Lecture 4

Math 178

Nonlinear Data Analytics

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Recall: Regular Surface (2-dimensional Manifold)

Definition

A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathbf{x}$ is differentiable (so we can use calculus).

2. $\mathbf{x}$ is a homeomorphism (so we can use analysis)

3. $\mathbf{x}$ is regular (so we can use linear algebra)

Remark

In contrast to our treatment of curves, we have *defined a surface as a subset* S of $\mathbb{R}^3$, and not as a map. This is achieved by covering S with the traces of parametrizations which satisfy conditions 1, 2, and 3.

Exact meanings:

$\mathbf{x}$ is differentiable

This means that if we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$

the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ have continuous partial derivatives of all orders.

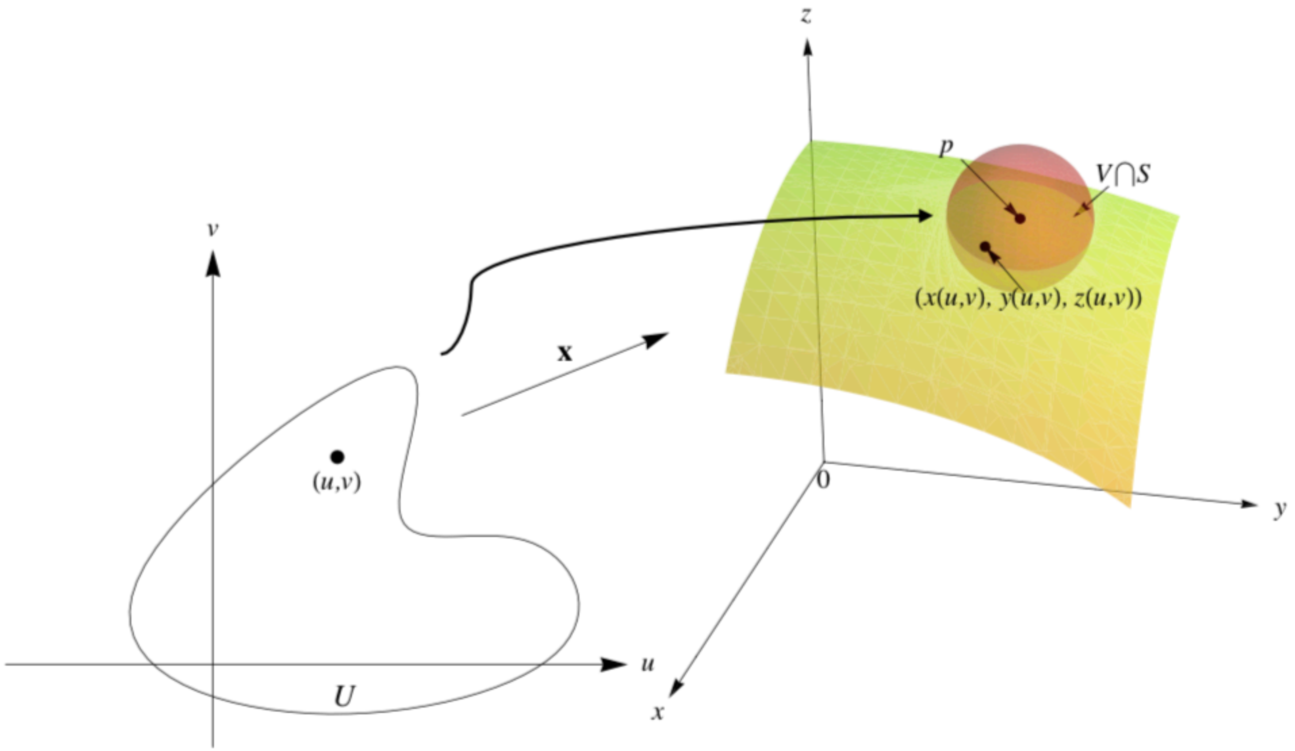
$\mathbf{x}$ is a homeomorphism

Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set W containing $V \cap S$.

$\mathbf{x}$ is regular

For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

A Parametrization and a coordinate neighborhood



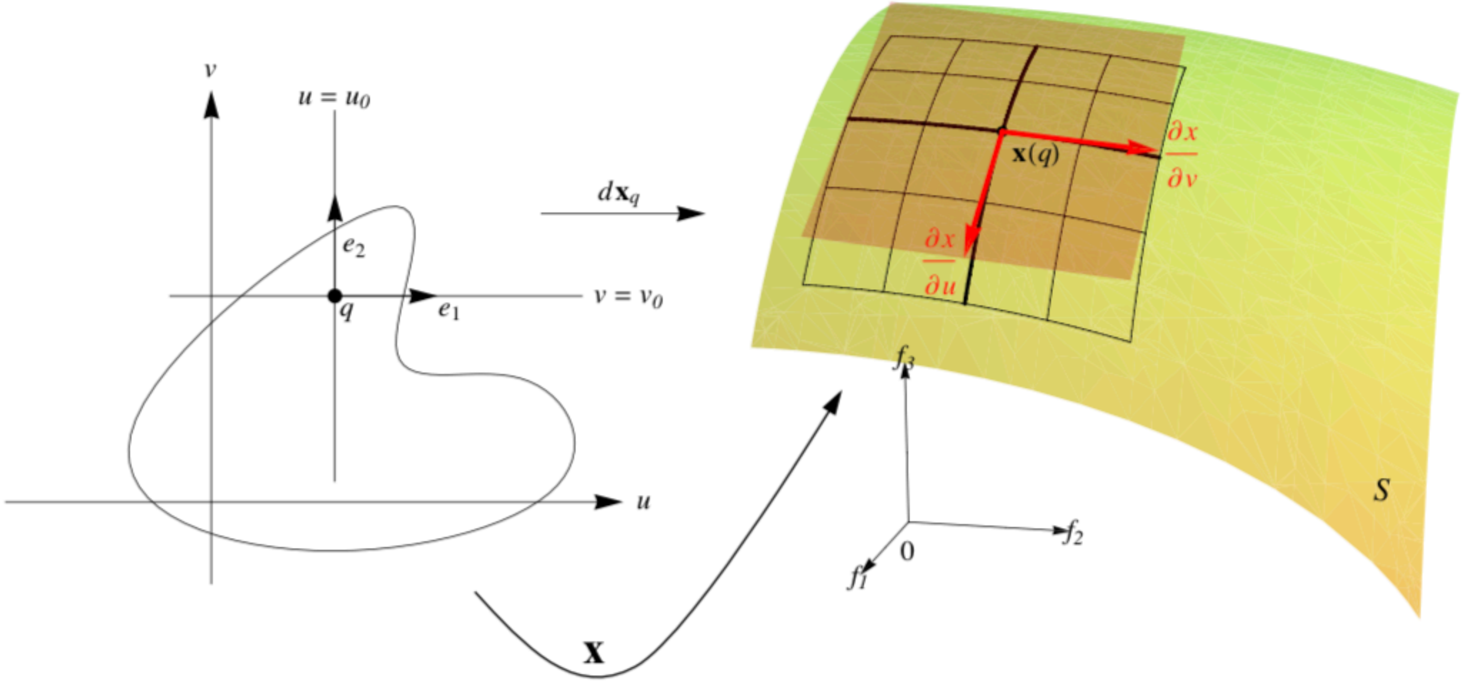
Definition

The mapping **x** is called a *parametrization* or a *system of (local) coordinates* in (a neighborhood of) p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood*.

The Regularity Condition

An Illustrative Example

To give condition 3 a more familiar form, let us compute the matrix of the linear map $d\mathbf{x}_q$ in the canonical bases $e_1 = (1,0)$, $e_2 = (0,1)$ of $\mathbb{R}^2$ with coordinates u, v and $f_1 = (1,0,0)$, $f_2 = (0,1,0)$, $f_3 = (0,0,1)$ of $\mathbb{R}^3$, with coordinates (x, y, z).



The Regularity Condition

An Illustrative Example (cont'd)

Thus, the matrix of the linear map $d\mathbf{x}_q$ in the referred (standard) basis is

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

Condition 3 may now be expressed by requiring the two column vectors of this matrix to be linearly independent; or, equivalently, that the vector product $\partial \mathbf{x}/\partial u \wedge \partial \mathbf{x}/\partial v \neq 0$; or, in still another way, that one of the minors of order 2 of the matrix $d\mathbf{x}_q$, that is, one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(x,z)}{\partial(u,v)},$$

be nonzero at q.

The Three Conditions

- Condition 1 is very natural if we expect to do some differential geometry on S.
- The one-to-oneness in condition 2 has the purpose of preventing self-intersections in regular surfaces. This is clearly necessary if we are to speak about, say, *the* tangent plane at a point $p \in S$. The continuity of the inverse in condition 2 has a more subtle purpose. For the time being, we shall mention that this condition is essential to proving that certain objects defined in terms of a parametrization do not depend on this parametrization but only on the set S itself.
- Finally, condition 3 will guarantee the existence of a "tangent plane" at all points of S.

Proving that a Set is a Regular Surface

Example

Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

Method 1: Using Cartesian Coordinates

We first verify that the map $\mathbf{x}_1 : U \in \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}_1(x, y) = (x, y, +\sqrt{1 - (x^2 + y^2)}), \quad (x, y) \in U,$$

where $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ and
$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is a parametrization of S^2.

Proving that a Set is a Regular Surface

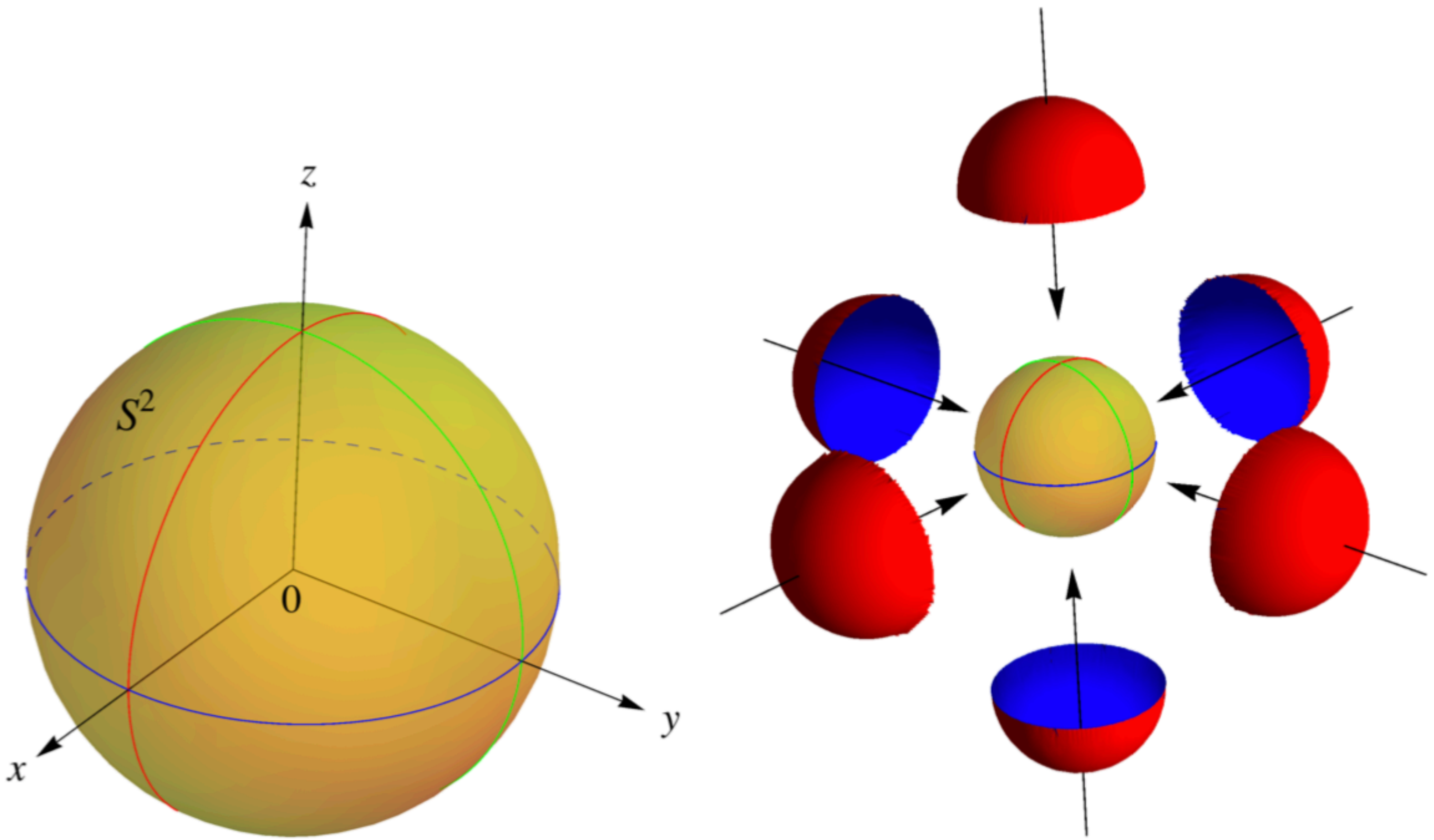
We shall now cover the whole sphere with similar parametrizations as follows. we define $\mathbf{x}_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{x}_2(x, y) = (x, y, -\sqrt{1 - (x^2 + y^2)}),$$

check that $\mathbf{x}_2$ is a parametrization, and observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$. Then, using the xz and zy planes, we define the parametrization

$$\begin{aligned}
\mathbf{x}_3(x, z) &= (x, +\sqrt{1 - (x^2 + z^2)}, z),\\
\mathbf{x}_4(x, z) &= (x, -\sqrt{1 - (x^2 + z^2)}, z),\\
\mathbf{x}_5(y, z) &= (+\sqrt{1 - (y^2 + z^2)}), y, z),\\
\mathbf{x}_6(y, z) &= (-\sqrt{1 - (y^2 + z^2)}), y, z),
\end{aligned}$$

which, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover S^2 completely and shows that S^2 is a regular surface.



z
S^2
0
x
y

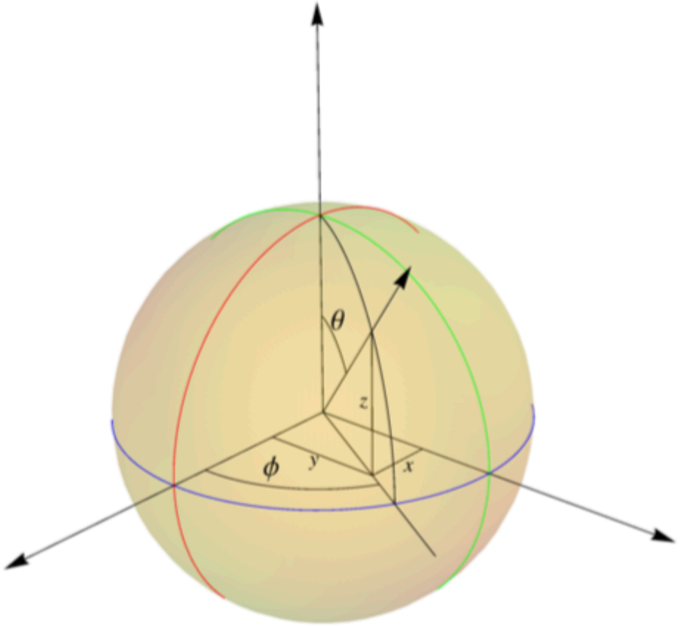
Proving that a Set is a Regular Surface

Method 2: Using Spherical Coordinates

For most applications, it is convenient to relate parametrizations to the geographical coordinates on S^2. Let $V = \{(\theta, \varphi) \mid 0 < \theta < \pi, 0 < \varphi < 2\pi\}$ and let $\mathbf{x} : V \to \mathbb{R}^3$ be given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Clearly, $\mathbf{x}(V) \subset S^2$.



Proving that a Set is a Regular Surface

We shall prove that $\mathbf{x}$ is a parametrization of S^2.

Next, we observe that given $(x, y, z) \in S^2 \setminus C$, where C is the semicircle $C = \{(x, y, z) \in S^2 \mid y = 0, x \geq 0\}$, θ is uniquely determined by $\theta = \cos^{-1} z$, since $0 < \theta < \pi$. By knowing θ, we find $\sin\varphi$ and $\cos\varphi$ from $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and this determines φ uniquely ($0 < \varphi < 2\pi$). It follows that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}$. To complete the verification of condition 2, we should prove that $\mathbf{x}^{-1}$ is continuous. However, since we shall soon prove that this verification is not necessary provided we already know that the set S is a regular surface, we shall not do that here.

We remark that $\mathbf{x}(V)$ only omits a semicircle of S^2 (including the two poles) and that S^2 can be covered with the coordinate neighborhoods of two parametrizations of this type.

HW1: Show that a sphere is a regular surface using spherical coordinates.

- Reference: Differential geometry of curves and surfaces, by do Carmo.

Two Shortcuts

The last example in the previous lecture shows that deciding whether a given subset of $\mathbb{R}^3$ is a regular surface directly from the definition may be quite tiresome.

Shortcut 1

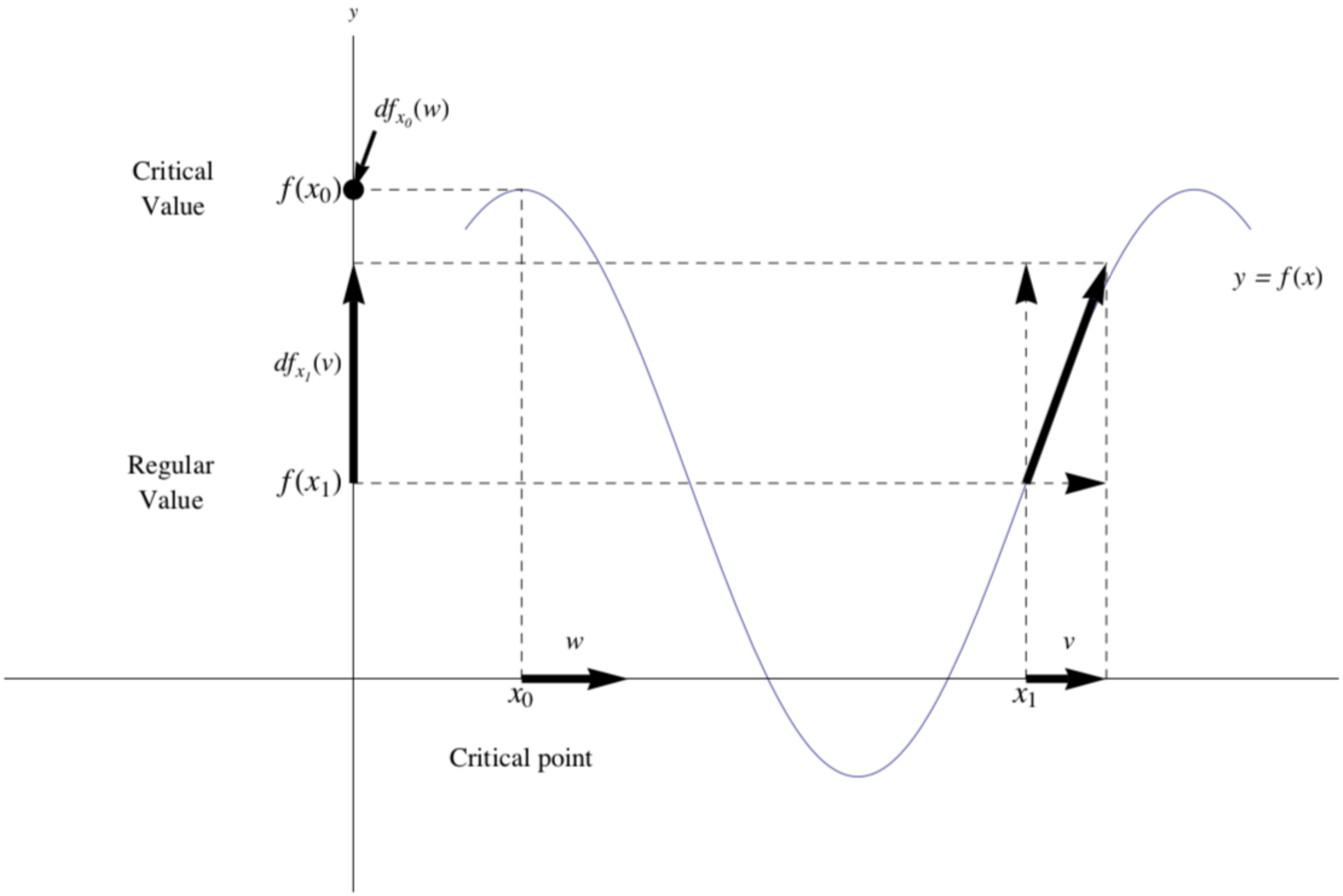
If $f : U \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface

Critical Points and Values

Definition

Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$ we say that $p \in U$ is a *critical point* of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective (or onto) mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a *critical value* of F. A point of $\mathbb{R}^m$ which is not a critical value is called a *regular value* of F.

The terminology is evidently motivated by the particular case in which $f : U \subset \mathbb{R} \to \mathbb{R}$ is a real-valued function of a real variable. A point $x_0 \in U$ is critical if $f'(x_0) = 0$, that is, if the differential df_{x_0} carries all the vectors in $\mathbb{R}$ to the zero vector. Notice that any point $a \notin f(U)$ is trivially a regular value of f.



y
$df_{x_0}(w)$
Critical
Value
$f(x_0)$
$y = f(x)$
$df_{x_1}(v)$
Regular
Value
$f(x_1)$
w
v
x_0
x_1
Critical point

Critical Points and Values

Remark

If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, then

$$df_p = (f_x, f_y, f_z).$$

Note, in this case, that to say that df_p is not surjective is equivalent to saying that $f_x = f_y = f_z = 0$ at p. Hence, $a \in f(U)$ is a regular value of $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ if and only if f_x, f_y, and f_z do not vanish simultaneously at any point in the inverse image

$$f^{-1}(a) = \{(x, y, z) \in U \mid f(x, y, z) = a\}.$$

Two Shortcuts

Shortcut 2

If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Examples

Example

The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface.

The examples of regular surfaces presented so far have been connected subsets of $\mathbb{R}^3$. A surface $S \subset \mathbb{R}^3$ if said to be *connected* if any two of its points can be joined by a continuous curve in S. In the definition of a regular surface we made no restrictions on the connectedness of the surfaces, and the following example shows that the regular surfaces given by Shortcut 2 may not be connected.

WTS:

$$f(x,y,z) = \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} = 1$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$

$$(x,y,z) \longmapsto f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right) = (0,0,0)$$

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \Rightarrow \text{the critical pt is only } (0,0,0).$$

The only

$\Rightarrow$ Critical value is $f(0,0,0) = \frac{0^2}{a^2} + \frac{0^2}{b^2} + \frac{0^2}{c^2} = \underline{\underline{0}}$

$\Rightarrow$ so all the values "a" s.t $a \neq 0$ are regular values. $\Rightarrow$ 1 is an regular value.

$$f^{-1}(1) = \Big\{ (x, y, z) \in \mathbb{R}^3 \;\Big|\; \underbrace{f(x, y, z)}_{x^2/a^2 + y^2/b^2 + z^2/c^2} = 1 \Big\}$$ be a regular surface.

Show S^3 is a manifold.

- Work out details with the students on the iPad.

Now we can show S^3 is a manifold in $\mathbb{R}^4$.

$$S^3 = \{ \underbrace{(x, y, z, w)}_{\in \mathbb{R}^4} \mid x^2+y^2+z^2+w^2=1 \}$$

Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$
$(x, y, z, w) \longmapsto x^2+y^2+z^2+w^2$

$$\nabla f = (2x, 2y, 2z, 2w) = (0, 0, 0, 0)$$

$\Rightarrow$ $(x, y, z, w) = (0, 0, 0, 0)$ is the only critical pt of f.

$\Rightarrow$ The only critical value is $f(0, 0, 0, 0) = 0$

$\Rightarrow$ The only critial value is $f(0,0,0,0) = 0$

$\Rightarrow$ 1 is a regular value

$$\Rightarrow f^{-1}(1) = \{ (x, y, z, w) \mid \underbrace{f(x, y, z, w)}_{x^2+y^2+z^2+w^2} = 1 \} = S^3$$

is a regular 3-D manifold.

Q: Why there are 3 variables x, y, and z, but S^2 is two dimensional?

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

3 (variables) − 1 (constraint) = 2

Next Lecture starts with What is SO(3)?

- The set of rotation matrices in R^3 is denoted by SO(3).