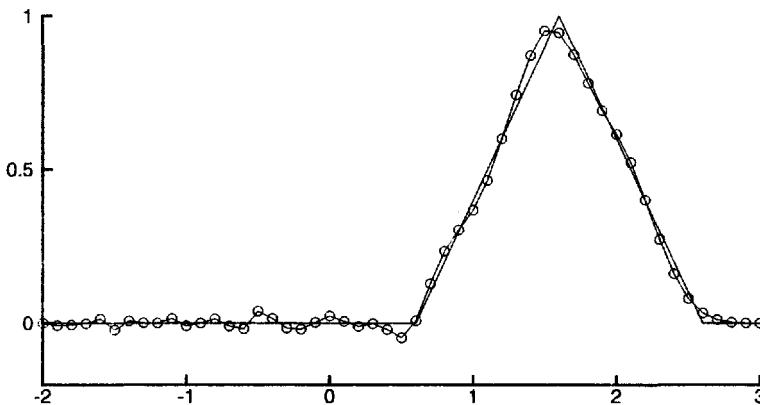


**Figure 1.6.** A solution of the Lax–Friedrichs scheme,  $\lambda = 0.8$ .

A similar calculation but using  $\lambda = 1.6$  is shown in Figure 1.7 at  $t = 0.8$ . The figure shows that for this case the computed solution is not well behaved. As the computation proceeds for larger values of  $t$ , the behavior becomes worse. Also, if the grid spacing is decreased, with  $\lambda$  fixed at 1.6, the behavior does not get better and in fact becomes worse. The explanation for this behavior is given in the next chapter.  $\square$



**Figure 1.7.** A solution of the Lax-Friedrichs scheme,  $\lambda = 1.6$ .



**Figure 1.8.** A solution computed with leapfrog scheme,  $\lambda = 0.8$ .

**Example 1.3.2.** The leapfrog scheme (1.3.4) with  $\lambda = 0.8$  gives much better results than does the Lax-Friedrichs scheme for the same initial-boundary value problem in Example 1.3.1. The computational results are displayed in Figure 1.8. Notice that the resolution of the peak in the solution is much better in Figure 1.8 than in Figure 1.6. The leapfrog scheme has a less smooth solution than does the Lax-Friedrichs; however the small oscillations do not detract significantly from the accuracy. In Section 5.1 we discuss methods of removing these oscillations. At the right-hand boundary,  $v_M^{n+1}$  is computed as it was for the Lax-Friedrichs scheme.

As discussed before, the leapfrog scheme requires that another scheme be used to calculate the values at the time level with  $n$  equal to 1. For the calculations shown in Figure 1.8, the forward-time central-space scheme (1.3.3) was used.  $\square$

## Computer Implementation of Finite Difference Schemes

To implement any of the finite difference schemes (1.3.1)–(1.3.5) or similar finite difference schemes in a computer program, values of the solution  $v_m^n$  should not be stored beyond the time steps in which they are needed. A simple way to do this is to use two one-dimensional arrays vold and vnew, each of which is indexed by the spatial grid indices. The values of vnew( $m$ ) and vold( $m$ ) correspond to  $v_m^{n+1}$  and  $v_m^n$ , respectively. For each value of  $n$ , vnew, corresponding to  $v^{n+1}$ , is computed using vold, corresponding to  $v^n$ . After vnew has been computed for all  $m$ , then vold must be reset to vnew, and the time step is incremented to the next value. For the leapfrog scheme the array vnew can be used to store both  $v^{n-1}$  and  $v^{n+1}$ .

Any values of the solution that are to be saved or plotted may be written to a file as they are computed. It is not advisable to save past values beyond the time they are needed in the computation.

A more convenient way to store the solution for schemes (1.3.1)–(1.3.5) is to use a two-dimensional array, such as  $v(nmod, m)$ , where nmod is equal to  $n$  modulo 2. The values of  $v(0, \cdot)$  are used to compute the values of  $v(1, \cdot)$ , which are used to compute  $v(0, \cdot)$ , and so on. This method avoids the need to reset arrays such as vold, which was set equal to vnew in the method described previously.

Here is a sample of pseudocode for the Lax–Friedrichs scheme.

```
# Supply initial data
now = 0
new = 1
time = 0
loop on m from 0 to M ! Set initial data
    v(now,m) = u0(x(m))
end of loop on m
loop for time < TIME_MAX
    time = time + k ! This is the time being computed.
    n_time = n_time + 1
    v(new,0) = beta(time) ! Set the boundary value.
    loop on m from 1 to M-1
        v(new,m) = (v(now, m-1) + v(now,m+1))/2
        - a*lambda*( v(now,m+1) - v(now,m-1))/2
    end of loop on m
    v(new,M) = v(new,M-1) ! Apply boundary condition.

    now = new ! Reset for the next time step.
    new = mod(n_time, 2)
end of loop on time
```

For periodic problems on the interval  $[0, 1]$  with  $h = 1/M$  and grid points  $x_m = mh$ , it is useful to store values at  $x_0$  and at  $x_M$ , even though these values represent the same point in the periodic problem.

## Exercises

- 1.3.1.** For values of  $x$  in the interval  $[-1, 3]$  and  $t$  in  $[0, 2.4]$ , solve the one-way wave equation

$$u_t + u_x = 0,$$

with the initial data

$$u(0, x) = \begin{cases} \cos^2 \pi x & \text{if } |x| \leq \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and the boundary data  $u(t, -1) = 0$ .

Use the following four schemes for  $h = 1/10$ ,  $1/20$ , and  $1/40$ .

- (a) Forward-time backward-space scheme (1.3.2) with  $\lambda = 0.8$ .
- (b) Forward-time central-space scheme (1.3.3) with  $\lambda = 0.8$ .
- (c) Lax–Friedrichs scheme (1.3.5) with  $\lambda = 0.8$  and  $1.6$ .
- (d) Leapfrog scheme (1.3.4) with  $\lambda = 0.8$ .

For schemes (b), (c), and (d), at the right boundary use the condition  $v_M^{n+1} = v_{M-1}^{n+1}$ , where  $x_M = 3$ . For scheme (d) use scheme (b) to compute the solution at  $n = 1$ .

For each scheme determine whether the scheme is a useful or useless scheme. For the purposes of this exercise *only*, a scheme will be useless if  $|v_m^n|$  is greater than 5 for any value of  $m$  and  $n$ . It will be regarded as a useful scheme if the solution looks like a reasonable approximation to the solution of the differential equations. Graph or plot several solutions at the last time they were computed. What do you notice about the “blow-up time” for the useless schemes as the mesh size decreases? Is there a pattern to these solutions? For the useful cases, how does the error decrease as the mesh decreases; i.e., as  $h$  decreases by one-half, by how much does the error decrease?

- 1.3.2.** Solve the system

$$\begin{aligned} u_t + \frac{1}{3}(t-2)u_x + \frac{2}{3}(t+1)w_x + \frac{1}{3}u &= 0, \\ w_t + \frac{1}{3}(t+1)u_x + \frac{1}{3}(2t-1)w_x - \frac{1}{3}w &= 0 \end{aligned}$$

by the Lax–Friedrichs scheme: i.e., each time derivative is approximated as it is for the scalar equation and the spatial derivatives are approximated by central differences. The initial values are

$$u(0, x) = \max(0, 1 - |x|),$$

$$w(0, x) = \max(0, 1 - 2|x|).$$

Consider values of  $x$  in  $[-3, 3]$  and  $t$  in  $[0, 2]$ . Take  $h$  equal to  $1/20$  and  $\lambda$  equal to  $1/2$ . At each boundary set  $u = 0$ , and set  $w$  equal to the newly computed value one grid point in from the boundary. Describe the solution behavior for  $t$  in the range  $[1.5, 2]$ . You may find it convenient to plot the solution. Solve the system in the form given; do not attempt to diagonalize it.

### 1.3.3. Solve the system

$$u_t + \frac{1}{3}(t-2)u_x + \frac{2}{3}(t+1)w_x = 0,$$

$$w_t + \frac{1}{3}(t+1)u_x + \frac{1}{3}(2t-1)w_x = 0$$

by the Lax–Friedrichs scheme as in Exercise 1.3.2, using the same initial data. An examination of the computed solution should show how to obtain the analytical solution to this problem.

- 1.3.4.** Numerically solve the equation in Exercise 1.1.5 using the initial data and intervals of Exercise 1.3.1. Use the leapfrog scheme with  $\lambda = 0.5$  and  $h = 1/10, 1/20$ , and  $1/40$ . Use the forward-time central-space scheme to compute the first time step. The boundary condition at  $x = -1$  is  $u(t, -1) = 0$ .

## 1.4 Convergence and Consistency

The most basic property that a scheme must have in order to be useful is that its solutions approximate the solution of the corresponding partial differential equation and that the approximation improves as the grid spacings,  $h$  and  $k$ , tend to zero. We call such a scheme a *convergent scheme*, but before formally defining this concept it is appropriate to extend our discussion to a wider class of partial differential equations than the hyperbolic equations. We consider linear partial differential equations of the form

$$P(\partial_t, \partial_x)u = f(t, x),$$

which are of first order in the derivative with respect to  $t$ . We also assume for such equations or systems of equations that the specification of initial data,  $u(0, x)$ , completely determines a unique solution. More is said about this in Chapter 9. The real variable  $x$  ranges over the whole real line or an interval. Examples of equations that are first order in time are the one-way wave equation (1.1.1) and the following three equations:

$$\begin{aligned} u_t - bu_{xx} + au_x &= 0, \\ u_t - cu_{txx} + bu_{xxxx} &= 0, \\ u_t + cu_{tx} + au_x &= 0. \end{aligned} \tag{1.4.1}$$

**Definition 1.4.1.** A one-step finite difference scheme approximating a partial differential equation is a *convergent scheme* if for any solution to the partial differential equation,

$u(t, x)$ , and solutions to the finite difference scheme,  $v_m^n$ , such that  $v_m^0$  converges to  $u_0(x)$  as  $mh$  converges to  $x$ , then  $v_m^n$  converges to  $u(t, x)$  as  $(nk, mh)$  converges to  $(t, x)$  as  $h, k$  converge to 0.

This definition is not complete until we clarify the nature of the convergence of  $v_m^n$ , defined on the grid, to  $u(t, x)$  defined for continuously varying  $(t, x)$ . We discuss this convergence completely in Chapter 10. For multistep schemes the definition assumes that some initializing procedure is used to compute the first several time levels necessary to employ the multistep scheme. For the case that the data are specified on these first time levels, the definition is altered to require  $v_m^j$  for  $0 \leq j \leq J$  to converge to  $u_0(x_m)$ .

As illustrated by Figures 1.6 and 1.8, the Lax–Friedrichs scheme and the leapfrog scheme with  $\lambda$  equal to 0.8 are convergent schemes. These figures show that the solution of the difference scheme is a reasonable approximation to the solution of the differential equation. As  $h$  and  $k$  are decreased, the solutions of the schemes become better approximations. The Lax–Friedrichs scheme with  $\lambda = 1.6$  is not convergent. As  $h$  and  $k$  decrease, with  $\lambda$  equal to 1.6, the solution of the scheme does not approach the solution of the differential equation in any sense. As can be seen in Figure 1.7, the behavior of a nonconvergent scheme can be quite poor.

The convergence of the Lax–Friedrichs scheme is also illustrated in Figure 1.9, which shows a portion of Figure 1.6 along with the results for  $h = 1/20$  and  $h = 1/40$ . The three plots show that as  $h$  gets smaller, with  $\lambda = 0.8$ , the solution of the finite difference scheme approaches the solution of the differential equation.

Proving that a given scheme is convergent is not easy in general, if attempted in a direct manner. However, there are two related concepts that are easy to check: consistency and stability. First, we define consistency.

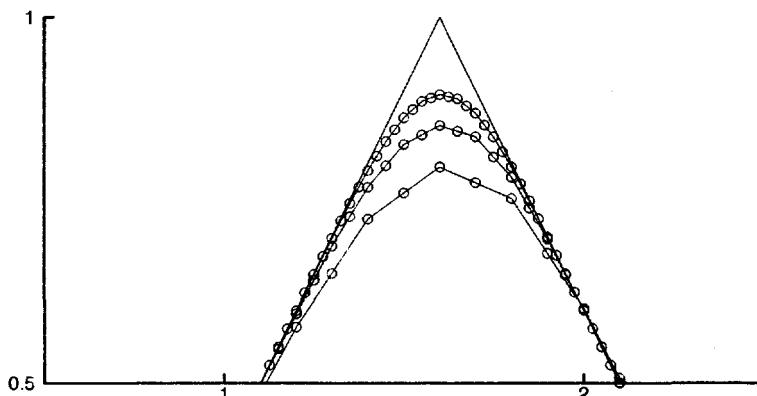


Figure 1.9. Lax–Friedrichs scheme convergence.

**Definition 1.4.2.** Given a partial differential equation,  $Pu = f$ , and a finite difference scheme,  $P_{k,h}v = f$ , we say that the finite difference scheme is consistent with the partial differential equation if for any smooth function  $\phi(t, x)$

$$P\phi - P_{k,h}\phi \rightarrow 0 \quad \text{as } k, h \rightarrow 0,$$

the convergence being pointwise convergence at each point  $(t, x)$ .

For some schemes we may have to restrict the manner in which  $k$  and  $h$  tend to zero in order for it to be consistent (see Example 1.4.2). When we refer to a smooth function we mean one that is sufficiently differentiable for the context.

Also, note that the difference operator  $P_{k,h}$  when applied to a function of  $(t, x)$  does not need to be restricted to grid points. Thus, a forward difference in  $x$  applied at a point  $(t, x)$  is

$$\frac{\phi(t, x + h) - \phi(t, x)}{h}.$$

We demonstrate the use of this definition and the notation by presenting two examples, showing that two of the schemes in the above list are consistent with the equation (1.1.1).

**Example 1.4.1. The Forward-Time Forward-Space Scheme.** For the one-way wave equation (1.1.1), the operator  $P$  is  $\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$  so that

$$P\phi = \phi_t + a\phi_x.$$

For the forward-time forward-space scheme (1.3.1), the difference operator  $P_{k,h}$  is given by

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^n - \phi_m^n}{h},$$

where

$$\phi_m^n = \phi(nk, mh).$$

We begin with the Taylor series of the function  $\phi$  in  $t$  and  $x$  about  $(t_n, x_m)$ . We have that

$$\phi_m^{n+1} = \phi_m^n + k\phi_t + \frac{1}{2}k^2\phi_{tt} + O(k^3),$$

$$\phi_{m+1}^n = \phi_m^n + h\phi_x + \frac{1}{2}h^2\phi_{xx} + O(h^3),$$

where the derivatives on the right-hand side are all evaluated at  $(t_n, x_m)$ , and so

$$P_{k,h}\phi = \phi_t + a\phi_x + \frac{1}{2}k\phi_{tt} + \frac{1}{2}ah\phi_{xx} + O(k^2) + O(h^2).$$

Thus

$$P\phi - P_{k,h}\phi = -\frac{1}{2}k\phi_{tt} - \frac{1}{2}ah\phi_{xx} + O(k^2) + O(h^2)$$

$$\rightarrow 0 \quad \text{as } (k, h) \rightarrow 0.$$

Therefore, this scheme is consistent.  $\square$

When analyzing consistency it is convenient to use the “big oh” and “little oh” notation, as we have done in the preceding example. In general, if  $F$  and  $G$  are functions of some parameter  $\alpha$ , we write

$$F = O(G) \quad \text{as} \quad \alpha \rightarrow 0,$$

if

$$\left| \frac{F}{G} \right| \leq K$$

for some constant  $K$  and all  $\alpha$  sufficiently small. We write

$$F = o(G) \quad \text{as} \quad \alpha \rightarrow 0,$$

if  $F/G$  converges to zero as  $\alpha$  tends to zero. In particular, a quantity is  $O(h^r)$  if it is bounded by a constant multiple of  $h^r$  for small  $h$ . A quantity is  $o(1)$  if it converges to zero at an unspecified rate.

**Example 1.4.2. The Lax–Friedrichs Scheme.** For the Lax–Friedrichs scheme the difference operator is given by

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \frac{1}{2}(\phi_{m+1}^n + \phi_{m-1}^n)}{k} + a \frac{\phi_{m+1}^n - \phi_{m-1}^n}{2h}.$$

We use the Taylor series

$$\phi_{m\pm 1}^n = \phi_m^n \pm h\phi_x + \frac{1}{2}h^2\phi_{xx} \pm \frac{1}{6}h^3\phi_{xxx} + O(h^4),$$

where, as before, the derivatives are evaluated at  $(t_n, x_m)$  and we have

$$\frac{1}{2}(\phi_{m+1}^n + \phi_{m-1}^n) = \phi_m^n + \frac{1}{2}h^2\phi_{xx} + O(h^4)$$

and

$$\frac{\phi_{m+1}^n - \phi_{m-1}^n}{2h} = \phi_x + \frac{1}{6}h^2\phi_{xxx} + O(h^4).$$

Substituting these expressions in the scheme, we obtain

$$\begin{aligned} P_{k,h}\phi &= \phi_t + a\phi_x + \frac{1}{2}k\phi_{tt} - \frac{1}{2}k^{-1}h^2\phi_{xx} \\ &\quad + \frac{1}{6}ah^2\phi_{xxx} + O\left(h^4 + k^{-1}h^4 + k^2\right). \end{aligned}$$

So  $P_{k,h}\phi - P\phi \rightarrow 0$  as  $h, k \rightarrow 0$ ; i.e., it is consistent, as long as  $k^{-1}h^2$  also tends to 0.  $\square$

Consistency implies that the solution of the partial differential equation, if it is smooth, is an approximate solution of the finite difference scheme. Similarly, convergence means that a solution of the finite difference scheme approximates a solution of the partial differential equation. It is natural to consider whether consistency is sufficient for a scheme to be convergent. Consistency is certainly necessary for convergence, but as the following example shows, a scheme may be consistent but not convergent.

**Example 1.4.3.** Consider the partial differential equation  $u_t + u_x = 0$  with the forward-time forward-space scheme (1.3.1):

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{v_{m+1}^n - v_m^n}{h} = 0.$$

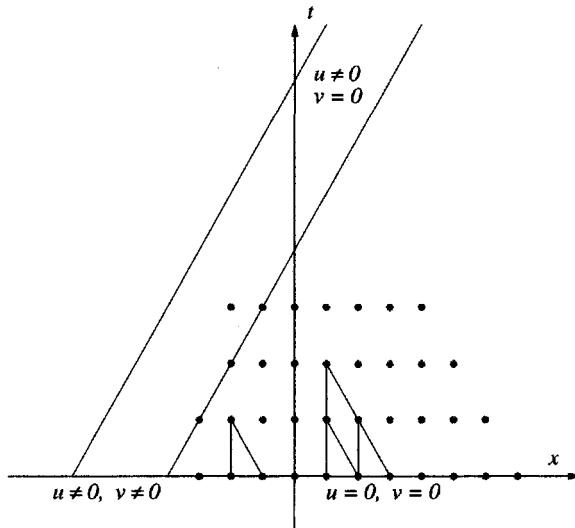
The scheme may be rewritten as

$$\begin{aligned} v_m^{n+1} &= v_m^n - \frac{k}{h}(v_{m+1}^n - v_m^n) \\ &= (1 + \lambda)v_m^n - \lambda v_{m+1}^n, \end{aligned} \tag{1.4.2}$$

where we have set  $\lambda = k/h$  as usual. In Example 1.4.1 this scheme was shown to be consistent. As initial conditions for the differential equation we take

$$u_0(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

The solution of the partial differential equation is a shift of  $u_0$  to the right by  $t$ . In particular, for  $t$  greater than 0, there are positive values of  $x$  for which  $u(t, x)$  is nonzero. This is illustrated in Figure 1.10.



**Figure 1.10.** Consistency does not imply convergence.

For the difference scheme take the initial data

$$v_m^0 = \begin{cases} 1 & \text{if } -1 \leq mh \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

As equation (1.4.2) shows, the solution of the difference scheme at  $(t_n, x_m)$  depends only on  $x_{m'}$  for  $m' \geq m$  at previous times. Thus we conclude that  $v_m^n$  is always 0 for points  $x_m$  to the right of 0, that is,

$$v_m^n = 0 \quad \text{for } m > 0, n \geq 0.$$

Therefore,  $v_m^n$  cannot converge to  $u(t, x)$ , since for positive  $t$  and  $x$ , the function  $u$  is not identically zero, yet  $v_m^n$  is zero.

Notice that we conclude that the scheme is nonconvergent without specifying the type of convergence, but clearly, a sequence of functions that are all zero—i.e., the  $v_m^n$  for  $m > 0$ —cannot converge, under any reasonable definition of convergence, to the nonzero function  $u$ .  $\square$

## Exercises

- 1.4.1.** Show that the forward-time central-space scheme (1.3.3) is consistent with equation (1.1.1).
- 1.4.2.** Show that the leapfrog scheme (1.3.4) is consistent with the one-way wave equation (1.1.1).
- 1.4.3.** Show that the following scheme is consistent with the one-way wave equation (1.1.5):

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{a}{2} \left( \frac{v_{m+1}^{n+1} - v_m^{n+1}}{h} + \frac{v_m^n - v_{m-1}^n}{h} \right) = f_m^n. \quad (1.4.3)$$

- 1.4.4.** Show that the following scheme is consistent with the equation  $u_t + cu_{tx} + au_x = f$ :

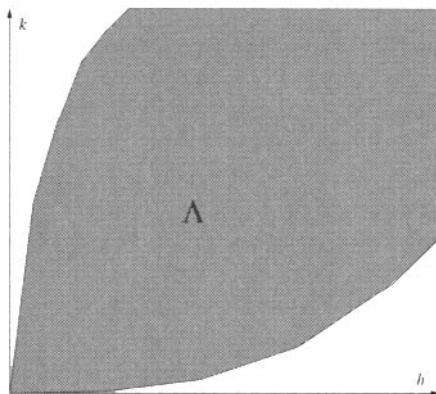
$$\frac{v_m^{n+1} - v_m^n}{k} + c \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} - v_{m+1}^n + v_{m-1}^n}{2kh} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} = f_m^n.$$

- 1.4.5.** Interpret the results of Exercise 1.3.1 in light of the definition of convergence. Based on the cases run in that exercise, decide which of the schemes are convergent.

## 1.5 Stability

Example 1.4.3 shows that a scheme must satisfy other conditions besides consistency before we can conclude that it is convergent. The important property that is required is stability. To introduce this concept we note that, if a scheme is convergent, as  $v_m^n$  converges to  $u(t, x)$ , then certainly  $v_m^n$  is bounded in some sense. This is the essence of stability. The following definition of stability is for the homogeneous initial value problem, that is, one in which the right-hand-side function  $f$  is 0.

Before giving the definition of stability we need to define a stability region. For many schemes there are restrictions on the way that  $h$  and  $k$  should be chosen so that the scheme is stable, and therefore useful in computation. A stability region is any bounded nonempty region of the first quadrant of  $R^2$  that has the origin as an accumulation point. That is, a stability region must contain a sequence  $(k_\nu, h_\nu)$  that converges to the origin as  $\nu$  tends to infinity. A common example is a region of the form  $\{(k, h) : 0 < k \leq ch \leq C\}$  for some positive constants  $c$  and  $C$ . An example of a stability region is displayed in Figure 1.11.



**Figure 1.11. Stability region.**

**Definition 1.5.1.** A finite difference scheme  $P_{k,h}v_m^n = 0$  for a first-order equation is stable in a stability region  $\Lambda$  if there is an integer  $J$  such that for any positive time  $T$ , there is a constant  $C_T$  such that

$$h \sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq C_T h \sum_{j=0}^J \sum_{m=-\infty}^{\infty} |v_m^j|^2 \quad (1.5.1)$$

for  $0 \leq nk \leq T$ , with  $(k, h) \in \Lambda$ .

Before proceeding with our discussion of stability, we introduce some notation that will be of use in understanding inequality (1.5.1). We first introduce the notation

$$\|w\|_h = \left( h \sum_{m=-\infty}^{\infty} |w_m|^2 \right)^{1/2} \quad (1.5.2)$$

for any grid function  $w$ . The quantity  $\|w\|_h$  is called the  $L^2$  norm of the grid function  $w$  and is a measure of the size of the solution (see Appendix B for a discussion of function norms). In many problems the  $L^2$  norm is a measure of a physically significant quantity such as the energy of the system. With this notation the inequality (1.5.1) can be written as

$$\|v^n\|_h \leq \left( C_T \sum_{j=0}^J \|v^j\|_h^2 \right)^{1/2},$$

which is equivalent to

$$\|v^n\|_h \leq C_T^* \sum_{j=0}^J \|v^j\|_h \quad (1.5.3)$$

for some constant  $C_T^*$ . Inequalities (1.5.1) and (1.5.3) express the idea that the norm of the solution at any time  $t$ , with  $0 \leq t \leq T$ , is limited in the amount of growth that can occur. The growth is at most a constant multiple of the sum of the norms of the solution on the first  $J + 1$  steps.

We may take  $J$  equal to zero for one-step schemes and also for multistep schemes incorporating an initializing procedure for computing the solution for the first several time steps, as discussed earlier in this section. We include the possibility of  $J$  being positive to include multistep schemes with data specified on the first  $J + 1$  levels. It will be shown that the stability of a multistep scheme is not dependent on the method of initialization.

To demonstrate whether or not the estimate (1.5.1) holds for a particular scheme can be quite formidable unless we use methods from Fourier analysis, which is discussed in the next chapter. In Section 2.2 a relatively simple procedure, von Neumann analysis, is presented for determining the stability of difference schemes.

For certain rather simple schemes we can determine sufficient conditions that ensure that the scheme is stable. This is done by establishing the stability estimate (1.5.1) directly.

**Example 1.5.1.** We will prove a sufficient condition for stability for the forward-time forward-space scheme (1.3.1) by considering schemes of the form

$$v_m^{n+1} = \alpha v_m^n + \beta v_{m+1}^n,$$

of which the forward-time forward-space scheme is a special case. We will show that the scheme is stable if  $|\alpha| + |\beta| \leq 1$ . The analysis is similar for the forward-time backward-space scheme (1.3.2). We have

$$\begin{aligned} \sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 &= \sum_{m=-\infty}^{\infty} |\alpha v_m^n + \beta v_{m+1}^n|^2 \\ &\leq \sum_{m=-\infty}^{\infty} |\alpha|^2 |v_m^n|^2 + 2|\alpha||\beta| |v_m^n| |v_{m+1}^n| + |\beta|^2 |v_{m+1}^n|^2 \\ &\leq \sum_{m=-\infty}^{\infty} |\alpha|^2 |v_m^n|^2 + |\alpha||\beta| (|v_m^n|^2 + |v_{m+1}^n|^2) + |\beta|^2 |v_{m+1}^n|^2, \end{aligned}$$

where we have used the inequality  $2xy \leq x^2 + y^2$ . The sum can be split over the terms with index  $m$  and those with index  $m + 1$  and the index can be shifted so that all terms

have the index  $m$ :

$$\begin{aligned}
 &= \sum_{m=-\infty}^{\infty} |\alpha|^2 |v_m^n|^2 + |\alpha||\beta||v_m^n|^2 + \sum_{m=-\infty}^{\infty} |\alpha||\beta||v_{m+1}^n|^2 + |\beta|^2 |v_{m+1}^n|^2 \\
 &= \sum_{m=-\infty}^{\infty} |\alpha|^2 |v_m^n|^2 + |\alpha||\beta||v_m^n|^2 + \sum_{m=-\infty}^{\infty} |\alpha||\beta||v_m^n|^2 + |\beta|^2 |v_m^n|^2 \\
 &= \sum_{m=-\infty}^{\infty} \left( |\alpha|^2 + 2|\alpha||\beta| + |\beta|^2 \right) |v_m^n|^2 \\
 &= (|\alpha| + |\beta|)^2 \sum_{m=-\infty}^{\infty} |v_m^n|^2.
 \end{aligned}$$

This shows that we have the relation

$$\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \leq (|\alpha| + |\beta|)^2 \sum_{m=-\infty}^{\infty} |v_m^n|^2,$$

and since this applies for all  $n$ , we have that

$$\sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq (|\alpha| + |\beta|)^{2n} \sum_{m=-\infty}^{\infty} |v_m^0|^2.$$

If  $|\alpha| + |\beta|$  is at most 1 in magnitude, then the scheme will be stable. Thus, schemes of the form given above are stable if  $|\alpha| + |\beta| \leq 1$ .

For the forward-time forward-space scheme (1.3.1) the condition  $|\alpha| + |\beta| \leq 1$  is that  $|1 + a\lambda| + |a\lambda|$  is at most 1. Thus we see that this scheme is stable if  $-1 \leq a\lambda \leq 0$ . In Section 2.2 we show that this is also a necessary condition.  $\square$

The concept of stability for finite difference schemes is closely related to the concept of well-posedness for initial value problems for partial differential equations. As before, we restrict our discussion to equations  $Pu = f$  that are of first order with respect to differentiation in time.

**Definition 1.5.2.** *The initial value problem for the first-order partial differential equation  $Pu = 0$  is well-posed if for any time  $T \geq 0$ , there is a constant  $C_T$  such that any solution  $u(t, x)$  satisfies*

$$\int_{-\infty}^{\infty} |u(t, x)|^2 dx \leq C_T \int_{-\infty}^{\infty} |u(0, x)|^2 dx \tag{1.5.4}$$

for  $0 \leq t \leq T$ .

A discussion of the concept of a well-posed initial value problem is given in Chapter 9. It is shown that only well-posed initial value problems can be used to model the evolution of physical processes. The methods of Fourier analysis that are introduced in the next chapter will be useful in the study of well-posed initial value problems.

In Chapter 9 we discuss stability and well-posedness for the inhomogeneous problems,  $P_{k,h}v = f$  and  $Pu = f$ , respectively. As we show, the inhomogeneous equations can be treated using the estimates (1.5.1) and (1.5.4) by use of Duhamel's principle. Thus a scheme is stable for the equation  $P_{k,h}v = f$  if it is stable for the equation  $P_{k,h}v = 0$ .

## The Lax–Richtmyer Equivalence Theorem

The importance of the concepts of consistency and stability is seen in the Lax–Richtmyer equivalence theorem, which is the fundamental theorem in the theory of finite difference schemes for initial value problems.

**Theorem 1.5.1. The Lax–Richtmyer Equivalence Theorem.** *A consistent finite difference scheme for a partial differential equation for which the initial value problem is well-posed is convergent if and only if it is stable.*

A proof of this theorem is given in Chapter 10. The Lax–Richtmyer equivalence theorem is a very useful theorem, since it provides a simple characterization of convergent schemes. As discussed earlier, determining whether a scheme is convergent or nonconvergent can be difficult if we attempt to verify Definition 1.4.1 in a rather direct way. However, the determination of the consistency of a scheme is quite simple, as we have seen, and determining the stability of a scheme is also quite easy, as we show in Section 2.2. Thus the more difficult result—convergence—is replaced by the equivalent and easily verifiable conditions of consistency and stability. It is also significant that the determination of the consistency and stability of schemes involves essentially algebraic manipulations. A computerized symbolic manipulation language can be useful in determining consistency and stability. By contrast, a direct proof of convergence would rely on concepts in analysis. Such a proof would have to begin by considering any solution  $u$  of the differential equation and then it would have to be shown that given any  $\epsilon$ , there exist  $h$  and  $k$  small enough that the solution of the scheme is within  $\epsilon$  of  $u$ . The Lax–Richtmyer theorem allows us to dispense with all this analysis.

The preceding discussion of Theorem 1.5.1 has focused on the half of the theorem that states that consistency and stability imply convergence. The theorem is useful in the other direction also. It states that we should not consider any unstable schemes, since none of these will be convergent. Thus the class of reasonable schemes is precisely delimited as those that are consistent and stable; no other schemes are worthy of consideration.

The Lax–Richtmyer equivalence theorem is an example of the best type of mathematical theorem. It relates an important concept that is difficult to establish directly with other concepts that are relatively easy to verify and establishes this relationship very precisely. Notice that if we had only the half of the theorem that showed that consistency and stability implied convergence, then it would be conceivable that there were unstable schemes that were also convergent. If we had only the other half of the theorem, stating that a consis-

tent convergent scheme is stable, then we would not know if a stable consistent scheme is convergent. The usefulness of the Lax–Richtmyer theorem arises both from the ease of verifying consistency and stability and from the precise relationship established between these concepts and the concept of convergence.

## Exercises

- 1.5.1.** Show that schemes of the form

$$v_m^{n+1} = \alpha v_{m+1}^n + \beta v_{m-1}^n$$

are stable if  $|\alpha| + |\beta|$  is less than or equal to 1. Conclude that the Lax–Friedrichs scheme (1.3.5) is stable if  $|a\lambda|$  is less than or equal to 1.

- 1.5.2.** By multiplying the leapfrog scheme (1.3.4) by  $v_m^{n+1} + v_m^{n-1}$  and summing over all values of  $m$ , obtain the relation

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 + |v_m^n|^2 + a\lambda(v_m^{n+1}v_{m+1}^n - v_{m+1}^{n+1}v_m^n) \\ &= \sum_{m=-\infty}^{\infty} |v_m^n|^2 + |v_m^{n-1}|^2 + a\lambda(v_m^n v_{m+1}^{n-1} - v_{m+1}^n v_m^{n-1}). \end{aligned}$$

Show that the leapfrog scheme is stable for  $|a\lambda| < 1$ .

- 1.5.3.** By multiplying scheme (1.4.3), with  $f_m^n$  equal to 0, by  $v_m^{n+1} + v_m^n$  and summing over all values of  $m$ , obtain the relation

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} \left(1 - \frac{a\lambda}{2}\right) |v_m^{n+1}|^2 + \frac{a\lambda}{2} v_m^{n+1} v_{m+1}^{n+1} \\ &= \sum_{m=-\infty}^{\infty} \left(1 - \frac{a\lambda}{2}\right) |v_m^n|^2 + \frac{a\lambda}{2} v_m^n v_{m+1}^n. \end{aligned}$$

Conclude that the scheme is stable for  $a\lambda < 1$ .

- 1.5.4.** By multiplying scheme (1.4.3), with  $f_m^n$  equal to 0, by  $v_{m+1}^{n+1} + v_{m-1}^n$  and summing over all values of  $m$ , obtain the relation

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} \frac{a\lambda}{2} |v_m^{n+1}|^2 + \left(1 - \frac{a\lambda}{2}\right) v_m^{n+1} v_{m+1}^{n+1} \\ &= \sum_{m=-\infty}^{\infty} \frac{a\lambda}{2} |v_m^n|^2 + \left(1 - \frac{a\lambda}{2}\right) v_m^n v_{m+1}^n. \end{aligned}$$

Conclude that the scheme is stable for  $a\lambda > 1$ .

## 1.6 The Courant–Friedrichs–Lewy Condition

The condition that the magnitude of  $a\lambda$  be at most 1 is the stability condition for many finite difference schemes for hyperbolic systems in one space dimension when  $\lambda$  is a constant. This has been the stability condition for the Lax–Friedrichs scheme (1.3.5) (see Exercise 1.5.1) and for the forward-time forward-space scheme (1.3.1) when  $a$  is negative and the forward-time backward-space scheme (1.3.2) when  $a$  is positive (see Example 1.5.1). We now show that this condition is a necessary condition for stability for many explicit schemes for the equation (1.1.1).

An explicit finite difference scheme is any scheme that can be written in the form

$$v_m^{n+1} = \text{a finite sum of } v_{m'}^{n'} \text{ with } n' \leq n.$$

All the schemes we considered so far are explicit; we examine implicit (i.e., nonexplicit) schemes later. We now prove the following result, which covers all the one-step schemes we have discussed.

**Theorem 1.6.1.** *For an explicit scheme for the hyperbolic equation (1.1.1) of the form  $v_m^{n+1} = \alpha v_{m-1}^n + \beta v_m^n + \gamma v_{m+1}^n$  with  $k/h = \lambda$  held constant, a necessary condition for stability is the Courant–Friedrichs–Lewy (CFL) condition,*

$$|a\lambda| \leq 1.$$

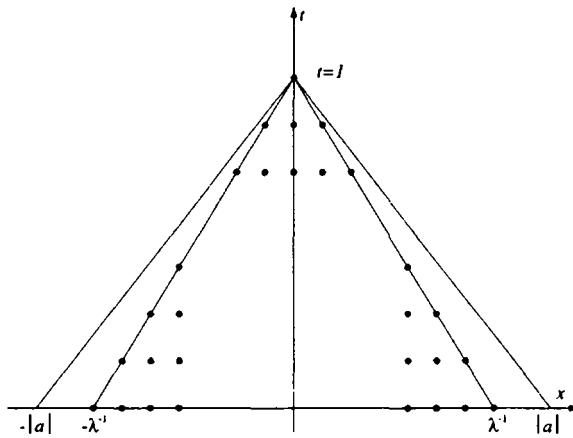
*For systems of equations for which  $v$  is a vector and  $\alpha$ ,  $\beta$ , and  $\gamma$  are matrices, we must have  $|a_i\lambda| \leq 1$  for all eigenvalues  $a_i$  of the matrix  $A$ .*

*Proof.* First consider the case of a single equation. If  $|a\lambda| > 1$ , then by considering the point  $(t, x) = (1, 0)$  we see that the solution to the partial differential equation depends on the values of  $u_0(x)$  at  $x = -a$ . But the finite difference scheme will have  $v_0^n$  depend on  $v_m^0$  only for  $|m| \leq n$ , by the form of the scheme. This situation is illustrated in Figure 1.12. Since  $h = \lambda^{-1}k$ , we have  $|m|h \leq \lambda^{-1}kn = \lambda^{-1}$ , since  $kn = 1$ . So  $v_0^n$  depends on  $x$  only for  $|x| \leq \lambda^{-1} < |a|$ . Thus  $v_0^n$  cannot converge to  $u(1, 0)$  as  $h \rightarrow 0$ . This proves the theorem in this case.

For the case of a system of equations, we have that  $u(1, x)$  depends on  $u_0(x)$  for  $x$  in the interval  $[-a, a]$ , where  $a$  is the maximum magnitude of the characteristic speeds  $a_i$ . If  $|a_i\lambda| > 1$  for some characteristic speed  $a_i$ , then we can take initial data that are zero in  $[-\lambda^{-1}, \lambda^{-1}]$  but not zero near  $a_i$ . Then  $u(1, x)$  will not be zero, in general, and yet  $v_0^n$  with  $nk = 1$  will be zero. Thus  $v^n$  cannot converge to  $u(1, \cdot)$ , and the theorem is proved.  $\square$

A similar argument can be used to show that there is no explicit, consistent scheme for hyperbolic partial differential equations that is stable for all values of  $\lambda$  (with  $\lambda$  constant as  $h, k \rightarrow 0$ ). We obtain the following theorem, first proved by Courant, Friedrichs, and Lewy [11].

**Theorem 1.6.2.** *There are no explicit, unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations.*



**Figure 1.12.** The grid for an unstable scheme.

The *numerical speed* of propagation for a scheme of the form considered in Theorem 1.6.1 is  $h/k = \lambda^{-1}$  since information can propagate one grid spacing in one time step. The CFL condition can be rewritten as

$$\lambda^{-1} \geq |a|,$$

which can be interpreted as stating that the numerical speed of propagation must be greater than or equal to the speed of propagation of the differential equation. This is the basic idea of these theorems. If the numerical scheme cannot propagate the solution at least as fast as the solution of the differential equation, then the solution of the scheme cannot converge to the solution of the partial differential equation.

We now present two implicit schemes for the one-way wave equation (1.1.1). These schemes are consistent and stable for all values of  $\lambda$  and thus illustrate that Theorem 1.6.2 does not extend to implicit schemes. The two schemes are the backward-time central-space scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = 0 \quad (1.6.1)$$

and the backward-time backward-space scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a \frac{v_m^{n+1} - v_{m-1}^{n+1}}{h} = 0 \quad (1.6.2)$$

for  $a$  positive. We are not concerned at this point with how to solve for the values  $v_m^{n+1}$  given the values at time level  $n$ ; this topic is considered in Section 3.5. It is easy to check that both of these schemes are consistent schemes for (1.1.1). In Section 2.2 we show that the scheme (1.6.1) is stable for all values of  $a$  and  $\lambda$ .

**Example 1.6.1.** We now show that the backward-time backward-space scheme (1.6.2) is stable when  $a$  is positive and  $\lambda$  is any positive number. This shows that Theorem 1.6.2 does not extend to implicit schemes.

We first write the scheme (1.6.2) as

$$(1 + a\lambda)v_m^{n+1} = v_m^n + a\lambda v_{m-1}^{n+1}.$$

If we take the square of both sides, we obtain

$$\begin{aligned}(1 + a\lambda)^2 |v_m^{n+1}|^2 &\leq |v_m^n|^2 + 2a\lambda|v_m^n||v_{m-1}^{n+1}| + (a\lambda)^2 |v_{m-1}^{n+1}|^2 \\ &\leq (1 + a\lambda)|v_m^n|^2 + (a\lambda + (a\lambda)^2)|v_{m-1}^{n+1}|^2.\end{aligned}$$

Taking the sum over all values of  $m$ , we obtain

$$(1 + a\lambda)^2 \sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \leq (1 + a\lambda) \sum_{m=-\infty}^{\infty} |v_m^n|^2 + (a\lambda + (a\lambda)^2) \sum_{m=-\infty}^{\infty} |v_{m-1}^{n+1}|^2.$$

Subtracting the last expression on the right-hand side from the left-hand side gives the estimate

$$\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \leq \sum_{m=-\infty}^{\infty} |v_m^n|^2,$$

showing that the scheme is stable for every value of  $\lambda$  when  $a$  is positive.  $\square$

We point out that even though we can choose  $\lambda$  arbitrarily large for scheme (1.6.2) and still have a stable scheme, the solution will not be accurate unless  $\lambda$  is restricted to reasonable values. We discuss the accuracy of solutions in Chapter 3, and in Section 5.2 we show that there are advantages to choosing  $|a\lambda|$  small.

## Exercises

- 1.6.1.** Show that the following modified Lax–Friedrichs scheme for the one-way wave equation,  $u_t + au_x = f$ , given by

$$v_m^{n+1} = \frac{1}{2} (v_{m+1}^n + v_{m-1}^n) - \frac{a\lambda}{1 + (a\lambda)^2} (v_{m+1}^n - v_{m-1}^n) + kf_m^n$$

is stable for all values of  $\lambda$ . Discuss the relation of this explicit and unconditionally stable scheme to Theorem 1.6.2.

- 1.6.2.** Modify the proof of Theorem 1.6.1 to cover the leapfrog scheme.

- 1.6.3.** Show that schemes of the form

$$\alpha v_{m+1}^{n+1} + \beta v_{m-1}^{n+1} = v_m^n$$

are stable if  $||\alpha| - |\beta||$  is greater than or equal to 1. Conclude that the reverse Lax–Friedrichs scheme,

$$\frac{\frac{1}{2} (v_{m+1}^{n+1} + v_{m-1}^{n+1}) - v_m^n}{k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = 0,$$

is stable if  $|a\lambda|$  is greater than or equal to 1.

## Chapter 2

# Analysis of Finite Difference Schemes

In this chapter we present and develop the basic properties of Fourier analysis, which is an important tool for analyzing finite difference schemes and their solutions. In this and subsequent chapters this tool is used to study many important properties of finite difference schemes and their solutions. We use Fourier analysis throughout this text to study both finite difference schemes and partial differential equations.

## 2.1 Fourier Analysis

The tool that we will use most extensively in our study of stability and well-posedness is Fourier analysis. We will use Fourier analysis on both the real line  $\mathbb{R}$  and on the grid of integers  $\mathbb{Z}$  or  $h\mathbb{Z}$ , which is defined by  $h\mathbb{Z} = \{hm : m \in \mathbb{Z}\}$ . For a function  $u(x)$  defined on the real line  $\mathbb{R}$ , its Fourier transform  $\hat{u}(\omega)$  is defined by

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x) dx. \quad (2.1.1)$$

The Fourier transform of  $u$  is a function of the real variable  $\omega$  and is uniquely defined by  $u$ . The function  $\hat{u}$  is an alternative representation of the function  $u$ . Information about certain properties of  $u$  can be inferred from the properties of  $\hat{u}$ . For example, the rate at which  $\hat{u}$  decays for large values of  $\omega$  is related to the number of derivatives that  $u$  has.

The Fourier inversion formula, given by

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega) d\omega, \quad (2.1.2)$$

shows how  $u$  can be recovered from  $\hat{u}$ . The Fourier inversion formula expresses the function  $u$  as a superposition of waves, given by  $e^{i\omega x}$ , with different amplitudes  $\hat{u}(\omega)$ . We will postpone for now the discussion of what conditions  $u(x)$  must satisfy so that (2.1.1) and (2.1.2) are well defined. Notice that  $\hat{u}(\omega)$  may be complex valued even if  $u(x)$  is real valued.

**Example 2.1.1.** As an example of the Fourier transform, consider the function

$$u(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$