Dot products ①

$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

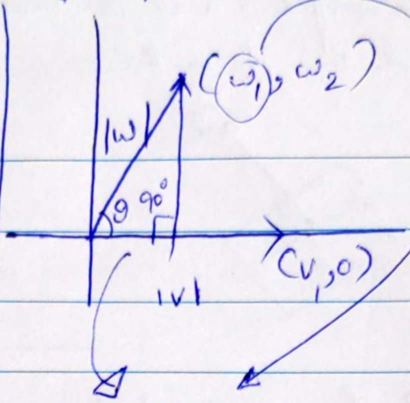
$\vec{v} \cdot \vec{w} = \vec{v}^T \cdot \vec{w}$

$= (v_1 \dots v_n)\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$= v_1 w_1 + \dots + v_n w_n$

$= \sum_{i=1}^{n} v_i w_i$

$\vec{v} = (v_1, 0)$ $\vec{w} = (w_1, w_2)$ ②



$\|\vec{w}\|\cos\theta = w_1$

$\theta = \arccos\left(\frac{w_1}{\|\vec{w}\|}\right)$

③

now,

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$

As per the example above $\vec{v} = (v_1, 0)$
$\vec{w} = (w_1, w_2)$

$\vec{v} \cdot \vec{w} = v_1 w_1 + 0 \cdot w_2$

$= v_1 w_1$

$= v_1 (\|\vec{w}\|\cos\theta)$

$\simeq \|\vec{v}\| \cdot \|\vec{w}\| \cos\theta$

$\Rightarrow \theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$

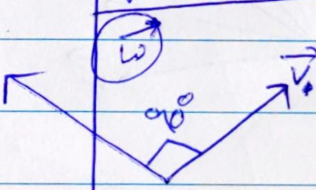
Dot product is same as the product of the norms & cosine of angle between them.

④

Orthogonality

$\vec{v} \cdot \vec{w} = 0$

given $\vec{v}$, what are the vectors such that $\vec{v} \cdot \vec{w} = 0$



we have derived from ③ that $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$.

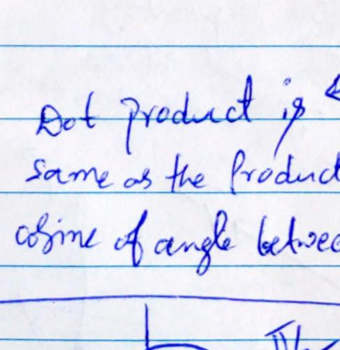
Let's Assume that $\|\vec{v}\|$ & $\|\vec{w}\| \neq 0$

then $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$

$\because \|\vec{v}\|, \|\vec{w}\| \neq 0$

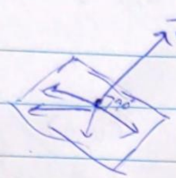
$\cos\theta = 0$

$\Rightarrow \theta = -90°$ (or) $+90°$



we say $\vec{w}$ is orthogonal to $\vec{v}$

4 (b) In 3rd Dimen space, a $\vec{w}$ is said to be orthogonal then it is



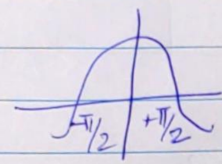
5 $\vec{v} \cdot \vec{w} > 0$

What are $\vec{w}$ such that $\vec{v} \cdot \vec{w} > 0$

$\|\vec{v}\| \|\vec{w}\| \cos\theta > 0$

$\vec{v}, \vec{w}$ are non-zero vectors

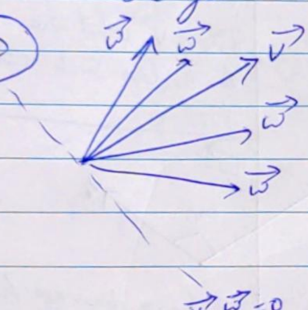
$\cos\theta > 0$



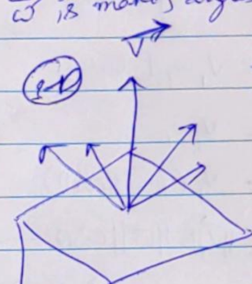
$\theta \in (-\pi/2, +\pi/2)$

If we now see geometrically, $\vec{w}$ is making angles less than 90°

2-D



3-D



6 $\vec{v} \cdot \vec{w} < 0$

$\|\vec{v}\| \|\vec{w}\| \cos\theta < 0$

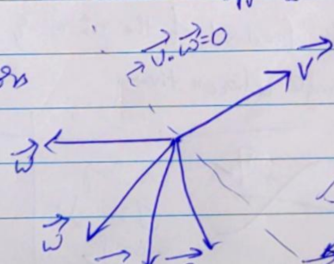
$\because \vec{v}, \vec{w}$ are non-zero vectors

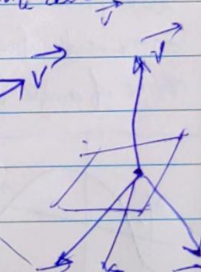
$\cos\theta < 0$

$|\theta| > \pi/2$

2D

Here $\vec{w}$ pointing somewhat in the opposite direction to $\vec{v}$





Dot products are essentially giving us how these vectors are oriented in a given space. If is

+ve, they are pointing in the same direction

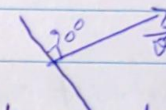
zero, they are in right angles to each other

-ve, they are pointing in the opposite direction.

7 Hyperplane Definition

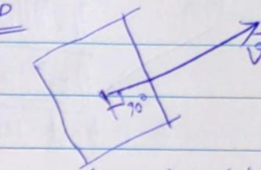
(a) It is the thing orthogonal to a given vector

In 2D



line orthogonal to the vector $\vec{v}$

In 3D



plane orthogonal to the vector $\vec{v}$

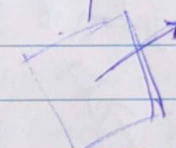
In Higher: Similar

In 4-D's, it is 3D-space orthogonal to the vector given

In 5-D's, it is 4D-space orthogonal to the vector given

All the hyper-planes pass through the point (0,0,0). wherever zero vector in that specific Dimension

(b) A Translate to a different point.



it can be a plane orthogonal to the vector not passing through the origin.

$\vec{0}$

Hyperplane is a (Sub-Space) of a higher dimen space that separates into 2 equal parts

(8) Decision plane

whether (or) not a feature passes a certain threshold

$\vec{w} = (w_1 \dots w_n)$

$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

we ask,
does this weight vector apply to the feature vector & exceeds some threshold c

$\vec{v} \cdot \vec{w} > c$

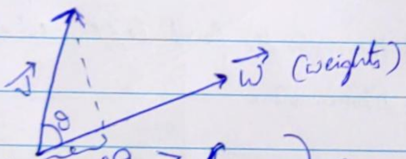
geometry

$\|\vec{v}\| \|\vec{w}\| \cos\theta > c$

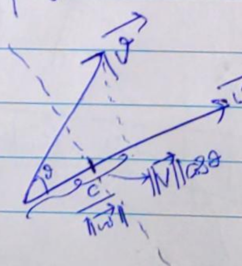
Since $\boxed{\|\vec{w}\| \text{ is fixed}}$, some weight vector we have been given to extract the feature.

Let us convert this $\vec{v} \cdot \vec{w} > c$ in terms of $\vec{w}$

$$\|\vec{v}\| \cos\theta > \frac{c}{\|\vec{w}\|}$$



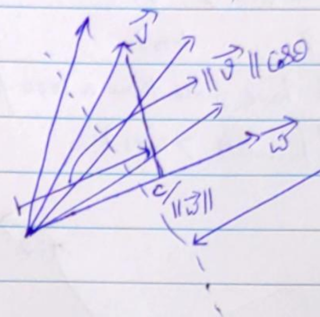
$\left(\|\vec{v}\| \cos\theta > \frac{C}{\|\vec{w}\|} \right)$ so, what we are asking is, is the length of the projection of $\vec{v}$ over $\vec{w}$ is greater than $c/\|\vec{w}\|$.



Let us say, we have point $\frac{c}{\|\vec{w}\|}$

if $\underbrace{\|\vec{v}\| \cos\theta}_{}$ projection of $\vec{v}$ $> \frac{C}{\|\vec{w}\|}$ then $\underline{\vec{v} \cdot \vec{w} > C}$

So, any vector $\vec{v}$ like this on the right hand side of the plane will have



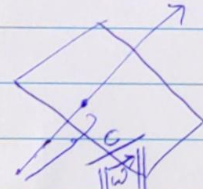
$\|\vec{v}\| \cos\theta > \frac{C}{\|\vec{w}\|}$

which implies
$\vec{w} \cdot \vec{v} > c$

this line $\frac{C}{\|\vec{w}\|}$ separates one from the other, & that is the decision plane.

So if you are towards the left of the line then $\vec{v} \cdot \vec{w} < C$

If we extend to higher dimensions, the decision plane lies orthogonal to the $\vec{w}$.



that is how we think about decision planes, they are ways of separating one side from another in a yes/no question.

* * *

Example:- If you are doing some fancy image analysis tasks, it might be the case that we want to find the decision plane that puts cats representing the vectors on one side (vs) the pictures of dogs representing the vectors on the other side.

In the housing problem from before

$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_{1000} \end{pmatrix}$ v_i = Building having i family homes

we saw that no. of families is $\vec{v} \cdot \vec{w}$

$\therefore \vec{w} = (1, 2, \ldots 1000)$

if we want to ask, can Seattle have more than a 1000 families

it is equivalent in asking if (# of families > 1000)

$$\vec{v} \cdot \vec{w} > 1000$$

nothing but $\vec{v} \cos\theta > \dfrac{1000}{\|\vec{w}\|}$

nothing but we look at the orthogonal decision plane that this problem defines.