Lecture 3

Math 178

Nonlinear Data Analytics

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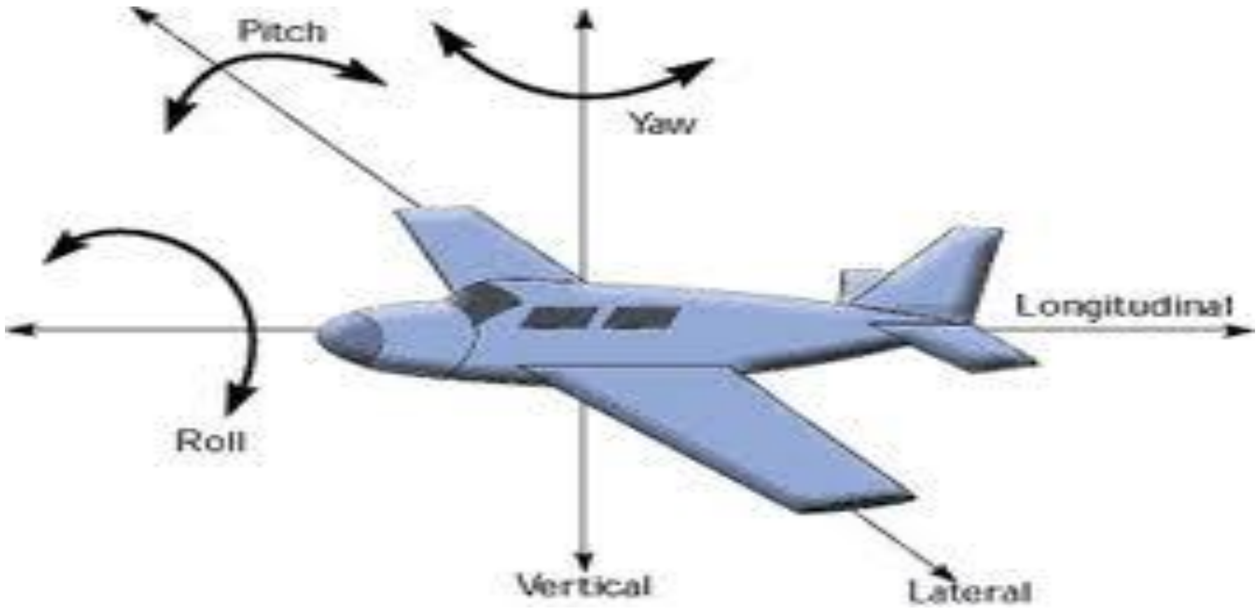
Configuration space of a robot

Definition: The **configuration space** of a **robot** is the **space** of possible positions the **robot** may attain.

- The configuration space of any auto car moving on R^2 consists two parts, 1) the set of translations in R^2 , and 2) the set of rotations in R^2
- The configuration space of any UVA consists two parts, 1) the set of translations in R^3 , and 2) the set of rotations in R^3
- The configuration space of any cell phone consists two parts, 1) the set of translations in R^3 , and 2) the set of rotations in R^3

- How to model and capture the dynamics and kinematics of an UAV?



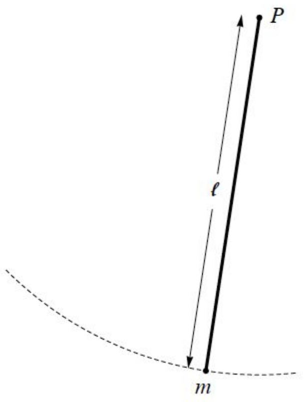
Example: Configuration space usually is a manifold

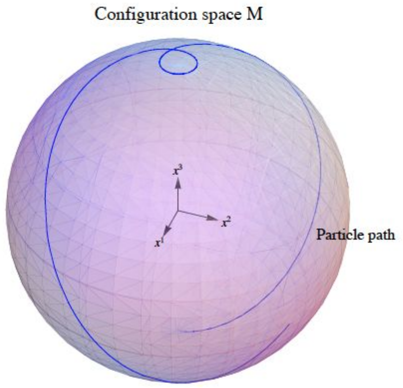
Mathematical Models and Physical Systems

When we wish to describe a physical system in a "mathematical" way we try to construct some sort of mathematical structure which, in some sense, "represents" those aspects of the system which are of interest to us. This structure is then a "mathematical model" of the physical system.

Example

A mass m is fixed on the end of a rigid rod of negligible mass having length ℓ. One end of the rod is fixed at a point P in space so that the mass can move about about P subject to the condition that it always be a distance ℓ from P. The sphere M (a *regular surface* or *manifold*) of all possible positions for m is called the *configuration space* of the system.





How to model the velocity of a robot?

Example (cont'd)

Suppose we are only interested in the motion of the particle. Then we take, as the state of the particle, the pair of three-dimensional vectors (x, v), $x = (x^1, x^2, x^3)$, $v = (v^1, v^2, v^3)$, where x is the position vector of m and v is the velocity vector of m (with respect to some Cartesian coordinate system).

Since the mass must stay on the sphere M, we see v must be tangent to M. Thus our *state space* S does not consist of all pairs of 3-vectors but, instead, we have the *tangent bundle* of M (which can also be viewed as a manifold);

$$S = \{(x, v) \mid x \in M \text{ and } v \text{ is tangent to } M \text{ at } x\}.$$

Although S is not a Euclidean space, nor an open set in one, we shall see that S is a space on which notions such as tangent vector, vector field, and time-dependent vector field have meaning. If we have a force field then the force field will determine a vector field on the state space S.

A simple anomaly detection example:

1. How do the “headings” of the following flight path look like?

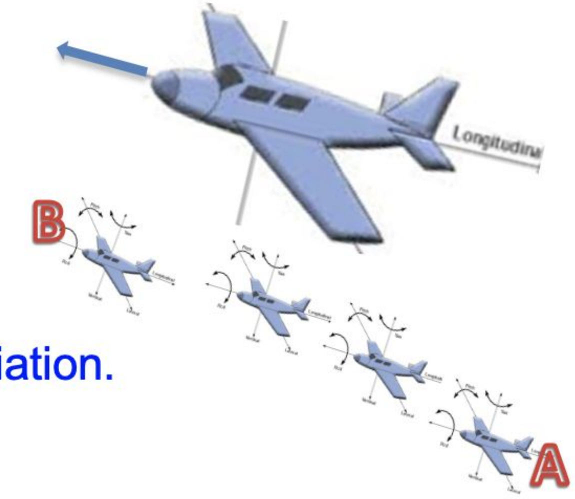
- Only consider UAV heading

1. How do the “headings” of the following flight path look like?

- Only consider UAV heading Directions (depend on time)

2. Mission Objective: Transport a missal from site *A* to site *B* as soon as possible.

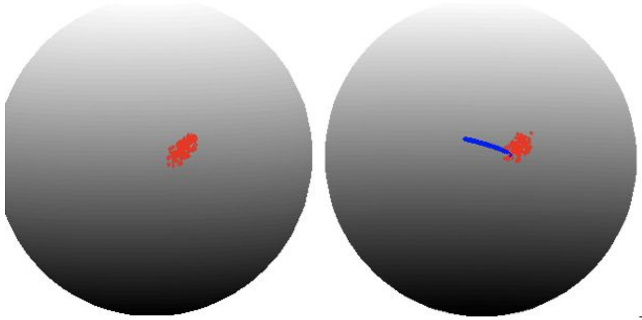
Q: *Dose the pilot have to constantly monitor the UAV?*



3. How to detect anomalies? (See figures below}.

4. Back-identify: a strong wind just started causing the deviation.

Figure L: Normal neighborhood of UAV headings in a specified direction;
Figure R: Deviation beyond bdry of normal nbhd considered anomalous



5. A warning auto issued

Note: Smaller the neighborhood, Less mission cost!

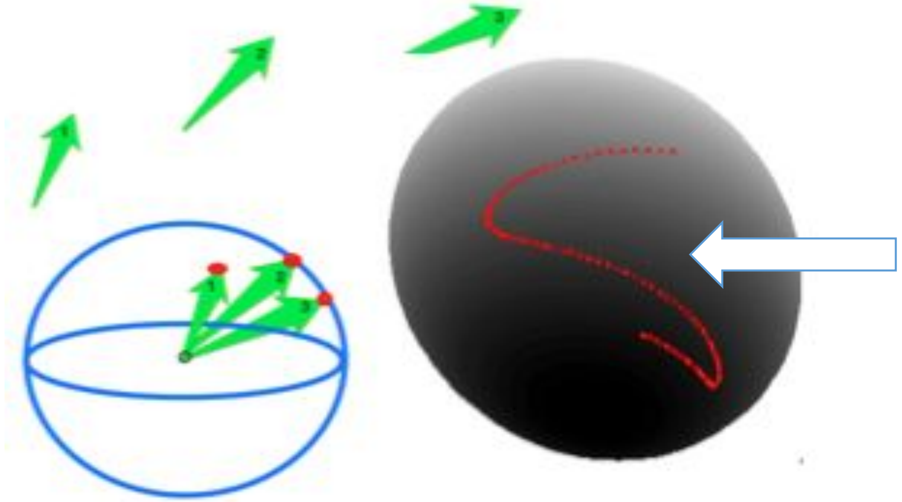
6. The operator corrected UAV’s deviation from its mission path.

You may wonder: How to use manifold to study UAV data?
Simplest case: drawing a curve on a sphere
Try to capture characteristics of flight controls

- **For example: Only look at UAV "headings"**
- **All possible headings for all UAVs form a sphere.**

Only consider UAV heading directions here, but works for any other UAV characteristics

- ***Key: Developed a dimension-reduction technique for large nonlinear data.***



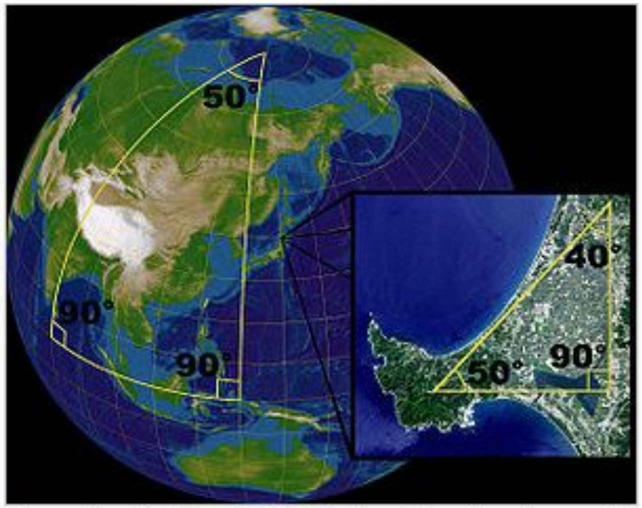
Just recording the heading while a UAV is flying gives a heading-behavior curve.

Rigid Body Kinematics

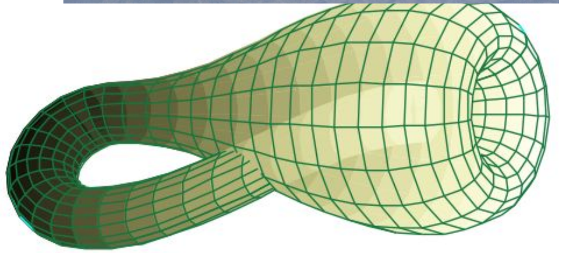
- The set of all 3-dimensional rotations is denoted by SO(3)
- Claim: SO(3) is a manifold, in fact SO(3) is also a group.
- A manifold structure + A group structure = Lie group
- Nonlinear data is in SO(3)
- Work out details with students on the board

What is a manifold?

- An n-dimensional manifold locally "looks like" a piece of $\mathbf{R}^n$.
- For examples, sphere and torus.
- Key features of a **manifold**: **curved**



The sphere (surface of a ball) is a two-dimensional manifold since it can be represented by a collection of two-dimensional maps.



- Only manifolds can capture UAV's dynamical behaviors

From Regular Surface to Manifold

Definition

A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathbf{x}$ is differentiable (so we can use calculus).

2. $\mathbf{x}$ is a homeomorphism (so we can use analysis)

3. $\mathbf{x}$ is regular (so we can use linear algebra)

Remark

In contrast to our treatment of curves, we have *defined a surface as a subset* S of $\mathbb{R}^3$, and not as a map. This is achieved by covering S with the traces of parametrizations which satisfy conditions 1, 2, and 3.

Exact meanings:

$\mathbf{x}$ is differentiable

This means that if we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$

the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ have continuous partial derivatives of all orders.

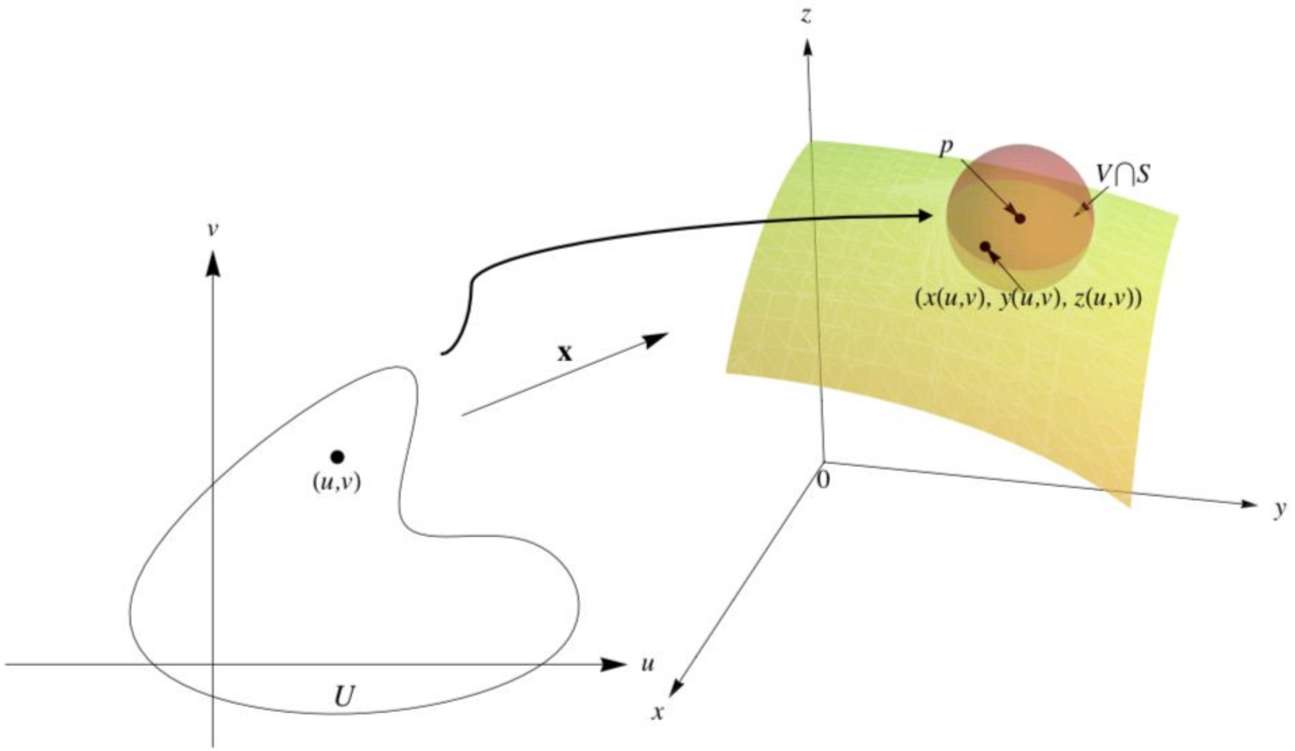
$\mathbf{x}$ is a homeomorphism

Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set W containing $V \cap S$.

$\mathbf{x}$ is regular

For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

A Parametrization and a coordinate neighborhood



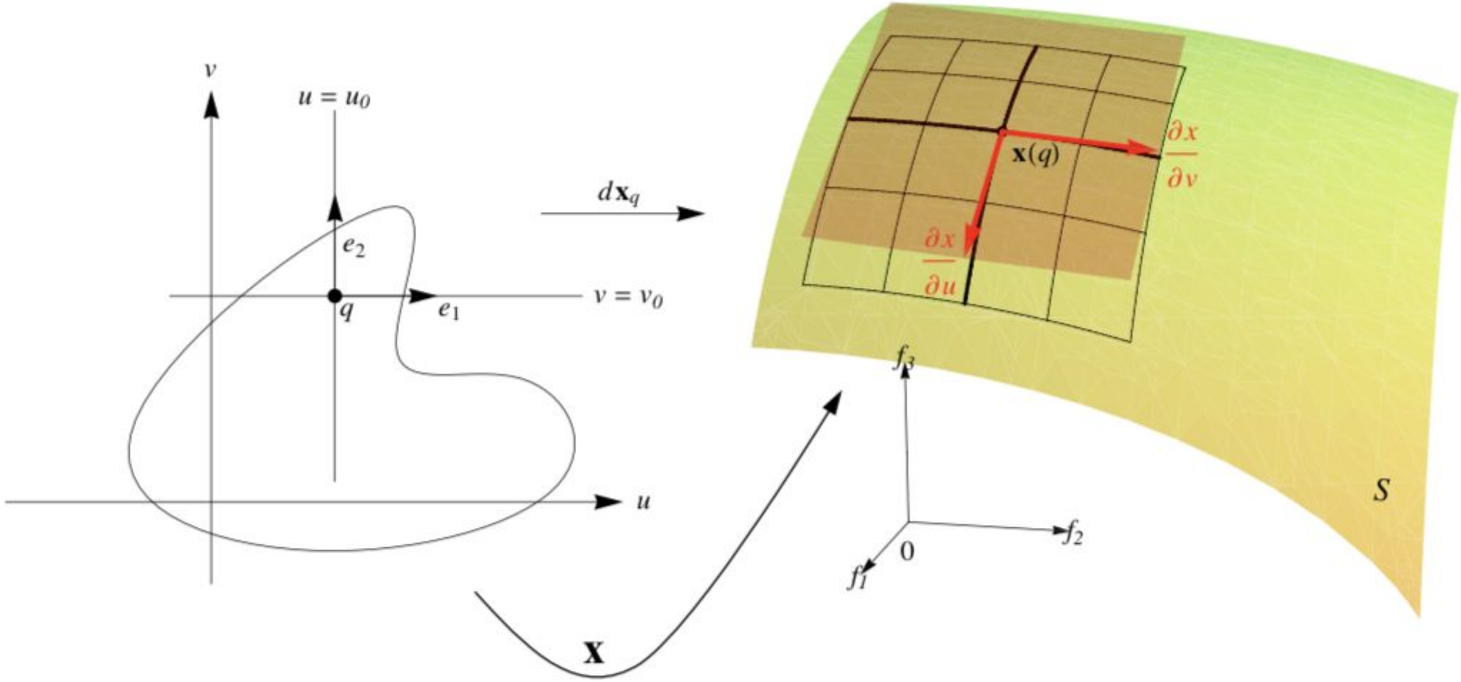
Definition

The mapping **x** is called a *parametrization* or a *system of (local) coordinates* in (a neighborhood of) p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood*.

The Regularity Condition

An Illustrative Example

To give condition 3 a more familiar form, let us compute the matrix of the linear map $d\mathbf{x}_q$ in the canonical bases $e_1 = (1,0)$, $e_2 = (0,1)$ of $\mathbb{R}^2$ with coordinates u, v and $f_1 = (1,0,0)$, $f_2 = (0,1,0)$, $f_3 = (0,0,1)$ of $\mathbb{R}^3$, with coordinates (x, y, z).



The Regularity Condition

An Illustrative Example (cont'd)

Thus, the matrix of the linear map $d\mathbf{x}_q$ in the referred (standard) basis is

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

Condition 3 may now be expressed by requiring the two column vectors of this matrix to be linearly independent; or, equivalently, that the vector product $\partial\mathbf{x}/\partial u \wedge \partial\mathbf{x}/\partial v \neq 0$; or, in still another way, that one of the minors of order 2 of the matrix $d\mathbf{x}_q$, that is, one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(x,z)}{\partial(u,v)},$$

be nonzero at q.

The Three Conditions

- Condition 1 is very natural if we expect to do some differential geometry on S.
- The one-to-oneness in condition 2 has the purpose of preventing self-intersections in regular surfaces. This is clearly necessary if we are to speak about, say, *the* tangent plane at a point $p \in S$. The continuity of the inverse in condition 2 has a more subtle purpose. For the time being, we shall mention that this condition is essential to proving that certain objects defined in terms of a parametrization do not depend on this parametrization but only on the set S itself.
- Finally, condition 3 will guarantee the existence of a "tangent plane" at all points of S.

Proving that a Set is a Regular Surface

Example

Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

Method 1: Using Cartesian Coordinates

We first verify that the map $\mathbf{x}_1 : U \in \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}_1(x, y) = (x, y, +\sqrt{1 - (x^2 + y^2)}), \quad (x, y) \in U,$$

where $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ and
$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is a parametrization of S^2.

Proving that a Set is a Regular Surface

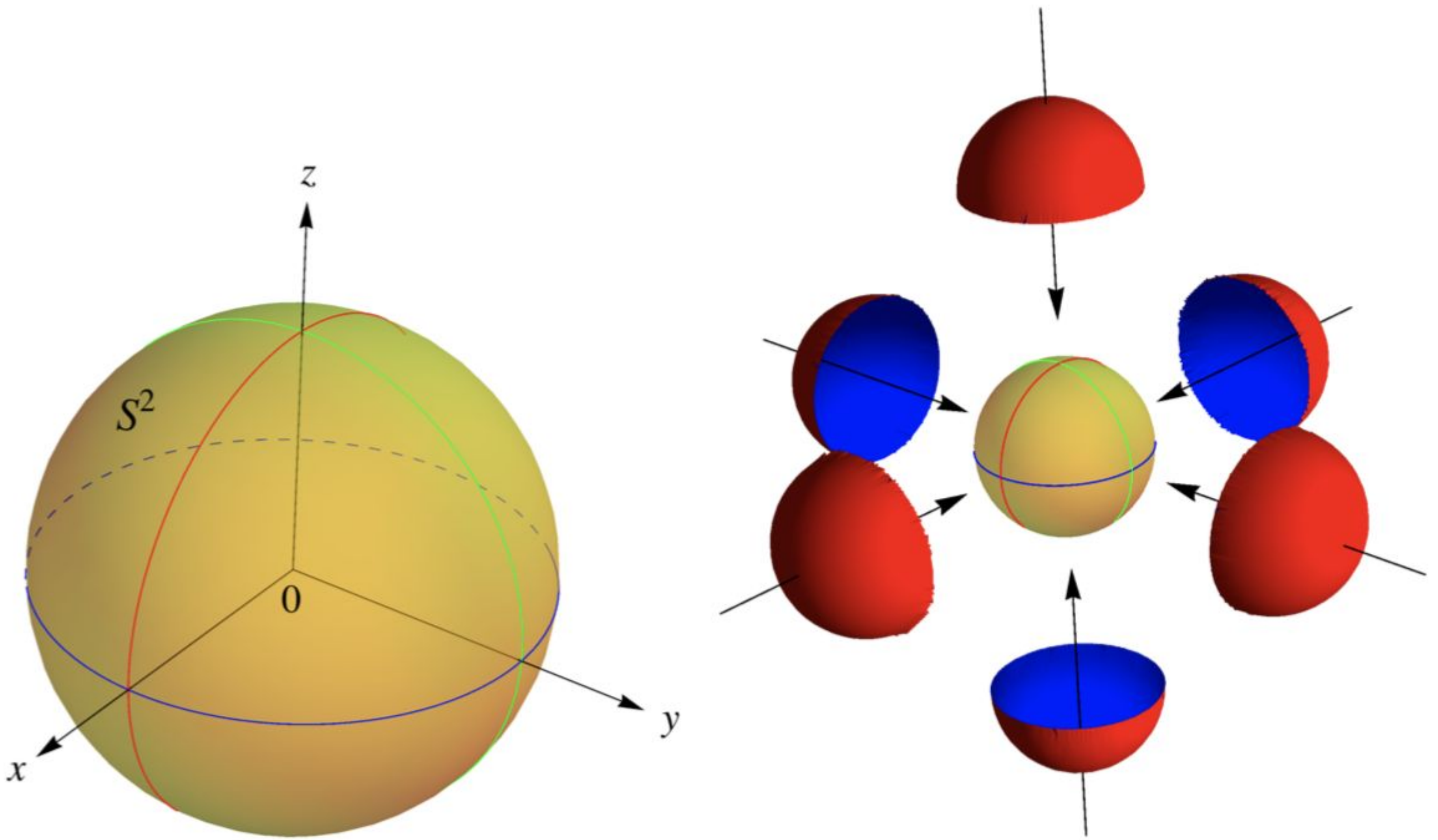
We shall now cover the whole sphere with similar parametrizations as follows. we define $\mathbf{x}_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{x}_2(x, y) = (x, y, -\sqrt{1 - (x^2 + y^2)}),$$

check that $\mathbf{x}_2$ is a parametrization, and observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$. Then, using the xz and zy planes, we define the parametrization

$$\begin{aligned}
\mathbf{x}_3(x, z) &= (x, +\sqrt{1 - (x^2 + z^2)}, z), \\
\mathbf{x}_4(x, z) &= (x, -\sqrt{1 - (x^2 + z^2)}, z), \\
\mathbf{x}_5(y, z) &= (+\sqrt{1 - (y^2 + z^2)}), y, z), \\
\mathbf{x}_6(y, z) &= (-\sqrt{1 - (y^2 + z^2)}), y, z),
\end{aligned}$$

which, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover S^2 completely and shows that S^2 is a regular surface.


z
S^2
0
x
y

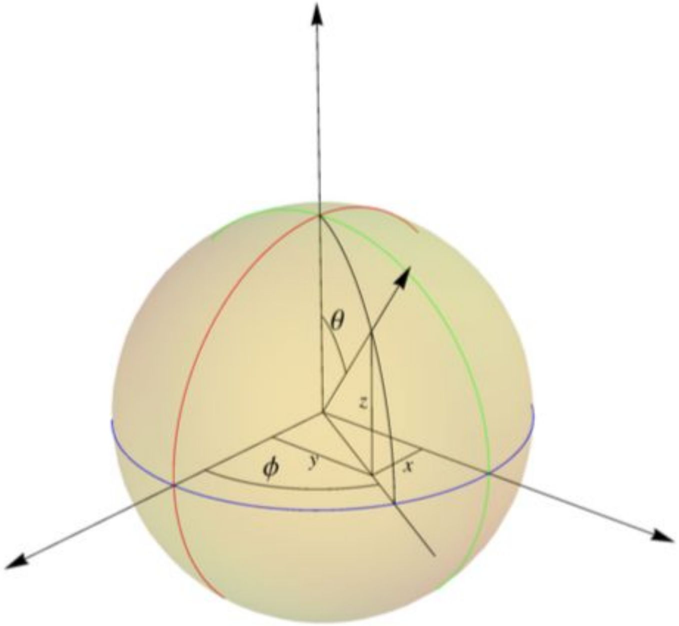
Proving that a Set is a Regular Surface

Method 2: Using Spherical Coordinates

For most applications, it is convenient to relate parametrizations to the geographical coordinates on S^2. Let $V = \{(\theta, \varphi) \mid 0 < \theta < \pi, 0 < \varphi < 2\pi\}$ and let $\mathbf{x} : V \to \mathbb{R}^3$ be given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Clearly, $\mathbf{x}(V) \subset S^2$.



Proving that a Set is a Regular Surface

We shall prove that $\mathbf{x}$ is a parametrization of S^2.

Next, we observe that given $(x, y, z) \in S^2 \setminus C$, where C is the semicircle $C = \{(x, y, z) \in S^2 \mid y = 0, x \geq 0\}$, θ is uniquely determined by $\theta = \cos^{-1} z$, since $0 < \theta < \pi$. By knowing θ, we find $\sin\varphi$ and $\cos\varphi$ from $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and this determines φ uniquely $(0 < \varphi < 2\pi)$. It follows that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}$. To complete the verification of condition 2, we should prove that $\mathbf{x}^{-1}$ is continuous. However, since we shall soon prove that this verification is not necessary provided we already know that the set S is a regular surface, we shall not do that here.

We remark that $\mathbf{x}(V)$ only omits a semicircle of S^2 (including the two poles) and that S^2 can be covered with the coordinate neighborhoods of two parametrizations of this type.

HW1: Show that a sphere is a regular surface using spherical coordinates.

- Reference: Differential geometry of curves and surfaces, by do Carmo.