Sage and Linear Algebra Worksheets

Overview

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Introduction. This is an overview, and Table of Contents, for the worksheets distributed here.

Section RREF: Reduced Row-Echelon Form. Construct two matrices, learning Sage syntax. Row-reduce each.
Short: 5-10 minutes, 2 exercises. [HTML] [PDF]

Section NM: Reduced Row-Echelon Form. Demonstrate Definition NM, Theorem NMRRI, Definition NSM, Theorem NMUS. Experiment with random vectors of constants. Foreshadow the column space of a matrix. Preliminary work with vector spaces (membership).
Medium to Long: 20 minutes, 6 exercises. [HTML] [PDF]

Section SS: Spanning Sets. For Section LC, I illustrate Theorem VFSLS by writing the row-reduced version of an augmented matrix with white chalk for the pivot columns (first columns of an identity matrix), white chalk for the zero rows, and then different colors for each of the remaining columns. Then the vectors that describe the solution get the same numbers, or their negatives, in the same colors. But first I place the "pattern of zeros and ones" in place in white. Part of the entries in white is that they are known from just knowledge of the sets D and F. This example gets recycled in this Sage worksheet on the following day. This is similar to Example VFSAI.

Null spaces, spans and membership round out the topics. There is some foreshadowing of bases for subspaces.
Medium: 15 minutes, 2 exercises. [HTML] [PDF]

Section MISLE: Matrix Inverses and Systems of Linear Equations. A computational illustration of Theorem CINM and its proof. A nonsingular matrix versus a singular matrix.
Medium: 15 minutes, 4 exercises. [HTML] [PDF]

Section CRS: Column and Row Space. A detailed demonstration of Theorem CSCS, in both directions. You could do this quickly, or you could linger a while and drive the point home.
Short to Medium: 10 minutes, 2 exercises. [HTML] [PDF]

Section FS: Four Subspaces. Computations of all the constituent matrices of extended echelon form, and exercises exploring the matrices J and L for an 8×10 matrix.

Medium: 15 minutes, 2 exercises. [HTML] [PDF]

Section B: Bases. Theme: express vectors from $\mathbb{C}^n$ as (unique) linear combinations of basis vectors. Techniques include solving systems, inverting matrices, obtaining a **coordinatization** from Sage, and Theorem COB for an orthonormal basis. The second and fourth exercises each has several parts.

Long: 20-25 minutes, 4 exercises. [HTML] [PDF]

Section PDM: Properties of Determinants. This worksheet is different from all the others. It uses elementary matrices and triangular matrices to construct the **LU decomposition** of a 5×5 matrix. It follows the logic of a constructive proof of the existence of such a decomposition, and uses some properties of determinants as checks along the way. The conclusion discusses how to solve a linear system by first constructing an LU decomposition, then **forward-solving** and **back-solving** .

This topic is not in the textbook, though all the prerequisites are. It is meant as a demonstration of the utility of elementary matrices and a glimpse at topics that might be part of a more advanced course.

Long: 20 minutes, no exercises. [HTML] [PDF]

Section EE: Eigenvalues and Eigenvectors. Just two exercises to experiment with, but lots of other things to demonstrate. The theme of this worksheet is really just to show all the computational tools for various computations related to eigenvalues and eigenvectors that we would never want to do by hand. The later sections have some impressive examples of how fast Sage can be, for both exact and numerical linear algebra.

Long: 15-20 minutes, two exercises. [HTML] [PDF]

Section SD: Similarity and Diagonalization. A similarity check, diagonalization of a diagonalizable matrix, an attempt to diagonalize a defective matrix, Jordan canonical form, and rational canonical form. The two canonical forms can be skipped, they are present just to help answer the question of what is possible for a matrix that does not diagonalize.

Medium: 15 minutes, four exercises. [HTML] [PDF]

Section LT: Linear Transformations. Three ways to create linear transformations from $\mathbb{Q}^n$ to $\mathbb{Q}^m$, in addition to basic queries and operations for Sage linear transformations.

I do not present any of the four linear transformation worksheets in class, but instead suggest students work through them themselves, if they are interested. So they are mostly not organized with exercises.

Medium: no exercises. [HTML] [PDF]

Section ILT: Injective Linear Transformations. Two linear transformations, one injective, the other non-injective. Checks on injectiveness and images, along with a demonstration of the non-injective linear transformation failing the definition.

Short: no exercises. [HTML] [PDF]

Section SLT: Surjective Linear Transformations. Two linear transformations, one surjective, the other non-surjective. Checks on surjectiveness and images, along with a demonstration of the non-injective linear transformation failing the definition.

Some work with pre-images, including three exercises.

Medium: three exercises. [HTML] [PDF]

Section IVLT: Invertible Linear Transformations. An invertible linear transformation, its inverse, and the composition of the two. Twice. There is some work with the rank and nullity of a Sage linear transformation.

Medium: no exercises. [HTML] [PDF]

Section VR: Vector Representations. Two examples, one in $\mathbb{Q}^6$, and the other in P_3.

Short, 10 minutes: no exercises. [HTML] [PDF]

Section MR: Matrix Representations. A random linear transformation from $\mathbb{Q}^6$ to $\mathbb{Q}^4$ is used to build a matrix representation with random bases of the domain and codomain. The first construction mirrors Definition MR, and uses it to demonstrate Theorem FTMR. The second construction is semi-automatic with the right linear transformation commands in Sage. Everthing is then in place for a sneak preview of Theorem MRCB of Section CB, so we go ahead and do the relevant computation.

Medium, 15 minutes: no exercises. [HTML] [PDF]

Section CB: Change of Basis. One random linear transformation (from $\mathbb{Q}^3$ to $\mathbb{Q}^7$), four random bases, two matrix representations, and two change-of-basis matrices, all rolled into a demonstration of Theorem MRCB.

Then an engineered linear transformation from $\mathbb{Q}^8$ to $\mathbb{Q}^8$ provides a diagonal representation with a basis of eigenvectors. Eigenvalues and eigenvectors are investigated as properties of the linear transformation (rather than as properties of a matrix). The connections to similarity and Section SD are made explicit.

Medium, 20 minutes: one simple exercise. [HTML] [PDF]