Proximal Interphalangeal Joints
Lifejoint Orthopedic Solutions

Computing of the Hessian Matrix . . .

~~Finding~~ Eigenvalues ~~in the 3-Dimensional Case~~

MAA Research Grant

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1 Introduction

As we have been studying CT scans, we have found that the millions of voxels that the image is comprised of can be represented by convolutions. These convolutions can be represented by a 3×3 matrix. In order to locate and use each voxel, it is critical that we find the eigenvalues of each of the matrices. This has been proven to be quite a challenge, as these eigenvalues produce cubic equations when solving for them. In this paper, we will share our work so far, including the matrices in question, and will document our strategy for finding the eigenvalues from a cubic equation.

2 Preliminaries

~~The concepts in this section are the basis of understanding for what will be further explained throughout this paper.~~

~~2.1 Cubic Roots~~

Let ω be a cubic root of unity. That is, if $\omega^3 = 1$, then:

Let $\omega = e^{2i\pi/3} = \frac{1+\sqrt{-3}}{2}$

$\omega^3 = 1$

$$0 = \omega^3 - 1$$

or

$$0 = (\omega - 1)(\omega^2 + \omega + 1).$$

Then we have and

We then use the quadratic formula to find the roots:

$$\omega = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{-1 \pm \sqrt{-3}}{2}.$$

This implies that:

$$\omega = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

We can also determine that: Notice

$$\omega + \omega^2 = -1$$
$$\omega - \omega^2 = \sqrt{-3}.$$

~~2.2 Complex Numbers~~

If we are given complex numbers z and ψ such that $z^3 = \psi$, we know that $(\omega z)^3 = \psi$ and $(\omega^2 z)^3 = \psi$.

by eq

~~Recall from arithmetic in the complex numbers that if z satisfies the equation:~~

$$\cancel{z^3 = \psi},$$

keep

in other words, z is a cubic root of ψ, then ωz and $\omega^2 z$ are also cubic roots of ψ.

3 Computing the Hessian

read Exploiting

We start by building a Hessian matrix that shows how the intensity in the CT scan changes in different directions. Following Canny [1], we compute second-order derivatives of the 3D Gaussian function:

$$G(x, y, z) = \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-\frac{x^2+y^2+z^2}{2\sigma^2}}.$$

We take the six second order partial derivatives of this function and ~~apply~~ convolve each one ~~to~~ with the CT image, denoted by I, ~~by convolution.~~ This gives us a 3×3 matrix:

We can write all off-diagonals in the P_{ij} when $i \le j$.

Since $G_{xy} = G_{yx}$ etc.,

$[P_{ij}]_{ij}$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} = \begin{bmatrix} G_{xx} * I & G_{xy} * I & G_{xz} * I \\ G_{xy} * I & G_{yy} * I & G_{yz} * I \\ G_{xz} * I & G_{yz} * I & G_{zz} * I \end{bmatrix}.$$

$$\det\left(\begin{bmatrix} P_{11}-\lambda & P_{12} & P_{13} \\ P_{12} & P_{22}-\lambda & P_{23} \\ P_{13} & P_{23} & P_{33}-\lambda \end{bmatrix}\right) = 0$$

?

This matrix is symmetric. It is used to describe how the intensity values change around each voxel in the image. In the CT scans, this is useful for finding the boundaries of bones and joints.
~~As denoted by Kronenburg~~ [3], T~~t~~his matrix has the characteristic polynomial:

$$= a\lambda^3 + b\lambda^2 + c\lambda + d = 0,$$

with coefficients:

$$\begin{aligned}
a &= 1, \\
b &= P_{11} + P_{22} + P_{33}, \\
c &= P_{11}P_{22} + P_{11}P_{33} + P_{22}P_{33} - P_{12}^2 - P_{13}^2 - P_{23}^2, \\
d &= P_{11}P_{23}^2 + P_{22}P_{13}^2 + P_{33}P_{12}^2 - P_{11}P_{22}P_{33} - 2P_{12}P_{13}P_{23}.
\end{aligned}$$

Notice that b is the trace of the matrix (~~the sum of the diagonal values~~) and d is the determinant of the matrix. When we solve this polynomial, the result is the three eigenvalues. ~~These eigenvalues tell us how the intensity is changing in three directions.~~

4 The General Cubic Equation With Real Roots

In this section ~~Throughout this paper,~~ we will demonstrate how to find the roots of a generic cubic polynomial with rational coefficients that factors over the real numbers. Let a, b, c, and d be real, radical numbers, where $a \neq 0$. The cubic is represented by:

$$ax^3 + bx^2 + cx + d = 0. \tag{1}$$

~~5 Our Strategy~~

To solve the general cubic equation, we use a substitution to depress the cubic equation into the form of $x^3 + px + q = 0$. ~~This enables for further calculations to be conducted such as using Cardano's cubic formula by radicals to solve the depressed cubic and attain our solution.~~

5.1 Substitution

~~We begin with a substitution to depress the general cubic equation.~~ Let $t \in \mathbb{R}$ such that $x = t - \frac{b}{3a}$. Substituting yields:

tion

$$a\left(t - \tfrac{b}{3a}\right)^3 + b\left(t - \tfrac{b}{3a}\right)^2 + c\left(t - \tfrac{b}{3a}\right) + d = 0.$$

Expanding and dividing by a results in:

$$t^3 - \tfrac{bt^2}{a} + \tfrac{b^2t}{3a^2} - \tfrac{b^3}{27a^3} + \tfrac{bt^2}{a} - \tfrac{2b^2t}{3a^2} + \tfrac{b^3}{9a^3} + \tfrac{ct}{a} - \tfrac{bc}{3a^2} + \tfrac{d}{a} = 0.$$

~~We then factor out t from all terms containing it, while excluding t^3.~~ Simplifying and combining like-powers of t generates:

$$t^3 + t\left(\frac{3ac - b^2}{3a^2}\right) + \left(\frac{2b^3 - 9abc + 27a^2d}{27a^3}\right) = 0. \tag{2}$$

Finally, we substitute $p = \frac{3ac-b^2}{3a^2}$ and $q = \frac{2b^3-9abc+27a^2d}{27a^3}$ into Equation (2). This results in the depressed cubic equation, as seen below:

$$t^3 + pt + q = 0. \tag{3}$$

5.2 Solving the Depressed Cubic Equation

To solve the depressed cubic equation, we will use Cardano's formula based on cubic roots of unity. We begin with Equation (3) and let α, β, and γ be the roots. Since we previously assumed that α, β, and γ are real, we can factor Equation (3), which gives us [2]:

$$(t-\alpha)(t-\beta)(t-\gamma) = 0. \tag{4}$$

Because Equations (3) and (4) are equal, we have:

$$\begin{aligned} 0 &= \alpha + \beta + \gamma, \\ p &= \alpha\beta + \alpha\gamma + \beta\gamma, \\ q &= -\alpha\beta\gamma. \end{aligned}$$

simplify English

From the three equations shown above, we can derive a number of related equations that will be used throughout our process to the solution. Two of these equations are described as:

consider idioms or metaphors

$$\begin{aligned} -2p &= -\alpha\beta - \alpha\gamma - \beta\gamma - \alpha\beta - \alpha\gamma - \beta\gamma \\ &= \alpha(-\beta-\gamma) + \beta(-\gamma-\alpha) + \gamma(-\alpha-\beta) \\ &= \alpha(\alpha) + \beta(\beta) + \gamma(\gamma) \\ &= \alpha^2 + \beta^2 + \gamma^2 \end{aligned}$$

Number these

and

regular align

$$\begin{aligned} p^2 &= (\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha\beta + \alpha\gamma + \beta\gamma) \\ &= \alpha^2\beta^2 + \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha^2\beta\gamma + \alpha^2\gamma^2 + \alpha\beta\gamma^2 + \alpha\beta^2\gamma + \alpha\beta\gamma^2 + \beta^2\gamma^2 \\ &= \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + 2\alpha^2\beta\gamma + 2\alpha\beta^2\gamma + 2\alpha\beta\gamma^2 \\ &= \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + 2\alpha\beta\gamma(\alpha+\beta+\gamma) \\ &= \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2. \end{aligned}$$

\notag

\notag

\label{ }

Furthermore, since α, β, and γ are real, we know $(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)$ is real as well. We will cunningly denote this with the symbol $\sqrt{D}$. This implies:

$$\begin{aligned} D &= ((\alpha-\beta)(\alpha-\gamma)(\beta-\gamma))^2 \\ &= (\alpha-\beta)(\alpha-\gamma)(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)(\beta-\gamma). \end{aligned}$$

Expanding yields:

$$D = (\alpha^2 - \alpha\beta - \alpha\gamma + \beta\gamma)(\alpha\beta - \beta^2 - \alpha\gamma + \beta\gamma)(\alpha\beta - \beta\gamma - \alpha\gamma + \gamma^2).$$

Rewriting terms based on p and simplifying:

$$\begin{aligned} D &= (\alpha^2 - \alpha\beta - \alpha\gamma + p - \alpha\beta - \alpha\gamma)(\beta^2 - \beta\gamma - \alpha\beta + \alpha\gamma + p - \alpha\beta - \beta\gamma)(\gamma^2 - \alpha\gamma - \beta\gamma + p - \alpha\gamma - \beta\gamma) \\ &= -(\alpha^2 + 2\alpha(-\beta-\gamma) + p)(\beta^2 + 2\beta(-\alpha-\gamma) + p)(\gamma^2 + 2\gamma(-\alpha-\beta) + p) \\ &= -(3\alpha^2 + p)(3\beta^2 + p)(3\gamma^2 + p). \end{aligned}$$

Expanding further:

$$\begin{aligned} D &= -(9\alpha^2\beta^2 + 3\alpha^2 p + 3\beta^2 p + p^2)(3\gamma^2 + p) \\ &= -(27\alpha^2\beta^2\gamma^2 + 9\alpha^2\gamma^2 p + 3\gamma^2 p^2 + 9\alpha^2\beta^2 p + 3\alpha^2 p^2 + 3\beta^2 p^2 + p^3). \end{aligned}$$

Factoring out terms of p and substituting using equations * and * gives us:

$$
\begin{aligned}
D &= -(27\alpha^2\beta^2\gamma^2 + 9p(\alpha^2\gamma^2 + \beta^2\gamma^2 + \alpha^2\beta^2) + 3p^2(\alpha^2+\beta^2+\gamma^2) + p^3) \\
&= -27q^2 - 9p(p^2) - 3p^2(-2p) - p^3 \\
&= -27q^2 - 4p^3.
\end{aligned}
$$

This is the discriminant of the depressed cubic equation.
Alternatively:

$$
\begin{aligned}
\sqrt{D} &= (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma) \\
&= (\alpha^2 - \alpha\beta - \alpha\gamma + \beta\gamma)(\beta-\gamma) \\
&= \alpha^2\beta - \alpha\beta^2 - \alpha\beta\gamma + \beta^2\gamma - \alpha^2\gamma + \alpha\beta\gamma + \alpha\gamma^2 - \beta\gamma^2 \\
&= \alpha^2\beta + \alpha\gamma^2 + \beta^2\gamma - \alpha\beta^2 - \alpha^2\gamma - \beta\gamma^2.
\end{aligned}
$$

Similarly, we define ~~another equation~~ S, which is ~~related~~ like to $\sqrt{D}$, except containing all positive terms:

$$
\begin{aligned}
S &= \alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 + \beta^2\gamma + \alpha\gamma^2 + \beta\gamma^2 \\
&= \alpha\beta(\alpha+\beta) + \alpha\gamma(\alpha+\gamma) + \beta\gamma(\beta+\gamma) \\
&= -\alpha\beta\gamma - \alpha\beta\gamma - \alpha\beta\gamma = 3q \\
&= 3q.
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
S + \sqrt{D} &= \alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 + \beta^2\gamma + \alpha\gamma^2 + \beta\gamma^2 + \alpha^2\beta + \alpha\gamma^2 + \beta^2\gamma - \alpha\beta^2 - \alpha^2\gamma - \beta\gamma^2 \\
&= 2\alpha^2\beta + 2\beta^2\gamma + 2\alpha\gamma^2
\end{aligned}
$$

and

$$
\begin{aligned}
S - \sqrt{D} &= \alpha^2\beta + \alpha^2\gamma + \alpha\beta^2 + \beta^2\gamma + \alpha\gamma^2 + \beta\gamma^2 - (\alpha^2\beta + \alpha\gamma^2 + \beta^2\gamma - \alpha\beta^2 - \alpha^2\gamma - \beta\gamma^2) \\
&= 2\alpha\beta^2 + 2\alpha^2\gamma + 2\beta\gamma^2.
\end{aligned}
$$

~~We then consider $\alpha^3+\beta^3+\gamma^3$, which we reconfigure as:~~ Here, we must once again pause to show another simplification:

$$
\begin{aligned}
\alpha^3+\beta^3+\gamma^3 &= \alpha^2(\alpha) + \beta^2(\beta) + \gamma^2(\gamma) \\
&= \alpha^2(-\beta-\gamma) + \beta^2(-\alpha-\gamma) + \gamma^2(-\alpha-\beta) \\
&= -\alpha^2\beta - \alpha^2\gamma - \alpha\beta^2 - \beta^2\gamma - \alpha^2\gamma - \beta\gamma^2 \\
&= -S.
\end{aligned}
$$

Since $\alpha^3+\beta^3+\gamma^3$ is equal to $-S$, this implies that $\alpha^3+\beta^3+\gamma^3$ is also equal to $-3q$.
We then let $A = \alpha + \omega\beta + \omega^2\gamma$ and let $B = \alpha + \omega^2\beta + \omega\gamma$. Recall ~~$\omega$~~ from the preliminaries ~~. We consider only the primitive roots~~ we defined $\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.
We now begin [verb] for relations between A and B.
Summing A and B results in:

$$
\begin{aligned}
A + B &= 2\alpha + \beta(\omega+\omega^2) + \gamma(\omega+\omega^2) \\
&= 2\alpha - \beta - \gamma \\
&= 2\alpha + \alpha \\
&= 3\alpha.
\end{aligned}
$$

Multiplying A and B produces:

$$\begin{aligned}
AB &= (\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma) \\
&= \alpha^2 + \omega^2\alpha\beta + \omega\alpha\gamma + \omega\alpha\beta + \beta^2 + \omega^2\beta\gamma + \omega^2\alpha\gamma + \omega\beta\gamma + \gamma^2 \\
&= (\alpha^2 + \beta^2 + \gamma^2) + \omega(\alpha\beta + \alpha\gamma + \beta\gamma) + \omega^2(\alpha\beta + \alpha\gamma + \beta\gamma) \\
&= -2p + \omega p + \omega^2 p \\
&= -2p + p(\omega + \omega^2) \\
&= -2p - p \\
&= -3p.
\end{aligned}$$

We can also sum $\omega^2 A$ with ωB as well as ωA with $\omega^2 B$ to obtain equations for β and γ, respectively. This yields:

$$\begin{aligned}
\omega^2 A + \omega B &= \omega^2(\alpha + \omega\beta + \omega^2\gamma) + \omega(\alpha + \omega^2\beta + \omega\gamma) \\
&= \omega^2\alpha + \beta + \omega\gamma + \omega\alpha + \beta + \gamma\omega^2 \\
&= \alpha(\omega^2 + \omega) + 2\beta + \gamma(\omega^2 + \omega) \\
&= -\alpha - \gamma + 2\beta \\
&= \beta + 2\beta \\
&= 3\beta
\end{aligned}$$

and

$$\begin{aligned}
\omega A + \omega^2 B &= \omega(\alpha + \omega\beta + \omega^2\gamma) + \omega^2(\alpha + \omega^2\beta + \omega\gamma) \\
&= \omega\alpha + \omega^2\beta + \gamma + \omega^2\alpha + \omega\beta + \gamma \\
&= \alpha(\omega + \omega^2) + \beta(\omega + \omega^2) + 2\gamma \\
&= -\alpha - \beta + 2\gamma \\
&= \gamma + 2\gamma \\
&= 3\gamma.
\end{aligned}$$

Cubing A yields:

$$\begin{aligned}
A^3 &= (\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega\beta + \omega^2\gamma) \\
&= (\alpha^2 + \omega\alpha\beta + \omega^2\alpha\gamma + \omega\alpha\beta + \omega^2\beta^2 + \omega^3\beta\gamma + \omega^2\alpha\gamma\omega^3\beta\gamma + \omega^4\gamma^2)(\alpha + \omega\beta + \omega^2\gamma) \\
&= \alpha^3 + \omega\alpha^2\beta + \omega^2\alpha^2\gamma + \omega\alpha^2\beta + \omega^2\alpha\beta^2 + \alpha\beta\gamma + \omega^2\alpha^2\gamma + \alpha\beta\gamma + \omega\alpha\gamma^2 + \omega\alpha^2\beta \\
&+ \omega^2\alpha\beta^2 + \alpha\beta\gamma + \omega^2\alpha\beta^2 + \omega\beta^2\gamma + \alpha\beta\gamma + \omega\beta^2\gamma + \omega^2\beta\gamma^2 + \omega^2\alpha^2\gamma + \alpha\beta\gamma + \omega\alpha\gamma^2 \\
&+ \alpha\beta\gamma + \omega\beta^2\gamma + \omega^2\beta\gamma^2 + \omega\alpha\gamma^2 + \omega^2\beta\gamma^2 + \gamma^3.
\end{aligned}$$

Factoring out like-terms of ω:

$$\begin{aligned}
A^3 \quad &= \alpha^3 + \beta^3 + \gamma^3 + 6(\alpha\beta\gamma) + \omega(\alpha^2\beta + \alpha^2\beta + \alpha\gamma^2 + \alpha^2\beta + \beta^2\gamma + \beta^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \alpha\gamma^2) \\
&+ \omega^2(\alpha^2\gamma + \alpha\beta^2 + \alpha^2\gamma + \alpha\beta^2 + \alpha\beta^2 + \beta\gamma^2 + \alpha^2\gamma + \beta\gamma^2 + \beta\gamma^2) \\
&= -3q - 6q + \omega(3\alpha^2\beta + 3\alpha\gamma^2 + 3\beta^2\gamma) + \omega^2(3\alpha^2\gamma + 3\alpha\beta^2 + 3\beta\gamma^2) \\
&= -9q + 3\omega(\alpha^2\beta + \alpha\gamma^2 + \beta^2\gamma) + 3\omega^2(\alpha^2\gamma + \alpha\beta^2 + \beta\gamma^2).
\end{aligned}$$

Substituting: reference equations

$$
\begin{aligned}
A^3 &= -9q + 3\omega\left(\frac{S+\sqrt{D}}{2}\right) + 3\omega^2\left(\frac{S-\sqrt{D}}{2}\right)\\
&= -9q + 3\omega\left(\frac{S}{2}\right) + 3\omega\left(\frac{\sqrt{D}}{2}\right) + 3\omega^2\left(\frac{S}{2}\right) + 3\omega^2\left(\frac{\sqrt{D}}{2}\right)\\
&= -9q + \frac{S}{2}(3\omega+3\omega^2) + \frac{\sqrt{D}}{2}(3\omega-3\omega^2)\\
&= -9q + \frac{3q}{2}(-3) + \frac{\sqrt{D}}{2}(3\sqrt{-3})\\
&= \frac{-27q}{2} + \frac{3}{2}\sqrt{-3D}. \qquad \frac{-27}{2} + \frac{3\sqrt{3D}}{2}i
\end{aligned}
$$

In a similar way, we cube B to obtain:

$$B^3 = \tfrac{-27q}{2} - \tfrac{3}{2}\sqrt{-3D}.$$

~~We then convert A and B into polar form.~~ $\boxed{\text{In our case, } D \geq 0,}$ ~~so we can define the radius as.~~

converting A^3 into polar form $A^3 = re^{i\theta}$ where

$$r = \sqrt{\left(\tfrac{27q}{2}\right)^2 + \left(\tfrac{3\sqrt{-3D}}{2}\right)^2}, = \text{simplify}$$

and the angle as:

$$\theta = \arccos\left(\tfrac{9q}{2\sqrt{-p^3}}\right). \quad \text{this come by trig after simplifying}$$

similarly for B

~~We then write A^3 and B^3 as $A^3 = re^{i\theta}$ and $B^3 = re^{-i\theta}$.~~ Since there are three choices for A, α: $r^{\frac{1}{3}}e^{\frac{i\theta}{3}}$, $\omega r^{\frac{1}{3}}e^{\frac{i\theta}{3}}$, and $\omega^2 r^{\frac{1}{3}}e^{\frac{i\theta}{3}}$. Recall that $\omega A = \omega\alpha + \omega^2\beta + \gamma$ and $\omega^2 A = \omega^2\alpha + \beta + \omega\gamma$. So without loss of ~~generalization,~~ generality, $A = r^{\frac{1}{3}}e^{\frac{i\theta}{3}}$. We would similarly have three choices for B, but ~~based on~~ $AB = -3p$ and ~~can once again~~ ~~assume that~~ $B = r^{\frac{1}{3}}e^{-\frac{i\theta}{3}}$. Solving for each variable gives us:

ref eq

ref Prelim

Since the order of α, β, and γ is arbitrary at this moment

$$
\begin{aligned}
\alpha &= \frac{r^{\frac{1}{3}}e^{\frac{i\theta}{3}} + r^{\frac{1}{3}}e^{-\frac{i\theta}{3}}}{3}\\
&= \frac{r^{\frac{1}{3}}}{3}\left(e^{\frac{i\theta}{3}} + e^{-\frac{i\theta}{3}}\right)\\
&= \frac{2r^{\frac{1}{3}}}{3}\cos\left(\frac{\theta}{3}\right),
\end{aligned}
$$

$$
\begin{aligned}
\beta &= \frac{r^{\frac{1}{3}}e^{\frac{i\theta}{3}}\cdot e^{\frac{4\pi i}{3}} + r^{\frac{1}{3}}e^{-\frac{i\theta}{3}}\cdot e^{\frac{2\pi i}{3}}}{3}\\
&= \frac{r^{\frac{1}{3}}}{3}\left(e^{\frac{i\theta}{3}+\frac{4\pi i}{3}} + e^{-\frac{i\theta}{3}+\frac{2\pi i}{3}}\right)\\
&= \frac{r^{\frac{1}{3}}}{3}\left(e^{\frac{i\theta}{3}+\frac{4\pi i}{3}} + e^{-(\frac{i\theta}{3}+\frac{4\pi i}{3})}\right)\\
&= \frac{2r^{\frac{1}{3}}}{3}\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right),
\end{aligned}
$$

$$
\begin{aligned}
\gamma &= \frac{r^{\frac{1}{3}}e^{\frac{i\theta}{3}} \cdot e^{\frac{2\pi i}{3}} + r^{\frac{1}{3}}e^{-\frac{i\theta}{3}} \cdot e^{\frac{4\pi i}{3}}}{3} \\
&= \frac{r^{\frac{1}{3}}}{3}\left(e^{\frac{i\theta}{3}+\frac{2\pi i}{3}} + e^{-\frac{i\theta}{3}+\frac{4\pi i}{3}}\right) \\
&= \frac{r^{\frac{1}{3}}}{3}\left(e^{\frac{i\theta}{3}+\frac{2\pi i}{3}} + e^{-(\frac{i\theta}{3}+\frac{2\pi i}{3})}\right) \\
&= \frac{2r^{\frac{1}{3}}}{3}\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right).
\end{aligned}
$$

By eq #

Furthermore, we can simplify the common $\frac{2r^{\frac{1}{3}}}{3}$ term in front of each of the solutions as such:

earlier and have eq #

$$
\begin{aligned}
\frac{2r^{\frac{1}{3}}}{3} &= \frac{2}{3}\sqrt[3]{\sqrt{\frac{729q^2}{4}+\frac{27}{4}(-27q^2-4p^3)}} \\
&= 2\sqrt[3]{\frac{1}{27}\sqrt{\frac{729q^2}{4}-\frac{729q^2}{4}-27p^3}} \\
&= 2\sqrt[3]{\frac{1}{27}\sqrt{-27p^3}} \\
&= 2\sqrt[3]{\sqrt{\frac{-27p^3}{729}}} \\
&= 2\sqrt[3]{\sqrt{\frac{-p^3}{27}}} \\
&= 2\left(\left(\left(\frac{-p}{3}\right)^3\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} \\
&= 2\left(\frac{-p}{3}\right)^{\frac{1}{2}} \\
&= 2\sqrt{\frac{-p}{3}}.
\end{aligned}
$$

Why is p negative?

5.3 Solutions to the Depressed Cubic Equation

Finally, plugging in the simplified term for $\frac{2r^{\frac{1}{3}}}{3}$ results in the three solutions to the depressed cubic equation:

$$\alpha = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\theta}{3}\right),$$

$$\beta = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right),$$

$$\gamma = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right).$$

~~6 Solutions to the General Cubic Equation~~

Based on the solutions to the depressed cubic equation, we can determine the solutions to the general cubic equation:

$$x_1 = \alpha - \frac{b}{3a}$$
$$x_2 = \beta - \frac{b}{3a}$$
$$x_3 = \gamma - \frac{b}{3a}.$$

7 Implementation Into Our Algorithm

later

At the beginning of this paper, we stated that each voxel of a CT scan is represented by a 3×3 matrix. Each of these matrices contains a characteristic polynomial, which can be represented by the general cubic equation (1). To solve this equation, we substituted for the result of the depressed cubic equation (3). To attain the solutions to the depressed cubic equation, we substituted, used the cubic roots of unity, and converted the roots to polar coordinates. Once we had solutions to the depressed cubic equation, we substituted them into the general cubic equation. These solutions to the general cubic equation can be used to find the eigenvalues of all voxel matrices. This information tells us the intensity of the voxels in the CT scans, which allows for increased clarification and definition in the boundaries of bones and joints. This process can be implemented into a computer algorithm and run automatically for the millions of voxels in the CT scans.

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[1] Canny, John Francis. *Finding Edges and Lines in Images*. Technical Report 720, MIT Artificial Intelligence Laboratory, 1983.

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[3] Kronenburg, M.J. *A Method for Fast Diagonalization of a 2×2 or 3×3 Real Symmetric Matrix*. Cornell University, 16 Feb 2015.