Beginning Activities for Section 4.2

Mathematical Reasoning: Writing and Proof, Version 3

Beginning Activity 1 (Exploring a Proposition about Factorials)

Notice how the numerical examples calculated in part 1 are used to explore the open sentence "$n! > 2^n$."

1.

n	1	2	3	4	5	6	7
2^n	2	4	8	16	32	64	128
$n!$	1	2	6	24	120	720	5040

2. $P(1)$, $P(2)$, $P(3)$ are false. $P(4)$, $P(5)$, $P(6)$, $P(7)$ are true.

3. Based on the evidence so far, the following proposition appears to be true: For each natural number n with $n \geq 4$, $2^n > n!$.

4. Since $(k+1)!$ is the product of the first $k+1$ natural numbers, we can think of this as the product of the first k natural numbers times $(k+1)$. This means that $(k+1)!$ is equal to $(k+1)$ times $k!$.

5. If we multiply both sides of the inequality $(k+1) > 2$ by 2^k, we obtain

$$\begin{aligned}(k+1)2^k &> 2 \cdot 2^k \\ (k+1)2^k &> 2^{k+1}\end{aligned}$$

6. We have

$$(k+1) \cdot k! > (k+1)2^k \tag{1}$$

and from part (5),

$$(k+1)2^k > 2^{k+1} \tag{2}$$

Inequalities (1) and (2) then imply that $(k+1) \cdot k! > 2^{k+1}$ or that $(k+1)! > 2^{k+1}$.

If the argument (and algebra) in parts 4 through 6 are hard to understand, first try to follow along with a specific example such as $k = 5$. We know that $5! > 2^5$. Although it is possible to do the calculation to show that $6! > 2^6$, pretend we cannot do this and use this numerical example to illustrate the algebraic argument. We would have

$$\begin{aligned}5! &> 2^5 \\ 6 \cdot 5! &> 6 \cdot 2^5 \\ 6! &> 6 \cdot 2^5 \qquad (1)\end{aligned}$$

In addition, $6 > 2$ and so

$$\begin{aligned}6 \cdot 2^5 &> 2 \cdot 2^5 \\ 6 \cdot 2^5 &> 2^6 \qquad (2)\end{aligned}$$

So using inequalities (1) and (2), we see that $6! > 2^6$.

Beginning Activities for Section 4.2

Mathematical Reasoning: Writing and Proof, Version 3

Beginning Activity 2 (Prime Factors of a Natural Number)

2.

$20 = 2^2 \cdot 5$	$40 = 2 \cdot 20$	$50 = 2 \cdot 5^2$	$150 = 3 \cdot 50$
	$40 = 2(2^2 \cdot 5)$		$150 = 3(2 \cdot 5^2)$
	$40 = 2^3 \cdot 5$		$150 = 2 \cdot 3 \cdot 5^2$

4. A natural number n is a composite number provided that there exists a natural number d such that d divides n and $d \neq 1$ and $d \neq n$. This means that there exists a natural number m such that $n = m \cdot d$, $1 < d < n$, and $1 < m < n$.

5. In this section, we will see how to use induction to prove that any composite number can be written as a product of primes. The idea will be to factor a composite number as $n = m \cdot d$, where $1 < d < n$, and $1 < m < n$. We will then use induction to conclude that m and d can be factored as a product of primes. (This was illustrated in Part (2).) We will need the Second Principle of Mathematical Induction, which is introduced in this section.
