Lecture 5

Math 178

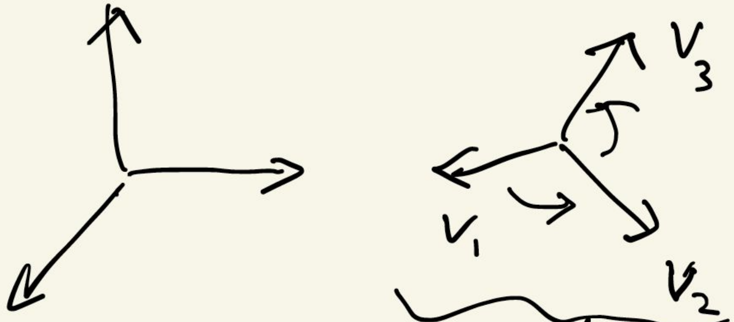
Nonlinear Data Analytics

Prof. Weiqing Gu

What is SO(3)?

- The set of rotation matrices in R^3 is denoted by SO(3).

$$SO(3) = \left\{ A = (\vec{v}_1, \vec{v}_2, \vec{v}_3) \;\middle|\; \begin{array}{c} A^T A = A A^T = I \\ \det A = 1 \end{array} \right\}$$



$$\left\{ \begin{array}{l} \|V_i\| = 1, \quad i = 1,2,3 \\ V_i \cdot V_j = \delta_{ij} \end{array} \right.$$

right hand sided $\Rightarrow \det A = 1$

$$\cancelto{0}{\vec{v}_1 \cdot \vec{v}_2}$$

$$\underbrace{A^T A}_{} = \underbrace{\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix}}_{A^T} \underbrace{[\vec{v}_1, \vec{v}_2, \vec{v}_3]}_{A} = \underbrace{\begin{bmatrix} \cancelto{1}{\vec{v}_1^T \vec{v}_1} & \cancelto{0}{\vec{v}_1^T \vec{v}_2} & \cancelto{0}{\vec{v}_1^T \vec{v}_3} \\ \cancelto{0}{\vec{v}_2^T \vec{v}_1} & \cancelto{1}{\vec{v}_2^T \vec{v}_2} & \cancelto{0}{\vec{v}_2^T \vec{v}_3} \\ \cancelto{0}{\vec{v}_3^T \vec{v}_1} & \cancelto{0}{\vec{v}_3^T \vec{v}_2} & \cancelto{1}{\vec{v}_3^T \vec{v}_3} \end{bmatrix}}_{I}$$

$$(a_1, b_1, c_1) \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = (a_1, b_1, c_1) \cdot (a_2, b_2, c_2)$$

Show SO(3) is a manifold.

- Work out details with the students on the iPad.

<u>Claim</u> : $SO(3) = S^3 / \{\pm 1\}$

<u>Pf</u> : Let $q \in \mathbb{H}$ w/ $\|q\| = 1$.

quaternion number

For each q, I am going to define a map, call R_q.

$$R_q : \text{Im}\,\mathbb{H} \to \text{Im}\,\mathbb{H}$$

R_q can be viewed from $\underbrace{\text{Im}\,\mathbb{H}}_{} \to \underbrace{\text{Im}\,\mathbb{H}}_{}$

$R_q : \mathbb{R}^3 \to \mathbb{R}^3$

by first define $R_q : \mathbb{H} \to \mathbb{H}$

$$x \longmapsto q x q^*$$

Recall : $q^* = a - bi - cj - dk$

$q = a + bi + cj + dk$

Note : R_q fixes all real numbers r.

$$\boxed{\begin{array}{l} r \longmapsto \underbrace{q r q^*} \\ \because r \in \mathbb{R} \quad \underbrace{r q q^*} \\ \Downarrow \qquad \underbrace{r\|q\|^2}_{1} \\ R_q(r) = r \end{array}}$$

Claim: R_q is a rotation in $\mathbb{R}^3$.

That is to say: we need to show

$R_q \in SO(3)$
1) R is Linear map $: \mathbb{R}^3 \to \mathbb{R}^3$
2) R keeps length.

} will imply that R is an orthogonal map.

3) $\det R = 1$.

To show i), let $R_q(ax+by) = q(ax+by)q^*$ $\quad a, b \in \mathbb{R}$

$$= qaxq^* + qbyq^*$$

$$= a\,qxq^* + b\,qyq^* = a R_q(x) + b R_q(y)$$

$\Rightarrow R$ is linear ✓

To show 2) $\|R_q(x)\| = \|qxq^*\| = \underbrace{\cancel{\|q\|}}_{1}\,\|x\|\,\underbrace{\cancel{\|q^*\|}}_{1}$

$$= \|x\|.$$

Note:

The set of orthonormal matrices have two components.

$$\left\{A\in M_3(\mathbb{R}) \;\middle|\; \underbrace{A^TA}_{\overset{\shortparallel}{AA^T}}=I\right\} = O(3)$$

If

$A^TA=I$, then determinant both sides

$$\det(A^TA)=\det I$$

$$\Rightarrow \underbrace{\det A^T}_{(\det A)}\det A = 1$$

$$\Rightarrow (\det A)^2 = 1$$

$$\Rightarrow \det A = 1$$

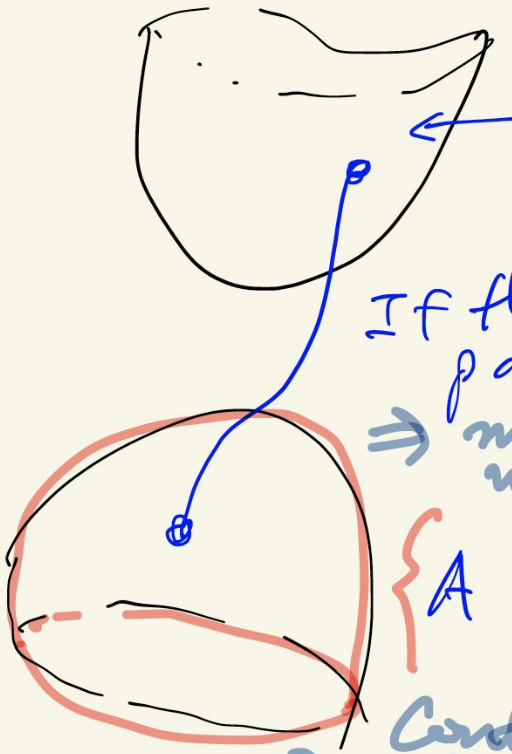
or $\det A = -1$

Can use analysis to prove it.

But det is a continuous fcn

$\Rightarrow O(3)$ must be not (Path) connected.

$\{A \mid \det A = 1\}$



If there were path connected

$\Rightarrow$ must $\exists$ some matrix B s.t. $\det B = 0$

$\{A \mid \det A = -1\}$

Contradiction

$\Rightarrow O(3)$ must disconnected.

Note: unit quaternion q and –q correspond to the same rotation.

Note:

$$R_q = R_{-q}$$

$\forall x \in \mathbb{H}$

$R_q(x) = q x q^*$, q^* is a conjugate of q.

$$R_{-q}(x) = (-q)\, x\, (-q)^* = q x q^*$$

$$\Rightarrow R_q = R_{-q}.$$

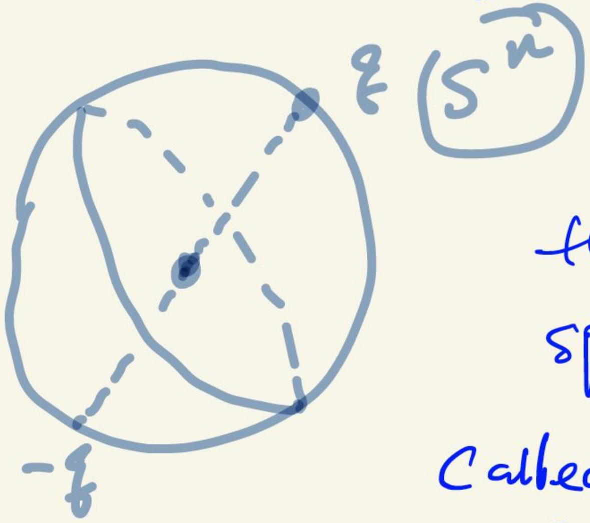
$q = \underline{a} + bi + cj + dk$

$\downarrow$

1

$\Rightarrow R_q \in SO(3).$

From manifold point view, SO(3) is the manifold RP^3



Identify ξ w/ $-\xi$ on S^n, the result space is called a real projective space, denoted by $\mathbb{RP}^n$.

$$S^n/\{\pm 1\} \cong \mathbb{RP}^n$$

$$SO(3) = S^3/\{\pm 1\} = \mathbb{RP}^3$$

Thm: Any Projective space $\mathbb{R}P^n$ is a manifold of dim n.

So $SO(3) = \mathbb{R}P^3$ is a <u>mfld</u>.

Why? locally it is just a piece of S^3 after identification under q w/ $-q$; any piece of S^3 locally "looks like" $\mathbb{R}^3$.
$\Rightarrow$ any piece of S^3 locally "looks like" $\mathbb{R}^3$.

Now Show SO(3) is also a group.

- Work out details with the students on the iPad.

In fact, SO(3) is also a group.

Now we show that SO(3) is also a gp. (HW).

Hint:

i) $\forall A, B \in SO(3)$

W.T.S. $AB \in SO(3)$.

Why: $(AB)(AB)^T$

$= A\underbrace{\cancel{BB^T}}_{I}A^T$

$= AA^T = I$ ✓

$\underbrace{\det(AB)}_{\det A \cdot \det B}=1$ ✓

$\det A = 1$, $\det B = 1$

$I \in SO(3)$? ✓

$A \in SO(3)$

$A^{-1} \overset{?}{\in} SO(3)$?

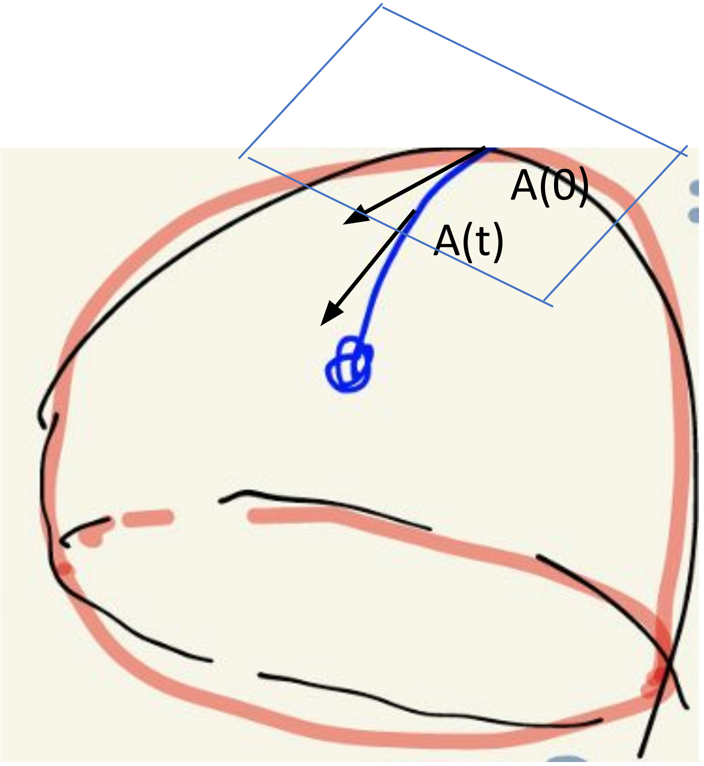
SO(3) has both manifold and group structures! Moreover those two structures are compatible! In such a case we say SO(3) is a Lie Group!

Now SO(3) is both a $\underbrace{\text{manifold}}_{\text{structure}}$, also a $\underbrace{\text{gp.}}_{\text{group structure}}$

We must make sure the two structures being compatible.

$$\left.\begin{array}{l} (AB)' \\ (A^{-1})' \end{array}\right\} \text{both exist.}$$

Find Lie algebra of the Lie group SO(3)



SO(3)

Dim (SO(3)) = 3
Since the dimension
of the tangent plane is 3.

Say take a curve $A(t)$ in SO(3) with $A(0) = I_{3x3}$
Want to find $A'(0)$.

We know that $A(t)A^T(t) = I$
Take the derivative both sides of the above equation.

$$A'(t)\, A^T(t) + A(t)(A^T(t))' = 0$$

Evaluate the above at the identity matrix I:

$$A'(0)\, [A(0)]^T + A(0)\, [A'(0)]^T = 0$$

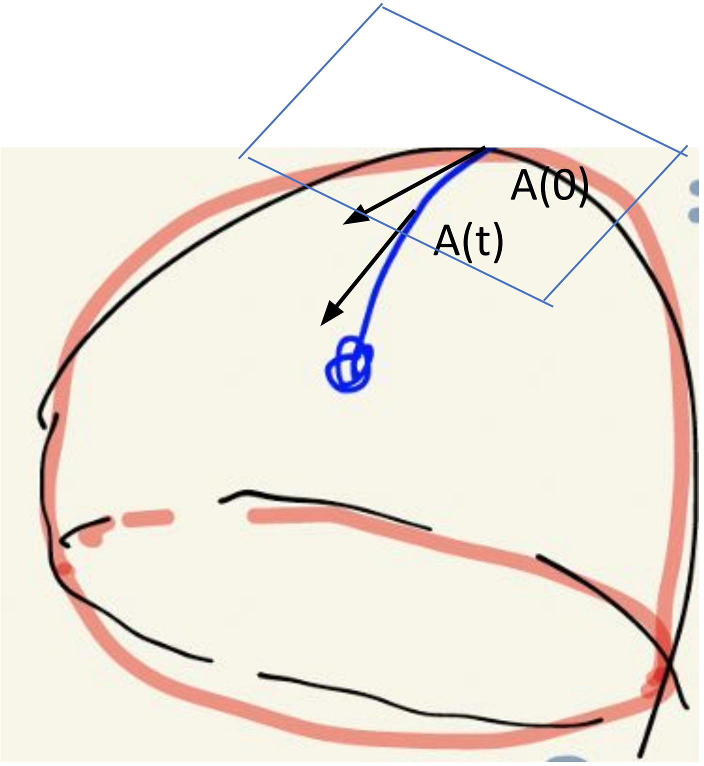
$$A'(0) + [A'(0)]^T = 0$$

$$[A'(0)]^T = -A'(0) \quad I$$

That is to say that $A'(0)$ is a skew symmetric matrix.

So, the tangent plane at I is the set of 3x3 skew symmetric matrices, which is 3 dimensional.
We define this tangent plane at the identy of SO(3) as **the Lie algebra** of the Lie group SO(3) with the Algebra structure defined as $[A, B] = AB-BA$.
Usually, denoted by so(3).

Find Lie algebra of the Lie group SO(n)



SO(n)

Say take a curve $A(t)$ in SO(n) with $A(0) = I_{nxn}$
Want to find $A'(0)$.

We know that $A(t)A^T(t) = I$
Take the derivative both sides of the above equation.

$$A'(t)\, A^T(t) + A(t)(A^T(t))' = 0$$

Evaluate the above at the identity matrix I:

$$A'(0)\, [\overset{I}{\cancel{A(0)}}]^T + \overset{I}{\cancel{A(0)}}\, [A'(0)]^T = 0$$

$$A'(0) + [A'(0)]^T = 0$$

$$[A'(0)]^T = -A'(0)$$

That is to say that $A'(0)$ is a skew symmetric matrix.

So, the tangent plane at I is the set of nxn skew symmetric matrices, which is n dimensional.
We define this tangent plane at the identy of SO(n) as **the Lie algebra** of the Lie group SO(n) with the Algebra structure defined as $[A, B] = AB-BA$.
Usually, denoted by so(n).

Dim (SO(n)) = n(n-1)/2
Since the dimension
of the tangent plane is (n-1) + (n-2) +…+1 = n(n-1)/2.
Check Dim (SO(3)) = 3(3-1)/2 = 2
Dim (SO(2)) = 2(2-1)/2 = 1

- Now let's define an exponential map:
- Given M, define $e^M = I + M + M^2/2+\ldots.$
- Now let M in the Lie algebra so(2). That is M is a skew symmetric 2x2 matrix as blow and try to find e^M

0 t

-t 0

- You will notice that you get rotation matrix with an element at (1, 1) position as
- 1-t^2/2 +t^4/4! - … = Cos t

Use Quaternions as Rotations in R^3 and Finding a Tangent Plane of SO(3)

Recall: We try to use a unit quaternion to define a rotation in $\mathbf{R}^3$

1. We first defined a map:

$R_q(p) = qpq^{-1}$

Note: For unit quaternion q,
Inverse (q) = conjugate (q).
Why?

2. We need to show R_q is an orthogonal map:

- R_q keeps the length.
- R_q is linear
- How to make R_q from $\mathbf{R}^3$ to $\mathbf{R}^3$?
- How to show R_q is in SO(3) not just in O(3)? (Is R_q continuous?)

Q: What else we must be shown? Why can't we define a rotation as $R_q(p) = qp$?

In other words…

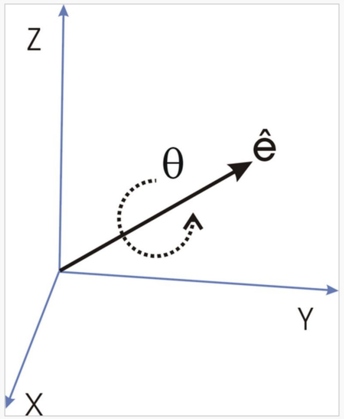
It can be shown that the desired rotation can be applied to an ordinary vector $\mathbf{p} = (p_x, p_y, p_z) = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}$ in 3-dimensional space, considered as a quaternion with a real coordinate equal to zero, by evaluating the conjugation of $\mathbf{p}$ by $\mathbf{q}$:

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$

using the Hamilton product, where $\mathbf{p}' = (p_x{}', p_y{}', p_z{}')$ is the new position vector of the point after the rotation. In a programmatic implementation, this is achieved by constructing a quaternion whose vector part is p and real part equals zero and then performing the quaternion multiplication. The vector part of the resulting quaternion is the desired vector p'.

Euler's Rotation Theorem

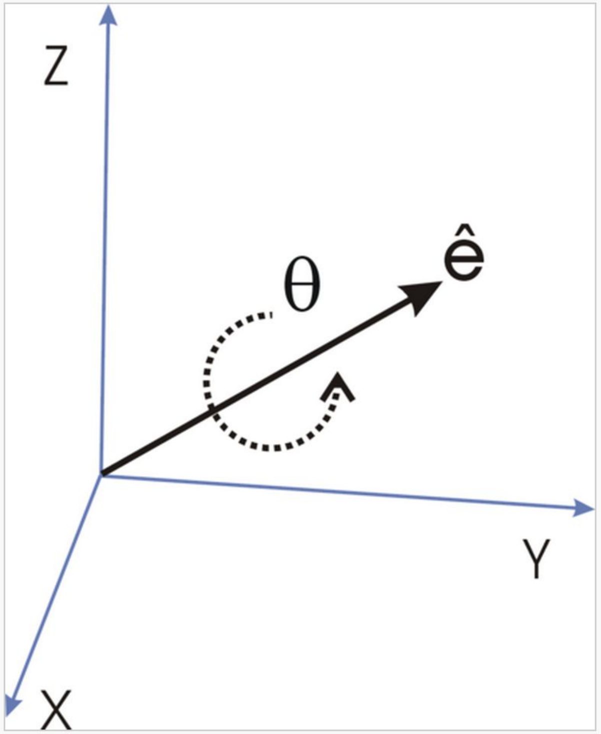
In geometry, **Euler's rotation theorem** states that, in three-dimensional space, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point. It also means that the composition of two rotations is also a rotation. Therefore the set of rotations has a structure known as a *rotation group*.



A visualization of a rotation represented by an Euler axis and angle.

How to find Euler axis?

- What is the dimension of this rotation matrix?
- Does it have a real eigenvalue?
- Is there any real eigenvector?



A visualization of a rotation represented by an Euler axis and angle.

In this topic, we always need to identify H with R^4

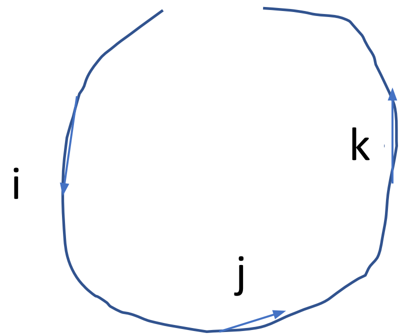
But more, now vectors in R^4 can multiply and get another vector!

Using the basis 1, *i*, *j*, *k* of **H** makes it possible to write **H** as a set of quadruples:

$$\mathbf{H} = \{(a, b, c, d) \mid a, b, c, d \in \mathbf{R}\}.$$

Then the basis elements are:

$$1 = (1, 0, 0, 0),$$
$$i = (0, 1, 0, 0),$$
$$j = (0, 0, 1, 0),$$
$$k = (0, 0, 0, 1),$$



$$\begin{aligned}(a_1,\ b_1,\ c_1,\ d_1)(a_2,\ b_2,\ c_2,\ d_2) &= \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2, \\ &\quad a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2, \\ &\quad a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2, \\ &\quad a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2).\end{aligned}$$

Quaternion multiplication in R^4 is related to dot product and cross product in R^3

- **Scalar and vector parts:**

  A number of the form a + 0i + 0j + 0k, where a is a real number, is called ***real***, and a number of the form 0 + bi + cj + dk, where b, c, and d are real numbers, and at least one of b, c or d is nonzero, is called pure ***imaginary***. If a + bi + cj + dk is any quaternion, then **a** is called its ***scalar part*** and ***bi + cj + dk*** is called its vector part.

$$q = (r,\ \vec{v}),\ q \in \mathbf{H},\ r \in \mathbf{R},\ \vec{v} \in \mathbf{R}^3$$

then the formulas for addition and multiplication are:

$$(r_1,\ \vec{v}_1) + (r_2,\ \vec{v}_2) = (r_1 + r_2,\ \vec{v}_1 + \vec{v}_2)$$
$$(r_1,\ \vec{v}_1)(r_2,\ \vec{v}_2) = (r_1 r_2 - \vec{v}_1 \cdot \vec{v}_2, r_1 \vec{v}_2 + r_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$$

Exactly How quaternion multiplications related to dot and cross product in R^3?

$$
\begin{aligned}
(a_1,\ b_1,\ c_1,\ d_1)(a_2,\ b_2,\ c_2,\ d_2) = \\
= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2, \\
a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2, \\
a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2, \\
a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2).
\end{aligned}
$$

$$(r_1,\ \vec{v}_1)(r_2,\ \vec{v}_2) = (r_1 r_2 - \vec{v}_1 \cdot \vec{v}_2, r_1 \vec{v}_2 + r_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$$

Extend 2D rotation using a unit complex number to 3D rotation using a unit quaternion

How?

A rotation through an angle of θ around the axis defined by a unit vector in R^3

$$\vec{u} = (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

can be represented by a unit quaternion. This can be done using an extension of Euler's formula:

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

Matrix Representation of Rotation using Unit Quaternion q

Alternatively, the rotation matrix can be expressed as

$$\begin{bmatrix} 1-2q_j^2-2q_k^2 & 2(q_iq_j-q_kq_r) & 2(q_iq_k+q_jq_r) \\ 2(q_iq_j+q_kq_r) & 1-2q_i^2-2q_k^2 & 2(q_jq_k-q_iq_r) \\ 2(q_iq_k-q_jq_r) & 2(q_jq_k+q_iq_r) & 1-2q_i^2-2q_j^2 \end{bmatrix}$$

How to find this matrix representation?

Key: Look at where the base vectors 1, i, j, and k go under the R_q

Proof of the quaternion rotation identity

Let $\vec{u}$ be a unit vector (the rotation axis) and let $q = \cos\frac{\alpha}{2} + \vec{u}\sin\frac{\alpha}{2}$. Our goal is to show that

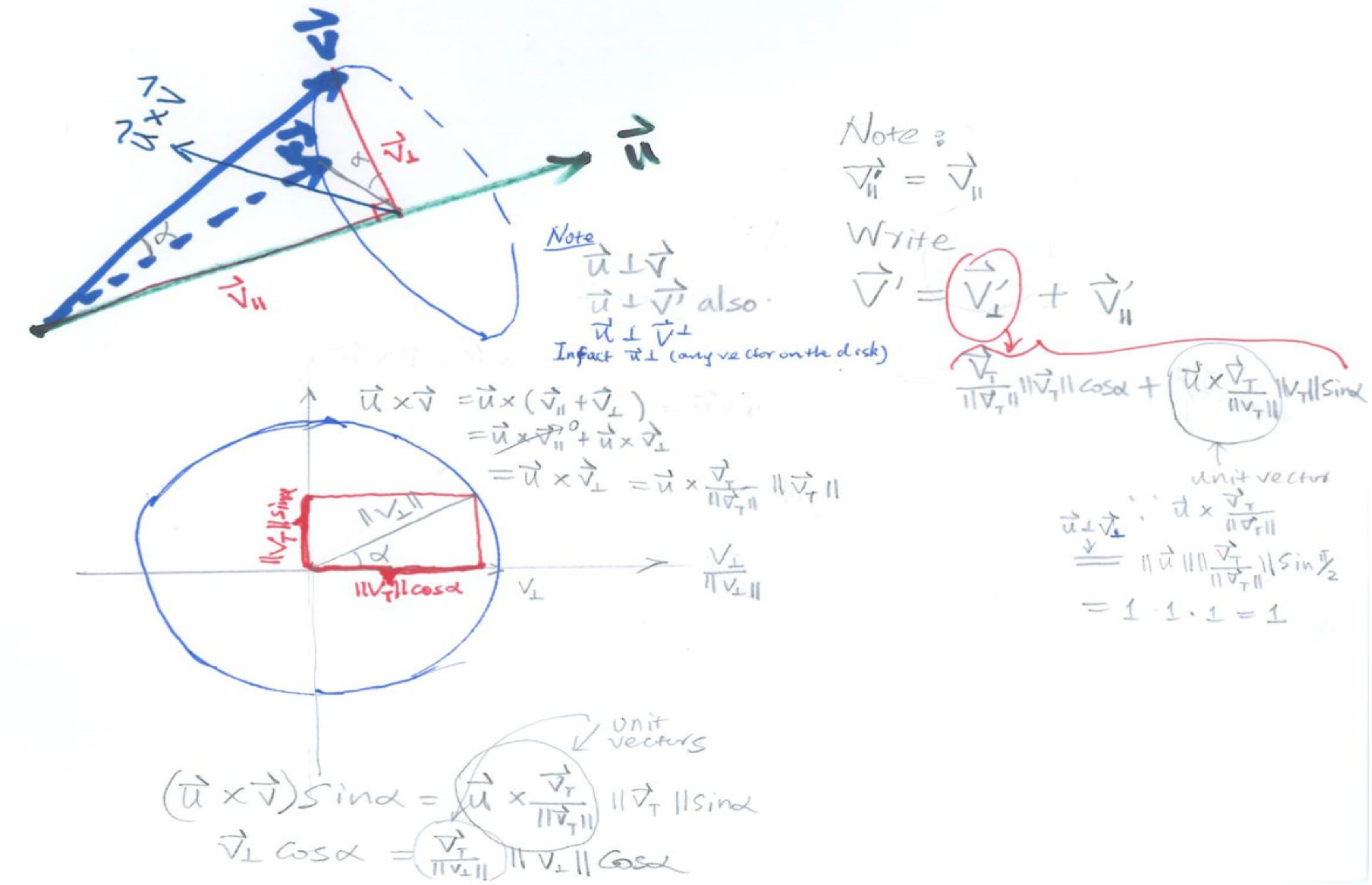
$$\vec{v'} = q\vec{v}q^{-1} = \left(\cos\frac{\alpha}{2} + \vec{u}\sin\frac{\alpha}{2}\right)\vec{v}\left(\cos\frac{\alpha}{2} - \vec{u}\sin\frac{\alpha}{2}\right)$$

yields the vector $\vec{v}$ rotated by an angle α around the axis $\vec{u}$. Expanding out, we have

$$\begin{aligned}
\vec{v'} &= \vec{v}\cos^2\frac{\alpha}{2} + (\vec{u}\vec{v} - \vec{v}\vec{u})\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \vec{u}\vec{v}\vec{u}\sin^2\frac{\alpha}{2} \\
&= \vec{v}\cos^2\frac{\alpha}{2} + 2(\vec{u}\times\vec{v})\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - (\vec{v}(\vec{u}\cdot\vec{u}) - 2\vec{u}(\vec{u}\cdot\vec{v}))\sin^2\frac{\alpha}{2} \\
&= \vec{v}(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}) + (\vec{u}\times\vec{v})(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}) + \vec{u}(\vec{u}\cdot\vec{v})(2\sin^2\frac{\alpha}{2}) \\
&= \vec{v}\cos\alpha + (\vec{u}\times\vec{v})\sin\alpha + \vec{u}(\vec{u}\cdot\vec{v})(1-\cos\alpha) \\
&= (\vec{v} - \vec{u}(\vec{u}\cdot\vec{v}))\cos\alpha + (\vec{u}\times\vec{v})\sin\alpha + \vec{u}(\vec{u}\cdot\vec{v}) \\
&= \vec{v}_{\perp}\cos\alpha + (\vec{u}\times\vec{v})\sin\alpha + \vec{v}_{\|}
\end{aligned}$$

where $\vec{v}_{\perp}$ and $\vec{v}_{\|}$ are the components of $\vec{v}$ perpendicular and parallel to $\vec{u}$ respectively. This is the formula of a rotation by α around the $\vec{u}$ axis.

Work out details on the board.



Local Canonical Form of Curves

- Using moving frame and view the curve in that frame locally.
- Using Taylor expansion (Note: The Reminder R is a vector and $limR/s^3=0$ as s□0.
- Plug Frenet formulas in.
- Reconcile
- Get Local Canonical Form of Curves.

See page 27, Do Carmo.

Think: Can we multiply two vectors in $\mathbf{R}^n$ for any n?

- Yes, in $\mathbf{R}^1$
- Yes, in $\mathbf{R}^2$ (using complex numbers)
- Yes, in $\mathbf{R}^3$ (using cross product)
- Yes, in $\mathbf{R}^4$ (using Quaternion numbers)

- Yes in $\mathbf{R}^8$ (using Octonion (i.e. Cayley) numbers)

Hint: please look at normed division algebra online.

We will give a rigorous definition of a Lie group in our next lecture.

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How to prove that the manifold assumption is correct? - Cross ...

Sep 12, 2014 — The more careful ones define it with a subtle but hugely important caveat: that the **data lie** on or close to a low-dimensional **manifold**. Even those ...

2 answers · 10 votes: It quickly becomes apparent, by looking at many accounts of the "manifo...

Computing low-dimensional representations of speech from socio-auditory structures for phonetic analyses

Andrew R. Plummer[a,*] and Patrick F. Reidy[b]

Abstract

Low-dimensional representations of speech data, such as formant values extracted by linear predictive coding analysis or spectral moments computed from whole spectra viewed as probability distributions, have been instrumental in both phonetic and phonological analyses over the last few decades. In this paper, we present a framework for computing low-dimensional representations of speech data based on two assumptions: that speech data represented in high-dimensional data spaces lie on shapes called *manifolds* that can be used to map speech data to low-dimensional coordinate spaces, and that manifolds underlying speech data are generated from a combination of language-specific lexical, phonological, and phonetic information as well as culture-specific socio-indexical information that is expressed by talkers of a given speech community. We demonstrate the basic mechanics of the framework by carrying out an analysis of children's productions of sibilant fricatives relative to those of adults in their speech community using the phoneigen package – a publicly available implementation of the framework. We focus the demonstration on enumerating the steps for constructing manifolds from data and then using them to map the data to a low-dimensional space, explicating how manifold structure affects the learned low-dimensional representations, and comparing the use of these representations against standard acoustic features in a phonetic analysis. We conclude with a discussion of the framework's underlying assumptions, its broader modeling potential, and its position relative to recent advances in the field of representation learning.

Keywords: manifold alignment, Laplacian Eigenmaps, socio-indexical, phonetic categories, low-dimensional representations of speech

https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6706093/