# Beginning Activities for Section 9.1

*Mathematical Reasoning: Writing and Proof, Version 3*

## Beginning Activity 1 (Equivalent Sets, Part 1)

As is indicated in the definition of equivalent sets, one of the fundamental tools for "comparing the sizes of sets" is the concept of a function. In particular, we need to thoroughly understand the concepts of injections, surjections, and bijections.

**1.** (**a**) The function $f$ is an injection provided that for all $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

(**b**) The function $f$ is not an injection provided that there exist $x, y \in A$, such that $x \neq y$ and $f(x) \neq f(y)$.

(**c**) The function $f$ is a surjection provided that for all $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

(**d**) The function $f$ is not a surjection provided that there exists a $y \in B$ such that for all $x \in A$ $f(x) \neq y$.

(**e**) The function $f$ is a bijection provided that $f$ is an injection and $f$ is a surjection.

**2.** (**a**) There are several bijections that can be used to prove that $A \approx B$. One is: $f : A \to B$ where $f(1) = a$, $f(2) = b$, and $f(3) = c$.

(**b**) The set $C$ is not equivalent to $B$. For any function $f : C \to B$, the range of the $f$ contains at most two elements. This means that $f$ cannot be a surjection and hence cannot be a bijection.

(**c**) The following function shows that $X = \{1, 2, 3, \ldots, 10\}$ is equivalent $Y = \{57, 58, 59, \ldots, 66\}$. $f : X \to Y$ by $f(x) = x + 56$, for all $x \in X$.

**3.** Let $f : \mathbb{N} \to D^+$ by $f(x) = 2x + 1$, for all $x \in \mathbb{N}$.

Let $x, t \in \mathbb{N}$ and assume that $f(x) = f(t)$. Then, $2x - 1 = 2t - 1$ and hence, $x = t$. Therefore, $f$ is an injection.

To prove that $f$ is a surjection, let $y \in D^+$. Since $y$ is odd, $y + 1$ is an even natural number, and hence $\dfrac{y+1}{2} \in \mathbb{N}$. Also,

$$f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1 = y.$$

Therefore, $f$ is a surjection and hence $f$ is a bijection. Thus, $\mathbb{N} \approx D^+$.

**4.** Let $g : \mathbb{R} \to \mathbb{R}^+$ by $g(x) = e^x$ for all $x \in \mathbb{R}$. If $a, b \in R$ and $g(a) = g(b)$, then $e^a = e^b$. Taking the natural logarithm of both sides of this equation yields

$$\begin{aligned} \ln\left(e^a\right) &= \ln\left(e^b\right) \\ a &= b \end{aligned}$$

Therefore, $g$ is an injection. To prove that $g$ is a surjection, let $y \in \mathbb{R}^+$. Then, $\ln y \in R$ and

$$\begin{aligned} g(\ln y) &= e^{\ln y} \\ &= y \end{aligned}$$

Hence, $g$ is a surjection. This proves that $g$ is a bijection and hence, $\mathbb{R} \approx \mathbb{R}^+$.

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## Beginning Activity 2 (Equivalent Sets, Part 2)

**Theorem 9.1** Let $A$, $B$, and $C$ be sets.

(**a**) For each set $A$, $A \approx A$.

(**b**) For all sets $A$ and $B$, if $A \approx B$, then $B \approx A$.

(**c**) For all sets $A$, $B$, and $C$, if $A \approx B$ and $B \approx C$, then $A \approx C$.

***Proof***. Let $A$ be a set. The identity mapping $I_A$ on the set $A$ is a bijection. Therefore, $A \approx A$. (Recall that $I_A(x) = x$ for all $x \in A$.)

Now, assume that $A$ and $B$ are sets and that $A \approx B$. Then, there exists a bijection $f : A \to B$. Hence, by Exercise (9) in Section 6.5, $f^{-1} : B \to A$ is a bijection. Therefore, $B \approx A$.

Now, assume that $A$, $B$, and $C$ are sets and that $A \approx B$ and $B \approx C$. Then, there exist bijections $f : A \to B$ and $g : B \to C$. By Theorem 6.20 in Section 6.4, $g \circ f : A \to C$ is a bijection. Therefore, $A \approx C$.

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## Proof of Exercise 9 from Section 6.5

***Proof***. Let $A$ and $B$ be sets and assume $A \approx B$. So there exists a bijection $f : A\, to B$ and hence, the inverse function $f^{-1} : B \to A$ is defined.

Now assume that $b_1, b_2 \in B$ and $f^{-1}(b_1) = f^{-1}(b_2)$. We can then apply the function $f$ to both sides of this equation and obtain

$$\begin{aligned} f\left(f^{-1}(b_1)\right) &= f\left(f^{-1}(b_1)\right) \\ b_1 &= b_2, \end{aligned}$$

This proves that for all $b_1, b_2 \in B$, if $f^{-1}(b_1) = f^{-1}(b_2)$, then $b_1 = b_2$. Hence, $f^{-1}$ is an injection.

Now, let $a \in A$ and let $f(a) = b$, where $b \in B$. This means that $f^{-1}(b) = a$, and hence, $f^{-1}$ is a surjection. Since $f^{-1}$ is both an injection and a surjection, we conclude that $f^{-1} : B \to A$ is a bijection, and hence, $B \approx A$. ■