Efficient Diagonalization of Hessian Matrices:

Applied to Segmentation of Phalanges from 3D CT Scans of Hands

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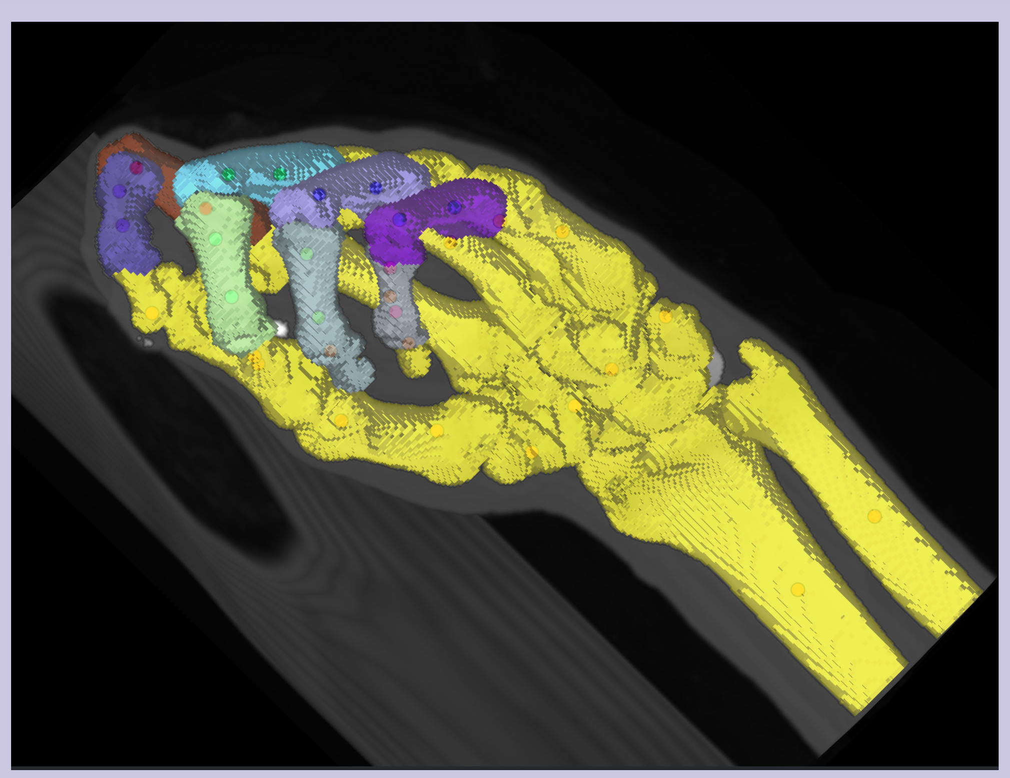
Introduction

We began this research to collaborate with Lifejoint in their ambition to develop improved finger joint implants because current implants are prone to breakage and provide a limited range of motion. So far, we managed to:

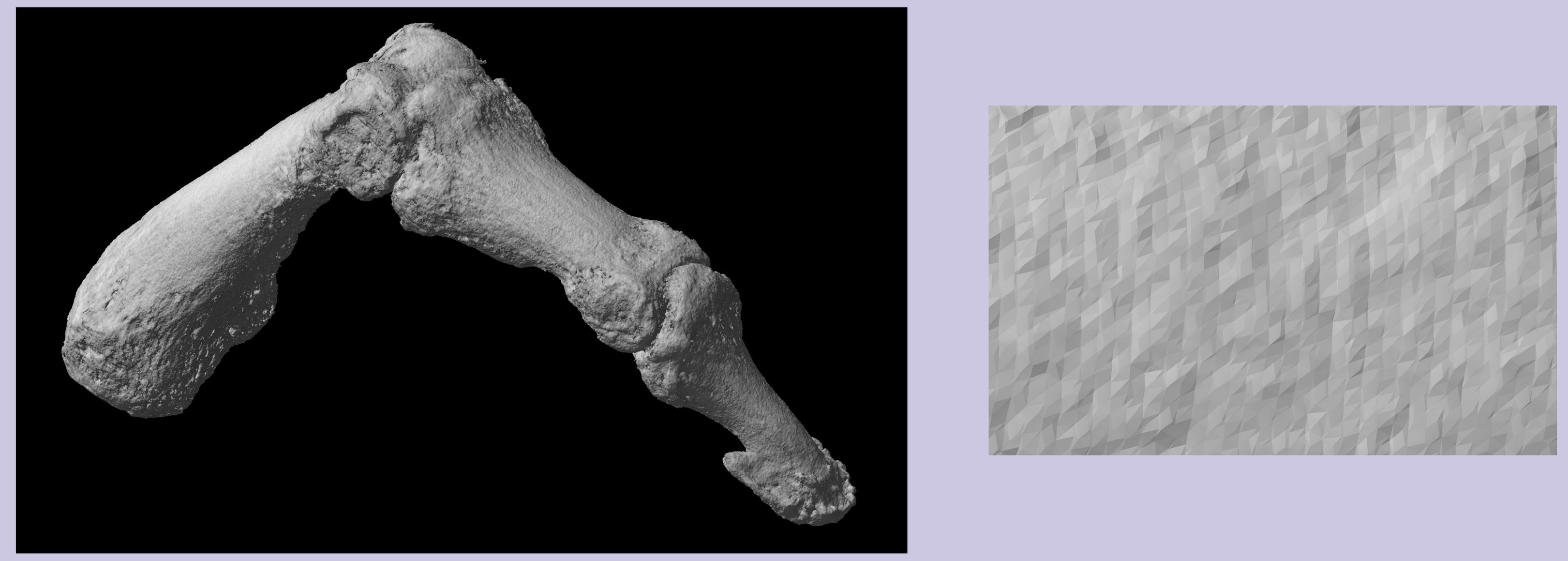
- Perform image analysis and segmentation of CT scans.
- Diagonalize Hessian matrices to find eigenvalues.
- Propose an algorithm for finding eigenvalues using Viete's formula.
- Compare our algorithm to the Jacobi algorithm, testing accuracy and efficiency.

Research

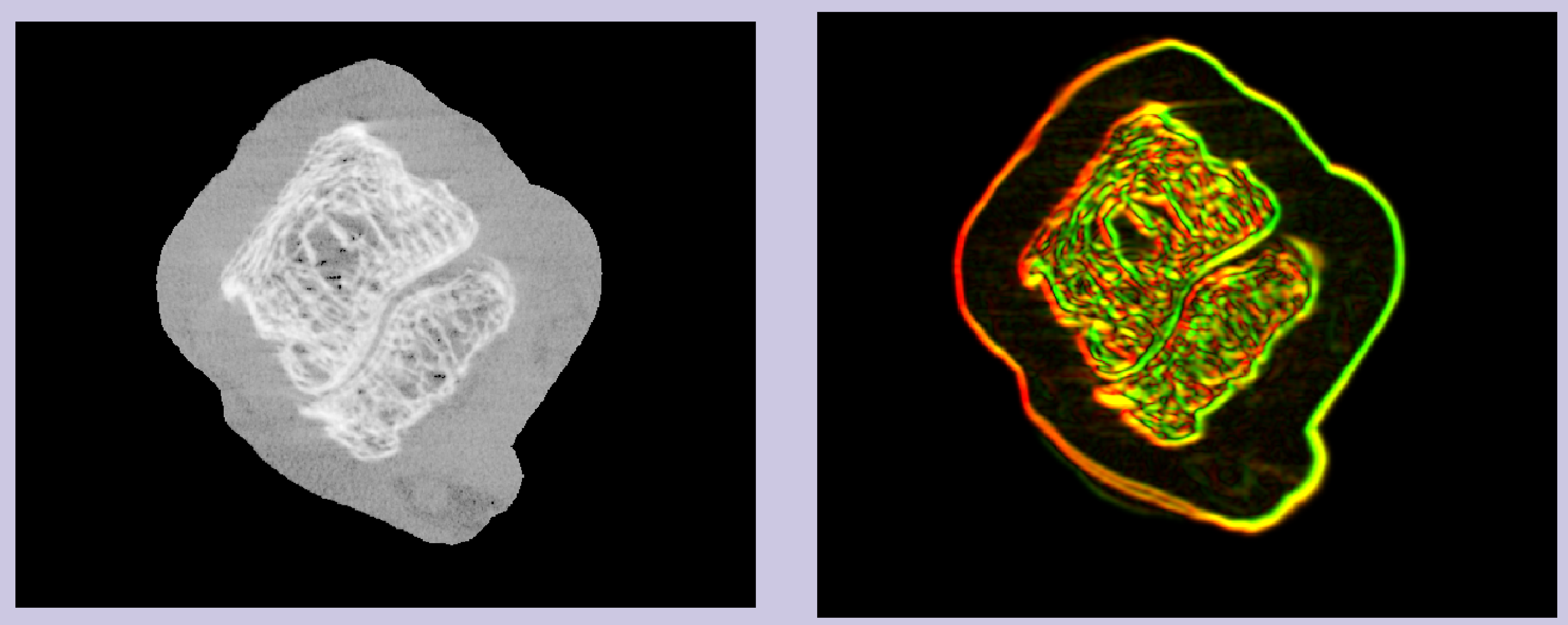
We implemented various segmentation methods to provide clarity to the proximal and medial phalanges. This process generates a defined contour around the edges of each region to accurately model the phalanges and attain reproducible results.



We have been able to computationally separate each finger from the hand at the metacarpal joint. This allows for further analysis and research to be conducted.



The image (left) features a segmented pinky thresholded at 500 Hounsfield units. The Marching Cubes algorithm is then applied to the pinky, creating a triangular mesh (right) to visualize the data.



The images show a 2-dimensional slice at the proximal interphalangeal joint. The variance of the three eigenvalues is represented by two colors when the first value exceeds the second (red), the second value exceeds the third (green), and where they overlap (yellow).
These eigenvalues exhibit the curvature of the intensity of each voxel in three orthogonal directions.

Hessian Matrix:

$$[P_{ij}]_{ij} = \begin{bmatrix} G_{xx} * I & G_{xy} * I & G_{xz} * I \\ G_{xy} * I & G_{yy} * I & G_{yz} * I \\ G_{xz} * I & G_{yz} * I & G_{zz} * I \end{bmatrix}$$

The six second-derivatives are convolved with the image.

General Cubic Solutions:

$$x_1 = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\theta}{3}\right) - \frac{b}{3a} \qquad \theta = \arccos\left(\frac{9q}{2\sqrt{-3p^3}}\right)$$

$$x_2 = \sqrt{\frac{-p}{3}}\left(-\cos\left(\frac{\theta}{3}\right) + \sqrt{3}\sin\left(\frac{\theta}{3}\right)\right) - \frac{b}{3a} \qquad p = \frac{3ac - b^2}{3a^2}$$

$$x_3 = \sqrt{\frac{-p}{3}}\left(-\cos\left(\frac{\theta}{3}\right) - \sqrt{3}\sin\left(\frac{\theta}{3}\right)\right) - \frac{b}{3a} \qquad q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}$$

GitHub: Proof and Code

Comparison

The Jacobi eigenvalue algorithm is known to be simple and reliable for $N \times N$ symmetric matrices. We proposed and developed an algorithm for finding eigenvalues in 3×3 matrices. In comparison to the Jacobi algorithm, the run time of our algorithm is 1.568 times as fast $\pm$ 1.667 seconds.

Verifications, such as dissections of cadaver hands, will be helpful in determining the accuracy of our algorithm and its functionality in comparison to our expectations.

Conclusions

The computation of eigenvalues has allowed for increased definition between phalanges. Clear separation at the joints is required for the next step - statistical shape modeling. The future of this research is promising and we hope others may build upon our work.