

Beginning Activities for Section 9.3

Mathematical Reasoning: Writing and Proof, Version 3

Beginning Activity 1 (The Game of Dodge Ball)

Player Two has a winning strategy.

- Player One must complete the first row of six boxes. Player Two looks only at the first box in this row. Whatever Player One put in the first box, Player Two puts the opposite symbol in the first box of Player Two's row.
- Player One then completes the second row of six boxes. Player Two looks only at the second box in this row. Whatever Player One put in the second box, Player Two puts the opposite symbol in the second box of Player Two's row.

This is the pattern. After Player One completes the k th row of six boxes, Player Two looks only at the k th box in this row. Whatever Player One put in the k th box, Player Two puts the opposite symbol in the k th box of Player Two's row. Doing this for all six boxes, Player Two will create a string of six symbols that will not match any of Player One's six rows.

Applying the Winning Strategy to Lists of Real Numbers

Following is a list of real numbers between 0 and 1. Each real number is written as a decimal number.

$$a_1 = 0.1234567890$$

$$a_6 = 0.0103492222$$

$$a_2 = 0.3216400000$$

$$a_7 = 0.0011223344$$

$$a_3 = 0.4321593333$$

$$a_8 = 0.7077700022$$

$$a_4 = 0.9120930092$$

$$a_9 = 0.2100000000$$

$$a_5 = 0.0000234102$$

$$a_{10} = 0.9870008943$$

We consider the numbers to the right of the decimal point to be like the boxes in the game of Dodgeball. So we will construct a number $0.b_1b_2b_3b_4b_5b_6b_7b_8b_9b_{10}$ so that:

- b_1 is not equal to the first digit to the right of the decimal in a_1 ;
- b_2 is not equal to the second digit to the right of the decimal in a_2 ;
- b_3 is not equal to the second digit to the right of the decimal in a_3 ;

and so on through b_{10} . We will do this by choosing b_k to be 3 if the k^{th} digit to the right of the decimal place in a_k is not equal to 3 and $b_k = 5$ if the k^{th} digit to the right of the decimal place in a_k is equal to 3. (There are many other ways to do this.) So we use

$$b_k = 0.3333335335.$$

We should be able to apply this method to longer lists of numbers.



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Beginning Activity 2 (Functions from a Set to Its Power Set)

The work in this beginning activity is intended to help understand the proof of one of the most profound results of number theory, namely, Cantor's Theorem (Theorem 9.27) on page 483.

1. (a) $1 \in f(1)$ $2 \notin f(2)$ $3 \notin f(3)$ $4 \in f(4)$
(b) $S = \{x \in A \mid x \notin f(x)\} = \{2, 3\}$
(c) There does not exist a $t \in A$ such that $f(t) = S$. So $S \notin \text{range}(f)$.
 2. (a) $f(1) = \{2, 3, 4\}$. So, $1 \notin f(1)$.
(b) $f(2) = \{1, 3, 4\}$. So, $2 \notin f(2)$.
(c) $f(3) = \{1, 2, 4\}$. So, $3 \notin f(3)$.
(d) $f(4) = \{1, 2, 3\}$. So, $4 \notin f(4)$.
(e) $S = \{x \in A \mid x \notin f(x)\} = \{1, 2, 3, 4\}$
(f) There does not exist a $t \in A$ such that $f(t) = S$. So $S \notin \text{range}(f)$.
 3. (a) $f(1) = \mathbb{N} - \{1\} = \{2, 3, 4, 5, \dots\}$. So, $1 \notin f(1)$.
 $f(2) = \mathbb{N} - \{4\} = \{2, 3, 5, 6, 7, \dots\}$. So, $2 \in f(2)$.
 $f(3) = \mathbb{N} - \{9, 3\}$. So, $3 \notin f(3)$.
 $f(4) = \mathbb{N} - \{16, 8\}$. So, $4 \in f(4)$.
(b) If $n > 3$, then $n^2 > 3n$ and $3n > n$. Therefore, $n^2 > n$. Also, since $n^2 > 3n$, we seen that $n^2 - 2n > 3n - 2n$ or $n^2 - 2n > n$. Therefore, $n \notin \{n^2, n^2 - 2n\}$, and hence $n \in f(n)$.
(c) A natural number x is not in $f(x)$ if and only if $x^2 = x$ or $x^2 - 2x = x$. The only natural number solution of the first equation is $x = 1$, and the only natural number solution of the second equation is $x = 3$. Consequently, $S = \{x \in \mathbb{N} \mid x \notin f(x)\} = \{1, 3\}$.
(d) For each $t \in \mathbb{N}$, $f(t)$ is an infinite set. Consequently, There does not exist a $t \in \mathbb{N}$ such that $f(t) = S$. So $S \notin \text{range}(f)$.
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