Revisão de Matrizes

- A **matrix A** is an *m* x *n* rectangular array of elements, arranged in *m* rows and *n* columns, denoted

$$A = \left(a_{ij} \right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- Some examples of 2 x 2 matrices are given below:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3-2i \\ 4+5i & 6-7i \end{pmatrix}$$

Transpose

- The **transpose** of $\mathbf{A} = (a_{ij})$ is $\mathbf{A}^T = (a_{ji})$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

- For example,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow B^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Conjugate

- The **conjugate** of $\mathbf{A} = (a_{ij})$ is $\overline{\mathbf{A}} = (\overline{a_{ij}})$.

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \bar{A}=\begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{m1} & \bar{a}_{m2} & \cdots & \bar{a}_{mn} \end{pmatrix}$$

- For example,

$$A=\begin{pmatrix} 1 & 2+3i \\ 3-4i & 4 \end{pmatrix} \Rightarrow \bar{A}=\begin{pmatrix} 1 & 2-3i \\ 3+4i & 4 \end{pmatrix}$$

Adjoint

- The **adjoint** of **A** is $\overline{\mathbf{A}}^{T}$, and is denoted by $\mathbf{A}^{*}$

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow A^{\text{¿}}=\begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{m1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{mn} \end{pmatrix}$$

- For example,

$$A=\begin{pmatrix} 1 & 2+3i \\ 3-4i & 4 \end{pmatrix} \Rightarrow A^{\text{¿}}=\begin{pmatrix} 1 & 3+4i \\ 2-3i & 4 \end{pmatrix}$$

Square Matrices

- A **square matrix A** has the same number of rows and columns. That is, **A** is *n* x *n*. In this case, **A** is said to have order *n*.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- For example,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Vectors

- A **column vector x** is an *n* x 1 matrix. For example,

$$x=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

- A **row vector x** is a 1 x *n* matrix. For example,

$$y=\begin{pmatrix}1 & 2 & 3\end{pmatrix}$$

- Note here that $\mathbf{y} = \mathbf{x}^T$, and that in general, if **x** is a column vector **x**, then $\mathbf{x}^T$ is a row vector.

The Zero Matrix

- The **zero matrix** is defined to be $\mathbf{0} = (0)$, whose dimensions depend on the context. For example,

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \ldots$$

Matrix Equality

- Two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are **equal** if $a_{ij} = b_{ij}$ for all i and j. For example,

$$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ B=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\ \Rightarrow A=B$$

Matrix – Scalar Multiplication

- The product of a matrix $\mathbf{A} = (a_{ij})$ and a constant k is defined to be $\mathrm{k}\mathbf{A} = (\mathrm{k}a_{ij})$. For example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow -5A = \begin{pmatrix} -5 & -10 & -15 \\ -20 & -25 & -30 \end{pmatrix}$$

Matrix Addition and Subtraction

- The **sum** of two m x n matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is defined to be $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$. For example,

$$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ B=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\ \Rightarrow A+B=\begin{pmatrix}6 & 8\\ 10 & 12\end{pmatrix}$$

- The **difference** of two m x n matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is defined to be $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$. For example,

$$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ B=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\ \Rightarrow A-B=\begin{pmatrix}-4 & -4\\ -4 & -4\end{pmatrix}$$

Matrix Multiplication

- The **product** of an m x n matrix $\mathbf{A} = (a_{ij})$ and an n x r matrix $\mathbf{B} = (b_{ij})$ is defined to be the matrix $\mathbf{C} = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Examples (note **AB** does not necessarily equal **BA**):

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\ B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow \mathbf{AB} = \begin{pmatrix} 1+4 & 3+8 \\ 3+8 & 9+16 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}$$

$$\Rightarrow \mathbf{BA} = \begin{pmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{pmatrix} = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},\ D = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} \Rightarrow \mathbf{CD} = \begin{pmatrix} 3+2+0 & 0+4-3 \\ 12+5+0 & 0+10-6 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 17 & 4 \end{pmatrix}$$

Vector Multiplication

- The **dot product** of two *n* x 1 vectors **x** & **y** is defined as

$$x^T y = \sum_{k=1}^{n} x_i y_j$$

- The **inner product** of two *n* x 1 vectors **x** & **y** is defined as

$$(x, y) = x^T \bar{y} = \sum_{k=1}^{n} x_i \bar{y}_j$$

- Example:

$$x = \begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ 2-3i \\ 5+5i \end{pmatrix} \Rightarrow x^T y = (1)(-1) + (2)(2-3i) + (3i)(5+5i) = -12 + 9i$$

$$\Rightarrow (x, y) = x^T \bar{y} = (1)(-1) + (2)(2+3i) + (3i)(5-5i) = 18 + 21i$$

Vector Length

- The **length** of an n x 1 vector **x** is defined as

$$\| x \| = (x, x)^{1/2} = \left[\sum_{k=1}^{n} x_k \bar{x}_k \right]^{1/2} = \left[\sum_{k=1}^{n} |x_k|^2 \right]^{1/2}$$

- Note here that we have used the fact that if $x = a + bi$, then

$$x \cdot \bar{x} = (a+bi)(a-bi) = a^2 + b^2 = |x|^2$$

- Example:

$$x = \begin{pmatrix} 1 \\ 2 \\ 3+4i \end{pmatrix} \Rightarrow \|x\| = (x, x)^{1/2} = \sqrt{(1)(1) + (2)(2) + (3+4i)(3-4i)}$$

$$= \sqrt{1+4+(9+16)} = \sqrt{30}$$

Orthogonality

- Two n x 1 vectors **x** & **y** are **orthogonal** if **(x,y)** = 0**.**
- Example:

$$x=\begin{pmatrix}1\\2\\3\end{pmatrix}\quad y=\begin{pmatrix}11\\-4\\-1\end{pmatrix}\quad \Rightarrow (x,\ y)=(1)(11)+(2)(-4)+(3)(-1)=0$$

Identity Matrix

- The multiplicative **identity matrix I** is an *n* x *n* matrix given by

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- For any square matrix **A**, it follows that **AI** = **IA** = **A**.
- The dimensions of **I** depend on the context. For example,

$$\mathbf{AI} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{IB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Inverse Matrix

- A square matrix **A** is **nonsingular**, or **invertible**, if there exists a matrix **B** such that that **AB** = **BA** = **I**. Otherwise **A** is **singular**.
- The matrix **B**, if it exists, is unique and is denoted by $\mathbf{A}^{-1}$ and is called the **inverse** of **A**.
- It turns out that $\mathbf{A}^{-1}$ exists iff det**A** $\neq 0$, and $\mathbf{A}^{-1}$ can be found using **row reduction** (also called Gaussian elimination) on the augmented matrix (**A**|**I**), see example on next slide.
- The three elementary row operations:
  - Interchange two rows.
  - Multiply a row by a nonzero scalar.
  - Add a multiple of one row to another row.

Example: Finding the Identity Matrix (1 of 2)

- Use row reduction to find the inverse of the matrix **A** below, if it exists.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$$

- Solution: If possible, use elementary row operations to reduce **(A|I)**,

$$\left(A|I\right) = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{pmatrix},$$

such that the left side is the identity matrix, for then the right side will be $\mathbf{A}^{-1}$. (See next slide.)

Example: Finding the Identity Matrix (2 of 2)

$$(A|I)=\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{pmatrix}$$

- Thus

$$A^{-1}=\begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}$$

Matrix Functions

- The elements of a matrix can be functions of a real variable. In this case, we write

$$x(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \quad A(t)=\begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

- Such a matrix is continuous at a point, or on an interval (a, b), if each element is continuous there. Similarly with differentiation and integration:

$$\frac{dA}{dt}=\left(\frac{da_{ij}}{dt}\right), \quad \int_a^b A(t)\,dt=\left(\int_a^b a_{ij}(t)\,dt\right)$$

Example & Differentiation Rules

- Example:

$$A(t)=\begin{pmatrix} 3t^2 & \sin t \\ \cos t & 4 \end{pmatrix} \Rightarrow \frac{dA}{dt}=\begin{pmatrix} 6t & \cos t \\ -\sin t & 0 \end{pmatrix},$$

$$\Rightarrow \int_0^{\pi} A(t)\,dt=\begin{pmatrix} \pi^3 & 0 \\ -1 & 4\pi \end{pmatrix}$$

- Many of the rules from calculus apply in this setting. For example:

$$\frac{d(\mathbf{CA})}{dt}=C\frac{dA}{dt},\quad \text{where } C \text{ is a constant matrix}$$

$$\frac{d(A+B)}{dt}=\frac{dA}{dt}+\frac{d\,B}{dt}$$

$$\frac{d(\mathbf{AB})}{dt}=\left(\frac{dA}{dt}\right)B+A\left(\frac{d\,B}{dt}\right)$$

Sistemas

- A system of n linear equations in n variables,

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n , \end{aligned}$$

  can be expressed as a matrix equation $\mathbf{Ax} = \mathbf{b}$:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

- If $\mathbf{b} = \mathbf{0}$, then system is **homogeneous**; otherwise it is **nonhomogeneous**.

Nonsingular Case

- If the coefficient matrix **A** is nonsingular, then it is invertible and we can solve $\mathbf{Ax} = \mathbf{b}$ as follows:

$$\mathbf{Ax} = \mathbf{b} \;\Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

- This solution is therefore unique. Also, if $\mathbf{b} = \mathbf{0}$, it follows that the unique solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$.
- Thus if **A** is nonsingular, then the only solution to $\mathbf{Ax} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$.

Example 1: Nonsingular Case (1 of 3)

- From a previous example, we know that the matrix **A** below is nonsingular with inverse as given.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}$$

- Using the definition of matrix multiplication, it follows that the only solution of $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 1: Nonsingular Case (2 of 3)

- Now let's solve the nonhomogeneous linear system $\mathbf{Ax} = \mathbf{b}$ below using $\mathbf{A}^{-1}$:

$$\begin{aligned} 0x_1 + x_2 + 2x_3 &= \ \ 2 \\ 1x_1 + 0x_2 + 3x_3 &= -2 \\ 4x_1 - 3x_2 + 8x_3 &= \ \ 0 \end{aligned}$$

- This system of equations can be written as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$$

- Then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -23 \\ -12 \\ 7 \end{pmatrix}$$

Example 1: Nonsingular Case (3 of 3)

- Alternatively, we could solve the nonhomogeneous linear system $\mathbf{Ax} = \mathbf{b}$ below using row reduction.

$$
\begin{aligned}
0x_1 + x_2 + 2x_3 &= \ \ 2 \\
1x_1 + 0x_2 + 3x_3 &= -2 \\
4x_1 - 3x_2 + 8x_3 &= \ \ 0
\end{aligned}
$$

- To do so, form the augmented matrix $(\mathbf{A}|\mathbf{b})$ and reduce, using elementary row operations.

$$
(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & -2 \\ 4 & -3 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 2 \\ 4 & -3 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -4 & 8 \end{pmatrix}
$$

$$
\rightarrow \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 14 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 7 \end{pmatrix} \rightarrow \begin{aligned} x_1 \qquad + 3x_3 &= -2 \\ x_2 + 2x_3 &= \ \ 2 \\ x_3 &= \ \ 7 \end{aligned} \rightarrow \mathbf{x} = \begin{pmatrix} -23 \\ -12 \\ 7 \end{pmatrix}
$$

Singular Case

- If the coefficient matrix $\mathbf{A}$ is singular, then $\mathbf{A}^{-1}$ does not exist, and either a solution to $\mathbf{Ax} = \mathbf{b}$ does not exist, or there is more than one solution (not unique).
- Further, the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has more than one solution. That is, in addition to the trivial solution $\mathbf{x} = \mathbf{0}$, there are infinitely many nontrivial solutions.
- The nonhomogeneous case $\mathbf{Ax} = \mathbf{b}$ has no solution unless $(\mathbf{b}, \mathbf{y}) = 0$, for all vectors $\mathbf{y}$ satisfying $\mathbf{A}^*\mathbf{y} = \mathbf{0}$, where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$.
- In this case, $\mathbf{Ax} = \mathbf{b}$ has solutions (infinitely many), each of the form $\mathbf{x} = \mathbf{x}^{(0)} + \boldsymbol{\xi}$, where $\mathbf{x}^{(0)}$ is a particular solution of $\mathbf{Ax} = \mathbf{b}$, and $\boldsymbol{\xi}$ is any solution of $\mathbf{Ax} = \mathbf{0}$.

Example 2: Singular Case (1 of 3)

- Solve the nonhomogeneous linear system $\mathbf{Ax} = \mathbf{b}$ below using row reduction.

$$\begin{aligned} 1x_1 - 2x_2 - 1x_3 &= 1 \\ -1x_1 + 5x_2 + 6x_3 &= 0 \\ 5x_1 - 4x_2 + 5x_3 &= -1 \end{aligned}$$

- To do so, form the augmented matrix $(\mathbf{A}|\mathbf{b})$ and reduce, using elementary row operations.

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & -2 & -1 & 1 \\ -1 & 5 & 6 & 0 \\ 5 & -4 & 5 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 3 & 5 & 1 \\ 0 & 6 & 10 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 3 & 5 & 1 \\ 0 & 3 & 5 & -3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 1 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{array}{rrrl} x_1 & -2x_2 & -x_3 & =1 \\ & 3x_2 & +5x_3 & =1 \to \text{no soln} \\ & & 0x_3 & =1 \end{array}$$

Example 2: Singular Case (2 of 3)

- Solve the nonhomogeneous linear system $\mathbf{Ax} = \mathbf{b}$ below using row reduction.

$$\begin{aligned} 1x_1 - 2x_2 - 1x_3 &= b_1 \\ -1x_1 + 5x_2 + 6x_3 &= b_2 \\ 5x_1 - 4x_2 + 5x_3 &= b_3 \end{aligned}$$

- Reduce the augmented matrix $(\mathbf{A}|\mathbf{b})$ as follows:

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & -2 & -1 & b_1 \\ -1 & 5 & 6 & b_2 \\ 5 & -4 & 5 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & b_1 \\ 0 & 3 & 5 & b_2 + b_1 \\ 0 & 6 & 10 & b_3 - 5b_1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & -1 & b_1 \\ 0 & 3 & 5 & b_2 + b_1 \\ 0 & 3 & 5 & \frac{1}{2}b_3 - \frac{5}{2}b_1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & b_1 \\ 0 & 3 & 5 & b_2 + b_1 \\ 0 & 0 & 0 & \frac{1}{2}b_3 - b_2 - \frac{7}{2}b_1 \end{pmatrix} \to b_3 - 2b_2 - 7b_1 = 0$$

Example 2: Singular Case (3 of 3)

- From the previous slide, we require

$$b_3 - 2b_2 - 7b_1 = 0$$

- Suppose $b_1 = 1,\ b_2 = -1,\ b_3 = 5$

- Then the reduced augmented matrix $(\mathbf{A}|\mathbf{b})$ becomes:

$$\begin{pmatrix} 1 & -2 & -1 & b_1 \\ 0 & 3 & 5 & b_2 + b_1 \\ 0 & 0 & 0 & \frac{1}{2}b_3 - b_2 - \frac{7}{2}b_1 \end{pmatrix} \to \begin{array}{llll} x_1 & -2x_2 & -1x_3 & = 1 \\ & 3x_2 & +5x_3 & = 0 \\ & & 0x_3 & = 0 \end{array}$$

$$\to \mathbf{x} = \begin{pmatrix} 1 - 7x_3/3 \\ -5x_3/3 \\ x_3 \end{pmatrix} \to \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -7/3 \\ -5/3 \\ 1 \end{pmatrix} \Longrightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 7 \\ 5 \\ -3 \end{pmatrix} = \mathbf{x}^{(0)} + \boldsymbol{\xi}$$

Linear Dependence and Independence

- A set of vectors $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$,…, $\mathbf{x}^{(n)}$ is **linearly dependent** if there exists scalars c_1, c_2,…, c_n, not all zero, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{0}$$

- If the only solution of

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{0}$$

is $c_1 = c_2 = \ldots = c_n = 0$, then $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$,…, $\mathbf{x}^{(n)}$ is **linearly independent.**

Example 3: Linear Independence (1 of 2)

- Determine whether the following vectors are linear dependent or linearly independent.

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix}$$

- We need to solve

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

or

$$c_1\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + c\begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 3: Linear Independence (2 of 2)

- We thus reduce the augmented matrix $(\mathbf{A}|\mathbf{b})$, as before.

$$
(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 0 \\ 4 & -3 & 8 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

$$
\to \begin{array}{rcl} c_1 \qquad + 3c_3 & = & 0 \\ c_2 + 2c_3 & = & 0 \\ c_3 & = & 0 \end{array} \to \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

- Thus the only solution is $c_1 = c_2 = \ldots = c_n = 0$, and therefore the original vectors are linearly independent.

Example 4: Linear Dependence (1 of 2)

- Determine whether the following vectors are linear dependent or linearly independent.

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}, \quad \mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$$

- We need to solve

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

or

$$c_1\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 4: Linear Dependence (2 of 2)

- We thus reduce the augmented matrix $(\mathbf{A}|\mathbf{b})$, as before.

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & -2 & -1 & 0 \\ -1 & 5 & 6 & 0 \\ 5 & -4 & 5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{array}{rrrl} c_1 & -2c_2 & -1c_3 & = 0 \\ & 3c_2 & +5c_3 & = 0 \\ & & 0c_3 & = 0 \end{array} \to \mathbf{c} = \begin{pmatrix} -7c_3/3 \\ -5c_3/3 \\ c_3 \end{pmatrix} \to \mathbf{c} = k\begin{pmatrix} 7 \\ 5 \\ -3 \end{pmatrix}$$

- Thus the original vectors are linearly dependent, with

$$7\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + 5\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Linear Independence and Invertibility

- Consider the previous two examples:
  - The first matrix was known to be nonsingular, and its column vectors were linearly independent.
  - The second matrix was known to be singular, and its column vectors were linearly dependent.
- This is true in general: the columns (or rows) of $\mathbf{A}$ are linearly independent iff $\mathbf{A}$ is nonsingular iff $\mathbf{A}^{-1}$ exists.
- Also, $\mathbf{A}$ is nonsingular iff $\det\mathbf{A} \neq 0$, hence columns (or rows) of $\mathbf{A}$ are linearly independent iff $\det\mathbf{A} \neq 0$.
- Further, if $\mathbf{A} = \mathbf{BC}$, then $\det(\mathbf{C}) = \det(\mathbf{A})\det(\mathbf{B})$. Thus if the columns (or rows) of $\mathbf{A}$ and $\mathbf{B}$ are linearly independent, then the columns (or rows) of $\mathbf{C}$ are also.

Linear Dependence & Vector Functions

- Now consider vector functions $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$, where

$$\mathbf{x}^{(k)}(t) = \begin{pmatrix} x_1^{(k)}(t) \\ x_2^{(k)}(t) \\ \vdots \\ x_m^{(k)}(t) \end{pmatrix}, \quad k = 1, 2, \ldots, n, \qquad t \in I = (\alpha, \beta)$$

- As before, $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$ is **linearly dependent** on I if there exists scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) = \mathbf{0}, \text{ for all } t \in I$$

- Otherwise $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$ is **linearly independent** on I

  See text for more discussion on this.

Eigenvalues and Eigenvectors

- The eqn. $\mathbf{A}\mathbf{x} = \mathbf{y}$ can be viewed as a linear transformation that maps (or transforms) $\mathbf{x}$ into a new vector $\mathbf{y}$.
- Nonzero vectors $\mathbf{x}$ that transform into multiples of themselves are important in many applications.
- Thus we solve $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ or equivalently, $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.
- This equation has a nonzero solution if we choose λ such that $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$.
- Such values of λ are called **eigenvalues** of $\mathbf{A}$, and the nonzero solutions $\mathbf{x}$ are called **eigenvectors**.

Example 5: Eigenvalues (1 of 3)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

- Solution: Choose λ such that det(**A**-λ**I**) = 0, as follows.

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \det\left(\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&= \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix} \\
&= (2-\lambda)(-6-\lambda) - (3)(3) \\
&= \lambda^2 + 4\lambda - 21 = (\lambda - 3)(\lambda + 7) \\
&\Rightarrow \ \lambda = 3, \ \lambda = -7
\end{aligned}$$

Example 5: First Eigenvector (2 of 3)

- To find the eigenvectors of the matrix **A**, we need to solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda = 3$ and $\lambda = -7$.
- Eigenvector for $\lambda = 3$: Solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 2-3 & 3 \\ 3 & -6-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 3 & -9 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{matrix} 1x_1 & -3x_2 & = 0 \\ & 0x_2 & = 0 \end{matrix}$$

$$\to \mathbf{x}^{(1)} = \begin{pmatrix} 3x_2 \\ x_2 \end{pmatrix} = c\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \ c \text{ arbitrary} \to \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Example 5: Second Eigenvector (3 of 3)

- Eigenvector for λ = -7: Solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0} \Leftrightarrow \begin{pmatrix} 2+7 & 3 \\ 3 & -6+7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/3 & 0 \\ 3 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{array}{lll} 1x_1 & +1/3x_2 & =0 \\ & 0x_2 & =0 \end{array}$$

$$\to \mathbf{x}^{(2)}=\begin{pmatrix} -1/3x_2 \\ x_2 \end{pmatrix}=c\begin{pmatrix} -1/3 \\ 1 \end{pmatrix},\ c \text{ arbitrary} \to \text{choose } \mathbf{x}^{(2)}=\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Normalized Eigenvectors

- From the previous example, we see that eigenvectors are determined up to a nonzero multiplicative constant.
- If this constant is specified in some particular way, then the eigenvector is said to be **normalized**.
- For example, eigenvectors are sometimes normalized by choosing the constant so that $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = 1$.

Algebraic and Geometric Multiplicity

- In finding the eigenvalues λ of an n x n matrix **A**, we solve $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$.
- Since this involves finding the determinant of an n x n matrix, the problem reduces to finding roots of an nth degree polynomial.
- Denote these roots, or eigenvalues, by $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- If an eigenvalue is repeated m times, then its **algebraic multiplicity** is m.
- Each eigenvalue has at least one eigenvector, and a eigenvalue of algebraic multiplicity m may have q linearly independent eigevectors, $1 \leq q \leq m$, and q is called the **geometric multiplicity** of the eigenvalue.

Eigenvectors and Linear Independence

- If an eigenvalue λ has algebraic multiplicity 1, then it is said to be **simple**, and the geometric multiplicity is 1 also.
- If each eigenvalue of an n x n matrix **A** is simple, then **A** has n distinct eigenvalues. It can be shown that the n eigenvectors corresponding to these eigenvalues are linearly independent.
- If an eigenvalue has one or more repeated eigenvalues, then there may be fewer than n linearly independent eigenvectors since for each repeated eigenvalue, we may have $q < m$. This may lead to complications in solving systems of differential equations.

Example 6: Eigenvalues (1 of 5)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- Solution: Choose λ such that det(**A**-λ**I**) = 0, as follows.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= (\lambda - 2)(\lambda + 1)^2$$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2 = -1,\ \lambda_2 = -1$$

Example 6: First Eigenvector (2 of 5)

- Eigenvector for $\lambda = 2$: Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{rrrl} 1x_1 & & -1x_3 & = 0 \\ & 1x_2 & -1x_3 & = 0 \\ & & 0x_3 & = 0 \end{array}$$

$$\to \mathbf{x}^{(1)} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ c \text{ arbitrary} \to \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example 6: 2nd and 3rd Eigenvectors

- Eigenvector for $\lambda = -1$: Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{cccc} 1x_1 & +1x_2 & +1x_3 & =0 \\ & 0x_2 & & =0 \\ & & 0x_3 & =0 \end{array}$$

$$\rightarrow \mathbf{x}^{(2)} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{ where } x_2, x_3 \text{ arbitrary}$$

$$\rightarrow \text{choose } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Example 6: Eigenvectors of A (4 of 5)

- Thus three eigenvectors of **A** are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

  where $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ correspond to the double eigenvalue $\lambda = -1$.

- It can be shown that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ are linearly independent.
- Hence **A** is a 3 x 3 **symmetric matrix** ($\mathbf{A} = \mathbf{A}^T$) with 3 real eigenvalues and 3 linearly independent eigenvectors.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Example 6: Eigenvectors of **A** (5 of 5)

- Note that we could have we had chosen

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

- Then the eigenvectors are orthogonal, since

$$\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = 0, \ \left(\mathbf{x}^{(1)}, \mathbf{x}^{(3)}\right) = 0, \ \left(\mathbf{x}^{(2)}, \mathbf{x}^{(3)}\right) = 0$$

- Thus **A** is a 3 x 3 symmetric matrix with 3 real eigenvalues and 3 linearly independent orthogonal eigenvectors.

Hermitian Matrices

- A **self-adjoint**, or **Hermitian** matrix, satisfies $\mathbf{A} = \mathbf{A}^*$, where we recall that $\mathbf{A}^* = \overline{\mathbf{A}}^T$.
- Thus for a Hermitian matrix, $a_{ij} = \overline{a}_{ji}$.
- Note that if **A** has real entries and is symmetric (see last example), then **A** is Hermitian.
- An n x n Hermitian matrix **A** has the following properties:
  - All eigenvalues of **A** are real.
  - There exists a full set of n linearly independent eigenvectors of **A**.
  - If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues of **A**, then $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
  - Corresponding to an eigenvalue of algebraic multiplicity m, it is possible to choose m mutually orthogonal eigenvectors, and hence A has a full set of n linearly independent orthogonal eigenvectors.