HOUND: High-Order Universal Numerical Differentiator for a Parameter-free Polynomial Online Approximation

Igor Katrichek

International Conference dedicated to Mathematics in Artificial Intelligence
MathAI 2025

Contents

1. From cumulative smoothing to differential equation
2. Cumulative Smoothing of the first derivative (equation)
3. Cumulative Smoothing of the first derivative (solution)
4. Cumulative Smoothing of the second derivative
5. High-Order Universal Numerical Differentiator (HOUND)
6. Solution of HOUND equations
7. Stochastic HOUND equations
8. Discretization of HOUND equations
9. High-Order Cumulative Smoothing
10. Polynomial approximation
11. Demo example (noised polynomial): HOUND
12. Demo example (noised polynomial): another methods
13. Demo example (noised polynomial): approximation
14. Demo example (noised polynomial): error analysis
15. Conclusion

1. From cumulative smoothing to differential equation

Series of values: $f[1], \quad f[2], \quad f[3], \quad \ldots, \quad f[t]$

Cumulative average: z_0

$$z_0[1] = f[1],\ z_0[2] = \frac{f[1]+f[2]}{2} = \frac{\frac{f[1]}{1}\cdot 1 + f[2]}{2} = \frac{z_0[1]\cdot 1 + f[2]}{2}$$

$$z_0[3] = \frac{f[1]+f[2]+f[3]}{3} = \frac{\frac{f[1]+f[2]}{2}\cdot 2 + f[3]}{3} = \frac{z_0[2]\cdot 2 + f[3]}{3}$$

$$z_0[t] = \frac{z_0[t-1]\cdot(t-1) + f[t]}{t} \qquad t = 1,2,3,\ldots \quad \Delta t = 1$$

$$z_0[t] - z_0[t-1] = -\frac{1}{t}\left(z_0[t-1] - f[t]\right)$$

$$\Longrightarrow \quad \frac{dz_0(t)}{dt} = -\frac{1}{t}\left(z_0(t) - f(t)\right) \text{ where } t > 0$$

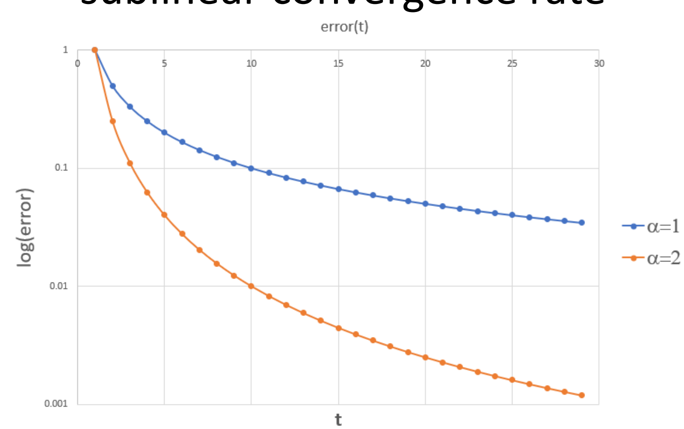
if $f(t) = K_0 = const$

then $z_0(t) = K_0 + \frac{c_1}{t} = f + small\_error$

when $\frac{dz_0(t)}{dt} = -\frac{\alpha}{t}\left(z_0(t) - f(t)\right)$

then $z_0(t) = K_0 + \frac{c_1}{t^{\alpha}} = f + smaller\_error$

sublinear convergence rate



2. Cumulative Smoothing of the first derivative (equation)

Linear trend: $f(t) = K_0 + K_1 \cdot t$

Evaluate of signal and derivative: $z_0 \approx f, \quad z_1 \approx \dfrac{df}{dt} \approx \dfrac{dz_0}{dt}$

Maybe? $\dfrac{dz_1}{dt} = -\dfrac{1}{t}\left(z_1 - \dfrac{df}{dt}\right)$ but unknown $\dfrac{df}{dt}$

Let's try: $\dfrac{dz_1}{dt} = -\dfrac{1}{t}\left(z_1 - \dfrac{dz_0}{dt}\right) \Longrightarrow \begin{cases} \dfrac{dz_0}{dt} = -\dfrac{1}{t}(z_0 - f) + z_1 \\ \dfrac{dz_1}{dt} = -\dfrac{1}{t^2}(z_0 - f) \end{cases}$

$$z_0(t) = (K_0 + K_1 \cdot t) + \underline{\big(c_1 + c_2(\ln t - 1)\big)} = f + big\_error$$

$$z_1(t) = \qquad K_1 \qquad + \underline{\frac{c_1 + c_2 \ln t}{t}} \qquad = \frac{df}{dt} + small\_error$$

Better:

$$\begin{cases} \dfrac{dz_0}{dt} = -\dfrac{\alpha}{t}(z_0 - f) + z_1 \\ \dfrac{dz_1}{dt} = -\dfrac{\beta}{t^2}(z_0 - f) \end{cases} \equiv \begin{cases} \dfrac{de_0}{dt} = -\dfrac{\alpha}{t} e_0 + e_1 \\ \dfrac{de_1}{dt} = -\dfrac{\beta}{t^2} e_0 - \dfrac{d^2 f}{dt^2} \end{cases} \text{ where } \begin{matrix} e_0 = z_0 - f \\ e_1 = z_1 - \dfrac{df}{dt} \end{matrix}$$

$$t^2 \frac{d^2 e_0}{dt^2} + \alpha \cdot t \frac{de_0}{dt} + (\beta - \alpha) e_0 = -t^2 \frac{d^2 f}{dt^2}$$

Cauchy-Euler ODE

3. Cumulative Smoothing of the first derivative (solution)

Cauchy-Euler ODE: $$t^2 \frac{d^2 e_0}{dt^2} + \alpha \cdot t \frac{de_0}{dt} + (\beta - \alpha) \cdot e_0 = -t^2 \frac{d^2 f}{dt^2}$$

Characteristic polynomial with substitution $u = \ln t$:

$$\lambda(\lambda - 1) + \alpha \cdot \lambda + (\beta - \alpha) = 0 \qquad \lambda^2 + (\alpha - 1) \cdot \lambda + (\beta - \alpha) = 0$$

If both roots are real and different: $e_0(t) = \frac{c_1}{t^{-\lambda_1}} + \frac{c_2}{t^{-\lambda_2}}$ else will be ln,sin,cos

For cumulative smoothing: $\lambda_1 = -1 \Longrightarrow \beta = 2(\alpha - 1) \qquad \lambda_2 = 2 - \alpha$

Let's roots are integers: $\lambda_2 = -2 \Longrightarrow \underline{\alpha = 4, \quad \beta = 6}$

$$\begin{cases} \dfrac{dz_0}{dt} = -\dfrac{4}{t}(z_0 - f) + z_1 \\ \dfrac{dz_1}{dt} = -\dfrac{6}{t^2}(z_0 - f) \end{cases} \Longrightarrow \begin{aligned} z_0(t) &= (K_0 + K_1 \cdot t) + \underline{\left(\frac{c_1}{t} + \frac{c_2}{t^2}\right)} = f + small\_error \\ z_1(t) &= K_1 + \underline{\left(3\frac{c_1}{t^2} + 2\frac{c_2}{t^3}\right)} = \frac{df}{dt} + small\_error \end{aligned}$$

4. Cumulative Smoothing of the second derivative

Parabolic trend: $f(t) = K_0 + K_1 \cdot t + K_2 \cdot t^2$

Evaluate of signal and derivatives: $z_0 \approx f, \quad z_1 \approx \frac{df}{dt}, \quad z_2 \approx \frac{d^2 f}{dt^2} \approx \frac{dz_1}{dt}$

$$\begin{cases} \frac{dz_0}{dt} = -\frac{\alpha}{t}(z_0 - f) + z_1 \\ \frac{dz_1}{dt} = -\frac{\beta}{t^2}(z_0 - f) + z_2 \\ \frac{dz_2}{dt} = -\frac{\gamma}{t^3}(z_0 - f) \end{cases} \equiv \begin{cases} \frac{de_0}{dt} = -\frac{\alpha}{t} e_0 + e_1 \\ \frac{de_1}{dt} = -\frac{\beta}{t^2} e_0 + e_2 \\ \frac{de_2}{dt} = -\frac{\gamma}{t^3} e_0 - \frac{d^3 f}{dt^3} \end{cases} \text{where} \begin{matrix} e_0 = z_0 - f \\ e_1 = z_1 - \frac{df}{dt} \\ e_2 = z_2 - \frac{d^2 f}{dt^2} \end{matrix}$$

$$t^3 \frac{d^3 e_0}{dt^3} + \alpha \cdot t^2 \frac{d^2 e_0}{dt^2} + (\beta - 2\alpha) \cdot t \frac{de_0}{dt} + (2\alpha - 2\beta + \gamma) \cdot e_0 = -t^3 \frac{d^3 f}{dt^3}$$

Cauchy-Euler ODE

Characteristic polynomial:

$$\lambda(\lambda - 1)(\lambda - 2) + \alpha \cdot \lambda(\lambda - 1) + (\beta - 2\alpha)\lambda + (2\alpha - 2\beta + \gamma) = 0$$

$$\lambda^3 + (\alpha - 3)\lambda^2 + (2 - 3\alpha + \beta)\lambda + (2\alpha - 2\beta + \gamma) = 0$$

$\lambda_1 \neq \lambda_2 \neq \lambda_3$
$\lambda_{1,2,3} < 0$
$\mathrm{Im}(\lambda_{1,2,3}) = 0$

Roots of characteristic polynomial:

$$\lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = -3 \implies \alpha = 9, \quad \beta = 36, \quad \gamma = 60$$

$$e_0(t) = \frac{c_1}{t} + \frac{c_2}{t^2} + \frac{c_3}{t^3}, \quad e_1(t) = 8\frac{c_1}{t^2} + 7\frac{c_2}{t^3} + 6\frac{c_3}{t^4}, \quad e_2(t) = 20\frac{c_1}{t^3} + 15\frac{c_2}{t^4} + 12\frac{c_3}{t^5}.$$

5. High-Order Universal Numerical Differentiator

n=1: $\frac{dz_0(t)}{dt} = -\frac{1}{t}(z_0(t) - f(t))$

n=2: $\begin{cases} \frac{dz_0}{dt} = -\frac{4}{t}(z_0 - f) + z_1 \\ \frac{dz_1}{dt} = -\frac{6}{t^2}(z_0 - f) \end{cases}$

n=3: $\begin{cases} \frac{dz_0}{dt} = -\frac{9}{t}(z_0 - f) + z_1 \\ \frac{dz_1}{dt} = -\frac{36}{t^2}(z_0 - f) + z_2 \\ \frac{dz_2}{dt} = -\frac{60}{t^3}(z_0 - f) \end{cases}$

...

HOUND: $\boxed{\frac{dz_{m-1}}{dt} = -\frac{(n+m-1)!}{m!(n-m)!}\frac{n}{t^m}(z_0 - f) + z_m}$, where m=1,2,…,n and $z_n \equiv 0$

$$\frac{d}{dt}\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{bmatrix} = \begin{pmatrix} -n^2/t & 1 & 0 & \cdots & 0 \\ -(n-1)n^2(n+1)/2t^2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -n^2(n+1)(n+2)\ldots(2n-2)/t^{n-1} & 0 & 0 & \cdots & 1 \\ -n(n+1)(n+2)\ldots(2n-1)/t^n & 0 & 0 & \cdots & 0 \end{pmatrix}\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{d^n f}{dt^n} \end{bmatrix}$$

$$t^n \frac{d^n e_0}{dt^n} + n^2 t^{n-1}\frac{d^{n-1}e_0}{dt^{n-1}} + \cdots + n!\, n \cdot t\frac{de_0}{dt} + n!\, e_0 = -t^n \frac{d^n f}{dt^n}$$ Euler ODE

Characteristic polynomial:

$(\lambda + 1)(\lambda + 2)\ldots(\lambda + n) = 0 \qquad \lambda_1 = -1, \quad \lambda_2 = -2, \ldots, \quad \lambda_n = -n$ if $\frac{d^n f}{dt^n} = 0$ then:

$$e_0(t) = \frac{c_{01}}{t} + \frac{c_{02}}{t^2} + \cdots + \frac{c_{0n}}{t^n}, \quad e_1(t) = \frac{c_{11}}{t^2} + \frac{c_{12}}{t^3} + \cdots + \frac{c_{1n}}{t^{n+1}}, \quad \ldots, \quad e_{n-1}(t) = \frac{c_{n1}}{t^n} + \frac{c_{n2}}{t^{n+1}} + \cdots + \frac{c_{nn}}{t^{2n-1}}.$$

6. Solution of HOUND equations

HOUND: $$\frac{dz_{m-1}}{dt} = -\frac{(n+m-1)!}{m!(n-m)!}\frac{n}{t^m}\big(z_0(t) - f(t)\big) + z_m(t),\ z_n(t) \equiv 0,\ t > 0$$

n=1: $$z_0(t) = f(t) + \frac{1}{t}\left(c_1 - \int_{t_0}^{t} \tau \frac{df(\tau)}{d\tau} d\tau\right) = f(t) + \frac{1}{t}\left(c - t \cdot f(t) + \int_{t_0}^{t} f(\tau)d\tau\right)$$

n=2: $$z_0(t) = f(t) + \left(\frac{1}{t}\left(c_1 - \int_{t_0}^{t} \tau^2 \frac{d^2 f(\tau)}{d\tau^2} d\tau\right) + \frac{1}{t^2}\left(c_2 - \int_{t_0}^{t} \tau^3 \frac{d^2 f(\tau)}{d\tau^2} d\tau\right)\right)$$

$$z_1(t) = \frac{df(t)}{dt} + \left(\frac{3}{t^2}\left(c_1 - \int_{t_0}^{t} \tau^2 \frac{d^2 f(\tau)}{d\tau^2} d\tau\right) + \frac{2}{t^3}\left(c_2 - \int_{t_0}^{t} \tau^3 \frac{d^2 f(\tau)}{d\tau^2} d\tau\right)\right)$$

Arbitrary n:

$$z_{m-1}(t) = \frac{d^{m-1} f(t)}{dt^{m-1}} + e_{m-1}(t)$$, where m=1,2,…,n

Particular solution:

$$e_{m-1}(t) = \sum_{d=1}^{n} \frac{a_{m,d,n}}{t^{d+m-1}}\left(c_d - \frac{(-1)^d}{b_{d,n}} \int_{t_0}^{t} \tau^{d+n-1} \frac{d^n f(\tau)}{d\tau^n} d\tau\right) =$$

$$\sum_{d=1}^{n} a_{m,d,n}\left(\frac{c_d}{t^{d+m-1}} - \frac{(-1)^d t^{n-m}}{b_{d,n}}\left(\frac{d^{n-1} f(t)}{dt^{n-1}} - \frac{d+n-1}{t^{d+n-1}} \int_{t_0}^{t} \tau^{d+n-2} \frac{d^{n-1} f(\tau)}{d\tau^{n-1}} d\tau\right)\right)$$

where $a_{m+1,d,n} = -(d+m-1)a_{m,d,n} + \frac{(n+m-1)!n}{m!(n-m)!},\ a_{1,d,n} = 1,\ b_{d,n} = (n-d)!\,(d-1)!$

if $\left|\frac{d^{n-1} f(x)}{dx^{n-1}}\right| < L$ where Lipschitz constant $L \geq 0$

then $|e_{m-1}(t)| < 2L + \frac{c_{m1}}{t^m} + \frac{c_{m2}}{t^{m+1}} + \cdots + \frac{c_{mn}}{t^{n+m-1}}$

$$\boxed{|e_{m-1}(t)| < 2L \text{ when } t \to \infty}$$

7. Stochastic HOUND equations

Ordindary differential equations:

$$\frac{dz_{m-1}(t)}{dt} = -\frac{(n+m-1)!}{m!(n-m)!}\frac{n}{t^m}\big(z_0(t) - f(t)\big) + z_m(t),\ z_n(t) \equiv 0,\ t > 0$$

Stochastic differential equations:

$$\mathrm{d}z_{m-1}(t) = z_{m-1}(t)\mathrm{d}t - \frac{(n+m-1)!}{m!(n-m)!}\frac{n}{t^m}\Big(\big(z_0(t) - f_0(t)\big)\mathrm{d}t - \sigma_0 \mathrm{d}W_0(t)\Big)$$

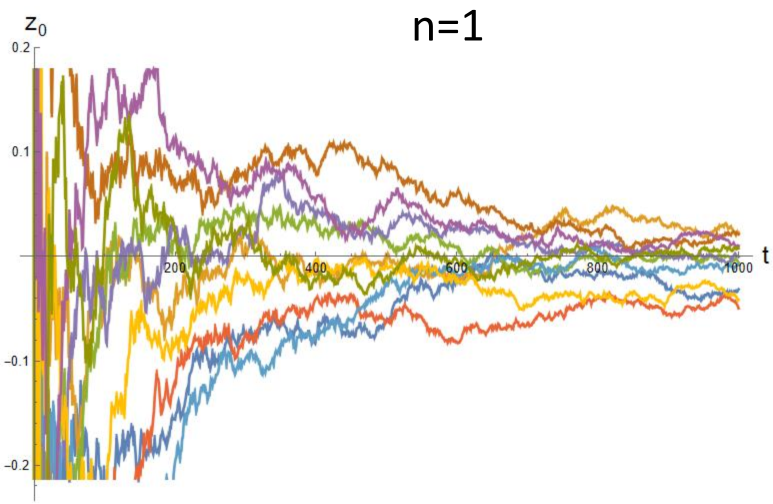
where $f(t) = f_0(t) + \eta_0(t)$, additive white Gaussian noise $\eta_0(t) \sim N(0, \sigma_0^2)$

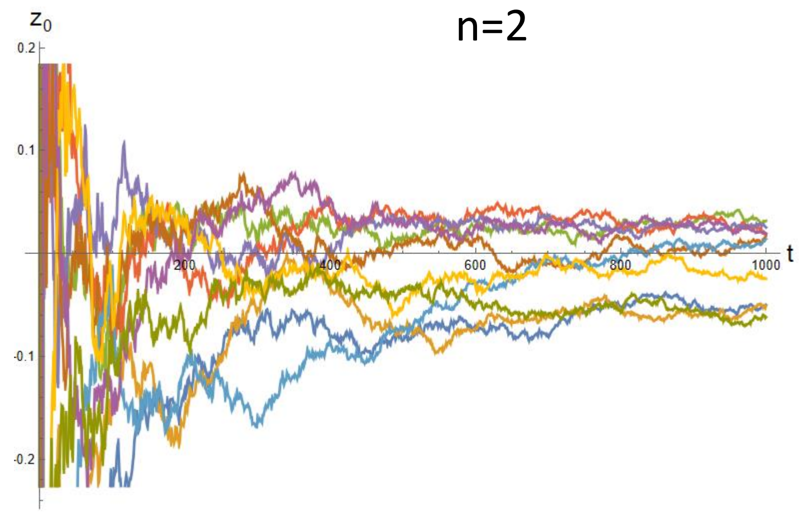
Mean value $z_{m-1}(t)$ coverges to $\frac{d^{m-1} f_0(t)}{dt^{m-1}}$, variance converges to 0:

$$Var\big(z_{m-1}(t)\big) = C_{m-1}\frac{\sigma_0^2}{t^{2m-1}} \to 0 \text{ when } t \to \infty$$

$$Var\big(z_0(t)\big) = C_0\frac{\sigma_0^2}{t}, Var\big(z_1(t)\big) = C_1\frac{\sigma_0^2}{t^3}, \ldots, Var\big(z_{n-1}(t)\big) = C_{n-1}\frac{\sigma_0^2}{t^{2n-1}}$$

Examples of solutions SDE with $f_0(t) = 0$ and $\sigma_0^2 = 1$:





8. Discretization of HOUND equations

Remember classic exponential smoothing.

Simple exponential smoothing (Brown, n=1):

$$z_0[t] = \alpha \cdot f[t] + (1-\alpha) \cdot z_0[t-1], \text{ where } 0 \le \alpha \le 1$$

Double exponential smoothing (Holt, n=2, redesignate $l = z_0$, $b = z_1$, $y = f$):

$$\begin{cases} z_0[t] = \alpha \cdot f[t] + (1-\alpha)(z_0[t-1] + z_1[t-1]) \\ z_1[t] = \beta^*(z_0[t] - z_0[t-1]) + (1-\beta^*) z_1[t-1] \end{cases} \text{where } 0 \le \alpha \le 1 \text{ and } 0 \le \beta^* \le 1$$

Analogous cumulative smoothing ($\Delta t = 1$).

Simple cumulative smoothing ($\alpha = \frac{1}{t}$):

n=1: $$\begin{cases} z_0[t] = z_0[t-1] + \frac{1}{t}\varepsilon[t] \\ \varepsilon[t] = f[t] - z_0[t-1] \end{cases} \equiv \quad z_0[t] = \frac{1}{t} f[t] + \left(1 - \frac{1}{t}\right) z_0[t-1]$$

Double cumulative smoothing ($\alpha = \frac{4}{t}$, $\beta^* = \frac{3}{2t}$):

n=2: $$\begin{cases} z_0[t] = z_0[t-1] + \frac{4}{t}\varepsilon[t] + z_1[t-1] \\ z_1[t] = z_1[t-1] + \frac{6}{t^2}\varepsilon[t] \\ \varepsilon[t] = f[t] - (z_0[t-1] + z_1[t-1]) \end{cases} \equiv \begin{cases} z_0[t] = \frac{4}{t} f[t] + \left(1 - \frac{4}{t}\right)(z_0[t-1] + z_1[t-1]) \\ z_1[t] = \frac{3}{2t}(z_0[t] - z_0[t-1]) + \left(1 - \frac{3}{2t}\right) z_1[t-1] \end{cases}$$

9. High-Order Cumulative Smoothing

Taylor: $z_{m-1}(t) \approx \sum_{k=m-1}^{n-1} \frac{z_k(t_{prev})}{(k-m+1)!}(t-t_{prev})^{k-m+1}$ with step $\Delta t = t - t_{prev}$

$$\begin{cases} z_{m-1}[t] = \left(\sum_{k=m-1}^{n-1} \frac{z_k[t-\Delta t]}{(k-m+1)!}\Delta t^{k-m+1}\right) + \Delta t \frac{(n+m-1)!}{m!(n-m)!}\frac{n}{t^m}\varepsilon[t] \\ \varepsilon[t] = \left(f[t] - \sum_{k=0}^{n-1}\frac{z_k[t-\Delta t]}{k!}\Delta t^k\right) \end{cases}$$

For example (n=5):

$$\begin{cases} z_0[t] = 25\frac{\Delta t}{t}\varepsilon[t] + z_0[t-\Delta t] + z_1[t-\Delta t]\cdot\Delta t + z_2[t-\Delta t]\frac{\Delta t^2}{2} + z_3[t-\Delta t]\frac{\Delta t^3}{6} + z_4[t-\Delta t]\frac{\Delta t^4}{24} \\ z_1[t] = 300\frac{\Delta t}{t^2}\varepsilon[t] + z_1[t-\Delta t] + z_2[t-\Delta t]\cdot\Delta t + z_3[t-\Delta t]\frac{\Delta t^2}{2} + z_4[t-\Delta t]\frac{\Delta t^3}{6} \\ z_2[t] = 2100\frac{\Delta t}{t^3}\varepsilon[t] + z_2[t-\Delta t] + z_3[t-\Delta t]\cdot\Delta t + z_4[t-\Delta t]\frac{\Delta t^2}{2} \\ z_3[t] = 8400\frac{\Delta t}{t^4}\varepsilon[t] + z_3[t-\Delta t] + z_4[t-\Delta t]\cdot\Delta t \\ z_4[t] = 15120\frac{\Delta t}{t^5}\varepsilon[t] + z_4[t-\Delta t] \\ \varepsilon[t] = f[t] - \left(z_0[t-\Delta t] + z_1[t-\Delta t]\cdot\Delta t + z_2[t-\Delta t]\frac{\Delta t^2}{2} + z_3[t-\Delta t]\frac{\Delta t^3}{6} + z_4[t-\Delta t]\frac{\Delta t^4}{24}\right) \end{cases}$$

Algorithm 1 High-Order Cumulative Smoothing

Input: Scalar data sequence $f[t]$ for $t = t_0+\Delta t, t_0+2\Delta t, t_0+3\Delta t, \ldots$, scalar variables z_{m-1} for $m = 1, 2, \ldots, n$ with initial values $z_0[t_0] = f[t_0]$ and $z_m[t_0] = 0$ where $t_0 > 0$
Output: Updated variables values $z_{m-1}[t]$ containing estimates of the signal and its derivatives
$t \leftarrow t_0$
for $i = 1, 2, 3, \ldots$ **do**
  $t \leftarrow t + \Delta t$
  $\varepsilon[t] \leftarrow f[t] - \sum_{k=0}^{n-1}\frac{z_k[t-\Delta t]}{k!}\Delta t^k$
  for $m = 1, 2, \ldots, n$ **do**
    $z_{m-1}[t] \leftarrow \left(\sum_{k=m-1}^{n-1}\frac{z_k[t-\Delta t]}{(k-m+1)!}\Delta t^{k-m+1}\right) + \Delta t\frac{(n+m-1)!}{m!(n-m)!}\frac{n}{t^m}\varepsilon[t]$
  end for
end for

10. Polynomial approximation

Using Taylor series expansion to approximate the signal and its derivatives:

$$\frac{d^{m-1}f}{dt^{m-1}}[\tau] \approx \widehat{\frac{d^{m-1}f}{dt^{m-1}}}[\tau] = \sum_{k=m-1}^{n-1} \frac{z_k[t]}{(k-m+1)!}(\tau - t)^{k-m+1}, \text{ where m=1,2,…,n}$$

Signal interpolation and extrapolation (m=1):

$$f[\tau] \approx \widehat{f[\tau]} = \sum_{k=0}^{n-1} \frac{z_k[t]}{k!}(\tau - t)^k = z_0[t] + z_1[t](\tau - t) + \frac{z_2[t]}{2}(\tau - t)^2 + \cdots$$

Approximating polynomial:

$$f[\tau] \approx \widehat{f[\tau]} = \sum_{j=0}^{n-1} K_j \tau^j = K_0 + K_1 \cdot \tau + K_2 \cdot \tau^2 + \cdots + K_{n-1} \cdot \tau^{n-1}$$

Coefficients of approximating polynomial:

$$K_{m-1} = \frac{1}{(m-1)!} \sum_{i=m-1}^{n-1} \frac{z_i[t]}{(i-m+1)!}(-t)^{i-m+1}$$

For example, first coefficient of approximating polynomial (m=1):

$$K_0 = z_0[t] - z_1[t] \cdot t + \frac{z_2[t]}{2} t^2 - \frac{z_3[t]}{6} t^3 + \cdots + \frac{z_{n-1}[t]}{(n-1)!}(-t)^{n-1}$$

Algorithm 2 Polynomial Approximation

Input: Updated scalar variables values $z_{m-1}[t]$ at the t for $m = 1, 2, \ldots, n$ from Algorithm 1

Output: Polynomial approximated values $\hat{f}^{(m-1)}[\tau]$ of the signal and its derivatives at the τ, coefficients K_{m-1} values of the approximating polynomial $\hat{f}[\tau] = \sum_{m=1}^{n} K_{m-1}\tau^{m-1}$

for $m = 1, 2, \ldots, n$ **do**

$\quad \hat{f}^{(m-1)}[\tau] := \sum_{k=m-1}^{n-1} \frac{z_k[t]}{(k-m+1)!}(\tau - t)^{k-m+1}$

$\quad K_{m-1} := \frac{1}{(m-1)!} \sum_{i=m-1}^{n-1} \frac{z_i[t]}{(i-m+1)!}(-t)^{i-m+1}$

end for

11. Demo example (noised polynomial): HOUND

Ground truth is polynomial $f_0(t)$ with additive white Gaussian noise $\eta_0(t) \sim N(0, \sigma_0^2)$:

$$f(t) = f_0(t) + \eta_0(t) =$$
$$= (5 - 0.004 \cdot t + 0.0003 \cdot t^2 - 0.00002 \cdot t^3 + 0.000001 \cdot t^4) + N(0, 0.7^2)$$

over range t from 0 to 20000

Using High-Order Cumulative Smoothing with $\mathrm{n} = 5$ and $\Delta\mathrm{t} = 1$:

$$\begin{cases} z_0[t] = \frac{25}{t}\varepsilon[t] + z_0[t-1] + z_1[t-1] + \frac{z_2[t-1]}{2} + \frac{z_3[t-1]}{6} + \frac{z_4[t-1]}{24} \\ z_1[t] = \frac{300}{t^2}\varepsilon[t] + z_1[t-1] + z_2[t-1] + \frac{z_3[t-1]}{2} + \frac{z_4[t-1]}{6} \\ z_2[t] = \frac{2100}{t^3}\varepsilon[t] + z_2[t-1] + z_3[t-1] + \frac{z_4[t-1]}{2} \\ z_3[t] = \frac{8400}{t^4}\varepsilon[t] + z_3[t-1] + z_4[t-1] \\ z_4[t] = \frac{15120}{t^5}\varepsilon[t] + z_4[t-1] \\ \varepsilon[t] = f[t] - \left(z_0[t-1] + z_1[t-1] + \frac{z_2[t-1]}{2} + \frac{z_3[t-1]}{6} + \frac{z_4[t-1]}{24}\right) \end{cases}$$

where $z_0[0] = f[0] \approx 5$ and $z_1[0] = z_2[0] = z_3[0] = z_4[0] = 0$

Estimation errors: $e_0[t] = z_0[t] - f_0[t]$, $e_1[t] = z_1[t] - \frac{df_0}{dt}[t]$,
$e_2[t] = z_2[t] - \frac{d^2 f_0}{dt^2}[t]$, $e_3[t] = z_3[t] - \frac{d^3 f_0}{dt^3}[t]$, $e_4[t] = z_4[t] - \frac{d^4 f_0}{dt^4}[t]$

12. Demo example (noised polynomial): another methods

Comparing with high-gain differentiator:

$$\begin{cases} x_0[t] = \varepsilon[t] + x_0[t-1] + x_1[t-1] + \frac{x_2[t-1]}{2} + \frac{x_3[t-1]}{6} + \frac{x_4[t-1]}{24} \\ x_1[t] = 0.4012 \cdot \varepsilon[t] + x_1[t-1] + x_2[t-1] + \frac{x_3[t-1]}{2} + \frac{x_4[t-1]}{6} \\ x_2[t] = 0.0744 \cdot \varepsilon[t] + x_2[t-1] + x_3[t-1] + \frac{x_4[t-1]}{2} \\ x_3[t] = 0.007312 \cdot \varepsilon[t] + x_3[t-1] + x_4[t-1] \\ x_4[t] = 0.000352 \cdot \varepsilon[t] + x_4[t-1] \\ \varepsilon[t] = f[t] - \left(x_0[t-1] + x_1[t-1] + \frac{x_2[t-1]}{2} + \frac{x_3[t-1]}{6} + \frac{x_4[t-1]}{24}\right) \end{cases}$$

where $x_0[0] = f[0] \approx 5$ and $x_1[0] = x_2[0] = x_3[0] = x_4[0] = 0$

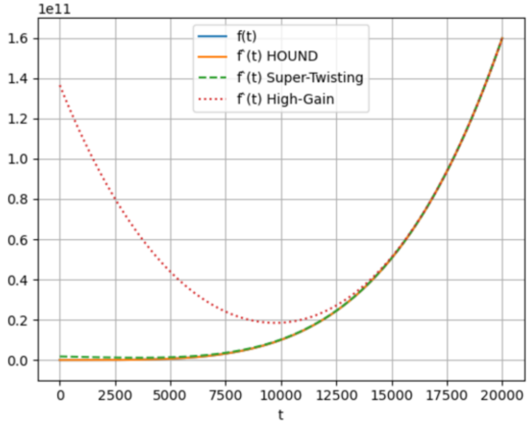
And comparing with super-twisting differentiator (sliding mode):

$$\begin{cases} y_0[t] = 3\sqrt[4]{L}\sqrt[4]{|\varepsilon[t]|^3} \cdot sign(\varepsilon[t]) + y_0[t-1] + y_1[t-1] + \frac{y_2[t-1]}{2} + \frac{y_3[t-1]}{6} \\ y_1[t] = 4.16\sqrt{L}\sqrt{|\varepsilon[t]|} \cdot sign(\varepsilon[t]) + y_1[t-1] + y_2[t-1] + \frac{y_3[t-1]}{2} \\ y_2[t] = 3.06\sqrt[4]{L^3}\sqrt[4]{|\varepsilon[t]|} \cdot sign(\varepsilon[t]) + y_2[t-1] + y_3[t-1] \\ y_3[t] = 1.1 \cdot L \cdot sign(\varepsilon[t]) + y_3[t-1] \\ \varepsilon[t] = f[t] - \left(y_0[t-1] + y_1[t-1] + \frac{y_2[t-1]}{2} + \frac{y_3[t-1]}{6}\right) \end{cases}$$

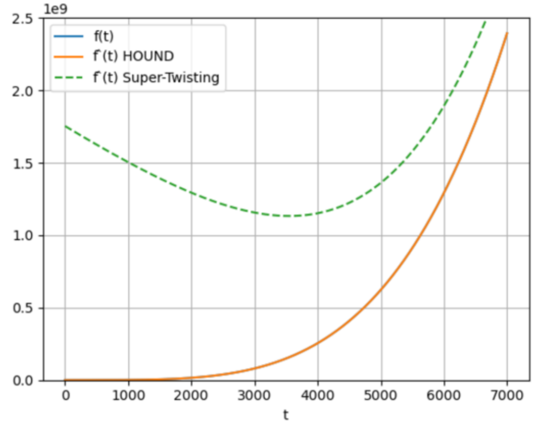
where $y_0[0] = f[0] \approx 5$ and $y_1[0] = y_2[0] = y_3[0] = 0$

and $L = \left|\frac{1}{1.1}\frac{d^4 f_0(t)}{dt^4}\right|$ =2.18182e-5

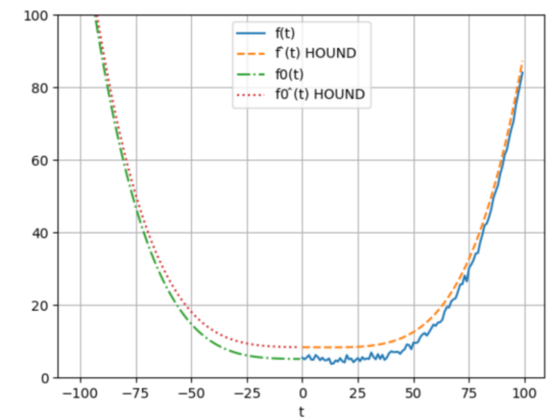
13. Demo example (noised polynomial): approximation



Interpolation (full range)



Interpolation (start range)



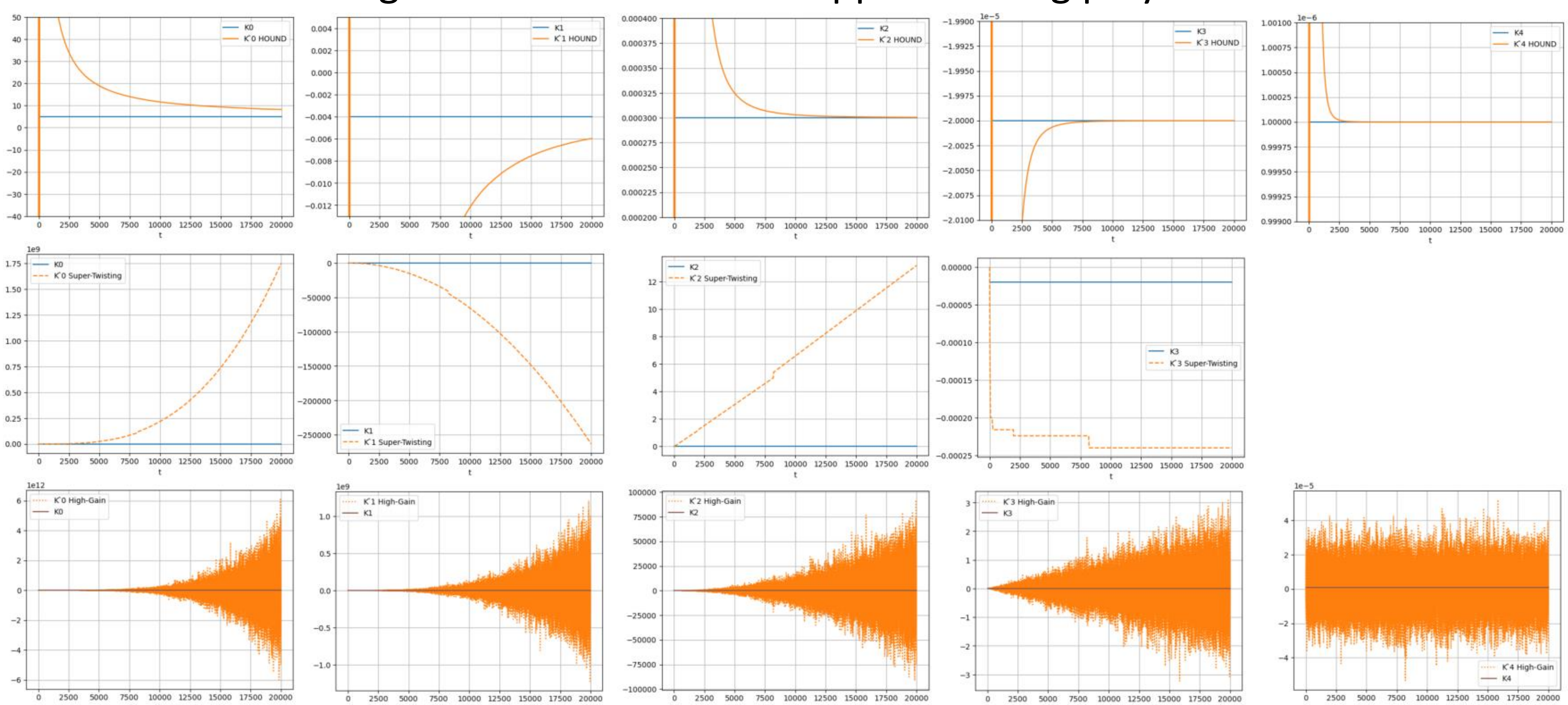
Extrapolation (range t < 0)

Convergence to coefficients of approximating polynomial:

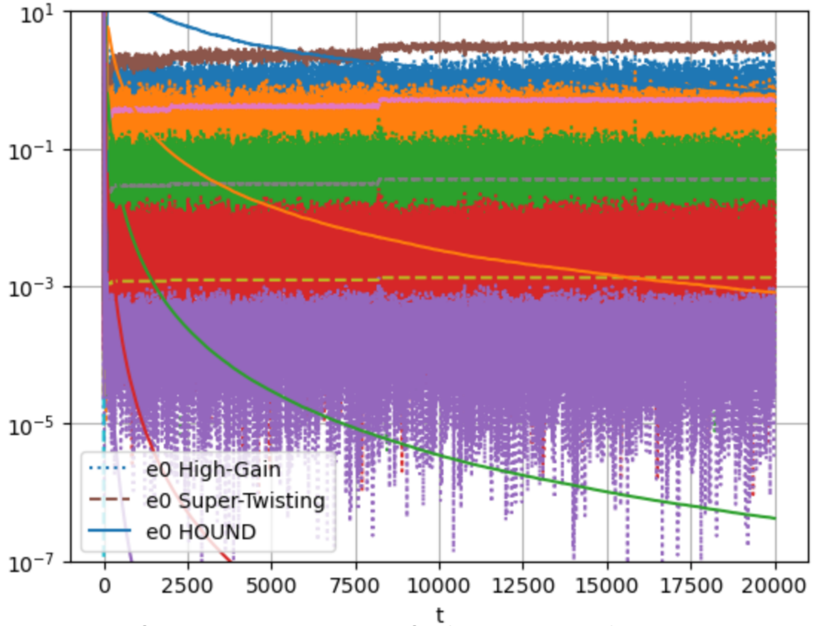
HOUND

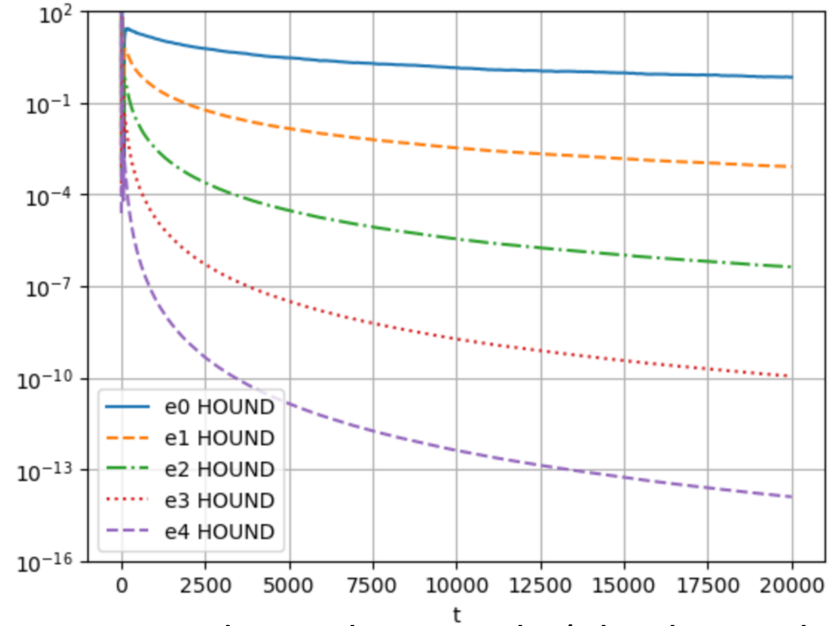
super-twisting

high-gain



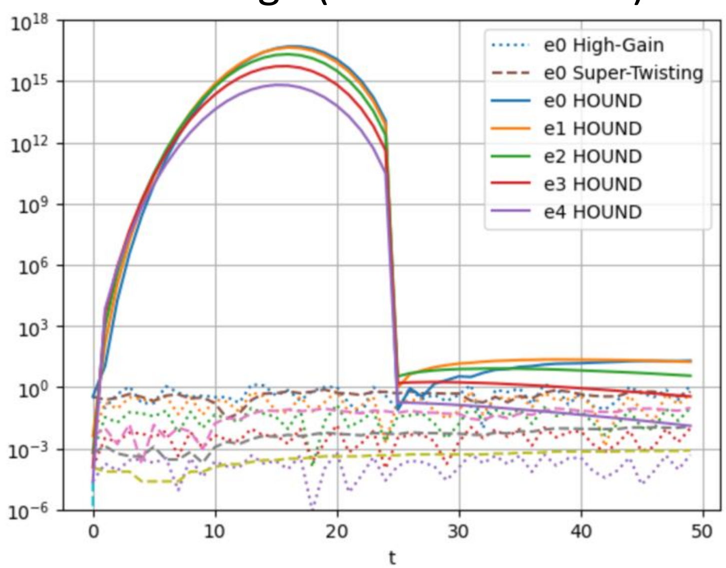
14. Demo example (noised polynomial): error analysis



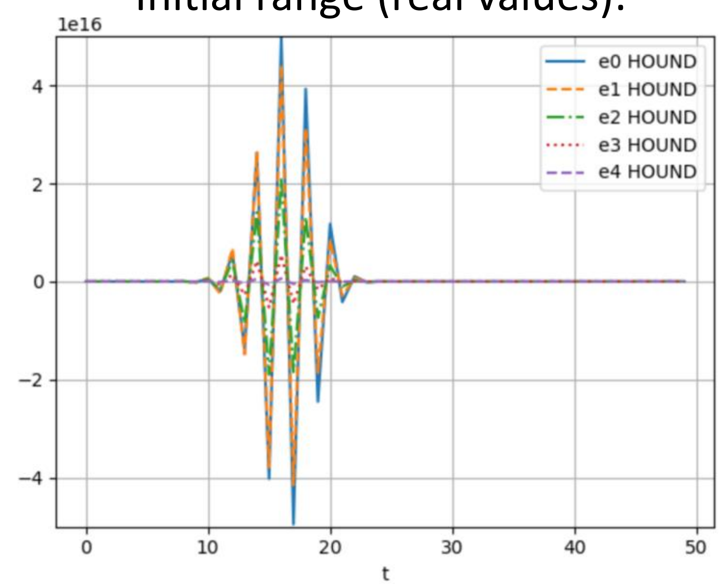


Errors of estimation of the signal and its derivatives in logarithmic scale (absolute values)

Initial range (absolute values):



Initial range (real values):



15. Conclusion

a. High-Order Universal Numerical Differentiator (HOUND) is a system of differential equations, or Cauchy-Euler ODE with roots of characteristic polynomial $\lambda = -1, -2, -3, \ldots - n$

b. The error of solution is bounded if the highest estimated derivative is bounded $\left|\frac{d^{n-1}f(x)}{dx^{n-1}}\right| < L$ (error converges to zero for polynomial signal)

c. Variance of additive white Gaussian noise converges to zero

d. Discretization of HOUND equations is High-Order Cumulative Smoothing

e. The HOUND key is not only cumulative smoothing of the signal itself, but cumulative smoothing of all estimated derivatives as well

f. Automatic convergence to coefficient values of approximating polynomial (interpolation and extrapolation)

g. HOUND is parameter-free online method: no need to accumulate data , no need to fit any coefficients to data