Intermediate Algebra

Functions and Graphs

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Preface

Intermediate Algebra is a textbook for students who have some acquaintance with the basic notions of variables and equations, negative numbers, and graphs, although we provide a ”Toolkit” to help the reader refresh any skills that may have gotten a little rusty. In this book we journey farther into the subject, to explore a greater variety of topics including graphs and modeling, curve-fitting, variation, exponentials and logarithms, and the conic sections. We use technology to handle data and give some instructions for using a graphing calculator, but these can easily be adapted to any other graphing utility.

We discuss functions and their applications, but not in the detail expected of a precalculus course; our intended audience includes students preparing for the many fields that may not use calculus but nonetheless require facility with quantitative reasoning. We aim to develop that facility in the context of modeling and problem solving.

- Each section of the text includes Examples followed by a similar Practice Exercise that students can try for themselves in WeBWorK.
- There are also ”QuickCheck” exercises, usually multiple choice or fill in the blanks, for students to check their understanding of the concepts presented.
- Homework problems come in three groups: a short Skills Warm Up that reviews prerequisite skills for the section, then Skills Practice with new skills, and finally Applications.
- We have included a variety of applied problems that we hope students (and their teachers) will find interesting.
- In addition, each Chapter begins with an Investigation that can be used as a group project or as a guided in-class activity.

Chapter Reviews include a Glossary and a Summary of key concepts as well as Review Problems. There is an Activity booklet available that provides an interactive lesson for each section of the text. The Activities can be completed by students in groups or with guidance from the instructor; or they can be used as support for a lecture format.

Katherine Yoshiwara
Atascadero, CA 2020

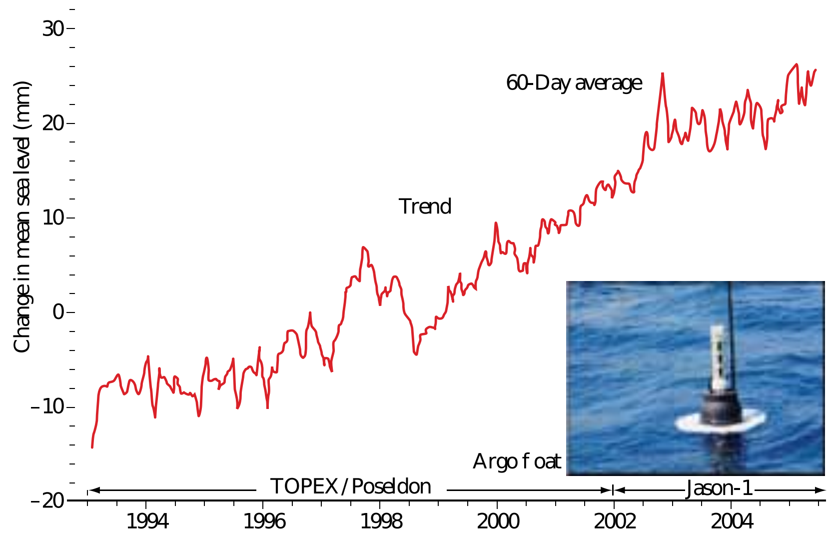
Contents

Chapter 1

Linear Models

In the debate over global warming, sea level is a reliable indicator of climate change, because it is affected by both melting glaciers and by the thermal expansion of sea water. During the 20th century, sea level rose by about 15 centimeters, compared to a fairly constant level for the previous 3,000 years.

Not surprisingly, scientists would like to understand the causes of sea-level change as thoroughly as possible, and to measure the rate of sea-level rise as accurately as possible. Using data from satellites and floats (mechanical devices drifting in the ocean), oceanographers at NASA's Jet Propulsion Laboratory have calculated that the sea level rose, on average, 3 millimeters (0.1 inches) per year between 1995 and 2005. The graph below shows the change in mean sea level, measured in millimeters, over that time period.



The Intergovernmental Panel on Climate Change (IPCC) predicts that sea level could rise as much as 1 meter during the 21st century. If that happens, low-lying, densely populated areas in China, Southeast Asia, and the Nile Delta would become uninhabitable, as well as the Gulf Coast and Eastern Seaboard of the United States.

1.1 Creating a Linear Model

To help them understand phenomena that affect our planet, scientists study the connections between relevant variables, such as sea level. In the next investigation we consider a simpler example and look at some ways to connect the variables involved in a career choice.

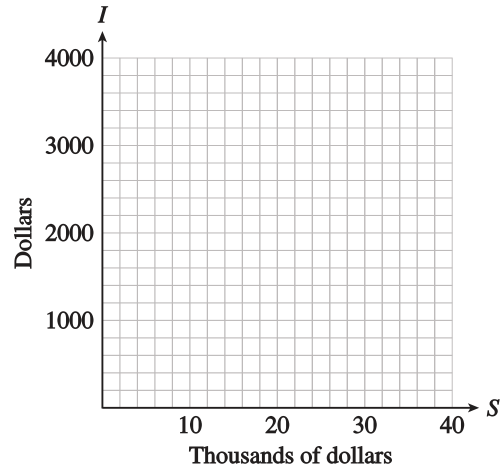
Investigation 1.1.1 Sales on Commission. Delbert is offered a part-time job selling restaurant equipment. He will be paid $1000 per month plus a 6% commission on his sales. The sales manager tells Delbert he can expect to sell about $8000 worth of equipment per month. To help him decide whether to accept the job, Delbert does a few calculations.

1. Based on the sales manager's estimate, what monthly income can Delbert expect from this job? What annual salary would that provide?

2. What would Delbert's monthly salary be if he sold only $5000 of equipment per month? What would his salary be if he sold $10,000 worth per month? Compute the monthly incomes for each sales totals shown in the table.

Sales	Income
5000	
$10,000$	
$15,000$	
$20,000$	
$25,000$	
$30,000$	

3. Plot your data points on a graph, using sales, S, on the horizontal axis and income, I, on the vertical axis, as shown in the figure. Connect the data points to show Delbert's monthly income for all possible monthly sales totals.



4. Add two new data points to the table by reading values from your graph.

5. Write an algebraic expression for Delbert's monthly income, I, in terms of his monthly sales, S. Use the description in the problem to help you:

   He will be paid: \$1000 plus a 6% commission on his sales.

   Income = $\boxed{}$

6. Test your formula from part (5) to see if it gives the same results as those you recorded in the table.

7. Use your formula to find out what monthly sales total Delbert would need in order to have a monthly income of \$2500.

8. Each increase of \$1000 in monthly sales increases Delbert's monthly income by $\boxed{}$.

9. Summarize the results of your work: In your own words, describe the relationship between Delbert's monthly sales and his monthly income. Include in your discussion a description of your graph.

1.1.1 Tables, Graphs, and Equations

In the Investigation you studied the connection between the variables *Sales* and *Income* for a part-time job. You created a **mathematical model** for that situation.

Definition 1.1.1 Mathematical Model. A **mathematical model** is a simplified description of reality that uses mathematics to help us understand a system or process. ◊

We can use a model to analyze data, identify trends, and predict the effects of change. The first step in creating a model is to describe relationships between the variables involved.

Checkpoint 1.1.2 QuickCheck 1. What is the first step in creating a mathematical model?

a. Solve the equation.

b. Analyze data, identify trends, and predict the effects of change.

c. Describe the relationships between the variables involved.

d. Make a table of values.

Starting from a description in words, we can represent the relationship by:

- a table of values
- a graph, or by
- an algebraic equation

Each of these mathematical tools is useful in a different way.

1. A **table of values** lists specific data points with precise numerical values.
2. A **graph** is a visual display of the data. It is easier to spot trends and describe the overall behavior of the variables from a graph.
3. An **algebraic equation** is a compact summary of the model. It can be used to analyze the model and to make predictions.

Look back at the Investigation to see how each of these modeling tools featured in your work.

Checkpoint 1.1.3 QuickCheck 2. Name three ways to represent a relationship between variables.

In the examples that follow, observe the interplay among the three modeling tools and how each contributes to the model.

Example 1.1.4 In May 2005, the city of Lyons, France, started a bicycle rental program. Over 3,000 bicycles are available at 350 computerized stations around the city. Each of the 52,000 subscribers pays an annual 5 euro fee (about $7.20) and gets a PIN to access the bicycles. The bicycles rent for 1 euro per hour and can be returned to any station.

Your community decides to set up a similar program, charging a $5 subscription fee and $3 an hour for rental. (A fraction of an hour is charged as the corresponding fraction of $3).

a. Make a table of values showing the cost, C, of renting a bike for various lengths of time, t.

b. Plot the points on a graph. Draw a curve through the data points. [TK]

c. Write an equation for C in terms of t.

Solution.

a. There is an initial fee of $5, and a rental fee of $3 per hour. To find the cost, we multiply the rental time by $3 per hour, and add the result to the $5 subscription fee. For example, the cost of a one-hour bike ride is

$$\begin{aligned} \text{Cost} &= (\$5 \text{ subscription fee}) + (\$3 \text{ per hour}) \times (\text{one hour}) \\ C &= 5 + 3(\mathbf{1}) = 8 \end{aligned}$$

A one-hour bike ride costs $8.

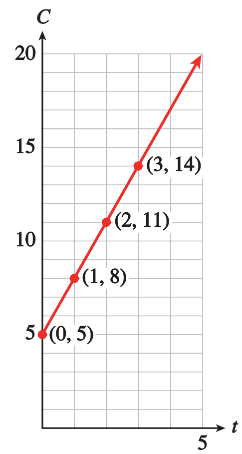
We calculate the cost for the other values of t and record the results in a table as shown below.

Length of rental (hours)	Cost of rental (dollars)
0	5
1	8
2	11
3	14

	(t, C)
$C = 5 + 3(\mathbf{0})$	$(0, 5)$
$C = 5 + 3(\mathbf{1})$	$(1, 8)$
$C = 5 + 3(\mathbf{2})$	$(2, 11)$
$C = 5 + 3(\mathbf{3})$	$(3, 14)$

b. Each pair of values represents a point on the graph. The first value gives the horizontal coordinate of the point, and the second value gives the vertical coordinate.

The points lie on a straight line, as shown in the figure. The line extends infinitely in only one direction, because negative values of t do not make sense here.



c. To write an equation, we let C represent the cost of the rental, and we use t for the number of hours:

$$\begin{aligned} \text{Cost} &= (\$5 \text{ subscription}) + \$3 \cdot (\text{number of hours}) \\ C &= 5 + 3 \cdot t \end{aligned}$$

□

[TK] For more help on plotting points, see Section 1.1.2 of the Toolkit.

Checkpoint 1.1.5 Practice 1. Write an equation for the cost C of renting a bicycle for t hours if the subscription fee is \$7 and the rental fee is \$2.50 per hour.

Hint: Fill in the correct numbers in the equation below.

$$\begin{aligned} \text{Cost} &= (\text{subscription fee}) + (\text{hourly rate}) \cdot (\text{number of hours}) \\ C &= \boxed{} + \boxed{} \cdot t \end{aligned}$$

1.1.2 Equations for Linear Models

The equation we wrote in Example 1.1.4, $C = 5 + 3t$, is an example of a **linear model**, which describes a variable that increases or decreases at a constant rate. The cost of renting a bicycle increased at a constant rate of \$3 every hour.

Definition 1.1.6 Linear Model. A **linear model** describes a variable that increases or decreases at a constant rate. It has the form

$$\boldsymbol{y} = (\text{starting value}) + (\text{rate}) \times \boldsymbol{t}$$

◊

Checkpoint 1.1.7 QuickCheck 3. What sort of variables can be described by a linear model?

a. Increasing variables

b. Variables that change at a constant rate

c. Variables that describe time

d. Variables that can be graphed

[TK] For more help on writing a linear model, see Section 1.1.1 of the Toolkit.

The connections between an equation and its graph can give us useful information about a model. In the next Example we see how to use this idea.

Example 1.1.8 Use the equation $C = 5 + 3 \cdot t$ you found in Example 1.1.4 to answer the following questions. Then show how to find the answers by using the graph.

a. How much will it cost to rent a bicycle for 6 hours?

b. How long can you bicycle for \$18.50?

Solution.

a. We substitute $t = \mathbf{6}$ into the equation to find

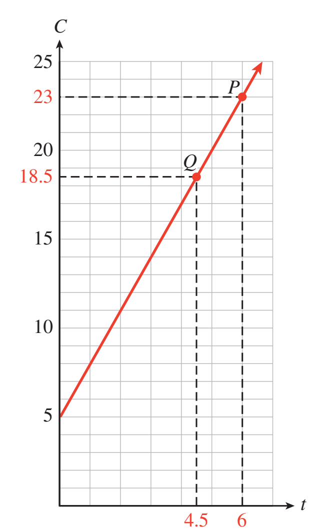
$$C = 5 + 3(\mathbf{6}) = 23$$

A 6-hour bike ride will cost \$23. The point P on the graph in the figure represents the cost of a 6-hour bike ride. The value on the C-axis at the same height as point P is 23, so a 6-hour bike ride costs \$23.

b. We substitute $C = \mathbf{18.50}$ into the equation and solve for t. [TK]

$$\begin{aligned} \mathbf{18.50} &= 5 + 3t \\ 13.50 &= 3t \\ t &= 4.5 \end{aligned}$$

For \$18.50 you can bicycle for $4\frac{1}{2}$ hours. The point Q on the graph represents an \$18.50 bike ride. The value on the t-axis below point Q is 4.5, so \$18.50 will buy a 4.5 hour bike ride.



□

[TK] For more help on solving linear equations, see Section 1.1.3 of the Toolkit.

Checkpoint 1.1.9 Practice 2. In the preceding Example, how long can you bicycle for \$9.50?

Hint: Start by finding \$9.50 on the Cost (vertical axis). Then find the point on the graph with C-coordinate \$9.50. Finally, find the t-coordinate of that point.

In Example 1.1.8, note the different algebraic techniques we used in parts (a) and (b):

- In part (a) we were given a value of t and we **evaluated the expression** $5 + 3t$ to find C.
- In part (b) we were given a value of C and we **solved the equation** $C = 5 + 3t$ to find t.

Checkpoint 1.1.10 QuickCheck 4. Consider the expression $C = 5 + 3t$. Finding a value of t when we know C is called _______________. Finding a value of t when we know C is called _______________.

In the examples above, the graph is **increasing** as t increases. In the next Example we consider a **decreasing graph**.

Example 1.1.11 Leon's camper has a 20-gallon gas tank, and he gets 12 miles to the gallon. (Note that getting 12 miles to the gallon is the same as using $\frac{1}{12}$ gallon of gas per mile.)

a. Write an equation for the amount of gasoline, g, left in Leon's tank after he has driven for d miles.

b. Make a table of values for the equation.

c. Graph the equation.

d. If Leon has less than 5 gallons of gas left, how many miles has he driven since his last fill-up? Illustrate on the graph.

Solution.

a. We use the form for a linear model,

$$y = (\text{starting value}) + (\text{rate}) \times t$$

However, in this problem, instead of variables y and t, we use g and d. Leon's fuel tank started with 20 gallons, and the amount of gasoline is decreasing at a rate of $\frac{1}{12}$ gallon for every mile that he drives. Thus,

$$g = 20 - \frac{1}{12}d$$

b. Every time Leon drives 12 miles, he uses another gallon of gasoline. So, to make the calculations easier, we choose values that are divisible by 12.

d (miles)	0	48	96	144
g (gallons)	20	16	12	8

c. We scale the values of d along the horizontal axis, and the values of g along the vertical axis. Then we plot the points from the table and connect with a line, as shown in the figure below.

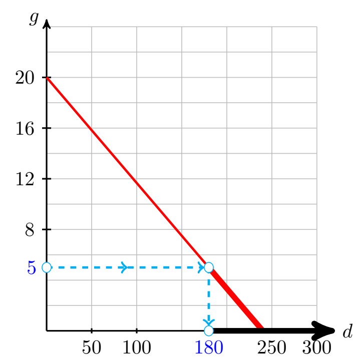
d. If Leon has less than 5 gallons of gasoline left, then $g < 5$, as shown on the g-axis. Using our model from part (a), we solve the inequality. [TK]

$$20 - \frac{1}{12}d < 5 \qquad \text{Subtract } \mathbf{20} \text{ from both sides.}$$

$$-\frac{1}{12}d < -15 \quad \text{Multiply by } \mathbf{-12} \text{ on both sides.}$$

$$d > 180 \quad \text{Note that we reversed the direction of the the inequality when we multiplied by } \mathbf{-12.}$$

Leon has driven at least 180 miles. The solution is shown on the graph below.



□

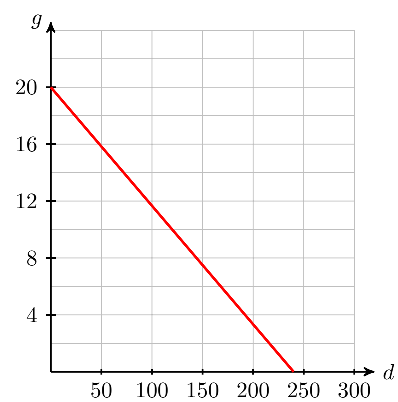
[TK] For more help on solving linear inequalities, see Section 1.1.4 of the Toolkit.

Note 1.1.12 In part (d) of the previous Example we used an **inequality** to answer the question. We use inequalities to model English phrases such as ”less than,” ”more than,” ”at least,” and ”at most.”

Checkpoint 1.1.13 Practice 3.

Leon forgot to reset his odometer after his last fill-up, but he thinks he has driven at least 150 miles. How much gas does he have left? Show this on the graph.

Hint: Locate 150 miles on the d-axis. What part of the axis represents ”at least” 150 miles? Find the points on the graph with d-coordinates at least 150. What are the g-coordinates of those points?



1.1.3 Problem Set 1.1

Warm Up

1. Solve the equation.

a. $8x - 18 = 26$

b. $14 - \frac{3}{4}x = 5$

2. Solve the inequality.

a. $8x - 18 < 16$

b. $14 - \frac{3}{4}x \geq 5$

3. April sells environmentally friendly cleaning products. Her income consists of \$200 per week plus a commission of 9% of her sales. Write an algebraic expression for April's weekly income, I, in terms of her sales, S.

Income	=	Flat Rate	+	Commission
______	=	________	+	__________

4. Trinh is bicycling down a mountain road that loses 500 feet in elevation for each 1 mile of road. She started at an elevation of 6300 feet. Write an expression for Trinh's elevation, h, in terms of the distance she has cycled, d.

Elevation	=	Initial Elevation	−	Decrease
______	=	____________	−	________

Skills Practice

Exercise Group. For problems 5 and 6, solve the equation.

5. $3y + 2y = 12 - 5y$

6. $0.8w - 2.6 = 1.4w + 0.3$

Exercise Group. For problems 7 and 8, solve the inequality.

7. $3y + 2y > 12 - 5y$

8. $0.8w - 2.6 \leq 1.4w + 0.3$

9. Bruce buys a 50-pound bag of rice and consumes about 0.4 pounds per week. Write an expression for the amount of rice, R, Bruce has left in terms of the number of weeks, w, since he bought the bag.

10. Delbert is offered a job as a salesman. He will be paid \$1000 per month plus a 6% commission on his sales. Write an expression for Delbert's monthly income, I, in terms of his sales, S.

Applications

11. Frank plants a dozen corn seedlings, each 6 inches tall. With plenty of water and sunlight they will grow approximately 2 inches per day. Complete the table of values for the height, h, of the seedlings after t days.

a. Complete the table of values for the height, h of the seedlings after t days.

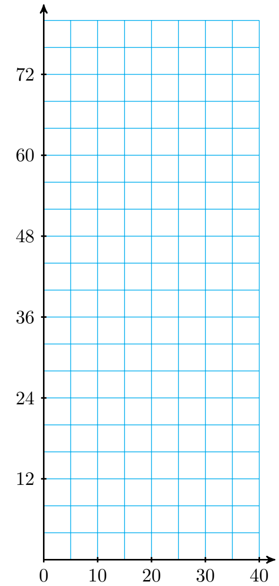
t (days)	0	5	10	15	20
h (inches)					

b. Write an equation for the height, h, of the seedlings in terms of the number of days, t, since they were planted.

c.

Graph the equation on the grid. Label the axes with the correct variables.

To answer questions (d) and (e), read the graph. Show your work on the graph:



d. How tall is the corn after 3 weeks?

e. When will the corn grow to 6 feet tall? (How many inches is 6 feet?)

f. Use algebra to find the answers for parts (d) and (e) above.

g. Frank also sets out some tomato plants. The height of the plants in inches after t days is:

$$h = 14 + 1.5t$$

What do the constants in this equation tell us about the tomato plants?

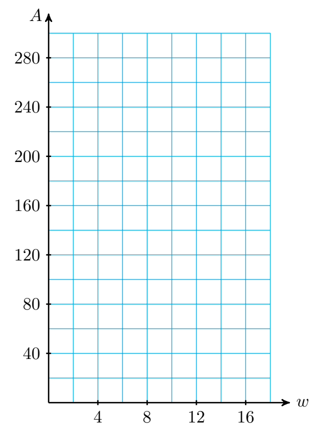
12. On October 31, Betty and Paul fill their 250-gallon oil tank for their heater. Beginning in November they use an average of 15 gallons of oil per week.

a. Complete the table of values for the amount of oil, A, left in the tank after w weeks.

w (weeks)	0	4	8	12	16
A (gallons)					

b. Write an equation that expresses the amount of oil, A, in terms of the number of weeks, w, since October 31.

c. Graph the equation.



d. How much did the amount of fuel oil in the tank decrease between the third week and the eighth week? Illustrate this amount on the graph.

e. During which weeks will the tank contain more than 175 gallons of fuel oil? Illustrate on the graph.

f. Write and solve an inequality to verify your answer to part (e).

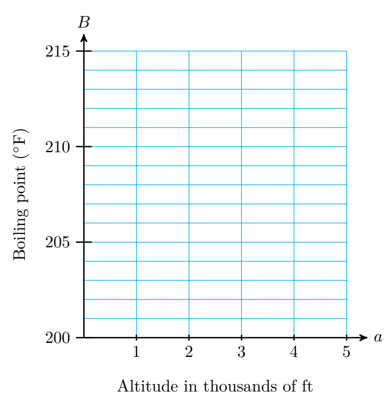
13. The boiling point of water changes with altitude. At sea level, water boils at 212°F, and the boiling point decreases by approximately 2°F for each 1000-foot increase in altitude.

a. Write an equation for the boiling point, B, in terms of a, the altitude in thousands of feet.

b. Complete the table of values.

Altitude (1000 ft)	0	1	2	3	4	5
Boiling point (°F)						

c. Graph the equation.



d. How much does the boiling point decrease when the altitude increases from 1000 to 3000 feet? Illustrate this amount on the graph.

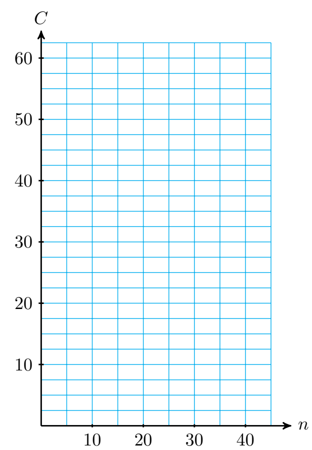
e. At what altitudes is the boiling point less than 204°F? Illustrate on the graph.

14. The taxi out of Dulles Airport charges a traveler with one suitcase an initial fee of \$2.00, plus \$1.50 for each mile traveled. Complete the table of values showing the charge, C, for a trip of n miles.

n	0	5	10	15	20	25
C						

a. Write an equation for the charge, C, in terms of the number of miles traveled, n.

b. Graph the equation.



c. What is the charge for a trip to Mount Vernon, 40 miles from the airport? Illustrate the answer on your graph.

d. If a ride to the National Institutes of Health (NIH) costs \$39.50, how far is it from the airport to the NIH? Illustrate the answer on your graph.

1.2 Graphs and Equations

1.2.1 Equations and Solutions

In the last section we studied some linear models, and in particular we looked at graphs and equations that described those models. In this section we review some techniques and terminology related to equations, inequalities, and their graphs.

Definition 1.2.1 Solution. A **solution** of an equation is a value of the variable that makes the equation true. ◊

For example, $x = \mathbf{-2}$ is a solution of the equation

$$4x^3 + 5x^2 - 8x = 4$$

because $4(\mathbf{-2})^3 + 5(\mathbf{-2})^2 - 8(\mathbf{-2}) = -2 + 20 + 16 = 4$.

Checkpoint 1.2.2 QuickCheck 1. Which of the following values are solutions of the equation

$$-2x^3 + x^2 + 16x = 15?$$

[TK]

a. $x = 2$

b. $x = -3$

c. $x = 3$

d. $x = 1$

[TK] For more help on checking for a solution, see Section 1.2.1 of the Toolkit.

A **linear equation** has no powers of the variable other than 1. It has at most one solution. We find the solution by transforming the equation into a simpler **equivalent equation** whose solution is obvious.

Example 1.2.3 Solve the equation $3(2x-5)-4x = 2x-(6-3x)$

Solution. We begin by simplifying each side of the equation.[TK]

$$\begin{aligned} 3(2x-5)-4x &= 2x-(6-3x) && \text{Apply the distrubutive law.}\\ 6x-15-4x &= 2x-6+3x && \text{Combine like terms.}\\ 2x-15 &= 5x-6 && \text{Add } \mathbf{-5x+15} \text{ to both sides}\\ -3x &= 9 && \text{Divide both sides by } \mathbf{-3.}\\ x &= -3 \end{aligned}$$

The solution is -3. You can check that substituting $x = -3$ into the original equation produces a true statement. □

[TK] For more help on solving linear equations, see Section 1.2.2 of the Toolkit.

Checkpoint 1.2.4 **Practice 1.** Find the solution of the equation $16-2(3x-1) = 4x+2(x-3)$.

1.2.2 Linear Inequalities

Although a linear equation can have at most one solution, a linear **inequality** can have many solutions. For example, complete the table of values for the expression $5-2x$:

x	-2	-1	0	1	2	3	4
$5-2x$							

Now use your table to list at least three solutions of the inequality $5-2x < 2$.

Example 1.2.5 Use algebra to solve the inequality $5-2x < 2$.

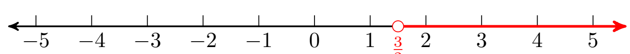
Solution. We begin by isolating the term containing the variable, just as we do when solving a linear equation. We subtract 5 from both sides to obtain

$$-2x < -3$$

Then we divide both sides by -2 to find

$$x > \frac{-3}{-2} = \frac{3}{2} \qquad \text{Reverse the direction of the inequality.}$$

Any value of x greater than $\frac{3}{2}$ is a solution of the inequality. We write the solutions as $x > \frac{3}{2}$. Because we cannot list all of these solutions, we often illustrate them as a graph on a **number line**, as shown below.



□

In the Example above, we used the following rule for solving linear inequalities.

Solving a Linear Inequality.

If we multiply or divide both sides of an inequality by a negative number, we must reverse the direction of the inequality.

Other than the rule stated in the box above, the rules for solving a linear inequality are the same as the rules for solving a linear equation. [TK]

[TK] For more help on solving linear inequalities, see Section 1.1.4 of the Toolkit.

Checkpoint 1.2.6 QuickCheck 2. Which of the following show the correct solutions of the inequality?

a. $2x > -8$ has solution $x < -4$

b. $-2x > -8$ has solution $x < 4$

c. $x + 2 > -8$ has solution $x < -10$

d. $x - 2 > -8$ has solution $x < -6$

Checkpoint 1.2.7 Practice 2. Solve the inequality $2x + 1 < 7$, and graph the solutions on a number line.

1.2.3 Equations in Two Variables

An **equation in two variables**, such as

$$-3x + 4y = 24$$

has many solutions. Each **solution** consists of an **ordered pair** of values, one for x and one for y, that together satisfy the equation (make the equation true.)

Checkpoint 1.2.8 QuickCheck 3. What is the solution of an equation in two variables?

a. A value of x that makes the equation true

b. An ordered pair of values (x, y) that satisfy the equation

c. Solving the equation for y in terms of x

d. The graph of the equation

Example 1.2.9

a. Is $(2, 7.5)$ a solution of the equation $-3x + 4y = 24$?

b. Is $(4, 3)$ a solution of the equation $-3x + 4y = 24$?

[TK]

Solution.

a. The ordered pair $(2, 7.5)$ is a solution of the equation above, because it satifies the equation.

$$-3(\mathbf{2}) + 4(\mathbf{7.5}) = -6 + 30 = 24$$

b. The ordered pair $(4, 3)$ is not a solution, because it does not satify the equation.

$$-3(\mathbf{4}) + 4(\mathbf{3}) = -12 + 12 = 0 \neq 24$$

□

[TK] For more help on finding solutions of an equation in two variables, see Section 1.2.3 of the Toolkit.

Checkpoint 1.2.10 Practice 3. Find some more solutions of the equation $-3x + 4y = 24$ and complete the table of values:

x	-12	-8		0	
y			3		9

Because an equation in two variables may have many solutions, we can use a graph to visualize those solutions.

Definition 1.2.11 Graph. The **graph** of an equation in two variables is just a picture of all its solutions. ◊

You might think it would be difficult to find *all* the solutions of an equation, but for a linear equation $Ax + By = C$, we can at least illustrate the solutions: all the solutions lie on a straight line. (Later on we can prove that this is true.)

Checkpoint 1.2.12 QuickCheck 4. True or false?

1. All the solutions of a linear equation in two variables lie on a straight line.
2. The equation $-3x + 4y = 24$ is a linear equation.
3. The graph of an equation in two variables is a picture of its solutions.
4. If a point lies on the graph of an equation, it is a solution of the equation.

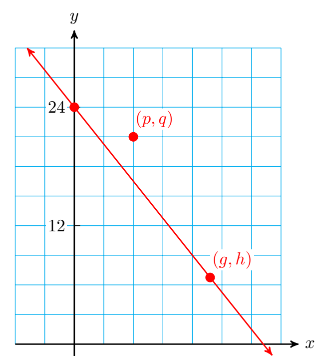
Example 1.2.13

The figure shows a graph of the equation $Ax + By = C$. Which of the following equations are true?

a. $24B = C$

b. $Ap + Bq = C$

c. $Ag + Bh = C$



Solution.

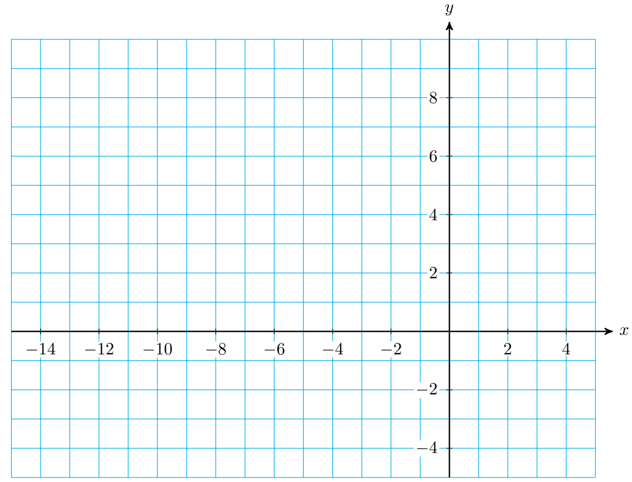
a. The point $(0, 24)$ lies on the graph, so $x = 0$, $y = 24$ is a solution of the equation. Thus, $A \cdot 0 + B \cdot 24 = C$, or $24B = C$ is a true statement.

b. The point (p, q) does not lie on the graph, so $x = p$, $y = q$ does not satisfy the equation, and $Ap + Bq = C$ is a not true.

c. The point (g, h) does lie on the graph, so $x = g$, $y = h$ does satisfy the equation, and $Ag + Bh = C$ is a true statement.

□

Checkpoint 1.2.14 Practice 4. On the grid below, plot the points you found in Practice 3. All the points should lie on a straight line; draw the line with a ruler or straightedge. Which of the following points lie on the graph?



a. $(-13, -4)$ b. $(-1.6, 4.8)$ c. $(1.25, 7)$

Which of the points above satisfy the equation in Practice 3?

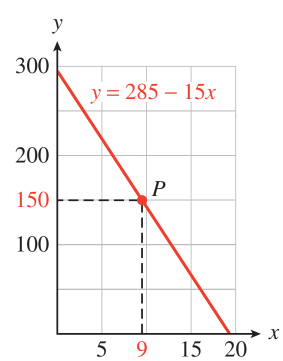
1.2.4 Graphical Solution of Equations and Inequalities

Here is a clever way to solve an equation in one variable by using a graph. Suppose we would like to solve the equation $150 = 285 - 15x$. We start by looking at the graph of $y = 285 - 15x$.

Example 1.2.15 Use the graph of $y = 285 - 15x$ to solve the equation

$$150 = 285 - 15x$$

Solution.



Compare the two equations in the problem. In the equation we want to solve, y has been replaced by **150**. We begin by locating the point P on the graph for which $y = 150$.

Next we find the x-coordinate of point P by drawing an imaginary line from P straight down to the x-axis. The x-coordinate of P is $x = 9$.

Thus, P is the point $(9, 150)$, and $x = 9$ when $y = 150$. The solution we seek is $x = 9$.

You can verify the solution algebraically by substituting $x = \mathbf{9}$ into the equation:

Does $150 = 285 - 15(\mathbf{9})$?

$$285 - 15(\mathbf{9}) = 285 - 135 = 150 \qquad \text{Yes}$$

□

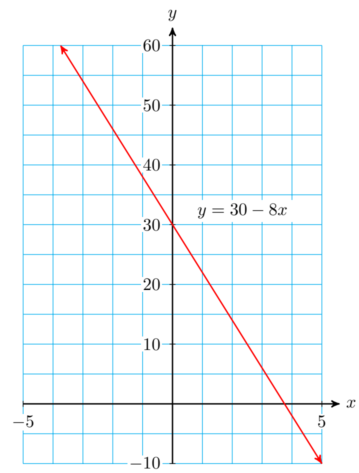
Note 1.2.16 The relationship between an equation and its graph is an important one. For the previous example, make sure you understand that the following three statements are equivalent.

1. The point $(9, 150)$ lies on the graph of $y = 285 - 15x$.
2. The ordered pair $(9, 150)$ is a solution of the equation $y = 285 - 15x$.
3. $x = 9$ is a solution of the equation $150 = 285 - 15x$.

Checkpoint 1.2.17 Practice 5.

a. Use the graph of $y = 30 - 8x$ to solve the equation $30 - 8x = 50$. Follow the steps:
   - Step 1: Locate the point P on the graph with $y = 50$.
   - Step 2: Find the x-coordinate of your point P.

b. Verify your solution algebraically.



We can also use graphs to solve inequalities. Consider the inequality

$$285 - 15x \geq 150$$

To solve this inequality means to find all values of x that make the expression $285 - 15x$ greater than or equal to 150. We could begin by trying some values of x. Here is a table obtained by evaluating $285—15x$.

x	0	2	4	6	8	10	12
$285 - 15x$	285	255	225	195	165	135	105

From the table, we see that values of x less than or equal to 8 are solutions of the inequality, but we have not checked *all* possible x-values. We can get a more complete picture from a graph.

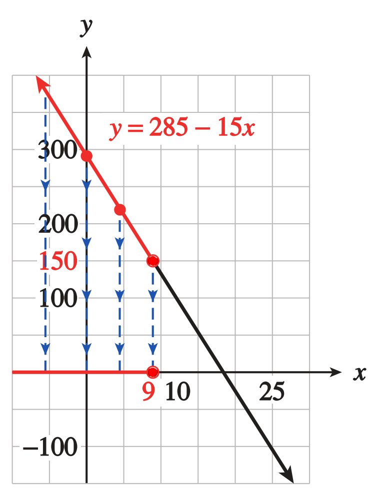
Example 1.2.18 Use the graph of the equation $y = 285 - 15x$ to solve the inequality

$$285 - 15x > 150$$

Solution.

We look for points on the graph with y-coordinates greater than or equal to 150. These points are shown in color. Which x-values produced these points?

We can read the x-coordinates by dropping straight down to the x-axis, as shown by the arrows. For example, the x-value corresponding to $y = 150$ is $x = 9$. For larger values of $285-15x$, we must choose x-values less than 9. Thus, all values of x less than or equal to 9 are solutions, as shown on the x-axis.



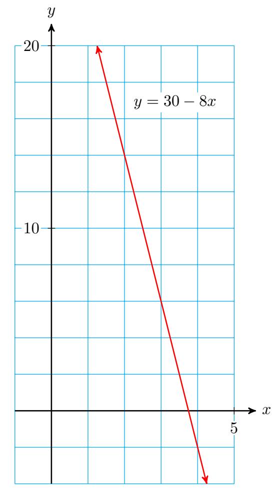
We write the solutions as $x \leq 9$. □

Checkpoint 1.2.19 Practice 6.

a. Use the graph of $y = 30 - 8x$ to solve the inequality $30 - 8x < 14$. Follow the steps:

- Step 1: Locate the point P on the graph with $y = 14$.
- Step 2: Find the x-coordinate of the point P.
- Step 3: Which points on the graph have $y < 14$? Mark them on the graph.
- Step 4: Find the x-coordinates of the points in Step 3. Mark them all on the x-axis.

b. Verify your solution algebraically.



1.2.5 Using a Graphing Utility

We can use a graphing utility to graph equations if they are written in the form $y = (\text{expression in } x)$. First, let's review how to solve an equation for y in terms of x.

Example 1.2.20 Solve the equation $6x - 5y = 90$ for y in terms of x.

Solution. To begin, we isolate the y-term by subtracting $6x$ from both sides of the equation.

$$\begin{aligned} -5y &= 90 - 6x && \text{Divid both sides by } \mathbf{-5.} \\ y &= \frac{90}{-5} - \frac{6x}{-5} && \text{Simplify.} \\ y &= -18 + \frac{6}{5}x \end{aligned}$$

Note that the equation now has the form of the linear models we saw in Section 1.1. □

Checkpoint 1.2.21 Practice 7. Solve for y in terms of x: $10y - 15x = 6$

Now we are ready to graph an equation with technology. For most graphing utilities, we follow three steps.

To Graph an Equation.

1. Enter the equation you wish to graph.
2. Select a graphing window.
3. Graph.

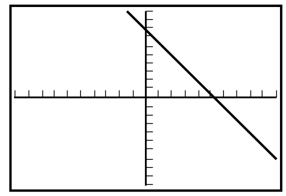
Choosing a graphing window corresponds to drawing the x- and y-axes and marking a scale on each axis when we graph by hand. The **standard graphing window** displays values from -10 to 10 on both axes. We can start with this window and then adjust it if necessary.

Example 1.2.22 Use a graphing utility to graph the equation $3x + 2y = 16$.

Solution. First, we solve the equation for y in terms of x.

$$\begin{aligned} 3x + 2y &= 16 && \text{Subtract } \mathbf{3x} \text{ from both sides.} \\ 2y &= -3x + 16 && \text{Divide both sides by 2.} \\ y &= \frac{-3x}{2} + \frac{16}{2} && \text{Simplify.} \\ y &= -1.5x + 8 \end{aligned}$$

Now enter $-1.5x + 8$ after $Y =$, and choose the standard graphing window. The graph is shown at right.



□

Don't forget to erase your graph when you are done.

Checkpoint 1.2.23 Practice 8.

a. Solve the equation $-3x + 4y = 24$ for y in terms of x.

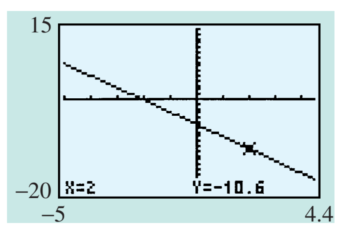
b. Graph the equation in the standard window.

We can use the TRACE feature to find the coordinates of points on a graph.

Example 1.2.24 Use a graph to find a solution to the equation $y = -2.6x - 5.4$ with y-coordinate -10.6.

Solution. First we graph the equation $y = -2.6x - 5.4$ in the window

Xmin= -5	Xmax= 4.4
Ymin= -20	Ymax= 15



We press TRACE, and a "bug" begins flashing on the display. The coordinates of the bug appear at the bottom of the display, as shown in the figure. We use the left and right arrows to move the bug along the graph. You can check that the coordinates of the point $(2, -10.6)$ do satisfy the equation $y = -2.6x - 5.4$.

□

Checkpoint 1.2.25 Practice 9.

a. Graph the equation $y = 32x - 42$ in the window:

Xmin= -4.7	Xmax= 4.7	Xscl= 1
Ymin= -250	Ymax= 50	Yscl= 25

b. Use the Trace feature to find the point that has y-coordinate -122.

c. Verify your answer algebraically by substituting your x-value into the equation.

1.2.6 Problem Set 1.2

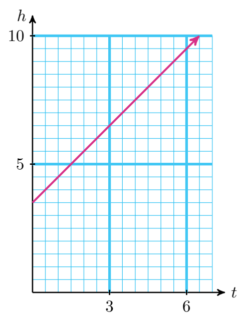
Warm Up

Exercise Group. For Problems 1–2, decide whether the ordered pairs are solutions of the equation whose graph is shown.

1.

a. $(6.5, 3)$ c. $(8, 2)$

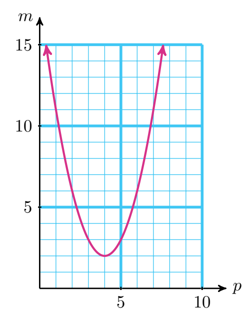
b. $(0, 3.5)$ d. $(4.5, 1)$



2.

a. $(2, 6)$ c. $(10, 0)$

b. $(4, 2)$ d. $(11, 7)$



Exercise Group. For Problems 3–6, decide whether the ordered pairs are solutions of the given equation.

3. $y = \dfrac{3}{4}x$

a. $(8, 6)$ c. $(2, 3)$

b. $(12, 16)$ d. $(6, \dfrac{9}{2})$

4. $y = \dfrac{x}{2.5}$

a. $(1, 2.5)$ c. $(5, 2)$

b. $(25, 10)$ d. $(8, 20)$

5. $w = z - 1.8$

a. $(10, 8.8)$ c. $(2, \dfrac{1}{5})$

b. $(6, 7.8)$ d. $(9.2, 7.4)$

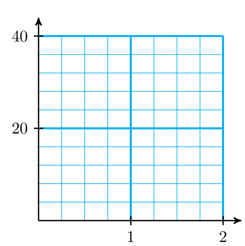
6. $w = 120 - z$

a. $(0, 120)$ c. $(150, 30)$

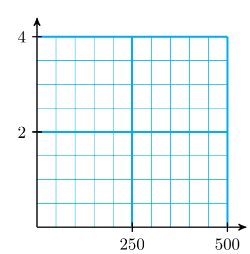
b. $(65, 55)$ d. $(9.6, 2.4)$

Exercise Group. For Problems 7–8, state the interval that each grid line represents on the horizontal and vertical axes.

7.

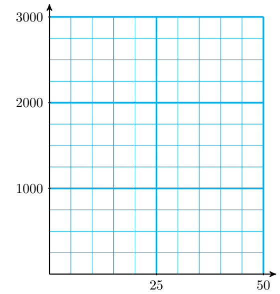


8.



9.

a. What interval does each grid line represent on the horizontal axis? On the vertical axis?

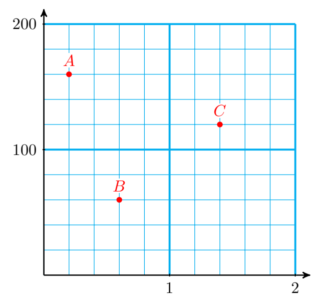


b. Plot the following points on the grid:

$$(0, 500),\ (20, 1750),\ (40, 250)$$

10.

a. What interval does each grid line represent on the horizontal axis? On the vertical axis?



b. Find the coordinates of each point.

Skills Practice

Exercise Group. For Problems 11–14, solve.

11. $5 - 2(3x - 4) = 28 - 2x$

12. $0.0048z - 0.12 = -0.08 + 0.0016z$

13. $0.25t + 0.10(t - 4) = 11.85$

14. $0.12t + 0.08(t + 10,000) = 12,000$

Exercise Group. For Problems 15–18, solve the inequality and graph the solutions on a number line.

15. $3x - 2 > 1 + 2x$

16. $\dfrac{-2x - 6}{-3} > 2$

17. $\dfrac{-2x - 3}{2} \le -5$

18. $\dfrac{2x - 3}{3} \le \dfrac{3x}{-2}$

Exercise Group. For Problems 19–24, solve for y in terms of x.

19. $2x - 3y = -72$

20. $4x + 75y = 60,000$

21. $7x = 91 - 13y$

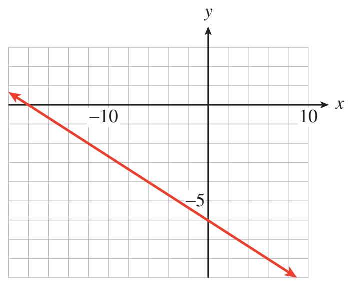
22. $\dfrac{x}{80} + \dfrac{y}{400} = 1$

23. $80x - 360y = 6120$

24. $3x + \dfrac{3}{4}y = \dfrac{1}{2}$

Applications

25. The figure shows a graph of $y = \dfrac{-x}{3} - 6$.



a. Use the graph to find all values of x for which

i. $y = -4$

ii. $y > -4$

iii. $y < -4$

b. Use the graph to solve

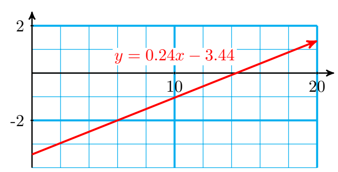
i. $\dfrac{-x}{3} - 6 = -4$

ii. $\dfrac{-x}{3} - 6 > -4$

iii. $\dfrac{-x}{3} - 6 < -4$

c. Explain why your answers to parts (a) and (b) are the same.

26. The figure shows a graph of $y = 0.24x - 3.44$.

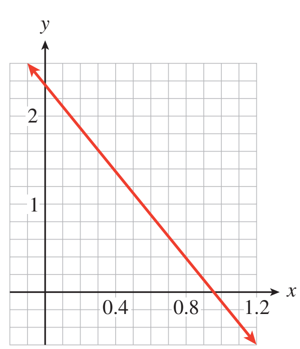


a. Use the graph to solve $0.24x - 3.44 = -2$.

b. Use the graph to solve $0.24x - 3.44 > -2$.

c. Solve the inequality in part (b) algebraically.

27. The figure shows the graph of $y = -2.4x + 2.32$. Use the graph to solve:



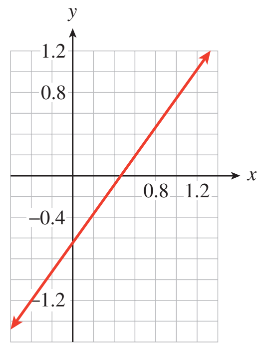
a. $1.6 = -2.4x + 2.32$

b. $-2.4x + 2.32 = 0.4$

c. $-2.4x + 2.32 \geq 1.6$

d. $0.4 \geq -2.4x + 2.32$

28. The figure shows the graph of $y = 1.4x - 0.64$. Use the graph to solve:



a. $1.4x - 0.64 = 0.2$

b. $-1.2 = 1.4x - 0.64$

c. $1.4x - 0.64 > 0.2$

d. $-1.2 > 1.4x - 0.64$

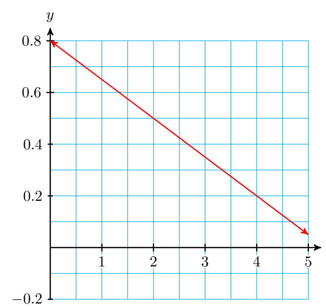
29. Here is a graph of

$$y = 0.8 - kx$$

Use the graph to solve:

a. $0.8 - kx \geq 0.2$

b. $0.5 > 0.8 - kx$



30. Here is a graph of

$$y = mx + b$$

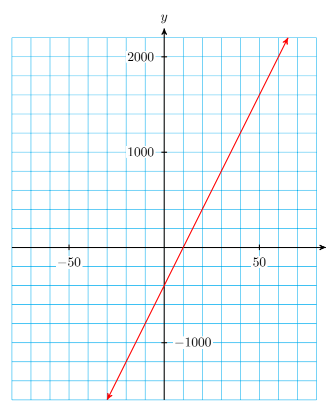
Use the graph to solve:

a. $mx + b = 1200$

b. $-800 = mx + b$

c. $mx + b > 400$

d. $-1200 \geq mx + b$



Exercise Group. In Problems 31 and 32, graph each equation in the window

Xmin= -47	Xmax= 47	Xscl= 10
Ymin= -31	Ymax= 31	Yscl= 10

31. Graph $y = -0.4x + 3.7$. Use the graph to solve the equation or inequality. Then check your answers algebraically.

a. Solve $-0.4x + 3.7 = 2.1$

b. Solve $-0.4x + 3.7 > -5.1$

32. Graph $y = 6.5 - 18x$. Use the graph to answer the questions. Then check your answers algebraically.

a. For what value of x is $y = -13.3$?

b. For what value of x is $y = 24.5$?

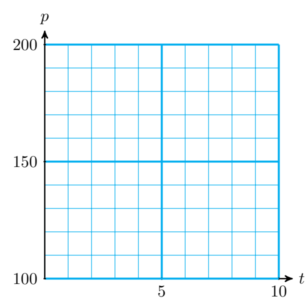
c. For what values of x is $y \leq 15.5$?

d. For what values of x is $y \geq -7.9$?

33. Kieran's resting blood pressure, in mm Hg, is 120, and it rises by 6 mm for each minute he jogs on a treadmill programmed to increase the level of intensity at a steady rate.

a. Find a formula for Kieran's blood pressure, p, in terms of time, t.

b. Graph the equation for p for $0 \leq t \leq 10$.



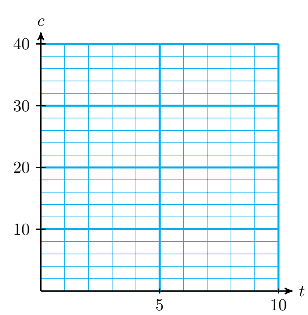
c. What is Kieran's blood pressure after 3.5 minutes? Label this point on the graph.

d. Kieran's blood pressure should not exceed 165 mm Hg. When will this level be reached? Label this point on the graph.

34. When Francine is at rest, her cardiac output is 5 liters per minute. The output increases by 3 liters per minute for each minute she spends on a cycling machine with increasing intensity.

a. Find a formula for Francine's cardiac output, c, in terms of time, t.

b. Graph the equation for c for $0 \leq t \leq 10$.



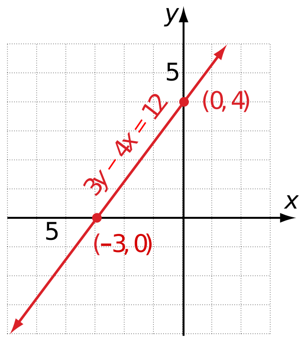
c. What will Francine's cardiac output be after 6 minutes? Label this point on the graph.

d. When will Francine's cardiac output exceed 14.5 liters per minute? Label this point on the graph.

1.3 Intercepts

1.3.1 Intercepts of a Graph

Here is a graph of the linear equation $3y - 4x = 12$. Notice the points where the graph crosses the x- and y-axes. These points are called the **intercepts** of the graph.



Definition 1.3.1 Intercepts. The points at which a graph crosses the axes are called the **intercepts** of the graph. ◊

The x-intercept of the graph shown above is $(-3, 0)$, and its y-intercept is $(0, 4)$. The intercepts can help us graph a linear equation and can help us interpret the meaning of a linear model.

Checkpoint 1.3.2 QuickCheck 1. What are the intercepts of a graph?

a. The variables displayed on the axes

b. Points where the graphs intersect

c. The highest and lowest points

d. Points where the graph intersects the axes

The coordinates of the intercepts are easy to find.

Intercepts of a Graph.

1. To find the x-intercept, we set $y = 0$ and solve for x.

2. To find the y-intercept, we set $x = 0$ and solve for y.

Example 1.3.3 In Example 1.1.11 of Section 1.1, we graphed an equation, $g = 20 - \frac{1}{12}d$, for the amount of gasoline, g, left in Leon's tank after he has driven for d miles. Find the intercepts of the graph.

Solution. To find the d-intercept, we set $g = 0$ and solve for d.

$$\begin{aligned} \mathbf{0} &= 20 - \frac{1}{12}d && \text{Add } \frac{1}{12}d \text{ to both sides.} \\ \frac{1}{12}d &= 20 && \text{Multiply both sides by 12.} \\ d &= 240 \end{aligned}$$

The d-intercept is $(240, 0)$. To find the g-intercept, we set $d = 0$ and solve for g.

$$g = 20 - \frac{1}{12}(\mathbf{0}) = 20$$

The g-intercept is $(0, 20)$. □

Checkpoint 1.3.4 Practice 1. Find the intercepts of the graph of $y = -9 - \frac{3}{2}x$.

1.3.2 Meaning of the Intercepts

The intercepts of a graph give us information about the situation it models.

Example 1.3.5 What do the intercepts of the graph in the Example above tell us about Leon's camper? [TK]

Solution. The d-intercept tells us that when $d = 240$, $g = 0$, or that when Leon has traveled 240 miles, he has 0 gallons of gasoline left; the fuel tank is empty.

The g-intercept tells us that when $d = 0$, $g = 20$, or that when Leon has traveled 0 miles, he has 20 gallons of gasoline left. The fuel tank holds 20 gallons when full. □

[TK] For more help on interpretineg the intercepts, see Section 1.3.2 of the Toolkit.

Checkpoint 1.3.6 Practice 2. The gas tank in Rosa's Toyota Prius holds 11 gallons, and she gets 48 miles to the gallon.

a. Write an equation for the amount of gasoline, g, left in the tank after Rosa has driven for d miles.

b. Find the intercepts of the graph. What do they tell us about the problem situation?

Hint: Rewrite the sentence with mathematical symbols:

(gasoline left) = (gallons in full tank) − (mileage rate) × (miles driven)

1.3.3 General Form for a Linear Equation

The order of the terms in a linear equation does not matter. For example, the equation in Example 1.1.4 of Section 1.1

$$C = 5 + 3t \qquad \text{can be written equivalently as} \quad -3t + C = 5$$

and the equation in Example 1.1.11 of that section,

$$g = 20 - \frac{1}{12}d \qquad \text{can be written as } \frac{1}{12}d + g = 20$$

This form of a linear equation, $Ax + By = C$, is called the **general form**.

Definition 1.3.7 General Form for a Linear Equation. The **general form** for a linear equation is

$$\boldsymbol{Ax + By = C}$$

(where A and B cannot both be 0). ◊

Checkpoint 1.3.8 QuickCheck 2. What is the general form of a linear equation?

a. $y = mx + b$

b. $Ax + By = C$

c. Any equation whose graph is a straight line

d. Set $x = 0$ and solve for y.

Some linear models are easier to use when they are written in the general form.

Example 1.3.9 The manager at Albert's Appliances has \$3000 to spend on advertising for the next fiscal quarter. A 30-second spot on television costs \$150 per broadcast, and a 30-second radio ad costs \$50.

a. The manager decides to buy x television ads and y radio ads. Write an equation relating x and y.

b. Make a table of values showing several choices for x and y.

c. Plot the points from your table, and graph the equation.

Solution.

a. Each television ad costs \$150, so x ads will cost \$$150x$. Similarly, y radio ads will cost \$$50y$. The manager has \$3000 to spend, so the sum of the costs must be \$3000. Thus,

$$150x + 50y = 3000$$

b. We choose some values of x, and solve the equation for the corresponding value of y. For example, if $x = \mathbf{10}$ then

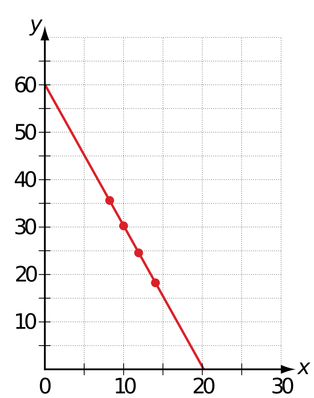
$$\begin{aligned} 150(\mathbf{10}) + 50y &= 300 \\ 1500 + 50y &= 3000 \\ 50y &= 1500 \\ y &= 30 \end{aligned}$$

If the manager buys 10 television ads, she can also buy 30 radio ads. You can verify the other entries in the table.

x	8	10	12	14
y	36	30	24	18

c.

We plot the points from the table.
All the solutions lie on a straight line.



□

Checkpoint 1.3.10 Practice 3. The manager at Breadbasket Bakery has \$120 to spend on advertising. An ad in the local newspaper costs \$15, and posters cost \$4 each. She decides to buy x ads and y posters. Write an equation relating x and y.

Hint: Use the general form for a linear equation. What is the total amount of money the manager will spend?

1.3.4 Intercept Method of Graphing

Because we really need only two points to graph a linear equation, we might as well find the intercepts and use them to draw the graph.

To Graph a Linear Equation by the Intercept Method.

a. Find the horizontal and vertical intercepts.

b. Plot the intercepts, and draw the line through the two points.

Example 1.3.11

a. Find the x- and y-intercepts of the graph of $150x + 50y = 3000$.

b. Use the intercepts to graph the equation. [TK]

Solution.

a. To find the x-intercept, we set $y = \mathbf{0}$.

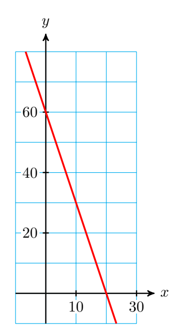
$$\begin{aligned} 150x - 50(\mathbf{0}) &= 3000 && \text{Simpify.} \\ 150x &= 3000 && \text{Divide both sides by 150.} \\ x &= 20 \end{aligned}$$

The x-intercept is the point $(20, 0)$. To find the y-intercept, we set $x = \mathbf{0}$.

$$\begin{aligned} 150(\mathbf{0}) - 50y &= 3000 && \text{Simpify.} \\ 506 &= 3000 && \text{Divide both sides by 50.} \\ y &= 60 \end{aligned}$$

The y-intercept is the point $(0, 60)$.

b. We scale both axes in intervals of 10 and then plot the two intercepts, $(20, 0)$ and $(0, 60)$. We draw the line through them, as shown.



□

[TK] For more help on graphing by the intercept method, see Section 1.3.1 of the Toolkit.

Checkpoint 1.3.12 QuickCheck 3. Describe the intercept method of graphing a linear equation.

a. Make a table of values and plot the points.

b. Extend the line until it crosses both axes.

c. Solve for y in terms of x.

d. Plot the points where $x = 0$ and where $y = 0$, then draw the line through them

Checkpoint 1.3.13 Practice 4. Find the x- and y-intercepts of the equation in Practice 3 (about the Breadbasket Bakery), and use the intercepts to graph the equation.

Hint: Choose convenient scales for the x- and y-axes.

1.3.5 Two Forms for Linear Equations

We have now seen two forms for linear equations: the general linear form,

$$\boldsymbol{Ax + By = C}$$

and the form for a linear model,

$$\boldsymbol{y} = (\text{starting value}) + (\text{rate}) \times \boldsymbol{t}$$

Sometimes it is useful to convert an equation from one form to the other.

Example 1.3.14

a. Write the equation $4x - 3y = 6$ in the form for a linear model.

b. Write the equation $y = -9 - \dfrac{3}{2}x$ in general linear form.

Solution.

a. We would like to solve for y in terms of x. We first isolate the y-term on one side of the equation.

$$\begin{aligned} 4x - 3y &= 6 && \text{Subtract } \boldsymbol{4x} \text{ from both sides.}\\ -3y &= 6 - 4x && \text{Divide both sides by } \boldsymbol{-3.}\\ \frac{-3y}{-3} &= \frac{6-4x}{-3} && \text{Simplify: divide each term by } \boldsymbol{-3.}\\ y - 2 &+ \frac{4}{3}x \end{aligned}$$

b. Write the equation in the form $Ax + By = C$.

$$\begin{aligned} y &= -9 - \frac{3}{2}x && \text{Add } \boldsymbol{\frac{3}{2}x} \text{ to both sides.}\\ \frac{3}{2}x + y &= -9 \end{aligned}$$

We can write the equation with integer coefficients by clearing the fractions. We multiply both sides of the equation by **2**.

$$\begin{aligned} \mathbf{2}\left(\frac{3}{2}x + y\right) &= -9(\mathbf{2})\\ 3x + 2y &= -18 \end{aligned}$$

□

[TK] For more help on solving for y, see Section 1.3.3 of the Toolkit.

Caution 1.3.15 Do not confuse solving for y in terms of x with finding the y-intercept. Compare:

a. In the Example above, we solved $4x-3y=6$ for y in terms of x to get

$$y=-2+\frac{4}{3}x.$$

This is still an equation in two variables; it is just another (equivalent) form of the original equation.

b. To find the y-intercept of the same equation, we first set $x=0$, then solve for y, as follows:

$$\begin{aligned}4(0)-3y&=6\\ y&=-2\end{aligned}$$

This gives us a **particular point** on the graph, namely, $(0,-2)$; the point whose x-coordinate is 0.

Checkpoint 1.3.16 Practice 5.

a. Write the equation $150x+50y=3000$ in the form for a linear model.

b. Write the equation $y=0.15x-3.8$ in general linear form with integer coefficients.

1.3.6 Problem Set 1.3

Warm Up

1. The owner of a movie theater needs to bring in \$1000 revenue at each screening in order to stay in business. He sells adults' tickets for \$5 each and children's tickets at \$2 each.

a. How much revenue does he earn from selling x adults' tickets?

b. How much revenue does he earn from selling y children's tickets?

c. Write an equation in x and y for the number of tickets he must sell at each screening

2. Karel needs 45 milliliters of a 40% solution of carbolic acid. He plans to mix some 20% solution with some 50% solution.

a. How much carbolic acid is in x milliliters of the 20% solution?

b. How much carbolic acid is in y milliliters of the 50% solution?

c. How much carbolic acid is in the solution Karel needs?

d. Write an equation in x and y for the amount of each solution Karel should mix.

Exercise Group. For Problems 3 and 4, solve the equation for y in terms of x.

3. $3x+5y=16$

4. $20x=30y-45,000$

Skills Practice

Exercise Group. For Problems 5-8,

a. Find the intercepts of the graph.

b. Graph the equation by the intercept method.

5. $9x - 12y = 36$

6. $\dfrac{x}{9} - \dfrac{y}{4} = 1$

7. $4y = 20 + 2.5x$

8. $30x = 45y + 60,000$

9. Find the intercepts of the graph for each equation.

a. $\dfrac{x}{3} + \dfrac{y}{5} = 1$

b. $2x - 4y = 1$

c. $\dfrac{2x}{5} - \dfrac{2y}{3} = 1$

d. $\dfrac{x}{p} + \dfrac{y}{q} = 1$

e. Why is the equation $\dfrac{x}{a} + \dfrac{y}{b} = 1$ called the **intercept form** for a line?

10. Write an equation in intercept form (see Problem 9) for the line with the given intercepts. Then write the equation in general form.

a. $(6, 0), (0, 2)$

b. $(-3, 0), (0, 8)$

c. $\left(\dfrac{3}{4}, 0\right), \left(0, \dfrac{-1}{4}\right)$

d. $(v, 0), (0, -w)$

e. $\left(\dfrac{1}{H}, 0\right), \left(0, \dfrac{1}{T}\right)$

11.

a. Find the y-intercept of the line $y = mx + b$.

b. Find the x-intercept of the line $y = mx + b$.

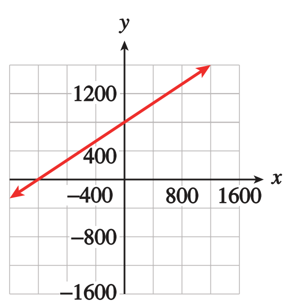
12.

a. Find the y-intercept of the line $Ax + By = C$.

b. Find the x-intercept of the line $Ax + By = C$.

Exercise Group. For Problems 13-16, write an equation in general form for the line.

13.



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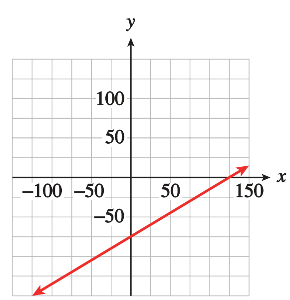

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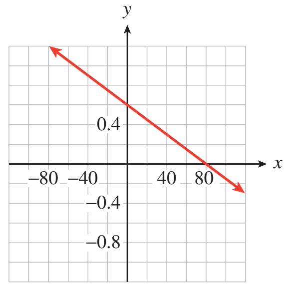
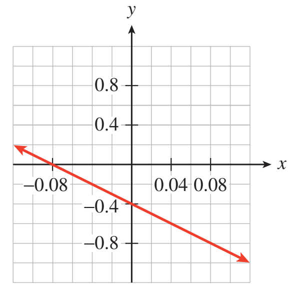


16.



Exercise Group. For Problems 17-20, write the equation in two standard forms:

a. the general linear form, $Ax + By = C$, with integer coefficients, and

b. the form for a linear model, $y = (\text{starting value}) + (\text{rate}) \times x$

17. $\dfrac{2x}{3} + \dfrac{3y}{11} = 1$

18. $\dfrac{8x}{7} + \dfrac{2y}{7} = 1$

19. $0.4x = 4.8 - 1.2y$

20. $-0.8y = 12.8 - 3.2x$

Applications

21. Delbert must increase his daily potassium intake by 1800 mg. He decides to eat a combination of figs and bananas. One gram of fig contains 9 mg of potassium, and one gram of banana contains 4 mg of potassium.

a. How many mg of potassium are in x grams of figs?

b. How many mg of potassium are in y grams of bananas?

c. Write an equation for the number of grams of fig, x, and the number of grams of banana, y, that Delbert needs to eat daily.

d. Find the intercepts of the graph. What do the intercepts tell us about Delbert's diet?

22. Five pounds of body fat is equivalent to approximately 16,000 calories. Carol can burn 600 calories per hour bicycling and 400 calories per hour swimming.

a. How many calories will Carol burn in x hours of cycling?

b. How many calories will she burn in y hours of swimming?

c. Write an equation that relates the number of hours, x, of cycling and y, of swimming Carol needs to perform in order to lose 5 pounds.

d. Find the intercepts of the graph. What do the intercepts tell us about Carol's exercise program?

23. A deep-sea diver is taking some readings at a depth of 400 feet. He begins rising at a rate of 20 feet per minute.

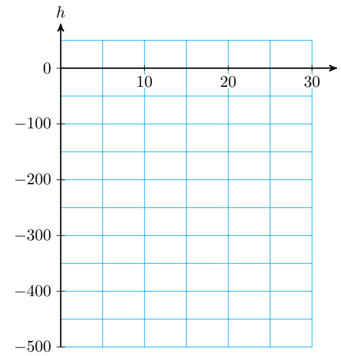
a. Complete the table of values for the diver's altitude h after t minutes. (A depth of 400 feet is the same as an altitude of -400 feet.)

t	0	5	10	15	20
h					

b. Write an equation for the diver's altitude, h, in terms of the number of minutes, t, elapsed.

c. Find the intercepts and sketch the graph.

t	h
0	
	0



d. Explain what each intercept tells us about this problem.

24. In central Nebraska, each acre of corn requires 25 acre-inches of water per year, and each acre of winter wheat requires 18 acre-inches of water. (An acre-inch is the amount of water needed to cover one acre of land to a depth of one inch.) A farmer can count on 9000 acre-inches of water for the coming year. (Source: Institute of Agriculture and Natural Resources, University of Nebraska)

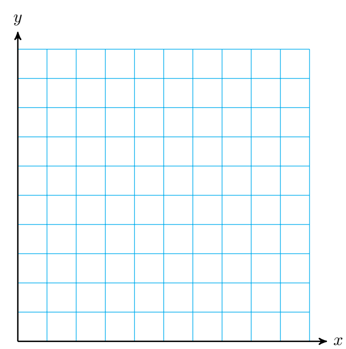
a. Write an equation relating the number of acres of corn, x, and the number of acres of wheat, y, that the farmer can plant.

b. Complete the table.

x	50	100	150	200
y				

c. Find the intercepts of the graph.

x	y
0	
	0



d. Use the intercepts to help you choose appropriate scales for the axes, and graph the equation.

e. What do the intercepts tell us about the problem?

f. What does the point $(288, 100)$ mean in this context?

25. The owner of a gas station has $19,200 to spend on unleaded gas this month. Regular unleaded costs him $2.40 per gallon, and premium unleaded costs $3.20 per gallon.

a. How much do x gallons of regular cost? How much do y gallons of premium cost?

b. Write an equation in general form that relates the amount of regular unleaded gasoline, x, the owner can buy and the amount of premium unleaded, y.

c. Find the intercepts and sketch the graph.

d. What do the intercepts tell us about the amount of gasoline the owner can purchase?

26. Leslie plans to invest some money in two CD accounts. The first account pays 3.6% interest per year, and the second account pays 2.8% interest per year. Leslie would like to earn \$500 per year on her investment.

a. If Leslie invests x dollars in the first account, how much interest will she earn? How much interest will she earn if she invests y dollars in the second account?

b. Write an equation in general form that relates x and y if Leslie earns \$500 interest.

c. Find the intercepts and sketch the graph.

d. What do the intercepts tell us about Leslie's investments?

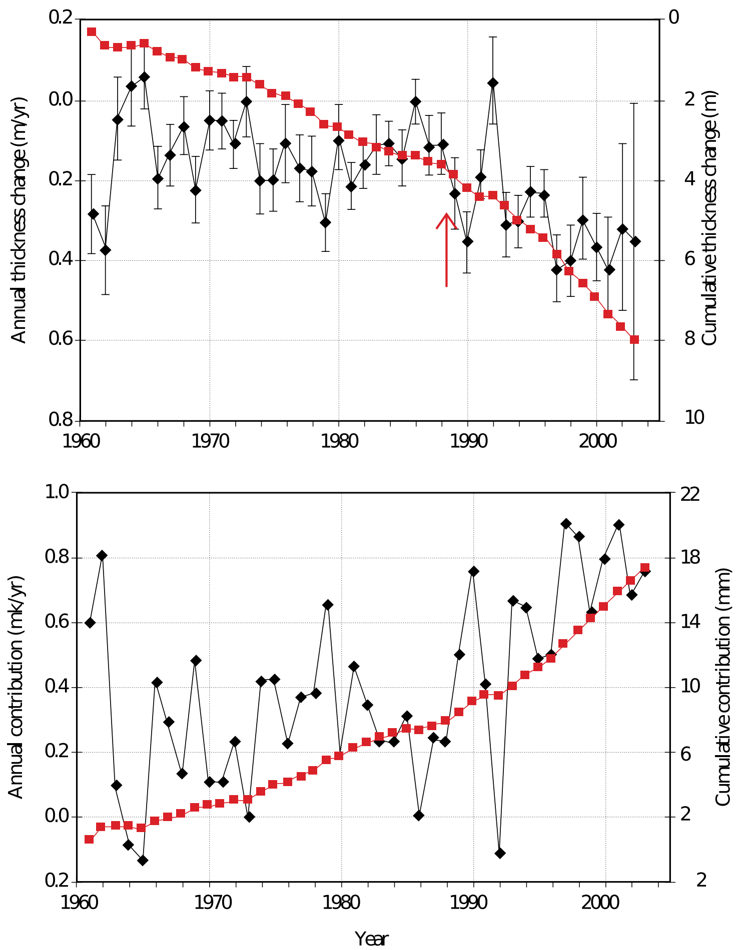
1.4 Slope

The introduction to Chapter 1 discussed the rise in sea level over the last few decades. One of the contributing factors to this rise is the melting of Earth's glaciers.

Glaciers around the world are retreating at accelerating rates, and many lower-latitude mountain glaciers may soon disappear entirely. The meltwater from these smaller glaciers contributed as much as 40% to the total rise in sea level over the 1990s.

By measuring the change in ice and snow height at fixed points and multiplying by the surface area of the glacier, scientists calculate the total volume of water lost from land-based glaciers. Dividing this volume by the surface area of the world's oceans gives the resulting change in sea level.

The graphs below show the change in thickness of the land-based glaciers overthe past 5 years, and the rise in sea level attributed to their melting. In this section we consider how to measure a rate of change.



1.4.1 Rate of Change

The equation we use to describe a linear model, $y = mx + b$, gives the starting value b of the variable y and the rate m at which y changes. Now we will look more closely at rates and how they appear on the graph of the model. First, let's review the notion of a ratio used as a rate. You are already familiar with several types of rates.

Definition 1.4.1 Rate. A **rate** is a type of ratio that compares two quantities with different units. ◊

Checkpoint 1.4.2 QuickCheck 1. A ratio that compares two quantities with different units is called a __________.

In the examples below, notice that each rate has units of the form $\dfrac{\text{something}}{\text{something else}}$, which we read as "something per something else."

Example 1.4.3 Caryn bought 7 used paperback novels for a total of \$2.45. Use a ratio to calculate the price per book. [TK]

Solution.

$$\frac{\text{total cost in dollars}}{\text{number of books}} = \frac{2.45 \text{ dollars}}{7 \text{ books}} = 0.35 \text{ dollars/book}$$

The novels were priced at a rate of \$0.35 per book. □

[TK] For more help on using ratios and rates, see Section 1.4.1 of the Toolkit.

Checkpoint 1.4.4 Practice 1. Delbert biked 34 miles in 5 hours. Use a ratio to calculate his average speed.

$$\frac{\text{distance in miles}}{\text{time in hours}} =$$

Delbert biked at a rate of __________.

The rate you calculated in Practice 1, Delbert's average speed, actually compares the *change* in two variables, his distance from his starting point, and the time elapsed. This type of rate appears often in linear models.

Definition 1.4.5 Rate of Change. A **rate of change** is a special kind of ratio that compares the change in two quantities or variables. ◊

In the next Example we calculate speed as a rate of change, and we introduce a new notation, $\boldsymbol{\Delta}$, to help with the calculation.

Example 1.4.6 Gregor is driving across Montana. At 1 pm his trip odometer reads 189 miles, and at 4 pm it reads 360 miles. Calculate Gregor's average speed as a rate of change.

Solution. We have two variables: time, t, and distance, d, and the following data points:

t	d
1	189
4	360

Gregor's speed is the ratio of the distance he traveled to the time it took. The distance he traveled is the change in his odometer reading (from 189 miles to 360 miles), and the time it took is the change in the clock reading (from 1 pm to 4 pm).The units of this ratio are miles per hour.

In mathematics, we use the symbol $\boldsymbol{\Delta}$ (delta) for **change in**. Thus

$$\text{distance traveled} = \Delta d = 360 - 189 = 171 \text{ miles}$$
$$\text{time elapsed} = \Delta t = 4 - 1 = 3 \text{ hours}$$

Gregor's average speed is the ratio $\dfrac{\text{distance traveled}}{\text{time elapsed}} = \dfrac{\Delta d}{\Delta t}$, so

$$\text{Speed} = \frac{\Delta d}{\Delta t} = \frac{171 \text{ miles}}{3 \text{ hours}} = 57 \text{ miles/hour}$$

□

Checkpoint 1.4.7 QuickCheck 2. What does the symbol Δ stand for?

a. A triangle

b. Speed

c. Change in

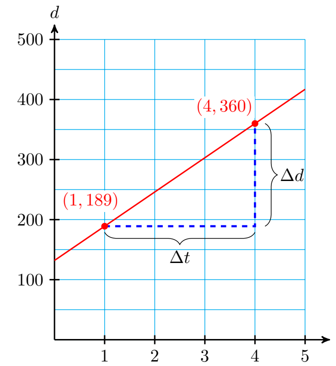
d. Coordinate

Checkpoint 1.4.8 Practice 2. Nelson is a long-distance truck driver. On a recent trip through the Midwest, he noted these odometer readings:

4 am	127 miles
10 am	421 miles

What was Nelson's average speed?

How do we see the rate of change on a graph? Let's consider the example above. The graph shows Gregor's distance, d, at time t. We plot the two data points, $(1, 189)$ and $(4, 360)$, and draw a straight line joining them. We can illustrate Δd and Δt by vertical and horizontal line segments, as shown on the graph.



The rate of change of distance with respect to time, or speed, is the ratio of Δd to Δt. It measures how much d changes for each unit increase in t, or how far Gregor travels in each hour. This quantity, the ratio $\dfrac{\Delta d}{\Delta t}$, is the *slope of the line*. [TK]

[TK] For more help calculating slope from a graph, see Section 1.4.2 of the Toolkit.

Checkpoint 1.4.9 QuickCheck 3. What feature of a graph illustrates rate of change?

a. The y-axis

b. Slope

c. Scales on the axes

d. The coordinate of a point

1.4.2 Review of Slope

You may have encountered the notion of slope in previous courses. Let us see how slope is related to rate of change.

Slope.

The **slope** of a line is a rate of change that measures the steepness of the line.

The slope tells us how much the y-coordinate changes for each unit of increase in the x-coordinate, as we move from one point to another along the line.

Checkpoint 1.4.10 QuickCheck 4. The slope of a line tells us how much the y-coordinate changes __________

a. from one end of the graph to the other.

b. as we move up the y-axis.

c. from one point to the next point.

d. for each unit of increase in the x-coordinate.

Using the notation Δ for "change in," we define slope as follows.

Definition 1.4.11 Slope of a Line.

$$\text{slope} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

or, in symbols

$$m = \frac{\Delta y}{\Delta x}$$

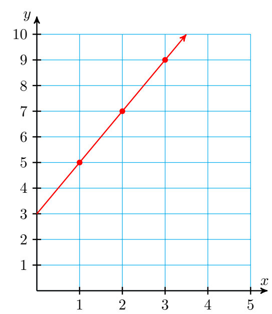
◊

In the definition of slope,

$$\Delta x \text{ is } \begin{cases} \text{positive if } x \text{ increases} & \text{(we move to the right)} \\ \text{negative if } x \text{ decreases} & \text{(we move to the left)} \end{cases}$$

$$\Delta y \text{ is } \begin{cases} \text{positive if } y \text{ increases} & \text{(we move up)} \\ \text{negative if } y \text{ decreases} & \text{(we move down)} \end{cases}$$

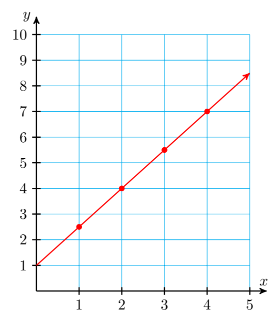
How does slope measure the steepness of a line? Study the three examples below and notice how y changes for each 1 unit increase in x.



A

y increases by 2 units

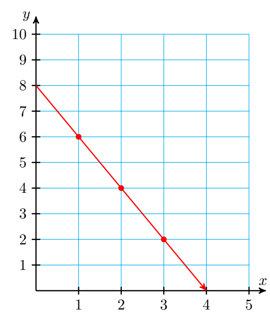
$$m = \frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$$



B

y increases by $\frac{3}{2}$ units

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{2}$$



C

y decreases by 2 units

$$m = \frac{\Delta y}{\Delta x} = \frac{-2}{1} = -2$$

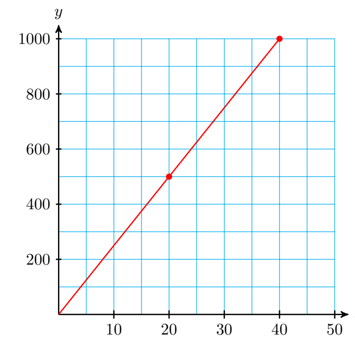
From these examples, we can make the following observations:

Slope and Steepness.

a. For positive slopes, the larger the value of m, the more the y-value increases for each unit increase in x, and the more we climb up as our location changes from left to right. (So graph A is steeper than graph B.)

b. If y decreases as we move from left to right, then Δy is negative when Δx is positive, so their ratio (the slope) is negative. (See graph C.)

Example 1.4.12



a. Compute the slope of the line. [TK]

b. Illustrate the slope on the graph by drawing a vertical segment of length Δy and a horizontal segment of length Δx.

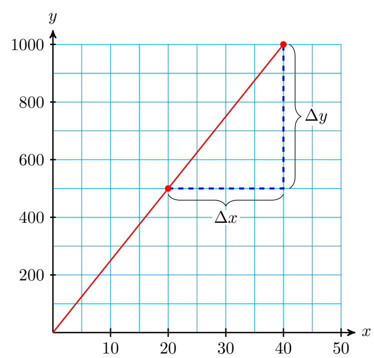
c. If $\Delta x = 1$, what is the length of the vertical segment?

Solution.

a. The points $(20, 500)$ and $(40, 1000)$ lie on the graph. As we move from the first point to the second point, x increases by 20 units, so $\Delta x = 20$, and y increases by 500 units, so $\Delta y = 500$. Thus

$$\frac{\Delta y}{\Delta x} = \frac{500}{20} = 25$$

b. The segments are shown in the graph below.

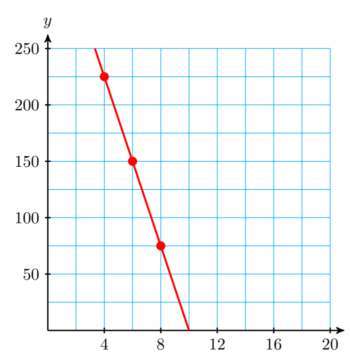


c. The slope is 25, which means that y increases 25 units for each 1-unit increase in x. So, if $\Delta x = 1$, then $\Delta y = 25$.

□

[TK] For more examples of calculating slope, see Section 1.4.2 of the Toolkit.

Checkpoint 1.4.13 Practice 3.



a. Compute the slope of the line.

b. Illustrate the slope on the graph by drawing a vertical segment for Δy and a horizontal segment for Δx.

c. If $\Delta x = 1$, what is the length of the vertical segment?

1.4.3 Interpreting Slope as a Rate

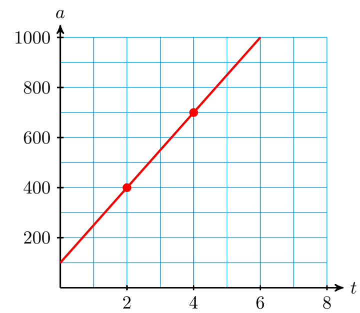
By putting together all of our discussion so far, we can now see that the slope of a line measures the rate of change of y with respect to x.

Example 1.4.14 The graph shows the altitude, a (in feet), of a skier t minutes after getting on a ski lift.

a. Choose two points on the graph and compute the slope, including units.

b. Explain what the slope measures in this problem. [TK]

c. Write a linear model for a in terms of t.



Solution.

a. We choose the points $(2, 400)$ and $(4, 700)$, as shown on the graph. Then

$$m = \frac{\Delta a}{\Delta t} = \frac{300 \text{ feet}}{2 \text{ minutes}} = 150 \text{ feet/minute}$$

b. The slope gives the rate of change of altitude with respect to time. The skier rises at a rate of 150 feet per minute.

c. A linear model has the form

$$y = (\text{starting value}) + (\text{rate}) \times t$$

From the graph, we see that at $t = 0$ the skier's altitude was $a = 100$, and we calculated the rate of change of altitude as 150 feet per minute. Substituting these values into the formula, we find

$$a = 100 + 150t$$

□

[TK] For more help interpretine slope, see Section 1.4.3 of the Toolkit.

Slope as a Rate of Change.

1. The slope of a line measures the **rate of change** of y with respect to x.

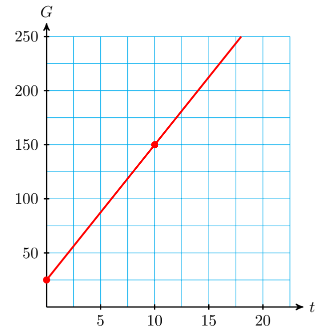
2. The units of Δy and Δx can help us interpret the slope as a rate.

Checkpoint 1.4.15 Practice 4. The graph shows the amount of garbage G (in tons) that has been deposited at a dumpsite t years after new regulations go into effect.

a. Choose two points on the graph and compute the slope, including units.

b. Explain what the slope measures in this problem.

c. Choose two different points and compute the slope again. Do you get the same value as before?



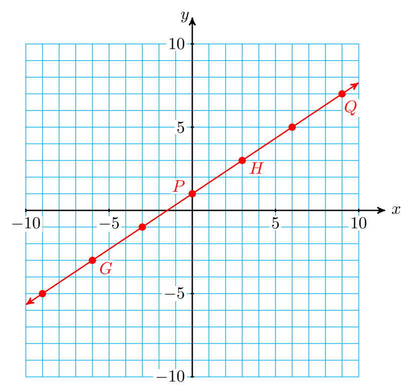
1.4.4 Lines Have Constant Slope

You have probably already noticed the following important fact about lines.

> **Lines Have Constant Slope.**
>
> The slope of a line is constant: no matter which two points you pick to compute the slope, you will always get the same value.

For the line shown in the figure, try computing the slope using the points P and Q, and then using the points G and H. In each case, you should find that the slope is $\frac{3}{2}$.

Here is another way to look at slopes. If we start at any point on the line shown and move 9 units to the right, what value of Δy will bring us back to the line? We can use the slope formula with $\Delta x = 9$.



$$m = \frac{\Delta y}{\Delta x} \qquad \text{Substitute the known values.}$$

$$\frac{2}{3} = \frac{\Delta y}{9} \qquad \text{Solve for } \boldsymbol{\Delta y}\textbf{.}$$

$$\Delta y = 9\left(\frac{2}{3}\right) = 6$$

You can use the graph to check this result for yourself; try starting at the point $(-6, -3)$.

The fact that lines have constant slope has two important consequences. First because m is constant for a given line, we can use the formula $m = \dfrac{\Delta y}{\Delta x}$ to find Δy when we know Δx, or to find Δx when we know Δy.

Checkpoint 1.4.16 QuickCheck 5. What formula can we use to find Δy when we know Δx, or Δx when we know Δy?

a. The equation of the line

b. The general linear formula

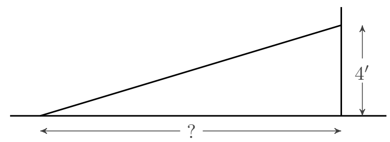
c. The slope formula

d. $D = RT$

Example 1.4.17 A wheelchair ramp can have a slope of no more than 24%, or 0.24. What horizontal distance is needed if the ramp must climb an elevation of 4 feet?

Solution.

We first draw a sketch of the wheelchair ramp and label Δx and Δy. We are given that $\Delta y = 4$ feet, and we are looking for Δx. We substitute the known values into the slope formula, and solve for Δx.



$$\begin{aligned} 0.24 &= \frac{4}{\Delta x} && \text{Multiply both sides by } \boldsymbol{\Delta x}. \\ 0.24\Delta x &= 4 && \text{Divide both sides by } \mathbf{0.24}. \\ \Delta x &= \frac{4}{0.24} = 16.\overline{6} \end{aligned}$$

The wheelchair ramp must have a horizontal length of $16\frac{2}{3}$ feet, or 16 feet 8 inches. □

Checkpoint 1.4.18 Practice 5. A wheelchair ramp can have a slope of no more than 24%, or 0.24. What height can the wheelchair ramp climb over a horizontal distance of 10 feet?

Hint: Do we know the value of Δy or of Δx?

Here is a second consequence of the fact that lines have constant slope: We can tell whether a collection of data points lies on a straight line by computing slopes. If the slopes between pairs of data points are all the same, the points lie on a straight line.

Checkpoint 1.4.19 QuickCheck 6. How can we can tell whether a collection of data points lies on a straight line?

a. Plot them and look at the graph.

b. Calculate the slopes between points.

c. Find an equation for the line.

d. It depends on the scales on the axes.

Example 1.4.20 Could this table represent a linear equation? Explain why or why not.

x	-6	-3	0	3	8
y	20	18	16	14	12

Solution. We compute the slope between each consecutive pair of points. In each case

$$m = \frac{\Delta y}{\Delta x} = \frac{-2}{3}$$

Because the slope is the same for all pairs of points, the table could be linear. □

Checkpoint 1.4.21 Practice 6. Could this table represent a linear equation? Explain why or why not.

t	5	10	15	20	25
P	0	3	6	12	24

Hint: Calculate the slopes between points.

To summarize, here are two ways that we can use the slope to study a model.

Two Uses for Slope.

- If we know the slope of a line, we can use the formula $m = \dfrac{\Delta y}{\Delta x}$ to find Δy when we know Δx, or to find Δx when we know Δy.
- To test whether a collection of data points lies on a straight line, we can compute slopes. If the slopes between all pairs of data points are the same, the points lie on a straight line.

1.4.5 Problem Set 1.4

Warm Up

Exercise Group. Compute ratios to answer the questions in Problems 1–4.

1. Carl runs 100 meters in 10 seconds. Anthony runs 200 meters in 19.6 seconds. Who has the faster average speed?

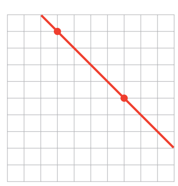
2. On his 512-mile round trip to Las Vegas and back, Corey needed 16 gallons of gasoline. He used 13 gallons of gasoline on a 429-mile trip to Los Angeles. On which trip did he get better fuel economy?

3. Grimy Gulch Pass rises 0.6 miles over a horizontal distance of 26 miles. Bob's driveway rises 12 feet over a horizontal distance of 150 feet. Which is steeper?

4. Which is steeper, the truck ramp for Acme Movers, which rises 4 feet over a horizontal distance of 9 feet, or a toy truck ramp, which rises 3 centimeters over a horizontal distance of 7 centimeters?

Exercise Group. In Problems 5-8, compute the slope of the line through the indicated points. On both axes, one square represents one unit.

5.



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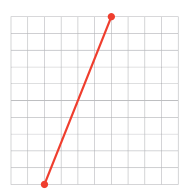

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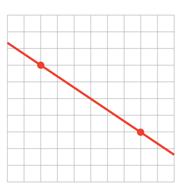

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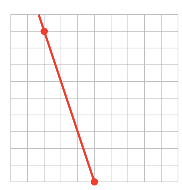
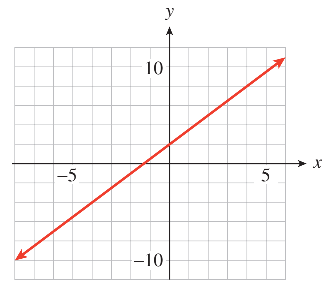


Skills Practice

9.

a. Compute the slope of the line.



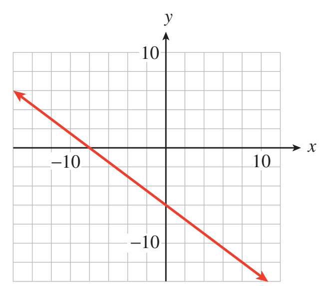
b. Start at point $(0, 2)$ and move 4 units in the positive x-direction. How many units must you move in the y-direction to get back to the line? What is the ratio of Δy to Δx?

c. Start at point $(0, 2)$ and move -6 units in the positive x-direction. How many units must you move in the y-direction to get back to the line? What is the ratio of Δy to Δx?

d. Suppose you start at any point on the line and move 18 units in the x-direction. How many units must you move in the y-direction to get back to the line? Use the equation $m = \dfrac{\Delta y}{\Delta x}$ to calculate your answer.

10.

a. Compute the slope of the line.



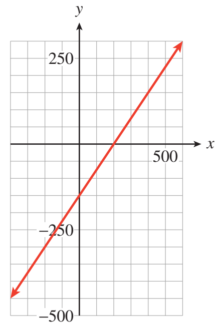
b. Start at point $(0, -6)$ and move -6 units in the y-direction (down). How many units must you move in the x-direction to get back to the line? What is the ratio of Δy to Δx?

c. Start at point $(0, -6)$ and move 9 units in the positive y-direction. How many units must you move in the x-direction to get back to the line? What is the ratio of Δy to Δx?

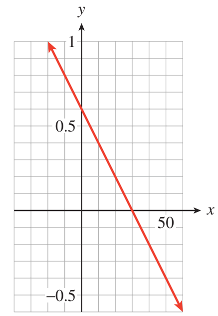
d. Suppose you start at any point on the line and move 24 units in the y-direction. How many units must you move in the x-direction to get back to the line? Use the equation $m = \dfrac{\Delta y}{\Delta x}$ to calculate your answer.

Exercise Group. For Problems 11–14, compute the slope of the line. Note the scales on the axes.

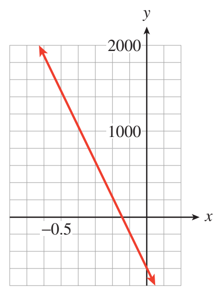
11.



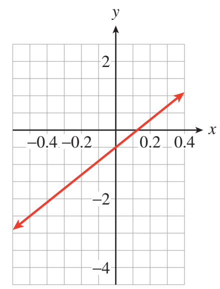
12.



13.



14.



Exercise Group. For Problems 15 and 16,

a. Graph the line by the intercept method.

b. Use the intercepts to compute the slope.

c. Use the intercepts to illustrate the slope on each graph. Put arrows on Δx and Δy to indicate the direction of motion.

15. $9x + 12y = 36$

16. $\dfrac{x}{7} - \dfrac{y}{4} = 1$

17. Residential staircases are usually built with a slope of 70%, or $\dfrac{7}{10}$. If the vertical distance between stories is 10 feet, how much horizontal space does the staircase require?

18. A line has slope $m = \dfrac{-4}{5}$. Use the equation $m = \dfrac{\Delta y}{\Delta x}$ to find the horizontal or vertical change along the line.

a. $\Delta x = -10$

b. $\Delta y = 2$

c. $\Delta x = 12$

d. $\Delta y = -6$

Exercise Group. For Problems 19 and 20, which tables represent variables that are related by a linear equation? (Hint: which relationships have constant slope?)

19.

a.

x	y
2	12
3	17
4	22
5	27
6	32

b.

t	P
2	4
3	9
4	16
5	25
6	36

20.

a.

h	w
-6	20
-3	18
0	16
3	14
6	12

b.

t	d
5	0
10	3
15	6
20	12
25	24

Applications

21. The population of Smallville grew from 7000 people in 1990 to 16,600 in 2002.

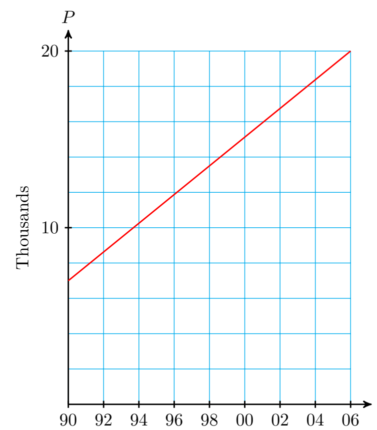
a. Use a rate of change to calculate the town's average rate of growth, in people per year. First, complete the table of values:

t	P

change in population $= \Delta P =$

time elapsed $= \Delta t =$

rate of growth $= \dfrac{\Delta P}{\Delta t} =$



b. Illustrate ΔP and Δt by line segments on the graph. (Note that the P-axis is labeled in thousands.)

c. How much did the town grow each year?

d. Write a linear model for P in terms of t.

22. A traditional first experiment for chemistry students is to make 98 observations about a burning candle. Delbert records the height, h, of the candle in inches at various times t minutes after he lit it.

t	0	10	30	45
h	12	11.5	10.5	9.75

a. Choose appropriate scales for the axes and plot the data. Do the points lie on a straight line?

b. Compute the slope of the graph, including units, and explain what the slope tells us about the candle.

c. Write a linear model for h in terms of t.

Exercise Group. For Problems 23 and 24,

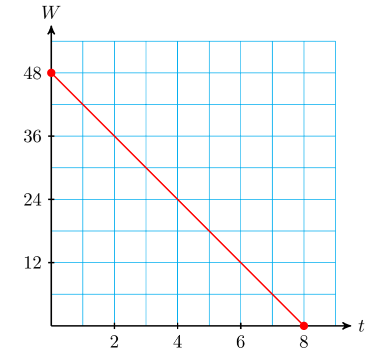
a. Choose two points and compute the slope, including units.

b. Write the slope as a rate of change, including units.

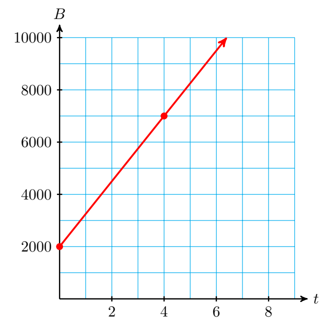
c. Illustrate the slope on the graph.

d. Write a linear model for the variables.

23. The graph shows the number of liters of emergency water W remaining in a southern California household t days after an earthquake.



24. The graph shows the number of barrels of oil, B, that have been pumped at a drill site t days after a new drill is installed.



25. A spring is suspended from the ceiling. The table shows the length of the spring, in centimeters, as it is stretched by hanging various weights from it.

Weight, kg	3	4	8	10	12	15	22
Length, cm	25.76	25.88	26.36	26.6	26.84	27.2	28.04

a. If you plot the data, will the points lie on a straight line? Why or why not?

b. Interpret the slope as a rate of change. Include units in your answer

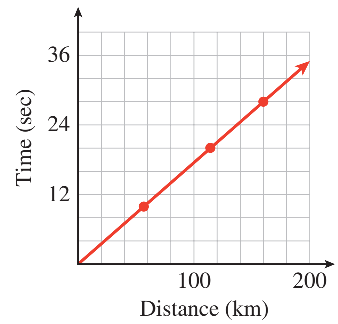
26. The table gives the radius and circumference of various circles, rounded to three decimal places.

r	C
4	25.133
6	37.699
10	62.832
15	94.248

a. If we plot the data, will the points lie on a straight line? Why or why not?

b. What familiar number does the slope turn out to be? (Hint: Recall a formula from geometry.) What does the slope tell us about circles?

27. Geologists calculate the speed of seismic waves by plotting the travel times for waves to reach seismometers at known distances from the epicenter. The speed of the wave can help them determine the nature of the material it passes through. The graph shows a travel time graph for P-waves from a shallow earthquake.



a. Why do you think the graph is plotted with distance as the independent variable?

b. Use the graph to calculate the speed of the wave.

28. Niagara Falls was discovered by Father Louis Hennepin in 1682. In 1952, much of the water of the Niagara River was diverted for hydroelectric power, but until that time erosion caused the Falls to recede upstream by 3 feet per year.

a. How far did the Falls recede from 1682 to 1952?

b. The Falls were formed about 12,000 years ago during the end of the last ice age. How far downstream from their current position were they then? (Give your answer in miles.)

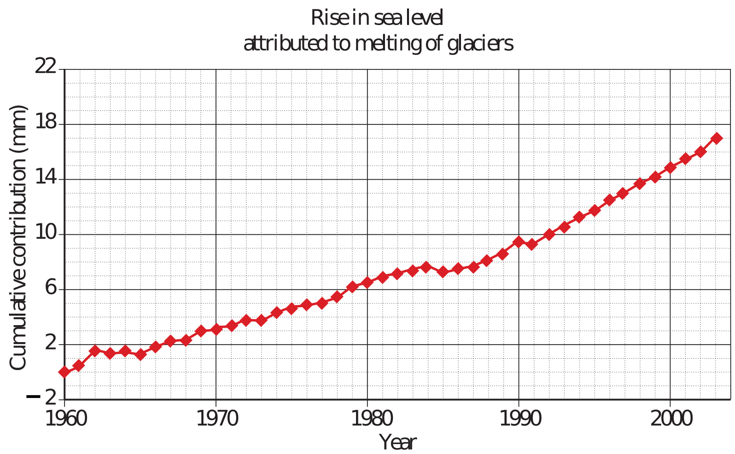
29. Naismith's Rule is used by runners and walkers to estimate journey times in hilly terrain. In 1892 Naismith wrote in the Scottish Mountaineering Club Journal that a person "in fair condition should allow for easy expeditions an hour for every three miles on the map, with an additional hour for every 2000 feet of ascent." (Source: Scarf, 1998.)

a. According to Naismith, 1 unit of ascent requires the same travel time as how many units of horizontall travel? (This is called Naismith's number.) Round your answer to one decimal place.

b. A walk in the Brecon Beacons in Wales covers 3.75 kilometers horizontally and climbs 582 meters. What is the equivalent flat distance?

c. If you can walk at a pace of 15 minutes per kilometer over flat ground, how long will the walk in the Brecon Beacons take you?

30. The graph shows the rise in sea level attributed to the melting of land-based glaciers from 1960 to 2003.



(a) The graph appears to be almost linear from 1992 to 2002. Read the graph to complete the table, then compute the slope of the graph over that time interval, including units. What does the slope mean in this situation?

Year	Sea level
1992	
2002	

(b) What was the total change in sea level from land-based glaciers over the time period from 1960 to 2003?

(c) Calculate the average rate of change of sea level from land-based glaciers from 1960 to 2003.

(d) From 1960 to 2003, the land-based glaciers decreased in thickness by about 8 meters (or 0.008 km). The total area of those glaciers is 785,000 square kilometers. Calculate the total volume of water released by melting. (Hint: Volume = area × thickness)

(e) The surface area of the world's oceans is 361.6 million square kilometers. When the meltwater from the land-based glaciers (that's the volume you calculated in part (d)) enters the oceans, how much will the sea level rise, in kilometers? Use the formula in part (d). Convert your answer to millimeters, and check your answer against your answer to part (b).

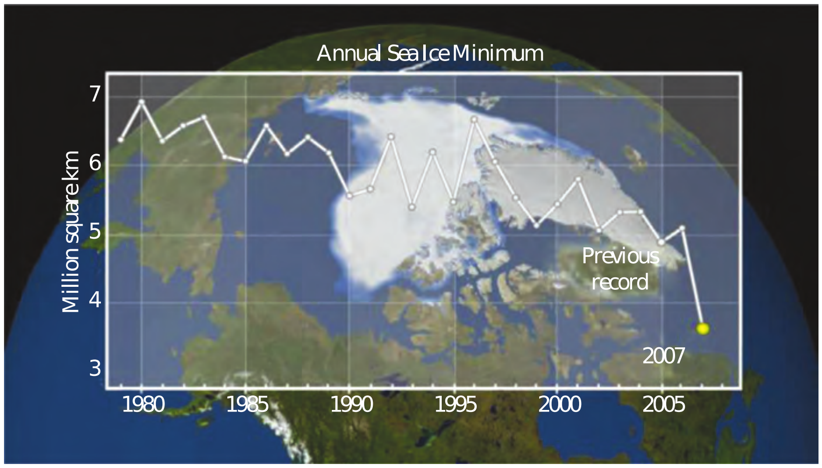
1.5 Equations of Lines

Three main factors influence the energy balance of the Earth and its temperature:

- The total energy influx from the sun
- The chemical composition of the atmosphere
- The ability of the Earth's surface to reflect light, or **albedo**

Because polar ice reflects light from the sun, the radiation balance over an ice-covered ocean is very different from the balance over an open ocean. The ice component of the climate system, called the **cryosphere**, plays an important role in the Earth's radiation balance.

Climate models predict that global warming over the next few decades will occur mainly in the polar regions. As polar ice begins to melt, less sunlight is reflected into space, which raises the overall temperature and fuels further melting. This process is called **ice albedo feedback**. Since satellite monitoring began in 1979, Arctic sea ice cover has decreased about 10% per decade, falling to a startling new low in 2007.



Numerous factors influence the freezing point of sea water, including its salinity, or mineral content. In this Lesson we'll develop a formula for the freezing temperature of water in terms of its salinity.

1.5.1 Slope-Intercept Form

In earlier sections we learned that:

- The y-intercept of a line gives the **initial value** of y.
- The **slope** of the line gives the **rate of change** of y with respect to x.

Comparing these observations with the form for a linear model, we see that

$$\begin{aligned} \boldsymbol{y} &= (\text{starting value}) + (\text{rate}) \cdot x \\ \boldsymbol{y} &= \qquad \boldsymbol{b} \qquad + \quad \boldsymbol{mx} \end{aligned}$$

Usually we write the terms in the opposite order, like this: $y = mx + b$. We call this last equation the **slope-intercept form** for a line.

> **Slope-Intercept Form.**
>
> We can write the equation of a line in the form
>
> $$\boldsymbol{y = mx + b}$$
>
> where m is the **slope** of the line, and b is the y**-intercept**.

Checkpoint 1.5.1 QuickCheck 1. Choose the correct statement about the equation $y = 8 - 6x$.

a. The slope is 8 and the y-intercept is -6.

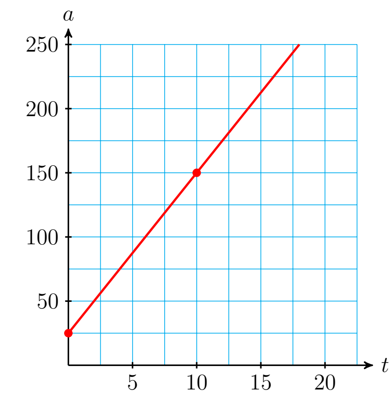
b. The slope is -6 and the y-intercept is 8.

c. The slope is $-6x$ and the y-intercept is -6.

d. The slope is 8 and the y-intercept is 6.

Example 1.5.2

Recall Practice 4 from Section 1.4: The graph shows the amount of garbage G, in tons, that has been deposited at a dumpsite t years after new regulations go into effect.



a. State the vertical intercept and the slope of the graph.

b. Find an equation for the graph shown. [TK]

c. State the meaning of the slope and the vertical intercept for this model.

Solution.

a. The vertical intercept of the graph is $(0, 25)$. Two points on the line are $(0, 25)$ and $(10, 150)$, so the slope is

$$m = \frac{\Delta G}{\Delta t} = \frac{125 \text{ tons}}{10 \text{ years}} = 12.5 \text{ tons/year}$$

b. $G = mt + b$, so the equation is $G = 12.5t + 25$

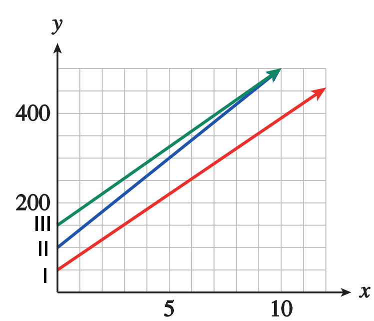
c. The slope tells us that garbage is accumulating at a rate of 12.5 tons per year. The vertical intercept tells us that when the new regulations went into effect, the dump held 25 tons of garbage.

□

[TK] For more help with slope-intercept form, see Section 1.5.1 of the Toolkit.

Checkpoint 1.5.3 Practice 1.

Delbert decides to use DSL for his Internet service. Earthlink charges a \$99 activation fee and \$39.95 per month, DigitalRain charges \$50 for activation and \$34.95 per month, and FreeAmerica charges \$149 for activation and \$34.95 per month.



a. Write a formula for Delbert's Internet costs under each plan.

b. Match Delbert's Internet cost under each company with its graph in the figure.

1.5.2 Coordinate Formula for Slope

Using the formula $m = \dfrac{\Delta y}{\Delta x}$, it is fairly easy to calculate the slope of a line from its graph if there are two obvious points to use. However, if the coordinates are not easy, or if we don't have a graph, we may need another method.

You know that to calculate the **net change** between two points on a number line, we can subtract their coordinates. That is,

$$\text{net change} = \text{final value} - \text{starting value}$$

For example, if you walk from 3^{rd} street to 8^{th} street, your distance, s, from the center of town has increased by 5 blocks, or

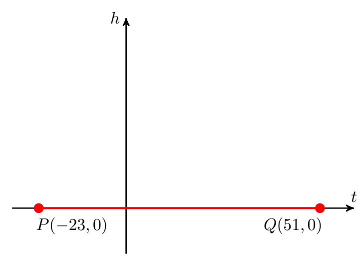
$$\Delta s = 8 - 3 = 5$$

If the temperature T drops from 28° to 22°, it has decreased by 6°, or

$$\Delta T = 22 - 28 = -6$$

The net change is positive if the variable increases, and negative if it decreases. For the graph shown at right, the net change in t-coordinate from P to Q is

$$\Delta t = 51 - (-23) = 74$$



We can use the notion of net change to write a coordinate formula for computing slope. We will need **subscripts** to designate the coordinates of two different points. We'll write (x_1, y_1) for the coordinates of the first point, and (x_2, y_2) for the coordinates of the second point.

Note 1.5.4 Do not confuse subscripts with exponents! An exponent changes the value of a variable, so that for instance if $x = 3$ then $x^2 = 3^2 = 9$, but a subscript merely tells us which point the variable comes from, so that x_2 just means "the x-coordinate of the second point."

Coordinate Formula for Slope.

If (x_1, y_1) and (x_2, y_2) are two points on a line, then the slope of the line is given by

$$\boldsymbol{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

as long as $x_1 \neq x_2$

Note 1.5.5 Notice that the numerator of the slope formula, $y_2 - y_1$, gives the net change in y, or Δy, and the denominator, $x_2 - x_1$, gives the net change in x, or Δx. The coordinate formula is equivalent to our definition of slope, $m = \dfrac{\Delta y}{\Delta x}$.

Checkpoint 1.5.6 QuickCheck 2. In the coordinate formula for slope, why do we require that $x_1 \neq x_2$?

a. The slope cannot be zero.

b. The two points cannot be the same.

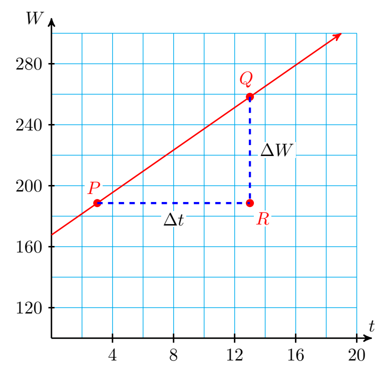
c. The denominator of a fraction cannot be zero.

d. The denominator must equal 1.

Example 1.5.7 The graph shows wine consumption, W, in the US, in millions of cases, starting in 1990. In 1993, Americans drank 188.6 million cases of wine. In 2003, Americans drank 258.3 million cases of wine.

a. Find the slope of the graph from 1993 to 2003. [TK]

b. State the slope as a rate of change. What does the slope tell us about wine consumption?



Solution.

a. If $t = 0$ in 1990, then in 1993, $t = 3$, and in 2003, $t = 13$. Thus, the points $P(3, 188.6)$ and $Q(13, 258.3)$ lie on the line. We want to compute the slope,

$$m = \frac{\Delta W}{\Delta t} = \frac{W_2 - W_1}{t_2 - t_1}$$

between these two points. Think of moving from P to Q in two steps, first moving horizontally to the right from P to the point R, and then vertically from R to Q. The coordinates of R are $(13, 188.6)$. (Do you see why?) Then

$$\begin{aligned} \Delta t &= t_2 - t_1 = 13 - 3 = 10 \\ \Delta W &= W_2 - W_1 = 258.3 - 188.6 = 69.7 \end{aligned}$$

and thus

$$m = \frac{W_2 - W_1}{t_2 - t_1} = \frac{258.3 - 188.6}{13 - 3} = 6.97$$

b. The slope gives us a rate of change, and the units of the variables help us interpret the slope in context.

$$\frac{\Delta W}{\Delta t} = \frac{258.3 - 188.6 \text{ million cases}}{13 - 3 \text{ years}} = 6.97 \text{ million cases/year}$$

Over the ten years between 1993 and 2003, wine consumption in the US increased at a rate of 6.97 million cases per year.

□

[TK] For more help calculating slope, see Section 1.5.2 of the Toolkit.

Checkpoint 1.5.8 Practice 2. In 1991, there were 64.6 burglaries per 1000 households in the United States. The number of burglaries reported annually declined at a roughly constant rate over the next decade, and in 2001 there were 28.7 burglaries per 1000 households. (Source: U.S. Department of Justice)

a. Give the coordinates of two points on the graph of $B = mt+b$, where $t = 0$ in 1990, and B stands for the number of burglaries per 1000 households.

b. Find the slope of the line, including units.

1.5.3 Point-Slope Formula

Now we'll consider using the slope formula for a different problem. If we know the slope of a line and the coordinates of one point on the line, we can use the coordinate formula for slope to find the y-coordinate of any other point on the line.

Instead of *evaluating* the formula to find m, we *substitute* the values we know for m and for (x_1, y_1). If we then plug in the x-coordinate of any unknown point, we can solve for y.

Checkpoint 1.5.9 QuickCheck 3. A line has slope $\dfrac{-3}{4}$ and passes through the point $(1, -4)$. Which equation can you use to find the y-coordinate of the point on the line with x-coordinate 6?

a. $\dfrac{3}{4} = \dfrac{1-6}{y+4}$

b. $\dfrac{-3}{4} = \dfrac{y+4}{1-6}$

c. $\dfrac{-3}{4} = \dfrac{y-6}{-4-1}$

d. $\dfrac{-3}{4} = \dfrac{y+4}{6-1}$

Remember that the equation for a line is really just a formula that gives the y-coordinate of any point on the line in terms of its x-coordinate. So, if we know the slope of a particular line and one point on the line, we can use the coordinate formula for slope to find its equation.

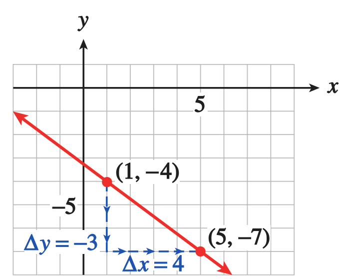
Example 1.5.10

a. Graph the line that passes through the point $(1, -4)$ and has slope $= \dfrac{-3}{4}$. [TK]

b. Find an equation for the line in part (a). [TK]

Solution.

a. We first plot the given point, $(1, -4)$, and then use the slope to find another point on the line. The slope is $m = \dfrac{-3}{4} = \dfrac{\Delta y}{\Delta x}$, so starting from $(1, -4)$ we move down 3 units and then 4 units to the right. This brings us to the point $(5, -7)$. We draw the line through these two points.



b. To find an equation for the line, we start with the slope formula,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

We substitute $\dfrac{-3}{4}$ for the slope, m, and $(1, -4)$ for (x_1, y_1). For the second point, (x_2, y_2), we substitute the variable point (x, y) to obtain

$$m = \frac{y+4}{x-1}$$

This is an equation for the line, but if we want to solve for y, we first multiply both sides by $x-1$.

$$\begin{aligned}
\mathbf{(x-1)}\frac{-3}{4} &= \frac{y+4}{x-1}\mathbf{(x-1)} && \\
\frac{-3}{4}(x-1) &= y+4 && \text{Apply the distributive law.} \\
\frac{-3}{4}x + \frac{3}{4} &= y+4 && \text{Subtract 4 from both sides.} \\
\frac{-3}{4}x - \frac{13}{4} &= y && \mathbf{\frac{3}{4} - 4 = \frac{3}{4} - \frac{16}{4} = \frac{-13}{4}}
\end{aligned}$$

In slope-intercept form, the equation of the line is $y = \frac{-3}{4}x - \frac{13}{4}$.

□

[TK] For more help graphing with the point-slope form, see Section 1.5.4 of the Toolkit.

When we use the slope formula to find the equation of a line, we substitute a variable point (x, y) for the second point. This version of the formula,

$$m = \frac{y-y_1}{x-x_1}$$

is called the **point-slope form** for a linear equation. It is sometimes stated in another version by clearing the fraction to get

$$\begin{aligned}
\mathbf{(x-x_1)}m &= \frac{y-y_1}{x-x_1}\mathbf{(x-x_1)} \\
(x-x_1)m &= y-y_1 \\
y &= y_1 + m(x-x_1)
\end{aligned}$$

Point-Slope Form.

The equation of the line that passes through the point (x_1, y_1) and has slope m is

$$\boldsymbol{y = y_1 + m(x-x_1)}$$

You may also see the formula written in an alternate version:

$$\boldsymbol{y - y_1 = m(x-x_1)} \qquad \text{or} \qquad \boldsymbol{\frac{y-y_1}{x-x_1} = m}$$

[TK] For more help using the point-slope form, see Section 1.5.3 of the Toolkit.

Checkpoint 1.5.11 Practice 3. Use the point-slope form to find the equation of the line that passes throughthe point $(-3, 5)$ and has slope -1.4.

1.5.4 Finding a Linear Model

Now we are ready to find the equation promised in the introduction to this section: a formula for the freezing temperature of water in terms of its salinity.

When we have two data points for a linear model, we can find its equation using two steps:

- first we compute the slope of the line,
- then we use the point-slope formula.

Example 1.5.12 Sea water does not freeze at exactly 32°F because of its salinity. The temperature at which water freezes depends on its dissolved mineral content. A common unit for measuring salinity is parts per thousand, or ppt. For example, salinity of 8 ppt means 8 grams of dissolved salts in each kilogram of water. Here are some data for the freezing temperature of water.

Salinity (ppt), S	8	12	20
Freezing temperature (°F), T	31.552	31.328	30.88

a. Do these data points describe a linear model? Why or why not?

b. Find a linear equation for freezing temperature, T, in terms of salinity, S.

   Step 1: Find the slope.

   Step 2: Use the point-slope formula.

c. What is the salinity of water that freezes at 32°F?

d. Sea water has an average salinity of 35 ppt. What is the freezing point of sea water?

Solution.

a. We check to see if the slope between each pair of points is the same.

$$\begin{aligned} m &= \frac{T_2 - T_1}{S_2 - S_1} = \frac{31.328 - 31.552}{12 - 8} = -0.056 \\ m &= \frac{T_3 - T_2}{S_3 - S_2} = \frac{30.88 - 31.328}{20 - 12} = -0.056 \end{aligned}$$

   Because the slopes are the same, the data could describe a linear model.

b. Step 1: In part (a) we found the slope: $m = -0.056$

   Step 2: We substitute the slope and either point into the point-slope formula.

$$\begin{aligned} T - T_1 &= m(S - S_1) \\ T - 30.88 &= -0.056(S - 20) \\ T - 30.88 &= -0.056S + 1.12 \\ T &= -0.056S + 32 \end{aligned}$$

c. If $T = 32$ in our equation, then S must be zero, so pure water (no salinity) freezes at 32°F.

d. We substitute 35 for S in our equation.

$$T = -0.056(35) + 32 = 30.04$$

   Sea water freezes at 30.04°F.

□

Checkpoint 1.5.13 Practice 4. The fee for registering a new car is given by a linear equation that depends on the car's value. The fee for a $15,000 car is $128.50, and the fee for a $25,000 car is $193.50.

a. Find a linear equation that gives the registration fee F for a new car that cost V dollars.

b. Use your equation to estimate the registration fee for a car that costs $22,000.

c. What is the slope of your line? What does the slope mean for registration fees?

1.5.5 Summary

In this section, we studied three different formulas associated with linear equations: the slope-intercept formula, the coordinate formula for slope, and the point-slope formula. How are these formulas related, and how are they different?

1. The **slope-intercept form**, $y = mx + b$, is just a special case of the **point-slope formula**. If the given point (x_1, y_1) happens to be the y-intercept $(0, b)$, then the point-slope formula reduces to the familiar form:

$$\begin{aligned} y &= y_1 + m(x - x_1) && \text{Substitute } \boldsymbol{b} \text{ for } \boldsymbol{y_1} \text{ and 0 for } \boldsymbol{x_1}. \\ y &= b + m(x - 0) && \text{Simplify.} \\ y &= mx + b \end{aligned}$$

   We can use the (shorter) slope-intercept form if we are lucky enough to know the y-intercept of the line.

2. What is the difference between the **slope formula**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

   and the **point-slope formula**

$$m = \frac{y - y_1}{x - x_1}?$$

   They are really the same formula, but they are used for different purposes:

   a. The slope formula is used to calculate the slope when we know two points. We know (x_1, y_1) and (x_2, y_2), and we are looking for m.

   b. The point-slope formula is used to find the equation of a line. We know (x_1, y_1) and m, and we are looking for $y = mx + b$.

1.5.6 Problem Set 1.5

Warm Up

Exercise Group. For Problems 1 and 2, complete the table of values and graph the line, then answer the questions below.

1. $y = 8 - 2x$

x	-1	0	2	3	4
y					

2. $y = 2 + \dfrac{1}{2}x$

x	-2	0	1	3	5
y					

a. What is the initial value for each line; that is, what is the y-value when $x = 0$?

Line 1: Line 2:

b. Look at the table for each line. How much does y increase or decrease for each 1-unit increase in x? What is this value called?

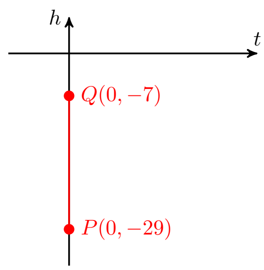
Line 1: Line 2:

c. Compare your answers to parts (a) and (b) with the equation for each line. What do you observe?

Exercise Group. For Problems 3 and 4, calculate the net change in coordinate from P to Q.

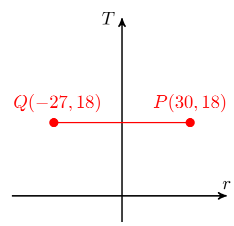
3.

a.



$\Delta h =$

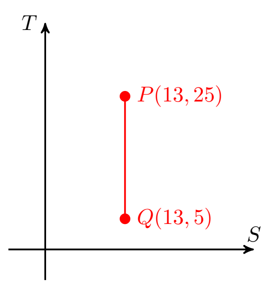
b.



$\Delta r =$

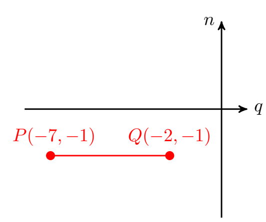
4.

a.



$\Delta T =$

b.



$\Delta q =$

Skills Practice

Exercise Group. For Problems 5–8,

a. Write the equation in slope-intercept form.

b. State the slope and the y-intercept of its graph.

5. $3x + 2y = 1$

6. $2x - \dfrac{3}{2}y = 3$

7. $4.2x - 0.3y = 6.6$

8. $5x - 4y = 0$

Exercise Group. For Problems 9 and 10,

a. Sketch the graph of the line with the given slope and y-intercept.

b. Write an equation for the line.

c. Find the x-intercept of the line.

9. $m = \dfrac{-5}{3}$ and $b = -6$

10. $m = \dfrac{3}{4}$ and $b = -2$

Exercise Group. In Problems 11 and 12, choose the correct graph for each equation. The scales on both axes are the same.

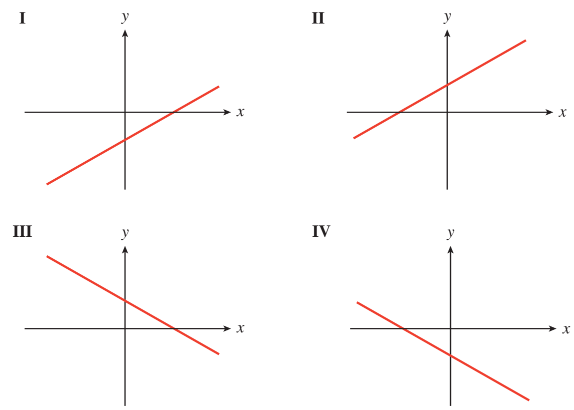
11.

a. $y = \dfrac{3}{4}x + 2$

b. $y = \dfrac{-3}{4}x + 2$

c. $y = \dfrac{3}{4}x - 2$

d. $y = \dfrac{-3}{4}x - 2$



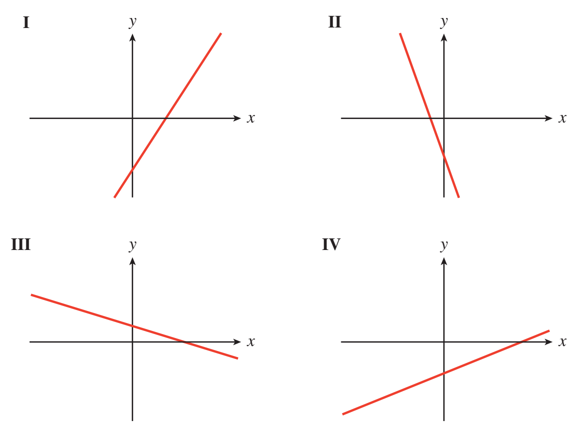
12.

a. $m < 0$, $b > 0$

b. $m > 1$, $b < 0$

c. $0 < m < 1$, $b < 0$

d. $m < -1$, $b < 0$



Exercise Group. For Problems 13 and 14,

a. Graph the line that passes through the given point and has the given slope.

b. Write an equation for the line in point-slope form.

c. Put your equation from part (b) into slope-intercept form.

13. $(2, -5)$; $m = -3$

14. $(2, -1)$; $m = \dfrac{5}{3}$

Exercise Group. For Problems 15 and 16, find an equation for the line that goes through the given points. Put your equation into slope-intercept form.

15. $(-16, -24)$, $(8, 72)$

16. $(-5, 65)$, $(20, -145)$

Exercise Group. In Problems 17 and 18, choose the correct graph for each equation. The scales on both axes are the same.

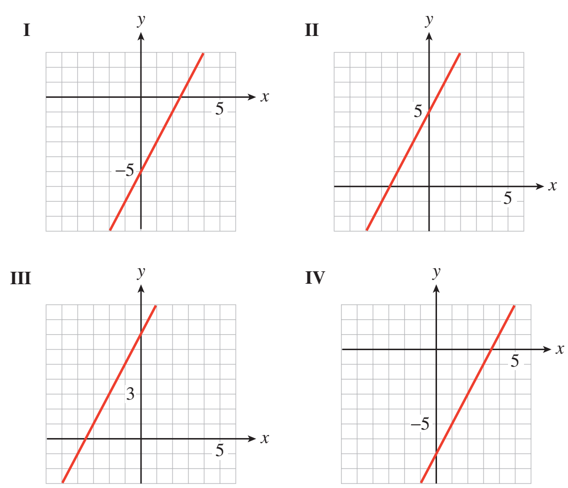
17.

a. $y = 1 + 2(x + 3)$

b. $y = -1 + 2(x - 3)$

c. $y = -1 + 2(x + 3)$

d. $y = 1 + 2(x - 3)$



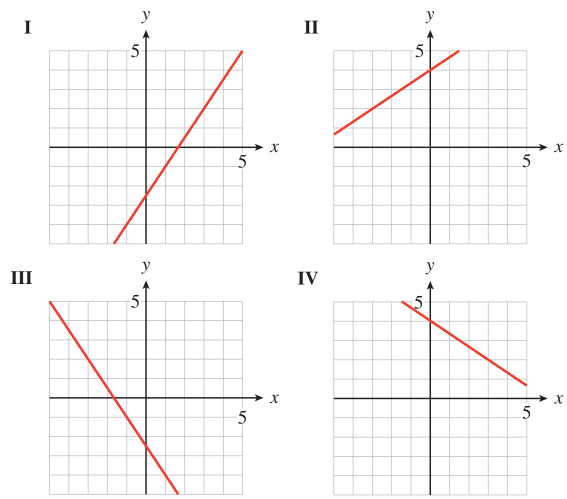
18.

a. $y = 2 - \dfrac{2}{3}(x - 3)$

b. $y = 2 - \dfrac{3}{2}(x + 3)$

c. $y = 2 + \dfrac{3}{2}(x - 3)$

d. $y = 2 + \dfrac{2}{3}(x + 3)$



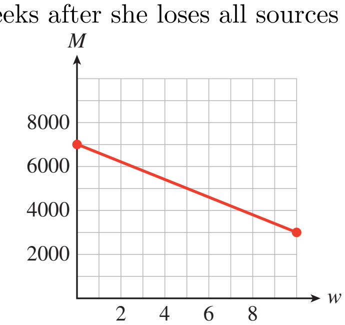
Applications

Exercise Group. In Problems 19 and 20,

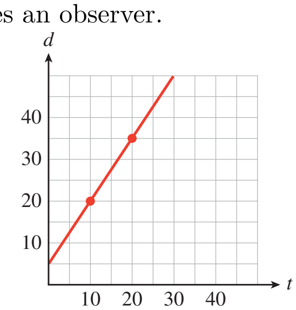
a. Find a formula for the function whose graph is shown.

b. Say what the slope and the vertical intercept tell us about the problem.

19. The graph shows the amount of money, M (in dollars), in Tammy's bank account w weeks after she loses all sources of income.



20. The graph shows the distance, d (in meters), traveled by a train t seconds after it passes an observer.



21. The boiling point of water changes with altitude and is approximated by the formula

$$B = 212 - 0.0018h$$

where B is in degrees and h is in feet. State the slope and vertical intercept of the graph, including units, and explain their meaning in this context.

22. The height of a woman in centimeters is related to the length of her femur (in centimeters) by the formula

$$H = 2.47x + 54.10$$

State the slope and the vertical intercept of the graph, including units, and explain their meaning in this context.

Exercise Group. In Problems 23–26, we find a linear model from two data points.

a. Make a table showing the coordinates of two data points for the model. (Which variable should be plotted on the horizontal axis?)

b. Find a linear equation relating the variables.

c. State the slope of the line, including units, and explain its meaning in the context of the problem.

23. Flying lessons cost \$645 for an 8-hour course and \$1425 for a 20-hour course. Both prices include a fixed insurance fee. Write an equation for the cost, C, of flying lessons in terms of the length, h, of the course in hours.

24. In the desert, the sun rose at 6 am. At 9 am the temperature was 80°, and at 3 pm the temperature was 110°. Write an equation for the temperature T, at h hours after sunrise.

25. A radio station in Detroit, Michigan, reports the high and low temperatures in the Detroit/Windsor area as 59°F and 23°F, respectively. A station in Windsor, Ontario, reports the same temperatures as 15°C and −5°C. Express the Fahrenheit temperature, F, in terms of the Celsius temperature, C.

26. The gas tank in Cicely's car holds 14 gallons. When the tank was half full her odometer read 308 miles, and when she filled her tank with 12 gallons of gasoline the odometer read 448. Express her odometer reading, m, in terms of the amount of gas, g, she used.

27. If the temperature on the ground is 70° Fahrenheit, the formula

$$T = 70 - \frac{3}{820}h$$

gives the temperature at an altitude of h feet.

a. What is the temperature at an altitude of 4100 feet?

b. At what altitude is the temperature 34° Fahrenheit?

c. Choose appropriate WINDOW settings and graph the equation $y = 70 - \frac{3}{820}x$.

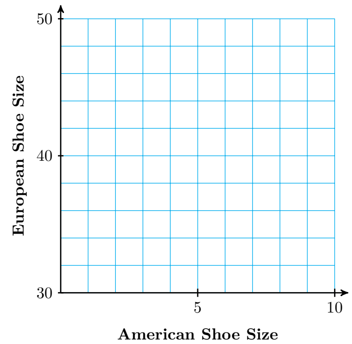
d. Find the slope and explain its meaning for this problem.

e. Find the intercepts and explain their meanings for this problem.

28. European shoe sizes are scaled differently than American shoe sizes. The table shows the European equivalents for various American shoe sizes.

American shoe size	5.5	6.5	7.5	8.5
Europoean shoe size	37	38	39	40

a. Use the grid below to plot the data and draw a line through the data points.



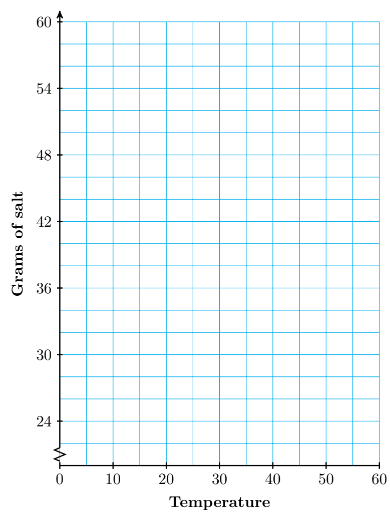
b. Calculate the slope of your line. Estimate the y-intercept from the graph.

c. Find an equation that gives the European shoe size, E, in terms of the American shoe size, A.

29. The table shows the amount of ammonium chloride salt, in grams, that can be dissolved in 100 grams of water at different temperatures.

Temperature (°C)	10	12	15	21	25	40	52
Salt (grams)	33	34	35.5	38.5	40.5	48	54

a. Use the grid below to plot the data and draw a straight line through the points. Estimate the y-intercept of your graph.



b. Calculate the slope of the line.

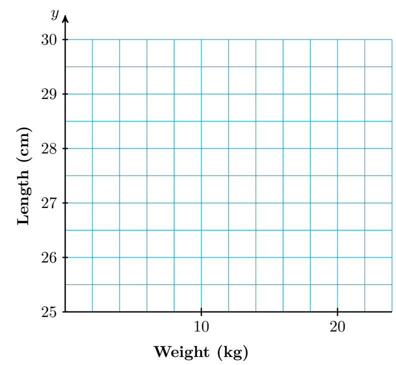
c. Use the point-slope formula to find an equation for the line.

d. At what temperature will 46 grams of salt dissolve?

30. A spring is suspended from the ceiling. The table shows the length of the spring in centimeters as it is stretched by hanging various weights from it.

Weight, kg	3	4	8	10	12	15	22
Length, cm	25.76	25.88	26.36	26.6	26.84	27.2	28.04

a. Plot the data on graph paper and draw a straight line through the points. Estimate the y-intercept of your graph.



b. Find an equation for the line.

c. If the spring is stretched to 27.56 cm, how heavy is the attached weight?

1.6 Chapter Summary and Review

1.6.1 Glossary

- mathematical model
- linear model
- evaluate an expressions
- solve an equation
- increasing graph
- decreasing graph
- inequality
- intercepts
- linear equation
- solution of an equation
- graph of an equation
- ordered pair
- rate
- rate of change
- net change
- slope
- slope-intercept form
- point-slope form

1.6.2 Key Concepts

1. A **mathematical model** is a simplified description of reality that helps us understand a system or process.
2. We can describe a relationship between variables with a table of values, a graph, or an equation.
3. Linear models have equations of the form:

$$y = (\text{starting value}) + (\text{rate of change}) \cdot x$$

4. The **general form** for a linear equation is: $Ax + By = C$.
5. We can use the **intercepts** to graph a line. The intercepts are also useful for interpreting a model.

6. The **graph** of an equation in two variables is just a picture of all its solutions.

7. Lines have constant slope.

8. The slope of a line gives us the **rate of change** of one variable with respect to another.

9. Formulas for Linear Models

   - **slope:** $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$
   - **slope-intercept form:** $y = b + mx$
   - **point-slopet form:** $y = y_1 + m(x - x_1)$

10. The **slope-intercept** form is useful when we know the initial value and the rate of change.

11. The **point-slope** form is useful when we know the rate of change and one point on the line.

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1.6.3 Chapter 1 Review Problems

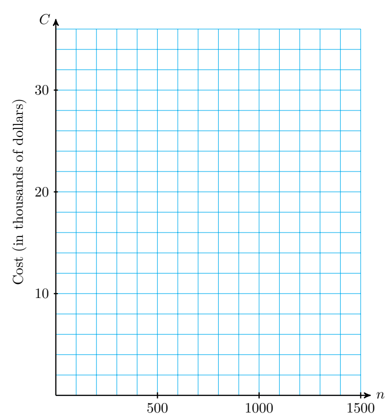
1. Last year, Pinwheel Industries introduced a new model calculator. It cost \$2000 to develop the calculator and \$20 to manufacture each one.

a. Complete the table of values showing the total cost, C, of producing n calculators.

n	100	500	800	1200	1500
C					

b. Write an equation that expresses C in terms of n.

c. Graph the equation by hand.



d. What is the cost of producing 1000 calculators? Illustrate this as a point on your graph.

e. How many calculators can be produced for \$10,000? Illustrate this as a point on your graph.

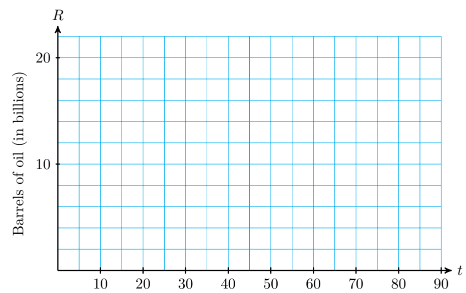
2. The world's oil reserves were 2100 billion barrels in 2005; total annual consumption is 28 billion barrels per year.

a. Complete the table of values that shows the remaining oil reserves R in terms of time t (in years since 2005).

t	5	10	15	20	25
R					

b. Write an equation that expresses R in terms of t.

c. Find the intercepts and graph the equation by hand.

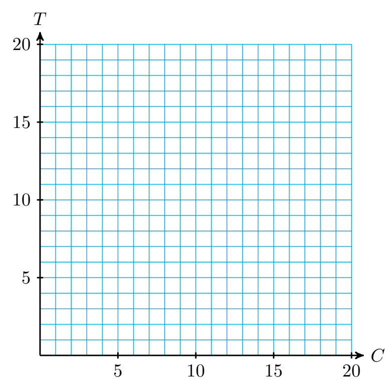


d. What do the intercepts tell us about the world's oil supply?

3. Alida plans to spend part of her vacation in Atlantic City and part in Saint-Tropez. She estimates that after airfare her vacation will cost \$60 per day in Atlantic City and \$100 per day in Saint-Tropez. She has \$1200 to spend after airfare.

a. Write an equation that relates the number of days, C, Alida can spend in Atlantic City and the number of days, T, in Saint-Tropez.

b. Find the intercepts and graph the equation by hand.



c. If Alida spends 10 days in Atlantic City, how long can she spend in Saint-Tropez?

d. What do the intercepts tell us about Alida's vacation?

Exercise Group. For Problems 4–9, graph the equation on graph paper. Use the most convenient method for each problem.

4. $\dfrac{x}{6} - \dfrac{y}{12} = 1$

5. $50x = 40y - 20,000$

6. $1.4x + 2.1y = 8.4$

7. $3x - 4y = 0$

8. $3x - 4y = 0$

9. $4x = -12$

10. The table shows the amount of oil, B (in thousands of barrels), left in a tanker t minutes after it hits an iceberg and springs a leak.

t	0	10	20	30
B	800	750	700	650

a. Write a linear function for B in terms of t.

b. Choose appropriate window settings on your calculator and graph your function.

c. Give the slope of the graph, including units, and explain the meaning of the slope in terms of the oil leak.

11. An interior decorator bases her fee on the cost of a remodeling job. The accompanying table shows her fee, F, for jobs of various costs, C, both given in dollars.

C	5000	10,000	20,000	50,000
F	1000	1500	2500	5500

a. Write a linear function for F in terms of C.

b. Choose appropriate window settings on your calculator and graph your function.

c. Give the slope of the graph, including units, and explain the meaning of the slope in terms of the the decorator's fee.

Exercise Group. For Problems 12–15, find the slope of the line segment joining the points.

12. $(-1,4),\ (3,-2)$

13. $(5,0),\ (2,-6)$

14. $(6.2,1.4),\ (-2.3,4,8)$

15. $(0,-6.4),\ (-5.6,3.4)$

Exercise Group. For Problems 16–17, which tables could describe variables related by a linear equation?

16.

a.

r	E
1	5
2	$\frac{5}{2}$
3	$\frac{5}{3}$
4	$\frac{5}{4}$
5	1

b.

s	t
10	6.2
20	9.7
30	12.6
40	15.8
50	19.0

17.

a.

w	A
2	-13
4	-23
6	-33
8	-43
10	-53

b.

x	G
0	0
2	5
4	10
8	20
16	40

Exercise Group. For Problems 18–19, the table gives values for a linear equation in two variables. Fill in the missing values.

18.

d	V
-5	-4.8
-2	-3
	-1.2
6	1.8
10	

19.

q	S
-8	-8
-4	36
3	
	200
9	264

20. The planners at AquaWorld want the small water slide to have a slope of 25%. If the slide is 20 feet tall, how far should the end of the slide be from the base of the ladder?

Exercise Group. For Problems 21–24, find the slope and y-intercept of the line.

21. $2x - 4y = 5$

22. $\dfrac{1}{2}x + \dfrac{2}{3}y = \dfrac{5}{6}$

23. $8.4x + 2.1y = 6.3$

24. $y - 3 = 0$

Exercise Group. For Problems 25 and 26,

a. Graph by hand the line that passes through the given point with the given slope.

b. Find the equation of the line.

25. $(-4, 6)$; $m = \dfrac{-2}{3}$

26. $(2, -5)$; $m = \dfrac{3}{2}$

27. The rate at which air temperature decreases with altitude is called the lapse rate. In the troposphere, the layer of atmosphere that extends from the Earth's surface to a height of about 7 miles, the lapse rate is about 3.6°F for every 1000 feet. (Source: Ahrens, 1998)

a. If the temperature on the ground is 62°F, write an equation for the temperature, T, at an altitude of h feet.

b. What is the temperature outside an aircraft flying at an altitude of 30,000 feet? How much colder is that than the ground temperature?

c. What is the temperature at the top of the troposphere?

Exercise Group. For Problems 28 and 29, find an equation for the line passing through the two given points.

28. $(3, -5)$, $(-2, 4)$

29. $(0, 8)$, $(4, -2)$

30. The population of Maple Rapids was 4800 in 2005 and had grown to 6780 by 2020. Assume that the population increases at a constant rate.

a. Make a table of values showing two data points.

b. Find a linear equation that expresses the population, P, of Maple Rapids in terms of the number of years, t, since 2005.

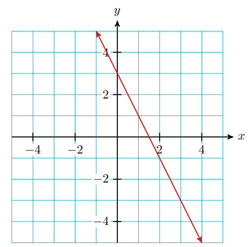
c. State the slope of the line, including units. What does it tell us about the population?

Exercise Group. For Problems 31 and 32,

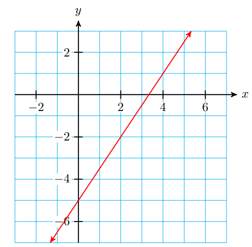
a. Find the slope and y-intercept of the line.

b. Write an equation for the line.

31.



32.



33. What is the slope of the line whose intercepts are $(-5, 0)$ and $(0, 3)$?

34.

a. Find the x- and y-intercepts of the line $\dfrac{x}{4} - \dfrac{y}{6} = 1$.

b. What is the slope of the line in part (a)?

35.

a. Find the x- and y-intercepts of the line $y = 2 + \dfrac{3}{2}(x - 4)$.

b. Find the point on the line whose x-coordinate is 4. Can there be more than one such point?

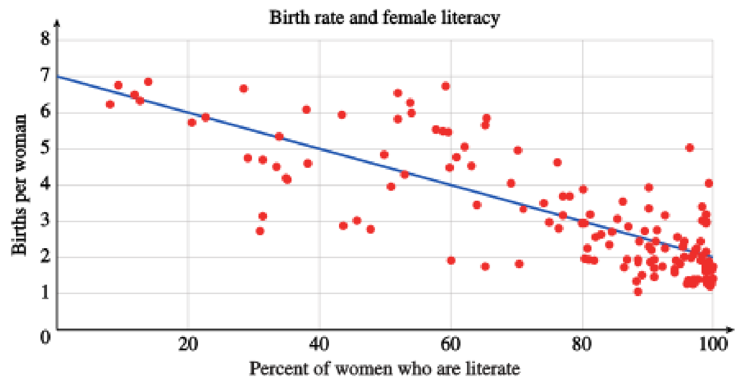
36. Find an equation in slope-intercept form for the line of slope $\dfrac{6}{5}$ that passes through $(-3, -4)$.

Chapter 2

Applications of Linear Models

You may have heard that mathematics is the language of science. In fact, professionals in nearly every discipline take advantage of mathematical methods to analyze data, identify trends, and predict the effects of change. This process is called **mathematical modeling**. A **model** is a simplified representation of reality that helps us understand a process or phenomenon. Because it is a simplification, a model can never be completely accurate. Instead, it should focus on those aspects of the real situation that will help us answer specific questions. Here is an example.

The world's population is growing at different rates in different nations. Many factors, including economic and social forces, influence the birth rate. Is there a connection between birth rates and education levels? The figure shows the birth rate plotted against the female literacy rate in 148 countries. Although the data points do not all lie precisely on a line, we see a generally decreasing trend: the higher the literacy rate, the lower the birth rate. The **regression line** provides a model for this trend, and a tool for analyzing the data. In this chapter we study the properties of linear models and some techniques for fitting a linear model to data.



2.1 Linear Regression

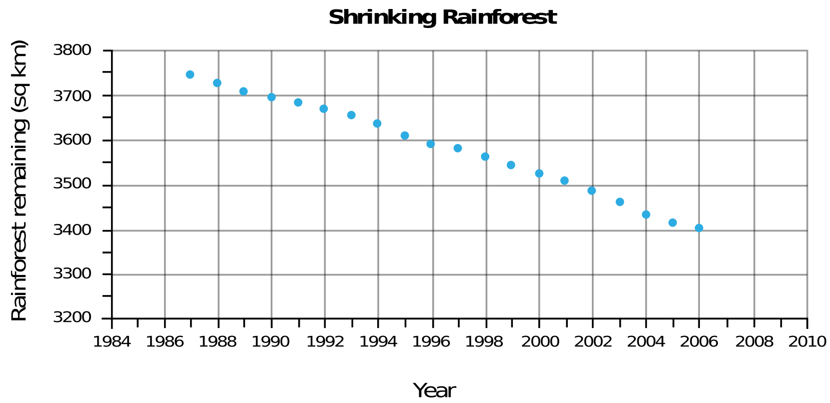
2.1.1 Shrinking Rain Forest

The Amazon Basin in South America contains over half of the planet's rain forest. The Amazon rain forest is home to the largest collection of plant and animal species in the world, including more than one-third of all living species. During the 1960s, colonists began cutting down the rain forest to clear land for agriculture. The construction of the Trans-Amazonian Highway in the early 1970s opened large forest areas to development by settlers and commercial interests, increasing the rate of deforestation.

Environmentalists are concerned about the loss of biodiversity which will result from destruction of the forest, and about the release of the carbon contained within the vegetation, which could accelerate global warming.

In Brazil, the Instituto Nacional de Pesquisas Espaciais (INPE, or National Institute of Space Research) uses Landsat satellite photos to monitor the pace of deforestation. According to their data, the original Amazon rain forest biome in Brazil of 4,100,000 square kilometers was reduced to 3,413,000 square kilometers by 2005, representing a loss of 16.8%. The figures for 1987 to 2006 are shown at right, and a plot of the data appears below. [TK]

Year	Remaining forest (thousands sq km)
1987	3745
1988	3274
1989	3706
1990	3692
1991	3681
1992	3667
1993	3652
1994	3637
1995	3608
1996	3590
1997	3577
1998	3560
1999	3542
2000	3524
2001	3506
2002	3485
2003	3460
2004	3432
2005	3413
2006	3400



[TK] To see more examples of scatterplots, see Section 2.1.1 of the Toolkit.

Although the data points do not all lie exactly on a straight line, they are very close. One question we might ask is: If deforestation continues at the same rate, when will the Amazon rain forest disappear completely? In this section we learn to find a linear model that approximates a data set.

2.1.2 Line of Best Fit

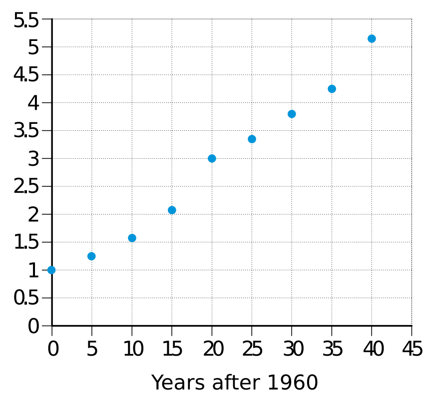
In most cases, a mathematical model is not a perfect description of reality. Many factors can affect empirical data, including measurement error, environmental conditions, and the influence of related variables. Nonetheless, we can often find an equation that approximates the data in a useful way.

Example 2.1.1 The table shows the minimum wage in the US at five-year intervals. (Source: Economic Policy Institute)

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000
Min. wage	1.00	1.25	1.60	2.10	3.10	3.35	3.80	4.25	5.15

a. Let t represent the number of years after 1960, and plot the data. Are the data linear?

b. Draw a line that "fits" the data. [TK]



Solution.

a. The graph shown is called a **scatterplot**. The data are not strictly linear, because the slope is not constant: from 1960 to 1965, the minimum wage increased at an average rate of

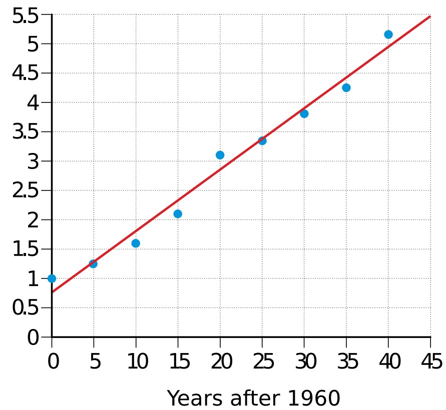
$$\frac{1.25 - 1.00}{5} = 0.05 \text{ dollars per year}$$

and from 1970 to 1975, the minimum wage increased at a rate of

$$\frac{2.10 - 1.60}{5} = 0.10 \text{ dollars per year}$$

However, the data points do appear to lie close to an imaginary line.

b. We would like to draw a line that comes as close as possible to all the data points, even though it may not pass precisely through any of them. In particular, we try to adjust the line so that we have the same number of points above the line and below the line. One possible solution is shown in the figure at right.



□

[TK] To see more examples of line of best fit, see Section 2.1.2 of the Toolkit.

A line that fits the data in a scatterplot is called a **regression line**. Drawing a regression line by eye is a subjective process. Using technology, we can compute a particular regression line called the **least-squares regression line**, which is widely used in statistics and modeling.

We can still find an equation for a line of best fit using the point-slope formula. (To review using the point-slope formula, see Finding a Linear Model in Section 1.5.) We choose two points on the line whose coordinates we can estimate fairly accurately. Note that these two points need not be any of the original data points.

Checkpoint 2.1.2 QuickCheck 1. Explain how to find the equation of a regression line.

a. Choose two data points and use the point-slope formula.

b. Use the point-slope formula with the first and last data points.

c. The regression line must pass through two data points, and we use the point-slope formula for those points.

d. Choose two points on the regression line and use the point-slope formula.

Checkpoint 2.1.3 Practice 1. The regression line in the Example above appears to pass through the points $(5, 1.25)$ and $(25, 3.35)$. Use those points to find an equation for the regression line. [TK]

Hint: Step 1 Calculate the slope.
Step 2 Use the point-slope formula.

[TK] To review finding the equation of a line, see Section 2.1.3 of the Toolkit.

2.1.3 Interpolation and Extrapolation

Example 2.1.4 An outdoor snack bar collected the following data showing the number of cups of cocoa, C, they sold when the high temperature for the day was $T°$ Celsius.

Temperature (°C), T	2	4	5	8	10	11	12	15	16	18
Cups of cocoa, C	45	42	42	35	25	25	17	16	15	6

a. Make a scatterplot of the data, and draw a regression line

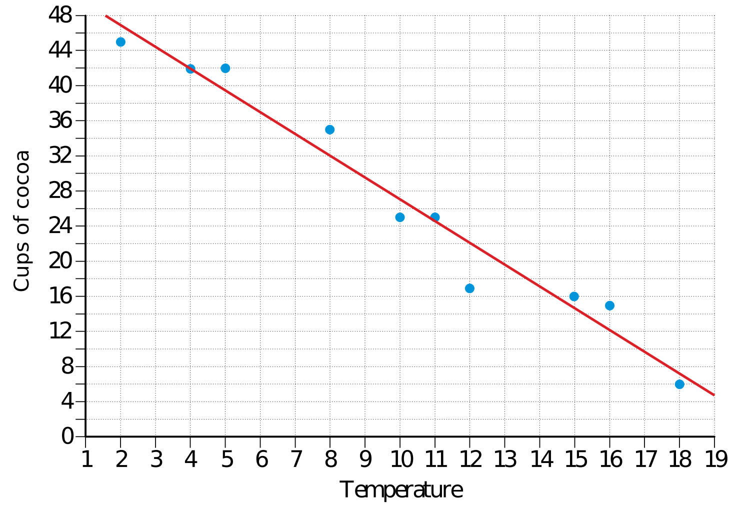
b. Read values from your line for the number of cups of cocoa that will be sold when the temperature is 8°C and when the temperature is 16°C.

c. Find an equation for the regression line.

d. Use your equation to predict the number of cups of cocoa that will be sold when the temperature is 9°C, and when the temperature is 24°C.

Solution.

a. The scatterplot and a regression line are shown in the figure.



The regression line need not pass through any of the data points, but it should be as close as possible. We try to draw the regression line so that there are an equal number of data points above and below the line.

b. The points $(8, 32)$ and $(16, 12)$ appear to lie on the regression line. According to this model, the snack bar will sell 32 cups of cocoa when the temperature is 8°C, and 12 cups when it is 16°C. These values are close to the actual data, but not exact.

c. To find an equation for the regression line, we use two points on the line—not data points! We will use $(8, 32)$ and $(16, 12)$. First we compute the slope

$$m = \frac{C_2 - C_1}{T_2 - T_1} = \frac{12 - 32}{16 - 8} = -2.5$$

Next, we apply the point-slope formula. We'll use the point $(16, 12)$.

$$\begin{aligned} C &= C_1 + m(T - T_1) \\ C &= 12 - 2.5(T - 16) \\ C &= -2.5T + 52 \end{aligned}$$

d. When $T = \mathbf{9}$,

$$C = -2.5(\mathbf{9}) + 52 = 29.5$$

We predict that the snack bar will sell 29 or 30 cups of cocoa when the temperature is 9°C. When $T = \mathbf{24}$,

$$C = -2.5(\mathbf{24}) + 52 = -8$$

Because the snack bar cannot sell -8 cups of cocoa, this prediction is not useful. (What is the Fahrenheit equivalent of 24°C?)

□

Using a regression line to estimate values between known data points is called **interpolation**. If the data points lie fairly close to the regression line, then interpolation will usually give a fairly accurate estimate. In the Example above, the estimate of 29 or 30 cups of cocoa at 9°C seems reasonable in the context of the data.

Making predictions beyond the range of known data is called **extrapolation**. Extrapolation can often give useful information, but if we try to extrapolate too far beyond our data, we may get unreasonable results. The conditions that produced the data may no longer hold, as in the Example above, or other unexpected conditions may arise to alter the situation.

Checkpoint 2.1.5 QuickCheck 2. True or false.

a. A scatterplot is a type of linear model.

b. A regression line should give the same y-values as the data points.

c. We use interpolation to estimate the y-value at a data point.

d. Extrapolation is usually more reliable than interpolation.

Checkpoint 2.1.6 Practice 2.

a. Use your regression equation from the previous Example to predict the number of cups of cocoa sold when the temperature is -10°C.

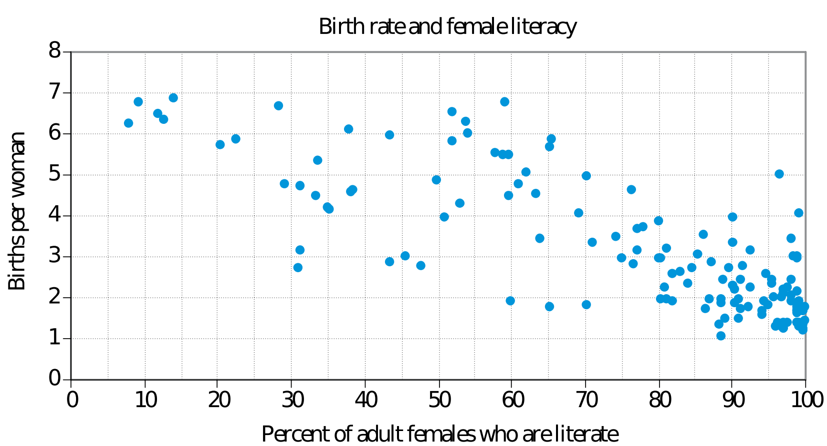
b. Predict the number of cups of cocoa sold when the temperature is 7°C.

c. Which prediction is more likely to be accurate? Why?

2.1.4 Scatterplots

The data in a scatterplot may show a linear trend, even though the individual points are not clustered closely around a line. Scattering of data is common in the social sciences, where many variables may influence a particular situation. Nonetheless, by analyzing the data, we may be able to detect a connection between some of the variables.

Example 2.1.7 The world's population is growing at different rates in different nations. Many factors, including economic and social forces, influence the birthrate. Is there a connection between birth rates and education levels? The figure below shows the birth rate plotted against the female literacy rate in 148 countries.



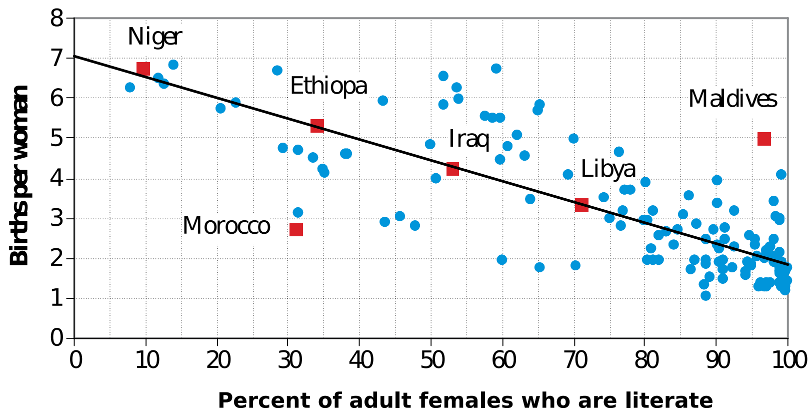
a. Draw a line of best fit for the data points.

b. Locate on the scatterplot the points representing the following nations

Country	Literacy rate	Birth rate
Ethiopa	33.8	5.33
Iraq	53.0	4.28
Libya	71.0	3.34
Maldives	96.4	5.02
Morocco	31.0	2.73
Niger	9.4	6.75

c. Data points that lie far from the regression line are called **outliers**. Which of the nations listed in part (b) could be considered outliers?

Solution.

a. 

b. The figure above shows the regression line and the data points for each of the nations in part (b).

c. Morocco and the Maldives could be considered outliers.

□

Checkpoint 2.1.8 Practice 3. The equation for the least-squares regression line in the previous Example is

$$y = 7.04 - 0.05x$$

a. What values for the input variable make sense for the model? What are the largest and smallest values predicted by the model for the output variable?

b. State the slope of the regression line, including units, and explain what it means in the context of the data.

2.1.5 Problem Set 2.1

Warm Up

1. Choose the correct graph for each equation. The scales on both axes are the same.

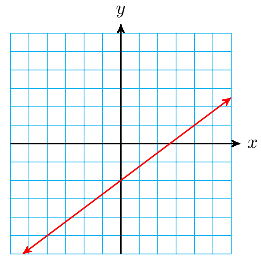
a. $y = \frac{3}{4}x + 2$

b. $y = \frac{-3}{4}x + 2$

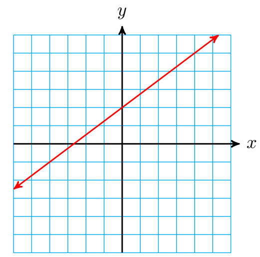
c. $y = \frac{3}{4}x - 2$

d. $y = \frac{-3}{4}x - 2$

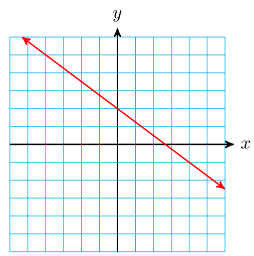
I



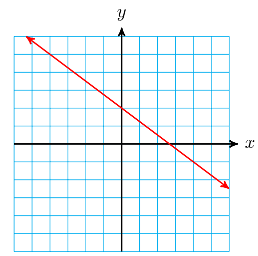
II



III



III



2. Choose the correct graph for each equation. The scales on both axes are the same.

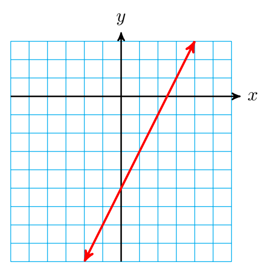
a. $y = 1 + 2(x + 3)$

b. $y = -1 + 2(x - 3)$

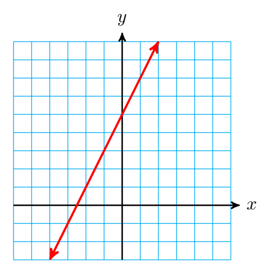
c. $y = -1 + 2(x + 3)$

d. $y = 1 + 2(x - 3)$

I



II



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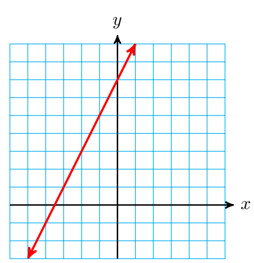

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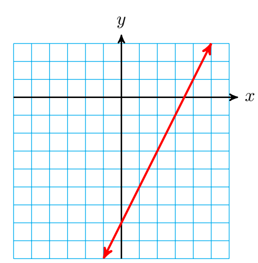


Exercise Group. In Problems 3 and 4, find a linear model from two data points as follows:

a. Make a table showing the coordinates of two data points for the model. (Which variable should be plotted on the horizontal axis?)

b. Write an equation for the line.

c. State the slope of the line, including units, and explain its meaning in the context of the problem.

3. On an international flight a passenger may check two bags each weighing 70 kilograms, or 154 pounds, and one carry-on bag weighing 50 kilograms, or 110 pounds. Express the weight, p, of a bag in pounds in terms of its weight, k, in kilograms.

4. Ms. Randolph bought a used car in 2010. In 2012 the car was worth \$9000, and in 2015 it was valued at \$4500. Express the value, V, of Ms. Randolph's car in terms of the number of years, t, she has owned it.

Skills Practice

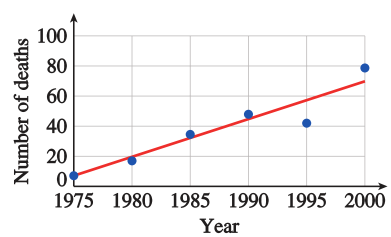
5.

a. Find the slope, the C-intercept, and the T-intercept for the regression line in Example 2.1.4.

b. What do the slope and the intercepts tell us about the sale of cocoa?

6. The number of manatees killed by watercraft in Florida waters has been increasing since 1975. Data are given at 5-year intervals in the table, and a scatterplot with regression line is shown below. (Source: Florida Fish and Wildlife Conservation Commission)

Year	Manatee deaths
1975	6
1980	16
1985	33
1990	47
1995	42
2000	78



An equation for the regression line is $y = 4.7 + 2.6t$

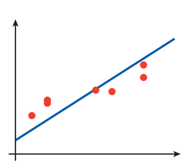
a. Use the regression equation to estimate the number of manatees killed by watercraft in 1998.

b. What does the slope of the regression line mean in this situation?

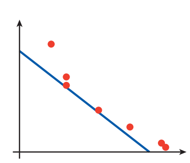
c. Which data point might be considered an outlier?

Exercise Group. In Problems 7 and 8, the regression lines can be improved by adjusting either m or b. Draw a line that fits the data points more closely.

7.



8.



Exercise Group. For Problems 9 and 10,

a. Use the point-slope formula to write an equation.

b. Use linear interpolation to give approximate answers.

c. What does the slope tell us about the problem setting?

9. Newborn blue whales are about 24 feet long and weigh 3 tons. The young whale nurses for 7 months, at which time it is 53 feet long. Estimate the length of a 1-year-old blue whale.

10. A truck on a slippery road is moving at 24 feet per second when the driver steps on the brakes. The truck needs 3 seconds to come to a stop. Estimate the truck's speed 2 seconds after the brakes were applied.

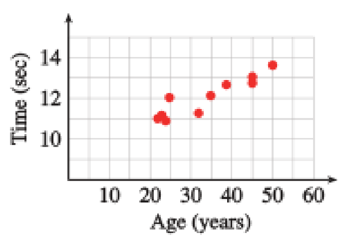
Exercise Group. In Problems 11 and 12, use linear interpolation or extrapolation to answer the questions.

11. The temperature of an automobile engine is 9° Celsius when the engine is started and 51°C seven minutes later. Use a linear model to predict the engine temperature for both 2 minutes and 2 hours after it started. Are your predictions reasonable?

12. The elephant at the City Zoo becomes ill and loses weight. She weighed 10,012 pounds when healthy and only 9641 pounds a week later. Predict her weight after 10 days of illness.

Applications

13. The scatterplot shows the ages of 10 army drill sergeants and the time it took each to run 100 meters, in seconds.



a. What was the hundred-meter time for the 25-year-old drill sergeant?

b. How old was the drill sergeant whose hundred-meter time was 12.6 seconds?

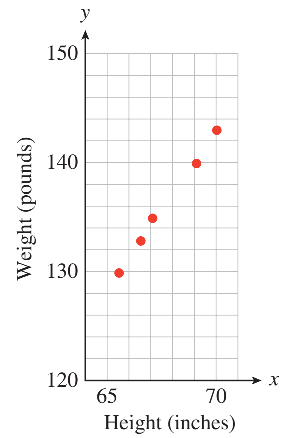
c. Use a straightedge to draw a line of best fit through the data points.

d. Use your line of best fit to predict the hundred-meter time of a 28-year-old drill sergeant.

e. Choose two points on your regression line and find its equation.

f. Use the equation to predict the hundred-meter time of a 40-year-old drill sergeant and a 12 year-old drill sergeant. Are these predictions reasonable?

14. The scatterplot shows the weights in pounds and the heights in inches of a team of distance runners.



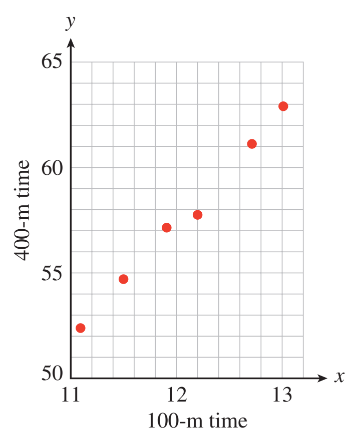
a. Use a straightedge to draw a line that fits the data.

b. Use your line to predict the weight of a 65-inch tall runner and the weight of a 71-inch tall runner.

c. Use your answers from part (b) to approximate the equation of your regression line.

d. Use your answer from part(c) to predict the weight of a runner who is 68 inches tall.

15. The scatterplot shows the best times for various women running the 400 meters and the 100 meters.



a. Use a straightedge to draw a line that fits the data.

b. Use your line to predict the 400-meter time of a woman who runs the 100-meter dash in 11.2 seconds, and the 400-meter time of a woman who runs the 100-meter dash in 13.2 seconds.

c. Use your answers from part (b) to approximate the equation of your regression line.

d. Use your answer from part (c) to predict the the 400-meter time of a woman who runs the 100-meter dash in 12.1 seconds.

16. The table shows the minimum wage in the United States at five-year intervals. (Source: Economic Policy Institute)

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005	2010
Minimum wage	1.00	1.25	1.60	2.10	3.10	3.35	3.80	4.25	5.15	5.15	7.25

a. Let t represent the number of years after 1960 and plot the data. Draw a line of best fit for the data points.

b. Find an equation for your regression line.

c. Estimate the minimum wage in 1972.

d. Predict the minimum wage in 2025.

17. With Americans' increased use of faxes, pagers, and cell phones, new area codes are being created at a steady rate. The table shows the number of areacodes in the US each year. (Source: USA Today, NeuStar, Inc.)

Year	1997	1998	1999	2000	2001	2002	2003
Number of area codes	151	186	204	226	239	262	274

a. Let t represent the number of years after 1995 and plot the data. Draw a line of best fit for the data points.

b. Find an equation for your regression line.

c. How many area codes do you predict for 2010?

18. The table shows the amount of carbon released into the atmosphere annually from burning fossil fuels, in billions of tons, at 5-year intervals from 1950 to 1995. (Source: www.worldwatch.org)

Year	1950	1955	1960	1965	1970	1975	1980	1985	1990	1995
Carbon emissions	1.6	2.0	2.5	3.1	4.0	4.5	5.2	5.3	5.9	6.2

a. Let t represent the number of years after 1950 and plot the data. Scale the t-axis from 0 to 50 by 5's, and the C-axis from 0 to 7 by 0.5's.

b. Draw a line of best fit for the data points.

c. Find an equation for your regression line.

d. Estimate the amount of carbon released in 1992.

19. Male birds with the largest repertoire of songs are the first to acquire mates in the spring. The table shows the number of different songs sung by several sedge warblers, and the days on which they acquired their mates, where day 1 is April 20. (Source: Krebs and Davies, 1993)

Number of songs, x	41	38	34	32	30	25	24	24	23	14
Pairing day, y	20	24	25	21	24	27	31	35	40	42

a. Plot the data points on graph paper, scale the x-axis from 0 to 65 by 5's, and the y-axis from 0 to 60 by 5's.

b. The least-squares regression line is

$$y = -0.85x + 53$$

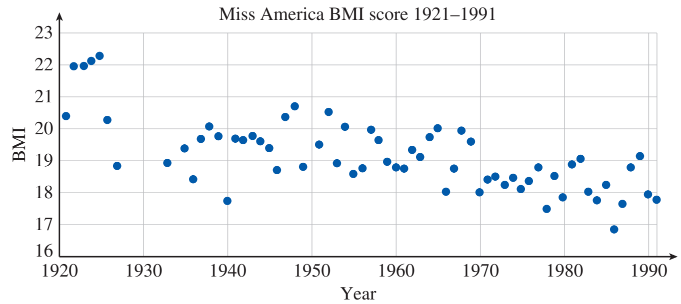
Graph this line on the same grid with the data. (Make a short table of values and plot the points.)

c. What does the slope of the regression line tell us about sedge warblers?

d. Use extrapolation to estimate when a sedge warbler that knows 10 songs can expect to find a mate.

e. What do the intercepts of the regression line represent? Do these values make sense for this situation?

20. One measure of a person's physical fitness is the body mass index, or BMI. Your body mass index is the ratio of your weight in kilograms to the square of your height in meters. The points on the scatterplot show the BMI of Miss America from 1921 to 1991.



a. Use a straightedge to draw a line of best fit on the scatterplot above.

b. The equation of the least-squares regression line for the data is

$$y = 20.69 - 0.04t$$

where t is the number of years since 1920. On the figure above, relabel the horizontal axis with values of t. Then graph this line and compare to your estimated line of best fit.

c. Do thinner people have higher or lower BMI scores than fatter people? Use the definition of BMI to explain your reasoning.

d. The Center for Disease Control considers a BMI between 18.5 and 24.9 to be healthy. In 2002, Miss America was 5'3" tall and weighed 110 pounds. Calculate her BMI. (You will need to convert inches to meters and pounds to kilograms.)

e. What BMI score does the regression line predict for Miss America 2002?

f. There are no data points for the years 1928 to 1932. What happened during those years that might cause this gap?

2.2 Linear Systems

For the rest of this chapter we'll consider problems that can be solved using two or more linear equations simultaneously. To begin, think about the two equations in the next Investigation.

Investigation 2.2.1 Water Level. When sailing upstream in a canal or a river that has rapids, ships must sometimes negotiate locks to raise them to a higher water level. Suppose your ship is in one of the lower locks, at an elevation of 20 feet. The next lock is at an elevation of 50 feet. Water begins to flow from the higher lock to the lower one, raising your level by 1 foot per minute, and simultaneously lowering the water level in the next lock by 1.5 feet per minute.

1. Fill in the table

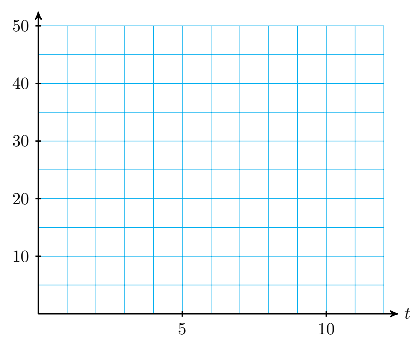
t (minutes)	Lower lock water level	Upper lock water level
0		
2		
4		
6		
8		
10		

2. Let t stand for the number of minutes the water has been flowing.

   a. Write an equation for L, the water level in the lower lock after t minues.

   b. Write an equation for U, the water level in the upper lock after t minues.

3. Graph both your equations on the grid.



4. When will the water level in the two locks be 10 feet apart?

5. When will the water level in the two locks be the same?

6. Write an equation you can use to verify your answer to part (5), and solve it.

2.2.1 Systems of Equations

A biologist wants to know the average weights of two species of birds in a wildlife preserve. She sets up a feeder whose platform is actually a scale and mounts a camera to monitor the feeder. She waits until the feeder is occupied only by members of the two species she is studying, robins and thrushes. Then she takes a picture, which records the number of each species on the scale and the total weight registered.

From her two best pictures, she obtains the following information. The total weight of 3 thrushes and 6 robins is 48 ounces, and the total weight of 5 thrushes and 2 robins is 32 ounces. The biologist writes two equations about the photos. [TK] She begins by assigning variables to the two unknown quantities:

$$\text{Average weight of a thrush: } t$$
$$\text{Average weight of a robin: } r$$

In each of the photos,

$$(\text{weight of thrushes}) + (\text{weight of robins}) = \text{total weight}$$

So the two equations are

$$\begin{aligned} 3t + 6r &= 48 \\ 5t + 2r &= 32 \end{aligned}$$

[TK] To review writing an expression for a linear model, see Section 2.2.1 of the Toolkit.

This pair of equations is an example of a **linear system of two equations in two unknowns** (or a 2×2 linear system, for short). A **solution** to the system is an ordered pair of numbers, (t, r), that satisfies both equations in the system.

Checkpoint 2.2.1 QuickCheck 1. Choose the solution of the system:

$$\begin{aligned} 3x - 2y &= 19 \\ 4x + 5y &= 10 \end{aligned}$$

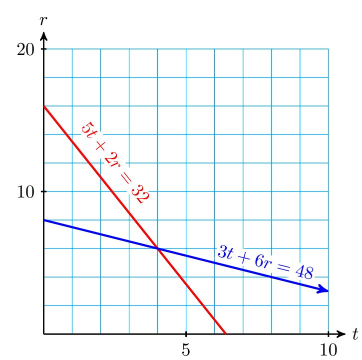
a. $(7, 1)$ b. $(0, 2)$ c. $(5, -2)$ d. $(2, 5)$

2.2.2 Solving Systems by Graphing

Recall that every point on the graph of an equation represents a solution to that equation, so a solution to both equations must be a point that lies on both graphs. Therefore, a solution to the system is a point where the two graphs intersect.

The figure at right shows a graph of the system about robins and thrushes. The two lines appear to intersect at the point $(4, 6)$, so we expect that the values $t = \mathbf{4}$ and $r = \mathbf{6}$ are the solution to the system. We can check by verifying that these values satisfy both equations in the system

$$3(\mathbf{4}) + 6(\mathbf{6}) \stackrel{?}{=} 48 \qquad \text{True}$$
$$5(\mathbf{4}) + 2(\mathbf{6}) \stackrel{?}{=} 32 \qquad \text{True}$$



Both equations are true, so we have found the solution: a thrush weighs 4 ounces on average, and a robin weighs 6 ounces. [TK]

[TK] To review solutions of linear systems, see Section 2.2.2 of the Toolkit.

Checkpoint 2.2.2 QuickCheck 2. True or false.

a. The point where two lines cross is called the **intercept**.

b. The **intercepts** of a line are the points where it intersects the axes.

c. The solution of a system may occur at an intercept.

d. The words **intercept** and **intersect** mean the same thing.

We can use a calculator or graphing utility to graph the equations in a system.

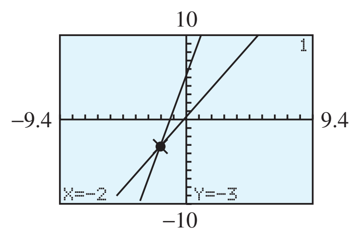
Example 2.2.3 Use your grapher to solve the system by graphing. [TK]

$$y = 1.7x + 0.4$$
$$y = 4.1x + 5.2$$

Solution. We set the graphing window to

$$\text{Xmin} = -9.4 \qquad \text{Ymin} = -10$$
$$\text{Xmax} = -9.4 \qquad \text{Ymax} = 10$$

and enter the two equations. We can see in the figure that the two lines intersect in the third quadrant. We use the TRACE feature to find the coordinates of the intersection point, $(-2, -3)$. The solution to the system is $x = -2$, $y = -3$.



□

[TK] To learn about graphing equations with GeoGebra, see Appendix B.

The values we obtain from a calculator may be only approximations, so it is a good idea to check the solution algebraically. In the example above, we find that both equations are true when we substitute $x = \mathbf{-2}$ and $y = \mathbf{-3}$.

$$\mathbf{-3} = 1.7(\mathbf{-2}) + 0.4 \qquad \text{True}$$
$$\mathbf{-3} = 4.2(\mathbf{-2}) + 5.2 \qquad \text{True}$$

Checkpoint 2.2.4 Practice 1.

a. Solve the system of equations

$$y = -0.7x + 6.9$$
$$y = 1.2x - 6.4$$

by graphing. Use the window

$$\text{Xmin} = -9.4 \qquad \text{Xmax} = 9.4$$
$$\text{Ymin} = -10 \qquad \text{Ymax} = 10$$

b. Verify algebraically that your solution satisfies both equations.

Because the TRACE feature does not show every point on a graph, we may not find the exact solution to a system by tracing the graphs.

Example 2.2.5 Solve the system

$$3x - 2.8y = 21.06$$
$$2x + 1.2y = 5.3$$

Solution. We can graph this system in the standard window by solving each equation for y. We enter

$$Y_1 = (21.06 - 3X)/-2.8$$
$$Y_2 = (5.3 - 2X)/1.2$$

and use the standard window. If we trace along the graph to the intersection point, we will not find the same coordinates on both lines. The intersection point is not displayed in this window. Instead, we can use the **intersect** feature of the grapher.

Follow the instructions for your grapher's intersect feature to find the intersection point, $x = 4.36$, $y = -2.85$.

We can substitute these values into the original system to check that they satisfy both equations.

$$3(\mathbf{4.36}) - 2.8(\mathbf{-2.85}) = 21.06 \qquad \text{True}$$
$$2(\mathbf{4.36}) + 1.2(\mathbf{-2.85}) = 5.3 \qquad \text{True}$$

□

Checkpoint 2.2.6 Practice 2. Solve the system of equations

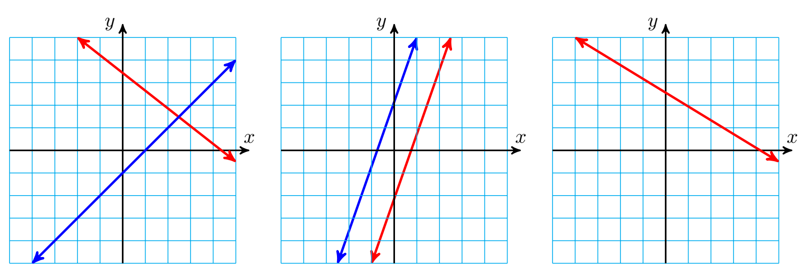
$$y = 47x - 1930$$
$$y + 19x = 710$$

by graphing. Estimate the intercepts of each graph to help you choose a suitable window, and use the **intersect** feature to locate the solution.

How does the calculator find the exact coordinates of the intersection point? In the next section we'll learn how to find the solution of a system using algebra.

2.2.3 Inconsistent and Dependent Systems

Because two straight lines do not always intersect at a single point, a 2×2 system of linear equations does not always have a unique solution. In fact, there are three possibilities, as illustrated below.



Consistent and independent: one solution | Inconsistent: no solution | Dependent: infinitely many solutions

Definition 2.2.7 Solutions of Linear Systems. There are three types of 2×2 linear system :

1. **Consistent and independent system**. The graphs of the two lines intersect in exactly one point. The system has exactly one solution.

2. **Inconsistent system**. The graphs of the equations are parallel lines and hence do not intersect. An inconsistent system has no solutions.

3. **Dependent system**. All the solutions of one equation are also solutions

to the second equation, and hence are solutions of the system. The graphs of the two equations are the same line. A dependent system has infinitely many solutions.

◊

Example 2.2.8 Solve each system.

a.

$$\begin{aligned} y &= -x+5 \\ 2x+2y &= 3 \end{aligned}$$

b.

$$\begin{aligned} x &= \frac{2}{3}y+3 \\ 3x-2y &= 9 \end{aligned}$$

Solution.

a. We can use technology to graph both equations on the same axes. First, we rewrite the second equation in slope-intercept form by solving for y.

$$\begin{aligned} 2x+2y &= 3 && \text{Substract } \mathbf{2x} \text{ from both sides.} \\ 2y &= -2x+3 && \text{Divide both sides by 2.} \\ y &= -x+1.5 \end{aligned}$$

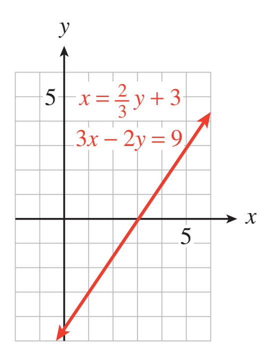
We enter the equations as

$$\begin{aligned} Y_1 &= -X+5 \\ Y_2 &= -X+1.5 \end{aligned}$$

You should see that the lines do not intersect within the viewing window; they appear to be parallel. If we look again at the equations of the lines, we recognize that both have slope -1 but different y-intercepts, so they are parallel. Because parallel lines never meet, there is no solution to the system.

b. This time we will graph by hand. We begin by writing each equation in slope-intercept form.

$$\begin{aligned} x &= \frac{2}{3}y+3 && \text{Subtract 3.} \\ x-3 &= \frac{2}{3}y && \text{Multiply by } \mathbf{\frac{3}{2}}. \\ \frac{3}{2}x-\frac{9}{2} &= y \end{aligned}$$



For the second equation,

$$\begin{aligned} 3x-2y &= 9 && \text{Subtract } \mathbf{3x}. \\ -2y &= -3x+9 && \text{Divide by } \mathbf{-2}. \\ y &= \frac{3}{2}x-\frac{9}{2} \end{aligned}$$

The two equations are actually different forms of the same equation. They are equivalent, so they share the same line as a graph. Every point on the first line is also a point on the second line, and hence a solution of the system. The system has infinitely many solutions.

□

Checkpoint 2.2.9 QuickCheck 3. Complete the statements.

a. If two lines have the same slope but different y-intercepts, the system is __________.

b. If two lines have the same slope and the same y-intercept, the system is __________.

c. If two lines have the same y-intercept but different slopes, the system is __________.

d. If both lines are horizontal, the system is __________.

Checkpoint 2.2.10 Practice 3.

a. Graph the system

$$\begin{aligned} y &= -3x + 6 \\ 6x + 2y &= 15 \end{aligned}$$

by hand, using either the intercept method or the slope-intercept method.

b. Identify the system as dependent, inconsistent, or consistent and independent.

2.2.4 Applications

Systems of equations are useful in many applied problems. Here are two examples from economics.

A business venture calculates its **profit** by subtracting its **costs** from its **revenue**, the amount of money it takes in from sales.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

Cost is usually calculated as the sum of fixed costs, or overhead, and variable costs, the cost of labor and materials to produce its product. Revenue is the product of the selling price per item times the number of items sold. If the company's revenue exactly equals its costs (so that their profit is zero), we say that the business venture will **break even**. [TK]

[TK] To see more examples of the profit equation, see Section 2.2.3 of the Toolkit.

Checkpoint 2.2.11 Practice 4. The Aquarius jewelry company determines that each production run to manufacture a pendant involves an initial setup cost of \$200 and \$4 for each pendant produced. The pendants sell for \$12 each.

a. Express the cost C of production in terms of the number x of pendants produced.

b. Express the revenue R in terms of the number x of pendants sold.

c. Graph the revenue and cost on the same set of axes. (Find the intercepts of each equation to help you choose a window for the graph.) State the solution of the system.

d. How many pendants must be sold for the Aquarius company to break even on a particular production run?

Another example involves supply and demand. The owner of a retail business must try to balance the demand for his product from consumers with the supply he can obtain from manufacturers. Supply and demand both vary with the price of the product: consumers usually buy fewer items if the price increases, but manufacturers will be willing to supply more units of the product if its price increases.

The **demand equation** gives the number of units of the product that consumers will buy at a given price. The **supply equation** gives the number of units that the producer will supply at that price. The price at which the supply and demand are equal is called the **equilibrium price**. This is the price at which the consumer and the producer agree to do business.

Example 2.2.12 A woolens mill can produce $400x$ yards of fine suit fabric if it can charge x dollars per yard. The mill's clients in the garment industry will buy $6000 - 100x$ yards of wool fabric at a price of x dollars per yard. Find the equilibrium price and the amount of fabric that will change hands at that price.

Solution.

- *Step 1.*

  We choose variables for the unknown quantities.

Price per yard:	x
Number of yards:	y

- *Step 2.*

  The supply equation tells us how many yards of fabric the mill will produce for a price of x dollars per yard.

  $$y = 400x$$

  The demand equation tells us how many yards of fabric the garment industry will buy at a price of x dollars per yard.

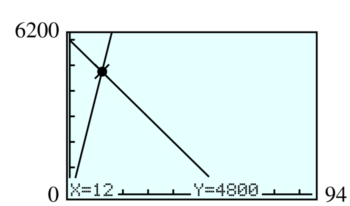
  $$y = 6000 - 100x$$

- *Step 3.*

  We graph the two equations on the same set of axes, as shown below. We set the window values to

Xmin = 0	Xmax = 94
Ymin = 0	Ymax = 6200

  and use the TRACE or the **intersect** command to locate the solution. The graphs intersect at the point $(12, 4800)$.

- *Step 4.*

  The equilibrium price is \$12 per yard, and the mill will sell 4800 yards of fabric at that price.

□

2.2.5 Problem Set 2.2

Warm Up

1. Graph by the intercept method.

$$4x = \frac{4}{3}y = 12$$

2. Graph by the slope-intercept method.

$$y = 8 - \frac{5}{2}x$$

3. Solve $0.4(30 - x) + 0.8x = 0.65(30)$

4. Write each equation in the form $ax + by = c$, where a, b, and c, are integers.

a. $4y = 6x - 300$

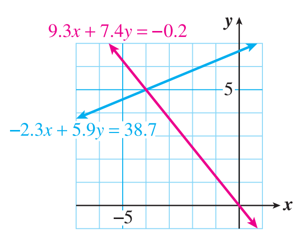
b. $24 - \frac{2}{3}y = \frac{3}{4}x$

Skills Practice

Exercise Group. For Problems 5 and 6, solve the system of equations using the graph. Then verify that your solution satisfies both equations.

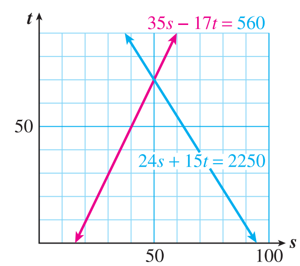
5.

$$\begin{aligned} -2.3x + 5.9y &= 38.7 \\ 9.3x + 7.4y &= -0.2 \end{aligned}$$



6.

$$\begin{aligned} 35s - 17t &= 560 \\ 24s + 15t &= 2250 \end{aligned}$$



Exercise Group. For Problems 7 and 8

a. Solve the system of equations by graphing. Use the "friendly" window

Xmin $= -9.4$	Ymin $= -10$
Xmax $= 9.4$	Ymax $= 10$

b. Verify algebraically that your solution satisfies both equations.

7.

$$\begin{aligned} y &= 2.6x + 8.2 \\ y &= 1.8 - 0.6x \end{aligned}$$

8.

$$\begin{aligned} y &= 7.2 - 2.1x \\ -2.8x + 3.7y &= 5.5 \end{aligned}$$

Exercise Group. For Problems 9–12,

a. Graph the system by hand.

b. Identify the system as dependent, inconsistent, or consistent and independent.

9.

$$\begin{aligned} 2x &= y + 4 \\ 8x - 4y &= 9 \end{aligned}$$

10.

$$\begin{aligned} 2t + 12 &= -6s \\ 12s + 4t &= 24 \end{aligned}$$

11.

$$\begin{aligned} -3x &= 4y + 12 \\ \frac{1}{2}x + 2 &= \frac{-2}{3}y \end{aligned}$$

12.

$$\begin{aligned} w - 3z &= 6 \\ 2w + x &= 8 \end{aligned}$$

Applications

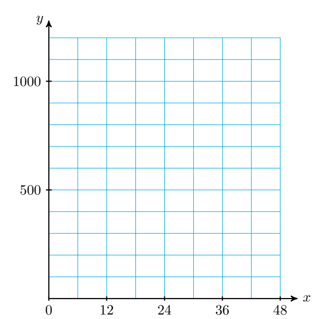
13. Francine wants to join a health club and has narrowed it down to two choices. The Sportshaus charges an initiation fee of \$500 and \$10 per month. Fitness First has an initiation fee of \$50 and charges \$25 per month.

a. Let x stand for the number of months Francine uses the health club. Write equations for the total cost of each health club for x months.

b. Complete the table for the total cost of each club.

x	Sporthaus total cost	Fitness First total cost
6		
12		
18		
24		
30		
36		
42		
48		

c. Graph both equations on the grid.



d. When will the total cost of the two health clubs be equal?

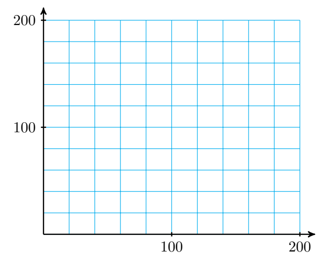
14. The Bread Alone Bakery has a daily overhead of \$90. It costs \$0.60 to bake each loaf of bread, and the bread sells for \$1.50 per loaf.

a. Write an equation for the cost C in terms of the number of loaves, x.

b. Write an equation for the revenue R in terms of the number of loaves, x.

c. Graph the revenue, R, and cost, C, on the same set of axes. State the solution of the system.

d. How many loaves must the bakery sell to break even on a given day?

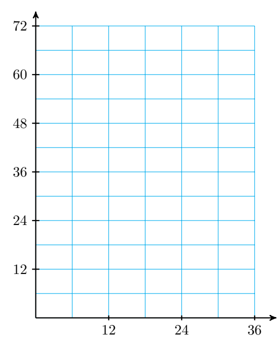


15. The manager for Books for Cooks plans to spend \$300 stocking a new diet cookbook. The paperback version costs her \$5, and the hardback costs \$10. She finds that she will sell three times as many paperbacks as hardbacks. How many of each should she buy?

a. Let x represent the number of hardbacks and y the number of paperbacks she should buy. Write an equation about the cost of the books.

b. Write a second equation about the number of each type of book.

c. Graph both equations and solve the system using the grid.



d. Answer the question in the problem.

16. There were 42 passengers on a local airplane flight. First-class fare was \$80, and coach fare was \$64. If the revenue for the flight totaled \$2880, how many first-class and how many coach passengers paid for the flight?

a. Write algebraic expressions to fill in the table.

	Number of tickets	Cost per ticket	Revenue
First-class	x		
Coach	y		
Total			

b. Write an equation about the number of tickets sold.

c. Write a second equation about the revenue from the tickets.

d. Graph both equations on graph paper and solve the system. (**Hint**:

Find the intercepts of each equation to help you choose scales for the axes.)

e. Answer the question in the problem.

17. Mel's Pool Service can clean $1.5x$ pools per week if it charges x dollars per pool, and the public will book $120 - 2.5x$ pool cleanings at x dollars per pool.

a. What is the supply equation?

b. What is the demand equation?

c. Graph both equations in the window

Xmin = 0	Ymin = 0
Xmax = 50	Ymax = 125

d. Find the equilibrium price and the number of pools Mel will clean at that price.

18.

a. Explain how you can tell, without graphing, that the system has no solution.

$$3x = y + 3$$
$$6x - 2y = 12$$

b. Explain how you can tell, without graphing, that the system has infinitely many solutions.

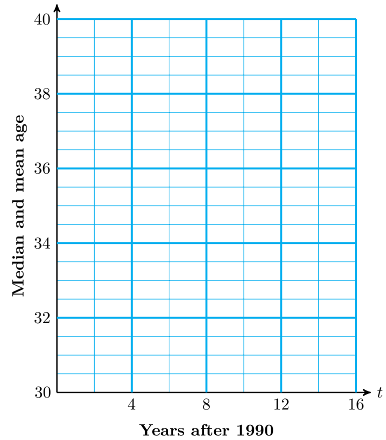
$$-x + 2y = 4$$
$$3x = 6y - 12$$

19. According to the Bureau of the Census, the average age of the U.S. population is steadily rising. The table gives data for two types of average, the mean and the median, for the ages of women. (The "mean" is the familiar average, and the "median" is the middle age: half of all women are older and half younger than the median.)

Date	1990	1992	1994	1996	1998
Median age	34.0	34.6	35.2	35.8	36.4
Mean age	36.6	36.8	37.0	37.3	37.5

a. Which is growing more rapidly, the mean age or the median age?

b. Plot the data for median age versus the date, using 1990 as $t = 0$. Draw a line through the data points.



c. What is the meaning of the slope of the line in part (b)?

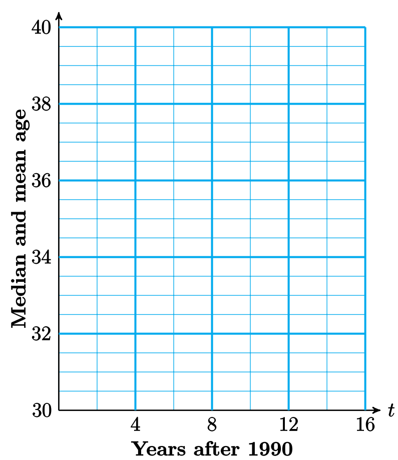
d. Plot the data for mean age versus date on the same grid. Draw a line that fits the data.

e. Use your graph to estimate when the mean age and the median age were the same.

f. If the median age of women in the United States is greater than the mean age, what does this tell you about the population of women?

20. The table below gives data for the mean and median ages for men in the United States. Repeat Problem 19 for these data.

Date	1990	1992	1994	1996	1998
Median age	31.6	32.2	32.9	33.5	34.0
Mean age	33.8	34.0	34.3	34.5	34.9



2.3 Algebraic Solution of Systems

In the previous section we used graphs to solve a system of two linear equations in two unknowns. Graphs help us visualize the system and its solution, but do not always provide the most efficient or most accurate way to find the solution. In this section we review two methods for solving a 2x2 linear system algebraically.

2.3.1 Substitution Method

You may be familiar with the **substitution method** for solving a system of equations. The idea is to write an expression for one of the variables in terms of the other variable, and then substitute that expression into the other equation.

Example 2.3.1 Solve the system

$$\begin{aligned} x + y &= 10 \\ 2x - y &= 2 \end{aligned}$$

Solution. If we solve the first equation for y, we get $y = 10 - x$. We can substitute the expression $\mathbf{10 - x}$ for y into the second equation to get

$$2x - (\mathbf{10 - x}) = 2$$

an equation in only one variable. Finally, we solve this equation:

$$\begin{aligned} 2x - 10 + x &= 2 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

And because $y = 10 - x$, we find $y = 10 - 4 = 6$. The solution is $x = 4, y = 6$. We may sometimes write the solution as an ordered pair, $(4, 6)$. □

Here is a summary of the steps for solving a system by substitution.

To Solve a System by Substitution.

1. Choose one of the variables in one of the equations. (It is best to choose a variable whose coefficient is 1 or −1.) Solve the equation for that variable.
2. Substitute the result of Step 1 into the other equation. This gives an equation in one variable.
3. Solve the equation obtained in Step 2. This gives the solution value for one of the variables.
4. Substitute this value into the result of Step 1 to find the solution value of the other variable.

Checkpoint 2.3.2 QuickCheck 1. What is the first step in the substitution method?

a. Solve both equations for y.

b. Substitute a value for y.

c. Solve one of the equations for one of the variables.

d. Add the equations together.

Checkpoint 2.3.3 Practice 1. Follow the suggested steps to solve the system by substitution:

$$\begin{aligned} 3y - 2x &= 3 \\ x - 2y &= -2 \end{aligned}$$

- Step 1: Solve the second equation for x in terms of y.

- Step 2: Substitute your expression for x into the first equation.
- Step 3: Solve the equation you got in Step 2.
- Step 4: You now have the solution value for y. Substitute that value into your result from Step 1 to find the solution value for x.

  As always, you should check that your solution values satisfy both equations in the system.

2.3.2 Elimination Method

A second algebraic method for solving systems is called **elimination**. As with the substitution method, we obtain an equation in a single variable, but we do it by eliminating one of the variables in the system. We must first put both equations into the general linear form $Ax + By = C$.

Example 2.3.4 Solve the system

$$\begin{aligned} 5x &= 2y + 21 \\ 2y &= 19 - 3x \end{aligned}$$

Solution. First, we rewrite each equation in the form $Ax + By = C$.

$$\begin{aligned} 5x &= 2y + 21 \\ \underline{\quad - 2y} &= \underline{-2y\quad} \\ 5x - 2y &= 21 \end{aligned} \qquad \text{Subtract } \mathbf{2y} \text{ from both sides.}$$

$$\begin{aligned} 2y &= 19 - 3x \\ \underline{\quad +3x} &= \underline{+3x\quad} \\ 3x + 2y &= 19 \end{aligned} \qquad \text{Add } \mathbf{3x} \text{ to both sides.}$$

We add the equations together by adding the left side of the first equation to the left side of the second equation, and then adding the two right sides together, as follows:

$$\begin{aligned} 5x - 2y &= 21 \\ \underline{3x + 2y} &= \underline{19} \\ 8x \qquad &= 40 \end{aligned}$$

Note that the y-terms canceled, or were **eliminated**. Solving the new equation, $8x = 40$, we find that $x = 5$. We are not finished yet, because we must still find the value of y. We can substitute our value for x into either of the original equations, and solve for y. We'll use the second equation, $3x + 2y = 19$:

$$\begin{aligned} 3(\mathbf{5}) + 2y &= 19 && \text{Subtract 15 from both sides.} \\ 2y &= 4 && \text{Divide by 2.} \\ y &= 2 \end{aligned}$$

Thus, the solution is the point $(5, 2)$. □

Checkpoint 2.3.5 QuickCheck 2. In order to use the elimination method, in what form should you write the equations?

a. Point-slope form

b. Slope-intercept form

c. General linear form

d. Coordinate form

In the previous Example, the elimination method worked because the coefficients of y in the two equations were opposites, 2 and -2. This caused the y-terms to cancel out when we added the two equations together. What if the coefficients of neither x nor y are opposites? Then we must multiply one or both of the equations in the system by a suitable constant.

Consider the system

$$\begin{aligned} 4x + 3y &= 7 \\ 3x + y &= -1 \end{aligned}$$

We can eliminate the y-terms if we multiply each term of the second equation by $\mathbf{-3}$, so that the coefficients of y will be opposites:

$$\mathbf{-3}(3x + y = -1) \longrightarrow -9x - 3y = 3$$

Be careful to multiply each term by -3, not just the y-term. We can now replace the second equation by its new version to obtain this system:

$$\begin{aligned} 4x + 3y &= 7 \\ -9x - 3y &= 3 \end{aligned}$$

Checkpoint 2.3.6 Practice 2. Follow the suggested steps to solve the system

$$\begin{aligned} 4x + 3y &= 7 \\ -9x - 3y &= 3 \end{aligned}$$

- Step 1: Add the equations together.
- Step 2: Solve the resulting equation for x.
- Step 3: Substitute your value for x into either original equation and solve for y.
- Step 4: Check that your solution satisfies both original equations.

Checkpoint 2.3.7 QuickCheck 3. Before adding the two equations together, we must arrange it so that the coefficients of one of the variables are __________.

When we add a multiple of one equation to the other we are making a **linear combination** of the equations. The method of elimination is also called the method of linear combinations. Sometimes we need to multiply both equations by suitable constants in order to eliminate one of the variables.

Example 2.3.8 Use linear combinations to solve the system

$$\begin{aligned} 5x - 2y &= 22 \\ 2x - 5y &= 13 \end{aligned}$$

Solution. This time we choose to eliminate the x-terms. We must arrange things so that the coefficients of the x-terms are opposites, so we look for the smallest integer that both 2 and 5 divide into evenly. (This number is called the **lowest common multiple**, or **LCM**, of 2 and 5.) The LCM of 2 and 5 is 10. We want one of the coefficients of x to be 10, and the other to be -10.

To achieve this, we multiply the first equation by 2 and the second equation by -5.

$$\begin{aligned} \mathbf{2}(5x - 2y = 22) &\quad\rightarrow\quad 10x - 4y = 44 \\ \mathbf{-5}(2x - 5y = 13) &\quad\rightarrow\quad -10x + 25y = -65 \end{aligned}$$

Adding these new equations eliminates the x-term and yields an equation in y.

$$\begin{aligned} 10x - 4y &= 44 \\ \underline{-10x + 25y} &= \underline{-65} \\ 21y &= -21 \end{aligned}$$

We solve for y to find $y = -1$. Finally, we substitute $y = \mathbf{-1}$ into the first equation and solve for x.

$$\begin{aligned} 5x - 2(\mathbf{-1}) &= 22 \\ 5x &= 20 \\ x &= 4 \end{aligned}$$

The solution of the system is the point $(4, -1)$. □

Checkpoint 2.3.9 QuickCheck 4. What is the name of the smallest integer that two integers, a and b, divide into evenly?

a. The divisor

b. The quotient

c. The lowest common multiple

d. The lowest common factor

Checkpoint 2.3.10 Practice 3. Follow the suggested steps to solve the system by elimination

$$\begin{aligned} \frac{3}{2}x &= y + \frac{13}{2} \\ y - 5 &= \frac{-7}{3}x \end{aligned}$$

First, clear the fractions: multiply the first equation by 2 and the second equation by 3.

- Step 1: Write each equation in the form $Ax + By = C$.
- Step 2: Eliminate the y-terms: Multiply each equation by an appropriate constant.
- Step 3: Add the new equations and solve the result for x.
- Step 4: Substitute your value for x into the second equation and solve for y.

Note 2.3.11 Which method should you choose to solve a particular system? Both methods work on any linear system, but the substitution method is easier if one of the variables in one of the equations has coefficient 1 or -1.

2.3.3 Problem Solving with Systems

If there are two unknown quantities in an applied problem, it is often more easily solved by writing a system of two equations.

Example 2.3.12 Staci stocks two kinds of sleeping bags in her sporting goods store, a standard model and a down-filled model for colder temperatures. From past experience, she estimates that she will sell twice as many of the standard variety as of the down-filled. She has room to stock 84 sleeping bags at a time. How many of each variety should Staci order?

Solution. We begin by choosing variables for the unknown quantities.

Number of standard sleeping bags: x
Number of down-filled sleeping bags: y

We write two equations about the variables.

Staci needs twice as many standard model as down-filled:	$x = 2y$
The total number of sleeping bags is 84:	$x + y = 84$

We will solve this system using substitution. Notice that the first equation is already solved for x in terms of y. We substitute $\boldsymbol{2y}$ for x in the second equation to obtain

$$\begin{aligned} \boldsymbol{2y} + y &= 84 \\ 3y &= 84 \end{aligned}$$

Solving for y, we find $y = \mathbf{28}$. Finally, substitute this value into the first equation to find

$$x = 2(\mathbf{28}) = 56$$

The solution to the system is $x = 56$, $y = 28$. Staci should order 56 standard sleeping bags and 28 down-filled bags. □

In the next Exercise, we'll use the formula $I = Pr$ for calculating the interest, I, earned on an investment. [TK]

[TK] To review the interest formula, see Section 2.3.1 of the Toolkit.

Checkpoint 2.3.13 Practice 4. Jerry invested \$2000, part in a CD at 4% interest and the remainder in a business venture at 9%. After one year, his income from the business venture was \$37 more than his income from the CD. How much did he invest at each rate?

a. Choose variables for the unknown quantities.

   Fill in the table.

	Principal	Interest Rate	Interest
First investment			
Second investment			

b. Write two equations; one about the principal, and one about the interest.

   Principal:

   Interest:

c. Solve the system. (Which method seems easiest, substitution or elimination?)

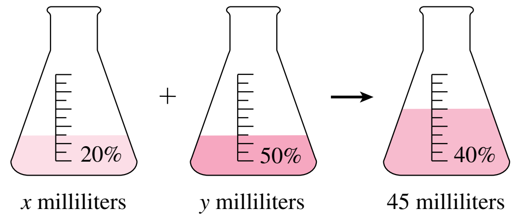
d. Write your answer to the question in the problem in a sentence.

A useful application of linear systems involves mixtures. In the next Example, we use the percent formula $P = rW$. [TK] For instance, suppose that 35% of a class of 60 students are math majors. Then the number of math majors is

$$\begin{aligned} \text{Part} &= \text{Percent rate} \times \text{Whole} \\ \text{Number of math majors} &= 0.35 \times 60 = 21 \end{aligned}$$

[TK] To review the percent formula, see Section 2.3.2 of the Toolkit.

Example 2.3.14 A chemist wants to produce 45 milliliters of a 40% solution of carbolic acid by mixing a 20% solution with a 50% solution. How many milliliters of each should she use?



Solution. We let x represent the number of milliliters of the 20% solution she needs and y the number of milliliters of the 50% solution. We can use a table to organize the information. The first two columns contain the variables and information given in the problem: the number of milliliters of each solution and its strength as a percent.

	Number of Milliliters (W)	Percent Acid (r)	Amount of Acid (P)
20% Solution	x	0.20	
50% Solution	y	0.50	
Mixture	45	0.40	

We fill in the last column of the table by using the formula $P = rW$. The entries in this last column give the amount of the important ingredient (in this case, milliliters of acid) in each component solution and in the mixture.

	Number of Milliliters (W)	Percent Acid (r)	Amount of Acid (P)
20% Solution	x	0.20	$\mathbf{0.20x}$
50% Solution	y	0.50	$\mathbf{0.50y}$
Mixture	45	0.40	$\mathbf{0.40(45)}$

Now we can write two equations about the mixture problem. The first equation is about the total number of milliliters mixed together. The chemist must mix x milliliters of one solution with y milliliters of the other solution and end up with 45 milliliters of the mixture, so

$$\text{Total amount of mixture:} \qquad x + y = 45$$

The second equation uses the fact that the acid in the mixture can only come from the acid in each of the two original solutions. We used the last column of the table to calculate how much acid was in each component, and we add these quantities to get the amount of acid in the mixture.

$$\text{Amount of acid:} \qquad 0.20x + 0.50y = 0.40(45)$$

These two equation make up a system:

$$\begin{aligned} x + y &= 45 \\ 0.20x + 0.50y &= 18 \end{aligned}$$

To simplify the system we first multiply the second equation by 100 to clear the decimals.

$$x + y = 45$$

$$20x + 50y = 1800$$

We'll solve this system by elimination. Multiply the first equation by -20, and add the equations together.

$$\begin{aligned} -20x - 20y &= -900 \\ \underline{20x + 50y} &\underline{= 1800} \\ 30y &= 900 \end{aligned}$$

Solving for y, we find $y = 30$. We substitute $y = \mathbf{30}$ into the first equation to find

$$x + \mathbf{30} = 45$$

or $x = 15$. The chemist needs 15 milliliters of the 20% solution and 30 milliliters of the 50% solution for the mixture. □

Checkpoint 2.3.15 Practice 5. Polls conducted by Senator Bluff's campaign manager show that he can win 60% of the rural vote in his state but only 45% of the urban vote. If 1,200,000 citizens vote, how many voters from rural areas and how many from urban areas must vote in order for the Senator to win 50% of the votes?

a. Let x represent the number of rural voters and y the number of urban voters. Fill in the table.

	Number of Voters (W)	Percent for Bluff (r)	Number for Bluff (P)
Rural			
Urban			
Total			

b. Add down the first and third columns to write a system of equations.

Number of voters:

Number for Bluff:

c. Solve the system.

d. Answer the question in the problem.

2.3.4 Inconsistent and Dependent Systems

Recall that a system of two linear equations does not always have a unique solution; it may be inconsistent or dependent. The elimination method will reveal whether the system falls into one of these two cases.

Example 2.3.16 Solve each system by elimination.

a.

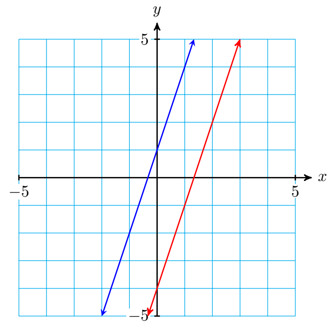
$$\begin{aligned} 3x - y &= 4 \\ -6x + 2y &= 2 \end{aligned}$$

b.

$$\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 6 \end{aligned}$$

Solution.

a.

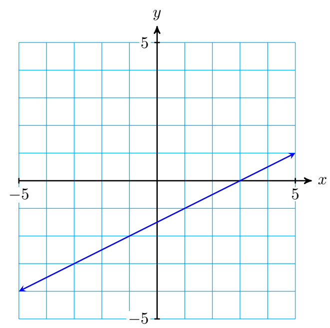


To eliminate the y-terms, we multiply the first equation by 2 and add:

$$\begin{array}{rcr} 6x - 2y &=& 8 \\ \underline{-6x + 2y} &=& \underline{2} \\ 0x + 0y &=& 10 \end{array}$$

Both variables are eliminated, and we are left with the false statement $0 = 10$. There are no values of x or y that will make this equation true, so the system has no solutions. The graph shows that the system is inconsistent.

b.



To eliminate the x-terms, we multiply the first equation by -2 and add:

$$\begin{array}{rcr} -2x + 4y &=& -6 \\ \underline{2x - 4y} &=& \underline{6} \\ 0x + 0y &=& 0 \end{array}$$

We are left with the true but unhelpful equation $0 = 0$. The two equations are in fact equivalent (one is a constant multiple of the other), so the system is dependent. The graph of both equations is shown in the figure.

□

Inconsistent and Dependent Systems.

When using elimination to solve a system:

1. If combining the two equations results in an equation of the form

$$0x + 0y = k \qquad\qquad (k \neq 0)$$

then the system is inconsistent.

2. If combining the two equations results in an equation of the form

$$0x + 0y = 0$$

then the system is dependent.

Checkpoint 2.3.17 Practice 6. Identify the system as dependent, inconsistent, or consistent and independent.

$$\begin{aligned} x + 3y &= 6 \\ 2x - 12 &= -6y \end{aligned}$$

2.3.5 Problem Set 2.3

Warm Up

1. Solve the system by **substitution**:

$$\begin{aligned} 2x &= y + 7 \\ 2y &= 14 - 3x \end{aligned}$$

- Step 1

  Choose one of the equations, and solve for one of the variables.

- Step 2

  Substitute this new expression for x or y into the other equation.

- Step 3

  Notice that the new equation has only one variable. Solve the equation.

- Step 4

  Substitute the answer to Step 3 into the result of Step 1 to find the other variable. Write your solution as an ordered pair.

2. Solve the system by **elimination**:

$$\begin{aligned} 3x + 5y &= 1 \\ 2x - 3y &= 7 \end{aligned}$$

- Step 1

  We choose to eliminate x. Multiply the first equation by 2 and the second equation by -3. (Why?)

  $$\begin{aligned} \mathbf{2}[3x + 5y = 1] &\rightarrow \\ \mathbf{-3}[2x - 3y = 7] &\rightarrow \end{aligned}$$

- Step 2

  The coefficients of x are now opposites. Add the two equations together.

- Step 3

  Solve the new equation for the remaining variable.

- Step 4

  Use substitution to find the other variable. Write your solution as an ordered pair.

Skills Practice

Exercise Group. For Problems 3–6, solve the system by substitution or by linear combinations.

3.

$$\begin{aligned} 3m + n &= 7 \\ 2m &= 5n - 1 \end{aligned}$$

4.

$$\begin{aligned} 2r &= s + 7 \\ 2s &= 14 - 3r \end{aligned}$$

5.

$$\begin{aligned} 2u - 3v &= -4 \\ 5u + 2v &= 9 \end{aligned}$$

6.

$$\begin{aligned} 3x + 5y &= 1 \\ 2x - 3y &= 7 \end{aligned}$$

Exercise Group. For Problems 7 and 8, clear the fractions in each equation first, then solve the system by substitution or by linear combinations.

7.

$$\begin{aligned} \frac{x}{4} &= \frac{y}{3} - \frac{5}{12} \\ \frac{y}{5} &= \frac{1}{2} - \frac{x}{10} \end{aligned}$$

8.

$$\begin{aligned} \frac{t}{3} &= \frac{w}{3} + 2 \\ \frac{w}{3} &= \frac{t}{6} - 1 \end{aligned}$$

Exercise Group. For Problems 9 and 10, use linear combinations to identify each system as dependent, inconsistent, or consistent and independent.

9.

$$\begin{aligned} 2n - 5p &= 6 \\ \frac{15p}{2} + 9 &= 3n \end{aligned}$$

10.

$$\begin{aligned} -3x &= 4y + 8 \\ \frac{1}{2}x + \frac{4}{3} &= \frac{-2}{3}y \end{aligned}$$

Applications

11. Benham will service $2.5x$ copy machines per week if he can charge x dollars per machine, and the public will pay for $350 - 4.5x$ jobs at x dollars per machine.

a. Write the supply and demand equations for servicing copy machines.

b. Find the equilibrium price and the number of copy machines Benham will service at that price.

12. Delbert answered 13 true-false and 9 fill-in questions correctly on his last test and got a score of 71. If he had answered 9 true-false and 13 fill-ins correctly, he would have made an 83. How many points was each type of problem worth?

13. Paul needs 40 pounds of 48% silver alloy to finish a collection of jewelry. How many pounds of 45% silver alloy should he melt with 60% silver alloy to obtain the alloy he needs?

a. Choose variables for the unknown quantities, and fill in the table.

	Pounds	% silver	Amount of silver
First alloy			
Second alloy			
Mixture			

b. Write one equation about the amount of alloy Paul needs.

c. Write a second equation about the amount of silver in the alloys.

d. Solve the system and answer the question in the problem.

14. Amal plans to make 10 liters of a 17% acid solution by mixing a 20% acid solution with a 15% acid solution. How much of each should she use?

a. Choose variables for the unknown quantities, and fill in the table.

	Liters	% acid	Amount of acid
First solution			
Second solution			
Mixture			

b. Write one equation about the amount of solution Amal needs.

c. Write a second equation about the acid in the solution.

d. Solve the system and answer the question in the problem.

15. Rani kayaks downstream for 45 minutes and travels a distance of 6000 meters. On the return journey upstream, she covers only 4800 meters in 45 minutes. How fast is the current in the river, and how fast would Rani kayak in still water? (Give your answers in meters per minute.)

a. Choose variables for the unknown quantities, and fill in the table. [TK]

	Rate	Time	Distance
Downstream			
Upstream			

b. Write one equation about the downstream trip.

c. Write a second equation about the return trip.

d. Solve the system and answer the question in the problem.

16. Because of prevailing winds, a flight from Detroit to Denver, a distance of 1120 miles, took 4 hours on Econoflite, and the return trip took 3.5 hours. What were the speed of the airplane and the speed of the wind?

a. Choose variables for the unknown quantities, and fill in the table.

	Rate	Time	Distance
Detroit to Denver			
Denver to Detroit			

b. Write one equation about the trip from Detroit to Ddenver.

c. Write a second equation about the return trip.

d. Solve the system and answer the question in the problem.

17. Geologists can calculate the distance from their seismograph to the epicenter of an earthquake by timing the arrival of the P and S waves. They know that, for this earthquake, P waves travel at about 5.4 miles per second and S waves travel at 3.0 miles per second. If the P waves arrived 3 minutes before the S waves, how far away is the epicenter of the quake?

18. A cup of rolled oats provides 310 calories. A cup of rolled wheat flakes provides 290 calories. A new breakfast cereal combines wheat and oats to provide 302 calories per cup. How much of each grain does 1 cup of the cereal include?

a. Choose variables for the unknown quantities, and fill in the table.

	Cups	Calories per cup	Calories
Rolled Oats			
Wheat Flakes			
Mixture			

b. Write one equation about the amounts of each grain.

c. Write a second equation about the number of calories.

d. Solve the system and answer the question in the problem.

19. Acme Motor Company is opening a new plant to produce chassis for two of its models, a sports coupe and a wagon. Each sports coupe requires a riveter for 3 hours and a welder for 4 hours; each wagon requires a riveter for 4 hours and a welder for 5 hours. The plant has available 120 hours of riveting and 155 hours of welding per day. How many of each model of chassis can it produce in a day?

a. Choose variables for the unknown quantities, and fill in the table.

	Sports coupes	Wagons	Total
Hours of riveting			
Hours of welding			

b. Write one equation about the hours of riveting.

c. Write a second equation about the hours of welding.

d. Solve the system and answer the question in the problem.

20. Carmella has \$1200 invested in two stocks; one returns 8% per year and the other returns 12% per year. The income from the 8% stock is \$3 more than the income from the 12% stock. How much did she invest in each stock?

a. Choose variables for the unknown quantities, and fill in the table.

	Principal	Interest Rate	Interest
First stock			
Second stock			

b. Write one equation about the amounts Carmella invested.

c. Write a second equation about Carmella's annual interest.

d. Solve the system and answer the question in the problem.

2.4 Gaussian Reduction

Some problems involve many variables and equations. For example, scheduling airline flights or planning delivery routes can involve hundreds of variables and linear equations. There are efficient computer algorithms to solve such large systems. In this section we look at a method that can be generalized to many variables, but we will apply it to 3x3 linear systems.

2.4.1 3x3 Linear Systems

A **solution** to an equation in three variables, such as

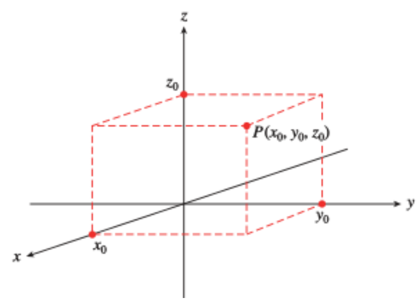
$$x + 2y - 3z = -4$$

is an **ordered triple** of numbers that satisfies the equation. For example, $(0, -2, 0)$ and $(-1, 0, 1)$ are solutions to the equation above, but $(1, 1, 1)$ is not. You can verify this by substituting the coordinates into the equation to see if a true statement results.

$$\begin{array}{llll} \text{For } (0,-2,0): & \mathbf{0} + 2(\mathbf{-2}) - 3(\mathbf{0}) = -4 & & \text{True} \\ \text{For } (-1,0,1): & \mathbf{-1} + 2(\mathbf{0}) - 3(\mathbf{1}) = -4 & & \text{True} \\ \text{For } (1,1,1): & \mathbf{1} + 2(\mathbf{1}) - 3(\mathbf{1}) = -4 & & \text{Not true} \end{array}$$

A linear equation in three variables has infinitely many solutions.

An ordered triple (x, y, z) can be represented geometrically as a point in space using a three-dimensional Cartesian coordinate system, as shown in the figure. The graph of a linear equation in three variables is a plane. A solution to a system of three linear equations in three variables is an ordered triple that satisfies each equation in the system. That triple represents a point where all three planes intersect.



- If the planes intersect in a single point, the system has a unique solution.
- If there is no point that lies on all three planes (for instance, if at least two of the planes are parallel), the system is inconsistent.
- If all three planes are the same, or if they intersect in a line, the system is dependent.

It is impractical to solve 3x3 systems by graphing. Even when technology for producing three-dimensional graphs is available, we cannot read coordinates on such graphs with any confidence. Thus, we will restrict our attention to algebraic methods of solving such systems.

Checkpoint 2.4.1 QuickCheck 1.

a. A solution to an equation in three variables is an __________.

b. The graph of a linear equation in three variables is a __________.

c. A linear 3x3 system has a unique solution if the three planes intersect in __________.

d. Three planes may also intersect in a __________ or a __________.

2.4.2 Back-Substitution

The strategy for solving a 3x3 system is the same as the strategy for 2x2 systems: we would like to reduce the system to an equation in a single variable. Once we have found the value for that variable, we substitute its value into the other equations to find the remaining unknowns.

A special case of this technique is called **back-substitution**. It works when one of the equations involves exactly one variable, and a second equation involves that same variable and just one other variable. A 3x3 system with these properties is said to be in **triangular form**.

Example 2.4.2 Solve the system

$$\begin{aligned} x + 2y + 3z &= 2 \\ -2y - 4z &= -2 \\ 3z &= -3 \end{aligned}$$

Solution. We begin by solving the third equation to find $z = -1$. Then we substitute $\mathbf{-1}$ for z in the second equation and solve for y.

$$\begin{aligned} -2y - 4(\mathbf{-1}) &= -2 \\ -2y + 4 &= -2 \\ -2y &= -6 \\ y &= 3 \end{aligned}$$

Finally, we substitute $\mathbf{-1}$ for z and $\mathbf{3}$ for y into the first equation to find x.

$$\begin{aligned} x + 2(\mathbf{3}) + 3(\mathbf{-1}) &= 2 \\ x + 6 - 3 &= 2 \\ x &= -1 \end{aligned}$$

The solution is the ordered triple $(-1, 3, -1)$. You should verify that this triple satisfies all three equations of the system. □

Checkpoint 2.4.3 Practice 1. Use back-substitution to solve the system

$$\begin{aligned} 2x + 2y + z &= 10 \\ y - 4z &= 9 \\ 3z &= -6 \end{aligned}$$

2.4.3 Gaussian Reduction

The method for solving a general 3×3 linear system is called **Gaussian reduction**, after the German mathematician Carl Gauss. We use linear combinations to reduce the system to triangular form, and then use back-substitution to find the solutions.

Checkpoint 2.4.4 QuickCheck 2.

a. The method for solving a 3x3 linear system is called __________.

b. A special case of this method is called __________.

c. The special case works on systems in __________.

d. To reduce a system to the special form, we use __________.

To obtain the triangular form, we eliminate one of the variables from each of the three equations by considering them in pairs. This results in a 2×2 system that we can solve using elimination. [TK]

[TK] To review the method of elimination, see Section 2.3 of the text.

Example 2.4.5 Solve the system:

$$\begin{aligned} x + 2y - 3z &= -4 && (1)\\ 2x - \ y + \ z &= \ 3 && (2)\\ 3x + 2y + \ z &= 10 && (3) \end{aligned}$$

Solution. We can choose any one of the three variables to eliminate first. For this example, we will eliminate x. Next, we choose two of the equations, say (1) and (2), and use a linear combination: We multiply Equation (1) by -2 and add the result to Equation (2) to produce Equation (4).

$$\begin{array}{rcrl} -2x - 4y + 6z &=& 8 & \quad (1a)\\ \underline{2x - \ y + \ z} &=& \underline{3} & \quad (2)\\ -5y + 7z &=& 11 & \quad (4) \end{array}$$

Now we have an equation involving only two variables. But we need *two* equations in two unknowns to find the solution. So we choose a different pair of equations, say (1) and (3), and eliminate x again. We multiply Equation (1) by -3 and add the result to Equation (3) to obtain Equation (5).

$$\begin{array}{rcrl} -3x - 6y + 9z &=& 12 & \quad (1b)\\ \underline{3x + 2y + \ z} &=& \underline{10} & \quad (3)\\ -4y + 10z &=& 22 & \quad (5) \end{array}$$

We now form a 2×2 system with our new Equations (4) and (5).

$$\begin{aligned} -5y + 7z &= 11 && (4)\\ -4y + 10z &= 22 && (5) \end{aligned}$$

Finally, we eliminate one of the remaining variables to obtain an equation in a single variable. We choose to eliminate y, so we add 4 times Equation (4) to -5 times Equation (5) to obtain Equation (6).

$$\begin{array}{rcrl} -20y + 28z &=& 44 & \quad (4a)\\ \underline{20y - 50z} &=& \underline{-110} & \quad (5a)\\ -22z &=& -66 & \quad (6) \end{array}$$

Now we are ready to form a triangular system. We choose one of the original equations (in three variables), one of the equations from our 2×2 system, and our final equation in one variable. We choose Equations (1), (4), and (6).

$$\begin{aligned} x + 2y - 3z &= -4 && (1)\\ -5y + 7z &= 11 && (4)\\ -22z &= -66 && (6) \end{aligned}$$

This new system has the same solutions as the original system, and we can solveit by back-substitution. We first solve Equation (6) to find $z = 3$. Substituting 3 for z in Equation (4), we find

$$-5y + 7(\mathbf{3}) = 11$$

$$\begin{aligned} -5y + 21 &= 11 \\ -5y &= -10 \end{aligned}$$

So $y = 2$. Finally, we substitute **3** for z and **2** for y into Equation (1) to find

$$\begin{aligned} x + 2(\mathbf{2}) - 3(\mathbf{3}) &= -4 \\ x + 4 - 9 &= -4 \\ x &= 1 \end{aligned}$$

The solution to the system is the ordered triple $(1, 2, 3)$. You should verify that this triple satisfies all three of the original equations. ☐

We summarize the method for solving a 3x3 linear system as follows.

Steps for Solving a 3x3 Linear System.

1. Clear each equation of fractions and put it in standard form.
2. Choose two of the equations and eliminate one of the variables by forming a linear combination.
3. Choose a different pair of equations and eliminate the *same* variable.
4. Form a 2×2 system with the equations found in steps (2) and (3). Eliminate one of the variables from this 2×2 system by using a linear combination.
5. Form a triangular system by choosing among the previous equations. Use back-substitution to solve the triangular system.

Checkpoint 2.4.6 QuickCheck 3. Suppose you eliminate y from two equations as Step 2 of Gaussian reduction. What should you do for Step 3?

a. Form a triangular system.

b. Eliminate either x or z.

c. Eliminate y again from a different pair of equations.

d. Substitute the value for $y$into the equations.

Checkpoint 2.4.7 Practice 2. Use Gaussian reduction to solve the system

$$\begin{aligned} x - 2y + z &= -1 && (1) \\ \frac{2}{3}x + \frac{1}{3}y - z &= -1 && (2) \\ 3x + 3y - 2z &= 10 && (3) \end{aligned}$$

Follow the steps:

- Step 1: Clear the fractions from Equation (2). [TK]
- Step 2: Eliminate z from Equations (1) and (2).
- Step 3: Eliminate z from Equations (1) and (3).
- Step 4: Eliminate x from your new 2×2 system.
- Step 5: Form a triangular system and solve by back-substitution.

[TK] For help on clearing fractions, see Section 2.4.2 of the Toolkit.

2.4.4 Inconsistent and Dependent Systems

If, at any step in forming linear combinations, we obtain an equation of the form

$$0x + 0y + 0z = k, \qquad (k \neq 0)$$

then the system is inconsistent and has no solution. If we don't obtain such an equation, but we do obtain one of the form

$$0x + 0y + 0z = 0$$

then the system is dependent and has infinitely many solutions. [TK]

[TK] To review the inconsistent and dependent systems, see Section 2.3 of the text.

Example 2.4.8 Solve the systems.

a. $$\begin{aligned} 3x + y - 2z &= 1 && (1)\\ 6x + 2y - 4z &= 5 && (2)\\ 2x - y + 3z &= -1 && (3) \end{aligned}$$

b. $$\begin{aligned} -x + 3y - z &= -2 && (1)\\ 2x + y - 4z &= -1 && (2)\\ 2x - 6y + 2z &= 4 && (3) \end{aligned}$$

Solution.

a. To eliminate y from Equations (1) and (2), we multiply Equation (1) by -2 and add the result to Equation (2).

$$\begin{array}{r} -6x - 2y + 4z = -2\\ \underline{6x + 2y - 4z} = \underline{5}\\ 0x + 0y + 0z = 3 \end{array}$$

Because the resulting equation has no solution, the system is *inconsistent*.

b. To eliminate x from Equations (1) and (3), we multiply Equation (1) by 2 and add Equation (3).

$$\begin{array}{r} -2x + 6y - 2z = -4\\ \underline{2x - 6y + 2z} = \underline{4}\\ 0x + 0y + 0z = 0 \end{array}$$

Because the resulting equation vanishes, the system is dependent and has infinitely many solutions.

□

Checkpoint 2.4.9 Practice 3. Decide whether the system is inconsistent, dependent, or consistent and independent.

$$\begin{aligned} x + 3y - z &= 4\\ -2x - 6y + 2z &= 4\\ x + 2y - z &= 3 \end{aligned}$$

Checkpoint 2.4.10 QuickCheck 4. Decide whether each statement is true or false.

a. If a system is dependent, it has no solutions.

b. The equation $0x + 0y + 0z = 0$ has no solution.

c. If a 3x3 system is dependent, all three equations are the same.

d. Gaussian reduction will reveal whether a system is dependent or inconsistent.

2.4.5 Applications

Here are some problems that can be modeled by a system of three linear equations.

Example 2.4.11 One angle of a triangle measures 4° less than twice the second angle, and the third angle is 20° greater than the sum of the first two. Find the measure of each angle.

Solution.

- *Step 1:*.

  We represent the measure of each angle by a separate variable.

First angle:	x
Second angle:	y
Third angle:	z

- *Step 2:*.

  We write the conditions stated in the problem as three equations.

x is 4° less than 2 times y :	$x = 2y - 4$
z is 20° more than $x + y$:	$z = x + y + 20$
the sum of the angles of a triangle is 180° :	$x + y + z = 180$

- *Step 3:*.

  We follow the steps for solving a 3×3 linear system.

  1. We write the three equations in standard form. [TK]

  $$\begin{aligned} x - 2y \phantom{{}-z} &= -4 && (1)\\ x + y - z &= -20 && (2)\\ x + y + z &= 180 && (3) \end{aligned}$$

  2–3. Because Equation (1) has no z-term, it will be most efficient to eliminate z from Equations (2) and (3). We add these two equations.

  $$\begin{array}{rcrl} x + y - z &=& -20 & (2)\\ \underline{x + y + z} &=& \underline{180} & (3)\\ 2x + 2y &=& 160 & (4) \end{array}$$

  4. We form a 2×2 system from Equations (1) and (4). We add the two equations to eliminate the variable y, yielding

  $$x - 2y = -4 \qquad (1)$$

$$\begin{aligned} \underline{2x + 2y} &= \underline{160} && (4) \\ 3x \qquad &= 156 && (5) \end{aligned}$$

5. We form a triangular system using Equations (2), (1), and (5). We use back-substitution to complete the solution.

$$\begin{aligned} x + \ \ y + z &= 180 && (2) \\ x - 2y \qquad &= -4 && (1) \\ 3x \qquad\qquad &= 156 && (5) \end{aligned}$$

We divide both sides of Equation (5) by 3 to find $x = 52$. We substitute 52 for x in Equation (1) and solve for y to find

$$\begin{aligned} \mathbf{52} - 2y &= -4 \\ y &= 28 \end{aligned}$$

We substitute **52** for x and **28** for y in Equation (3) to find

$$\begin{aligned} \mathbf{52} + \mathbf{28} + z &= 180 \\ z &= 100 \end{aligned}$$

- *Step 4:*.

  The angles measure 52°, 28°, and 100°.

□

[TK] For help writing an equation in standard form, see Section 2.4.1 of the Toolkit.

Checkpoint 2.4.12 Practice 4. A farmer has 1300 acres on which to plant wheat, corn, and soybeans. The seed costs \$6 for an acre of wheat, \$4 for an acre of corn, and \$5 for an acre of soybeans. An acre of wheat requires 5 acre-feet of water during the growing season, while an acre of corn requires 2 acre-feet and an acre of soybeans requires 3 acre-feet. If the farmer has \$6150 to spend on seed and can count on 3800 acre-feet of water, how many acres of each crop should he plant in order to use all his resources? [TK]

[TK] For help writing equations, see Section 2.4.3 of the Toolkit.

2.4.6 Problem Set 2.4

Warm Up

Exercise Group. For Problems 1 and 2, solve the system by elimination.

1. $$\begin{aligned} 4x - 3 &= 3y \\ 25 + 5x &= -2y \end{aligned}$$

2. $$\begin{aligned} \frac{2x}{3} + \frac{8y}{9} &= \frac{4}{3} \\ \frac{x}{2} + 2 &= \frac{y}{3} \end{aligned}$$

3. Karen has \$2000, part of it invested in bonds paying 10%, and the rest in a certificate account at 8%. Her annual income from the two investments is \$184. How much did Karen invest at each rate?

a. Choose variables for the unknown quantities, and fill in the table.

	Principal	Interest rate	Interest
Bonds			
Certificate			
Total		——	

b. Write one equation about the amount Karen invested.

c. Write a second equation about Kaaren's annual interest.

4. The pharmacist at Glenoaks Hospital was asked to supply 10 liters of a 20% solution of iodine. She has on hand iodine solutions in 15% strength and 40% strength. How much of each should she mix to make the required solution?

a. Choose variables for the unknown quantities, and fill in the table.

	Liters	Strength	Amount of iodine
15% Solution			
40% solution			
20% solution			

b. Write one equation about the number of liters of solution.

c. Write a second equation about the amount of iodine.

Skills Practice

Exercise Group. For Problems 5 and 6, use back-substitution to solve the system.

5.
$$\begin{aligned} 2x + 3y - z &= -7 \\ y - 2z &= -6 \\ 5z &= 15 \end{aligned}$$

6.
$$\begin{aligned} 2x + z &= 5 \\ 3y + 2z &= 6 \\ 5x &= 20 \end{aligned}$$

Exercise Group. For Problems 7–12, use Gaussian reduction to solve the system.

7.
$$\begin{aligned} x + y + z &= 0 \\ 2x - 2y + z &= 8 \\ 3x + 2y + z &= 2 \end{aligned}$$

8.
$$\begin{aligned} x - 2y + 4z &= -3 \\ 3x + y - 2z &= 12 \\ 2x + y - z &= 11 \end{aligned}$$

9.
$$\begin{aligned} 3x - 4y + 2z &= 20 \\ 4x + 3y - 3z &= -4 \\ 2x - 5y + 5z &= 24 \end{aligned}$$

10.
$$\begin{aligned} 4x + z &= 3 \\ 2x - y &= 2 \\ 3y + 2z &= 0 \end{aligned}$$

11.
$$\begin{aligned} 4x + 6y + 3z &= -3 \\ 2x - 3y - 2z &= 5 \\ -6x + 6y + 2z &= -5 \end{aligned}$$

12.
$$\begin{aligned} x - \frac{1}{2}y - \frac{1}{2}z &= 4 \\ x - \frac{3}{2}y - \phantom{\frac{1}{2}}2z &= 3 \\ \frac{1}{4}x + \frac{1}{4}y - \frac{1}{4}z &= 0 \end{aligned}$$

Exercise Group. For Problems 13 and 14, when you decide which variable to eliminate first, take advantage of the fact that one of the variables is already missing from each equation.

13.
$$x = -y$$
$$x + z = \frac{5}{6}$$
$$y - 2z = -\frac{7}{6}$$

14.
$$x = y + \frac{1}{2}$$
$$y = z + \frac{5}{4}$$
$$2z = x - \frac{7}{4}$$

Exercise Group. For Problems 15 and 16, decide whether the system is inconsistent or dependent.

15.
$$3x - 2y + z = 6$$
$$2x + y - z = 2$$
$$4x + 2y - 2z = 3$$

16.
$$x = 2y - 7$$
$$y = 4z + 3$$
$$x - 8z = -1$$

Applications

Exercise Group. Use a system of equations to solve Problems 17-22.

17. The perimeter of a triangle is 155 inches. Side x is 20 inches shorter than side y, and side y is 5 inches longer than side z. Find the lengths of the sides of the triangle.

18. One angle of a triangle measures 10° more than a second angle, and the third angle is 10° more than six times the measure of the smallest angle. Find the measure of each angle.

19. The Java Shoppe sells a house brand of coffee that is only 2.25% caffeine for \$6.60 per pound. The house brand is a mixture of Colombian coffee that sells for \$6 per pound and is 2% caffeine, French roast that sells for \$7.60 per pound and is 4% caffeine, and Sumatran that sells for \$6.80 per pound and is 1% caffeine. How much of each variety is in a pound of house brand?

20. Vegetable Medley is made of carrots, green beans, and cauliflower. The package says that 1 cup of Vegetable Medley provides 29.4 milligrams of vitamin C and 47.4 milligrams of calcium. One cup of carrots contains 9 milligrams of vitamin C and 48 milligrams of calcium. One cup of green beans contains 15 milligrams of vitamin C and 63 milligrams of calcium. One cup of cauliflower contains 69 milligrams of vitamin C and 26 milligrams of calcium. How much of each vegetable is in 1 cup of Vegetable Medley?

21. Reliable Auto Company wants to ship 1700 Status Sedans to three major dealers in Los Angeles, Chicago, and Miami. From past experience Reliable figures that it will sell twice as many sedans in Los Angeles as in Chicago. It costs \$230 to ship a sedan to Los Angeles, \$70 to Chicago, and \$160 to Miami. If Reliable Auto has \$292,000 to pay for shipping costs, how many sedans should it ship to each city?

22. Office Depot makes three types of file cabinets: two-drawer, four-drawer, and horizontal. The building process is divided into three phases: assembly, painting, and finishing. A two-drawer cabinet requires 3 hours to assemble, 1 hour to paint, and 1 hour to finish. The four-drawer model takes 5 hours to assemble, 90 minutes to paint, and 2 hours to finish. The horizontal cabinet takes 4 hours to assemble, 1 hour to paint, and 3 hours to finish. Office Depot employs enough workers for 500 hours of assembly time, 150 hours of painting, and 230 hours of finishing each week. How many of each type of file cabinet should they make in order to use all the hours available?

2.5 Linear Inequalities in Two Variables

To conclude this chapter on applications of linear models, we look at some uses of linear inequalities and their graphs.

2.5.1 Graphs of Inequalities in Two Variables

Recall that a linear inequality can have many solutions. [TK] For example, the solutions of $x + 2 < 5$ are all numbers less than 3. So it should not be surprising that inequalities in two variables also have many solutions.

[TK] To review solving linear inequalities, see Section 2.5.1 of the Toolkit.

Definition 2.5.1 A **solution** to an inequality in two variables is an ordered pair of numbers that satisfies the inequality. ♢

Example 2.5.2 Find a solution to the inequality $x + y \geq 10,000$ for $x = 2000$

Solution. For $x = 2000$, we have

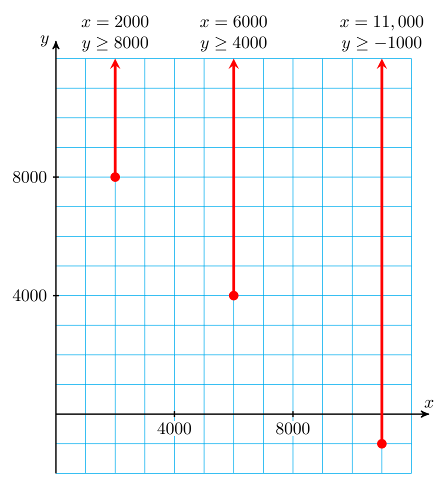
$$2000 + y \geq 10,000 \qquad \text{or} \qquad y \geq 8000$$

You can check that any value of y greater than or equal to 8000 provides a solution when $x = 2000$. For example, $(2000, 9000)$ is a solution because $2000 + 9000 > 10,000$. □

We can gain some insight into the nature of the solutions if we rewrite the inequality as

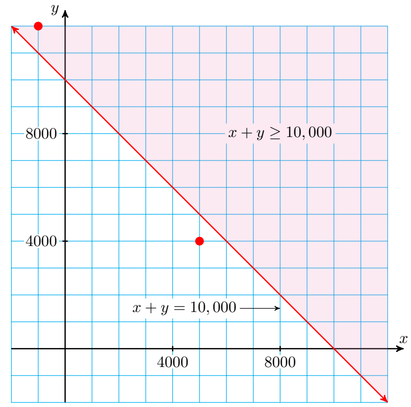
$$y \geq -x + 10,000$$

This inequality says that for each x-value, we must choose points with y-values greater than or equal to $-x + 10,000$. As we saw in Example 2.5.2 above, when $x = 2000$, the solutions have y-values greater than or equal to 8000. Solutions for $x = 2000$, $x = 6000$, and $x = 11,000$ are shown in the figure. Do you see a pattern emerging?



The graph of the inequality must show all the points whose coordinates are solutions. We found some solutions to the inequality $x+y \geq 10,000$ in Example 2.5.2 above by choosing a value for x and solving for y. A more efficient way to find all the solutions of the inequality is to start with the graph of the corresponding equation

$$y = -x + 10,000$$



The graph is a straight line with slope -1. Any point *above* this line has a y-coordinate greater than $-x + 10,000$ and hence satisfies the inequality. Thus, the graph of the inequality includes all the points on or above the line $y = -x + 10,000$, as shown by the shaded region in the figure. Of course, the shaded points are also solutions to the original inequality, $x + y \geq 10,000$.

For example, the point $(-1000, \ 12,000)$ is a solution, because

$$-1000 + 12,000 \geq 10,000$$

On the other hand, the point $(5000, 4000)$ does not lie in the shaded region because its coordinates do not satisfy the inequality.

So, just as the solutions of a linear inequality in one variable can be a portion of the x-axis, the solutions of a linear inequality in two variables can be a portion of the xy-plane.

Checkpoint 2.5.3 Practice 1.

a. Find one y-value that satisfies the inequality $y - 3x < 6$ for each of the x-values in the table.

x	1	0	-2
y			

b. Graph the line $y - 3x = 6$. Then plot your solutions from part (a) on the same grid.

2.5.2 Finding the Solutions

Let us organize our observations from the discussion above to develop a method for finding the solutions of a linear inequality.

- A **linear inequality** can be written in the form

  $$ax + by + c \leq 0 \qquad \text{or} \qquad ax + by + c \geq 0$$

  The solutions consist of the line $ax + by + c = 0$ and a **half-plane** on one side of that line. We shade the half-plane to show that all its points are included in the solution set.

- To decide which side of the line to shade, we can solve the inequality for y in terms of x. If we obtain

  $$y \geq mx + b \qquad (\text{ or } y > mx + b)$$

then we shade the half-plane *above* the line. If the inequality is equivalent to

$$y \leq mx + b \qquad (\text{ or } y < mx + b)$$

then we shade the half-plane *below* the line.

Example 2.5.4 Graph the solutions of $4x - 3y \geq 12$

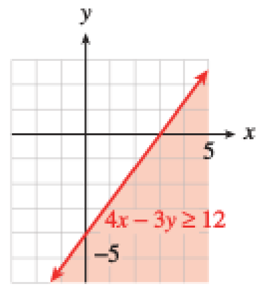
Solution. First we solve the inequality for y.

$$\begin{aligned} 4x - 3y &\geq 12 && \text{Subtract } \boldsymbol{4x} \text{ from both sides.} \\ -3y &\geq -4x + 12 && \text{Divide both sides by } \boldsymbol{-3.} \\ y &\leq \frac{4}{3}x - 4 \end{aligned}$$

Then we graph the corresponding line [TK]

$$y = \frac{4}{3}x - 4$$

with y-intercept is -4 and slope $m = \dfrac{4}{3}$. Finally, we shade the half-plane below the line. The completed graph is shown at right.



□

[TK] To review graphing lines, see Section 2.5.2 of the Toolkit.

Caution 2.5.5 Be careful when isolating y. Remember to reverse the direction of the inequality whenever we multiply or divide by a negative number. For example, the inequality

$$-2y > 6x - 8$$

is equivalent to

$$y < -3x + 4$$

An inequality that uses $>$ or $<$ instead of $\geq$ or $\leq$ is called **strict**. The graph of a strict inequality includes only the half-plane and not the line. In that case we use a dashed line for the graph of the equation $ax + by + c = 0$ to show that it is not part of the solution.

Checkpoint 2.5.6 Practice 2. Graph the solutions of the inequality $4x - 2y < -8$.

Checkpoint 2.5.7 QuickCheck 1. Complete each statement by filling in the blank.

a. The solutions of a linear inequality in two variables consist of a boundary line and a __________.

b. An inequality that uses $>$ or $<$ instead of $\geq$ or $\leq$ is called __________.

c. If we multiply or divide an inequality by a negative number we must __________ the direction of the inequality symbol.

d. The shaded region shows all the __________ of the inequality.

2.5.3 Using a Test Point

A second method for graphing inequalities does not require us to solve for y. Because all the solutions lie on one side of the boundary line, we only need to find one soution! Once we have graphed the boundary line, we can decide which half-plane to shade by using a **test point**. The test point can be any point that is not on the boundary line itself.

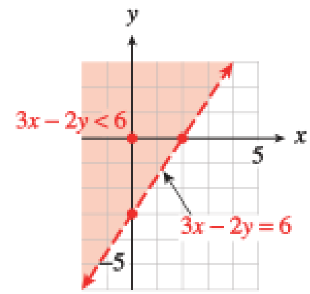
Example 2.5.8 Graph the solutions of the inequality $3x - 2y < 6$

Solution. First, we graph the line $3x - 2y = 6$, as shown below. We will use the intercept method. The intercepts are $(2, 0)$ and $(0, -3)$, so we sketch the boundary line through those points.

Next, we choose a test point. Because $(0, 0)$ does not lie on the line, we choose it as our test point. We substitute the coordinates of the test point into the inequality to obtain

$$3(\mathbf{0}) - 2(\mathbf{0}) < 6$$

Because this is a true statement, $(0, 0)$ is a solution of the inequality. Since all the solutions lie on the same side of the boundary line, we shade the half-plane that contains the test point. In this example, the boundary line is a dashed line because the original inequality was strict.



□

Here is a summary of our test point method for graphing inequalities.

To Graph an Inequality Using a Test Point.

1. Graph the corresponding equation to obtain the boundary line.
2. Choose a test point that does not lie on the boundary line.
3. Substitute the coordinates of the test point into the inequality.
   a. If the resulting statement is true, shade the half-plane that includes the test point.
   b. If the resulting statement is false, shade the half-plane that does not include the test point.
4. If the inequality is strict, make the boundary line a dashed line.

We can choose any point for the test point as long as it does not lie on the boundary line. We chose $(0, 0)$ in Example 2.5.8 because the coordinates are easy to substitute into the inequality. Because $(0, 0)$ is a solution, we shaded the half-plane including that point. If, for example, we choose $(5, 0)$ as the test point, we find

$$3(\mathbf{5}) - 2(\mathbf{0}) < 6$$

which is a false statement. Thus, $(5, 0)$ is not a solution to the inequality, so the solutions must lie on the other side of the boundary line. Note that using $(5, 0)$ as the test point gives us the same solutions we found in Example 2.5.8.

Checkpoint 2.5.9 Practice 3. Graph the solutions of the inequality $y > \dfrac{-3}{2}x$

1. Graph the line $y = \dfrac{-3}{2}x$. (Use the slope-intercept method.)

2. Choose a test point. (Do not choose $(0, 0)$!)

3. Decide which side of the line to shade.

4. Should the boundary line be dashed or solid?

Caution 2.5.10 We cannot choose a test point that lies on the boundary line. In Practice 3 above, we cannot use $(0, 0)$ as a test point, because it lies on the line $y = \frac{-3}{2}x$. In this case, we should choose some other point, such as $(0, 1)$ or $(1, 2)$.

A simple example of of our method involves horizontal and vertical lines. Recall that the equation of a vertical line has the form

$$x = k$$

where k is a constant, and a horizontal line has an equation of the form

$$y = k$$

Even though only one variable appears in the equation $y = k$, if we think of it as $0x + 1y = k$, it represents a line in the plane. Similarly, the inequality $x \geq k$ may represent the inequality in two variables

$$x + 0y \geq k$$

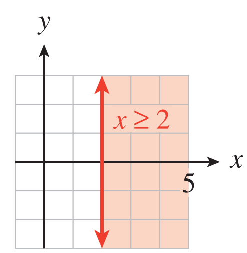
Its graph is then a region in the plane.

Example 2.5.11 Graph $x \geq 2$ in the plane.

Solution. First, we graph the equation $x = 2$; its graph is a vertical line. Because the origin does not lie on this line, we can use it as a test point. Substitute 0 for x (there is no y) into the inequality to obtain

$$0 \geq 2$$

Because this is a false statement, we shade the half-plane that does not contain the origin. We see that the graph of the inequality contains all points whose x-coordinates are greater than or equal to 2. (Or we can use common sense to shade all points with x-coordinates greater than or equal to 2.)

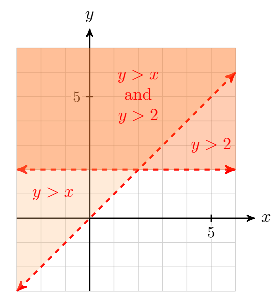


□

Checkpoint 2.5.12 **Practice 4.** Graph $-1 < y \leq 4$ in the plane.

2.5.4 Systems of Inequalities

Some applications are best described by a system of two or more inequalities. The solutions to a system of inequalities include all points that are solutions to every inequality in the system. The graph of the system is the intersection of the shaded regions for each inequality in the system. For example, the figure at right shows the solutions of the system



$$y > x \qquad \text{and} \qquad y > 2$$

Example 2.5.13 Laura's diet prescribes 500 milligrams of calcium from a combination of broccoli, at 160 milligrams per serving, and zucchini, at 30 milligrams per serving. Draw a graph representing the possible combinations of broccoli and zucchini that fulfill Laura's calcium requirements.

Solution.

- *Step 1.*

  Number of servings of broccoli: x

  Number of servings of zucchini: y

- *Step 2.*

  To get enough calcium, Laura must choose x and y so that

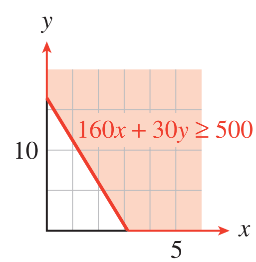
  $$160x + 30y \geq 500$$

  It makes no sense to consider negative values of x or of y, since Laura cannot eat a negative number of servings. Thus, we have two more inequalities to satisfy:

  $$x \geq 0 \quad \text{and} \quad y \geq 0$$

- *Step 3.*

  We graph all three inequalities on the same axes. The inequalities $x \geq 0$ and $y \geq 0$ restrict the solutions to lie in the first quadrant. The solutions common to all three inequalities are shown at right.



- *Step 4.*

  Laura can choose any combination of broccoli and zucchini represented by points in the shaded region. For example, the point $(3, 1)$ is a solution to the system of inequalities, so Laura could choose to eat 3 servings of broccoli and 1 serving of zucchini.

□

Checkpoint 2.5.14 Practice 5. Use the following steps to graph the solutions of the system

$$\begin{aligned} 3y - 2x &\leq 2 \\ y &> x - 1 \end{aligned}$$

1. Graph the boundary line $3y - 2x = 2$.
2. Lightly shade the solutions of the inequality $3y - 2x \leq 2$.
3. Graph the boundary line $y = x - 1$.
4. Lightly shade the solutions of $y > x - 1$.
5. Shade the intersection of the two solutions sets.

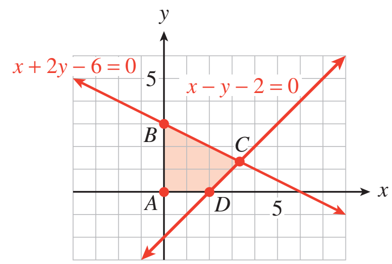
To describe the solutions of a system of inequalities, it is useful to locate the **vertices**, or corner points, of the boundary.

Example 2.5.15 Graph the solution set of the system below and find the coordinates of its vertices.

$$\begin{aligned} x - y - 2 &\le 0 \\ x + 2y - 6 &\le 0 \\ x \ge 0, \qquad y &\le 0 \end{aligned}$$

Solution.

The last two inequalities, $x \ge 0$ and $y \ge 0$, restrict the solutions to the first quadrant. We graph the line $x - y - 2 = 0$, and use the test point $(0, 0)$ to shade the half-plane including the origin. Finally we graph the line $x - 2y - 6 = 0$ and again use the test point $(0, 0)$ to shade the half-plane below the line. The intersection of the shaded regions is shown at right.



To find the coordinates of the vertices A, B, C, and D, we solve simultaneously the equations of the two lines that intersect at each vertex. [TK]

For A, we solve the system $\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$ to find $(0, 0)$

For B, we solve the system $\begin{aligned} x &= 0 \\ x + 2y &= 6 \end{aligned}$ to find $(0, 3)$

For C, we solve the system $\begin{aligned} x + 2y &= 6 \\ x - y &= 2 \end{aligned}$ to find $\left(\dfrac{10}{3}, \dfrac{4}{3}\right)$

For D, we solve the system $\begin{aligned} y &= 0 \\ x - y &= 2 \end{aligned}$ to find $(2, 0)$

The vertices are the points $(0, 0)$, $(0, 3)$, $\left(\dfrac{10}{3}, \dfrac{4}{3}\right)$, and $(2, 0)$. □

[TK] To review solving a linear 2x2 system, see Section 2.5.3 of the Toolkit.

Checkpoint 2.5.16 Practice 6.

a. Graph the system of inequalities

$$\begin{aligned} 5x + 4y &< 40 \\ -3x + 4y &< 12 \\ x < 6, \qquad y &> 2 \end{aligned}$$

b. Find the coordinaes of the vertices of the solution set.

Checkpoint 2.5.17 QuickCheck 2. Decide whether each statement is true or false.

a. We cannot choose a test point that lies on __________.

b. The soution set of a system of inequalities is the ________ of the solutions to each inequality in the system.

c. The corner points of the solution set are called the __________.

d. The graph of $x > 2$ is the set of all points that lie ________ the boundary line.

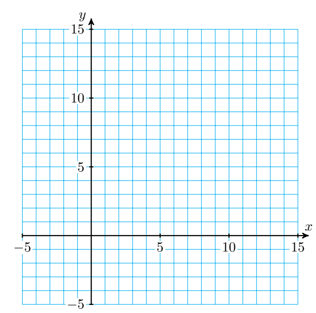
2.5.5 Problem Set 2.5

Warm Up

1. Ivana owns two hotels, that earn annual profits of x and y dollars, respectively. She would like her total annual profit from the two hotels to be (exactly) $10,000.

a. Fill in the table with some possible values for x and y, in thousands of dollars. (Note that it is possible for one of the hotels to sustain a loss, or negative profit.)

x				
y				



b. Plot the ordered pairs from your table on the grid.

c. Write an equation that describes this problem.

d. Graph all solutions of the equation.

2. Now suppose that Ivana would like her profit to be greater than $10,000.

a. List four pairs of values for x and y that achieve this goal.

x				
y				

b. Plot the four points on the grid in Problem 1.

c. Write an inequality for x and y that describes this situation.

d. Shade in the region that contains all solutions of the inequality. Note that all the solutions of the inequality lie on one side of the line.

3.

a. Explain how the graph of the equation $x + y = 10,000$ in Problem 1 is different from the graph of the inequality $x + y > 10,000$ in Problem 2.

b. Describe the graph of the inequality $x + y \geq 10,000$.

4. Graph the solutions of the inequality

$$3x - y \geq -2$$

a. Write the equation $3x - y = -2$ in slope-intercept form

b. Use the slope-intercept method to graph the line.

$b =$

$m = \dfrac{\Delta y}{\Delta x} =$

c. Use a test point to locate the solutions of the inequality. We can use any point for a test point, as long as it does not lie on the line! We'll use $(1, -4)$. Plot this point on your graph. Use algebra to decide whether $(1, -4)$ is a solution of the inequality $3x - y \le -2$.

d. Which side of the line includes all the solutions of the inequality? Shade that side of the line.

e. Can you suggest an easier test point instead of $(1, -4)$?

Skills Practice

Exercise Group. For Problems 5–14, graph the inequality.

5. $y > 2x + 4$

6. $y > \dfrac{4}{3}x$

7. $y < 9 - 3x$

8. $2x + 5y \ge 10$

9. $x + 4y \ge -6$

10. $x > 2y - 5$

11. $y < \dfrac{1}{2}x$

12. $0 \ge x + 3y$

13. $x \ge -3$

14. $-1 < y \le 4$

Exercise Group. For Problems 15 and 16, graph the system of inequalities.

15. $y > 2, \quad x \ge -2$

16. $y \le -x, \quad y < 2$

Exercise Group. For Problems 17 and 18:

a. Graph the system of inequalities

b. Find the coordinates of the vertices

17. $\begin{aligned} 2x + 3y - 6 &< 0 \\ x \ge 0,\ y &\ge 0 \end{aligned}$

18. $\begin{aligned} 3x + 2y &< 6 \\ x \ge 0,\ y &\ge 0 \end{aligned}$

Exercise Group. Graph each system of inequalities.

19. $\begin{aligned} x + y &\le 6 \\ x + y &\ge 4 \end{aligned}$

20. $\begin{aligned} 2x - y &\le 4 \\ x + 2y &> 6 \end{aligned}$

Exercise Group. Graph each system of inequalities and find the coordinates of the vertices.

21. $\begin{aligned} 5y - 3x &\le 15 \\ x + y &\le 11 \\ x \ge 0,\ y &\ge 0 \end{aligned}$

22. $\begin{aligned} 2y &\le x \\ 2x &\le y + 12 \\ x \ge 0,\ y &\ge 0 \end{aligned}$

23. $\begin{aligned} x + y &\ge 3 \\ 2y &\le x + 8 \\ 2y + 3x &\le 24 \\ x \ge 0,\ y &\ge 0 \end{aligned}$

24. $\begin{aligned} 3y - x &\ge 3 \\ y - 4x &\ge -10 \\ y - 2 &\le x \\ x \ge 0,\ y &\ge 0 \end{aligned}$

Applications

Exercise Group. For Problems 25–28, graph the set of solutions to the problem. For each system, $x \geq 0$ and $y \geq 0$.

25. Vassilis plans to invest at most \$10,000 in two banks. One bank pays 6% annual interest and the other pays 5% annual interest. Vassilis wants at least \$540 total annual interest from his two investments. Write a system of four inequalities for the amount Vassilis can invest in the two accounts, and graph the system.

26. Jeannette has 180 acres of farmland for growing wheat or soy. She can get a profit of \$36 per acre for wheat and \$24 per acre for soy. She wants to have a profit of at least \$5400 from her crops. Write a system of four inequalities for the number of acres she can use for each crop, and graph the solutions.

27. Gary's pancake recipe includes corn meal and whole wheat flour. Corn meal has 2.4 grams of linoleic acid and 2.5 milligrams of niacin per cup. Whole wheat flour has 0.8 grams of linoleic acid and 5 milligrams of niacin per cup. These two ingredients should not exceed 3 cups total. The mixture should provide at least 3.2 grams of linoleic acid and at least 10 milligrams of niacin. Write a system of five inequalities for the amount of corn meal and the amount of whole wheat flour Gary can use, and graph the solutions.

28. Cho and his brother go into business making comic book costumes. They need 1 hour of cutting and 2 hours of sewing to make a Batman costume. They need 2 hours of cutting and 1 hour of sewing to make a Wonder Woman costume. They have available at most 10 hours per day for cutting and at most 8 hours per day for sewing. They must make at least one costume each day to stay in business. Write a system of five inequalities for the number of each type of costume Cho can make, and graph the solutions.

2.6 Chapter Summary and Review

2.6.1 Glossary

- scatterplot
- regression line
- interpolation
- extrapolation
- linear system
- solution of a system
- ordered pair
- inconsistent system
- dependent system
- equilibrium price
- substitution method
- elimination method
- linear combination
- ordered triple
- back-substitution
- triangular form
- Gaussian reduction

2.6.2 Key Concepts

1. We can approximate a linear pattern in a **scatterplot** using a **regression line**.
2. We can use **interpolation** or **extrapolation** to make estimates and predictions.
3. If we extrapolate too far beyond the known data, we may get unreasonable results.
4. A solution to a 2×2 linear system is an ordered pair that satisfies both equations.
5. A solution to a 2×2 linear system is a point where the two graphs intersect.
6. The graphs of the equations in an **inconsistent system** are parallel lines and hence to do not intersect.
7. The graphs of the two equations in a **dependent system** are the same line.
8. If a company's revenue exactly equals its costs (so that their profit is zero), we say that the business venture will **break even**.
9. For solving a 2×2 linear system, the **substitution method** is easier if one of the variables in one of the equations has a coefficient of 1 or -1.
10. If a **linear combination** of the equations in a system results in an equation of the form

$$0x + 0y = k \qquad\qquad (k \neq 0)$$

then the system is **inconsistent**. If an equation of the form

$$0x + 0y = 0$$

results, then the system is **dependent**

11. The **point-slope** form is useful when we know the rate of change and one point on the line.
12. The solutions of the **linear inequality**

$$ax + by + c \leq 0 \qquad \text{or} \qquad ax + by + c \geq 0$$

consists of the line $ax + by + c = 0$ and a **half-plane** on one side of that line.

13. **To Graph an Inequality Using a Test Point.**

    1. Graph the corresponding equation to obtain the boundary line.
    2. Choose a test point that does not lie on the boundary line.
    3. Substitute the coordinates of the test point into the inequality.

        a. If the resulting statement is true, shade the half-plane that includes the test point.

b. If the resulting statement is false, shade the half-plane that does not include the test point.

4. If the inequality is strict, make the boundary line a dashed line.

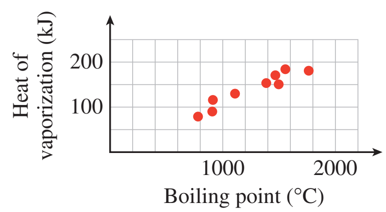
14. The solutions to a system of inequalities includes all points that are solutions to all the inequalities in the system. The graph of the system is the intersection of the shaded regions for each inequality in the system.

15. A solution to an equation in three variables is an **ordered triple** of numbers that satisfies the equation.

16. To solve a 3×3 linear system, we use linear combinations to reduce the system to **triangular form**, and then use **back-substitution** to find the solutions.

2.6.3 Chapter 2 Review Problems

1. The scatterplot shows the boiling temperature of various substances on the horizontal axis and their heats of vaporization on the vertical axis. (The heat of vaporization is the energy needed to change the substance from liquid to gas at its boiling point.)



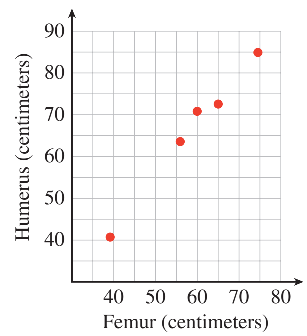
a. Use a straightedge to estimate a line of best fit for the scatterplot.

b. Use your line to predict the heat of vaporization of silver, whose boiling temperature is 2160°C.

c. Find the equation of the regression line.

d. Use the regression line to predict the heat of vaporization of potassium bromide, whose boiling temperature is 1435°C.

2. **Archaeopteryx** is an extinct creature with characteristics of both birds and reptiles. Only six fossil specimens are known, and only five of those include both a femur (leg bone) and a humerus (forearm bone) The scatterplot shows the lengths of femur and humerus for the five Archaeopteryx specimens. Draw a line of best fit for the data points.



a. Predict the humerus length of an Archaeopteryx whose femur is 40 centimeters

b. Predict the humerus length of an Archaeopteryx whose femur is 75 centimeters

c. Use your answers from parts (b) and (c) to approximate the equation of a regression line.

d. Use your answer to part (d) to predict the humerus length of an Archaeopteryx whose femur is 60 centimeters.

e. Use your calculator and the given points on the scatterplot to find the least squares regression line. Compare the score this equation gives for part (d) with what you predicted earlier. The ordered pairs defining the data are $(38, 41)$, $(56, 63)$, $(59, 70)$, $(64, 72)$, $(74, 84)$.

3. In 1986 the space shuttle Challenger exploded because of O-ring failure on a morning when the temperature was about 30°F. Previously there had been 1 incident of O-ring failure when temperature was 70°F and 3 incidents when the temperature was 54°F. Use linear extrapolation to estimate the number of incidents of O-ring failure you would expect when the temperature is 30°F.

4. Thelma typed a 19-page technical report in 40 minutes. She required only 18 minutes for an 8-page technical report. Use linear interpolation to estimate how long Thelma would require to type a 12-page technical report.

Exercise Group. For problems 5–6, solve the system by graphing. Use the **ZDecimal** window.

5. $$\begin{aligned} y &= -2.9x - 0.9 \\ y &= 1.4 - 0.6x \end{aligned}$$

6. $$\begin{aligned} y &= 0.6x - 1.94 \\ y &= -1.1x + 1.29 \end{aligned}$$

Exercise Group. For problems 7–10, solve the system by using substitution or elimination.

7. $$\begin{aligned} x + 5y &= 18 \\ x - y &= -3 \end{aligned}$$

8. $$\begin{aligned} x + 5y &= 11 \\ 2x + 3y &= 8 \end{aligned}$$

9. $$\begin{aligned} \frac{2}{3}x - 3y &= 8 \\ x - \frac{3}{4}y &= 12 \end{aligned}$$

10. $$\begin{aligned} 3x &= 5y - 6 \\ 3y &= 10 - 11x \end{aligned}$$

Exercise Group. For problems 11–14, decide whether the system is inconsistent, dependent, or consistent and independent.

11. $\begin{aligned} 2x - 3y &= 4 \\ x + 2y &= 7 \end{aligned}$

12. $\begin{aligned} 2x - 3y &= 4 \\ 6x - 9y &= 4 \end{aligned}$

13. $\begin{aligned} 2x - 3y &= 4 \\ 6x - 9y &= 12 \end{aligned}$

14. $\begin{aligned} x - y &= 6 \\ x + y &= 6 \end{aligned}$

Exercise Group. For problems 15–20, solve the system using Gaussian reduction.

15. $\begin{aligned} x + 3y - z &= 3 \\ 2x - y + 3z &= 1 \\ 3x + 2y + z &= 5 \end{aligned}$

16. $\begin{aligned} x + y + z &= 2 \\ 3x - y + z &= 4 \\ 2x + y + 2z &= 3 \end{aligned}$

17. $\begin{aligned} x + z &= 5 \\ y - z &= -8 \\ 2x + z &= 7 \end{aligned}$

18. $\begin{aligned} x + 4y + 4z &= 0 \\ 3x + 2y + z &= -4 \\ 2x - 4y + z &= -11 \end{aligned}$

19. $\begin{aligned} \frac{1}{2}x + y + z &= 3 \\ x - 2y - \frac{1}{3}z &= -5 \\ \frac{1}{2}x - 3y - \frac{2}{3}z &= -6 \end{aligned}$

20. $\begin{aligned} \frac{3}{4}x - \frac{1}{2}y + 6z &= 2 \\ \frac{1}{2}x + y - \frac{3}{4}z &= 0 \\ \frac{1}{4}x + \frac{1}{2}y - \frac{1}{2}z &= 0 \end{aligned}$

Exercise Group. Solve Problems 21–26 by writing and solving a system of linear equations in twoor three variables.

21. A math contest exam has 40 questions. A contestant scores 5 points for each correct answer, but loses 2 points for each wrong answer. Lupe answered all the questions and her score was 102. How many questions did she answer correctly?

22. A game show contestant wins \$25 for each correct answer he gives but loses \$10 for each incorrect response. Roger answered 24 questions and won \$355. How many answers did he get right?

23. Barbara wants to earn \$500 a year by investing \$5000 in two accounts, a savings plan that pays 8% annual interest and a high-risk option that pays 13.5% interest. How much should she invest in each account?

24. An investment broker promises his client a 12% return on her funds. If the broker invests \$3000 in bonds paying 8% interest, how much must he invest in stocks, with an estimated rate of return equal to 15% interest, to keep his promise?

25. The perimeter of a triangle is 30 centimeters. The length of one side is 7 centimeters shorter than the second side, and the third side is 1 centimeter longer than the second side. Find the length of each side.

26. A company ships its product to three cities: Boston, Chicago, and Los Angeles. The cost of shipping is \$10 per crate to Boston, \$5 per crate to Chicago, and \$12 per crate to Los Angeles. The company's shipping budget for April is \$445. It has 55 crates to ship, and demand for their product is twice as high in Boston as in Los Angeles. How many crates should the company ship to each destination?

Exercise Group. For Problems 27–30, graph the inequality.

27. $3x - 4y < 12$

28. $x > 3y - 6$

29. $y < \dfrac{-1}{2}$

30. $-4 \le x < 2$

Exercise Group. For Problems 31–34, graph the solutions to the system of inequalities.

31. $y > 3, \quad x \le 2$

32. $y \ge x, \quad x > 2$

33. $3x - 6 < 6, \quad x + 2y > 6$

34. $x - 3y > 3, \quad y < x + 2$

Exercise Group. For Problems 35–38

a. Graph the solutions to the system of inequalities.

b. Find the coordinates of the vertices.

35. $3x - 4y \le 12$
$x \ge 0, \quad y \le 0$

36. $x - 2y \le 6$
$y \le x$
$x \ge 0, \quad y \ge 0$

37. $x + y \le 5$
$y \ge x$
$y \ge 2, \quad x \ge 0$

38. $x - y \le -3$
$x + y \le 6$
$x \le 4$
$x \ge 0, \quad y \ge 0$

39. Ruth wants to provide cookies for the customers at her bookstore. It takes 20 minutes to mix the ingredients for each batch of peanut butter cookies and 10 minutes to bake them. Each batch of granola cookies takes 8 minutes to mix and 10 minutes to bake. Ruth does not want to use the oven more than 2 hours a day, or to spend more than 2 hours a day mixing ingredients. Write a system of inequalities for the number of batches of peanut butter cookies and of granola cookies that Ruth can make in one day, and graph the solutions.

40. A vegetarian recipe calls for 32 ounces of a combination of tofu and tempeh. Tofu provides 2 grams of protein per ounce and tempeh provides 1.6 grams of protein per ounce. Graham would like the dish to provide at least 56 grams of protein. Write a system of inequalities for the amount of tofu and the amountof tempeh for the recipe, and graph the solutions.

Chapter 3

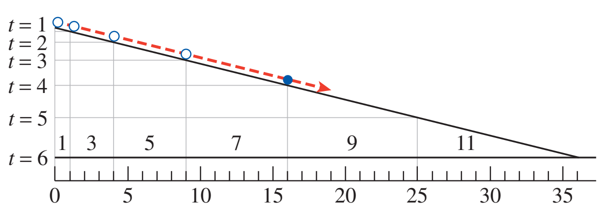
Quadratic Models

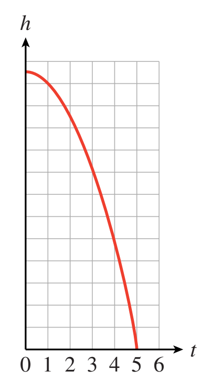
The models we have explored so far are linear models; their graphs are straight lines. In this chapter, we investigate problems where the graph may change from increeasing to decreasing, or vice versa. The simplest sort of function that models this behavior is a quadratic function, one that involves the square of the variable.

Around 1600, Galileo began to study the motion of falling objects. He used a ball rolling down an inclined plane or ramp to slow down the motion, but he had no accurate way to measure time; clocks had not been invented yet. So he used water running into a jar to mark equal time intervals.

After many trials, Galileo found that the ball traveled 1 unit of distance down the plane in the first time interval, 3 units in the second time interval, 5 units in the third time interval, and so on, as shown in the figure, with the distances increasing through odd units of distance as time went on.

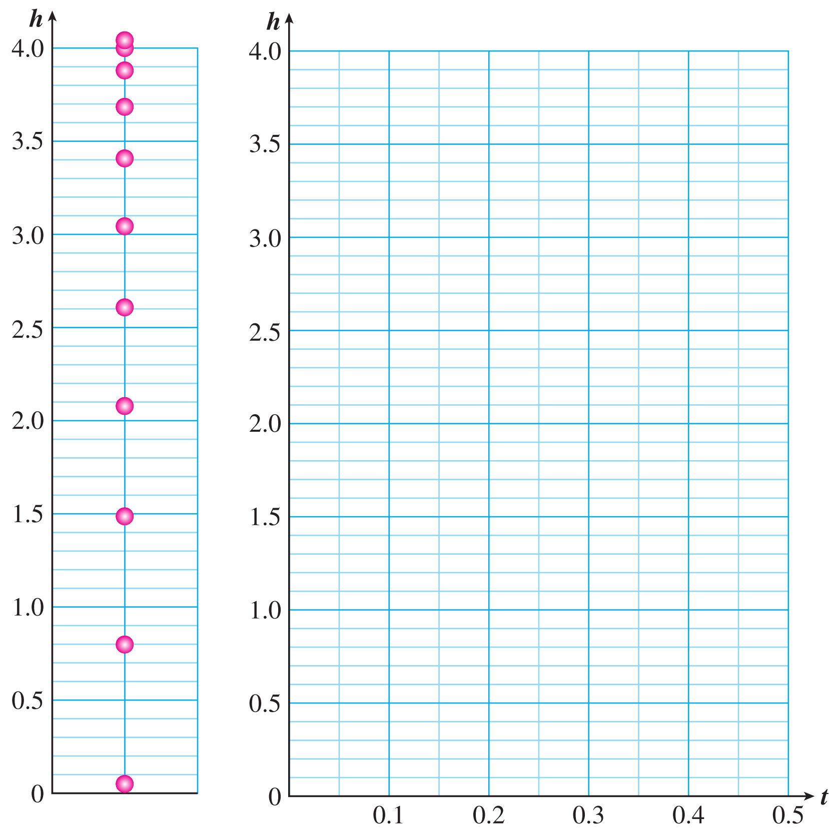
Time	Distance traveled	Total distance
1	1	1
2	3	5
3	5	9
4	7	16
5	9	25





The total distance traveled by the ball can be modeled by the equation, $d = kt^2$, where k is a constant. Galileo found that this relationship holds no matter how steep he made the ramp. If we plot the height of the ball (rather than the distance traveled) versus time, we obtain a portion of the graph of a quadratic equation.

Investigation 3.0.1 Falling. Suppose you drop a small object from a height and let it fall under the influence of gravity. Does it fall at the same speed throughout its descent? The diagram shows a sequence of photographs of a steel ball falling onto a table. The photographs were taken using a stroboscopic flash at intervals of 0.05 second, and a scale on the left side shows the height of the ball, in feet, at each interval.



1. Complete the table showing the height of the ball at each 0.05-second interval. Measure the height at the bottom of the ball in each image. The first photo was taken at time $t = 0$.

t	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
h											

2. Plot the points in the table. Connect the points with a smooth curve to sketch a graph of height versus elapsed time. Is the graph linear?

3. Use your graph to estimate the time elapsed when the ball is 3.5 feet high, and when it is 1 foot high.

4. What was the change in the ball's height during the first quarter second, from $t = 0$ to $t = 0.25$? What was the change in the ball's height from $t = 0.25$ to $t = 0.50$?

5. Add to your graph a line segment connecting the points at $t = 0$ and $t = 0.25$, and a second line segment connecting the points at $t = 0.25$ and $t = 0.50$. Compute the slope of each line segment.

6. What do the slopes in part (5) represent in terms of the problem?

7. Use your answers to part (4) to verify algebraically that the graph is not linear.

3.1 Extraction of Roots

3.1.1 Introduction

So far you have learned how to solve linear equations. In linear equations, the variable cannot have any exponent other than 1, and for this reason such equations are often called **first-degree**. Now we'll consider second-degree equations, or **quadratic** equations. A quadratic equation includes the square of the variable.

Here are some examples.

$$2x^2 + 5x - 3 = 0, \qquad 7t - t^2 = 0, \qquad \text{and} \qquad 3w^2 = 16$$

Some familiar geometric formulas are quadratic equations, such as the formula for the area of a circle, $A = \pi r^2$.

Definition 3.1.1 Quadratic Equation. A **quadratic equation** can be written in the standard form

$$ax^2 + bx + c = 0$$

where $a,$ $b,$ and c are constants, and a is not zero. ◊

Checkpoint 3.1.2 QuickCheck 1. Which of the following equations are quadratic?

a. $3x + 2x^2 = 1$

b. $4z^2 - 2z^3 + 2 = 0$

c. $36y - 16 = 0$

d. $v^2 = 6v$

We would like to be able to solve quadratic equations, use them in applications, and graph quadratic equations in two variables. Let's begin by considering some simple examples.

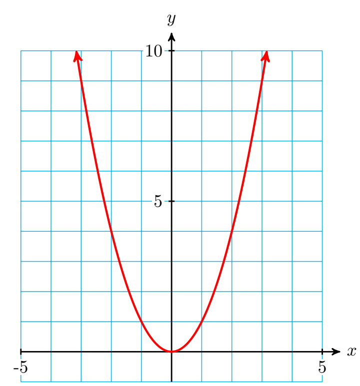
3.1.2 Graphs of Quadratic Equations

The simplest quadratic equation in two variables is

$$y = x^2$$

Its graph is not a straight line, but a curve called a **parabola**, shown in the figure. You can verify the table of values below for this parabola.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



$y = x^2$

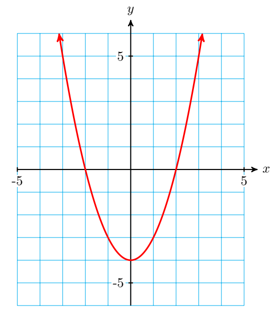
Caution 3.1.3 Be careful when squaring negative numbers. To evaluate the square of a negative number on a calculator, we must enclose the number in parentheses to show that the negative sign is included in the expression to be squared. For example,

$$(-3)^2 = (-3)(-3) = 9, \qquad \text{but} \qquad -3^2 = -3 \cdot 3 = -9$$

Example 3.1.4 Graph the parabola $y = x^2 - 4$

Solution. We make a table of values and plot the points. [TK] The graph is shown below.

x	-3	-2	-1	0	1	2	3
y	5	0	-3	-4	-3	0	5



□

[TK] To review evaluating quadratic expressions, see Section 3.1.1 of the Toolkit.

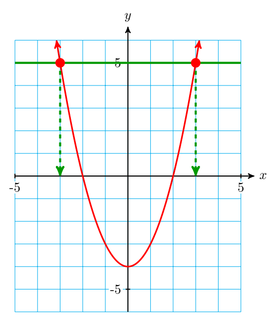
Checkpoint 3.1.5 Practice 1. Graph the parabola $y = 4 - x^2$ on the same grid in the Example above.

3.1.3 Solving Quadratic Equations

How can we solve a quadratic equation? Consider the equation

$$x^2 - 4 = 5$$

First, we can solve it graphically. Look again at the graph of $y = x^2 - 4$ from Example 1.



We would like to find the x-values that make $y = 5$. The horizontal line $y = 5$ intersects the graph at two points with y-coordinate 5, and their x-coordinates are the solutions of the equation. Thus, there are two solutions, namely 3 and -3.

Algebraically, we solve the equation as follows. [TK]

- First, we isolate the variable. We add 4 to both sides, yielding $x^2 = 9$.
- Because x is squared in this equation, we perform the opposite operation, or take square roots, in order to solve for x.

$x^2 = 9$	Take square roots of both sides.
$x = \pm\sqrt{9} = \pm 3$	Remember that every positive number has two square roots.

The solutions are 3 and -3, as we saw on the graph.

[TK] For more on square roots, see Section 3.1.2 of the Toolkit.

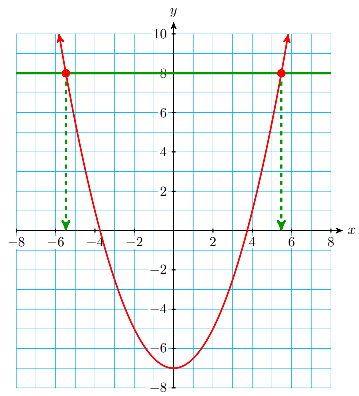
Note 3.1.6 Notice that we have found two solutions for this quadratic equation, whereas linear equations have at most one solution. (Sometimes they have no solution at all.) We shall see that every quadratic equation has two solutions, which may be equal. The solutions may also be complex numbers, which we'll study in Chapter 4.

Example 3.1.7 Solve the equation

$$\frac{1}{2}x^2 - 7 = 8$$

graphically and algebraically.

Solution. The figure shows the graph of $y = \dfrac{1}{2}x^2 - 7$.



We would like to find the x-values that make $y = 8$. The horizontal line $y = 8$ intersects the graph at two points with x-coordinates approximately 5.5 and -5.5. These are the solutions of the equation.

Algebraically, we solve the equation as follows.

- First, we isolate the variable. We add 7 to both sides, then multiply by 2, yielding $x^2 = 30$.
- Because x is squared in this equation, we perform the opposite operation, or take square roots, in order to solve for x.

$$\begin{aligned} x^2 &= 30 && \text{Take square roots of both sides.} \\ x &= \pm\sqrt{30} && \text{Remember that every positive number} \\ & && \text{has two square roots.} \end{aligned}$$

- We use a calculator to find that $\sqrt{30}$ is approximately 5.477, or about 5.5, as we saw on the graph.

□

Caution 3.1.8 Many square roots are **irrational numbers**, which means that their decimal form never ends. When we round off the decimal form we have an approximation to the square root, but not its exact value. It is important to make a distinction betweeen exact values and decimal approximations.

- For the example above, the exact solutions are $\pm\sqrt{30}$.
- The values from the calculator, ± 5.477, are decimal approximations to the solutions, rounded to thousandths.

Checkpoint 3.1.9 QuickCheck 2. Which solutions are exact values, and which are approximations?

a. $x^2 = 40,\qquad x = \pm 6.32455532$

b. $t^2 = \dfrac{81}{64},\qquad t = \pm 1.125$

c. $w^2 = 50,\qquad w = \pm 5\sqrt{2}$

d. $b^2 = (0.632)^2,\qquad b = \pm 0.632$

3.1.4 Extracting Roots

We can now solve quadratic equations of the form $ax^2+c=0$, where the linear term bx is missing, by isolating x on one side of the equation, and then taking the square root of each side. This method for solving quadratic equations is called **extraction of roots**.

Extraction of Roots.

To solve a quadratic equation of the form

$$ax^2 + c = 0$$

1. Isolate x on one side of the equation.
2. Take the square root of each side.

Checkpoint 3.1.10 Practice 2. Solve by extracting roots $\dfrac{3x^2-8}{4} = 10$

In the next Example, we compare the steps for *evaluating a quadratic expression* and for *solving a quadratic equation*.

Example 3.1.11 Tux the cat falls off a tree branch 20 feet above the ground. His height t seconds later is given by $h = 20 - 16t^2$.

a. How high is Tux above the ground 0.5 second later?

b. How long does Tux have to get in position to land on his feet before he reaches the ground?

Solution.

a. We evaluate the formula for $t = 0.5$. We substitute **0.5** for t into the formula, and simplify.

$$\begin{aligned} h &= 20 - 16(\mathbf{0.5})^2 && \text{Compute the power.}\\ &= 20 - 16(0.25) && \text{Multiply, then subtract.}\\ &= 20 - 4 = 16 \end{aligned}$$

Tux is 16 feet above the ground after 0.5 second. You can also use your calculator to simplify the expression for h by entering

20 [−] 16 [×] 0.5 [x^2] [ENTER]

b. We would like to find the value of t when Tux's height, h, is zero. We substitute $h = \mathbf{0}$ into the equation to obtain

$$\mathbf{0} = 20 - 16t^2$$

To solve this equation we use extraction of roots. We first isolate t^2 on one side of the equation.

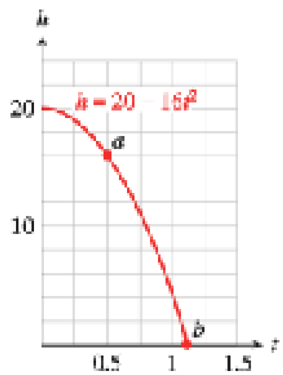
$$\begin{aligned} 16t^2 &= 20 && \text{Divide by 16.} \\ t^2 &= \frac{20}{16} = 1.25 \end{aligned}$$

Next, we take the square root of both sides of the equation to find

$$t = \pm\sqrt{1.25} \approx \pm 1.118$$

Only the positive solution makes sense here, so Tux has approximately 1.12 seconds to be in position for landing.

A graph of the Tux's height after t seconds is shown below. The points corresponding to parts (a) and (b) are labeled.

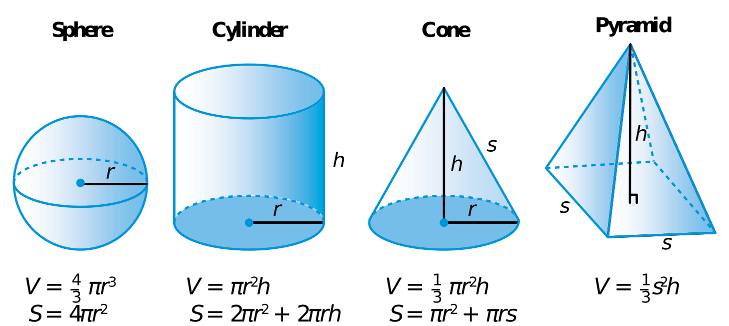


□

3.1.5 Geometric Formulas

The formulas for the volume and surface area of some everyday objects, such as cylinders and cones, involve quadratic expressions. We can use extraction of roots to solve problems involving these objects.

Formulas for Volume and Surface Area.



Example 3.1.12 The volume of a can of soup is 582 cubic centimeters, and its height is 10.5 centimeters. What is the radius of the can, to the nearest tenth of a centimeter? [TK]

Solution. The volume of a cylinder is given by the formula $V = \pi r^2 h$. We substitute **582** for V and **10.5** for h, then solve for r.

$$\begin{aligned} \mathbf{582} &= \pi r^2 (\mathbf{10.5}) && \text{Divide both sides by } \mathbf{10.5\pi}. \\ 17.643 &\approx r^2 && \text{Take square roots.} \\ 4.200 &\approx r \end{aligned}$$

The radius of the can is approximately 4.2 centimeters. □

[TK] For more examples using geometric formulas, see Section 3.1.3 of the Toolkit.

Checkpoint 3.1.13 Practice 3. The glass pyramid at the Louvre in Paris has a square base, is 21.64 meters tall, and encloses a volume of 9049.68 cubic meters. Use the formula $V = \dfrac{1}{3}s^2 h$ to find the length of the base. Round your answer to hundredths.

3.1.6 Solving Formulas

Sometimes it is useful to solve a formula for one variable in terms of the others. You might want to know what radius you need to build cones of various fixed volumes. In that case, it is more efficient to solve the volume formula for r in terms of v.

Example 3.1.14 The formula $V = \dfrac{1}{3}\pi r^2 h$ gives the volume of a cone in terms of its height and radius. Solve the formula for r in terms of V and h.

Solution. Because the variable we want is squared, we use extraction of roots. First, we multiply both sides by 3 to clear the fraction.

$$\begin{aligned} 3V &= \pi r^2 h && \text{Divide both sides by } \boldsymbol{\pi h}. \\ \frac{3V}{\pi h} &= r^2 && \text{Take square roots.} \\ \pm\sqrt{\frac{3V}{\pi h}} &= r \end{aligned}$$

Because the radius of a cone must be a positive number, we use only the positive square root: $r = \sqrt{\dfrac{3V}{\pi h}}$. □

Checkpoint 3.1.15 Practice 4. Find a formula for the radius of a circle in terms of its area, A.

Hint: Start with the formula for the area of a circle: $A =$
Solve for r in terms of A.

Checkpoint 3.1.16 QuickCheck 3. Match each quantity with the appropriate units.

a. Height of a cylinder	I. Square meters
b. Volume of a cone	II. Feet
c. Surface area of a sphere	III. Cubic centimeters
d. Area of a triangle	IV. Kilograms

3.1.7 More Extraction of Roots

We can also use extraction of roots to solve quadratic equations of the form

$$a(x-p)^2 = q$$

We start by isolating the squared expression, $(x-p)^2$.

Example 3.1.17 Solve the equation $3(x-2)^2 = 48$.

Solution. First, we isolate the perfect square, $(x-2)^2$.

$$\begin{aligned} 3(x-2)^2 &= 48 && \text{Divide both sides by 3.} \\ (x-2)^2 &= 16 && \text{Take the square root of each side.} \\ x-2 &= \pm\sqrt{16} = \pm 4 \end{aligned}$$

This gives us two equations for x,

$$\begin{aligned} x-2 &= 4 \quad \text{or} \quad x-2 = -4 && \text{Solve each equation.} \\ x &= 6 \quad \text{or} \quad x = -2 \end{aligned}$$

The solutions are 6 and -2. You can check that both of these solutions satisfy the original equation. □

Checkpoint 3.1.18 Practice 5. Solve $2(5x+3)^2 = 38$ by extracting roots.

a. Give your answers as exact values.

b. Find approximations for the solutions to two decimal places.

Checkpoint 3.1.19 QuickCheck 4. True or false.

a. The first step in extraction of roots is to take square roots.

b. The solutions of a quadratic equation are always of the form $\pm k$.

c. Your calculator gives exact decimal values for square roots of integers.

d. The coefficients of a quadratic equation are called parabolas.

3.1.8 An Application: Compound Interest

Many savings accounts offer interest compounded annually: at the end of each year the interest earned is added to the principal, and the interest for the next year is computed on this larger sum of money. [TK] After n years, the amount of money in the account is given by the formula

$$A = P(1+r)^n$$

where P is the original principal and r is the interest rate, expressed as a decimal fraction.

[TK] We'll see more about compound interest in Section 7.1.

Example 3.1.20 Carmella invests \$3000 in an account that pays an interest rate r compounded annually.

a. Write an expression for the amount of money in Carmella's account after two years.

b. What interest rate would be necessary for Carmella's account to grow to \$3500 in two years?

Solution.

a. We use the formula $A = P(1+r)^n$ with $P = 3000$ and $n = 2$. Carmella's account balance will be

$$A = 3000(1+r)^2$$

b. We substitute **3500** for A in the equation.

$$\mathbf{3500} = 3000(1+r)^2$$

This is a quadratic equation in the variable r, which we can solve by extraction of roots. First, we isolate the perfect square.

$$\begin{aligned} 3500 &= 3000(1+r)^2 && \text{Divide both sides by 3000.}\\ 1.1\overline{6} &= (1+r)^2 && \text{Take square roots.}\\ \pm 1.0801 &\approx 1+r && \text{Subtract 1 from both sides.}\\ r \approx 0.0801\text{or} &\quad r \approx -2.0801 \end{aligned}$$

Because the interest rate must be a positive number, we discard the negative solution. Carmella needs an account with interest rate $r \approx 0.0801$, or over 8%, in order to have an account balance of \$3500 in two years.

□

The formula for compound interest also applies to calculating the effects of inflation. For instance, if there is a steady inflation rate of 4% per year, then in two years the price of an item that costs \$100 now will be

$$\begin{aligned} A &= P(1+r)^2\\ 100 &= (1+0.04)^2 = 108.16 \end{aligned}$$

Checkpoint 3.1.21 Practice 6. The average cost of dinner and a movie two years ago was \$36. This year the average cost is \$38.16. What was the rate of inflation over the past two years? (Round to two decimal places.)

3.1.9 Problem Set 3.1

Warm Up

1. Simplify.

a. $4 - 2\sqrt{64}$ b. $\dfrac{4-\sqrt{64}}{2}$ c. $\sqrt{9-4(-18)}$

2. Give a decimal approximation rounded to thousandths.

a. $5\sqrt{3}$ b. $\dfrac{-2}{3}\sqrt{21}$ c. $-3 + 2\sqrt{6}$

3. Use the definition of square toot to simplify the expression.

a. $\sqrt{29}(\sqrt{29})$ b. $(\sqrt{7})^2$ c. $\dfrac{6}{\sqrt{6}}$

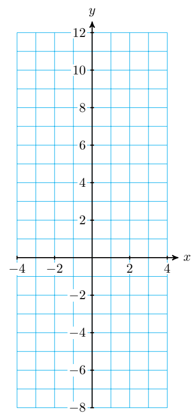
4. Solve. Remember that every positive number has two square roots.

a. $3x^2 = 147$ b. $4x^2 = 25$ c. $3x^2 = 15$

5.

a. Complete the table and graph $y = 2x^2 - 5$.

x	-3	-2	-1	0	1	2	3
y							

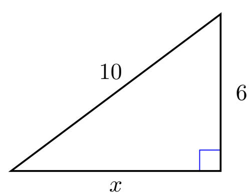


b. Use the graph to solve the equation $2x^2 - 5 = 7$. Show your work on the graph. How many solutions did you find?

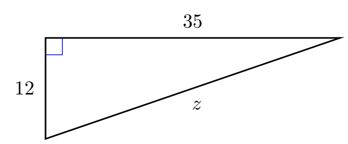
c. Solve the equation $2x^2 - 5 = 7$ algebraically, by "undoing" each operation.

6. Use the Pythagorean theorem to find the unknown side. [TK] To review the Pythagorean theorem, see Section 3.1.4 of the Toolkit.

a.



b.



Skills Practice

Exercise Group. For problems 7–14, Solve by extracting roots. Give exact values for your answers.

7. $3x^2 - 9 = 0$

8. $\dfrac{3x^2}{5} = 6$

9. $(2x - 1)^2 = 16$

10. $4(x - 1)^2 = 12$

11. $\left(x - \dfrac{2}{3}\right)^2 = \dfrac{5}{9}$

12. $81\left(x + \dfrac{1}{3}\right)^2 = 1$

13. $3(8x - 7)^2 = 24$

14. $2(5x - 12)^2 = 48$

Exercise Group. For problems 15 and 16, solve by extracting roots. Round your answers to two decimal places.

15. $5x^2 - 97 = 3.2x^2 - 38$

16. $17 - \dfrac{x^2}{4} = 43 - x^2$

Exercise Group. For problems 17 and 18,

a. Use technology to graph the quadratic equation in the suggested window.

b. Use your graph to find two solutions for the equation in part (b).

c. Check your solutions algebraically, using mental arithmetic.

17.

a. $y = 3(x-4)^2$

Xmin $= -5$ Ymin $= -20$
Xmax $= 15$ Ymax $= 130$

b. $3(x-4)^2 = 108$

18.

a. $y = \dfrac{1}{2}(x+3)^2$

Xmin $= -15$ Ymin $= -5$
Xmax $= 5$ Ymax $= 15$

b. $\dfrac{1}{2}(x+3)^2 = 8$

Exercise Group. For problems 19–22, solve the formula for the specified variable.

19. $F = \dfrac{mv^2}{r}$, for v

20. $S = 4\pi r^2$, for r

21. $L = \dfrac{8}{\pi^2}T^2$, for T

22. $s = \dfrac{1}{2}gt^2$, for t

Applications

Exercise Group. For problems 23 and 24,

a. Make a sketch of the situation described, and label a right triangle.

b. Use the Pythagorean theorem to solve each problem.

23. The size of a TV screen is the length of its diagonal. If the width of a 35-inch TV screen is 28 inches, what is its height?

24. A 30-meter pine tree casts a shadow of 30 meters, how far is the tip of the shadow from the top of the tree?

25. You plan to deposit your savings of \$1600 in an account that compounds interest annually.

a. Write a formula for the amount, A, in your savings account after two years in terms of the interest rate, r.

b. Complete the table showing your account balance after two years for various interest rates.

r	0.02	0.04	0.06	0.08
A				

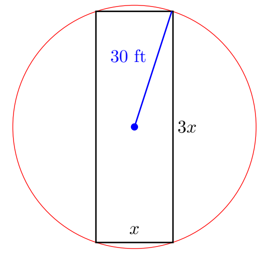
c. To the nearest tenth of a percent, what interest rate will you require if you want your \$1600 to grow to \$2000 in two years?

d. Use your calculator to graph the formula for the account balance. Locate the point on the graph that corresponds to the amount in part (c).

26. Two years ago Carol's living expenses were \$1200 per month. This year the same items cost Carol \$1400 per month. What was the annual inflation rate for the past two years?

27.

What size rectangle will fit inside a circle of radius 30 feet if the length of the rectangle must be three times its width?



28. A storage box for sweaters is constructed from a square sheet of cardboard measuring x inches on a side. The volume of the box, in cubic inches, is

$$V = 10(x - 20)^2$$

If the box should have a volume of 1960 cubic inches, what size cardboard square is needed?

29. A large bottle of shampoo is 20 centimeters tall and cylindrical in shape.

a. Write a formula for the volume of the bottle in terms of its radius.

b. Complete the table of values for the volume equation. If you cut the radius of the bottle in half, by what factor does the volume decrease?

r	1	2	3	4	5	6	7	8
V								

c. What radius should the bottle have if it must hold 240 milliliters of shampoo? (A milliliter is equal to one cubic centimeter.)

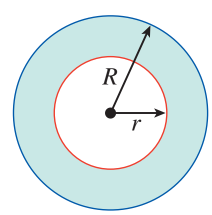
d. Use your calculator to graph the volume equation. (Use the table to help you choose a suitable window.) Locate the point on the graph that corresponds to the bottle in part (c). Make a sketch of your graph, and lable the scales on the axes.

30.

The area of a ring is given by the formula

$$A = \pi R^2 - \pi r^2$$

where R is the radius of the outer circle, and r is the radius of the inner circle.



a. Suppose the inner radius of the ring is kept fixed at $r = 4$ centimeters, but the radius of the outer circle, R, is allowed to vary. Find the area of the ring when the outer radius is 6 centimeters, 8 centimeters, and 12 centimeters.

b. Graph the area equation, with $r = 4$, in the window

Xmin $= 0$ Xmax $= 14.1$
Ymin $= 0$ Ymax $= 400$

Use the Trace to verify your answers to part (a).

c. Trace along the curve to the point $(9.75, 248.38217)$. What do the coordinates of this point represent?

d. Use your graph to estimate the outer radius of the ring when its area is 100 square centimeters.

e. Write and solve an equation to answer part(d).

Exercise Group. For Problems 31 and 32, solve for x in terms of a, b, and c.

31.

a. $\dfrac{ax^2}{b} = c$ b. $\dfrac{bx^2}{c} - a = 0$

32.

a. $(x-a)^2 = 16$ b. $(ax+b)^2 = 9$

3.2 Intercepts, Solutions, and Factors

In the last section, we used extraction of roots to solve quadratic equations of the form

$$a(x-p)^2 = q$$

But this technique will not work on quadratic equations that also include a linear term, bx. Recall that the most general type of quadratic equation looks like

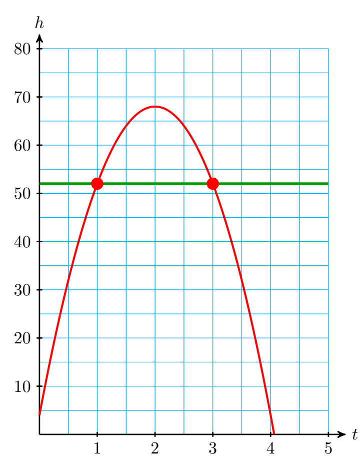
$$ax^2 + bx + c = 0$$

Here is an example.

Suppose a baseball player pops up, that is, she hits the baseball straight up into the air. The height, h, of the baseball after t seconds is given by a formula from physics. This formula takes into account the initial speed of the ball (64 feet per second) and its height when it was hit (4 feet).

$$h = -16t^2 + 64t + 4$$

The graph of this equation is shown below.



We would like to know when the baseball was exactly 52 feet high. To find out, we must solve the equation

$$-16t^2 + 64t + 4 = 52$$

where we have substituted 52 for the height, h. We can use the graph to solve this equation, by finding points with h-coordinate 52. You can see that there are two such points, with t-coordinates 1 and 3, so the baseball is 52 feet high at 1 second, and again on the way down at 3 seconds.

Can we solve the equation algebraically? Not with the techniques we know, because there are two terms containing the variable t, and they cannot be combined. We will need a new method. [TK] To find this method, we are going to study the connection between:

1. the **factors** of $ax^2 + bx + c$,
2. the **solutions** of the quadratic equation $ax^2 + bx + c = 0$, and
3. the x-intercepts of the graph of $y = \; ax^2 + bx + c = 0$.

[TK] If you would like to review multiplying binomials using the "FOIL" method, see Section 3.2.1 of the Toolkit.

3.2.1 Zero-Factor Principle

The method we will learn now is not like extraction of roots, or like solving linear equations, where we "undid" in reverse order each operation performed on the variable, like peeling an onion. This new method will seem less direct. It relies on applying a property of our number system.

Can you multiply two numbers together and obtain a product of zero? Only if one of the two numbers happens to be zero. (Try it yourself.)

Zero-Factor Principle.

The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0 \quad \text{(or both)}$$

Checkpoint 3.2.1 QuickCheck 1. Fill in the blanks.

a. If the sum of two numbers is zero, the numbers must be __________.

b. If a fraction equals zero, the __________ must be zero.

c. If the product of two numbers is zero, one of the numbers must be __________.

d. If the divisor in a quotient is zero, the quotient is __________.

Here is the simplest possible application of the Zero-Factor Principle (ZFP): For what value(s) of x is the equation $3x = 0$ true? You could divide both sides by 3, but you can also see that the product $3x$ can equal zero only if one of its factors is zero, so x must be zero!

The ZFP is true even if the numbers a and b are represented by algebraic expressions, such as $x - 6$ or $x + 2$. For example, if

$$(x - 6)(x + 2) = 0$$

then it must be true that either $x - 6 = 0$ or $x + 2 = 0$. This is how we can use the ZFP to solve quadratic equations.

Example 3.2.2 Solve the equation $x^2 - 4x - 12 = 0$

Solution. We can factor the expression $x^2 - 4x - 12$, and write the equation as [TK]

$$(x-6)(x+2) = 0$$

Now it is in the form $ab = 0$, with $a = x - 6$ and $b = x + 2$, so the ZFP tells us that either $x - 6 = 0$ or $x + 2 = 0$. We solve each of these equations.

$$\begin{aligned} x - 6 &= 0 \quad \text{or} \quad x + 2 = 0 \qquad \text{Solve each equation.} \\ x &= 6 \quad \text{or} \quad x = -2 \end{aligned}$$

Once again we see that a quadratic equation has two solutions. You can check that both of these values satisfy the original equation. □

[TK] To review factoring quadratic trinaomials, see Section 3.2.2 of the Toolkit.

Checkpoint 3.2.3 Practice 1. Solve the equation $x^2 - 11x + 24 = 0$

3.2.2 X-Intercepts of a Parabola

Recall that the x-intercept of a line is the point where $y = 0$, or where the line crosses the x-axis. We find the x-intercept by setting $y = 0$ in the equation of the line, and solving for x. We can find the x-intercepts of a parabola the same way.

Example 3.2.4 Find the x-intercepts of the graph of $y = x^2 - 4x - 12$

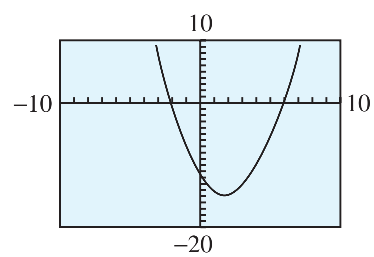
Solution. To find the x-intercepts of the graph, we set $y = \mathbf{0}$ and solve the equation

$$\mathbf{0} = x^2 - 4x - 12$$

But this is the same equation we solved in the last Example, because

$$x^2 - 4x - 12 = (x-6)(x+2)$$

The solutions of that equation were 6 and -2, so the x-intercepts of the graph are $(6, 0)$ and $(-2, 0)$. You can see this by graphing the equation on your calculator, as shown in the figure.



□

We can state a general result: The x-intercepts of the graph of

$$y = ax^2 + bx + c$$

are the solutions of the equation

$$0 = ax^2 + bx + c$$

So we can always solve a quadratic equation to find the x-intercepts of a parabola (if there are any).

And we can use this relationship the other way round, too: If we know the x-intercepts of the graph of $y = ax^2 + bx + c$, we also know the solutions of the equation $ax^2 + bx + c = 0$.

Checkpoint 3.2.5 Practice 2. Use technology to graph the equation

$$y = (x-3)(2x+3)$$

and find the x-intercepts of the graph. Use your answers to solve the equation

$$(x-3)(2x+3) = 0$$

Check your solutions by applying the ZFP.

3.2.3 Solving Quadratic Equations by Factoring

Now we'll consider some other quadratic equations. Before we apply the ZFP, we must write the equation so that one side is zero.

Example 3.2.6 Solve $3x(x+1) = 2x+2$

Solution. First, we write the equation in standard form.

$$\begin{aligned} 3x(x+1) &= 2x+2 && \text{Apply the distributive law to the left side.} \\ 3x^2+3x &= 2x+2 && \text{Subtract } \mathbf{2x+2} \text{ from both sides.} \\ 3x^2+x-2 &= 0 \end{aligned}$$

Now we factor the left side to obtain

$$\begin{aligned} (3x-2)(x+1) &= 0 && \text{Apply the zero-factor principle.} \\ 3x-2=0 \quad \text{or} \quad x+1 &= 0 && \text{Solve each equation.} \\ x = \frac{2}{3} \quad \text{or} \quad x &= -1 \end{aligned}$$

The solutions are $\dfrac{2}{3}$ and -1. □

Caution 3.2.7 When we apply the zero-factor principle, one side of the equation must be zero. For example, to solve the equation

$$(x-2)(x-4) = 15$$

it is *incorrect* to set each factor equal to 15! (There are many ways that the product of two numbers can equal 15; it is not necessary that one of the numbers be 15.)

We must first simplify the left side and write the equation in standard form. (The correct solutions are 7 and -1; check that you can find these solutions.)

We summarize the factoring method for solving quadratic equations as follows.

To Solve a Quadratic Equation by Factoring.

1. Write the equation in standard form.
2. Factor the left side of the equation.
3. Apply the zero-factor principle: Set each factor equal to zero.

4. Solve each equation. There are two solutions (which may be equal).

Checkpoint 3.2.8 Practice 3. Solve by factoring: $(t-3)^2 = 3(9-t)$
Hint: Begin by multiplying out each side of the equation.

Checkpoint 3.2.9 QuickCheck 2. Which technique, extracting roots or factoring, is better-suited to each equation?

a. $4x^2 - 12x = 0$

b. $6(4x-1)^2 = 18$

c. $(x+4)^2 = 16x$

d. $9x^2 - 42 = 0$

Now we can use factoring to solve the opening problem in this section.

Example 3.2.10 The height, h, of a baseball t seconds after being hit is given by

$$h = -16t^2 + 64t + 4$$

When will the baseball reach a height of 64 feet?

Solution. We substitute **64** for h in the formula, and solve for t.

$$\begin{aligned} -16t^2 + 64t + 4 &= \mathbf{64} && \text{Write the equation in standard form.} \\ 16t^2 - 64t + 60 &= 0 && \text{Factor 4 from the left side.} \\ 4(4t^2 - 16t + 15) &= 0 && \text{Factor the quadratic expression.} \\ 4(2t-3)(2t-50) &= 0 && \text{Set each variable factor equal to zero.} \\ 2t - 3 = 0 \quad \text{or} \quad 2t - 5 &= 0 && \text{Solve each equation.} \\ t = \frac{3}{2} \quad \text{or} \quad t &= \frac{5}{2} \end{aligned}$$

There are two solutions. At $t = \dfrac{3}{2}$ seconds, the ball reaches a height of 64 feet on the way up, and at $t = \dfrac{5}{2}$ seconds, the ball is 64 feet high on its way down. □

Caution 3.2.11 In the Example above, the factor of 4 does not affect the solutions of the equation at all. You can understand why this is true by looking at some graphs. Use technology to graph the equation

$$y_1 = x^2 - 4x + 3$$

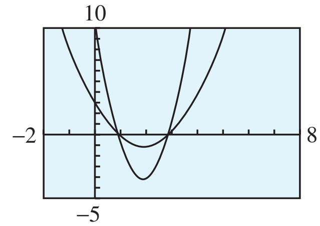
in the window

$$\begin{aligned} \text{Xmin} &= -2 \quad & \text{Ymin} &= -5 \\ \text{Xmax} &= 8 \quad & \text{Ymax} &= 10 \end{aligned}$$

Notice that when $y = 0$, $x = 1$ or $x = 3$. These two points are the x-intercepts of the graph. Now on the same window graph

$$y_2 = 4(x^2 - 4x + 3)$$

as shown below.



This graph has the same x-values when $y = 0$. The factor of 4 makes the graph "skinnier," but does not change the location of the x-intercepts.

Checkpoint 3.2.12 Practice 4.

a. Solve by factoring $4t - t^2 = 0$.

b. Solve by factoring $20t - 5t^2 = 0$.

c. Graph $y = 4t - t^2$ and $y = 20t - 5t^2$ together in the window

$$\begin{array}{ll} \text{Xmin} = -2 & \text{Ymin} = -20 \\ \text{Xmax} = 6 & \text{Ymax} = 25 \end{array}$$

and locate the horizontal intercepts on each graph.

Checkpoint 3.2.13 QuickCheck 3. Match each equation with its solutions.

a. $3t(t-2) = 0$	i. $-1, 2$
b. $t^2 - t = 2$	ii. $0, 2$
c. $3t^2 = 12$	iii. $2, 3$
d. $t(t-3) = 2(t-3)$	iv. $-2, 2$

3.2.4 An Application

Here is another example of how quadratic equations arise in applications.

Example 3.2.14 The size of a rectangular computer monitor screen is taken to be the length of its diagonal. If the length of the screen should be 3 inches greater than its width, what are the dimensions of a 15-inch monitor?

Solution. We express the two dimensions of the screen in terms of a single variable:

$$\text{Width of screen: } w$$
$$\text{Length of screen: } w + 3$$

We apply the Pythagorean theorem to write an equation [TK]:

$$w^2 + (w+3)^2 = 15^2$$

To solve this equation, we begin by simplifying the left side.

$$\begin{aligned} w^2 + w^2 + 6w + 9 &= 225 && \text{Write the equation in standard form.} \\ 2w^2 + 6w - 216 &= 0 && \text{Factor 2 from the left side.} \\ 2(w^2 + 3w - 108) &= 0 && \text{Factor the quadratic expression.} \\ 2(w-9)(w+12) &= 0 && \text{Set each factor equal to zero.} \\ w - 9 = 0 \quad \text{or} \quad w + 12 &= 0 && \text{Solve each equation.} \end{aligned}$$

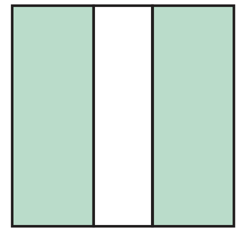
$$w = 9 \qquad \text{or} \qquad w = -12$$

Because the width of the screen cannot be a negative number, the width is 9 inches, and the length is $w + 3 = 12$ inches. ☐

[TK] To review the Pythagorean theorem, see Section 3.1.4 of the Toolkit.

Checkpoint 3.2.15 Practice 5.

Francine is designing the layout for a botanical garden. The plan includes a square herb garden, with a path 5 feet wide through the center of the garden, as shown at right. To include all the species of herbs, the planted area must be 300 square feet. Find the dimensions of the herb garden.



3.2.5 More About Solutions of Quadratic Equations

As we have seen in ehe examples above, the solutions of the quadratic equation

$$a(x - r_1)(x - r_2) = 0$$

are r_1 and r_2. This is called the **factored form** of the quadratic equation. Thus, if we know the two solutions of a quadratic equation, we can work backwards to reconstruct the equation.

Example 3.2.16 Find a quadratic equation whose solutions are $\dfrac{1}{2}$ and -3.

Solution. Each solution corresponds to a factor of the equation, so the equation must look like this:

$$\left(x - \frac{1}{2}\right)(x - (-3)) = 0$$

or, simplifying:

$$\left(x - \frac{1}{2}\right)(x + 3) = 0$$

We multiply the factors together to obtain

$$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

This is an equation that works, but we can make a "nicer" one if we clear the fractions. We can multiply both sides of the equation by 2. We know that multiplying by a constant does not change the solutions of the equation.

$$\begin{gathered}\mathbf{2}\left(x^2 + \frac{5}{2}x - \frac{3}{2}\right) = \mathbf{2}(0) \\ 2x^2 + 5x - 3 = 0\end{gathered}$$

By factoring, we can check that this equation really does have the given solutions.

$$0 = 2x^2 + 5x - 3 = (2x - 1)(x + 3)$$

From here, you can see that the solutions are indeed $\dfrac{1}{2}$ and -3. ☐

Checkpoint 3.2.17 Practice 6. Find a quadratic equation with integer coefficients whose solutions are $\dfrac{2}{3}$ and -5.

A quadratic equation in one variable always has two solutions. In some cases, the solutions may be equal. For example, the equation

$$x^2 - 2x + 1 = 0$$

can be solved by factoring as follows:

$$\begin{aligned} (x-1)(x-1) &= 0 && \text{Apply the zero-factor principle.} \\ x-1=0 \quad \text{or} \quad x-1 &= 0 \end{aligned}$$

Both of these equations have solution $x = 1$. We say that 1 is a solution of **multiplicity two**, meaning that it occurs twice as a solution of the quadratic equation.

Checkpoint 3.2.18 QuickCheck 4. True or false.

a. We find the x-intercepts of a graph by setting $x = 0$.

b. If z is a solution of a quadratic equation, then $(x - z)$ is a factor of the left side in standard form.

c. We can factor a constant from both sides of a quadratic equation without changing its solutions.

d. To solve a quadratic equation by factoring, we should factor each side of the equation.

3.2.6 Problem Set 3.2

Warm Up

Exercise Group. For Problems 1-4, write each product as a polynomial in simplest form.

1.

a. $(b + 6)(2b - 3)$

b. $(3z - 8)(4z - 1)$

2.

a. $(4z - 3)^2$

b. $(2d + 8)^2$

3.

a. $3p(2p - 5)(p - 3)$

b. $2v(v + 4)(3v - 4)$

4.

a. $-50(1 + r)^2$

b. $12(1 - t))^2$

5. Factor if possible.

a. $x^2 - 16$

b. $x^2 - 16x$

c. $x^2 - 8x + 16$

d. $x^2 + 16$

6. Solve if possible.

a. $x^2 - 16 = 0$

b. $x^2 - 16x = 0$

c. $x^2 - 8x + 16 = 0$

d. $x^2 + 16 = 0$

Exercise Group. For Problems 7-12, factor completely.

7. $x^2 - 7x + 10$

8. $x^2 - 225$

9. $w^2 - 4w - 32$

10. $2z^2 + 11z - 40$

11. $9n^2 + 24n + 16$

12. $4n^2 - 28n + 49$

Skills Practice

Exercise Group. For Problems 13–20, solve by factoring.

13. $2a^2 + 5a - 3 = 0$

14. $3b^2 - 4b - 4 = 0$

15. $2x^2 - 6x = 0$

16. $3y^2 - 6y = -3$

17. $x(2x - 3) = -1$

18. $t(t - 3) = 2(t - 3)$

19. $z(3z + 2) = (z + 2)^2$

20. $(v + 2)(v - 5) = 8$

Exercise Group. For problems 21-24, solve by extracting roots.

21. $3(8x - 7)^2 = 24$

22. $81\left(x + \dfrac{1}{3}\right)^2 = 1$

23. $(ax - b)^2 = 25$

24. $100 = \pi x^2 - 16\pi$

Exercise Group. For problems 25 and 26, graph the equation and locate the x-intercepts of the graph. Use the x-intercepts to write the quadratic expression in factored form.

25. $y = 0.1(x^2 - 3x - 270)$

26. $y = -0.06(x^2 - 22x - 504)$

27. Use technology to graph all three equations in the same window. What do you notice about the x-intercepts?

a. $y = x^2 + 2x - 15$

b. $y = 3(x^2 + 2x - 15)$

c. $y = 0.2(x^2 + 2x - 15)$

28. Write a quadratic equation with the given solutions. Give your answers in standard form with integer coefficients.

a. -2 and 1

b. -3 and $\dfrac{1}{2}$

c. $\dfrac{-1}{4}$ and $\dfrac{3}{2}$

Applications

29. Delbert stands at the top of a 300-foot cliff and throws his algebra book directly upward with a velocity of 20 feet per second. The height of his book above the ground t seconds later is given by the equation

$$h = -16t^2 + 20t + 300$$

where h is in feet.

a. Use your graphing utility to make a table of values for the height equation, with increments of 0.5 second.

b. Graph the height equation. Use your table of values to help you choose appropriate $\boxed{\text{WINDOW}}$ settings.

c. What is the highest altitude Delbert's book reaches? When does it reach that height? Use the $\boxed{\text{TRACE}}$ feature to find approximate answers first. Then use the $\boxed{\text{Table}}$ feature to improve your estimate.

d. When does Delbert's book pass him on its way down? (Delbert is standing at a height of 300 feet.) Use the intersect command.

e. Write and solve an equation to answer the question: How long will it take Delbert's book to hit the ground at the bottom of the cliff?

30. The annual increase I in the deer population in a national park is given by the formula

$$I = 1.2x - 0.0002x^2$$

where x is the size of the population that year.

a. Make a table of values for I for $0 \leq x \leq 7000$. Use increments of 500 in x.

b. How much will a population of 2000 deer increase? A population of 5000 deer? A population of 7000 deer?

c. Use your calculator to graph the annual increase versus the size of the population, x, for $0 \leq x \leq 7000$. Use your table from part (b) to help you choose appropriate values for Ymin and Ymax.

d. What do the x-intercepts tell us about the deer population?

e. Estimate the population size that results in the largest annual increase. What is that increase?

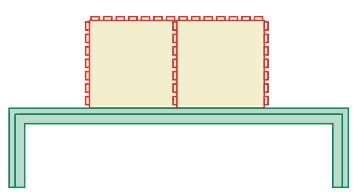
31. One end of a ladder is 10 feet from the base of a wall, and the other end reaches a window in the wall. The ladder is 2 feet longer than the height of the window.

a. Choose a variable for the height of the window. Make a sketch of the situation described, and label the sides of a right triangle.

b. Write a quadratic equation about the height of the window.

c. Solve your equation to find the height of the window.

32. Irene would like to enclose two adjacent chicken coops of equal size against the henhouse wall. She has 66 feet of chicken wire fencing, and she would like the total area of the two coops to be 360 square feet. What should the dimensions of the chicken coops be?



We'll use three methods to solve this problem: a table of values, a graph, and an algebraic equation.

a. Make a table by hand that shows the areas of coops of various widths, as shown below.

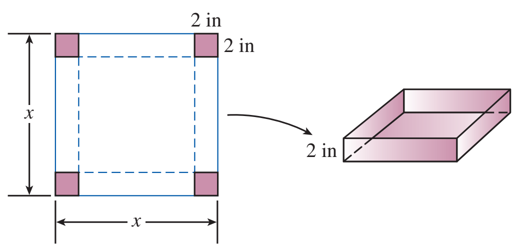
Width	Length	Area
4	27	216
6	24	288

Continue the table until you find a pair of chicken coops whose total area is 360 square feet. (Be careful computing the length of each chicken coop: look at the diagram above.)

b. Write an expression for the length of each of the two coops if their width is x. Then write an expression for the combined area of the coops if their width is x. Graph the equation for A, and use the graph to find the pair of coops whose combined area is 360 square feet. (Is there more than one solution?)

c. Write an equation for the area A of the two coops in terms of their width, x. Solve your equation algebraically for $A = 360$.

33. A box is made from a square piece of cardboard by cutting 2-inch squares from each corner and then turning up the edges.



a. If the piece of cardboard is x inches square, write expressions for the length, width, and height of the box. Then write an expression for the volume, V, of the box in terms of x.

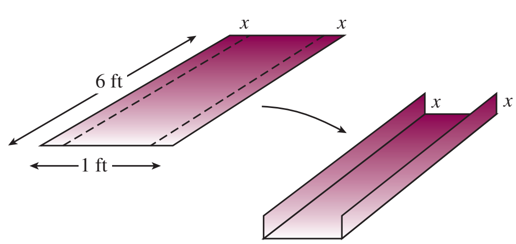
b. Use your calculator to make a table of values showing the volumes of boxes made from cardboard squares of side 4 inches, 5 inches, and so on.

c. Graph your expression for the volume on your calculator. What is the smallest value of x that makes sense for this problem?

d. Use your table or your graph to find what size cardboard you need to make a box with volume 50 cubic inches.

e. Write and solve a quadratic equation to answer part (d).

34. A length of rain gutter is made from a piece of aluminum 6 feet long and 1 foot wide.



a. If a strip of width x is turned up along each long edge, write expressions for the length, width and height of the gutter. Then write an expression for the volume V of the gutter in terms of x.

b. Use your calculator to make a table of values showing the volumes of various rain gutters formed by turning up edges of 0.1 foot, 0.2 foot, and so on.

c. Graph your expression for the volume. What happens to V as x increases?

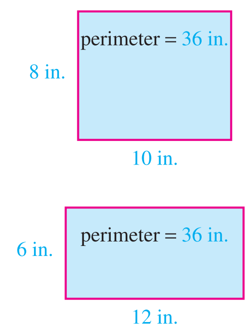
d. Use your table or your graph to discover how much metal should be turned up along each long edge so that the gutter has a capacity of $\frac{3}{4}$ cubic foot of rainwater.

e. Write and solve a quadratic equation to answer part (d).

3.3 Graphing Parabolas

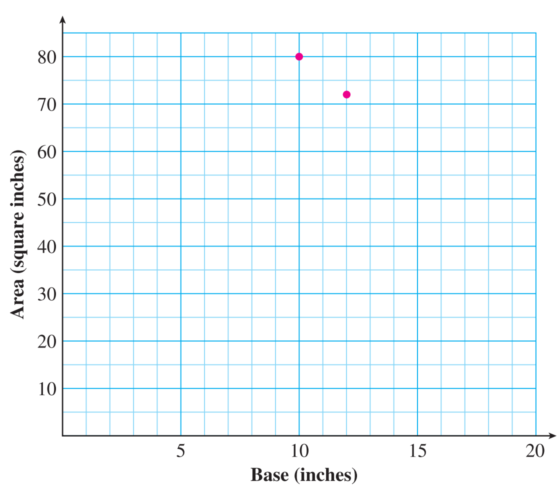
Investigation 3.3.1 Perimeter and Area.

Do all rectangles with the same perimeter, say 36 inches, have the same area? Two different rectangles with perimeter 36 inches are shown. The first rectangle has base 10 inches and height 8 inches, and its area is 80 square inches. The second rectangle has base 12 inches and height 6 inches. Its area is 72 square inches.



1. The table shows the bases of various rectangles, in inches. Each rectangle has a perimeter of 36 inches. Fill in the height and the area of each rectangle. (To find the height of the rectangle, reason as follows: The base plus the height makes up half of the rectangle's perimeter.)

Base	Height	Area
10	8	80
12	6	72
3		
14		
5		
17		
19		
2		
11		
4		
16		
15		
1		
6		
8		
13		
7		



2. What happens to the area of the rectangle when we change its base? On the grid above, plot the points with coordinates (Base, Area). (For this graph we will not use the heights of the rectangles.) The first two points, $(10, 80)$ and $(12, 72)$, are shown. Connect your data points with a smooth curve.

3. What are the coordinates of the highest point on your graph?

4. Each point on your graph represents a particular rectangle with perimeter 36 inches. The first coordinate of the point gives the base of the rectangle, and the second coordinate gives the area of the rectangle. What is the largest area you found among rectangles with perimeter 36 inches? What is the base for that rectangle? What is its height?

5. Describe the rectangle that corresponds to the point $(13, 65)$.

6. Find two points on your graph with vertical coordinate 80.

7. If the rectangle has area 80 square inches, what is its base? Why are there two different answers here? Describe the rectangle corresponding to each answer.

8. Now we'll write an algebraic expression for the area of the rectangle in terms of its base. Let x represent the base of the rectangle. First, express the height of the rectangle in terms of x. (Hint: If the perimeter of the rectangle is 36 inches, what is the sum of the base and the height?) Now write an expression for the area of the rectangle in terms of x.

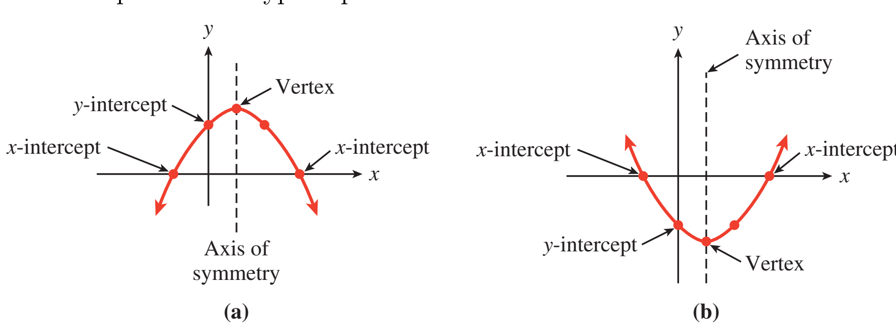
9. Use your formula from part (8) to compute the area of the rectangle when the base is 5 inches. Does your answer agree with the values in your table and the point on your graph?

10. Use your formula to compute the area of the rectangle when $x = 0$ and when $x = 18$. Describe the "rectangles" that correspond to these data points.

11. Continue your graph to include the points corresponding to $x = 0$ and to $x = 18$.

3.3.1 Introduction

The graph of any quadratic equation $y = ax^2 + bx + c$ is a **parabola** (if $a \neq 0$). In this section we'll make a systematic study of the parabolas we've met earlier in the chapter. Some typical parabolas are shown below.



To begin, notice that these parabolas share certain features.

- The graph has either a highest point (if the parabola opens downward, as in figure (a) or a lowest point (if the parabola opens upward, as in figure (b). This high or low point is called the **vertex** of the graph.

- The parabola is symmetric about a vertical line, called the **axis of symmetry**, that runs through the vertex.

- A parabola has a y-intercept, and it may have zero, one, or two x-intercepts.
- If there are two x-intercepts, they are equidistant from the axis of symmetry.

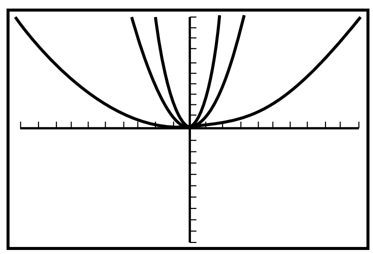
The values of the constants a, b, and c determine the location and orientation of the parabola. We'll consider each of these constants separately.

Checkpoint 3.3.1 QuickCheck 1. Which point on a parabola always lies on the axis of symmetry?

3.3.2 The Graph of $y = ax^2$

Use your calculator to graph the following three equations in the standard window, as shown below:

$$y = x^2$$
$$y = 3x^2$$
$$y = 0.1x^2$$



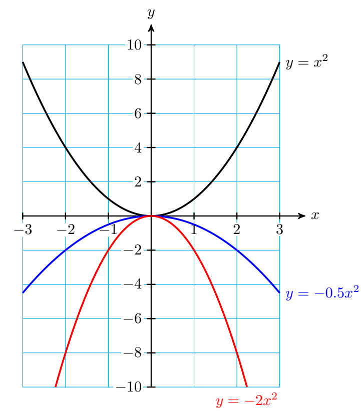
You can see that the graph of $y = 3x^2$ is narrower than the basic parabola, and the graph of $y = 0.1x^2$ is wider. As x increases, the graph of $y = 3x^2$ increases faster than the basic parabola, and the graph of $y = 0.1x^2$ increases more slowly. Compare the corresponding x-values for the three graphs shown in the table.

x	$y = x^2$	$y = 3x^2$	$y = 0.1x^2$
-2	4	12	0.4
1	1	3	0.1
3	9	27	0.9

For each x-value, the points on the graph of $y = 3x^2$ are higher than the points on the basic parabola, while the points on the graph of $y = 0.1x^2$ are lower. Multiplying by a positive constant greater than 1 stretches the graph vertically, and multiplying by a positive constant less than 1 squashes the graph vertically.

What about negative values for a? Consider the graphs of

$$y = x^2$$
$$y = -2x^2$$
$$y = -0.5x^2$$



We see that multiplying x^2 by a negative constant reflects the graph about the x-axis. These parabolas open downward.

The Graph of $y = ax^2$.

- The parabola opens upward if $a > 0$
- The parabola opens downward if $a < 0$
- The magnitude of a determines how wide or narrow the parabola is.
- The vertex, the x-intercepts, and the y-intercept all coincide at the origin.

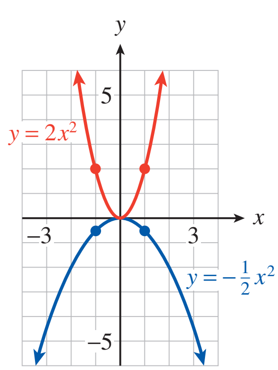
Example 3.3.2 Sketch by hand a graph of each quadratic equation.

a. $y = 2x^2$

b. $y = -\dfrac{1}{2}x^2$

Solution. Both equations have the form $y = ax^2$. The graph of $y = 2x^2$ opens upward because $a = 2 > 0$, and the graph of $y = -\dfrac{1}{2}x^2$ opens downward because $a = -\dfrac{1}{2} < 0$.

To make a reasonable sketch by hand, it is enough to plot a few *guidepoints*; the points with x-coordinates 1 and -1 are easy to compute.



x	$y = 2x^2$	$y = -\frac{1}{2}x^2$
-1	2	$-\dfrac{1}{2}$
0	0	0
1	2	$-\dfrac{1}{2}$

We sketch parabolas through each set of guidepoints, as shown at left.

□

Checkpoint 3.3.3 QuickCheck 2. True or false.

a. If $a > 1$, the graph of $y = ax^2$ is wider than the basic parabola.

b. The vertex of a parabola lies on its axis of symmetry.

c. If the y-intercept is negative, the parabola opens downward.

d. A parabola may have one, two, or three x-intercepts.

Checkpoint 3.3.4 Practice 1.

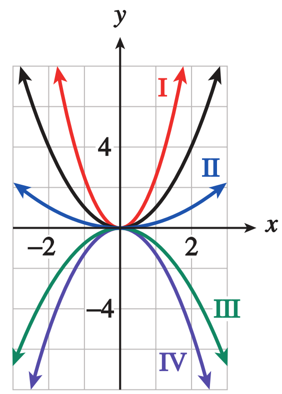
Match each parabola with its equation. The basic parabola is shown in black.

a. $y = -\frac{3}{4}x^2$

b. $y = \frac{1}{4}x^2$

c. $y = \frac{5}{2}x^2$

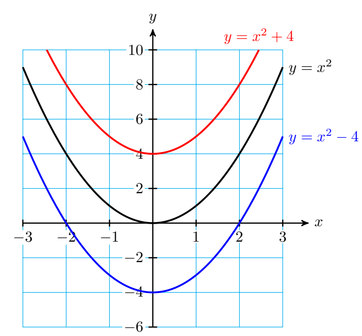
d. $y = -\frac{5}{4}x^2$



3.3.3 The Graph of $y = x^2 + c$

Next, we consider the effect of the constant term, c, on the graph. Compare the graphs of

$$y = x^2$$
$$y = x^2 + 4$$
$$y = x^2 - 4$$



The graph of $y = x^2 + 4$ is shifted upward four units compared to the basic parabola, and the graph of $y = x^2 - 4$ is shifted downward four units. Look at the table, which shows the y-values for the three graphs.

x	$y = x^2$	$y = x^2 + 4$	$y = x^2 - 4$
-1	1	5	-3
0	0	4	-4
2	4	8	0

Each point on the graph of $y = x^2 + 4$ is four units higher than the corresponding point on the basic parabola, and each point on the graph of $y = x^2 - 4$ is four units lower. In particular, the vertex of the graph of $y = x^2 + 4$ is the point $(0, 4)$, and the vertex of the graph of $y = x^2 - 4$ is the point $(0, -4)$.

The Graph of $y = x^2 + c$.

Compared to the graph of $y = x^2$, the graph of $y = x^2 + c$

- is shifted upward by c units if $c > 0$.
- is shifted downward by c units if $c < 0$.

Example 3.3.5 Sketch graphs for the following quadratic equations.

a. $y = x^2 - 2$

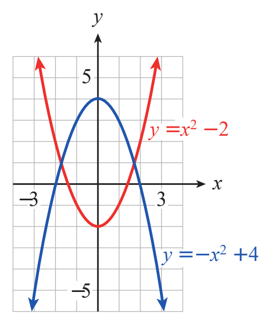
b. $y = -x^2 + 4$

Solution.

a. The graph of $y = x^2 - 2$ is shifted downward by two units, compared to the basic parabola. The vertex is the point $(0, -2)$ and the x-intercepts are the solutions of the equation

$$0 = x^2 - 2$$

or $\sqrt{2}$ and $-\sqrt{2}$. The graph is shown below.



b. The graph of $y = -x^2 + 4$ opens downward and is shifted 4 units up, compared to the basic parabola. Its vertex is the point $(0, 4)$. Its x-intercepts are the solutions of the equation

$$0 = -x^2 + 4$$

or 2 and -2. You can verify both graphs with your graphing calculator.

□

Checkpoint 3.3.6 QuickCheck 3. Match each equation with the description of its graph.

a. $y = x^2 - 3$

b. $y = -3x^2$

c. $y = 3 - x^2$

d. $y = \dfrac{1}{3}x^2$

i. Wider than the basic parabola.

ii. Shifted down 3 units from the basic parabola.

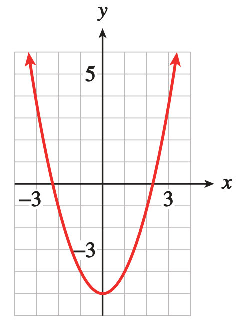
iii. Opens downward, narrower than the basic parabola.

iv. Opens downward, shifted up 3 units.

Checkpoint 3.3.7 Practice 2.

a. Find an equation for the parabola shown at right.

b. Give the x- and y-intercepts of the graph.



3.3.4 The Graph of $y = ax^2 + bx$

How does the linear term, bx, affect the graph?

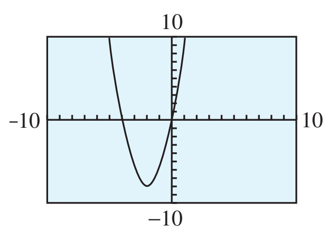
Example 3.3.8 Describe the graph of the equation

$$y = 2x^2 + 8x$$

Solution. The graph in the standard window is shown below. We see that the axis of symmetry for this parabola is not the y-axis: the graph is shifted to the left, compared to the basic parabola. We find the y-intercepts of the graph by setting y equal to zero: [TK]

$$\begin{aligned} 0 &= 2x^2 + 8x \\ &= 2x(x+4) \end{aligned}$$

The solutions of this equation are 0 and -4, so the x-intercepts are the points $(0, 0)$ and $(-4, 0)$.



□

[TK] To review solving quadratic equations, see Section 3.3.2 of the Toolkit.

We can find the vertex of the graph by using the symmetry of the parabola. The x-coordinate of the vertex lies exactly half-way between the x-intercepts, so we average their x-coordinates to find: [TK]

$$x = \frac{1}{2}[0 + (-4)] = -2$$

[TK] To review the average of two numbers, see Section 3.3.1 of the Toolkit.

To find the y-coordinate of the vertex, substitute $x = -2$ into the equation for the parabola:

$$\begin{aligned} y &= 2(\mathbf{-2})^2 + 8(\mathbf{-2}) \\ &= 8 - 16 = -8 \end{aligned}$$

Thus, the vertex is the point $(-2, -8)$.

Checkpoint 3.3.9 Practice 3.

a. Find the x-intercepts and the vertex of the parabola $y = 6x - x^2$.

b. Verify your answers by graphing the function in the window

$$\begin{array}{ll} \text{Xmin} = -9.4 & \text{Xmax} = 9.4 \\ \text{Ymin} = -10 & \text{Ymax} = 10 \end{array}$$

Checkpoint 3.3.10 QuickCheck 4. True or false.

a. The axis of symmetry of $y = ax^2 + bx$ is shifted horizontally, compared to the basic parabola.

b. The x-coordinate of the vertex is the average of the x-intercepts.

c. The y-coordinate of the vertex is the same as the y-intercept.

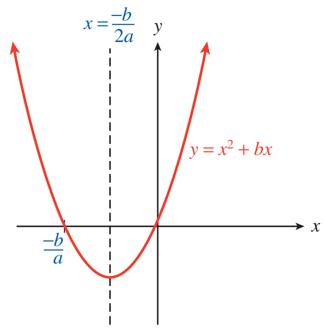
d. We can use extraction of roots to solve $ax^2 + bx = 0$.

3.3.5 A Formula for the Vertex

We can find a formula for the vertex of any parabola of the form

$$y = ax^2 + bx$$

First, find the x-intercepts of the graph by setting y equal to zero and solving for x.



$$\begin{aligned} 0 &= ax^2 + bx && \text{Factor.} \\ &= x(ax + b) && \text{Set each factor equal to zero.} \\ x = 0 \quad &\text{or} \quad ax + b = 0 && \text{Solve for } \boldsymbol{x}\text{.} \\ x = 0 \quad &\text{or} \quad x = \frac{-b}{a} \end{aligned}$$

The x-intercepts are the points $(0, 0)$ and $(\frac{-b}{a}, 0)$.

Next, we find the x-coordinate of the vertex by taking the average of the two x-intercepts.

$$x = \frac{1}{2}\left[0 + \left(\frac{-b}{a}\right)\right] = \frac{-b}{2a}$$

Now we have a formula for the x-coordinate of the vertex.

Vertex of a Parabola.

For the graph of $y = ax^2 + bx$, the x-coordinate of the vertex is

$$\boldsymbol{x_v = \frac{-b}{2a}}$$

We find the y-coordinate of the vertex by substituting its x-coordinate into the equation for the parabola. [TK]

[TK] For more examples of finding points on a parabola, see Section 3.3.3 of the Toolkit.

Example 3.3.11

a. Find the vertex of the graph of $f(x) = -1.8x^2 - 16.2x$.

b. Find the x-intercepts of the graph.

Solution.

a. The x-coordinate of the vertex is

$$x_v = \frac{-b}{2a} = \frac{-(-16.2)}{2(-1.8)} = -4.5$$

To find the y-coordinate of the vertex, evaluate $f(x)$ at $x = \mathbf{-4.5}$.

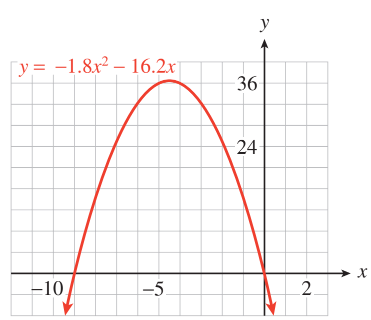
$$y_v = -1.8(\mathbf{-4.5})^2 - 16.2(\mathbf{-4.5}) = 36.45$$

The vertex is $(-4.5, 36.45)$.

b. To find the x-intercepts of the graph, set $y = 0$ and solve.

$$\begin{aligned} -1.8x^2 - 16.2x &= 0 && \text{Factor.} \\ -x(1.8x + 16.2) &= 0 && \text{Set each factor equal to zero.} \\ -x = 0 \qquad 1.8x + 16.2 &= 0 && \text{Solve each equation.} \\ x = 0 \qquad\qquad x &= -9 \end{aligned}$$

The x-intercepts of the graph are $(0, 0)$ and $(-9, 0)$. The graph is shown below.



□

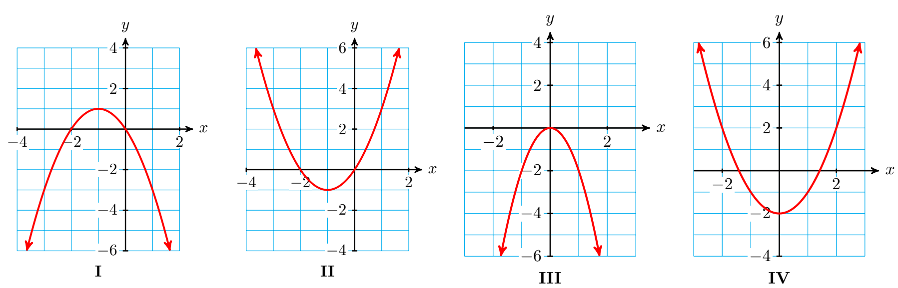
Checkpoint 3.3.12 QuickCheck 5. Match each equation with its graph.

a. $y = x^2 - 2$

b. $y = -x^2 - 2x$

c. $y = x^2 + 2x$

d. $y = -2x^2$



Checkpoint 3.3.13 Practice 4. Find the intercepts and the vertex of the graph of $y = 6.4x - 3.6x^2$.

3.3.6 Problem Set 3.3

Warm Up

Exercise Group. For problems 1 and 2, evaluate.

1. $2x^2 - 3x - 1$, for $x = -2$

2. $3 + 4x - 3x^2$, for $x = -3$

Exercise Group. For problems 3 and 4, which technique would you use to solve the equation, extracting roots or factoring? Then solve the equation.

3.

a. $3x^2 - 15 = 0$

b. $3x^2 - 15x = 0$

c. $(2x - 3)^2 = 9$

d. $2x^2 - 3x = 9$

4.

a. $20x = 2x^2$

b. $20 = 2x^2$

c. $4x^2 = 2 + 2x$

d. $4(x + 2)^2 - 1 = 0$

Exercise Group. For Problems 5 and 6, factor the right side of the formula.

5. $A = \dfrac{1}{2}bh + \dfrac{1}{2}h^2$

6. $S = 2\pi R^2 + 2\pi RH$

Exercise Group. For Problems 7 and 8, solve the equation.

7. $-9x = 81x^2$

8. $0 = -140x - 4x^2$

Skills Practice

9. Match each equation with its graph. In each equation, $k > 0$.

a. $y = x^2 + k$

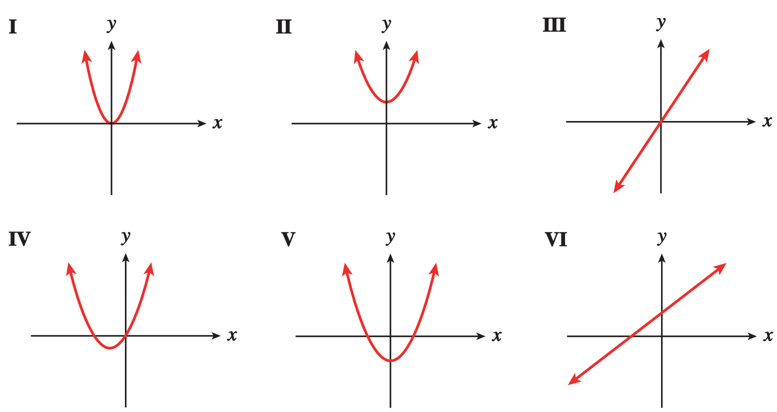
b. $y = x^2 + kx$

c. $y = kx^2$

d. $y = kx$

e. $y = x + k$

f. $y = x^2 - k$



10. Match each equation with its graph. In each equation, $k > 0$.

a. $y = -kx$

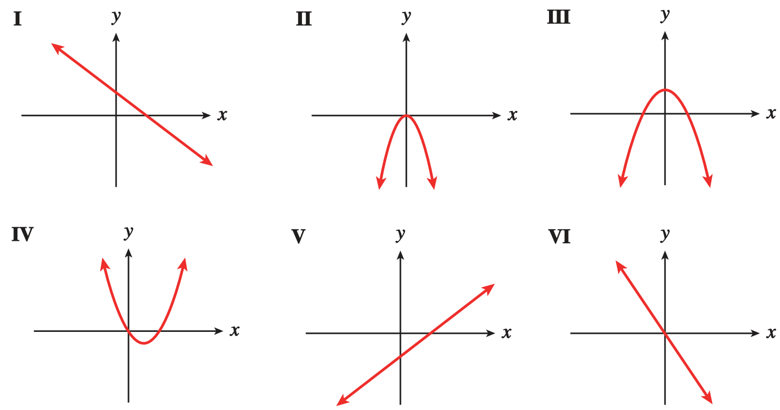
b. $y = -kx^2$

c. $y = k - x^2$

d. $y = x - k$

e. $y = k - x$

f. $y = x^2 - kx$



Exercise Group. For Problems 11 and 12:

a. Describe what each graph will look like compared to the basic parabola.

b. Sketch the graph by hand and label the coordinates of three points on the graph.

11.

i. $y = 2x^2$

ii. $y = \dfrac{1}{2}x^2$

iii. $y = -x^2$

12.

i. $y = x^2 + 2$

ii. $y = x^2 - 9$

iii. $y = 100 - x^2$

Exercise Group. For Problems 13–16, find the x-intercepts and the vertex of the graph. Then sketch the graph by hand.

13. $y = x^2 - 4x$

14. $y = x^2 + 2x$

15. $y = 3x^2 + 6x$

16. $y = -2x^2 + 5x$

17. $y = 40x - 2x^2$

18. $y = 144 - 12x^2$

Applications

19. Commercial fishermen rely on a steady supply of fish in their area. To avoid overfishing, they adjust their harvest to the size of the population. The formula

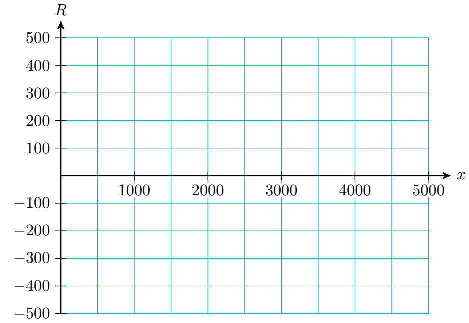
$$R = 0.4x - 0.001x^2$$

gives the annual rate of growth, in tons per year, of a fish population of biomass x tons.

a. Find the vertex of the graph. What does it tell us about the fish population?

b. Find the x-intercepts of the graph. What do they tell us about the fish population?

c. Sketch the graph for $0 \le x \le 5000$.



d. For what values of x does the fish population decrease rather than increase? Suggest a reason why the population might decrease.

20. After it lands on Earth, the distance the space shuttle travels is given by

$$d = vT + \frac{v^2}{2a}$$

where v is the shuttle's velocity in ft/sec at touchdown, T is the pilot's reaction time before the brakes are applied, and a is the shuttle's deceleration.

a. Suppose that for a particular landing, $T = 0.5$ seconds and $a = -12$ ft/sec^2. Write the equation for d in terms of v, using these values.

b. Find the coordinates of the vertex and the horizontal intercepts of the graph of your equation. Use these points to help you graph the equation.

c. Explain the meaning of the vertex and the intercepts, if any, in this context. As v increases, what happens to d?

d. The runway at Edwards Air Force base is 15,000 feet long. Graph your equation again in an appropriate (larger) window, and use it to estimate the answer to the question: What is the maximum velocity the shuttle can have at touchdown and still stop on the runway?

Exercise Group. For problems 21–24, graph the set of equations in the standard window on your calculator. Describe how the **parameters** (constants) a,b, and c in each equation transform the graph, compared to the basic parabola.

21.

a. $y = x^2$

b. $y = 3x^2$

c. $y = \dfrac{1}{4}x^2$

d. $y = -2x^2$

22.

a. $y = x^2$

b. $y = x^2 + 1$

c. $y = x^2 + 3$

d. $y = x^2 - 6$

23.

a. $y = x^2 - 4x$

b. $y = x^2 + 4x$

c. $y = 4x - x^2$

d. $y = -x^2 - 4x$

24.

a. $y = x^2 - 4x$

b. $y = 2x^2 - 8x$

c. $y = \frac{1}{2}x^2 - 2x$

d. $y = -x^2 + 4x$

3.4 Completing the Square

Not every quadratic equation can be solved easily by factoring or by extraction of roots. If the solutions are not integers, the expression $x^2 + bx + c$ may be very difficult to factor. For example, the equation $x^2 + 3x - 1 = 0$ cannot be solved easily by either of these methods. In this section we learn a method that will solve *any* quadratic equation.

3.4.1 Squares of Binomials

We can use extraction of roots to solve equations of the form

$$a(x + p)^2 + q = 0$$

where the left side of the equation includes the square of a binomial, or a **perfect square**. It turns out that we can write any quadratic equation in this form.

Study the following squares of binomials.

Square of binomial $(x + p)^2$	p	$2p$	p^2
1. $(x + \mathbf{5})^2 = x^2 + 10x + 25$	$\mathbf{5}$	$2(\mathbf{5}) = 10$	$\mathbf{5}^2 = 25$
2. $(x \mathbf{- 3})^2 = x^2 - 6x + 9$	$\mathbf{-3}$	$2(\mathbf{-3}) = -6$	$(\mathbf{-3})^2 = 9$
3. $(x \mathbf{- 12})^2 = x^2 - 24x + 144$	$\mathbf{-12}$	$2(\mathbf{-12}) = -24$	$(\mathbf{-12})^2 = 144$

In each case, the square of the binomial is a **quadratic trinomial**,

$$(x + p)^2 = x^2 + 2px + p^2$$

The coefficient of the linear term, $2p$, is twice the constant in the binomial, and the constant term of the trinomial, p^2, is its square. [TK]

[TK] For more examples of squares of binomials, see Section 3.4.1 of the Toolkit.

Checkpoint 3.4.1 QuickCheck 1. What is the linear term of $(x + 6)^2$?

We would like to reverse the process and write a quadratic expression as the square of a binomial. For example, what constant term can we add to

$$x^2 - 16x$$

to produce a perfect square trinomial? Compare the expression to the formula above:

$$\begin{aligned} x^2 + \mathbf{2p}x + p^2 &= (x + p)^2 \\ x^2 - \mathbf{16}x + \ ? \ &= (x + \ ? \ )^2 \end{aligned}$$

We see that $2p = -16$, so

$$p = \frac{1}{2}(-16) = \mathbf{-8} \quad \text{and} \quad p^2 = (-8)^2 = \mathbf{64}$$

We substitute these values for p^2 and p into the equation to find

$$x^2 - 16x + \mathbf{64} = (x \mathbf{-8})^2$$

You can check that in the resulting trinomial, the constant term is equal to *the square of one-half the coefficient of* x. In other words, we can find the constant term by taking one-half the coefficient of x and then squaring the result. Adding a constant term obtained in this way is called **completing the square**.

Example 3.4.2 Complete the square by adding an appropriate constant; write the result as the square of a binomial.

a. $x^2 - 12x + \boxed{\quad\quad}$

b. $x^2 + 5x + \boxed{\quad\quad}$

Solution.

a. One-half of -12 is -6, so the constant term is $(-6)^2$, or 36. We add 36 to obtain

$$x^2 - 12x + \mathbf{36} = (x-6)^2$$

$$\boldsymbol{p = \frac{1}{2}(-12) = -6}$$

$$\boldsymbol{p^2 = (-6)^2 = 36}$$

b. One-half of 5 is $\frac{5}{2}$, so the constant term is $\left(\frac{5}{2}\right)^2$, or $\frac{25}{4}$. [TK]

We add $\frac{25}{4}$ to obtain

$$x^2 + 5x + \mathbf{\frac{25}{4}} = \left(x + \frac{5}{2}\right)^2$$

$$\boldsymbol{p = \frac{1}{2}(5) = \frac{5}{2}}$$

$$\boldsymbol{p^2 = (\tfrac{5}{2})^2 = \tfrac{25}{4}}$$

□

[TK] To review multiplying and adding fractions, see Section 3.4.2 and Section 3.4.3 of the Toolkit.

Checkpoint 3.4.3 QuickCheck 2. True or False.

a. Every quadratic equation can be solved by factoring.

b. Every expression of the form $x^2 + bx$ can be turned into a perfect square by adding an appropriate constant.

c. The coefficient of the linear term in the expansion of $(x+p)^2$ is twice the constant term in the binomial.

d. To complete the square means to square the expression.

Checkpoint 3.4.4 Practice 1. Complete the square by adding an appropriate constant; write the result as the square of a binomial.

a. $x^2 - 18x + \boxed{\quad\quad} = (x + \boxed{\quad\quad})^2$

b. $x^2 + 9x + \boxed{} = (x + \boxed{})^2$

Hint:

a. $p = \frac{1}{2}(-18) =$ ________ , $p^2 =$ _________

b. $p = \frac{1}{2}(9) =$ ________ , $p^2 =$ _________

3.4.2 Solving Quadratic Equations by Completing the Square

Now we will use completing the square to solve quadratic equations. First, we will solve equations in which the coefficient of the squared term is 1. Consider the equation

$$x^2 - 6x - 7 = 0$$

Step 1 To begin, we move the constant term to the other side of the equation, to get

$$x^2 - 6x \quad \boxed{} = 7$$

Step 2 Next, we complete the square on the left. Because

$$p = \frac{1}{2}(-6) = -3 \quad \text{and} \quad p^2 = (-3)^2 = 9$$

we add 9 to *both* sides of our equation to get

$$x^2 - 6x \mathbf{+ 9} = 7 \mathbf{+ 9}$$

Step 3 The left side of the equation is now the square of a binomial, namely $(x - 3)^2$. We write the left side in its square form and simplify the right side, which gives us

$$(x - 3)^2 = 16$$

Step 4 We can now use extraction of roots to find the solutions. [TK] Taking square roots of both sides, we get

$$\begin{aligned} x - 3 &= 4 & \text{or} \quad x - 3 &= -4 & &\text{Solve each equation.} \\ x &= 7 & \text{or} \quad x &= -1 & & \end{aligned}$$

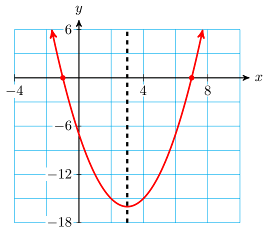
The solutions are 7 and -1.

[TK] To review extracting roots, see Section 3.1.4.

In Step 3, you can check that this equation is equivalent to the original one; if you expand the left side and collect like terms, you will return to the original equation:

$$\begin{aligned} (x - 3)^2 &= 16 & &\text{Expand the square.} \\ x^2 - 6x + 9 &= 16 & &\text{Subtract 16 fron both sides.} \\ x^2 - 6c - 7 &= 0 & & \end{aligned}$$

Note 3.4.5



The graph of $y = x^2 - 6x - 7$ is shown at left. Notice that the x-intercepts of the graph are $x = 7$ and $x = -1$, and the parabola is symmetric about the vertical line halfway between the intercepts, at $x = 3$.

We can also solve $x^2 - 6x - 7 = 0$ by factoring instead of completing the square. Of course, we get the same solutions by either method. In the next Example, we solve an equation that cannot be solved by factoring.

Example 3.4.6 Solve $x^2 - 4x - 3 = 0$ by completing the square.

Solution.

1. We write the equation with the constant term on the right side.

$$x^2 - 4x \boxed{} = 3$$

2. We complete the square on the left side. The coefficient of x is -4, so

$$p = \frac{1}{2}(-4) = -2 \quad \text{and} \quad p^2 = (-2)^2 = 4$$

   We add 4 to both sides of our equation:

$$x^2 - 4x + 4 = 3 + 4$$

3. We write the left side as the square of a binomial, and combine terms on the right side:

$$(x - 2)^2 = 7$$

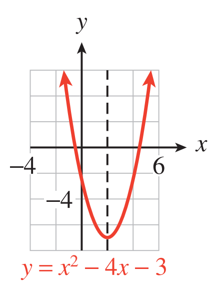
4. Finally, we use extraction of roots to obtain

$$\begin{aligned} x - 2 &= \sqrt{7} \qquad \text{or} \qquad x - 2 = -\sqrt{7} \qquad \text{Solve each equation.} \\ x &= 2 + \sqrt{7} \qquad \text{or} \qquad x = 2 - \sqrt{7} \end{aligned}$$

   These are the exact values of the solutions. We use a calculator to find decimal approximations for each solution:

$$2 + \sqrt{7} \approx 4.646 \quad \text{and} \quad 2 - \sqrt{7} \approx -0.646$$

   These values are the x-intercepts of the graph of $y = x^2 - 4x - 3$, as shown below.



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Checkpoint 3.4.7 QuickCheck 3. Put the steps for completing the square in the correct order:

a. Add to both sides of the equation.

b. Use extraction of roots.

c. Write the left side as a perfect square.

d. Write the equation with the constant term on the right side.

Checkpoint 3.4.8 Practice 2.

a. Follow the steps to solve by completing the square: $x^2 - 1 = 3x$.

 1. Write the equation with the constant on the right.
 2. Complete the square on the left:
 $p = \frac{1}{2}(-3) =$ ________ , ~~~$p^2 =$ ________
 Add p^2 to both sides.
 3. Write the left side as a perfect square; simplify the right side.
 4. Solve by extracting roots.

b. Find approximations to two decimal places for the solutions.

c. Graph the parabola $y = x^2 - 3x - 1$ in the window

$$\begin{aligned} &\text{Xmin} = -4.7 \quad &&\text{Xmax} = 4.7 \\ &\text{Ymin} = -5 &&\text{Ymax} = 5 \end{aligned}$$

3.4.3 The General Case

Our method for completing the square works only if the coefficient of x^2 is 1. If we want to solve a quadratic equation whose lead coefficient is not 1, we first divide each term of the equation by the lead coefficient.

Example 3.4.9 Solve $2x^2 - 6x - 5 = 0$.

Solution.

1. Because the coefficient of x^2 is 2, we must divide each term of the equation by 2.

$$x^2 - 3x - \frac{5}{2} = 0$$

 Now we proceed as before. Rewrite the equation with the constant on the right side.

$$x^2 - 3x \boxed{} = \frac{5}{2}$$

2. Complete the square:

$$p = \frac{1}{2}(-3) = \frac{-3}{2} \quad \text{and} \quad p^2 = \left(\frac{-3}{2}\right)^2 = \frac{9}{4}$$

 Add $\frac{9}{4}$ to both sides of our equation:

$$x^2 - 3x + \frac{9}{4} = \frac{5}{2} + \frac{9}{4}$$

3. Rewrite the left side as the square of a binomial and simplify the right side to get
$$\left(x-\frac{3}{2}\right)^2=\frac{19}{4}$$
4. Finally, extract roots and solve each equation for x.
$$x-\frac{3}{2}=\sqrt{\frac{19}{4}} \quad \text{or} \quad x-\frac{3}{2}=-\sqrt{\frac{19}{4}}$$
The solutions are $\frac{3}{2}+\sqrt{\frac{19}{4}}$ and $\frac{3}{2}-\sqrt{\frac{19}{4}}$.

Using a calculator, we can find decimal approximations for the solutions: 3.679 and -0.679.

□

Caution 3.4.10 In Example 3.4.9, it is essential that we first divide each term of the equation by 2, the coefficient of x^2. The following attempt at a solution is *incorrect*.

$$\begin{aligned} 2x^2-6x &= 5\\ 2x^2-6x+9 &= 5+9\\ (2x-3)^2 &= 14 \quad \rightarrow \quad \text{Incorrect!} \end{aligned}$$

You can check that $(2x-3)^2$ is not equal to $2x^2-6x+9$. We have not written the left side of the equation as a perfect square, so the solutions we obtain by extracting roots will not be correct.

Checkpoint 3.4.11 Practice 3.

a. Follow the steps to solve by completing the square:
$$-4x^2-36x-65=0.$$
   1. Divide each term by -4. Write the equation with the constant on the right.
   2. Complete the square on the left:
      $p=\frac{1}{2}(9)=$ ________ , $p^2=$ ________
      Add p^2 to both sides.
   3. Write the left side as a perfect square; simplify the right side.
   4. Solve by extracting roots.

b. Graph $y=-4x^2-36x-65$ in the window

$$\begin{aligned} \text{Xmin} &= -9.4 & \text{Xmax} &= 0\\ \text{Ymin} &= -10 & \text{Ymax} &= 20 \end{aligned}$$

Here is a summary of the steps for solving quadratic equations by completing the square.

To Solve a Quadratic Equation by Completing the Square.

1. a. Write the equation in standard form.
   b. Divide both sides of the equation by the coefficient of the quadratic term, and subtract the constant term from both sides.
2. Complete the square on the left side:
   a. Multiply the coefficient of the first-degree term by one-half, then square the result.
   b. Add the value obtained in (a) to both sides of the equation.
3. Write the left side of the equation as the square of a binomial. Simplify the right side.
4. Use extraction of roots to finish the solution.

3.4.4 Problem Set 3.4

Warm Up

Exercise Group.

1. Solve by factoring

$$2x^2 + 3x - 20 = 0$$

2. Solve by extraction of roots

$$5(2x + 3)^2 = 80$$

3. Expand each square.

a. $(x + 5)^2 =$

b. $(x - 6)^2 =$

c. $(x - 12)^2 =$

d. $(x + 15)^2 =$

Skills Practice

4. Add a term to make a square of a binomial.

a. $x^2 - 14x + \boxed{}$

b. $x^2 + 16x + \boxed{}$

c. $x^2 + \boxed{} + 36$

d. $x^2 - \boxed{} + 9$

Exercise Group. For Problems 5 and 6, which of the expressions are squares of binomials?

5.

a. $x^2 + 4x + 16$

b. $x^2 - 5x + \dfrac{25}{4}$

c. $x^2 + 16x + 169$

6.

a. $x^2 - 6x + 12$

b. $x^2 + 9x + 81$

c. $x^2 - \dfrac{3}{2}x + \dfrac{9}{4}$

Exercise Group. For Problems 7–10, solve by completing the square.

7. $x^2 + 9x + 20 = 0$

8. $x^2 = 3 - 3x$

9. $x^2 + 5x = 5$

10. $3x^2 + 12x + 2 = 0$

11. $2x^2 - 4x - 3 = 0$

12. $3x^2 + x = 4$

Exercise Group. For Problems 13–16, choose the best method, then solve the equation

13. $x^2 - 3x = 40$

14. $(x - 3)^2 = 40$

15. $3x^2 + 5x = 12$

16. $3x^2 + 6x = 15$

Applications

Exercise Group. For Problems 17–20,

a. Use completing the square to find the x-intercepts of the graph.

b. Find the vertex of the graph

17. $y = 4x^2 - 2x - 3$

18. $y = 2x^2 - 3x - 5$

19. $y = 5x^2 + 8x - 4$

20. $y = 3x^2 - x - 4$

21. The diagonal of a rectangle is 20 inches. The width of the rectangle is 4 inches shorter than its length.

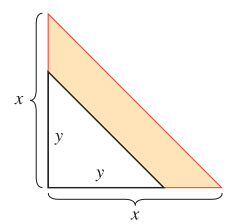
a. Write a quadratic equation about the length of the rectangle.

b. Solve your equation to find the dimensions of the rectangle.

22. The city park used 136 meters of fence to enclose its rectangular rock garden. The diagonal path across the middle of the garden is 52 meters long. What are the dimensions of the garden?

23.

The sail pictured is a right triangle of base and height x. It has a colored stripe along the hypotenuse and a white triangle of base and height y in the lower corner.



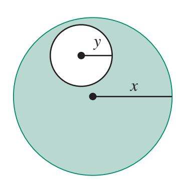
a. Write an expression for the area of the colored stripe.

b. Express the area of the stripe in factored form.

c. The sail is $7\frac{1}{2}$ feet high and the white triangle is $4\frac{1}{2}$ feet high. Use your answer to part (b) to calculate mentally the area of the stripe.

24.

An hors d'oeuvres tray has radius x, and the dip container has radius y.



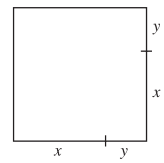
a. Write an expression for the area for the chips (the shaded region).

b. Express the area in factored form.

c. The tray has radius $8\frac{1}{2}$ inches and the space for the dip has radius

$2\frac{1}{2}$ inches. Use your answer to part (b) to calculate mentally the area for the chips. Express your answer as a multiple of π.

25.

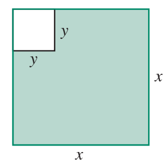


a. Write an expression for the area of the square.

b. Express the area as a polynomial.

c. Divide the square into four pieces whose areas are given by the terms of your answer to part (b).

26.



a. Write an expression for the area of the shaded region.

b. Express the area in factored form.

c. By making one cut in the shaded region, rearrange the pieces into a rectangle whose area is given by your answer to part (b).

Exercise Group. For Problems 27–30, solve by completing the square. Your answers will involve a, b, or c (or a combination of these).

27. $x^2 + 2x + c = 0$

28. $x^2 + bx - 4 = 0$

29. $x^2 + bx + c = 0$

30. $ax^2 - 4x + 9 = 0$

Exercise Group. For Problems 31–36, solve the formula for the indicated variable.

31. $V = \pi(r-3)^2 h$, for r

32. $E = \frac{1}{2}mv^2 + mgh$, for v

33. $V = 2(s^2 + t^2)w$, for t

34. $x^2 y - y^2 = 0$, for y

35. $(2y + 3x)^2 = 9$, for y

36. $4x^2 - 9y^2 = 36$, for y

3.5 Chapter 3 Summary and Review

3.5.1 Glossary

- quadratic equation
- parabola
- extraction of roots
- compound interest
- factor
- multiplicity
- vertex
- axis of symmetry
- quadratic trinomial
- complete the square

3.5.2 Key Concepts

1. A **quadratic** equation has the standard form $ax^2 + bx = c = 0$, where $a,\ b,$ and c are constants and a is not equal to zero.

2. The graph of a quadratic equation $y = ax^2 + bx = c$ is called a **parabola**. The **basic parabola** is the graph of $y = x^2$.

3. **Extraction of Roots.**

   To solve a quadratic equation of the form

   $$ax^2 + c = 0$$

   1. Isolate x on one side of the equation.
   2. Take the square root of each side.

4. Every quadratic equation has two solutions, which may be the same.

5. Solutions of a quadratic equation can be given as exact values or as decimal approximations.

6. **Formulas for Volume and Surface Area.**

   1. Sphere $\quad V = \dfrac{4}{3}\pi r^3 \qquad S = 4\pi r^2$
   2. Cylinder $\quad V = \pi r^2 h \qquad S = 2\pi r^2 + 2\pi rh$
   3. Cone $\quad V = \dfrac{1}{3}\pi r^2 h \qquad S = \pi r^2 + \pi rs$
   4. Square Pyramid $\quad V = \dfrac{1}{3}s^2 h$

7. The formula for interest compounded annually is

   $$A = P(1 + r)^n$$

8. **Zero-Factor Principle**: The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

   $$ab = 0 \text{ if and only if } a = 0 \text{ or } b = 0 \text{ or both}$$

9. **To Solve a Quadratic Equation by Factoring.**

   1. Write the equation in standard form.
   2. Factor the left side of the equation.
   3. Apply the zero-factor principle: Set each factor equal to zero.
   4. Solve each equation. There are two solutions (which may be equal).

10. Each solution of a quadratic equation corresponds to a non-constant factor in the factored form.

11. The value of the constant a in the factored form of a quadratic equation does not affect the solutions.

12. The x-intercepts of the graph of $y = ax^2 + bx + c$ are the solutions of the equation $ax^2 + bx + c = 0$.

13. **The Graph of $y = ax^2$.**

    - The parabola opens upward if $a > 0$
    - The parabola opens downward if $a < 0$
    - The magnitude of a determines how wide or narrow the parabola is.
    - The vertex, the x-intercepts, and the y-intercept all coincide at the origin.

14. **The Graph of $y = x^2 + c$.**

    Compared to the graph of $y = x^2$, the graph of $y = x^2 + c$
    - is shifted upward by c units if $c > 0$
    - is shifted downward by c units if $c < 0$

15. For the graph of $y = ax^2 + bx + c$, the x-coordinate of the vertex is

$$x_v = \frac{-b}{2a}$$

16. The square of a binomial is a quadratic trinomial,

$$(x + p)^2 = x^2 + 2px + p^2$$

17. **To Solve a Quadratic Equation by Completing the Square.**

    1. a. Write the equation in standard form.
       b. Divide both sides of the equation by the coefficient of the quadratic term, and subtract the constant term from both sides.
    2. Complete the square on the left side:
       a. Multiply the coefficient of the first-degree term by one-half, then square the result.
       b. Add the value obtained in (a) to both sides of the equation.
    3. Write the left side of the equation as the square of a binomial. Simplify the right side.
    4. Use extraction of roots to finish the solution.

3.5.3 Chapter 3 Review Problems

Exercise Group. For Problems 1–6, solve by extraction of roots.

1. $x^2 + 7 = 13 - 2x^2$

2. $\dfrac{2x^2}{5} - 7 = 9$

3. $3(x+4)^2 = 60$

4. $(7x-1)^2 = 15$

5. $(2x-5)^2 = 9$

6. $(3.5 - 0.2x)^2 = 1.44$

Exercise Group. For Problems 7–10, solve by factoring.

7. $(w+1)(2w-3) = 3$

8. $6y = (y+1)^2 + 3$

9. $4x - (x+1)(x+2) = -8$

10. $3(x+2)^2 = 15 + 12x$

Exercise Group. For Problems 11 and 12, write a quadratic equation with integer coefficients and with the given solutions.

11. $\dfrac{-3}{4}$ and 8

12. $\dfrac{5}{3}$ and $\dfrac{5}{3}$

Exercise Group. For Problems 13 and 14, graph the equation using the ZDecimal setting. Locate the x-intercepts, and use them to write the quadratic expression in factored form.

13. $y = x^2 - 0.6x - 7.2$

14. $y = -x^2 + 0.7x + 2.6$

Exercise Group. For Problems 15–18,

a. Find the coordinates of the vertex and the intercepts.

b. Sketch the graph.

15. $y = \dfrac{1}{2}x^2$

16. $y = x^2 - 4$

17. $y = x^2 - 9x$

18. $y = -2x^2 - 4x$

Exercise Group. For Problems 19–22, solve by completing the square.

19. $x^2 - 4x - 6 = 0$

20. $x^2 + 3x = 3$

21. $2x^2 + 3 = 6x$

22. $3x^2 = 2x + 3$

Exercise Group. For Problems 23–26, solve the formula for the indicated variable.

23. $K = \dfrac{1}{2}mv^2$, for v

24. $a^2 + b^2 = c^2$, for b

25. $V = \dfrac{1}{3}s^2h$, for h

26. $A = P(1+r)^2$, for r

27. In a tennis tournament among n competitors, $\dfrac{n(n-1)}{2}$ matches must be played. If the organizers can schedule 36 matches, how many players should they invite?

28. The formula $S = \dfrac{n(n-1)}{2}$ gives the sum of the first positive integers. How many consecutive integers must be added to make a sum of 91?

29. Lewis invested \$2000 in an account that compounds interest annually. He made no deposits or withdrawals after that. Two years later he closed the account, withdrawing \$2464.20. What interest rate did Lewis earn?

30. Earl borrowed \$5500 from his uncle for 2 years with interest compounded annually. At the end of two years he owed his uncle \$6474.74. What was the interest rate on the loan?

31. The perimeter of an equilateral triangle is 36 inches. Find its altitude. (Hint: The altitude is the perpendicular bisector of the base.)

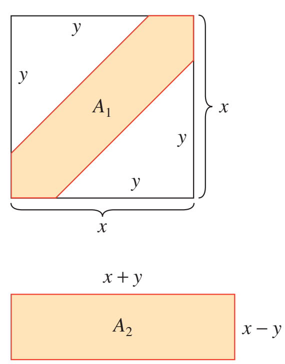
32. The base of an isosceles triangle is one inch shorter than the equal sides, and the altitude of the triangle is two inches shorter than the equal sides. What is the length of the equal sides?

33. A car traveling at 50 feet per second (about 34 miles per hour) can stop in 2.5 seconds after applying the brakes hard. The distance the car travels, in feet, t seconds after applying the brakes is $d = 50t - 10t^2$. How long does it take the car to travel 40 feet?

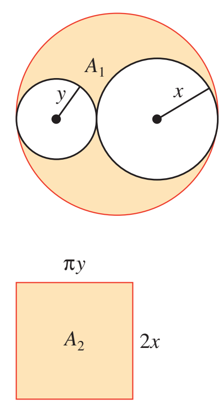
34. You have 300 feet of wire fence to mark off a rectangluar Christmas tree lot with a center-divider, using a brick wall as one side of the lot. If you would like to enclose a total area of 7500 square feet, what should be the dimensions of the lot?

Exercise Group. For Problems 35 and 36, show that the shaded areas are equal.

35.

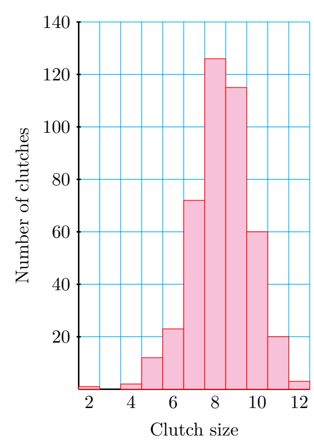


36.



Chapter 4

Applications of Quadratic Models

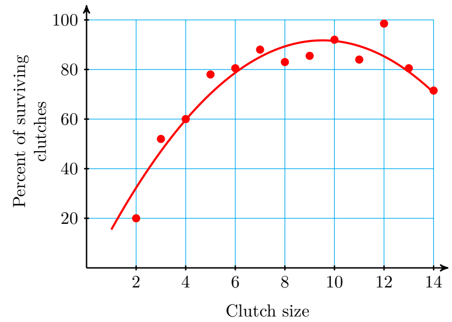


An important part of modeling is optimization, finding the best possible solution to a particular problem. For qudaratic models, this often involves finding the vertex of the graph where the maximum or minumum value of the output variable occurs.

For example, biologists conducted a four-year study of the nesting habits of a species of wrens, The bar graph shows the clutch size (the number of eggs) in 433 nests. Why is 8 or 9 eggs the most common clutch size? Does that number increase the birds' chances of survival?

The average weight of the young birds decrreases as the size of the brood increases, and the survival of individual nestlings is linked to their weight. Which clutch size produces the largest average number of survivors?

The graph shows the number of survivors for each clutch size in the study, along with the curve of best fit. The equation for the curve is $y = -0.0105x^2 + 0.2x - 0.035$. Looking at the graph, the optimum clutch size for maximizing the survival of the nestlings is about 9 eggs. How does this optimum clutch size compare with the most frequently observed clutch size in the study?



4.1 Quadratic Formula

4.1.1 A New Formula

Instead of completing the square every time we solve a new quadratic equation, we can complete the square on the general quadratic equation,

$$ax^2 + bx + c = 0, \qquad (a \neq 0)$$

and obtain a formula for the solutions of any quadratic equation. Here is the resulting formula.

The Quadratic Formula.

The solutions of the equation $\quad ax^2 + bx + c = 0, \quad (a \neq 0) \quad$ are

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Note 4.1.1 The formula gives us the solutions of a particular quadratic equation in terms of its coefficients, a, b, and c. We know that there should be two solutions, and the symbol $\pm$ is used to represent the two expressions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

in a single formula. [TK]

[TK] To review some properties of radicals, see Section 4.1.1 of the Toolkit.

Checkpoint 4.1.2 QuickCheck 1. Which of the following is a correct statement of the quadratic formula?

a. $x = -b \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$

b. $x = \dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$

c. $x = -b \pm \dfrac{b - \sqrt{4ac}}{2a}$

d. $x = -b \pm \dfrac{\sqrt{b^2 - ac}}{a}$

To solve a quadratic equation using the quadratic formula, all we have to do is substitute the coefficients a, b, and c into the formula.

Example 4.1.3 Solve $\quad 2x^2 + 1 = 4x$.

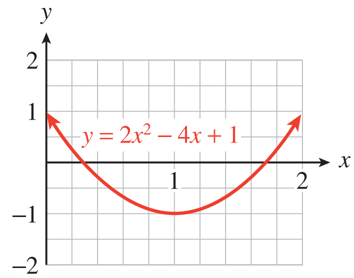
Solution. Write the equation in standard form as

$$2x^2 - 4x + 1 = 0$$

We substitute $\mathbf{2}$ for a, $\mathbf{-4}$ for b, and $\mathbf{1}$ for c into the quadratic formula, then simplify.

$$\begin{aligned} x &= \frac{-(\mathbf{-4}) \pm \sqrt{(\mathbf{-4})^2 - 4(\mathbf{2})(\mathbf{1})}}{2(\mathbf{2})} \\ &= \frac{4 \pm \sqrt{8}}{4} \end{aligned}$$

Remeber that we cannot cancel the 4's in this expression! [TK] Using a calculator, we find that the solutions are approximately 1.7 and 0.3. These values are the x-intercepts of the graph of $y = 2x^2 - 4x + 1$, as shown in the figure.



□

[TK] For more about the order of operations with radicals, see Section 4.1.2 of the Toolkit.

Checkpoint 4.1.4 Practice 1. Use the quadratic formula to solve $x^2 - 3x = 1$. [TK]

Hint: Write the equation in standard form.
Substitute $a = 1$, $b = -3$, $c = -1$ into the quadratic formula.
Simplify.

[TK] To see more examples of using the quadratic formula, see Section 4.1.3 of the Toolkit.

4.1.2 Applications

We have now seen four different algebraic methods for solving quadratic equations:

1. Factoring

2. Extraction of roots

3. Completing the square

4. Quadratic formula

Factoring and extraction of roots are relatively fast and simple, but they do not work on all quadratic equations. The quadratic formula will work on any quadratic equation.

Checkpoint 4.1.5 QuickCheck 2. Match each equation with the most efficient method of solution.

a. $6(x-4)^2 = 120$

b. $x^2 - 9x + 20 = 0$

c. $1.4x^2 - 6.2x + 2.5 = 0$

d. $x^2 - 20x = 44$

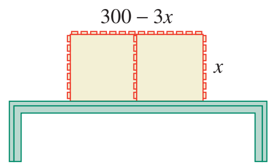
I. Factoring

II. Extraction of roots

III. Completing the square

IV. Quadratic formula

Example 4.1.6 The owners of a day-care center plan to enclose a divided play area against the back wall of their building, as shown below. They have 300 feet of picket fence and would like the total area of the playground to be 6000 square feet. Can they enclose the playground with the fence they have, and if so, what should the dimensions of the playground be?



Solution. Suppose the width of the play area is x feet. Because there are three sections of fence along the width of the play area, that leaves $300 - 3x$ feet of fence for its length.

The area of the play area should be 6000 square feet, so we have the equation

$$\begin{aligned} \text{width} \times \text{length} &= \text{Area} \\ x(300 - 3x) &= 6000 \end{aligned}$$

This is a quadratic equation. We use the distributive law and write the equation in standard form.

$$\begin{aligned} 3x^2 - 300x + 6000 &= 0 && \text{Divide each term by 3.} \\ x^2 - 100x + 2000 &= 0 \end{aligned}$$

The left side of this equation cannot be factored, so we use the quadratic formula with $a = 1$, $b = -100$, and $c = 2000$.

$$\begin{aligned} x &= \frac{-(-100) \pm \sqrt{(-100)^2 - 4(1)(2000)}}{2(1)} \\ &= \frac{100 \pm \sqrt{2000}}{2} \approx \frac{100 \pm 44.7}{2} \end{aligned}$$

When we evalute this last expression, we get two different positive values for the width of the play area, $x = 72.4$ or $x = 27.6$. Both values give solutions to the problem. To find the length of the playground in each case, we substitute x into $300 - 3x$.

- If the width of the play area is 72.4 feet, the length is $300 - 3(72.4)$, or 82.8 feet.
- If the width is 27.6 feet, the length is $300 - 3(27.6)$, or 217.2 feet.

The dimensions of the play area can be 72.4 feet by 82.8 feet, or it can be 27.6 feet by 217.2 feet. □

Checkpoint 4.1.7 Practice 2. The height of a baseball is given by the equation

$$h = -16t^2 + 64t + 4$$

where t is the time in seconds. Find two times when the ball is at a height of 20 feet. Round your answers to two decimal places.

Hint: Step 1: Set $h = 20$, then write the equation in standard form.
Step 2: Divide each term by -16.
Step 3: Use the quadratic formula to solve.

4.1.3 Complex Numbers

Not all quadratic equations have solutions that are real numbers. For example, when we try to solve the equation $x^2 + 4 = 0$, , we find

$$\begin{aligned} x^2 &= -4 \\ x &= \pm\sqrt{-4} \end{aligned}$$

Although square roots of negative numbers such as $\sqrt{-4}$ are not real numbers, they occur frequently in mathematics and its applications. Mathematicians in the sixteenth century gave them the name **imaginary numbers**, which reflected the mistrust with which they were viewed at the time. Today, however, such numbers are well understood and are used routinely by scientists and engineers.

To help us work with imaginary numbers, we define a new number, i, the **imaginary unit**, whose square is -1.

Definition 4.1.8 Imaginary Unit.

$$\boldsymbol{i = \sqrt{-1}} \quad \text{or} \quad \boldsymbol{i^2 = -1}$$

◊

With this new number we define the principal square root of any negative real number as follows.

Imaginary Numbers.

If $a \geq 0$,

$$\boldsymbol{\sqrt{-a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}}$$

Thus, the square root of any negative real number can be written as the product of a real number and i. Every negative real number has two imaginary square roots. For example, the square roots of -9 are $3i$ and $-3i$. You can verify that

$$(3i)^2 = 9i^2 = 9(-1) = -9 \quad \text{and} \quad (-3i)^2 = (-3)^2 i^2 = 9(-1) = -9$$

Example 4.1.9

a.

$$\begin{aligned} \sqrt{-4} &= \sqrt{-1}\sqrt{4} \\ &= i\sqrt{4} = 2i \end{aligned}$$

b.

$$\begin{aligned} \sqrt{-3} &= \sqrt{-1}\sqrt{3} \\ &= i\sqrt{3} \end{aligned}$$

□

Checkpoint 4.1.10 Practice 3. Simplify.

a. $-\sqrt{-36}$

b. $(5i)^2$

The solutions of many quadratic equations involve imaginary numbers.

Example 4.1.11 Solve $2x^2 - x + 2 = 0$.

Solution. For this equation, $a = \mathbf{2}$, $b = \mathbf{-1}$, and $c = \mathbf{2}$. We substitute these values into the quadratic formula to obtain

$$x = \frac{-(\mathbf{-1}) \pm \sqrt{(\mathbf{-1})^2 - 4(\mathbf{2})(\mathbf{2})}}{2(\mathbf{2})} = \frac{1 \pm \sqrt{-15}}{4}$$

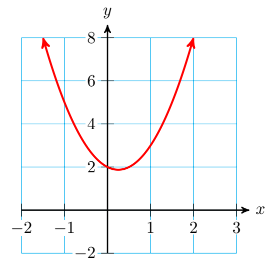
We write the solutions as

$$\frac{1 \pm i\sqrt{15}}{4} \qquad \text{or} \qquad \frac{1}{4} \pm \frac{\sqrt{15}}{4}i$$

Because the solutions are not real numbers, the graph of

$$y = 2x^2 - x + 2$$

has no x-intercepts, as shown below.



□

The sum of a real number and an imaginary number is called a **complex number**.

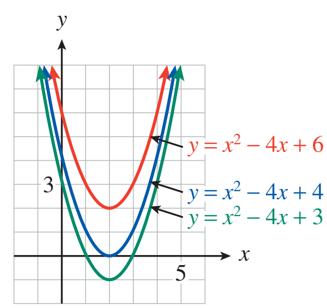
Checkpoint 4.1.12 Practice 4. Use extraction of roots to solve $(2x+1)^2 + 9 = 0$. Write your answers as complex numbers.

4.1.4 Number of x-Intercepts

The graph of the quadratic equation

$$y = ax^2 + bx + c$$

may have two, one, or no x-intercepts, according to the number of distinct real-valued solutions of the equation $ax^2 + bx + c = 0$. For example, consider the three graphs shown at right.



- The graph of
$$y = x^2 - 4x + 3$$
has two x-intercepts, because the equation
$$x^2 - 4x + 3 = 0$$
has two real-valued solutions, $x = 1$ and $x = 3$.
- The graph of
$$y = x^2 - 4x + 4$$
has only one x-intercept, because the equation
$$x^2 - 4x + 4 = 0$$
has only one (repeated) real-valued solution, $x = 2$.

- The graph of
$$y = x^2 - 4x + 6$$
has no x-intercepts, because the equation
$$x^2 - 4x + 6 = 0$$
has no real-valued solutions.

A closer look at the quadratic formula reveals useful information about the solutions of quadratic equations. The sign of the number under the radical determines how many solutions the equation has. For the three equations above, we calculate as follows:

$y = x^2 - 4x + 3$	$y = x^2 - 4x + 4$	$y = x^2 - 4 + 6$
two $\boldsymbol{x}$-intercepts	one $\boldsymbol{x}$-intercept	no $\boldsymbol{x}$-intercepts
$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2}$	$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(4)}}{2}$	$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(6)}}{2}$
$= \dfrac{4 \pm \sqrt{\mathbf{4}}}{2}$	$= \dfrac{4 \pm \sqrt{\mathbf{0}}}{2}$	$= \dfrac{4 \pm \sqrt{\mathbf{-12}}}{2}$
two real solutions	one repeated solution	no real solutions

Note 4.1.13 From these examples, we see that the solutions of a quadratic equation always occur in **conjugate pairs**,

$$\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

For example, if we know that one solution of a particular quadratic equation is $3 + \sqrt{2}$, the other solution must be $3 - \sqrt{2}$. If one solution is $5 - 3i$, the other solution must be $5 + 3i$.

The expression $b^2 - 4ac$, which appears under the radical in the quadratic formula, is called the **discriminant**, D, of the equation. The value of the discriminant determines the nature of the solutions of the equation. In particular, if the discriminant is negative, the equation has no real-valued solutions; the solutions are complex numbers.

The Discriminant.

The **discriminant** of a quadratic equation is

$$\boldsymbol{D = b^2 - 4ac}$$

1. If $D > 0$, there are two unequal real solutions.
2. If $D = 0$, there is one solution of multiplicity two.
3. If $D < 0$, there are two complex conjugate solutions.

Example 4.1.14 Use the discriminant to discover how many x-intercepts the graph has.

a. $y = x^2 - x - 3$

b. $y = 2x^2 + x + 1$

c. $y = x^2 - 6x + 9$

Solution. We set $y = 0$ and compute the discriminant for the resulting equation.

a. $D = b^2 - 4ac = (-1)^2 - 4(1)(-3) = 13 > 0$. The equation has two real, unequal solutions, and the graph has two x-intercepts.

b. $D = b^2 - 4ac = 1^2 - 4(2)(1) = -7 < 0$.

   The equation has no real solutions, so the graph has no x-intercepts.

c. $D = b^2 - 4ac = (-6)^2 - 4(1)(9) = 0$.

   The equation has one real solution of multiplicity two, and the graph has a single x-intercept.

□

Checkpoint 4.1.15 QuickCheck 3. True or False.

a. The discriminant is part of the quadratic formula.

b. We use the discriminant to calculate the solutions of a quadratic equation.

c. If the discriminant is negative, both x-intercepts of the graph are negative.

d. If a quadratic equation won't factor, its graph has no x-intercepts.

Checkpoint 4.1.16 Practice 5. Use the discriminant to discover how many x-intercepts the graph of each equation has.

a. $y = x^2 + 5x + 7$

b. $y = -\dfrac{1}{2}x^2 + 4x - 8$

4.1.5 Solving Formulas

Sometimes it is useful to solve a quadratic equation for one variable in terms of the others.

Example 4.1.17 Solve $x^2 - xy + y = 2$ for x in terms of y.

Solution. We first write the equation in standard form as a quadratic equation in the variable x.

$$x^2 - yx + (y - 2) = 0$$

Expressions in y are treated as constants with respect to x, so that $a = \mathbf{1}$, $b = \boldsymbol{-y}$, and $c = \boldsymbol{y - 2}$. We substitute these expressions into the quadratic formula.

$$\begin{aligned} x &= \frac{-(\boldsymbol{-y}) \pm \sqrt{(\boldsymbol{-y})^2 - 4(\mathbf{1})(\boldsymbol{y-2})}}{2(\mathbf{1})} \\ &= \frac{y \pm \sqrt{y^2 - 4y + 8}}{2} \end{aligned}$$

□

Checkpoint 4.1.18 Practice 6. Solve $2x^2 + kx + k^2 = 1$ for x in terms of k.

4.1.6 Problem Set 4.1

Warm Up

Exercise Group. For Problems 1 and 2, simplify according to the order of operations.

1.

a. $8-2\sqrt{36-16}$

b. $\dfrac{4+4\sqrt{4(16)}}{4}$

2.

a. $\dfrac{6+3\sqrt{36-4(3)}}{2(3)}$

b. $\dfrac{\sqrt{10^2+4(11)}-\sqrt{9+3(6)}}{12-\sqrt{1+5(16)}}$

3.

a. Write the formulas for the area and perimeter of a rectangle.

b. Find the area and perimeter of each rectangle:

i. 4 ft by 6 ft

ii. 3 ft by 8 ft

4.

a. Do all rectangles with the same area have the same perimeter?

b. Write expressions for the area and perimeter of each rectangle:

i. w in by $w-3$ in

ii. w cm by $160-2w$ cm

Skills Practice

Exercise Group. For Problems 5-10, use the quadratic formula to solve. Round your answers to three decimal places.

5. $0=x^2+x-1$

6. $3z^2=4z+1$

7. $0=x^2-\dfrac{5}{3}x+\dfrac{1}{3}$

8. $-5.2z^2+176z+1218=0$

9. $2x^2=7.5x-6.3$

10. $x=\dfrac{1}{2}x^2-\dfrac{3}{4}$

11.

a. Graph $y=6x^2-7x-3$ in the window

$$\begin{array}{ll} \text{Xmin} = -2 & \text{Ymin} = -6 \\ \text{Xmax} = 3 & \text{Ymax} = 4 \end{array}$$

b. Estimate the x-intercepts of the graph.

c. Use the quadratic formula to solve the equation

$$6x^2-7x-3=0$$

How do the solutions compare to your estimates in part (b)?

12.

a. Solve by the quadratic formula

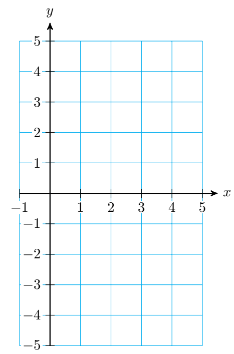
$$2x^2 - 1 = 5x$$

Find decimal approximations for your answers. Round to hundredths.

b. Graph $y = 2x^2 - 5x - 1$ in the window

$$\begin{array}{ll} \text{Xmin} = -1 & \text{Ymin} = -5 \\ \text{Xmax} = 5 & \text{Ymax} = 5 \end{array}$$

Sketch your graph on the grid.



c. How are your solutions from part (a) related to the graph?

13.

a. Graph the three equations

$$y = x^2 - 6x + 5$$
$$y = x^2 - 6x + 9$$
$$y = x^2 - 6x + 12$$

in the window

$$\begin{array}{ll} \text{Xmin} = -2 & \text{Ymin} = -5 \\ \text{Xmax} = 7.4 & \text{Ymax} = 15 \end{array}$$

Use the Trace to locat the x-intercepts of each graph.

b. Use the quadratic formula to find the solutions of each equation.

$$x^2 - 6x + 5 = 0$$
$$x^2 - 6x + 9 = 0$$
$$x^2 - 6x + 12 = 0$$

How are your answers related to the graphs in part (a)?

14. Use the discriminant to determine the nature of the solutions of each equation.

a. $3x^2 + 26 = 17x$

b. $4x^2 + 23x = 19$

c. $16x^2 - 712x + 7921 = 0$

d. $0.03x^2 = 0.05x - 0.12$

15. Here is one solution of a quadratic equation. Find the other solution, then write a quadratic equation in standard form that has those solutions.

a. $2+\sqrt{5}$

b. $4-3i$

16.

a. What is the sum of the two solutions of the quadratic equation $ax^2+bx+c=0$? (Hint: The two solutions are given by the quadratic formula.)

b. What is the product of the two solutions of the quadratic equation $ax^2+bx+c=0$? (Hint: Do not try to multiply the two solutions given by the quadratic formula! Think about the factored form of the equation.)

Exercise Group. For Problems 17-22, use the quadratic formula to solve the equation for the indicated variable.

17. $h=4t-16t^2$, for t

18. $A=2w^2+4lw$, for w

19. $s=vt-\dfrac{1}{2}at^2$, for t

20. $3x^2+xy+y^2=2$, for y

21. $S=\dfrac{n^2+n}{2}$, for n

22. $A=\pi r^2+\pi rs$, for r

Applications

23. A car traveling at s miles per hour on a wet road surface requires approximately d feet to stop, where d is given by the equation

$$d=\frac{s^2}{12}+\frac{s}{2}$$

a. Make a table showing the stopping distance, d, for speeds of 10, 10, . . ., 100 miles per hour. (Use the **Table** feature of your calculator.)

b. Graph the equation for d in terms of s. Use your table values to help you choose appropriate window settings.

c. Insurance investigators at the scene of an accident find skid marks 100 feet long leading up to the point of impact. Write and solve an equation to discover how fast the car was traveling when it put on the brakes. Verify your answer on your graph.

24. A high diver jumps from the 10-meter springboard. His height in meters above the water t seconds after leaving the board is given by

$$h=-4.9t^2+8t+10$$

a. Make a table of values showing the diver's altitude at 0.25-second intervals after he jumps from the springboard. (Use the Table feature of your calculator.)

b. Graph the equation. Use your table of values to choose appropriate window settings.

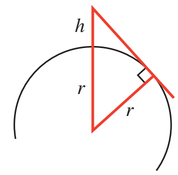
c. How long is it before the diver passes the board on the way down?

d. How long is it before the diver hits the water?

e. Find points on your graph that correspond to your answers to parts (c) and (d).

25.

When you look down from a height, say a tall building or a mountain peak, your line of sight is tangent to the Earth at the horizon, as shown in the figure.



a. The radius of the earth is 6370 kilometers. How far can you see from an airplane at an altitude of 10,000 meters? (You will need to use the Pythagorean theorem.)

b. How high would the airplane have to be in order for you to see a distance of 300 kilometers?

26. Refer to the figure in Problem 25.

a. Suppose you are standing on top of the Petronas Tower in Kuala Lumpur, 1483 feet high. How far can you see on a clear day? (The radius of the Earth is 3960 miles. Don't forget to convert the height of the Petronas Tower to miles.)

b. How tall a building should you stand on in order to see 100 miles?

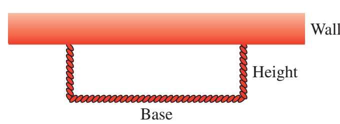
27. The volume of a large aquarium at the zoo is 2160 cubic feet. The tank is 10 feet wide, and its length is 6 feet less than twice its height.

a. Sketch the aquarium, and label its dimensions in terms of a variable.

b. Write an equation for the the variable.

c. Find the dimensions of the aquarium.

28. You have 72 feet of rope to enclose a rectangular display area against one wall of an exhibit hall. The area enclosed depends upon the dimensions of the rectangle you make. Because the wall makes one side of the rectangle, the length of the rope accounts for only three sides, as shown below.



a. Let h stand for the height of a rectangle, and write an algebraic expression for the base of the rectangle.

b. Write an expression for the area of the rectangle.

c. If you would like to enclose 640 square feet of display space, what should the dimensions of the rectangle be? (There are two possible solutions.)

d. What should the dimensions be if you would like to enclose exactly 600 square feet of space?

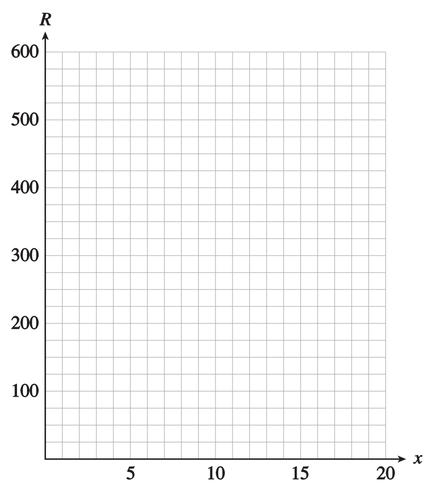
4.2 The Vertex

Investigation 4.2.1 Revenue from Theater Tickets. The local theater group sold tickets to its opening-night performance for $5 and drew an audience of 100 people. The next night they reduced the ticket price by $0.25 and 10 more people attended; that is, 110 people bought tickets at $4.75 apiece. In fact, for each $0.25 reduction in ticket price, 10 additional tickets can be sold.

1. Fill in the table

No. of price reductions	Price of ticket	No. of tickets sold	Total revenue
0	5.00	100	500.00
1	4.75	110	522.50
2			
3			
4			
5			
6			
8			
10			

2. On the grid below, plot *Total revenue* on the vertical axis versus *Number of price reductions* on the horizontal axis. Use the data from your table.



3. Let x represent the *number of price reductions*, as in the first column of your table. Write algebraic expressions in terms of x for

   The *price of a ticket* after x price reductions:

   $$\text{Price} =$$

   The *number of tickets* sold at that price:

   $$\text{Number} =$$

   The *total revenue* from ticket sales:

   $$\text{Revenue} =$$

4. Enter your expressions for the price of a ticket, the number of tickets sold, and the total revenue into the calculator as Y_1, Y_2, and Y_3. Use the Table feature to verify that your algebraic expressions agree with your table from part (1).

5. Use your calculator to graph your expression for total revenue in terms of x. Use your table to choose appropriate window settings that show the high point of the graph and both x-intercepts.

6. What is the maximum revenue possible from ticket sales? What price should the theater group charge for a ticket to generate that revenue? How many tickets will the group sell at that price?

4.2.1 Finding the Vertex

The vertex of a parabola $y = ax^2 + bx$ is an interesting point. It is the highest or lowest point on the parabola, and it "anchors" the parabola's position in the plane. As we saw in the Investigation, the vertex may represent the maximum or minimum possible value for a variable, so it has important implications for applications. Thus, it will be very useful to have a way to find the vertex easily.

In Section 3.3.5 we saw that the vertex of the graph of $y = ax^2 + bx$ has x-coordinate given by

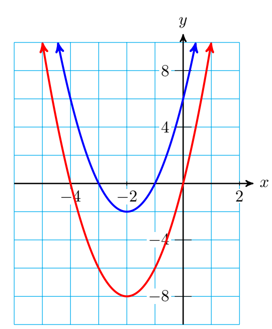
$$x_v = \frac{-b}{2a}$$

Now we'll see that the same formula holds for any parabola. Use your calculator to graph the two parabolas

$$y = 2x^2 + 8x$$
$$y = 2x^2 + 8x + 6$$

in the window

$$\text{Xmin} = -6 \quad \text{Ymin} = -6$$
$$\text{Xmax} = 4 \quad \text{Ymax} = 10$$



The graphs are shown at right. Now compare the two graphs. You should notice that:

- The second graph is identical to the first, except shifted upward by 6 units.
- For both graphs, the x-coordinate of the vertex is $x_v = -2$.

We see that the x-coordinate of the vertex is not affected by an upward shift. The formula for the x-coordinate of the vertex still holds.

Vertex of a Parabola.

For the graph of $y = ax^2 + bx + c = 0$, the x-coordinate of the vertex is

$$\boldsymbol{x_v = \frac{-b}{2a}}$$

Example 4.2.1 Find the vertex of the graph of $f(x) = -2x^2 + x + 1$.

Solution. For this equation, $a = -2$, $b = 1$, and $c = 1$. The x-coordinate of the vertex is given by

$$x_v = \frac{-b}{2a} = \frac{-1}{2(-2)} = \frac{1}{4}$$

To find the y-coordinate of the vertex, we substitute $x = \mathbf{\frac{1}{4}}$ into the equation. We can do this by hand to find

$$\begin{aligned} y_v &= -2\left(\mathbf{\frac{1}{4}}\right)^2 + \mathbf{\frac{1}{4}} + 1 \\ &= -2\left(\frac{1}{16}\right) + \frac{4}{16} + \frac{16}{16} = \frac{18}{16} = \frac{9}{8} \end{aligned}$$

So the coordinates of the vertex are $\left(\frac{1}{4}, \frac{9}{8}\right)$. Alternatively, we can use the calculator to evaluate $-2x^2 + x + 1$ for $x = 0.25$. We enter

$\boxed{\text{(-)}}$ 2 $\boxed{\times}$ 0.25 $\boxed{x\hat{}2}$ $\boxed{+}$ 0.25 $\boxed{+}$ 1

and press $\boxed{\text{ENTER}}$. The calculator returns the y-value 1.125. Thus, the vertex is the point $(0.25, 1.125)$, which is the decimal equivalent of $\left(\frac{1}{4}, \frac{9}{8}\right)$.

□

Checkpoint 4.2.2 Practice 1.

a. Find the vertex of the graph of $y = 3x^2 - 6x + 4$.

b. Decide whether the vertex is a maximum point or a minimum point of the graph.

   Hint: Does the parabola open up or down?

Once we know the vertex of a parabola, we can make a quick sketch of the graph.

Example 4.2.3 Sketch a graph of $y = x^2 + 3x + 1$

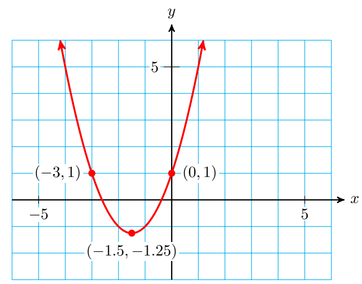
Solution. The vertex of the graph is given by

$$\begin{aligned} x_v &= \frac{-b}{2a} = \frac{-3}{2(1)} = -1.5 \\ y_v &= (-1.5)^2 + 3(-1.5) + 1 = -1.25 \end{aligned}$$

The vertex is the point $(-1.5, -1.25)$.

We set $x = 0$ to find the y-intercept, $(0, 1)$. Now, the axis of symmetry of the parabola is the vertical line that passes through its vertex. That line is $x = -1.5$, so the y-intercept lies 1.5 units to the right of the axis. There must be another point on the parabola with the same y-coordinate as the intercept but 1.5 units to the left of the axis. This point is $(-3, 1)$.

By plotting the vertex, the y-intercept, and its symmetric point, we can make a quick sketch of the parabola, as shown below.



If we need more accuracy in our graph, we can find and plot more points, including the x-intercepts. [TK] ☐

[TK] To review finding the x-intercepts of a parabola, see Section 3.2.2.

Checkpoint 4.2.4 Practice 2. Find the vertex and sketch a graph of $y = x^2 - 5x + 3$

4.2.2 Maximum or Minimum Values

Many quadratic models arise as the product of two variables, one of which increases while the other decreases. For example, in the Investigation for Section 3.3 we looked at the areas of different rectangles with the same perimeter. The area of a rectangle is the product of its length and its width, or $A = lw$. If we require that the rectangle have a certain perimeter, then as we increase its length, we must also decrease its width.

Another example is the formula for the revenue from sales of an item:

$$\text{Revenue} = (\text{price of one item}) \times (\text{number of items sold})$$

Usually, when the price of an item increases, the number of items sold decreases.

Finding the maximum or minimum value for a variable expression is a common problem in applications. For example, if you own a company that manufactures blue jeans, you might like to know how much to charge for your jeans in order to maximize your revenue. As you increase the price of the jeans, your revenue may increase for a while. But if you charge too much for the jeans, consumers will not buy as many pairs, and your revenue may actually start to decrease. Is there some optimum price you should charge for a pair of jeans in order to achieve the greatest revenue?

Example 4.2.5 Late Nite Blues finds that it can sell $600 - 15x$ pairs of jeans per week if it charges x dollars per pair. (Notice that as the price increases, the number of pairs of jeans sold decreases.)

a. Write an equation for the revenue as a function of the price of a pair of jeans.

b. Graph the function.

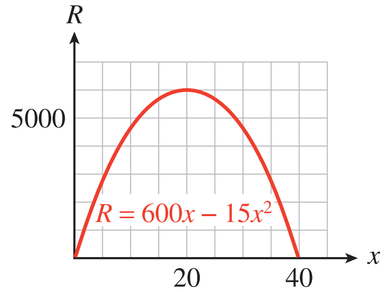
c. How much should Late Nite Blues charge for a pair of jeans in order to maximize its revenue?

Solution.

a. Using the formula for revenue stated above, we find

$$\begin{aligned}\text{Revenue} &= (\text{price of one item})(\text{number of items sold})\\ R &= x(600 - 15x)\\ R &= 600x - 15x^2\end{aligned}$$

b. We recognize the equation as quadratic, so the graph is a parabola. You can use your calculator to verify the graph below.



c. As you can see from the graph, the maximum value of R occurs at the vertex of the parabola. Thus,

$$x_v = \frac{-b}{2a} = \frac{-600}{2(-15)} = 20$$
$$y_v = 600(\mathbf{20}) - 15(\mathbf{20})^2 = 6000$$

The revenue takes on its maximum value when $x = 20$, and the maximum value is $R = 6000$. This means that Late Nite Blues should charge \$20 for a pair of jeans in order to maximize revenue at \$6000 a week.

□

Note 4.2.6 If the equation relating two variables is quadratic, then the maximum or minimum value is easy to find: It is the value at the vertex. If the parabola opens downward, as in Example 4.2.5, there is a maximum value at the vertex. If the parabola opens upward, there is a minimum value at the vertex.

Checkpoint 4.2.7 QuickCheck 1. To find the maximum or minimum value of a quadratic expression, we should:

a. Set $y = 0$ and solve for x.

b. Factor the expression.

c. Use the quadratic formula.

d. Find the vertex.

Checkpoint 4.2.8 Practice 3. The Metro Rail service sells $1200 - 80x$ tickets each day when it charges x dollars per ticket.

a. Write an equation for the revenue, R, as a function of the price of a ticket.

b. What ticket price will return the maximum revenue?

4.2.3 The Vertex Form for a Parabola

Because the vertex is such a useful tool, we introduce a new form for a quadratic equation. Consider the equation

$$y = 2(x-3)^2 - 8$$

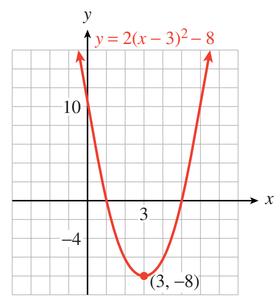
By expanding the squared expression [TK] and collecting like terms, we can rewrite the equation in standard form as

$$y = 2(x^2 - 6x + 9) - 8$$
$$y = 2x^2 - 12x + 10$$

The vertex of this parabola is

$$x_v = \frac{-(-12)}{2(2)} = 3$$
$$y_v = 2(3)^2 - 12(3) + 10 = -8$$

and its graph is shown at right.



[TK] To review multiplying binomials, see Section 4.2.2 of the Toolkit.

Now notice that the coordinates of the vertex, $(3, -8)$, are apparent in the original equation; we don't need to do any computation to find the vertex.

$$y = 2(x - \underset{x_v}{\mathbf{3}})^2 \quad \underset{y_v}{\mathbf{-8}}$$

This equation is an example of the **vertex form** for a quadratic function.

Vertex Form for a Quadratic Equation.

A quadratic equation $y = ax^2 + bx + c$, $a \neq 0$, can be written in the vertex form

$$\boldsymbol{y = a(x - x_v)^2 + y_v}$$

where the vertex of the graph is (x_v, y_v).

Example 4.2.9 Find the vertex of the graph of $y = -3(x-4)^2 + 6$. Is the vertex a maximum or a minimum point of the graph?

Solution. We compare the equation to the vertex form to see that the coordinates of the vertex are $(4, 6)$. For this equation, $a = -3 < 0$, so the parabola opens downward. The vertex is the maximum point of the graph. □

To understand why the vertex form works, substitute $x_v = \mathbf{4}$ into $y = -3(x-4)^2 + 6$ from Example 4.2.9 to find

$$y = -3(\mathbf{4} - 4)^2 + 6 = 6$$

which confirms that when $x = 4$, $y = 6$. Next, notice that if x is any number except 4, the expression $-3(x-4)^2$ is negative, so $y < 6$. Therefore, 6 is the maximum value for y on the graph, so $(4, 6)$ is the high point or vertex.

You can also rewrite $y = -3(x-4)^2 + 6$ in standard form and use the formula $x_v = \dfrac{-b}{2a}$ to confirm that the vertex is the point $(4, 6)$.

Checkpoint 4.2.10 QuickCheck 2. Which of the following is the vertex form for a parabola with vertex $(-2, 3)$?

a. $y = 4(x-2) + 3$

b. $y = 4(x-2)^2 + 3$

c. $y = 4(x+2)^2 + 3$

d. $y = 4(x-3)^2 + 2$

Checkpoint 4.2.11 Practice 4.

a. Find the vertex of the graph of $y = 5 - \dfrac{1}{2}(x+2)^2$.

b. Write the equation of the parabola in standard form.

Any quadratic equation in vertex form can be written in standard form by expanding, and any quadratic equation in standard form can be put into vertex form by completing the square. [TK]

[TK] To review completing the square, see Section 3.4.2.

Example 4.2.12 Write the equation $y = 3x^2 - 6x - 1$ in vertex form and find the vertex of its graph.

Solution. We factor the lead coefficient, 3, from the variable terms, leaving a space to complete the square.

$$y = 3(x^2 - 2x\boxed{}) - 1$$

Next, we complete the square inside parentheses. Take half the coefficient of x and square the result:

$$p = \frac{1}{2}(-2) = -1, \quad \text{and} \quad p^2 = (-1)^2 = 1$$

We must add 1 to complete the square. However, we are really adding 3(1) to the right side of the equation, so we must also subtract 3 to compensate:

$$y = 3(x^2 - 2x \mathbf{+ 1}) - 1 \mathbf{- 3}$$

The expression inside parentheses is now a perfect square, and the vertex form is

$$y = 3(x-1)^2 - 4$$

The vertex of the parabola is $(1, -4)$. □

Checkpoint 4.2.13 Practice 5. Write the equation $y = 2x^2 + 12x + 13$ in vertex form, and find the vertex of its graph.

Hint:

1. Factor 2 from the variable terms.
2. Complete the square inside parentheses.
3. Subtract $2p^2$ outside parentheses.
4. Write the vertex form.

4.2.4 Using the Vertex Form

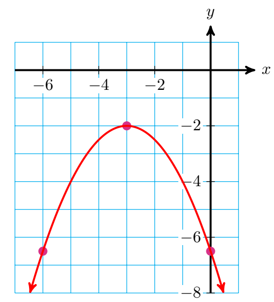
We can make a quick sketch of a parabola whose equation is given in vertex form.

Example 4.2.14 Graph the equation $y = \dfrac{-1}{2}(x+3)^2 - 2$

Solution. The vertex is the point $(-3, 2)$. We can find the y-intercept by setting $x = 0$.

$$y = \frac{-1}{2}(\mathbf{0} + 3)^2 - 2 = \frac{-9}{2} - 2 = -6\frac{1}{2}$$

The y-intercept is the point $\left(0, -6\frac{1}{2}\right)$. The axis of symmetry is the vertical line $x = -3$, and there is a symmetric point equidistant from the axis, namely $\left(-6, -6\frac{1}{2}\right)$. We plot these three points and sketch the parabola through them. [TK]



□

[TK] To see more examples of sketching from the vertex form, see Section 4.2.1 of the Toolkit.

Checkpoint 4.2.15 Practice 6. Find the vertex, the y-intercept, and the x-intercepts of $y = (x-2)^2 - 5$, and sketch its graph.

If we know the vertex of a parabola and one other point, we can use the vertex form to find its equation. [TK]

Example 4.2.16 When Andre practices free-throws at the park, the ball leaves his hands at a height of 7 feet, and reaches the vertex of its trajectory 10 feet away at a height of 11 feet.

a. Find a quadratic equation for the ball's trajectory.

b. Do you think Andre's free-throw would score on a regulation basketball court, where the hoop is 15 feet from the shooter and 10 feet high?

Solution.

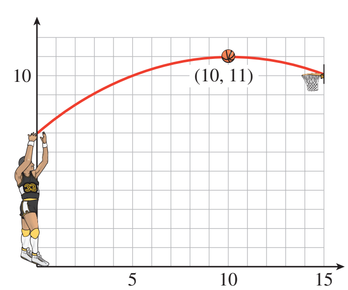
a. If Andre's feet are at the origin, then the vertex of the ball's trajectory is the point $(\mathbf{10}, \mathbf{11})$, and its y-intercept is $(0, 7)$. We start with the vertex form for a parabola.

$$y = a(x - x_v)^2 + y_v$$
$$y = a(x - \mathbf{10})^2 + \mathbf{11}$$

We use the point $(0, 7)$ to find the value of a.

$$\mathbf{7} = a(\mathbf{0} - 10)^2 + 11$$
$$7 = 100a + 11$$
$$a = -0.04$$

The equation of the trajectory is $y = -0.04(x-10)^2 + 11$.



b. We'd like to know if the point $(15, 10)$ is on the trajectory of Andre's free-throw. We substitute $x = \mathbf{15}$ into the equation.

$$y = -0.04(\mathbf{15} - 10)^2 + 11$$

$$y = -0.04(25) + 11 = 10$$

From our computations, we see that the point $(15, 10)$ is indeed on the trajectory. However, because Andre's shot will probably hit the backboard just where the hoop attaches and bounce off, so it is unlikely that his shot will score.

□

[TK] For more examples of finding the vertex form, see Section 4.2.3 of the Toolkit.

Checkpoint 4.2.17 QuickCheck 3. Why do we need to know a second point besides the vertex to find the equation of a parabola?

a. We need two points to find the equation of a line.

b. To find the value of a.

c. We must know one of the x-intercepts.

d. We must know the y-intercept.

Checkpoint 4.2.18 Practice 6. A parabola has its vertex at $(16, 80)$ and one of its x-intercepts at $(40, 0)$. Find an equation for the parabola.

4.2.5 Problem Set 4.2

Warm Up

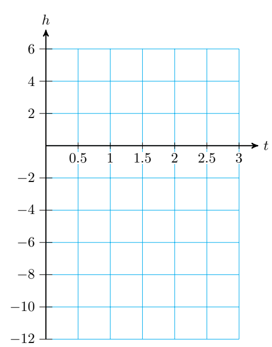
1. Francine throws a wrench into the air from the bottom of a trench 12 feet deep. Its height t seconds later is given in feet by

$$h = -12 + 32t - 16t^2$$

a. Complete the table of values.

t	0	0.25	0.5	0.75	1	0.25	1.5
h							

b. Graph the equation.



c. Find the vertex of the graph.

d. What does the h-coordinate of the vertex tell us about the wrench?

e. What does the t-coordinate of the vertex tell us about the wrench?

2.

a. Write an equation for a parabola that has x-intercepts at $(2, 0)$ and $(-3, 0)$.

b. Write an equation for another parabola that has the same x-intercepts.

3.

a. Write an equation for a parabola that opens upward with x-intercepts $(-1, 0)$ and $(4, 0)$.

b. Write an equation for a parabola that opens downward with x-intercepts $(-1, 0)$ and $(4, 0)$.

4.

a. The math club has \$68 in the treasury. Annual dues are \$4. If x more students join, write an expression for the amount of money in the treasury.

b. The monthly dues for Rafael's condo association are \$120. However, for each new tenant who moves in, the dues will be reduced by \$5. Write an expression for the dues if x new tenants move in.

Skills Practice

Exercise Group. For Problems 5 and 6:

a. Find the vertex of the parabola.

b. Sketch the graph.

5. $y = x^2 + 4x + 7$

6. $y = x^2 - 6x + 10$

Exercise Group. For Problems 7 and 8, sketch a graph of the parabola. What is the vertex of each graph?

7.

a. $y = (x - 3)^2$

b. $y = -(x - 3)^2$

c. $y = -(x - 3)^2 + 4$

8.

a. $y = (x + 4)^2$

b. $y = \dfrac{1}{2}(x + 4)^2$

c. $y = 3 + \dfrac{1}{2}(x + 4)^2$

Exercise Group. For Problems 9 and 10:

a. Find the vertex of the parabola.

b. Sketch the graph.

c. Write the equation in standard form.

9. $y = 2(x - 3)^2 + 4$

10. $y = -\dfrac{1}{2}(x + 4)^2 - 3$

11.

a. Write an equation for a parabola whose vertex is the point $(-2, 6)$.

(Many answers are possible.)

b. Find the value of a if the y-intercept of the parabola in part (a) is 18.

12.

a. Write an equation for a parabola with vertex at $(0,-3)$ and one x-intercept at $(2,0)$.

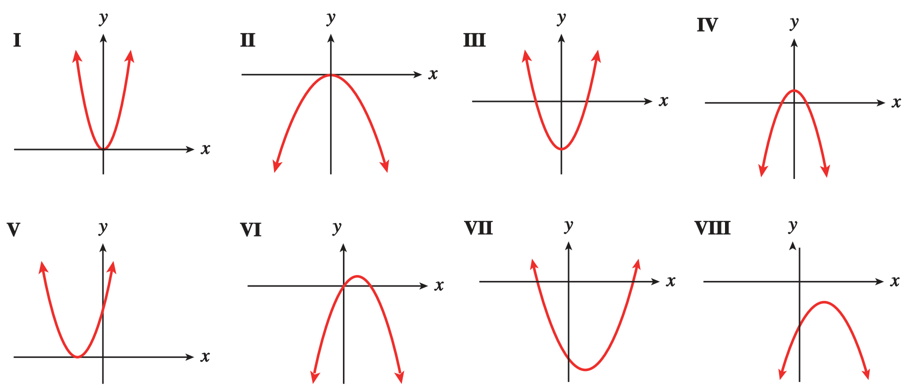
b. Write an equation for a parabola with vertex at $(0,-3)$ and no x-intercepts.

Exercise Group. For Problems 13 and 14, write the equation in the form $y=a(x-p)^2+q$ by completing the square.

13. $y=3x^2+6x-2$

14. $y=-2x^2-8x+3$

Exercise Group. For Problems 15 and 16, match each equation with one of the eight graphs shown.



15.

a. $y=1-x^2$

b. $y=(x+2)^2$

c. $y=2x^2$

d. $y=(x-4)(x+2)$

16.

a. $y=-2-(x-2)^2$

b. $y=x-x^2$

c. $y=x^2-4$

d. $y=-0.5x^2$

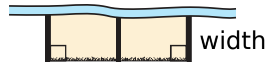
Applications

17. Gavin has rented space for a booth at the county fair. As part of his display, he wants to rope off a rectangular area with 80 yards of rope.

a. Let w represent the width of the roped-off rectangle, and write an expression for its length. Then write an expression in terms of w for the area A of the roped-off space.

b. What is the largest area that Gavin can rope off? What will the dimensions of the rectangle be?

18. A breeder of horses wants to fence two adjacent rectangular grazing areas along a river with 600 meters of fence.



a. Write an expression for the total area, A, of the grazing land in terms of the width, w, of the rectangles.

b. What is the largest area she can enclose?

19. The owner of a motel has 60 rooms to rent. She finds that if she charges \$20 per room per night, all the rooms will be rented. For every \$2 that she increases the price of a room, three rooms will stand vacant.

a. Complete the table. The first two rows are filled in for you.

No. of price increases	Price of a room	No. of rooms rented	Total revenue
0	20	60	1200
1	22	57	1254
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			

b. Let stand for the number of \$2 price increases the owner makes. Write algebraic expressions for the price of a room, the number of rooms that will be rented, and the total revenue earned at that price.

c. Use your calculator to make a table of values for your algebraic expressions. Let Y_1 stand for the price of a room, Y_2 for the number of rooms rented, and Y_3 for the total revenue. Verify the values you calculated in part (a).

d. Use your table to find a value of x that causes the total revenue to be zero.

e. Use your graphing calculator to graph your formula for total revenue.

f. What is the lowest price that the owner can charge for a room if she wants her revenue to exceed \$1296 per night? What is the highest price she can charge to obtain this revenue?

g. What is the maximum revenue the owner can earn in one night? How much should she charge for a room to maximize her revenue? How many rooms will she rent at that price?

20. A travel agent offers a group rate of \$2400 per person for a week in London if 16 people sign up for the tour. For each additional person who signs up, the price per person is reduced by \$100.

a. Let x represent the number of additional people who sign up. Write expressions for the number of people signed up, the price per person, and the total revenue.

b. How many people must sign up for the tour in order for the travel agent to maximize her revenue?

21. In skeet shooting, the clay pigeon is launched from a height of 4 feet and reaches a maximum height of 164 feet at a distance of 80 feet from the launch site.

a. Write an equation for the height of the clay pigeon in terms of the horizontal distance it has traveled.

b. If the shooter misses the clay pigeon, how far from the launch site will it hit the ground?

22. The batter in a softball game hits the ball when it is 4 feet above the ground. The ball reaches the greatest height on its trajectory, 35 feet, directly above the head of the left-fielder, who is 200 feet from home plate.

a. Write an equation for the height of the softball in terms of its horizontal distance from home plate.

b. Will the ball clear the left-field wall, which is 10 feet tall and 375 feet from home plate?

23. The rate at which an antigen precipitates during an antigen–antibody reaction depends on the amount of antigen present. For a fixed quantity of antibody, the time required for a particular antigen to precipitate is given in minutes by

$$t = 2w^2 - 20w + 54$$

where w is the quantity of antigen present, in grams. For what quantity of antigen will the reaction proceed most rapidly, and how long will the precipitation take?

24.

a. Graph in the standard window two lines and a parabola:

$$\begin{aligned} Y_1 &= x + 2 \\ Y_2 &= 4 - x \\ Y_3 &= (x + 2)(4 - x) \end{aligned}$$

b. What do you notice about the x-intercepts of the graphs?

c. What do you notice about the vertex of the parabola and the intersection point of the two lines?

25.

a. Graph in the standard window two lines and a parabola:

$$\begin{aligned} Y_1 &= x + 4 \\ Y_2 &= x - 2 \\ Y_3 &= (x + 4)(x - 2) \end{aligned}$$

b. What are the x-intercepts of the parabola?

c. By referring to your graph, complete the table showing whether the y-values are positive or negative in each region.

	$x < -4$	$-4 < x < 2$	$x > 2$
Y_1			
Y_2			
Y_3			

4.3 Curve Fitting

Earlier, we used linear regression to fit a line to a collection of data points. In this section we'll see how to fit a quadratic equation to a collection of data points.

4.3.1 Finding a Quadratic Equation through Three Points

Every linear equation can be written in the form

$$y = mx + b$$

To find a specific line we must find values for the two **parameters** (constants) m and b. We need two data points in order to find those two parameters. A quadratic equation, however, has three parameters, a, b, and c:

$$y = ax^2 + bx + c$$

To find these parameters we need three data points.

[TK] To review points on a graph, see Section 4.3.1 of the Toolkit.

Example 4.3.1 Find values for a, b, and c so that the points $(1, 3)$, $(3, 5)$, and $(4, 9)$ lie on the graph of $y = ax^2 + bx + c$. [TK]

Solution. We substitute the coordinates of each of the three points into the equation of the parabola to obtain three equations:

$$\begin{aligned} \mathbf{3} &= a(\mathbf{1})^2 + b(\mathbf{1}) + c \\ \mathbf{5} &= a(\mathbf{3})^2 + b(\mathbf{3}) + c \\ \mathbf{9} &= a(\mathbf{4})^2 + b(\mathbf{4}) + c \end{aligned}$$

or, equivalently,

$$\begin{aligned} a + b + c &= 3 && (1) \\ 9a + 3b + c &= 5 && (2) \\ 16a + 4b + c &= 9 && (3) \end{aligned}$$

This is a system of three equations in the three unknowns a, b, and c. To solve the system, we use Gaussian reduction. [TK] We first eliminate c. We subtract Equation (1) from Equation (2) to obtain

$$8a + 2b = 2 \qquad (4)$$

and then subtract Equation (1) from Equation (3) to get

$$15a + 3b = 6 \qquad (5)$$

We now have a system of two linear equations in two variables:

$$8a + 2b = 2 \qquad (4)$$

$$15a + 3b = 6 \qquad (5)$$

Next, we eliminate b from Equations (4) and (5): we add -3 times Equation (4) to 2 times Equation (5) to get

$$\begin{array}{rcrcr} -24a & - & 6b & = & -6 \\ 30a & + & 6b & = & 12 \\ \hline 6a & & & = & 6 \end{array} \qquad \begin{array}{r} -3 \times (4) \\ 2 \times (5) \\ \end{array}$$

or $a = 1$. We substitute 1 for a in Equation (4) to find

$$\begin{aligned} 8(\mathbf{1}) + 2b &= 2 \qquad \text{Solve for } \boldsymbol{b}. \\ b &= -3 \end{aligned}$$

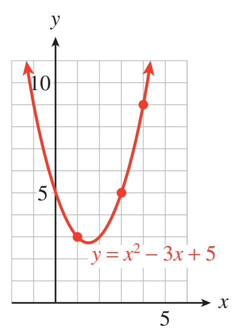
Finally, we substitute -3 for b and 1 for a in Equation (1) to find

$$\begin{aligned} \mathbf{1} + (\mathbf{-3}) + c &= 3 \qquad \text{Solve for } \boldsymbol{c}. \\ c &= 5 \end{aligned}$$

Thus, the equation of the parabola is

$$y = x^2 - 3x + 5$$

The parabola and the three points are shown below.



□

[TK] To review Gaussian reduction, see Section 4.3.3. To see more examples of elimination, see Section 4.3.2 of the Toolkit.

Checkpoint 4.3.2 QuickCheck 1. Fill in the blanks. To find the equation for a parabola:

a. We can use the formula $y = a(x - x_v)^2 + y_v$ if we know the __________ and one other point.

b. We can use the formula $y = a(x - r_1)(x - r_2)$ if we know the __________ and one other point.

c. Otherwise, we must know at least __________ points on the graph.

d. In that case, we use _________ to solve for the parameters of the equation.

Checkpoint 4.3.3 Practice 1.

a. Find the equation of a parabola

$$y = ax^2 + bx + c$$

that passes through the points $(0, 80)$, $(15, 95)$, and $(25, 55)$.

b. Plot the data points and sketch the parabola.

4.3.2 Applications

The simplest way to fit a parabola to a set of data points is to pick three of the points and find the equation of the parabola that passes through those three points.

Example 4.3.4 Major Motors Corporation is testing a new car designed for in-town driving. The data below show the cost of driving the car at different speeds. The speeds, v, are given in miles per hour, and the cost, C, includes fuel and maintenance for driving the car 100 miles at that speed.

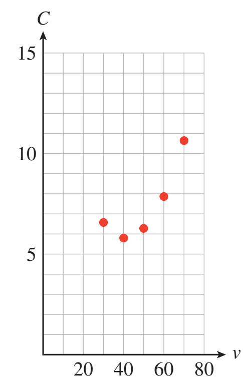
v	30	40	50	60	70
C	6.50	6.00	6.20	7.80	10.60

Find a possible quadratic model, $C = av^2 + bv + c$, that expresses C in terms of v.

Solution.

When we plot the data, it is clear that the relationship between v and C is not linear, but it may be quadratic, as shown at right.

We will use the last three data points, $(50, 6.20)$, $(60, 7.80)$, and $(70, 10.60)$, to fit a parabola to the data. We would like to find the coefficients a, b, and c of a parabola $C = av^2 + bv + c$ that includes the three data points. This gives us a system of equations:



$$
\begin{aligned}
2500a + 50b + c &= 6.20 & (1)\\
3600a + 60b + c &= 7.8 & (2)\\
4900a + 70b + c &= 10.6 & (3)
\end{aligned}
$$

Eliminating c from Equations (1) and (2) yields Equation (4), and eliminating c from Equations (2) and (3) yields Equation (5).

$$
\begin{aligned}
1100a + 10b &= 1.60 & (4)\\
1300a + 10b &= 2.8 & (5)
\end{aligned}
$$

Eliminating b from Equations (4) and (5) gives us

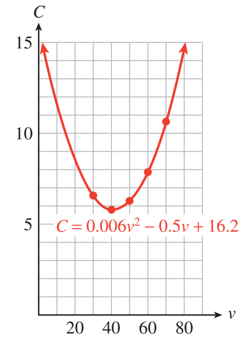
$$
\begin{aligned}
200a &= 1.20\\
a &= 0.006
\end{aligned}
$$

We substitute this value into Equation (4) to find $b = -0.5$, then substitute both values into Equation (1) to find $c = 16.2$.

Thus, our quadratic model is

$$C = 0.006v^2 - 0.5v + 16.2$$

The graph of this equation, along with the data points, is shown at right.



□

As was the case with linear regression, the graph of the regression equation may not pass through all of the data points, but it should be close to most of them.

Checkpoint 4.3.5 Practice 2. The data below show Americans' annual per capita consumption of chicken for several years since 1985.

Year	1986	1987	1988	1989	1990
Pounds of chicken	51.3	55.5	57.4	60.8	63.6

a. Use the values for 1987 through 1989 to fit a quadratic equation to the data, $C = at^2 + bt + c$, where t is measured in years since 1985.

b. What does your equation predict for per capita chicken consumption in 1990?

c. Sketch the graph of your equation and the given data. Does your model provide a good fit for the data?

Checkpoint 4.3.6 QuickCheck 2. True or False

a. We can plot the data points to see what type of curve is appropriate as a model.

b. We write a system of equations in which the y-coordinates of the data points are the unknown values.

c. A good regression equation will pass through all of the data points.

d. According to the model in the previous Example (involving a quadratic model for the cost of driving at different speeds), higher speeds always result in higher driving costs.

4.3.3 Using a Calculator for Quadratic Regression

We can use a graphing calculator or other technology to find an approximate quadratic fit for a set of data. The procedure is similar to the steps for linear regression.

Example 4.3.7

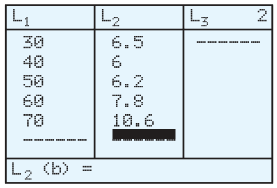
a. Use your calculator to find a quadratic fit for the data in Example 4.

b. How many of the given data points actually lie on the graph of the quadratic approximation?

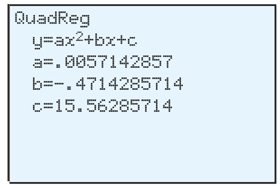
Solution.

a. We press STAT ENTER and enter the data under columns L_1 and L_2, as shown below. Next, we calculate the quadratic regression equation and store it in Y_1 by pressing STAT → 5 VARS → 1 1 ENTER.

The regression equation has the form $y = ax^2 + bx + c$, where $a = 0.0057$, $b = -0.47$, and $c = 15.56$. Notice that a, b, and c are all close to the values we computed in Example 4.3.4.



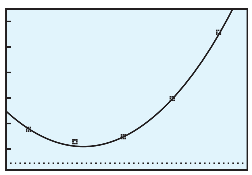
(a)



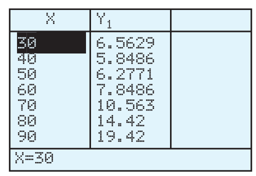
(b)

b. Next, we will graph the data and the regression equation. We press Y= and select *Plot1*, then press ZOOM 9 to see the graph shown below. The parabola seems to pass close to all the data points.

However, try using either the *value* feature or a table to find the y-coordinates of points on the regression curve. By comparing these y-coordinates with our original data points, we find that none of the given data points lies precisely on the parabola.



(a)



(b)

□

Checkpoint 4.3.8 Practice 3. To test the effects of radiation, a researcher irradiated male mice with various dosages and bred them with unexposed female mice. The table below shows the fraction of fertilized eggs that survived, as a function of the radiation dosage. (Source: Strickberger, Monroe W., 1976)

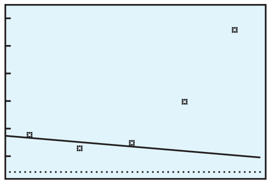
Radiation (rems)	100	300	500	700	900	1100	1500
Relative survival of eggs	0.94	0.700	0.544	0.424	0.366	0.277	0.195

a. Enter the data into your calculator and create a scatterplot. Does the graph appear to be linear? Does it appear to be quadratic?

b. Fit a quadratic regression equation to the data and graph the equation on the scatterplot.

4.3.4 Choosing an Appropriate Model

We must be careful that our data set gives a complete picture of the situation we want to model. A regression equation may fit a particular collection of data and still be a poor model if the rest of the data diverge from the regression graph.



In Example 4.3.4, suppose Major Motors had collected only the first three data points and fit a line through them, as shown at left. This regression line gives poor predictions for the cost of driving at 60 or 70 miles per hour.

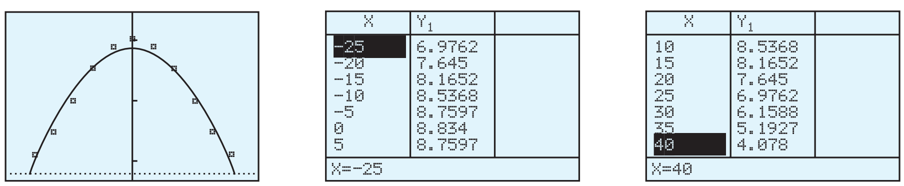
Example 4.3.9 Delbert records the height of the tip of the minute hand on the classroom's clock at different times. The data are shown in the table, where time is measured in minutes since noon. (A negative time indicates a number of minutes before noon.) Find a quadratic regression equation for the data and use it to predict the height of the minute hand's tip at 40 minutes past noon. Do you believe this prediction is valid?

Time (minutes)	-25	-20	-15	-10	-5	0	5	10	15	20	25
Height (feet)	7.13	7.50	8.00	8.50	8.87	9.00	8.87	8.50	8.80	7.50	7.13

Solution. We enter the time data under L_1 and the height data under L_2. Then we calculate and store the quadratic regression equation in Y_1, as we did in Example 4.3.7. The regression equation is

$$y = -0.00297x^2 + 0x + 8.834$$

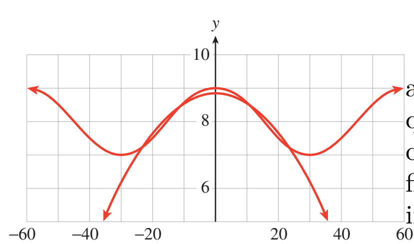
From either the graph of the regression equation or from the table (see figure below), we can see that the fit is not perfect, although the curve certainly fits the data better than any straight line could.



(a) (b) (c)

If we scroll down the table, we find that this equation predicts a height of approximately 4.08 feet at time 40 minutes. (See figure (c).) This is a preposterous estimate! The position of the minute hand at 40 minutes after noon should be the same as it was exactly one hour earlier (at 20 minutes before noon), when it was 7.50 feet. □

Caution 4.3.10 Using the wrong type of function to fit the data is a common error in making predictions. In the Example above, we know that the minute hand of a clock repeats its position every 60 minutes. The graph of the height of its tip oscillates up and down, repeating the same pattern over and over. We cannot describe such a graph using either a linear or a quadratic function.



The graph of the height is shown at left, along with the graph of our quadratic regression equation. You can see that the regression equation fits the actual curve only on a small interval.

Even though your calculator can always compute a regression equation,that equation is not necessarily appropriate for your data. Choosing a reasonable type of regression equation for a particular data set requires knowledge of different kinds of models and the physical or natural laws that govern the situation at hand.

Checkpoint 4.3.11 QuickCheck 3. True or False

a. Your calculator can choose the correct type of regression equation for a data set.

b. It is only necessary to use the first and last data points to compute a regression equation.

c. A regression equation may fit some of the data but still be a poor model.

d. A good regression equation should fit all the data points exactly.

Checkpoint 4.3.12 Practice 4. A speeding motorist slams on the brakes when she sees an accident directly ahead of her. The distance she has traveled t seconds after braking is shown in the table.

Time (seconds)	0	0.5	1.0	1.5	2.0	2.5
Distance (feet)	0	51	95	131	160	181

a. Enter the data into your calculator and create a scatterplot. Fit a quadratic regression equation to the data and graph the equation on the scatterplot.

b. Use your regression equation to find the vertex of the parabola. What do the coordinates represent in terms of the problem?

4.3.5 Problem Set 4.3

Warm Up

Exercise Group. For Problems 1 and 2, state the vertex, the y-intercept, and the x-intercepts of the parabola.

1. $y = -2(x-4)^2 + 10$

2. $y = \dfrac{1}{3}(x+1)^2 - 4$

3. You are in charge of selling tickets to a one-woman show at a local art gallery. Tickets to the opening night were priced at \$25, and you sold 30 tickets.

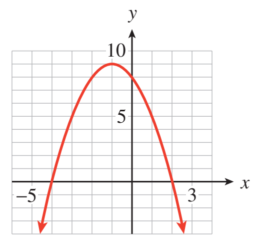
a. Every night after the opening, you reduce the ticket price by \$2. What is the ticket price after x nights?

b. Every night after the opening, you sell 4 more tickets than the previous night. How many tickets did you sell x nights after the opening?

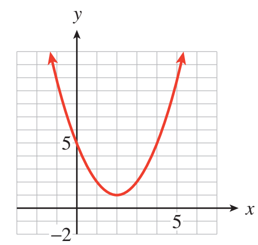
4. The point $(-3, 8)$ lies on the graph of $y = ax^2 + bx + c$. Write an equation involving a, b, and c.

Exercise Group. For Problems 5–8, find an equation for the parabola. Use the vertex form or the factored form of the equation, whichever is more appropriate.

5.



6.



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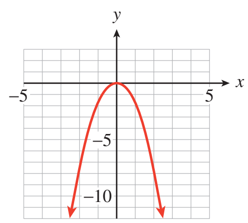

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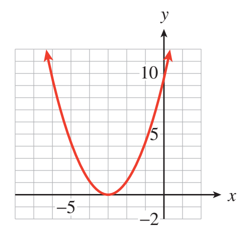


Skills Practice

Exercise Group. For Problems 9–12, solve the system by elimination. Begin by eliminating c.

9.

$$\begin{aligned} a+b+c &= -3 \\ a-b+c &= -9 \\ 4a+2b+c &= -6 \end{aligned}$$

10.

$$\begin{aligned} a+b+c &= 10 \\ 4a+2b+c &= 19 \\ 9a+3b+c &= 38 \end{aligned}$$

11.

$$\begin{aligned} a-b+c &= 12 \\ 4a-2b+c &= 19 \\ 9a+3b+c &= 4 \end{aligned}$$

12.

$$\begin{aligned} 4a+2b+c &= 14 \\ 9a-3b+c &= -41 \\ 16a-4b+c &= -70 \end{aligned}$$

13. Find values for $a,$ $b,$ and c so that the graph of $y = ax^2 + bx + c$ includes the points $(-1, 0),$ $(2, 12),$ and $(-2, 8).$

14. Find values for $a,$ $b,$ and c so that the graph of $y = ax^2 + bx + c$ includes the points $(-1, 2),$ $(1, 6),$ and $(2, 11).$

Applications

15. The data show the number of people of certain ages who were the victims of homicide in a large city last year.

Age	10	20	30	40
Number of victims	12	62	72	40

a. Use the first three data points to fit a quadratic equation to the data, $N = ax^2 + bx + c$, where x represents age.

b. What does your equation predict for the number of 40-year-olds who were the victims of homicide?

c. Sketch the graph of your quadratic equation and the given data on the same axes.

16. Sara plans to start a side business selling eggs. She finds that the total number of eggs produced each day depends on the number of hens confined in the henhouse, as shown in the table.

Number of hens, n	15	20	25	30	36	39
Number of eggs, E	15	18	20	21	21	20

a. Use the first three data points to find a quadratic model, $E = an^2 + bn + c$.

b. Plot the data and sketch the model on the same axes.

c. What does the model predict for the number of eggs produced when 39 hens are confined in the henhouse?

17. Find a quadratic model for the number of diagonals that can be drawn in a polygon of n sides. Some data are provided.

Sides	4	5	6	7
Diagonals	2	5	9	14

18. You are driving at a speed of 60 miles per hour when you step on the brakes. Find a quadratic model for the distance in feet that your car travels in t seconds after braking. Some data are provided.

Seconds	1	2	3	4
Feet	81	148	210	267

19. In the 1990's, an outbreak of mad cow disease (Creutzfeldt-Jakob disease) alarmed health officials in England. The table shows the number of deaths each year from the disease.

Year	94	95	96	97	98	99	00	01	02	03	04
Deaths	0	3	10	10	18	15	28	20	17	19	9

(Source: www.cjd.ed.ac.uk/vcjdqsep05)

a. The Health Protection Agency determined that a quadratic model was the best-fitting model for the data. Find a quadratic regression equation for the data.

b. Use your model to estimate when the peak of the epidemic occurred, and how many deaths from mad cow disease were expected in 2005.

20. The table shows the height in kilometers of a star-flare at various times after it exploded from the surface of a star.

Time (seconds)	0.2	0.4	0.6	0.8	1.0	1.2
Height (kilometers)	6.8	12.5	17.1	20.5	22.8	23.9

a. Find the equation of the least-squares regression line for the height of the flare in terms of time.

b. Use the regression line to predict the height of the flare 1.4 seconds after it exploded.

c. Make a scatterplot of the data and draw the regression line on the same axes.

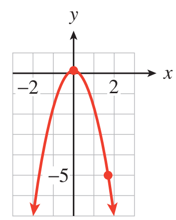
d. Find the quadratic regression equation for the height in terms of time.

e. Use the quadratic regression equation to predict the height of the flare 1.4 seconds after it exploded.

f. Draw the quadratic regression curve on the graph from part (c).

g. Which model is more appropriate for the height of the star-flare, linear or quadratic? Why?

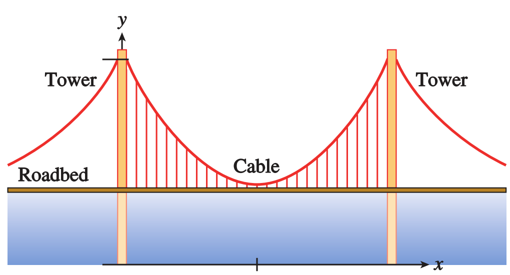
21. Some comets move about the sun in parabolic orbits. In 1973 the comet Kohoutek passed within 0.14 AU (**astronomical units**), or 21 million kilometers of the sun. Imagine a coordinate system superimposed on a diagram of the comet's orbit, with the sun at the origin, as shown below. The units on each axis are measured in AU.



a. The comet's closest approach to the sun (called perihelion) occurred at the vertex of the parabola. What were the comet's coordinates at perihelion?

b. When the comet was first discovered, its coordinates were $(1.68, -4.9)$. Find an equation for comet Kohoutek's orbit in vertex form.

22. The Akashi Kaikyo bridge in Japan is the longest suspension bridge in the world, with a main span of 1991 meters. Its main towers are 297 meters tall. The roadbed of the bridge is 14 meters thick and clears the water below by 65 meters. The cables on a suspension bridge hang in the shape of parabolas. Imagine a coordinate system superimposed on the diagram of the bridge, as shown in the figure.



a. Find the coordinates of the vertex and one other point on the cable.

b. Use the points from part (a) to find an equation for the shape of the cable in vertex form.

4.4 Quadratic Inequalities

In this section we'll see how to solve a quadratic inequality, that is, an inequality that can be written in the form

$$ax^2 + bx + c < 0 \quad \text{or} \quad ax^2 + bx + c > 0$$

[TK] Before we begin, you can review solving linear inequalities in Section 4.4.1 of the Toolkit.

4.4.1 Solving Inequalities Graphically

The easiest way to solve a quadratic inequality is with a graph. [TK]

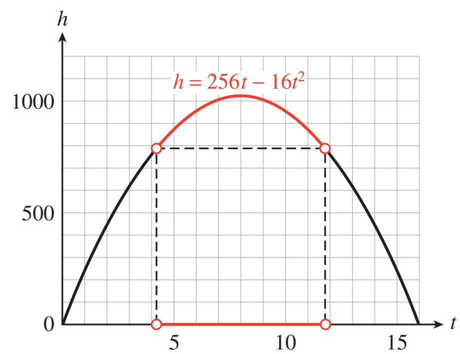
[TK] To review solving with a graph, see Section 1.2.4

Example 4.4.1 The Chamber of Commerce in River City plans to put on a Fourth of July fireworks display. City regulations require that fireworks at public gatherings explode higher than 800 feet above the ground. The mayor particularly wants to include the Freedom Starburst model, which is launched from the ground. Its height after t seconds is given by

$$h = 256t - 16t^2$$

When should the Starburst explode in order to satisfy the safety regulation?

Solution. We can get an approximate answer to this question by looking at the graph of the rocket's height, shown below.



We would like to know when the rocket's height is greater than 800 feet, or, in mathematical terms, for what values of t is $h > 800$? The answer to this question is the solution of the inequality

$$256t - 16t^2 > 800$$

Points on the graph with $h > 800$ are shown in color, and the t-coordinates of those points are marked on the horizontal axis. If the Freedom Starburst explodes at any of these times, it will satisfy the safety regulation.

From the graph, the safe time interval runs from approximately 4.25 seconds to 11.75 seconds after launch. The solution of the inequality is the set of all t-values greater than 4.25 but less than 11.75. □

Note 4.4.2 The solution set in Example 4.4.1 is called a **compound inequality**, because it involves more than one inequality symbol. We write this inequality as

$$4.25 < t < 11.75$$

and read "t greater than 4.25 but less than 11.75."

Checkpoint 4.4.3 QuickCheck 1. Which is the correct way to write "x is greater than 3 and less than 8"?

a. $8 > x < 3$

b. $3 < x < 8$

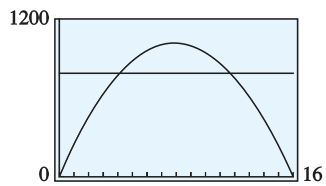
c. $8 < x < 3$

d. $3 < x > 8$

Technology 4.4.4 Solving an Inequality With a Graphing Calculator. You can use your graphing calculator to solve the problem in Example 4.4.1. Graph the two functions

$$Y_1 = 256X - 16X^2$$
$$Y_2 = 800$$

on the same screen. Use WINDOW settings to match the graph in Example 4.4.1.



Then use the intersect feature to find the points where the two graphs intersect, at about $x = 4.26$ and $x = 11.74$. Both these points y-coordinate 800, and between the points the parabola is above the line, so $h > 800$ when $4.26 < t < 11.74$.

Checkpoint 4.4.5 Practice 1.

a. Graph the function $y = x^2 - 2x - 9$ in the window

$$\text{Xmin} = -9.4 \qquad \text{Xmax} = 9.4$$
$$\text{Ymin} = -10 \qquad \text{Ymax} = 10$$

b. Use the graph to solve the inequality $x^2 - 2x - 9 \le 6$.

4.4.2 Using the x-Intercepts

Because it is easy to decide whether the y-coordinate of a point on a graph is positive or negative (the point lies above the x-axis or below the x-axis), we often rewrite a given inequality so that one side is zero.

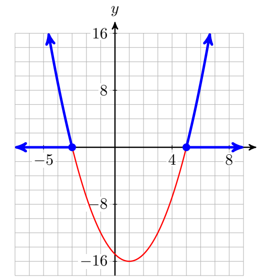
Example 4.4.6 Use a graph to solve $x^2 - 2x - 3 \ge 12$

Solution. We first write the inequality with zero on one side:

$$x^2 - 2x - 15 \ge 0.$$

We would like to find points on the graph of $y = x^2 - 2x - 15$ that have y-coordinates greater than or equal to zero. A graph of the equation is shown below.

You can check that the x-intercepts of the graph are -3 and 5. [TK] The points shown in blue on the graph lie above the x-axis and have $y \geq 0$, so the x-coordinates of these points are the solutions of the inequality.



Note that the solutions lie in two intervals, less than -3 or greater than 5. Thus, the solution is $x \leq -3$ or $x \geq 5$. ☐

[TK] To review finding the x-intercepts of a parabola, see Section 4.4.2 of the Toolkit.

Caution 4.4.7 In Example 4.4.6 above, the solution of the inequality $x^2 - 2x - 15 \geq 0$ is the set of values

$$x \leq -3 \quad \text{or} \quad x \geq 5$$

This set is another type of compound inequality, and its graph consists of two pieces. Therefore, we cannot write the solution as a single inequality. For instance, it would be *incorrect* to describe the solution set as $-3 \geq x \geq 5$, because this notation implies that $-3 \geq 5$. We must write the solution as two parts: $x \leq -3$ or $x \geq 5$.

Checkpoint 4.4.8 QuickCheck 2. Which is the correct way to write "x is less than 6 or greater than 10"?

a. $6 < x > 10$

b. $10 < x < 6$

c. $x < 6$ or $x > 10$

d. $10 > x < 6$

Checkpoint 4.4.9 Practice 2. Follow the steps below to solve the inequality $36 + 6x - x^2 \leq 20$.

1. Rewrite the inequality so that the right side is zero.
2. Graph the equation $y = 16 + 6x - x^2$.
3. Locate the points on the graph with y-coordinate less than zero, and mark the x-coordinates of the points on the x-axis.

4.4.3 Interval Notation

The solution set in Example 4.4.1, namely $4.26 < t < 11.74$, is called an interval. An **interval** is a set that consists of all the real numbers between two numbers a and b. An interval may include one or both of its endpoints.

Interval Notation.

1. The **closed interval** $[a, b]$ is the set $a \leq x \leq b$.
2. The **open interval** (a, b) is the set $a < x < b$.
3. Intervals may also be **half-open** or **half-closed**.
4. The **infinite interval** $[a, \infty)$ is the set $x \geq a$.
5. The **infinite interval** $(-\infty, a]$ is the set $x \leq a$.

The symbol ∞, infinity, does not represent a specific real number; it indicates that the interval continues forever along the real line. A set consisting of two or more intervals is called the **union** of the intervals. For example, the solution to Example 4.4.6 is denoted in interval notation by $(-\infty, -3) \cup (5, \infty)$.

Many solutions of inequalities are intervals or unions of intervals.

Example 4.4.10 Write the solution sets with interval notation, and graph the solution set on a number line.

a. $3 \le x < 6$

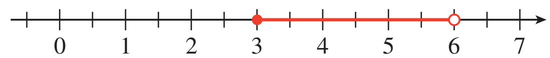
b. $x \ge -9$

c. $x \le 1$ or $x > 4$

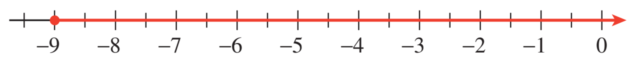
d. $-8 < x \le -5$ or $-1 \le x < 3$

Solution.

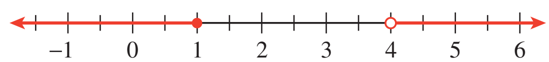
a. $[3, 6)$. This is called a half-open or half-closed interval.



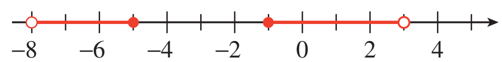
b. $[-9, \infty)$. We always use round brackets next to the symbol ∞ because ∞ is not a specific number and is not included in the set.



c. $(-\infty, 1] \cup (4, \infty)$. The word "or" describes the union of two sets.



d. $(-8, -5] \cup [-1, 3)$.



□

Checkpoint 4.4.11 Practice 3. Write the solutions to Practice 1 and Practice 2 in interval notation.

Checkpoint 4.4.12 QuickCheck 3. True or False

a. An open interval includes only the numbers between its endpoints.

b. Infinite intervals include the number ∞.

c. The union of two intervals includes all numbers that lie in at least one of the intervals.

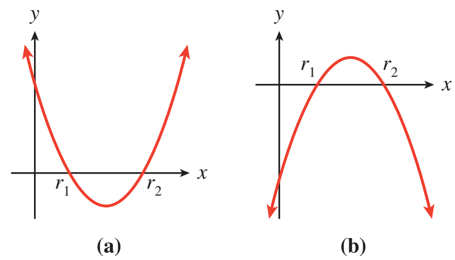
d. An interval must be either closed or open.

4.4.4 Solving Quadratic Inequalities Algebraically

Although a graph is very helpful in solving inequalities, it is not completely necessary. Every quadratic inequality can be put into one of the forms

$$ax^2 + bx + c < 0, \qquad ax^2 + bx + c > 0$$
$$ax^2 + bx + c \le 0, \qquad ax^2 + bx + c \ge 0$$

All we really need to know is whether the corresponding parabola $y = ax^2 + bx + c$ opens upward or downward. Consider the parabolas shown below.



The parabola in figure (a) opens upward. It crosses the x-axis at two points, $x = r_1$ and $x = r_2$. At these points, $y = 0$.

- The graph lies below the x-axis between r_1 and r_2, so the solutions to the inequality $y < 0$ lie between r_1 and r_2.
- The graph lies above the x-axis for x-values less than r_1 or greater than r_2, so the solutions to the inequality $y > 0$ are $x < r_1$ or $x > r_2$.

If the parabola opens downward, as in figure (b), the situation is reversed. The solutions to the inequality $y > 0$ lie between the x-intercepts, and the solutions to $y < 0$ lie outside the x-intercepts.

From the graphs, we see that the x-intercepts are the boundary points between the portions of the graph with positive y-coordinates and the portions with negative y-coordinates. To solve a quadratic inequality, we need only locate the x-intercepts of the corresponding graph and then decide which intervals of the x-axis produce the correct sign for y.

To solve a quadratic inequality algebraically.

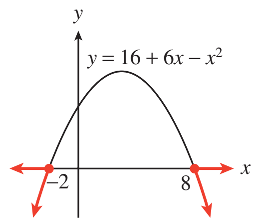
1. Write the inequality in standard form: One side is zero, and the other has the form $ax^2 + bx + c$.
2. Find the x-intercepts of the graph of $y = ax^2 + bx + c$ by setting $y = 0$ and solving for x.
3. Make a rough sketch of the graph, using the sign of a to determine whether the parabola opens upward or downward.
4. Decide which intervals on the x-axis give the correct sign for y.

Example 4.4.13 Solve the inequality $36 + 6x - x^2 \leq 20$ algebraically.

Solution.

1. We subtract 20 from both sides of the inequality so that we have 0 on the right side.
$$16 + 6x - x^2 \leq 0$$
2. Consider the equation $y = 16 + 6x - x^2$. To locate the x-intercepts, we set $y = 0$ and solve for x.
$$\begin{aligned} 16 + 6x - x^2 &= 0 && \text{Multiply each term by } \mathbf{-1.} \\ x^2 - 6x - 16 &= 0 && \text{Factor the left side.} \\ (x-8)(x+2) &= 0 && \text{Apply the zero-factor principle.} \\ x - 8 = 0 \quad \text{or} \quad x + 2 &= 0 \\ x = 8 \quad \text{or} \quad x &= -2 \end{aligned}$$
The x-intercepts are $x = -2$ and $x = 8$.

3. Make a rough sketch of the graph of $y = 16 + 6x - x^2$, as shown below. Because $a = -1 < 0$, the graph is a parabola that opens downward.



4. We are interested in points on the graph for which $y \leq 0$. The points with negative y-coordinates (that is, points below the x-axis) lie outside the x-intercepts of the graph, so the solution of the inequality is $x \leq -2$ or $x \geq 8$. Or, using interval notation, the solution is $(-\infty, -2] \cup [8, \infty)$.

□

Caution 4.4.14 Many people think that the inequality signs in the solution should point in the same direction as the sign in the original problem, and hence would incorrectly write the solution to Example 4.4.13 as $x \leq -2$ or $x \leq 8$. However, you can see from the graph that this is incorrect. Remember that the graph of a quadratic equation is a parabola, not a straight line!

Checkpoint 4.4.15 Practice 4. Solve $x^2 < 20$.
Hint:

1. Write the inequality in standard form.
2. Find the x-intercepts of the corresponding graph. Use extraction of roots.
3. Make a rough sketch of the graph.
4. Decide which intervals on the x-axis give the correct sign for y.

Checkpoint 4.4.16 QuickCheck 4. Which is the correct solution for $x^2 > 16$?

a. $x > 4$

b. $x > 4$ or $x > -4$

c. $4 < x < -4$

d. $x < -4$ or $x > 4$

4.4.5 Applications

If we cannot find the x-intercepts of the graph by factoring or extraction of roots, we can use the quadratic formula.

Example 4.4.17 TrailGear, Inc. manufactures camping equipment. The company finds that the profit from producing and selling x alpine parkas per month is given, in dollars, by

$$P = -0.8x^2 + 320x - 25,200$$

How many parkas should the company produce and sell each month if it must keep the profits above $2000?

Solution.

1. We would like to solve the inequality

$$-0.8x^2 + 320x - 25,200 > 2000$$

or, subtracting 2000 from both sides,

$$-0.8x^2 + 320x - 27,200 > 0$$

2. Consider the equation

$$y = -0.8x^2 + 320x - 27,200$$

We locate the x-intercepts of the graph by setting $y = 0$ and solving for x. We will use the quadratic formula to solve the equation

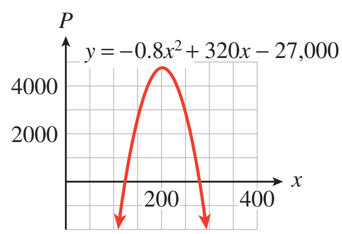
$$-0.8x^2 + 320x - 27,200 = 0$$

so $a = \mathbf{-0.8}$, $b = \mathbf{320}$, and $c = \mathbf{-27,200}$. We substitute these values into the quadratic formula to obtain [TK]

$$\begin{aligned} x &= \frac{-(\mathbf{320}) \pm \sqrt{(\mathbf{320})^2 - 4(-0.8)(\mathbf{-27,200})}}{2(\mathbf{-0.8})} \\ &= \frac{-320 \pm \sqrt{102,400 - 87,040}}{-1.6} \\ &= \frac{-320 \pm \sqrt{15,360}}{-1.6} \end{aligned}$$

To two decimal places, the solutions to the equation are 122.54 and 277.46.

3. The graph of the equation is a parabola that opens downward, because the coefficient of x^2 is negative.



4. The graph lies above the x-axis, and hence $y > 0$, for x-values between the two x-intercepts, that is, for $122.54 < x < 277.46$. Because we cannot produce a fraction of a parka, we restrict the interval to the closest whole number x-values included, namely 123 and 277.

Thus, TrailGear can produce as few as 123 parkas or as many as 277 parkas per month to keep its profit above \$2000.

□

[TK] To review the quadratic formula, see Section 4.1.

Checkpoint 4.4.18 Practice 5. Solve the inequality $10 - 8x + x^2 > 4$.
Hint:

1. Write the inequality in standard form.

2. Find the x-intercepts of the corresponding graph. Use extraction of roots.

3. Make a rough sketch of the graph.

4. Decide which intervals on the x-axis give the correct sign for y.

Checkpoint 4.4.19 QuickCheck 5. If you cannot find the x-intercepts of a parabola by factoring, what should you do?

a. Panic

b. Use the y-intercept instead

c. Use the quadratic formula

d. Go on to the next problem

4.4.6 Problem Set 4.4

Warm Up

Exercise Group. For Problems 1–4,

a. Find the x-intercepts of the parabola.

b. Decide whether the parabola opens up or down.

1. $y = x^2 - 2x - 24$

2. $y = 40 - x^2$

3. $y = 12 - (x - 3)^2$

4. $y = x^2 + 3x + 1$

Exercise Group. For Problems 5 and 6, write the set with interval notation, and graph the set on a number line.

5.

a. $0 \le x < 4$

b. $8 > x > 5$

6.

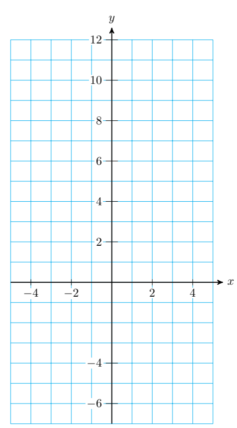
a. $x \le 1$

b. $x \ge 3$ or $x \le -3$

Skills Practice

7.

a. Graph the equation $y = x^2 - 2x - 3$ on the grid.

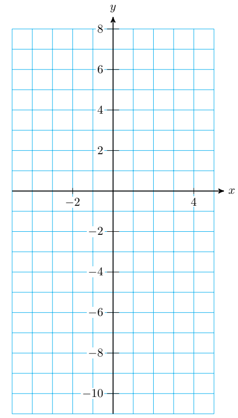


b. Darken the portion of the x-axis for which $y > 0$.

c. Solve the inequality $x^2 - 2x - 3 > 0$

8.

a. Graph the equation $y = x^2 + 2x - 8$ on the grid.

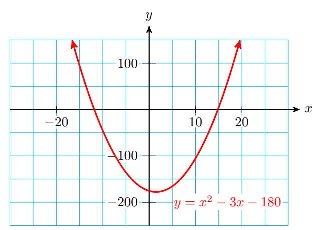


b. Darken the portion of the x-axis for which $y < 0$.

c. Solve the inequality $x^2 + 2x - 8 < 0$

Exercise Group. For Problems 9 and 10, use the graph to solve the equation and the inequality.

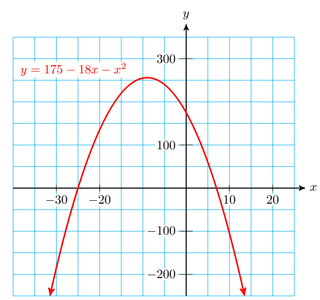
9.



a. $x^2 - 3x - 180 = 0$

b. $x^2 - 3x - 180 > 0$

10.



a. $175 - 18x - x^2 = 0$

b. $175 - 18x - x^2 < 0$

Exercise Group. For Problems 11 and 12, graph the parabola in the window

Xmin $= -9.4$	Xmax $= 9.4$
Ymin $= -25$	Ymax $= 25$

Then use the graph to solve the inequalities. Write your answers in interval notation.

11. $y = x^2 - 3x - 18$

For parts (c) and (d), it may be helpful to graph $Y_2 = -8$ as well.

a. $x^2 - 3x - 18 > 0$

b. $x^2 - 3x - 18 < 0$

c. $x^2 - 3x - 18 \leq -8$

d. $x^2 - 3x - 18 \geq -8$

12. $y = 16 - x^2$

For parts (c) and (d), it may be helpful to graph $Y_2 = 7$ as well.

a. $16 - x^2 > 0$

b. $16 - x^2 < 0$

c. $16 - x^2 \leq 7$

d. $16 - x^2 \geq 7$

Exercise Group. For Problems 13–16, solve the inequality. It may be helpful to sketch a rough graph.

13. $(x-3)(x+2) > 0$

14. $(x+3)(x-4) \le 0$

15. $k(k-4) \ge 0$

16. $t^2 - 36 < 0$

Exercise Group. For Problems 17–24, solve the inequality algebraically. Write your answers in interval notation, and round to two decimal places if necessary.

17. $q^2 + 9q + 18 < 0$

18. $28 - 3x - x^2 \ge 0$

19. $2z^2 - 7z > 4$

20. $4x^2 + x \ge -2x^2 + 2$

21. $5 - v^2 < 0$

22. $x^2 - 4x + 1 \ge 0$

23. $-3 - m^2 < 0$

24. $w^2 - w + 4 \le 0$

Exercise Group. For Problems 25 and 26, solve the inequality by graphing. Use the window setting

$$\begin{aligned} \text{Xmin} &= -9.4 \qquad & \text{Xmax} &= 9.4 \\ \text{Ymin} &= -64 \qquad & \text{Ymax} &= 62 \end{aligned}$$

25. $x^2 - 1.4x - 20 < 9.76$

26. $-6x^2 - 36x - 20 \le 25.36$

Applications

27. The x-intercepts of the graph of $y = 2x^2 + bx + c$ are -4.2 and 2.6. What are the solutions of the inequality $2x^2 + bx + c > 0$?

28. The x-intercepts of the graph of $y = -x^2 + bx + c$ are $-2 \pm \sqrt{17}$. Which of the following are solutions of the inequality $-x^2 + bx + c \ge 0$?

a. 2

b. -6

c. $\sqrt{17}$

d. $2\sqrt{17}$

Exercise Group. For Problems 29 and 30,

a. Solve the problem by writing and solving an inequality.

b. Graph the equation and verify your solution on the graph.

29. A fireworks rocket is fired from ground level. Its height in feet t seconds after launch is given by

$$h = 320t - 16t^2$$

During what time interval is the rocket higher than 1024 feet?

30. The volume of a cylindrical can should be between 21.2 and 21.6 cubic inches. If the height of the can is 5 inches, what values for the radius (to the nearest hundredth of an inch) will produce an acceptable can?

Exercise Group. For Problems 31 and 32, recall that

$$\text{Revenue} = (\text{number of items sold}) \cdot (\text{ price per item})$$

31. Green Valley Nursery sells $120 - 10p$ boxes of rose food per month at a price of p dollars per box. It would like to keep its monthly revenue from rose food over \$350. In what range should it price a box of rose food?

32. The Locker Room finds that it sells $1200-30p$ sweatshirts each month when it charges p dollars per sweatshirt. It would like its revenue from sweatshirts to be over \$9000 per month. In what range should it keep the price of a sweatshirt?

33. A farmer inherits an apple orchard on which 60 trees are planted per acre. Each tree yields 12 bushels of apples. Experimentation has shown that for each tree removed per acre, the yield per tree increases by $\frac{1}{2}$ bushel.

a. Write algebraic expressions for the number of trees per acre and for the yield per tree if x trees per acre are removed.

b. Write a quadratic equation for the total yield per acre if x trees are removed per acre.

c. What is the maximum yield per acre that can be achieved by removing trees? How many trees per acre should be removed to achieve this yield?

d. How many trees should the farmer remove per acre in order to harvest at least 850 bushels per acre?

e. Graph your equation for total yield in the window

$$\begin{array}{ll} \text{Xmin} = 0 & \text{Xmax} = 94 \\ \text{Ymin} = 0 & \text{Ymax} = 1000 \end{array}$$

and use your graph to verify your answers to parts (e) and (f).

4.5 Chapter 4 Summary and Review

4.5.1 Glossary

- quadratic formula
- imaginary unit
- imaginary number
- conjugate pair
- complex number
- maximum value
- minimum value
- vertex form
- compound inequality
- interval notation
- open interval
- closed interval

4.5.2 Key Concepts

1.

The Quadratic Formula.

The solutions of the equation $ax^2+bx+c=0$, $(a \neq 0)$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

2. We have four methods for solving quadratic equations: extraction of roots, factoring, completing the square, and the quadratic formula. The

first two methods are faster, but they do not work on all equations. The last two methods work on any quadratic equation.

3. The **imaginary unit**, i, is defined by $i = \sqrt{-1}$.

4. The square root of any negative number can be written as the product of a real number (the square root of its absolute value) and i.

5. The sum of a real number and an **imaginary number** is called a **complex number**.

6. The graph of the quadratic equation $y = ax^2 + bx + c$ may have two, one, or no x-intercepts, according to the number of distinct real-valued solutions of the equation $ax^2 + bx + c = 0$.

7. **The Discriminant.**

   The **discriminant** of a quadratic equation is

   $$D = b^2 - 4ac$$

   1. If $D > 0$, there are two unequal real solutions.
   2. If $D = 0$, there is one solution of multiplicity two.
   3. If$D < 0$, there are two complex conjugate solutions.

8. Every quadratic equation has two solutions, which may be the same.

9. For the graph of $y = ax^2 + bx + c$, the x-coordinate of the vertex is

   $$x_v = \frac{-b}{2a}$$

   To find the -coordinate of the vertex, we substitute x_v into the formula for the parabola.

10. Quadratic models may arise as the product of two variables.

11. The maximum or minimum of a quadratic equation occurs at the vertex of its graph.

12. **Vertex Form for a Quadratic Equation.**

    A quadratic equation $y = ax^2 + bx + c,\ a \neq 0,$ can be written in the vertex form

    $$\boldsymbol{y = a(x - x_v)^2 + y_v}$$

    where the vertex of the graph is (x_v, y_v).

13. We can convert a quadratic equation to vertex form by completing the square.

14. We need the coordinates of three points to find the equation of a parabola.

15. We can use the method of elimination to find the equation of a parabola through three points.

16. If we know the vertex of a parabola, we need only one other point to find its equation.

17. We can use quadratic regression to fit a parabola to a collection of data points from a quadratic model.

18. We can use a graphical technique to solve quadratic inequalities.

To solve a quadratic inequality algebraically.

19.
   1. Write the inequality in standard form: One side is zero, and the other has the form $ax^2 + bx + c$.
   2. Find the x-intercepts of the graph of $y = ax^2 + bx + c$ by setting $y = 0$ and solving for x.
   3. Make a rough sketch of the graph, using the sign of a to determine whether the parabola opens upward or downward.
   4. Decide which intervals on the x-axis give the correct sign for y.

20. We can write the solution set to a quadratic inequality in interval notation.

Interval Notation.

21.
   1. The **closed interval** $[a, b]$ is the set $a \leq x \leq b$.
   2. The **open interval** (a, b) is the set $a < x < b$.
   3. Intervals may also be **half-open** or **half-closed**.
   4. The **infinite interval** $[a, \infty)$ is the set $x \geq a$.
   5. The **infinite interval** $(\infty, a]$ is the set $x \leq a$.

To graph a quadratic equation $y = ax^2 + bx + c$.

22.
   1. Determine whether the parabola opens upward (if $a > 0$) or downward (if $a < 0$).
   2. Locate the vertex of the parabola.
      a. The x-coordinate of the vertex is $x_v = \dfrac{-b}{2a}$.
      b. Find the y-coordinate of the vertex by substituting x_v into the equation of the parabola.
   3. Locate the x-intercepts (if any) by setting $y = 0$ and solving for x.
   4. Locate the y-intercept by evaluating y for $x = 0$.
   5. Locate the point symmetric to the y-intercept across the axis of symmetry.

4.5.3 Chapter 4 Review Problems

Exercise Group. For Problems 1–4, solve by using the quadratic formula.

1. $\frac{1}{2}x^2 + 1 = \frac{3}{2}x$

2. $x^2 - 3x + 1 = 0$

3. $x^2 - 4x + 2 = 0$

4. $2x^2 + 2x = 3$

Exercise Group. For Problems 5 and 6, solve the formula for the indicated variable.

5. $h = 6t + 3t^2$, for t

6. $D = \frac{n^2 - 3n}{2}$, for n

Exercise Group. For Problems 7–10, use the discriminant to determine the nature of the solutions of the equation.

7. $4x^2 - 12x + 9 = 0$

8. $2t^2 + 6t + 5 = 0$

9. $2y^2 = 3y - 4$

10. $\frac{x^2}{4} = x + \frac{5}{4}$

11. The height, h, of an object t seconds after being thrown from ground level is given by

$$h = v_0 t - \frac{1}{2}gt^2$$

where v_0 is its starting velocity and g is a constant that depends on gravity. On the moon, the value of g is approximately 5.6. Suppose you hit a golf ball on the moon with an upwards velocity 100 feet per second.

a. Write an equation for the height of the golf ball t seconds after you hit it.

b. Graph your equation in the window

Xmin = 0	Xmax = 47
Ymin = 0	Ymax = 1000

c. Use the TRACE key to estimate the maximum height the golf ball reaches.

d. Use your equation to calculate when the golf ball will reach a height of 880 feet.

12. An acrobat is catapulted into the air from a springboard at ground level. His height h in meters is given by the formula

$$h = -4.9t^2 + 14.7t$$

where t is the time in seconds from launch. Use your calculator to graph the acrobat's height versus time. Set the WINDOW values on your calculator to

Xmin = 0	Xmax = 4.7
Ymin = 0	Ymax = 12

a. Use the TRACE key to find the coordinates of the highest point on the graph. When does the acrobat reach his maximum height, and what is that height?

b. Use the formula to find the height of the acrobat after 2.4 seconds.

c. Use the TRACE key to verify your answer to part (b). Find another time when the acrobat is at the same height.

d. Use the formula to find two times when the acrobat is at a height of 6.125 meters. Verify your answers on the graph.

e. What are the coordinates of the horizontal intercepts of your graph? What do these points have to do with the acrobat?

Exercise Group. For Problems 13-16,

a. Find the coordinates of the vertex and the intercepts.

b. Sketch the graph.

13. $y = x^2 - x - 12$

14. $y = -2x^2 + x - 4$

15. $y = -x^2 + 2x + 4$

16. $y = x^2 - 3x + 4$

17. Find the equation for a parabola whose vertex is $(15, -6)$ and that passes through the point $(3, 22.8)$.

18.

a. Find the vertex of the graph of $y = -2(x - 1)^2 + 5$

b. Write the equation of the parabola in standard form.

19. The total profit Kiyoshi makes from producing and selling x floral arrangements is

$$P = -0.4x^2 + 36x$$

a. How many floral arrangements should Kiyoshi produce and sell to maximize his profit?

b. What is his maximum profit?

c. Verify your answers on a graph.

20. The Metro Rail service sells $1200 - 80x$ fares each day when it charges x dollars per fare.

a. Write an equation for the revenue in terms of the price of a fare.

b. What fare will return the maximum revenue?

c. What is the maximum revenue?

d. Verify your answers on a graph.

21. A beekeeper has beehives distributed over 60 square miles of pastureland. When she places four hives per square mile, each hive produces about 32 pints of honey per year. For each additional hive per square mile, honey production drops by 4 pints per hive.

a. Write an equation for the total production of honey, in pints, in terms of the number of additional hives per square mile.

b. How many additional hives per square mile should the beekeeper install in order to maximize honey production?

22. A small company manufactures radios. When it charges \$20 for a radio, it sells 500 radios per month. For each dollar the price is increased, 10 fewer radios are sold per month.

a. Write an equation for the monthly revenue in terms of the price increase over \$20.

b. What should the company charge for a radio in order to maximize its monthly revenue?

23. Find values of a , b, and c so that the graph of the parabola $y = ax^2+bx+c$ contains the points $(-1, -4)$, $(0, -6)$ and $(4, 6)$.

24. Find a parabola that fits the following data points.

x	-8	-4	2	4
y	10	18	0	-14

25. The height of a cannonball was observed at 0.2-second intervals after the cannon was fired, and the data recorded in the table below.

Time (seconds)	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
Height(metes)	10.2	19.2	27.8	35.9	43.7	51.1	58.1	64.7	71.0	76.8

a. Find the equation of the least-squares regression line for height in terms of time.

b. Use the linear regression equation to predict the height of the cannonball at 3 seconds and at 4 seconds after it was fired.

c. Make a scatterplot of the data and draw the regression line on the same axes.

d. Find the quadratic regression equation for height in terms of time.

e. Use quadratic regression equation to predict the height of the cannonball at 3 seconds and at 4 seconds after it was fired.

f. Make a scatterplot of the data and draw the regression curve on the same axes.

g. Which model is more appropriate for the height of the cannonball, linear or quadratic? Why?

26. Max took a sequence of photographs of an explosion spaced at equal time intervals. From the photographs he was able to estimate the height and vertical velocity of some debris from the explosion, as shown in the table. (Negative velocities indicate that the debris is falling back to earth.)

Velocity (meters/second)	67	47	27	8	-12	-31
Height (meters)	8	122	196	232	228	185

a. Enter the data into your calculator and create a scatterplot. Fit a quadratic regression equation to the data, and graph the equation on the scatterplot.

b. Use your regression equation to find the vertex of the parabola. What do the coordinates represent, in terms of the problem? What should the velocity of the debris be at its maximum height?

Exercise Group. For Problems 27–32, solve the inequality algebraically, and give your answers in interval notation. Verify your solutions by graphing.

27. $(x - 3)(x + 2) > 0$

28. $y^2 - y - 12 \leq 0$

29. $2y^2 - y \leq 3$

30. $3z^2 - 5z > 2$

31. $s^2 \leq 4$

32. $4t^2 > 12$

33. The Sub Station sells $220 - \frac{1}{4}p$ submarine sandwiches at lunchtime if it sells them at p cents each.

a. Write a formula for the Sub Station's daily revenue in terms of p.

b. What range of prices can the Sub Station charge if it wants to keep its daily revenue from subs over \$480?

34. When it charges p dollars for an electric screwdriver, Handy Hardware will sell $30 - \frac{1}{2}p$ screwdrivers per month.

a. Write a formula in terms of p for Handy Hardware's monthly revenue from screwdrivers.

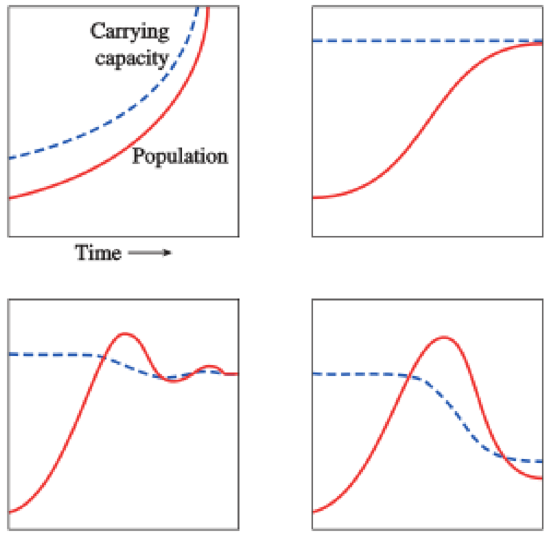
b. How much should Handy charge per screwdriver if it wants the monthly revenue from the screwdrivers to be over \$400?

Chapter 5

Functions and Their Graphs

World3 is a computer model developed by a team of researchers at MIT. The model tracks population growth, use of resources, land development, industrial investment, pollution, and many other variables that describe human impact on the planet.

The figure below is taken from *Limits to Growth: The 30-Year Update*. The graphs represent four possible answers to World3's core question: How may the global population and economy interact with and adapt to Earth's limited carrying capacity (the maximum it can sustain) over the coming decades?



Source: Meadows, Randers, and Meadows, 2004

Each of the graphs represents a nonlinear function. A **function** is a special relationship between two variables, and we have already encountered linear and quadratic funtions. In this chapter we examine the properties and features of some basic nonlinear functions, and how they may be used as mathematial models.

5.1 Functions

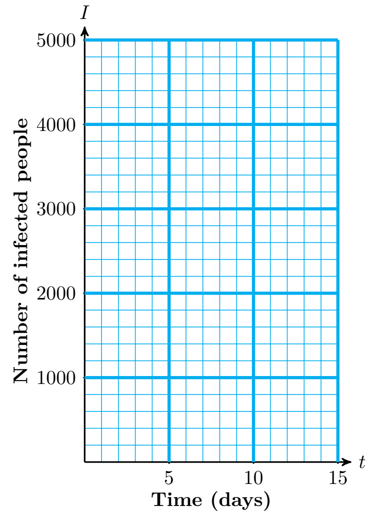
Investigation 5.1.1 Epidemics. A contagious disease whose spread is unchecked can devastate a confined population. For example, in the early 16th-century, Spanish troops introduced smallpox into the Aztec population in Central America, and the resulting epidemic contributed significantly to the fall of Montezuma's empire.

Suppose that an outbreak of cholera follows severe flooding in an isolated town of 5000 people. Initially (Day 0), 40 people are infected. Every day after that, 25% of those still healthy fall ill.

1. At the beginning of the first day (Day 1), how many people are still healthy? How many will fall ill during the first day? What is the total number of people infected after the first day?

2. Check your results against the table below. Subtract the total number of infected residents from 5000 to find the number of healthy residents at the beginning of the second day. Then fill in the rest of the table for 10 days. (Round off decimal results to the nearest whole number.)

Day	Number healthy	New patients	Total infected
0	5000	40	40
1	4960	1240	1280
2			
3			
4			
5			
6			
7			
8			
9			
10			



3. Use the last column of the table to plot the total number of infected residents, I, against time, t, on the grid. Connect your data points with a smooth curve.

4. Do the values of I approach some largest value? Draw a dotted horizontal line at that value of I. Will the values of I ever exceed that value?

5. What is the first day on which at least 95% of the population is infected?

6. Look back at the table. What is happening to the number of new patients each day as time goes on? How is this phenomenon reflected in the graph? How would your graph look if the number of new patients every day were a constant?

7. Summarize your work: In your own words, describe how the number of residents infected with cholera changes with time. Include a description of your graph.

5.1.1 Definitions and Notation

We often want to predict values of one variable from the values of a related variable. For example, when a physician prescribes a drug in a certain dosage, she needs to know how long the dose will remain in the bloodstream. A sales manager needs to know how the price of his product will affect its sales. A **function** is a special type of relationship between variables that allows us to make such predictions.

We have already seen some examples of functions. For instance, suppose it costs \$800 for flying lessons, plus \$30 per hour to rent a plane. If we let C represent the total cost for t hours of flying lessons, then

$$C = 800 + 30t \qquad (t \geq 0)$$

The variable t is called the **input** variable, and C is the **output** variable. Given a value of the input, we can calculate the corresponding output value using the formula for the function. Thus, for example

$$\begin{array}{lll} \text{when} & t = \mathbf{0}, & C = 800 + 30(\mathbf{0}) = 800 \\ \text{when} & t = \mathbf{4}, & C = 800 + 30(\mathbf{4}) = 920 \\ \text{when} & t = \mathbf{10}, & C = 800 + 30(\mathbf{10}) = 1100 \end{array}$$

We can display the relationship between two variables by a table or by ordered pairs. The input variable is the first component of the ordered pair, and the output variable is the second component. For the example above we have:

t	C	(t, C)
0	800	$(0, 800)$
4	920	$(4, 920)$
10	1100	$(10, 1100)$

Note that there can be only one value of C for each value of t. We say that "C is a **function** of t."

Definition 5.1.1 Definition of Function. A **function** is a relationship between two variables for which exactly one value of the **output** variable is determined by each value of the **input** variable. ◊

Checkpoint 5.1.2 QuickCheck 1. What distinguishes a function from other variable relationships?

a. There cannot be two output values for a single input value.

b. We can display the variables as ordered pairs.

c. The variables are related by a formula.

d. The values of the input and output variables must be different

Example 5.1.3

a. The distance, d, traveled by a car in 2 hours is a function of its speed, r. If we know the speed of the car, we can determine the distance it travels by the formula $d = r \cdot 2$.

b. The cost of a fill-up with unleaded gasoline is a function of the number of gallons purchased. The gas pump represents the function by displaying the corresponding values of the input variable (number of gallons) and the output variable (cost).

c. Score on the Scholastic Aptitude Test (SAT) is not a function of score on an IQ test, because two people with the same score on an IQ test may score differently on the SAT; that is, a person's score on the SAT is not uniquely determined by his or her score on an IQ test.

□

Checkpoint 5.1.4 Practice 1.

a. As part of a project to improve the success rate of freshmen, the counseling department studied the grades earned by a group of students in English and algebra. Do you think that a student's grade in algebra is a function of his or her grade in English? Explain why or why not.

b. Phatburger features a soda bar, where you can serve your own soft drinks in any size. Do you think that the number of calories in a serving of Zap Kola is a function of the number of fluid ounces? Explain why or why not.

A function can be described in several different ways. In the following examples, we consider functions defined by tables, by graphs, and by equations.

5.1.2 Functions Defined by Tables

When we use a table to describe a function, the first variable in the table (the left column of a vertical table or the top row of a horizontal table) is the input variable, and the second variable is the output. We say that the output variable *is a function of* the input.

Example 5.1.5

a. The table below shows data on sales compiled over several years by the accounting office for Eau Claire Auto Parts, a division of Major Motors. In this example, the year is the input variable, and total sales is the output. We say that total sales, S, *is a function of* t.

Year (t)	Total sales (S)
2000	\$612,000
2001	\$663,000
2002	\$692,000
2003	\$749,000
2004	\$904,000

b. The table below gives the cost of sending a letter by first-class mail in 2022.

Weight in ounces (w)	Postage (P)
$0 < w \leq 1$	\$0.60
$1 < w \leq 2$	\$0.84
$2 < w \leq 3$	\$1.08
$3 < w \leq 4$	\$1.32
$4 < w \leq 5$	\$1.56
$5 < w \leq 6$	\$1.80
$6 < w \leq 7$	\$2.04

If we know the weight of the article being mailed, we can find the postage from the table. For instance, a catalog weighing 4.5 ounces would require \$1.56 in postage. In this example, w is the input variable and p is the output variable. We say that p *is a function of* w.

c. The table below records the age and cholesterol count for 20 patients tested in a hospital survey.

Age	Cholesterol count		Age	Cholesterol count
53	217		**51**	**209**
48	232		53	241
55	198		49	186
56	238		**51**	**216**
51	**227**		57	208
52	264		52	248
53	195		50	214
47	203		56	271
48	212		53	193
50	234		48	172

According to these data, cholesterol count is *not* a function of age, because several patients who are the same age have different cholesterol levels. For example, three different patients are 51 years old but have cholesterol counts of 227, 209, and 216, respectively. Thus, we cannot determine a *unique* value of the output variable (cholesterol count) from the value of the input variable (age). Other factors besides age must influence a person's cholesterol count.

□

Note 5.1.6 Note that several different inputs for a function can have the same output. For example, the inputs 4.5 and 4.25 in part (b) of Example 5.1.5 above have output \$1.56. However, a single input cannot have more than one output, as illustrated in part (c) of the Example.

Checkpoint 5.1.7 Practice 2. Decide whether each table describes y as a function of x. Explain your choice.

a.

x	3.5	2.0	2.5	3.5	2.5	4.0	2.5	3.0
y	2.5	3.0	2.5	4.0	3.5	4.0	2.0	2.5

b.

x	-3	-2	-1	0	1	2	3
y	17	3	0	-1	0	3	17

Checkpoint 5.1.8 QuickCheck 2. How would you know if a table of values does not come from a function?

a. The output values are all the same.

b. The input values are not evely spaced.

c. Two different input values have the same output value.

d. Two different output values have the same input value

5.1.3 Functions Defined by Graphs

We can also use a graph to define a function. The input variable is displayed on the horizontal axis, and the output variable on the vertical axis.

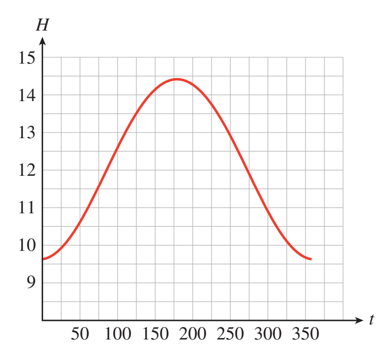
Example 5.1.9 The graph shows the number of hours, H, that the sun is above the horizon in Peoria, Illinois, on day t, where $t = 0$ on January 1.

a. Which variable is the input, and which is the output?

b. How many hours of sunlight are there in Peoria on day 150?

c. On which days are there 12 hours of sunlight?

d. What are the maximum and minimum values of H, and when do these values occur?



Solution.

a. The input variable, t, appears on the horizontal axis. The number of daylight hours, H, is a function of the date. The output variable appears on the vertical axis.

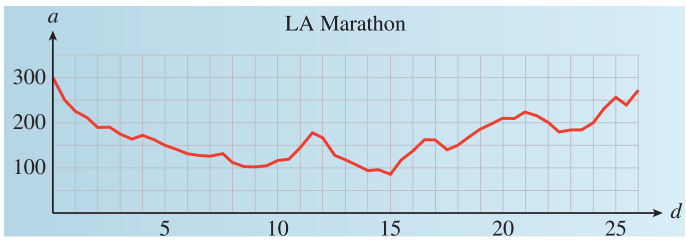
b. The point on the curve where $t = 150$ has $H \approx 14.1$, so Peoria gets about 14.1 hours of daylight when $t = 150$, which is at the end of May.

c. $H = 12$ at the two points where $t \approx 85$ (in late March) and $t \approx 270$ (late September).

d. The maximum value of 14.4 hours occurs on the longest day of the year, when $t \approx 170$, about three weeks into June. The minimum of 9.6 hours occurs on the shortest day, when $t \approx 355$, about three weeks into December.

□

Checkpoint 5.1.10 Practice 3. The graph shows the elevation in feet, a, of the Los Angeles Marathon course at a distance d miles into the race. (Source: *Los Angeles Times*, March 3, 2005)



a. Which variable is the input, and which is the output?

b. What is the elevation at mile 20?

c. At what distances is the elevation 150 feet?

d. What are the maximum and minimum values of a, and when do these values occur?

e. The runners pass by the Los Angeles Coliseum at about 4.2 miles into the race. What is the elevation there?

5.1.4 Functions Defined by Equations

Example 5.1.11 illustrates a function defined by an equation.

Example 5.1.11 As of 2020, One World Trade Center in New York City is the nation's tallest building, at 1776 feet. If an algebra book is dropped from the top of One World Trade Center, its height above the ground after t seconds is given by the equation

$$h = 1776 - 16t^2$$

Thus, after **1** second the book's height is

$$h = 1776 - 16(\mathbf{1})^2 = 1760 \text{ feet}$$

After **2** seconds its height is

$$h = 1776 - 16(\mathbf{2})^2 = 1712 \text{ feet}$$

For this function, t is the input variable and h is the output variable. For any value of t, a unique value of h can be determined from the equation for h. We say that *h is a function of t*. □

Checkpoint 5.1.12 Practice 4. Write an equation that gives the volume, V, of a sphere as a function of its radius, r.

Checkpoint 5.1.13 QuickCheck 3. Name three ways to describe a function.

a. By inputs, outputs, or evaluation

b. By tables, equations, or graphs

c. By the intercepts, the slope, or the vertex

d. By numbers, letters, or diagrams

5.1.5 Function Notation

There is a convenient notation for discussing functions. First, we choose a letter, such as f, g, or h (or F, G, or H), to name a particular function. (We can use any letter, but these are the most common choices.)

For instance, in Example 5.1.11, the height, h, of a falling algebra book is a function of the elapsed time, t. We might call this function f. In other words, f is the name of the relationship between the variables h and t. We write

$$h = f(t)$$

which means "h is a function of t, and f is the name of the function."

Caution 5.1.14 The new symbol $f(t)$, read "f of t," is another name for the height, h. The parentheses in the symbol $f(t)$ do not indicate multiplication. (It would not make sense to multiply the name of a function by a variable.) Think of the symbol $f(t)$ as a single variable that represents the output value of the function.

With this new notation we may write

$$h = f(t) = 1776 - 16t^2$$

or just

$$f(t) = 1776 - 16t^2$$

instead of

$$h = 1776 - 16t^2$$

to describe the function.

Note 5.1.15 Perhaps it seems complicated to introduce a new symbol for h, but the notation $f(t)$ is very useful for showing the correspondence between specific values of the variables h and t.

Example 5.1.16 In Example 5.1.11, the height of an algebra book dropped from the top of One World Trade Center is given by the equation

$$h = 1776 - 16t^2$$

We see that

$$\begin{aligned} &\text{when } t = 1, \quad h = 1760 \\ &\text{when } t = 2, \quad h = 1712 \end{aligned}$$

Using function notation, these relationships can be expressed more concisely as

$$f(1) = 1760 \quad \text{and} \quad f(2) = 1712$$

which we read as "f of 1 equals 1760" and "f of 2 equals 1712." The values for the input variable, t, appear *inside* the parentheses, and the values for the output variable, h, appear on the other side of the equation. □

Remember that when we write $y = f(x)$, the symbol $f(x)$ is just another name for the output variable. [TK]

[TK] For more examples of function terminology, see Section 5.1.1 of the Toolkit.

Function Notation.

Input variable

$$f(x) = y$$

Output variable

Checkpoint 5.1.17 QuickCheck 4. True or False.

a. The notation $f(t)$ indicates the product of f and t.

b. If $y = f(x)$, then $f(x)$ gives the value of the input variable.

c. If Q is a function of M, we may write $M = f(Q)$.

d. In the equation $d = g(n)$, the letters d, g, and n are variables.

Checkpoint 5.1.18 Practice 5. Let F be the name of the function defined by the graph in Example 5.1.9, the number of hours of daylight in Peoria t days after January 1.

a. Use function notation to state that H is a function of t.

b. What does the statement $F(15) = 9.7$ mean in the context of the problem?

Checkpoint 5.1.19 QuickCheck 5. Use function notation to write the statement "L defines w as a function of p."

a. $L = w(p)$

b. $w = L(p)$

c. $p = L(w)$

d. $L = p(w)$

5.1.6 Using Function Notation

Finding the value of the output variable that corresponds to a particular value of the input variable is called **evaluating the function**.

Example 5.1.20 Let g be the name of the postage function defined by the table in Example 5.1.5 b. Find $g(1)$, $g(3)$, and $g(6.75)$.

Solution. According to the table,

$$\begin{array}{llll} \text{when } w = 1, & p = 0.60 & \text{so} & g(1) = 0.60 \\ \text{when } w = 3, & p = 1.00 & \text{so} & g(3) = 1.08 \\ \text{when } w = 6.75, & p = 2.04 & \text{so} & g(6.75) = 2.04 \end{array}$$

Thus, a letter weighing 1 ounce costs \$0.60 to mail, a letter weighing 3 ounces costs \$1.08, and a letter weighing 6.75 ounces costs \$1.04. □

We can also find the input (or inputs) corresponding to a given output. For example, if $p = g(w)$ is the postage function, we solve the equation $g(w) = 0.84$ by finding all input values, w, that correspond to the output \$0.84. According to the table in Example 5.1.5 part (b), any value of w greater than 1 but less than or equal to 2 is a solution.

Checkpoint 5.1.21 Practice 6. When you exercise, your heart rate should increase until it reaches your target heart rate. The table shows target heart rate, $r = f(a)$, as a function of age.

a	20	25	30	35	40	45	50	55	60	65	70
r	150	146	142	139	135	131	127	124	120	116	112

a. Find $f(25)$ and $f(50)$.

b. Find a value of a for which $f(a) = 135$.

Checkpoint 5.1.22 QuickCheck 6. If $n = f(a)$, what are the input and output variables?

a. f is the output and n is the input

b. a is the input and f is the output

c. a is the input and$n$ is the output

d. $f(a)$ is the input and n is the output

To evaluate a function described by an equation, we simply substitute the given input value into the equation to find the corresponding output, or function value.

Example 5.1.23 The function H is defined by $H = f(s) = \dfrac{\sqrt{s+3}}{s}$. Evaluate the function at the following values.

a. $s = 6$

b. $s = -1$

Solution.

a. $f(\mathbf{6}) = \dfrac{\sqrt{\mathbf{6}+3}}{\mathbf{6}} = \dfrac{\sqrt{9}}{6} = \dfrac{3}{6} = \dfrac{1}{2}$. Thus, $f(6) = \dfrac{1}{2}$.

b. $f(\mathbf{-1}) = \dfrac{\sqrt{\mathbf{-1}+3}}{\mathbf{-1}} = \dfrac{\sqrt{2}}{-1} = -\sqrt{2}$. Thus, $f(-1) = -\sqrt{2}$.

□

Checkpoint 5.1.24 Practice 7. Complete the table displaying ordered pairs for the function $f(x) = 5 - x^3$. Evaluate the function to find the corresponding $f(x)$-value for each value of x.

x	$f(x)$
-2	
0	
1	
3	

$f(\mathbf{-2}) = 5 - (\mathbf{-2})^3 =$

$f(\mathbf{0}) = 5 - \mathbf{0}^3 =$

$f(\mathbf{1}) = 5 - \mathbf{1}^3 =$

$f(\mathbf{3}) = 5 - \mathbf{3}^3 =$

5.1.7 Problem Set 5.1

Warm Up

Exercise Group. For Problems 1-4, evaluate.

1. $2x - x^2$ for $x = -4$

2. $\dfrac{2z-3}{z+2}$ for $z = \dfrac{1}{2}$

3. $\sqrt{36 - (r+1)^2}$ for $r = 3$

4. $-t^3 + 3t^2$ for $t = -2$

Exercise Group. For Problems 5-8, solve.

[TK] To review solving nonlinear equations, see Section 5.1.2 of the Toolkit.

5. $4 - 5x - 2x^2 = 1$

6. $6(2x-8)^2 = 20$

7. $\dfrac{1}{2x-9} = 3$

8. $5\sqrt{8+x} = 20$

Skills Practice

9. $x = h(v) = 2v^2 - 3v + 1$

a. Which variable is the input, and which is the output?

b. Evaluate $h(-2)$.

c. Solve $h(v) = 6$.

10. $A = g(r) = 750(1+r)^2$

a. Which variable is the input, and which is the output?

b. Evaluate $g(0.04)$.

c. Solve $g(r) = 874.80$.

Exercise Group. For Problems 11 and 12, evaluate the function.

11. $F(x) = \dfrac{1-x}{2x-3}$

a. $F(0)$

b. $F(-3)$

c. $F\left(\dfrac{5}{2}\right)$

d. $F(9.8)$

12. $E(t) = \sqrt{t-4}$

a. $E(16)$

b. $E(4)$

c. $E(7)$

d. $E(4.2)$

Applications

13. Which of the following tables define the second variable as a function of the first variable? Explain why or why not.

a.

x	t
-1	2
0	9
1	-2
0	-3
-1	5

b.

y	w
0	8
1	12
3	7
5	-3
7	4

c.

x	y
-3	8
-2	3
-1	0
0	-1
1	0
2	3
3	8

d.

s	t
2	5
4	10
6	15
8	20
6	25
4	30
2	35

e.

r	-4	-2	0	2	4
v	6	6	3	6	8

f.

p	-5	-4	-3	-2	-1
d	-5	-4	-3	-2	-1

14. Which of the following tables define the second variable as a function of the first variable? Explain why or why not.

a.

Pressure (p)	Volume (v)
15	100.0
20	75.0
25	60.0
30	50.0
35	42.8
40	37.5
45	33.3
50	30.0

b.

Frequency (f)	Wavelength (w)
5	60.0
10	30.0
20	15.0
30	10.0
40	7.5
50	6.0
60	5.0
70	4.3

c.

Temperature (T)	Humidity (h)
Jan. 1 34°F	42%
Jan. 2 36°F	44%
Jan. 3 35°F	47%
Jan. 4 29°F	50%
Jan. 5 31°F	52%
Jan. 6 35°F	51%
Jan. 7 34°F	49%

d.

Inflation rate (I)	Unemployment rate (U)
1972 5.6%	5.1%
1973 6.2%	4.5%
1974 10.1%	4.9%
1975 9.2%	7.4%
1976 5.8%	6.7%
1977 5.6%	6.8%
1978 6.7%	7.4%

15. The function described in Problem 14(a) is called g, so that $v = g(p)$. Find the following:

a. $g(25)$

b. $g(40)$

c. x so that $g(x) = 50$

16. The function described in Problem 14(b) is called h, so that $w = h(f)$. Find the following:

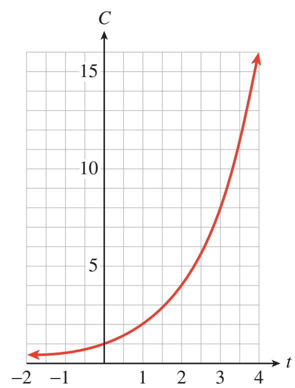
a. $h(20)$

b. $h(60)$

c. x so that $h(x) = 10$

Exercise Group. For Problems 17—24, use the graph of the function to answer the questions.

17. The graph shows $C = h(t)$, where C stands for the number of customers (in thousands) signed up for a new movie streaming service, measured in months after their advertising campaign at $t = 0$ in January.



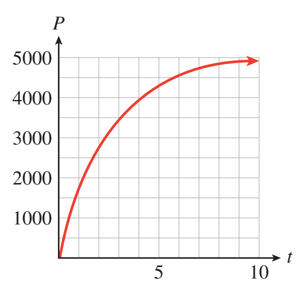
a. When did the service have 2000 customers? Write your answer with function notation.

b. How long did it take that number to double?

c. How long did it take for the number to double again?

d. How many customers signed up between March and April (months 2 and 3)?

18. The graph shows P as a function of t. P is the number of houses in Cedar Grove who have had solar panels installed t years after 2000.



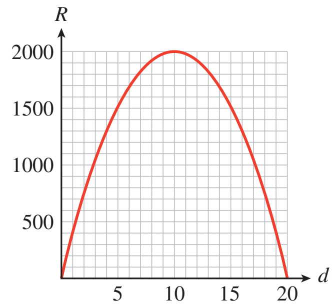
a. When did 3500 houses have solar panels? Write your answer using function notation.

b. How many houses had solar panels in 2005? Write your answer using function notation.

c. The number of houses with solar panels in Cedar Grove seems to be leveling off at what number?

d. How many houses had solar panels installed between 2001 and 2004?

19. The graph shows the revenue, $R = f(d)$, a movie theater collects as a function of the price, d, it charges for a ticket.



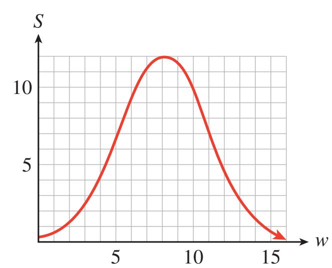
a. Estimate the revenue if the theater charges \$12.00 for a ticket.

b. What should the theater charge for a ticket in order to collect \$1500 in revenue?

c. Write your answers to parts (a) and (b) using function notation.

d. For what values of d is $R > 1800$?

20. The graph shows $S = g(w)$. S represents the weekly sales of a best-selling book, in thousands of dollars, w weeks after it is released.



a. In which weeks were sales over \$7000?

b. In which week did sales fall below \$5000 on their way down?

c. For what values of w is $S > 3.4$?

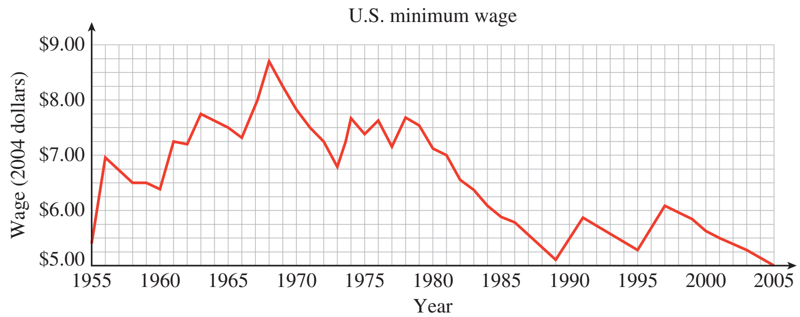
21. The graph shows the U.S. unemployment rate, $U = F(t)$, where t, represents years. Give your answers to the questions below in function notation. (Source: U.S. Bureau of Labor Statistics)

a. When did the unemployment rate reach its highest value, and what was its highest value?

b. When did the unemployment rate fall to its lowest value, and what was its lowest value?

c. Give two years in which the unemployment rate was 4.5%.

22. The graph shows the federal minimum wage, M, over the past five decades, adjusted for inflation to reflect its buying power in 2004 dollars. (Source: www.infoplease.com)

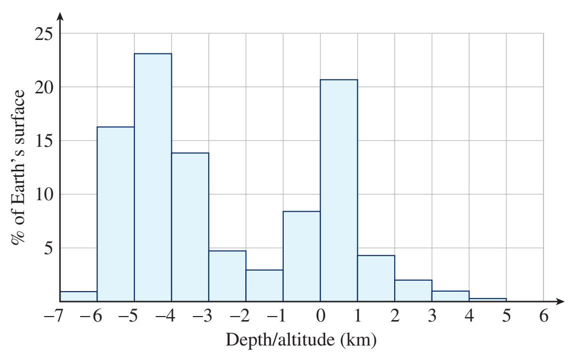


a. Is M a function of t? Support your answer.

b. What is the largest function value on the graph, and when did it occur? Write your answer with function notation, and explain what it means about the federal minimum wage.

c. Give two years in which the minimum wage was worth \$8 in 2004 dollars. Does this fact mean that M is not a function of t? Why or why not?

23. The bar graph shows the percent of Earth's surface that lies at various altitudes or depths below the surface of the oceans. (Depths are given as negative altitudes.) (Source: Open University)



a. Read the graph and complete the table.

Altitude (km)	Percent of Earth's surface
-7 to -6	
-6 to -5	
-5 to -4	
-4 to -3	
-3 to -2	
-2 to -1	
-1 to 0	
0 to 1	
1 to 2	
2 to 3	
3 to 4	
4 to 5	

b. What is the most common altitude? What is the second most common altitude?

c. Approximately what percent of the Earth's surface is below sea level?

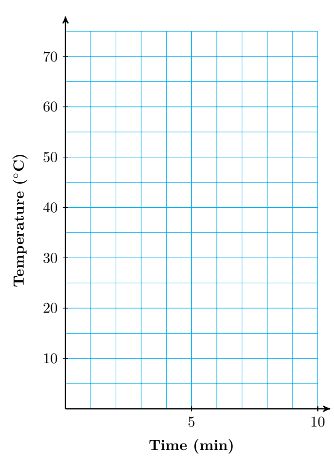
d. The height of Mt. Everest is 8.85 kilometers. Can you think of a reason why it is not included in the graph?

24. Energy is necessary to raise the temperature of a substance, and it is also needed to melt a solid substance to a liquid. The table shows data from heating a solid sample of stearic acid. Heat was applied at a constant rate throughout the experiment.

Time (minutes)	0	0.5	1.5	2	2.5	3	4	5	6	7	8	8.5	9	9.5	10
Temperature (deg C)	19	29	40	48	53	55	55	55	55	55	55	64	70	73	74

a. Did the temperature rise at a constant rate? Describe the temperature as a function of time.

b. Graph the temperature as a function of time.



c. What is the melting point of stearic acid? How long did it take the sample to melt?

25. The number of compact cars that a large dealership can sell at price p is given by

$$N(p) = \frac{12,000,000}{p}$$

a. Evaluate $N(6000)$ and explain what it means.

b. As p increases, does $N(p)$ increase or decrease? Support your answer with calculations.

c. Solve the equation $F(p) = 400$, and explain what it means.

26. The distance, d, in miles that a person can see on a clear day from a height, h, in feet is given by

$$d = G(h) = 1.22\sqrt{h}$$

a. Evaluate $G(20, 320)$ and explain what it means.

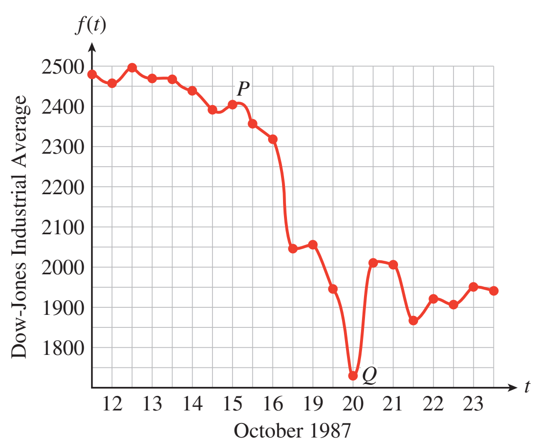
b. As h increases, does d increase or decrease? Support your answer with calculations.

c. Estimate the height you need in order to see 100 miles. Write your answer with function notation.

5.2 Graphs of Functions

5.2.1 Reading Function Values from a Graph

The graph below shows the Dow-Jones Industrial Average (the average value of the stock prices of 500 major companies) recorded at noon each day during the stock market correction around October 10, 1987 ("Black Monday").



The graph describes a function because there is only one value of the output, DJIA, for each value of the input, t. There is no formula that gives the DJIA for a particular day; but it is still a function, defined by its graph. The value of $f(t)$ is specified by the vertical coordinate of the point with the given t-coordinate.

Example 5.2.1

a. The coordinates of point P on the DJIA graph are $(15, 2412)$. What do the coordinates tell you about the function f?

b. If the DJIA was 1726 at noon on October 20, what can you say about the graph of f?

Solution.

a. The coordinates of point P tell us that $f(15) = 2412$, so the DJIA was 2412 at noon on October 15.

b. We can say that $f(20) = 1726$, so the point $(20, 1726)$ lies on the graph of f. This point is labeled Q in the figure above.

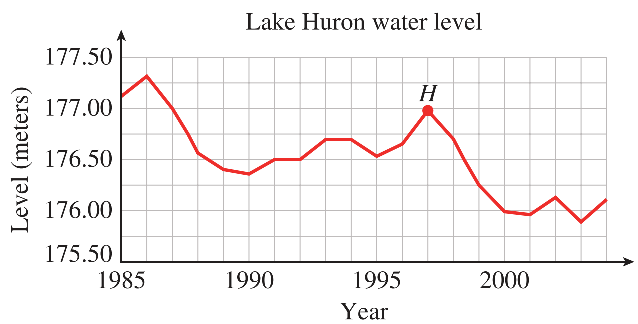
□

The coordinates of each point on the graph of the function give the output for a specific input.

Graph of a Function.

The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

Checkpoint 5.2.2 Practice 1. The water level in Lake Huron alters unpredictably over time. The graph below gives the average water level, $L = f(t)$, in meters in the year t over a 20-year period. (Source: The Canadian Hydrographic Service)



a. The coordinates of point H on the graph are $(1997, 176.98)$. What do the coordinates tell you about the function f?

b. The average water level in 2004 was 176.11 meters. Write this fact in function notation. What can you say about the graph of f?

The second coordinate of a point on the graph is the function value for the first coordinate.

Functions and Coordinates.

Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

Checkpoint 5.2.3 QuickCheck 1. Decide whether each statement is true or false.

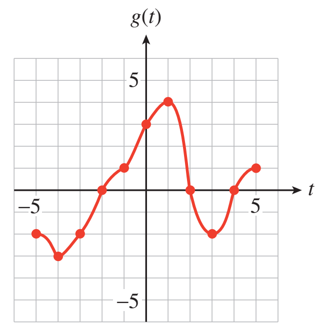
a. If $(12, 5)$ lies on the graph of f, then $f(5) = 12$.

b. If $g(p) = w$, then (p, w) lies on the graph of g.

c. We can find the function value at x by finding the x-coordinate of the corresponding point on the graph of f.

d. Every y-coordinate on the graph of F represents a function value for F.

Example 5.2.4 The figure shows the graph of a function g.



a. Find $g(-2)$ and $g(5)$. [TK]

b. For what value(s) of t is $g(t) = -2$?

c. What is the largest, or maximum, value of $g(t)$? For what value of t does the function take on its maximum value?

d. On what intervals is g increasing?

Solution.

a. To find $g(-2)$, we look for the point with t-coordinate -2. The point $(-2, 0)$ lies on the graph of g, so $g(-2) = 0$. Similarly, the point $(5, 1)$ lies on the graph, so $g(5) = 1$.

b. We look for points on the graph with y-coordinate -2. Because the points $(-5, -2)$, $(-3, -2)$, and $(3, -2)$ lie on the graph, we know that $g(-5) = -2$, $g(-3) = -2$, and $g(3) = -2$. Thus, the t-values we want are -5, -3, and 3.

c. The highest point on the graph is $(1, 4)$, so the largest y-value is 4. Thus, the maximum value of $g(t)$ is 4, and it occurs when $t = 1$.

d. A graph is increasing if the y-values get larger as we read from left to right. The graph of g is increasing for t-values between -4 and 1, and between 3 and 5. Thus, g is increasing on the intervals $(-4, 1)$ and $(3, 5)$.

□

[TK] For more practice with function notation, see Section 5.2.1 of the Toolkit.

Checkpoint 5.2.5 QuickCheck 2. Decide whether each statement is true or false.

a. A graph is called increasing if its x-values increase.

b. The maximum function value is the y-coordinate of the highest point on the graph.

c. If we say that f is increasing on the interval $(2, 7)$, we mean that the function values increased from 2 to 7.

d. It is not possible for a function of x to take on the same value at two different x-values.

Checkpoint 5.2.6 Practice 2. Refer to the graph of the function g shown in Example 5.2.4.

a. Find $g(0)$.

b. For what value(s) of t is $g(t) = 0$?

c. What is the smallest, or minimum, value of $g(t)$? For what value of t does the function take on its minimum value?

d. On what intervals is g decreasing?

5.2.2 Constructing the Graph of a Function

We can construct a graph for a function described by a table or an equation. We make these graphs the same way we graph equations in two variables: by plotting points whose coordinates satisfy the equation.

Example 5.2.7 Graph the function $f(x) = \sqrt{x+4}$

Solution. We choose several convenient values for x and evaluate the function to find the corresponding $f(x)$-values. For this function we cannot choose x-values less than -4, because the square root of a negative number is not a real number.

$$f(\mathbf{-4}) = \sqrt{\mathbf{-4}+4} = \sqrt{0} = 0$$

$$f(\mathbf{-3}) = \sqrt{\mathbf{-3}+4} = \sqrt{1} = 1$$

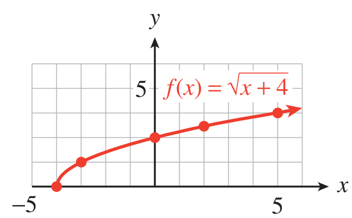
$$f(\mathbf{0}) = \sqrt{\mathbf{0}+4} = \sqrt{4} = 2$$

$$f(\mathbf{2}) = \sqrt{\mathbf{2}+4} = \sqrt{6} \approx 2.45$$

$$f(\mathbf{5}) = \sqrt{\mathbf{5}+4} = \sqrt{9} = 3$$

The results are shown in the table.

x	$f(x)$
-4	0
-3	1
0	2
2	$\sqrt{6}$
5	3



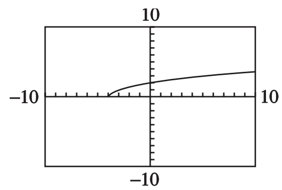
Points on the graph have coordinates $(x, f(x))$, so the vertical coordinate of each point is given by the value of $f(x)$. We plot the points and connect them with a smooth curve, as shown in the figure. Notice that no points on the graph have x-coordinates less than -4. ☐

Checkpoint 5.2.8 QuickCheck 3. How do we find the value of $f(3)$ from a graph of f?

Technology 5.2.9 Using Technology to Graph a Function. We can also use a graphing utility to obtain a table and graph for the function in Example 5.2.7. We graph a function just as we graphed an equation. For this function, we enter

$$Y_1 = \sqrt{}(X+4)$$

and press ZOOM 6 for the standard window. Your calculator does not use the $f(x)$ notation for graphs, so we will continue to use Y_1, Y_2, etc. for the output variable. Don't forget to enclose $x+4$ in parentheses, because it appears under a radical. The graph is shown below.



Checkpoint 5.2.10 Practice 3. $f(x) = x^3 - 2$

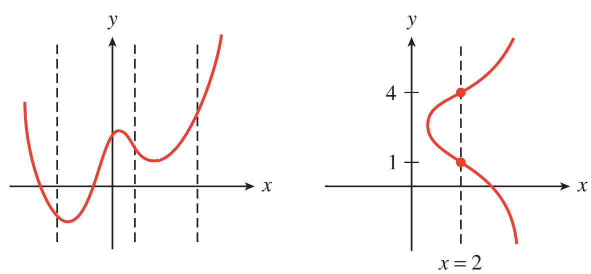
Complete the table of values and sketch a graph of the function by hand. Then use technology to make a table of values and graph the function.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$f(x)$							

5.2.3 The Vertical Line Test

In a function, two different outputs cannot be related to the same input. This restriction means that two different ordered pairs cannot have the same first coordinate. What does it mean for the graph of the function?

Consider the graph shown in Figure (a). Every vertical line intersects the graph in at most one point, so there is only one point on the graph for each x-value. This graph represents a function.



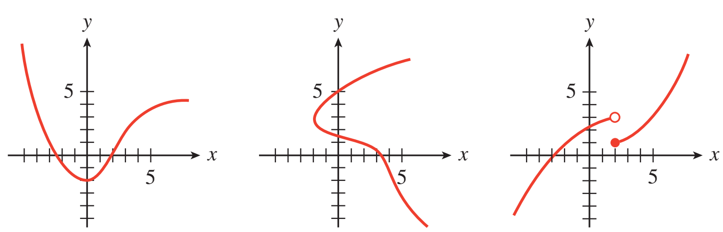
(a) (b)

In Figure (b), however, the line $x = 2$ intersects the graph at two points, $(2, 1)$ and $(2, 4)$. Two different y-values, 1 and 4, are related to the same x-value, 2. This graph cannot be the graph of a function.

The Vertical Line Test.

A graph represents a function if and only if every vertical line intersects the graph in at most one point.

Example 5.2.11 Use the vertical line test to decide which of the graphs in the figure represent functions.



(a) (b) (c)

Solution.

- Graph (a) represents a function, because it passes the vertical line test.

- Graph (b) is not the graph of a function, because the vertical line at (for example) $x = 1$ intersects the graph at two points.
- For graph (c), notice the break in the curve at $x = 2$: The solid dot at $(2, 1)$ is the only point on the graph with $x = 2$; the open circle at $(2, 3)$ indicates that $(2, 3)$ is not a point on the graph. Thus, graph (c) is a function, with $f(2) = 1$.

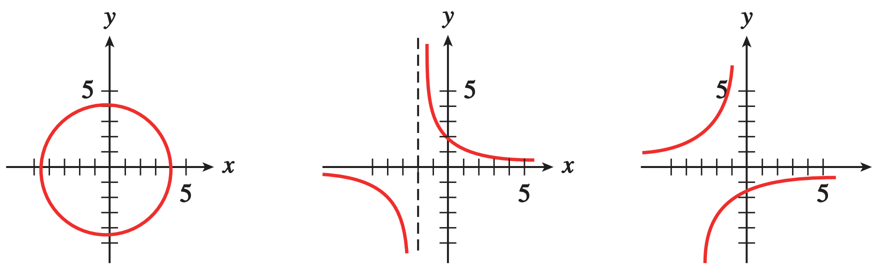
□

Checkpoint 5.2.12 QuickCheck 4. Fill in the blanks to coplete the statements.

a. The vertical line test is used to decide whether a graph represents a __________.

b. If it is, there cannot be two points on the graph with the same __________.

Checkpoint 5.2.13 Practice 4. Use the vertical line test to determine which of the graphs below represent functions.



(a) (b) (c)

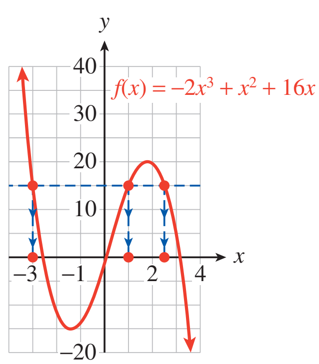
5.2.4 Graphical Solution of Equations and Inequalities

We have used graphs to solve linear and quadratic equations and inequalities. We can also use a graphical technique to solve equations and inequalities involving other functions.

Example 5.2.14 Use a graph of $f(x) = -2x^3 + x^2 + 16x$ to solve the equation

$$-2x^3 + x^2 + 16x = 15$$

Solution. If we sketch in the horizontal line $y = 15$, we can see that there are three points on the graph of f that have y-coordinate 15, as shown below. The x-coordinates of these points are the solutions of the equation.



From the graph, we see that the solutions are $x = -3$, $x = 1$, and approximately $x = 2.5$. We can verify each solution algebraically.

For example, if $x = \mathbf{-3}$, we have

$$\begin{aligned} f(\mathbf{-3}) &= -2(\mathbf{-3})3 + (\mathbf{-3})^2 + 16(\mathbf{-3}) \\ &= -2(-27) + 9 - 48 \\ &= 54 + 9 - 48 = 15 \end{aligned}$$

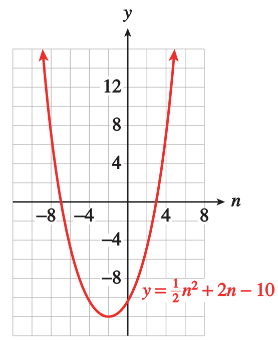
so -3 is a solution. Similarly, you can check that $x = 1$ and $x = 2.5$ are solutions. □

Checkpoint 5.2.15 Practice 5.

Use the graph of $y = \frac{1}{2}n^2 + 2n - 10$ shown at right to solve

$$\frac{1}{2}n^2 + 2n - 10 = 6$$

Verify your solutions algebraically.



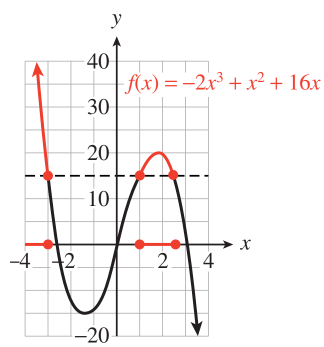
Checkpoint 5.2.16 QuickCheck 5. To solve an equation graphically, it is helpful to sketch in a __________ line. We are looking for __________ values that give the required __________ value.

Example 5.2.17 Use the graph in Example 5.2.14 to solve the inequality [TK]

$$-2x^3 + x^2 + 16x \geq 15$$

Solution. We first locate all points on the graph that have y-coordinates greater than or equal to 15. The x-coordinates of these points are the solutions of the inequality.

The figure below shows the points in red, and their x-coordinates as intervals on the x-axis. The solutions are $x \leq -3$ and $1 \leq x \leq 2.5$, or in interval notation, $(-\infty, -3] \cup [1, 2.5]$.



□

[TK] For more practice solving equatins and inequalities with a graph, see Section 5.2.2 of the Toolkit.

Checkpoint 5.2.18 Practice 6. Use the graph above from Checkpoint 5.2.15 to solve the inequality

$$\frac{1}{2}n^2 + 2n - 10 < 6$$

5.2.5 More about Notation

To simplify the notation, we sometimes use the same letter for the output variable and for the name of the function. In the next example, C is used in this way.

Example 5.2.19 TrailGear decides to market a line of backpacks. The cost, C, of manufacturing backpacks is a function of the number of backpacks produced, x, given by the equation

$$C = C(x) = 3000 + 20x$$

where $C(x)$ is in dollars. Find the cost of producing 500 backpacks.

Solution. To find the value of C that corresponds to $x = \mathbf{500}$, we evaluate $C(\mathbf{500})$:

$$C(\mathbf{500}) = 3000 + 20(\mathbf{500}) = 13,000$$

The cost of producing 500 backpacks is \$13,000. □

Checkpoint 5.2.20 **Practice 7.** The volume of a sphere of radius r centimeters is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

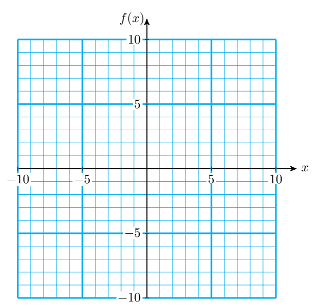
Evaluate $V(10)$ and explain what it means.

5.2.6 Problem Set 5.2

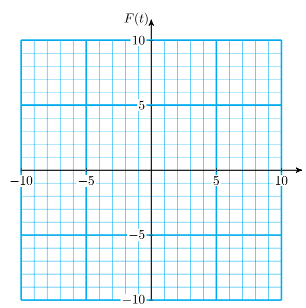
Warm Up

Exercise Group. For problems 1–4, sketch a graph of the linear or quadratic function by hand, and label the significant points.

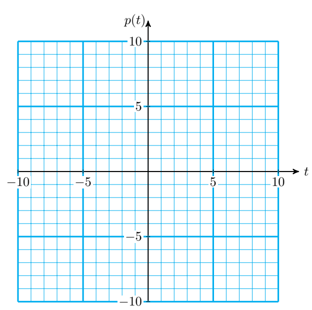
1. $f(x) = 3x - 4$



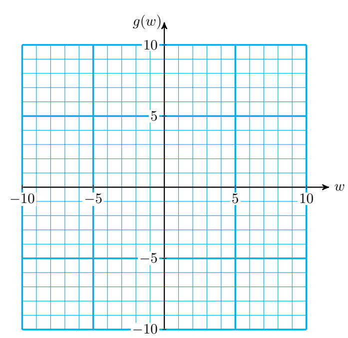
2. $F(t) = \dfrac{-3}{4}t + 6$



3. $p(t) = 3 - t^2$

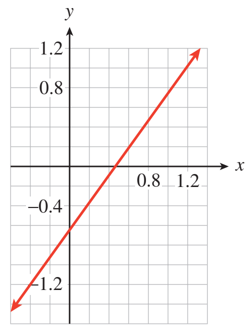


4. $g(w) = (w + 2)^2$

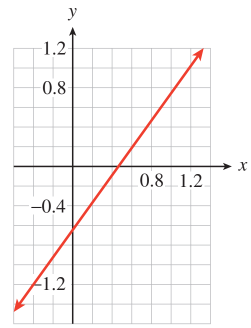


5. Use the graph of $y = 1.4x - 0.64$ to solve the inequalities. Show your method on the graph.

a. $1.4x - 0.64 > 0.2$

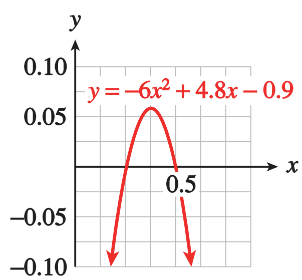


b. $-1.2 > 1.4x - 0.64$

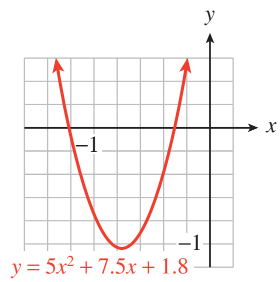


6. Use the graphs to solve the inequalities. Show your method on the graph.

a. $-6x^2 + 4.8x - 0.9 \geq 0$



b. $5x^2 + 7.5x + 1.8 \leq 0$



Skills Practice

Exercise Group. For Problems 7–10,

a. Make a table of values and sketch a graph of the function by plotting points. (Use the suggested x-values.)

b. Use your calculator to make a table of values and to graph the function. Compare the calculator's graph with your sketch.

7. $g(x) = x^3 + 4$
$x = -2, -1, \ldots, 2$

8. $h(x) = 2 + \sqrt{x}$
$x = 0, 1, \ldots, 9$

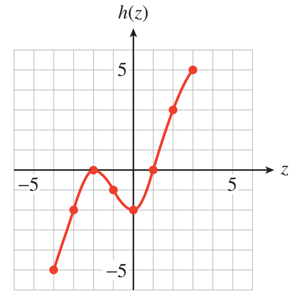
9. $G(x) = \sqrt{4 - x}$
$x = -5, -4, \ldots, 4$

10. $w(x) = x^3 - 8x$
$x = -4, -3, \ldots, 4$

Applications

Exercise Group. For Problems 11–14, use the graph.

11.



a. Find $h(-3)$, $h(1)$, and $h(3)$.

b. For what value(s) of z is $h(z) = 3$?

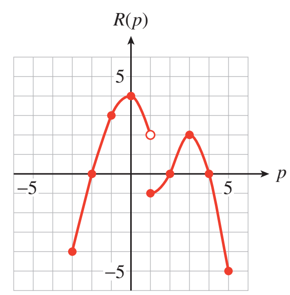
c. Find the intercepts of the graph. List the function values given by the intercepts.

d. What is the maximum value of $h(z)$?

e. For what value(s) of z does h take on its maximum value?

f. On what intervals is the function increasing? Decreasing?

12.



a. Find $R(1)$ and $R(3)$.

b. For what value(s) of p is $R(p) = 2$?

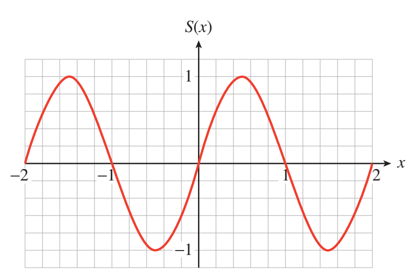
c. Find the intercepts of the graph. List the function values given by the intercepts.

d. Find the maximum and minimum values of $R(p)$.

e. For what value(s) of p does R take on its maximum and minimum values?

f. On what intervals is the function increasing? Decreasing?

13.



a. Find $S(0)$, $S\left(\frac{1}{6}\right)$, and $S(-1)$.

b. Estimate the value of $S\left(\frac{1}{3}\right)$ from the graph.

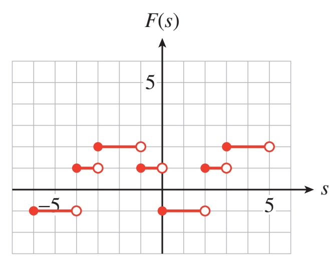
c. For what value(s) of x is $S(x) = \frac{-1}{2}$?

d. Find the maximum and minimum values of $S(x)$.

e. For what value(s) of x does S take on its maximum and minimum values?

f. On what intervals is the function increasing? Decreasing?

14.



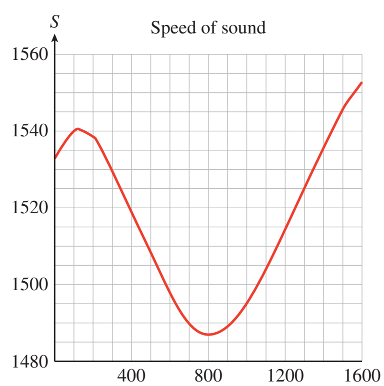
a. Find $F(-3)$, $F(-2)$ and $F(2)$.

b. For what value(s) of s is $F(s) = -1$?

c. Find the maximum and minimum values of $F(s)$.

d. For what value(s) of s does F take on its maximum and minimum values?

15. The graph shows the speed of sound in the ocean as a function of depth, $S = f(d)$. The speed of sound is affected both by increasing water pressure and by dropping temperature. (Source: Scientific American)



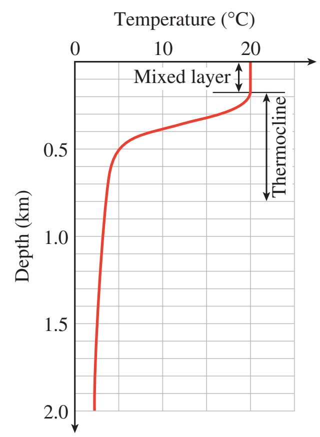
a. Evaluate $f(1000)$ and explain its meaning.

b. Solve $f(d) = 1500$ and explain its meaning.

c. At what depth is the speed of sound the slowest, and what is the speed? Write your answer with function notation.

d. Describe the behavior of $f(d)$ as d increases.

16. The figure shows the temperature of the ocean at various depths.



a. Is depth a function of temperature?

b. Is temperature a function of depth?

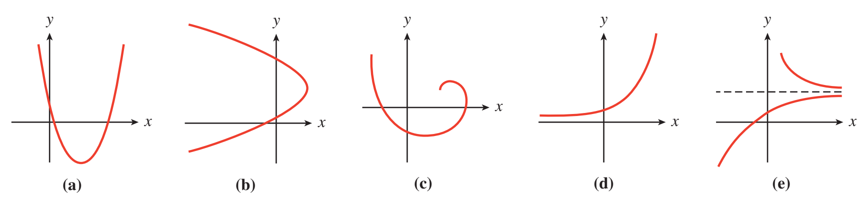
c. The axes in this figure are scaled in an unusual way. Why is it useful to present the graph in this way?

d. What is the difference in temperature between the surface of the ocean and the deepest level shown?

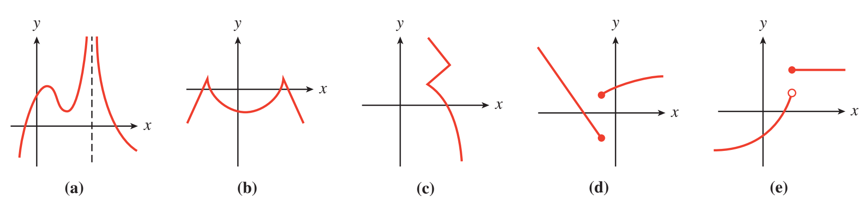
e. Over what depths does the temperature change most rapidly?

f. What is the average rate of change of temperature with respect to depth in the region called the thermocline?

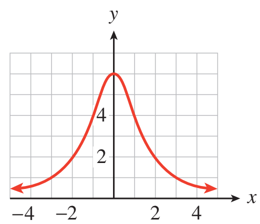
17. Which of the following graphs represent functions?



18. Which of the following graphs represent functions?



19. The figure shows the graph of $g(x) = \dfrac{12}{2+x^2}$. Use the graph to solve the following equations and inequalities. Show your work on the graph. Write your answers in interval notation.



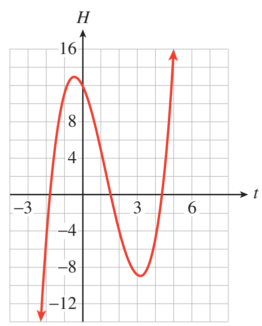
a. $\dfrac{12}{2+x^2} = 4$

b. $\dfrac{12}{2+x^2} > 4$

c. $1 \leq \dfrac{12}{2+x^2} \leq 2$

d. $\dfrac{12}{2+x^2} \leq 6$

20. The figure shows the graph of $H(t) = t^3 - 4t^2 - 4t + 12$. Use the graph to solve the following equations and inequalities. Show your work on the graph. Write your answers to the inequalities in interval notation.



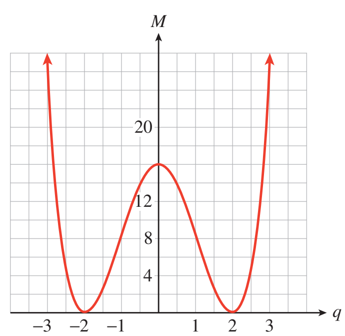
a. $t^3 - 4t^2 - 4t - 12 = -4$

b. $t^3 - 4t^2 - 4t - 12 = 17$

c. $t^3 - 4t^2 - 4t - 12 < -4$

d. $t^3 - 4t^2 - 4t - 12 > 6$

21. The figure shows a graph of $M = g(q)$.



a. Find all values of q for which

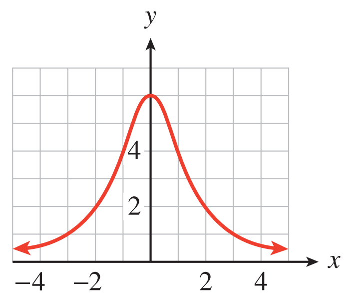
I. $g(q) = 0$

II. $g(q) = 16$

III. $g(q) < 6$

b. For what values of q is $g(q)$ increasing?

22. The figure shows a graph of $P = f(x)$.



a. Find all values of x for which

I. $f(x) = 3$

II. $f(x) > 4.5$

III. $2 \leq f(x) \leq 4$

b. For what values of x is $f(x)$ increasing?

Exercise Group. For Problems 23 and 24, graph each function in the "friendly" window

$$\text{Xmin} = -9.4 \qquad \text{Xmax} = 9.4$$
$$\text{Ymin} = -10 \qquad \text{Ymax} = 10$$

Then answer the questions about the graph.

23. $g(x) = \sqrt{36 - x^2}$

a. Find the intercepts of the graph. Write your answers in function notation.

b. Find all points on the graph for which $g(x) = 6.4$.

c. Explain why there are no points on the graph with $x > 6$ or $x < -6$.

24. $F(x) = 0.5x^3 - 4x$

a. Estimate the coordinates of the turning points of the graph, where the graph changes from increasing to decreasing or vice versa.

b. Estimate the coordinates of the x-intercepts.

c. Write an equation of the form $F(a) = b$ for each turning point and each x-intercept.

Exercise Group. For Problems 25 and 26,

a. Compute $f(0)$ and $g(0)$.

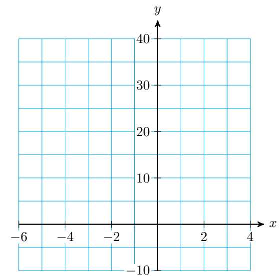
b. Find all values of x for which $f(x) = 0$.

c. Find all values of x for which $g(x) = 0$.

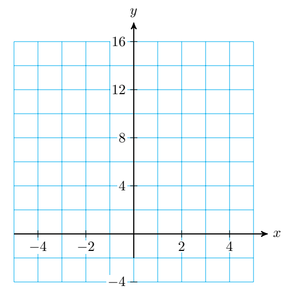
d. Find all values of x for which $f(x) = g(x)$.

e. Graph f and g on the same axes, and illustrate your answers to (a)–(d) as points on the graph.

25. $f(x) = 2x^2 + 3x, \quad g(x) = 5 - 6x$



26. $f(x) = 2x^2 - 2x, \quad g(x) = x^2 + 3$



5.3 Some Basic Graphs

In this section we study the graphs of some important basic functions. Many functions fall into families or classes of similar functions, and recognizing the appropriate family for a given situation is an important part of modeling.

We'll need two new algebraic operations.

5.3.1 Cube Roots

You are familiar with square roots. Every non-negative number has two square roots, defined as follows.

$$\boldsymbol{s} \quad \text{is a square root of} \quad \boldsymbol{n} \quad \text{if} \quad \boldsymbol{s^2 = n}$$

There are several other kinds of roots, one of which is called the **cube root**. We define the cube root as follows.

Definition 5.3.1 Cube Root. b is the **cube root** of a if b cubed equals a. In symbols, we write

$$\boldsymbol{b = \sqrt[3]{a}} \quad \text{if} \quad \boldsymbol{b^3 = a}$$

◊

Square roots of negative numbers are not real (they are complex), but every real number has a real cube root. For example,

$$\begin{array}{rcl} 4 = \sqrt[3]{64} & \text{because} & 4^3 = 64 \\ -3 = \sqrt[3]{-27} & \text{because} & (-3)^3 = -27 \end{array}$$

Simplifying radicals occupies the same position in the order of operations as computing powers: after parentheses, and before products and quotients.

Example 5.3.2 Simplify each expression. [TK]

a. $3\sqrt[3]{-8}$

b. $2 - \sqrt[3]{-125}$

Solution.

a. $3\sqrt[3]{-8} = 3(-2) = -6$

b. $2 - \sqrt[3]{-125} = 2 - (-5) = 7$

□

[TK] For more examples of cube roots, see Section 5.3.1 and Section 5.3.3 of the Toolkit.

Checkpoint 5.3.3 Practice 1. Simplify each expression.

a. $5 - 3\sqrt[3]{64}$

b. $\dfrac{6 - \sqrt[3]{-27}}{2}$

Checkpoint 5.3.4 QuickCheck 1. Decide whether each statement is true or false.

a. A negative number has a negative cube root.

b. A negative number has a negative square root.

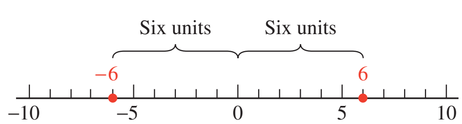
c. A positive number has a negative square root

d. A positive number has a negative cube root.

Note 5.3.5 We can use the calculator to find cube roots as follows. Press the MATH key to get a menu of options. Option 4 is labeled $\sqrt[3]{\ }$; this is the cube root key. To find the cube root of, say, 15.625, we key in
MATH 4 15.625) ENTER and the calculator returns the result, 2.5. Thus, $\sqrt[3]{15.625} = 2.5$. You can check this result by verifying that $2.5^3 = 15.625$.

5.3.2 Absolute Value

We use the absolute value to discuss problems involving distance. For example, consider the number line below. Starting at the origin, we travel in opposite *directions* to reach the two numbers 6 and -6, but the *distance* we travel in each case is the same.



The distance from a number c to the origin is called the **absolute value** of c, denoted by $|c|$. Because distance is never negative, the absolute value of a number is always positive (or zero). Thus, $|6| = 6$ and $|-6| = 6$. In general, we define the absolute value of a number x as follows.

Definition 5.3.6 Absolute Value. The **absolute value** of x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

◊

This definition is called **piecewise**, because the formula has two pieces. It says that the absolute value of a positive number (or zero) is the same as the number. To find the absolute value of a negative number, we take the opposite of the number, which is then positive. For instance,

$$|-6| = -(-6) = 6$$

Checkpoint 5.3.7 QuickCheck 2. If $|x| = -x$, what can you say about x?

a. x must be zero.

b. x must be negative.

c. x must be zero or negative.

d. This cannot happen for any value of x.

Absolute value bars act like grouping devices in the order of operations: you should complete any operations that appear inside absolute value bars before you compute the absolute value.

Example 5.3.8 Simplify each expression. [TK]

a. $|3 - 8|$

b. $|3| - |8|$

Solution.

a. We simplify the expression inside the absolute value bars first.

$$|3 - 8| = |-5| = 5$$

b. We simplify each absolute value; then subtract.

$$|3| - |8| = 3 - 8 = -5$$

□

[TK] For more examples using absolute value, see Section 5.3.2 and Section 5.3.3 of the Toolkit.

Checkpoint 5.3.9 Practice 2. Simplify each expression.

a. $12 - 3\,|-6| =$

b. $-7 - 3\,|2 - 9| =$

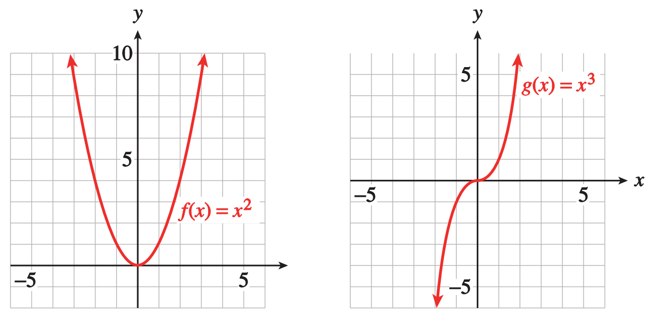
Checkpoint 5.3.10 QuickCheck 3. True or False: $|a + b| = |a| + |b|$
Hint: Try some values of a and b.

5.3.3 Eight Basic Graphs

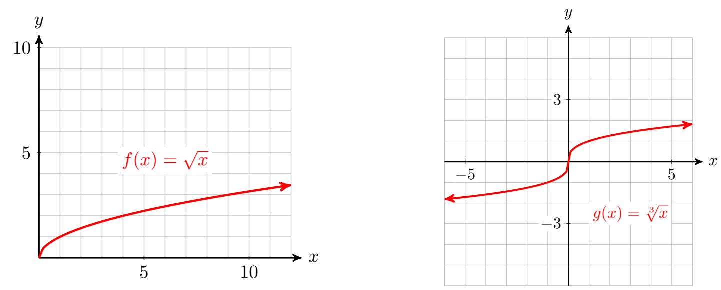
Most of the graphs in this section will be new to you, but many useful graphs are variations of the eight basic functions shown below.

Consider the first pair of graphs. You have already studied the graph of $f(x) = x^2$, the basic parabola. Compare that graph with the graph of $g(x) = x^3$. Notice several differences in the shape of the two graphs. Once you have a good idea of the shape of a graph, up can make a quick sketch with just a few "guide points." For these two graphs, complete a short table of values to find useful guide points:

x	-2	-1	0	1	2
$f(x)$					
$g(x)$					

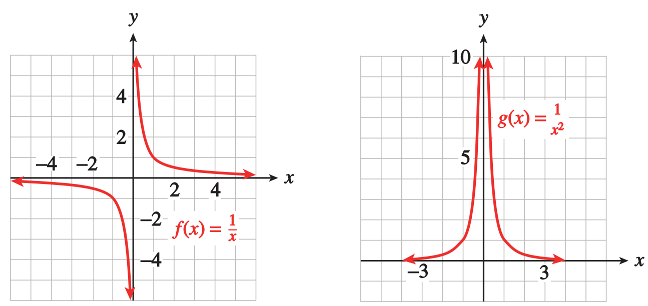


The next pair of graphs are $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$. Once again, notice the differences in the two graphs. For example, we cannot take the square root of a negative number, but we can take its cube root. How is this reflected in the graphs?

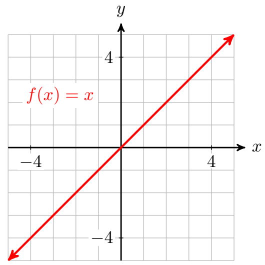


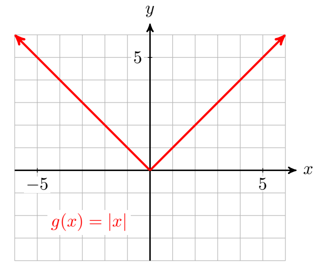
The next pair of functions, $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$, are both undefined at $x = 0$, so thier graphs do not include any points with x-coordinate zero. For very small positive values of x, both $f(x)$ and $g(x)$ get very large. As x gets closer to zero, the graphs approach the vertical line $x = 0$ (the y-axis). This line is called a **vertical asymptote** for the graph.

Also, notice that for very large values of x, both $f(x)$ and $g(x)$ get very close to zero. Their graphs approach the horizontal line $y = 0$ (the x-axis). This line is called the **horizontal asymptote** for the graph.



Finally, compare the familiar graph of $f(x) = x$ with the graph of $g(x) = |x|$. The piecewise definition of $|x|$ means that we graph $y = x$ in the first quadrant (where $x \geq 0$), and $y = -x$ in the first quadrant $x < 0$). The result is the V-shaped graph shown below.





Because they are fundamental to further study of mathematics and its applications, you should become familiar with the properties of these eight graphs, and be able to sketch them easily from memory, using their basic shapes and a few guidepoints.

5.3.4 Problem Set 5.3

Warm Up

1. Evaluate each function.

a. $f(x) = -2x^3 - 3x^2;\quad f(-2)$

b. $g(x) = \dfrac{x-1}{x^2+2x};\quad g(-1)$

Exercise Group. For Problems 2 and 3, compute each cube root. Round your answers to three decimal places if necessary. Verify your answers by cubing them.

2.

a. $\sqrt[3]{512}$

b. $\sqrt[3]{-125}$

c. $\sqrt[3]{-0.064}$

d. $\sqrt[3]{1.728}$

3.

a. $\sqrt[3]{9}$

b. $\sqrt[3]{258}$

c. $\sqrt[3]{-0.02}$

d. $\sqrt[3]{-3.1}$

Exercise Group. For Problems 4–6, simplify each by following the order of operations.

4.

a. $\dfrac{6-2\sqrt[3]{64}}{2}$

b. $2\sqrt[3]{-125} - \sqrt[3]{6^2-3^2}$

5.

a. $\sqrt[3]{\dfrac{8-1}{64-8}}$

b. $\dfrac{4+\sqrt[3]{-216}}{8-\sqrt[3]{8}}$

6.

a. $\sqrt[3]{3^3+4^3+5^3}$

b. $\sqrt[3]{9^3+10^3-1^3}$

Exercise Group. For problems 7–10, simplify the expression according to the order of operations.

7.

a. $-|-9|$

b. $-(-9)$

c. $2-(-6)$

d. $2-|-6|$

8.

a. $|-8|-|12|$

b. $|-8-12|$

c. $|-3|+|-5|$

d. $\big|-3+(-5)\big|$

9.

a. $4-9\,|2-8|$

b. $2-5\,|-6-3|$

c. $|-4-5|\,\big|1-3(-5)\big|$

10.

a. $|-3+7|\,\big|-2(6-10)\big|$

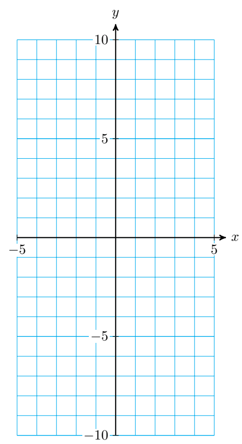
b. $\big||-5|-|-6|\big|$

c. $\big||4|-|-6|\big|$

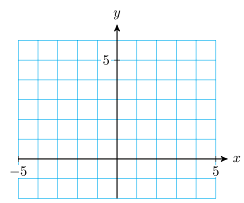
Skills Practice

Exercise Group. For Problems 11–16, sketch the graph of the function by hand, paying attention to the shape of the graph. Carefully plot at least three “guide points” to ensure accuracy. If possible, plot the points with x-coordinates -1, 0, and 1.

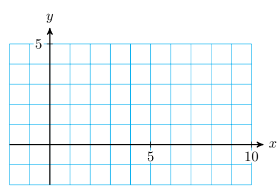
11. $f(x)=x^3$



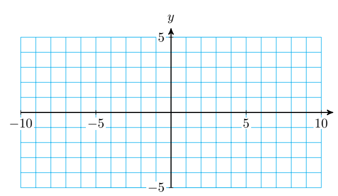
12. $f(x)=|x|$



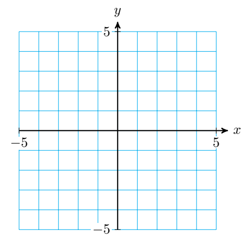
13. $f(x)=\sqrt{x}$



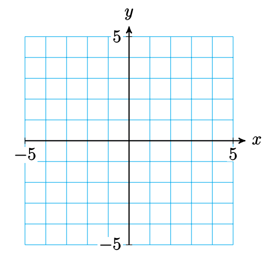
14. $f(x)=\sqrt[3]{x}$



15. $f(x) = \dfrac{1}{x}$



16. $f(x) = \dfrac{1}{x^2}$



17.

a. Use your calculator to graph $f(x) = x^2$ and $g(x) = x^3$ on the same axes for $0 \le x \le 1$. Which function is greater on that interval?

b. Use your calculator to graph $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$ on the same axes for $0 \le x \le 1$. Which function is greater on that interval?

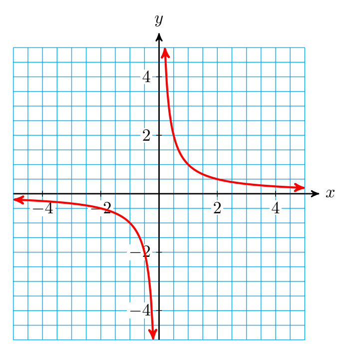
18.

a. Use your calculator to graph $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$ on the same axes for $0 \le x \le 1$. Which function is greater on that interval?

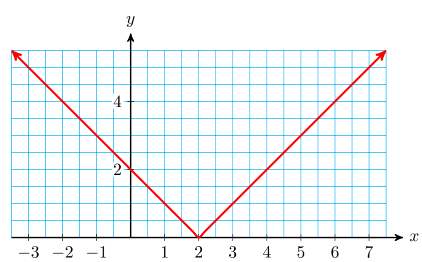
b. Use your calculator to graph $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$ on the same axes for $1 \le x \le 4$. Which function is greater on that interval?

Applications

19. Use the graph of $y = \dfrac{1}{x}$ to solve the inequality $\dfrac{1}{x} \le 2$.



20. Use the graph of $y = |x - 2|$ to solve the inequality $|x - 2| > 1$.



Exercise Group. For Problems 21–26, graph the functions in the same window on your calculator. Describe how the graphs in parts (b) and (c) are different from the basic graph.

21.

a. $f(x) = x^3$

b. $g(x) = x^3 - 2$

c. $h(x) = x^3 + 1$

22.

a. $f(x) = |x|$

b. $g(x) = |x - 2|$

c. $h(x) = |x + 1|$

23.

a. $f(x) = \dfrac{1}{x}$

b. $g(x) = \dfrac{1}{x + 1.5}$

c. $h(x) = \dfrac{1}{x - 1}$

24.

a. $f(x) = \dfrac{1}{x^2}$

b. $g(x) = \dfrac{1}{x^2} + 2$

c. $h(x) = \dfrac{1}{x^2} - 1$

25.

a. $f(x) = \sqrt{x}$

b. $g(x) = -\sqrt{x}$

c. $h(x) = \sqrt{-x}$

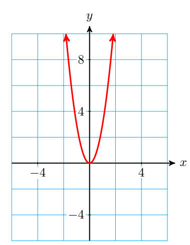
26.

a. $f(x) = \sqrt[3]{x}$

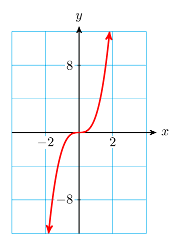
b. $g(x) = -\sqrt[3]{x}$

c. $h(x) = \sqrt[3]{-x}$

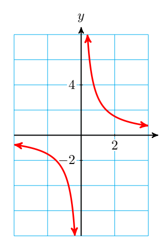
27. Match each graph with its equation.



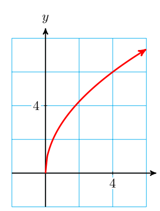
(a)



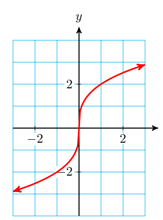
(b)



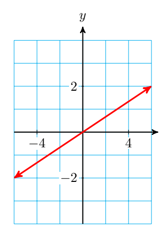
(c)



(d)



(e)



(f)

i. $f(x) = 3\sqrt{x}$

ii. $f(x) = 2x^3$

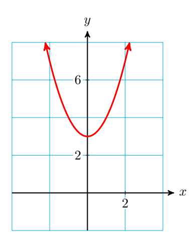
iii. $f(x) = \dfrac{x}{3}$

iv. $f(x) = \dfrac{3}{x}$

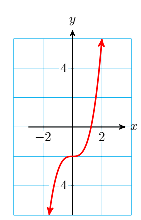
v. $f(x) = 2\sqrt[3]{x}$

vi. $f(x) = 3x^2$

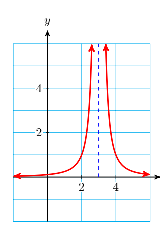
28. Match each graph with its equation.



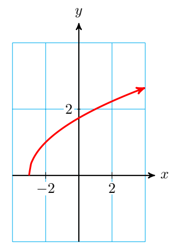
(a)



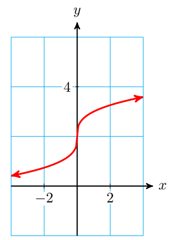
(b)



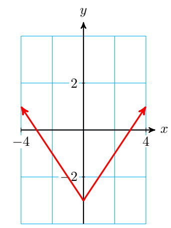
(c)



(d)



(e)



(f)

i. $f(x) = x^3 - 2$

ii. $f(x) = \sqrt[3]{x} + 2$

iii. $f(x) = \dfrac{1}{(x-3)^2}$

iv. $f(x) = |x| - 3$

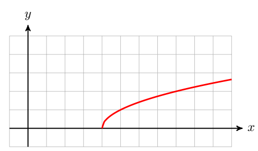
v. $f(x) = x^2 + 3$

vi. $f(x) = \sqrt{x+3}$

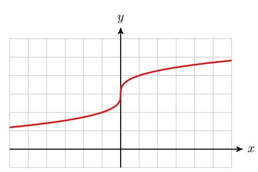
Exercise Group. For Problems 29 and 30, each graph is a variation of one of the eight basic graphs. Identify the basic graph for each.

29.

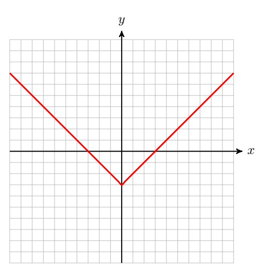
a.



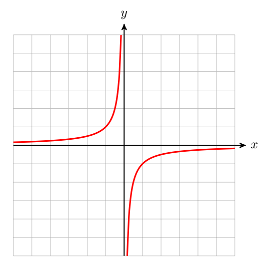
b.



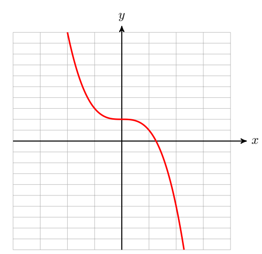
c.



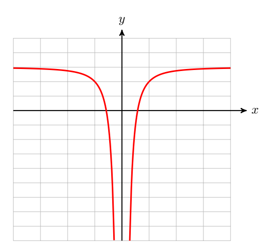
d.



e.

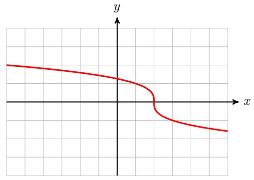


f.

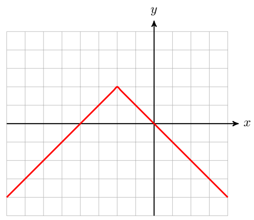


30.

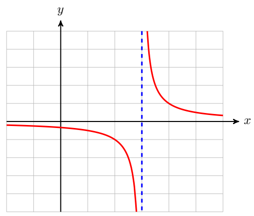
a.



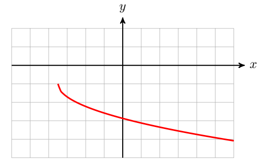
b.



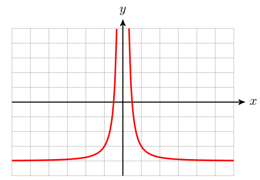
c.



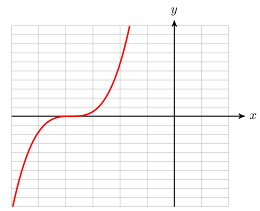
d.



e.

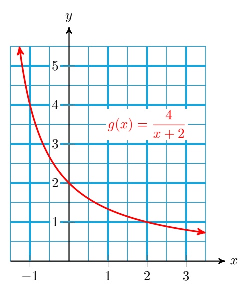


f.



Exercise Group. For Problems 31 and 32, use the graph to estimate the solutions to the inequalities.

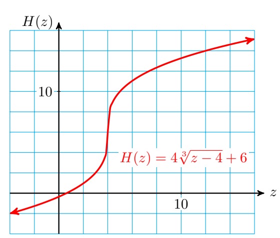
31. The figure shows a graph of $g(x) = \dfrac{4}{x+2}$.



a. $g(x) > 1$

b. $g(x) < 3$

32. The figure shows a graph of $H(z) = 4\sqrt[3]{z-4} + 6$.



a. $H(z) > 14$

b. $H(z) < 6$

Exercise Group. For Problems 33 and 34, graph the function in the window

$$\text{Xmin} = -47 \qquad \text{Xmax} = 47$$
$$\text{Ymin} = -31 \qquad \text{Ymax} = 31$$

Use the graph to solve each equation or inequality. Check your solutions algebraically.

33. Graph $F(x) = 4\sqrt{x-25}$.

a. Solve $4\sqrt{x-25} = 16$

b. Solve $4\sqrt{x-25} = -16$

c. Solve $8 < 4\sqrt{x-25} \leq 24$

34. Graph $G(x) = 20 - 0.001(x-8)^4$.

a. Solve $20 - 0.001(x-8)^4 = 26.2$

b. Solve $20 - 0.001(x-8)^4 = -8.561$

c. Solve $20 - 0.001(x-8)^4 \geq 10$

5.4 Direct Variation

Two types of functions are widely used in modeling and are known by special names: **direct variation** and **inverse variation**.

5.4.1 Proportion and Variation

Two variables are **directly proportional** (or just **proportional**) if the ratios of their corresponding values are always equal. Consider the functions described in the tables below. The first table shows the price of gasoline as a function of the number of gallons purchased.

Gallons of gasoline	Total price	Price/ Gallons
4	\$9.60	$\frac{9.60}{4} = 2.40$
6	\$14.40	$\frac{14.40}{6} = 2.40$
8	\$19.20	$\frac{19.20}{8} = 2.40$
12	\$28.80	$\frac{28.80}{12} = 2.40$
15	\$36.00	$\frac{36.00}{15} = 2.40$

Years	Population	People/Years
10	432	$\frac{432}{10} \approx 43$
20	932	$\frac{932}{20} \approx 47$
30	2013	$\frac{2013}{30} \approx 67$
40	4345	$\frac{4345}{40} \approx 109$
50	9380	$\frac{9380}{50} \approx 188$
60	20, 251	$\frac{20,251}{60} \approx 338$

The ratio $\frac{\text{total price}}{\text{number of gallons}}$, or price per gallon, is the same for each pair of values in the first table. This agrees with everyday experience: The price per gallon of gasoline is the same no matter how many gallons you buy. Thus, the total price of a gasoline purchase is directly proportional to the number of gallons purchased.

The second table shows the population of a small town as a function of the town's age. The ratio $\frac{\text{number of people}}{\text{number of years}}$ gives the average rate of growth of the population in people per year. You can see that this ratio is *not* constant; in fact, it increases as time goes on. Thus, the population of the town is *not* proportional to its age.

Checkpoint 5.4.1 QuickCheck 1. When does a table represent proportional variables?

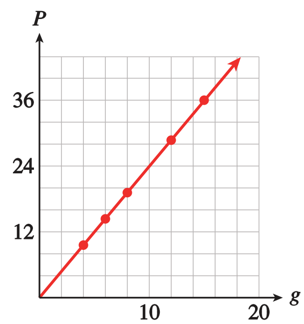
a. If it has a constant slope

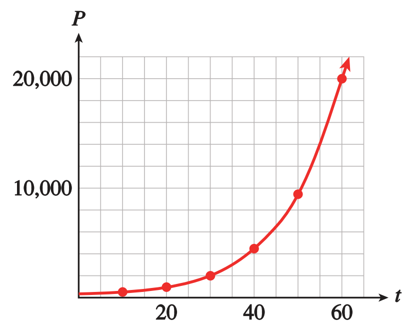
b. If it includes the point $(0, 0)$

c. If the ratio output/input is constant

d. If each output is double the previous one

The graphs of the two functions described above by tables (the gasoline price and the small town population tables) are shown below.





We see that the price, P, of a fill-up is a linear function of the number of gallons, g, purchased. This should not be surprising if we write an equation relating the variables g and P. Because the ratio of their values is constant, we can write

$$\frac{P}{g} = k$$

where k is a constant. In this example, the constant k is 2.40, the price of gasoline per gallon. Solving for P in terms of g, we have

$$P = kg = 2.40g$$

which we recognize as the equation of a line through the origin.

If y is proportional to x, we also say that y varies directly with x, and we make the following definition.

Direct Variation.

y **varies directly** with x if

$$y = kx$$

where k is a positive constant called the **constant of variation**.

Example 5.4.2

a. The circumference, C, of a circle varies directly with its radius, r, because

$$C = 2\pi r$$

The constant of variation is 2π, or about 6.28.

b. The amount of interest, I, earned in one year on an account paying 7% simple interest, varies directly with the principal, P, invested, because

$$I = 0.07P$$

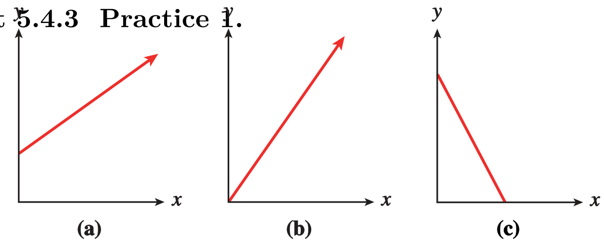
□

In the opening example, we saw that the price of gasoline, P, is a linear function of the number of gallons purchased. From the definition above, we see that any direct variation defines a linear function of the form

$$y = f(x) = kx$$

Compareing this equation to the standard form for a linear function, $y = b+mx$, we see that the constant term, b, is zero, so the graph of a direct variation passes through the origin. The positive constant k in the equation $y = kx$ is just the slope of the graph, so it tells us how rapidly the graph increases.

Checkpoint 5.4.3 Practice 1.



Which of the graphs above could represent direct variation? Explain why.

a. (a). The graph is a straight line that increases.

b. (b). The graph is a straight line through the origin.

c. (c). The graph is a straight line that decreases.

d. None of the above

Checkpoint 5.4.4 QuickCheck 2.

a. A table describes direct variation if the ratio of corresponding entries is __________.

b. A graph describes direct variation if it is a __________.

c. An equation describes direct variation if it has the form __________.

d. If two variables vary directly, we may also say that they are __________.

5.4.2 The Scaling Property of Direct Variation

Because the graph of $y = kx$ passes through the origin, direct variation has the following **scaling** property: if we double the value of x, then the value of y will double also. In fact, increasing x by any factor causes y to increase by the same factor. Look again at the table for the price of buying gasoline. Doubling the number of gallons of gas purchased, say, from 4 gallons to 8 gallons or from 6 gallons to 12 gallons, causes the total price to double also.

Checkpoint 5.4.5 QuickCheck 3. You invest \$800 for one year at 7% simple interest. The interest earned is

$$I = 0.07(800) = \boxed{}$$

If you increase the investment by a factor of, say, 1.6, to 1.6(800) or \$1280, the interest will be

$$I = 0.07(1280) = \boxed{}$$

The original interest is increased by a factor of __________.

Example 5.4.6

a. Tuition at Woodrow University is \$400 plus \$30 per unit. Is the tuition proportional to the number of units you take?

b. Imogen makes a 15% commission on her sales of environmentally friendly products marketed by her co-op. Do her earnings vary directly with her sales?

Solution.

a. Let u represent the number of units you take, and let $T(u)$ represent your tuition. Then

$$T(u) = 400 + 30u$$

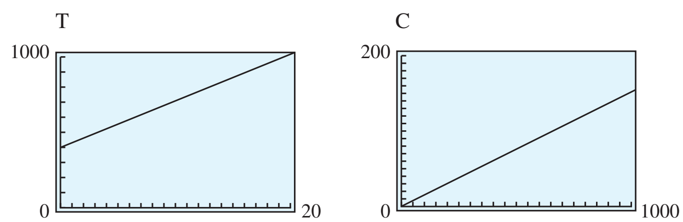
Thus, $T(u)$ is a linear function of u, but the T-intercept is 400, not 0. Your tuition is *not* proportional to the number of units you take, so this is not an example of direct variation. You can check that doubling the number of units does not double the tuition. For example,

$$T(6) = 400 + 30(6) = 580$$

and

$$T(12) = 400 + 30(12) = 760$$

Tuition for 12 units is not double the tuition for 6 units. The graph of $T(u)$ in figure (a) does not pass through the origin.



(a) (b)

b. Let S represent Imogen's sales, and let $C(S)$ represent her commission. Then

$$C(S) = 0.15S$$

Thus, $C(S)$ is a linear function of S with a C-intercept of zero, so Imogen's earnings do vary directly with her sales.

□

Checkpoint 5.4.7 Practice 2. Which table could represent direct variation? Explain why. (Hint: What happens to y if you multiply x by a constant?)

a.

x	1	2	3	6	8	9
y	2.5	5	7.5	15	20	22.5

b.

x	2	3	4	6	8	9
y	2	3.5	5	7	8.5	10

5.4.3 Finding a Formula for Direct Variation

If we know any one pair of values for the variables in a direct variation, we can find the constant of variation. Then we can use the constant to write a formula for one of the variables as a function of the other.

Example 5.4.8 If you kick a rock off the rim of the Grand Canyon, its speed, v, varies directly with the time, t, it has been falling. The rock is falling at a speed of 39.2 meters per second when it passes a lizard on a ledge 4 seconds later.

a. Express v as a function of t.

b. What is the speed of the rock after it has fallen for 6 seconds?

c. Sketch a graph of $v(t)$ versus t.

Solution.

a. Because v varies directly with t, there is a positive constant k for which

$$v = kt.$$

We substitute $v = \mathbf{39.2}$ and $t = \mathbf{4}$ and solve for k to find

$$\begin{aligned} 39.2 &= k(4) && \text{Divide both sides by 4.} \\ k &= 9.8 \end{aligned}$$

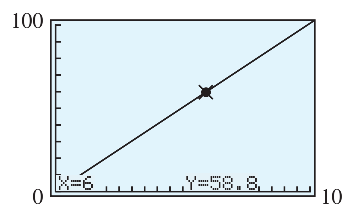
Replacing k by 9.8, we find the formula $v(t) = 9.8t$.

b. We evaluate the function found in part (a) for $t = 6$.

$$v = 9.8(\mathbf{6}) = 58.8$$

At $t = 6$ seconds, the rock is falling at a speed of 58.8 meters per second.

c. You can use your calculator to graph the function $v = 9.8t$. The graph is shown below.



□

Checkpoint 5.4.9 Practice 3. The volume of a bag of rice, in cups, is directly proportional to the weight of the bag. A 2-pound bag contains 3.5 cups of rice.

a. Express the volume, V, of a bag of rice as a function of its weight, w.

b. How many cups of rice are in a 15-pound bag?

Checkpoint 5.4.10 QuickCheck 4. How can you find the value of the constant of variation?

a. Double the value of x

b. Find the y-intercept of the graph

c. Substitute a pair of related values into $y = kx$

d. All of the above

5.4.4 Direct Variation with a Power of x

In many situations, y is proportional to a power of x, instead of x itself.

Direct Variation with a Power.

y **varies directly** with a power of x if

$$y = kx^n$$

where k and n are positive constants.

Example 5.4.11 The surface area of a sphere varies directly with the *square* of its radius. A balloon of radius 5 centimeters has surface area 100π square centimeters, or about 314 square centimeters.

a. Find a formula $S = f(r)$ for the surface area of a sphere as a function of its radius [TK].

b. What is the radius of a sphere whose surface area is 200π square centimeters,or about 628 square centimeters?

c. Sketch a graph of the function.

Solution.

a. If S stands for the surface area of a sphere of radius r, then

$$S = f(r) = kr^2$$

To find the constant of variation, k, we substitute the values of S and r.

$$\begin{aligned} \mathbf{100\pi} &= k(\mathbf{5})^2 \\ 4\pi &= k \end{aligned}$$

Thus, $S = f(r) = 4\pi r^2$.

b. We solve the variation equation for r when $S = 200\pi$.

$$\begin{aligned} \mathbf{200\pi} &= 4\pi r^2 \\ 50 &= r^2 \\ \sqrt{50} &= r \end{aligned}$$

The radius is $\sqrt{50}$ centimeters, or about 7.1 centimeters.

c. The graph has the shape of the basic function $y = x^2$, so all we need are a few points to "anchor" the graph. We know that $f(5) = 314$, and $f(1) = 4\pi \cdot 1^2$ or about 12.6. A graph is shown below.

(Need graph here)

□

[TK] For more examples of direct variation, see Section 5.4.1 , Section 5.4.2, and Section 5.4.3 of the Toolkit.

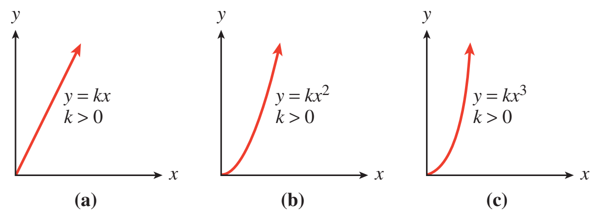
Checkpoint 5.4.12 Practice 4. The volume of a sphere varies directly with the *cube* of its radius. A balloon of radius 5 centimeters has volume $\dfrac{500\pi}{3}$ cubic centimeters, or about 524 cubic centimeters.

a. Find a formula for the volume, V, of a sphere as a function of its radius, r.

b. What is the radius of a balloon whose volume is $\dfrac{1000\pi}{3}$ cubic centimeters, or about 1028 cubic centimeters?

c. Sketch a graph of the function.

In any example of direct variation, as the input variable increases through positive values, the output variable increases also. Thus, a direct variation is an increasing function, as we can see when we consider the graphs of some typical direct variations shown below.



Caution 5.4.13 The graph of a direct variation always passes through the origin, so when the input is zero, the output is also zero. Thus, the functions $y = 3x + 2$ and $y = 0.4x^2 - 2.3$, for example, are not direct variation, even though they are increasing functions for positive x.

Checkpoint 5.4.14 QuickCheck 5. Decide whether each statement is true or false.

a. Every increasing function is a direct variation.

b. Every direct variation is an increasing function.

c. The graph of every direct variation is a straight line.

d. The graph of every direct variation passes through the origin.

Even without an equation, we can check whether a table of data describes direct variation or merely an increasing function. If y varies directly with x^n, then $y = kx^n$, or, equivalently, $\dfrac{y}{x^n} = k$.

Test for Direct Variation.

If the ratio $\dfrac{y}{x^n}$ is constant, then y varies directly with x^n.

Example 5.4.15 Delbert collects the following data and would like to know if y varies directly with the square of x. What should he calculate?

x	2	5	8	10	12
y	6	16.5	36	54	76

Solution. If y varies directly with x^2, then $y = kx^2$, or $\dfrac{y}{x^2} = k$. Delbert should calculate the ratio $\dfrac{y}{x^2}$ for each data point.

x	2	5	8	10	12
y	6	16.5	36	54	76
$\dfrac{y}{x^2}$	1.5	0.66	0.56	0.54	0.53

Because the ratio $\dfrac{y}{x^2}$ is not constant, y does not vary directly with x^2. □

5.4.5 Scaling

If y varies directly with x, then doubling x causes y to double also. But what if y varies directly with a *power* of x :

- Is the area of a 16-inch circular pizza double the area of an 8-inch pizza?
- If you double the dimensions of a model of a skyscraper, will its weight double also?

You probably know that the answer to both of these questions is *No.* The area of a circle is proportional to the *square* of its radius, and the volume (and hence the weight) of an object is proportional to the *cube* of its linear dimension. Variation with a power of x produces a different scaling effect.

Example 5.4.16 The Taipei 101 building is 1671 feet tall, and in 2006 it was the tallest skyscraper in the world. Show that doubling the dimensions of a model of the Taipei 101 building produces a model that weighs 8 times as much.

Solution. The Taipei 101 skyscraper is approximately box shaped, so its volume is given by the product of its linear dimensions, $V = lwh$. The weight of an object is proportional to its volume, so the weight, W, of the model is

$$W = klwh$$

where the constant k depends on the material of the model. If we double the length, width, and height of the model, then

$$\begin{aligned} W_{\text{new}} &= k(2l)(2w)(2h) \\ &= 2^3(klwh) = 8W_{\text{old}} \end{aligned}$$

The weight of the new model is $2^3 = 8$ times the weight of the original model. □

Checkpoint 5.4.17 Practice 6. Use the formula for the area of a circle to show that doubling the diameter of a pizza quadruples its area.

Step 1: The formula for the area of a circle of radius r is $A =$ $\boxed{}$

Step 2: If we double the diameter, the new radius is $\boxed{}$

Step 3: Substitute the new expression for the radius into the area formula to get the area of the new circle.

In general, if y varies directly with a power of x, that is, if $y = kx^n$, then doubling the value of x causes y to increase by a factor of 2^n. In fact, if we multiply x by any positive number c, then

$$\begin{aligned} y_{\text{new}} &= k(cx)^n \\ &= c^n(kx^n) = c^n(y_{\text{old}}) \end{aligned}$$

so the value of y is multiplied by c^n.

We will call n the **scaling exponent**, and you will often see variation described in terms of scaling. For example, we might say that ”the area of a circle scales as the square of its radius.” (In many applications, the power n is called the *scale factor*, even though it is not a factor but an exponent.)

5.4.6 Problem Set 5.4

Warm Up

1. Solve

a. $\dfrac{a}{a-2} = \dfrac{25}{30}$

b. $\dfrac{7}{90} = \dfrac{126}{5(t-2)^2}$

2. Sketch a graph in the first quadrant.

a. $f(x) = 2x$

x	0	$\frac{1}{2}$	1	2
$f(x)$				

b. $g(x) = 2x^2$

x	0	$\frac{1}{2}$	1	2
$g(x)$				

c. $h(x) = \dfrac{2}{x}$

x	0	$\frac{1}{2}$	1	2
$h(x)$				

Skills Practice

Exercise Group. For Problems 3 and 4,

a. Use the values in the table to find the constant of variation, k, and write y as a function of x.

b. Fill in the rest of the table with the correct values.

c. What happens to y when you double the value of x?

3. y varies directly with x.

x	2	5		12	
y		1.5	2.4		4.5

4. y varies directly with the square of x.

x	3	6		12	
y		24	54		150

Exercise Group. For Problems 5–8, decide whether

a. y varies directly with x

b. y varies directly with x^2

c. y does not vary directly with a power of x

Explain why your choice is correct. If your choice is (a) or (b), find the constant of variation, k.

5.

x	2	3	5	8
y	2.0	4.5	12.5	32.0

6.

x	2	4	6	9
y	12	28	44	68

7.

x	1.5	2.4	5.5	8.2
y	3.0	7.2	33	73.8

8.

x	1.2	2.5	6.4	12
y	7.20	31.25	204.80	720.00

Applications

9. Delbert's credit card statement lists three purchases he made while on a business trip in the Midwest. His company's accountant would like to know the sales tax rate on the purchases.

Price of item	18	28	12
Tax	1.17	1.82	0.78
Tax/Price			

a. Compute the ratio of the tax to the price of each item. Is the tax proportional to the price? What is the tax rate?

b. Express the tax, T, as a function of the price, p, of the item.

c. Sketch a graph of the function by hand, and label the scales on the axes.

10. At constant acceleration from rest, the distance traveled by a race car is proportional to the square of the time elapsed. The highest recorded road-tested acceleration is 0 to 60 miles per hour in 3.07 seconds, which produces the following data.

Time (seconds)	2	2.5	3
Distance (feet)	57.32	89.563	128.97
Distance/Time^2			

a. Compute the ratios of the distance traveled to the square of the time elapsed. What was the acceleration, in feet per second squared?

b. Express the distance traveled, d, as a function of time in seconds, t.

c. Sketch a graph of the function by hand, and label the scales on the axes.

11. The weight of an object on the Moon varies directly with its weight on Earth. A person who weighs 150 pounds on Earth would weigh only 24.75 pounds on the Moon.

a. Find a function that gives the weight m of an object on the Moon in terms of its weight w on Earth. Complete the table and graph your function in a suitable window.

w	50	100	200	400
m				

b. How much would a person weigh on the Moon if she weighs 120 pounds on Earth?

c. A piece of rock weighs 50 pounds on the Moon. How much will it weigh back on Earth?

d. If you double the weight of an object on Earth, what will happen to its weight on the Moon?

12. The length of a rectangle is 10 inches, and its width is 8 inches. Suppose we increase the length of the rectangle while holding the width constant.

a. Fill in the table.

Length	Width	Perimeter	Area
10	8		
12	8		
15	8		
20	8		

b. Does the perimeter vary directly with the length?

c. Write a formula for the perimeter of the rectangle in terms of its length.

d. Does the area vary directly with the length?

e. Write a formula for the area of the rectangle in terms of its length.

13. Hubble's law says that distant galaxies are receding from us at a rate that varies directly with their distance. (The speeds of the galaxies are measured using a phenomenon called redshifting.) A galaxy in the constellation Ursa Major is 980 million light-years away and is receding at a speed of 15,000 kilometers per second.

a. Find a function that gives the speed, v, of a galaxy in terms of its distance, d, from Earth. Complete the table and graph your function in a suitable window. (Distances are given in millions of light-years.)

d	500	1000	2000	4000
m				

b. How far away is a galaxy in the constellation Hydra that is receding at 61,000 kilometers per second?

c. A galaxy in Leo is 1240 million light-years away. How fast is it receding from us?

14. The length, L, of a pendulum varies directly with the square of its period, T, the time required for the pendulum to make one complete swing back and forth. The pendulum on a grandfather clock is 3.25 feet long and has a period of 2 seconds.

a. Express L as a function of T. Complete the table and graph your function in a suitable window.

T	1	5	10	20
L				

b. How long is the Foucault pendulum in the Pantheon in Paris, which has a period of 17 seconds?

c. A hypnotist uses a gold pendant as a pendulum to mesmerize his clients. If the chain on the pendant is 9 inches long, what is the period of its swing?

d. In order to double the period of a pendulum, how must you vary its length?

15. The amount of power, P, generated by a windmill varies directly with the cube of the wind speed, w. A windmill on Oahu, Hawaii, produces 7300 kilowatts of power when the wind speed is 32 miles per hour.

a. Express the power as a function of wind speed. Complete the table and graph your function in a suitable window.

w	10	20	40	80
P				

b. How much power would the windmill produce in a light breeze of 15 miles per hour?

c. What wind speed is needed to produce 10,000 kilowatts of power?

d. If the wind speed doubles, what happens to the amount of power generated?

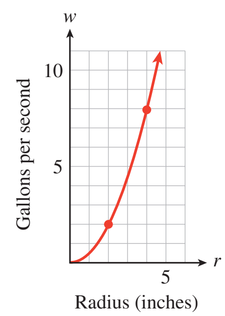
Exercise Group. The functions described by a table or a graph in Problems 16-19 are examples of direct variation.

a. Find an algebraic formula for the function, including the constant of variation, k.

b. Answer the question in the problem.

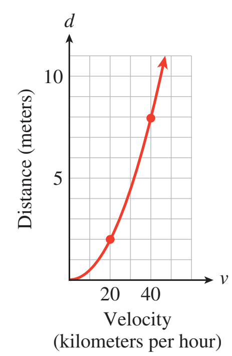
16.

A wide pipe can handle a greater water flow than a narrow pipe. The graph shows the water flow through a pipe, w, as a function of its radius, r. How great is the water flow through a pipe of radius of 10 inches?



17.

The faster a car moves, the more difficult it is to stop. The graph shows the distance, d, required to stop a car as a function of its velocity, v, before the brakes were applied. What distance is needed to stop a car moving at 100 kilometers per hour?



18. The maximum height attained by a cannonball depends on the speed at which it was shot. The table shows maximum height as a function of initial speed. What height is attained by a cannonball whose initial upward speed was 100 feet per second?

Speed (ft/sec)	Height (ft)
40	200
50	31.25
60	450
70	612.5

19. The strength of a cylindrical rod depends on its diameter. The greater the diameter of the rod, the more weight it can support before collapsing. The table shows the maximum weight supported by a rod as a function of its diameter. How much weight can a 1.2-centimeter rod support before collapsing?

Diameter (cm)	Weight (newtons)
0.5	150
1.0	600
1.5	1350
2.0	2400

20. The wind resistance, W, experienced by a vehicle on the freeway varies directly with the square of its speed, v.

a. If you double your speed, what happens to the wind resistance?

b. If you drive one-third as fast, what happens to the wind resistance?

c. If you decrease your speed by 10%, what happens to the wind resistance?

5.5 Inverse Variation

5.5.1 Inverse Variation

How long does it take to travel a distance of 600 miles? The answer depends on your average speed. If you are on a bicycle trip, your average speed might be 15 miles per hour. In that case, your traveling time will be

$$T = \frac{D}{R} = \frac{600}{15} = 40 \text{ hours}$$

(Of course, you will have to add time for rest stops; the 40 hours are just your travel time.)

If you are driving your car, you might average 50 miles per hour. Your travel time is then

$$T = \frac{D}{R} = \frac{600}{50} = 12 \text{ hours}$$

If you take a commercial air flight, the plane's speed might be 400 miles per hour, and the flight time would be

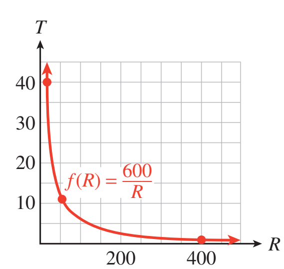
$$T = \frac{D}{R} = \frac{600}{400} = 1.5 \text{ hours}$$

You can see that for higher average speeds, the travel time is shorter. In other words, the time needed for a 600-mile journey is a decreasing function of average speed. In fact, a formula for the function is

$$T = f(R) = \frac{600}{R}$$

This function is an example of **inverse variation**. A table of values and a graph of the function are shown below.

R	T
10	60
15	40
20	30
50	12
200	3
400	1.5



Inverse Variation.

y **varies inversely** with x if

$$y = \frac{k}{x},\ x \neq 0$$

where k is a positive constant.

Caution 5.5.1 Inverse variation describes a decreasing function, but not every decreasing function represents inverse variation. People sometimes mistakenly use the phrase *varies inversely* to describe any decreasing function, but if y varies inversely with x, the variables must satisfy an equation of the form $y = \frac{k}{x}$, or $xy = k$.

To decide whether two variables truly vary inversely, we can check whether their product is constant. For instance, in the preceding travel-time example, we see from the table that $RT = 600$.

R	10	15	20	50	200	400
T	60	40	30	12	3	1.5
RT	600	600	600	600	600	600

Checkpoint 5.5.2 QuickCheck 1. How can you test whether a table for $y = f(x)$ represents inverse variation?

a. Check whether xy is a constant.

b. Check whether the function is decreasing.

c. Check whether y is the reciprocal of x.

d. Check whether y/x is a constant.

5.5.2 Finding a Formula for Inverse Variation

If we know that two variables vary inversely and we can find one pair of corresponding values for the variables, we can determine k, the constant of variation.

Example 5.5.3 The amount of current, I, that flows through a circuit varies inversely with the resistance, R, on the circuit. An iron with a resistance of 12 ohms draws 10 amps of current.

a. Write a formula that gives current as a function of the resistance. [TK]

b. Complete the table and graph your function in a suitable window.

R	1	2	10	20
I				

c. How much current is drawn by a light bulb with a resistance of 533.3 ohms?

d. What is the resistance of a toaster that draws 12.5 amps of current?

Solution.

a. Because I varies inversely with R, we know that $I = \frac{k}{R}$. To find the constant k, we substitute **10** for I and **12** for R.

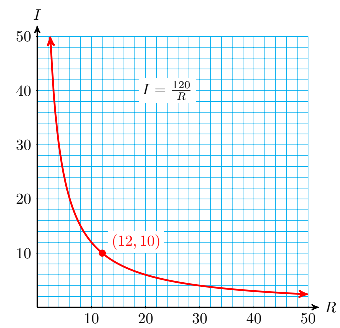
$$\mathbf{10} = \frac{k}{\mathbf{12}} \qquad \text{Solve for } \boldsymbol{k}.$$
$$k = 10 \cdot 12 = 120$$

So the formula is $I = \dfrac{120}{R}$.

b. We use the formula to complete the table.

R	1	2	10	20
I	120	60	12	6

A graph of I as a function of R is shown below.



c. We substitute **533.3** for R in the formula.

$$I = \frac{120}{\mathbf{533.3}} = 0.225$$

The light bulb draws 0.225 amps of current.

d. We substitute **12.5** for I in the formula and solve for R.

$$\mathbf{12.5} = \frac{120}{R} \qquad \text{Solve for } \boldsymbol{R}.$$
$$R = \frac{120}{12.5} = 9.6$$

The toaster has a resistance of 9.6 ohms.

□

[TK] For more examples of inverse variation, see Section 5.5.1 and Section 5.5.3 of the Toolkit.

Checkpoint 5.5.4 **Practice 1.** Delbert's officemates want to buy a $120 gold watch for a colleague who is retiring. The cost per person is inversely proportional to the number of people who contribute.

a. Express the cost per person, C, as a function of the number of people, p, who contribute.

b. Sketch the function for $0 \le p \le 20$. [TK]

[TK] For more graphs of inverse variation, see Section 5.5.2

5.5.3 Inverse Variation with a Power

We can also define inverse variation with a power of the variable.

Inverse Variation with a Power.

y **varies inversely** with x^n if

$$y = \frac{k}{x^n},\ x \neq 0$$

where k and n are positive constants.

We may also say that y is **inversely proportional** to x^n.

Example 5.5.5 The intensity of electromagnetic radiation, such as light or radio waves, varies inversely with the square of the distance from its source. Radio station KPCC broadcasts a signal that is measured at 0.016 watt per square meter by a receiver 1 kilometer away.

a. Write a formula that gives signal strength as a function of distance.

b. If you live 5 kilometers from the station, what is the strength of the signal you will receive?

c. Graph your function in the window

$$\begin{array}{ll} \text{Xmin} = 0 & \text{Xmax} = 6 \\ \text{Ymin} = 0 & \text{Ymax} = 0.02 \end{array}$$

Solution.

a. Let I stand for the intensity of the signal in watts per square meter, and d for the distance from the station in kilometers. Then $I = \frac{k}{d^2}$. To find the constant k, we substitute **0.016** for I and 1 for d. Solving for k gives us

$$\begin{aligned} \mathbf{0.016} &= \frac{k}{1^2} \\ k &= 0.016(1^2) = 0.016 \end{aligned}$$

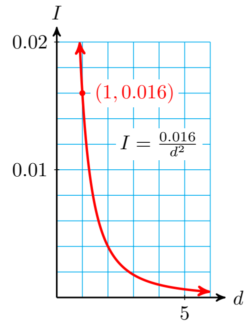
Thus, $I = \frac{0.016}{d^2}$.

b. Now we can substitute **5** for d and solve for I.

$$I = \frac{0.016}{\mathbf{5}^2} = 0.00064$$

At a distance of 5 kilometers from the station, the signal strength is 0.00064 watt per square meter.

c. The graph is shown below.



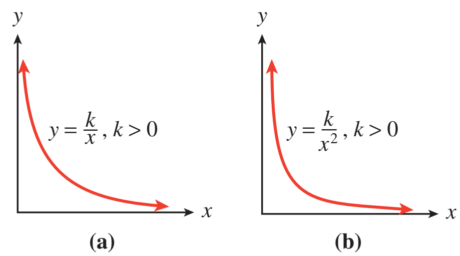
□

To summarize, an inverse variation is an example of a decreasing function, but not every decreasing function describes an inverse variation.

Test for Inverse Variation.

If the product yx^n is constant and n is positive, then y varies inversely with x^n.

The graphs of some typical inverse variations are shown below. They are versions of the basic graphs you studied in Section 5.3, but restricted to positive x-values only.



Checkpoint 5.5.6 QuickCheck 2. Use the table to decide whether H could vary inversely with m^2.

m	0.05	0.20	0.25	0.4
H	240	15	9.6	3.75

In Section 5.4, we considered the scaling property of direct variation. If $y = kx$ and you double the value of x, then the value of y doubles also. If $y = kx^2$ and you double the value of x, then the value of y is multiplied by a factor of $2^2 = 4$.

What happens when you double the input of an inverse variation?

Example 5.5.7 The weight, w, of an object varies inversely with the square of its distance, d, from the center of the Earth. Thus,

$$w = \frac{k}{d^2}$$

If you double your distance from the center of the Earth, what happens to your weight? What if you triple the distance?

Solution. Suppose you weigh W pounds at distance D from the center of the Earth. Then $W = \dfrac{k}{D^2}$. At distance $2D$, your weight will be

$$w = \frac{k}{(2D)^2} = \frac{k}{4D^2} = \frac{1}{4} \cdot \frac{k}{D^2} = \frac{1}{4}W$$

Your new weight will be $\frac{1}{4}$ of your old weight. By a similar calculation, you can check that by tripling the distance, your weight will be reduced to $\frac{1}{9}$ of its original value. □

Checkpoint 5.5.8 Practice 2. The amount of force, F, (in pounds) needed to loosen a rusty bolt with a wrench is inversely proportional to the length, l, of the wrench. Thus,

$$F = \frac{k}{l}$$

If you increase the length of the wrench by 50% so that the new length is $1.5l$, what happens to the amount of force required to loosen the bolt?

Checkpoint 5.5.9 QuickCheck 3. Match each formula with its description below.

a. Direct variation

b. Direct variation with a power

c. Inverse variation

d. Inverse variation with a power

i. $I = \frac{50}{d^2}$

ii. $S = 15.3h^2$

iii. $N = \frac{2600}{P}$

iv. $C = 85A$

5.5.4 Problem Set 5.5

Warm Up

Exercise Group. For Problems 1-4, choose variables and write an equation relating them. Which equations describe inverse variation?

1. Rachel spends one-third of her income on rent.

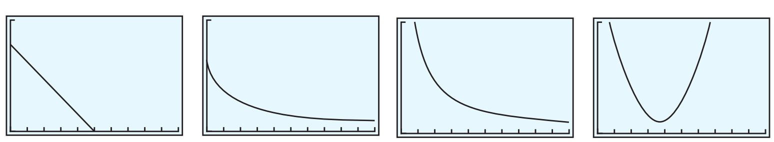
2. There were two candidates in the election for mayor of Centerville, Smith and Jones. A total of 4800 votes were cast (no write-in votes).

3. Water is leaking from a 2000-gallon tank at a rate of one cup per day. (There are 16 cups in a gallon.) The amount of water left in the tank is a function of the amount leaked out.

4. Craig is planning to tile the floor of a 250-square foot room. He is deciding what size tile to use, and how many tiles he will need.

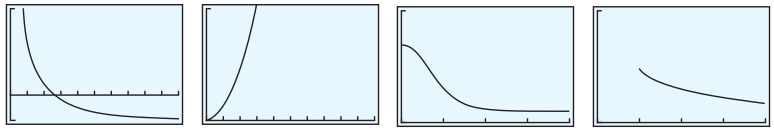
Skills Practice

5. Which of these graphs could describe inverse variation?



(a) (b) (c) (d)

6. Which of these graphs could describe inverse variation?



(a) (b) (c) (d)

Exercise Group. For Problems 7 and 8,

a. Use the values in the table to find the constant of variation, k, and write y as a function of x.

b. Fill in the rest of the table with the correct values.

c. What happens to y when you double the value of x?

7. y varies inversely with x.

x	4		20	30	
y		15	6		3

8. y varies inversely with the square of x.

x	0.2		2	4	
y		80		1.25	0.8

Exercise Group. For Problems 9–12, decide whether

a. y varies inversely with x

b. y varies inversely with x^2

c. y does not vary inversely with a power of x

Explain why your choice is correct. If your choice is (a) or (b), find the constant of variation, k.

9.

x	0.5	2	3	6
y	288	18	8	2

10.

x	0.5	2	4	5
y	100.0	25.0	12.5	10.0

11.

x	1	1.3	3	4
y	4.0	3.7	2.0	1.0

12.

x	0.5	2	3	5
y	180.00	11.25	5.00	1.80

Applications

13. The marketing department for a paper company is testing wrapping paper rolls in various dimensions to see which shape consumers prefer. All the rolls contain the same amount of wrapping paper.

Width (feet)	2	2.5	3
Length (feet)	12	9.6	8
Length × Width			

a. Compute the product of the length and width for each roll of wrapping paper. What is the constant of inverse proportionality?

b. Express the length, L, of the paper as a function of the width, w, of the roll.

c. Sketch a graph of the function by hand, and label the scales on the axes.

14. The force of gravity on a 1-kilogram mass is inversely proportional to the square of the object's distance from the center of the Earth. The table shows the force on the object, in newtons, at distances that are multiples of the Earth's radius.

Distance (Earth radii)	1	2	4
Force (newtons)	9.8	2.45	0.6125
Force $\times$ distance^2			

a. Compute the products of the force and the square of the distance. What is the constant of inverse proportionality?

b. Express the gravitational force, F, on a 1-kilogram mass as a function of its distance, r, from the Earth's center, measured in Earth radii.

c. Sketch a graph of the function by hand, and label the scales on the axes.

15. Computer monitors produce a magnetic field. The effect of the field, B, on the user varies inversely with his or her distance, d, from the screen. The field from a certain color monitor was measured at 22 milligauss 4 inches from the screen.

a. Express the field strength as a function of distance from the screen. Complete the table and graph your function in a suitable window.

d	1	2	12	24
B				

b. What is the field strength 10 inches from the screen?

c. An elevated risk of cancer can result from exposure to field strengths of 4.3 milligauss. How far from the screen should the computer user sit to keep the field level below 4.3 milligauss?

d. If you double your distance from the screen, how does the field strength change?

16. Boyle's law says that the pressure on a gas is inversely proportional to the volume it occupies. For example, deep-sea divers who return to the surface too rapidly get "the bends" when nitrogen bubbles in the blood expand. Suppose a submarine at a depth of 100 meters, where the pressure is 10.7 atmospheres, releases a bubble of volume 1.5 cubic centimeters.

a. Find a formula for the volume of the bubble as a function of the pressure.

b. What will the volume of the bubble be when it reaches the surface, where the pressure is 1 atmosphere?

c. Graph your function.

17. After the 2017 wildfires, California needs to replant 129,000,000 trees. The amount of time this will take is inversely proportional to the number of workers planting trees. On average, one worker can plant 2000 tree seedlings each day.

a. How many days would it take 100 workers to plant the trees?

b. Write a formula for the number of working days, D, it will take n

workers to plant the trees.

c. How many workers would be needed to plant the trees in 300 working days?

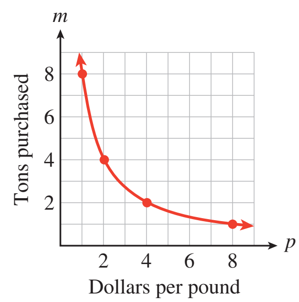
Exercise Group. The functions described by a table or a graph in Problems 18–21 are examples of inverse variation.

a. Find a formula for the function, including the constant of variation, k.

b. Answer the question in the problem.

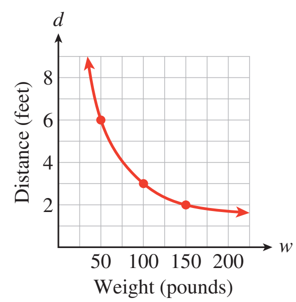
18.

If the price of mushrooms goes up, the amount consumers are willing to buy goes down. The graph shows the number of tons of shiitake mushrooms, m, sold in California each week as a function of their price, p. If the price of shiitake mushrooms rises to \$10 per pound, how many tons will be sold?



19.

When an adult plays with a small child on a seesaw, the adult must sit closer to the pivot point to balance the seesaw. The graph shows this distance, d, as a function of the adult's weight, w. How far from the pivot must Kareem sit if he weighs 280 pounds?



20. The thermocline is a layer of ocean water where the temperature changes rapidly. The table shows the temperature of the water as a function of depth in the thermocline. What is the ocean temperature at a depth of 500 meters?

Depth (m)	Temperature (°C)
200	20
400	10
1000	4

21. The shorter the length of a vibrating guitar string, the higher the frequency of the vibrations. The fifth string is 65 centimeters long and is tuned to A (with a frequency of 220 vibrations per second). The placement of the fret relative to the bridge changes the effective length of the guitar string. The table shows frequency as a function of effective length. How far from the bridge should the fret be placed for the note C (256 vibrations per second)?

Length (cm)	Frequency
55	260
57.2	250
65	220
71.5	200

22. The intensity of illumination, I, from a lamp varies inversely with the square of your distance, d, from the lamp.

a. If you double your distance from a reading lamp, what happens to the illumination?

b. If you triple the distance, what happens to the illumination?

c. If you increase the distance by 25%, what happens to the illumination?

5.6 Functions as Models

5.6.1 The Shape of the Graph

To create a good model we first decide what kind of function to use. What sort of function has the right shape to describe the process we want to model? Should it be increasing or decreasing, or some combination of both? Is the slope constant or is it changing?

In Examples 5.6.1 and 5.6.3, we investigate how the shape of a graph illustrates the nature of the process it models.

Example 5.6.1 Forrest leaves his house to go to school. For each of the following situations, sketch a possible graph of Forrest's distance from home as a function of time.

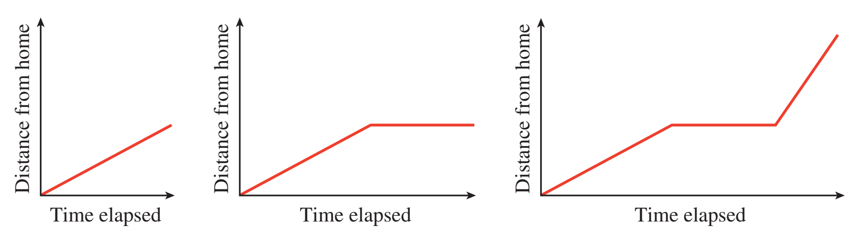
a. Forrest walks at a constant speed until he reaches the bus stop.

b. Forrest walks at a constant speed until he reaches the bus stop; then he waits there until the bus arrives.

c. Forrest walks at a constant speed until he reaches the bus stop, waits there until the bus arrives, and then the bus drives him to school at a constant speed.

Solution.

a. The graph is a straight-line segment, as shown in figure (a). It begins at the origin because at the instant Forrest leaves the house, his distance from home is 0. (In other words, when $t = 0, y = 0$.) The graph is a straight line because Forrest has a constant speed. The slope of the line is equal to Forrest's walking speed.



(a) **(b)** **(c)**

b. The first part of the graph is the same as part (a). But while Forrest waits for the bus, his distance from home remains constant, so the graph at that time is a horizontal line, as shown in figure (b). The line has slope 0 because while Forrest is waiting for the bus, his speed is 0.

c. The graph begins like the graph in part (b). The last section of the graph represents the bus ride. It has a constant slope because the bus is moving at a constant speed. Because the bus (probably) moves faster than Forrest walks, the slope of this segment is greater than the slope for the walking section. The graph is shown in figure (c).

□

Checkpoint 5.6.2 Practice 1. Erin walks from her home to a convenience store, where she buys some cat food, and then walks back home. Sketch a possible graph of her distance from home as a function of time.

The graphs in Example 5.6.1 are portions of straight lines. We can also consider graphs that bend upward or downward. The bend is called the **concavity** of the graph.

Example 5.6.3 The two functions described in this example are both increasing functions, but they increase in different ways. Match each function to its graph and to the appropriate table of values.

a. The number of flu cases reported at an urban medical center during an epidemic is an increasing function of time, and it is growing at a faster and faster rate.

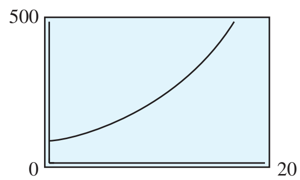
b. The temperature of a potato placed in a hot oven increases rapidly at first, then more slowly as it approaches the temperature of the oven.

(1)

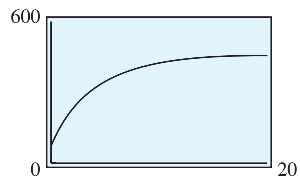
x	0	2	5	10	15
y	70	89	123	217	383

(2)

x	0	2	5	10	15
y	70	219	341	419	441



(A)



(B)

Solution.

a. The number of flu cases is described by graph (A) and table (1). The function values in table (1) increase at an increasing rate. We can see this by computing the rate of change over successive time intervals.

$$x = 0 \text{ to } x = 5: \qquad m = \frac{\Delta y}{\Delta x} = \frac{123 - 70}{5 - 0} = 10.6$$

$$x = 5 \text{ to } x = 10: \qquad m = \frac{\Delta y}{\Delta x} = \frac{217 - 123}{10 - 5} = 18.8$$

$$x = 10 \text{ to } x = 15: \qquad m = \frac{\Delta y}{\Delta x} = \frac{383 - 217}{15 - 10} = 33.2$$

The increasing rates can be seen inthe figure below; the graph bends upward as the slopes increase.