Beginning Activities for Section 7.2

Mathematical Reasoning: Writing and Proof, Version 3

Beginning Activity 1 (Properties of Relations)

1. The relation R is not reflexive on A provided that there exists an $x \in A$ such that $(x, x) \notin R$ (or equivalently, $x \not R\ x$).

2. The relation R is not symmetric provided that there exist $x, y \in A$ such that $(x, y) \in R$ and $(y, x) \notin R$ (or $x\ R\ y$ and $y \not R\ x$).

3. The relation R is not transitive provided that there exist $x, y, z \in A$ such that $(x, y) \in R$, $(y, z) \in R$, and $(x, z) \notin R$ (or $x\ R\ y$, $y\ R\ z$, and $x \not R\ z$).

4. The directed graph should have a loop at each of the four vertices, an arrow from 1 to 3, and an arrow from 3 to 2.

   The loops at each vertex show that for each $x \in A$, $x\ R\ x$ and so the relation R is reflexive.

   The relation R is not symmetric since $1\ R\ 3$ but $3 \not R\ 1$, and the relation R is not transitive since $1\ R\ 3$ and $3\ R\ 2$ but $1 \not R\ 2$.

5. The directed graph should have a loop at vertex 2, an arrow from 1 to 4, an arrow from 4 to 1, an arrow from 2 to 4 and an arrow from 4 to 2. The relation T is not reflexive since $a \in A$ and $a \not T\ a$.

   The relation T is symmetric since whenever there is an arrow from a vertex a to a vertex b, there is an arrow from b to a. This means that for all $a, b \in A$, if $a\ T\ b$, then $b\ T\ a$ is a true conditional statement.

   The relation T is not transitive since $1\ T\ 4$ and $4\ T\ 2$ but $1 \not T\ 2$.

Note: There are other properties of relations that we will not study in this text. Two such properties are:

- A relation R on a set A is **irreflexive** provided that for all $x, y \in A$, if $x\ R\ y$, then $x \neq y$. This means that no element in A is related to itself. One such example is the "less than" relation on the real numbers.

- A relation R on a set A is **antisymmetric** provided that for all $x, y \in A$, if $x\ R\ y$ and $y\ R\ x$, then $x = y$. One such example is the "less than or equal to" relation on the real numbers.

Beginning Activities for Section 7.2

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Beginning Activity 2 (Review of Congruence Modulo n)

1. Let $n \in \mathbb{N}$. If a and b are integers, then we say that a is congruent to b modulo n provided that n divides $a - b$. So,

$$a \equiv b \pmod{n} \text{ means } (\exists k \in \mathbb{Z})\,(a - b = nk)$$
$$a \equiv b \pmod{n} \text{ means } (\exists k \in \mathbb{Z})\,(a = b + nk)$$

2. Let $a, b \in \mathbb{Z}$ and let $n \in \mathbb{N}$. If $a \equiv b \pmod{n}$, then we have determined an ordered pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. So, congruence modulo n determines the following subset of $\mathbb{Z} \times \mathbb{Z}$:

$$\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \,(\text{mod} n)\}.$$

So, this is a relation on $\mathbb{Z}$.

3. Theorem 3.4 states that the relation of congruence modulo n is reflexive, symmetric, and transitive.

4. **Theorem 3.31**. Let $n \in \mathbb{N}$ and let $a \in \mathbb{Z}$. If $a = nq + r$ and $0 \leqslant r < n$ for some integers q and r, then $a \equiv r \pmod{n}$.

Corollary 3.32. If $n \in \mathbb{N}$, then each integer is congruent, modulo n, to precisely one of the integers $0, 1, 2, \ldots, n - 1$.

5. **Symmetric Property**: Let $a, b \in \mathbb{Z}$ and let $n \in \mathbb{N}$. If $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.

Proof. Let $a, b \in \mathbb{Z}$ and let $n \in \mathbb{N}$ and assume that $a \equiv b \pmod{n}$. We will prove that $b \equiv a \pmod{n}$. Since $a \equiv b \pmod{n}$, we know that $n \mid (a - b)$ and hence that there exists an integer k such that

$$a - b = nk.$$

If we multiply both sides of this equation by -1, we get

$$\begin{aligned}(-1)\,(a - b) &= (-1)\,nk\\ b - a &= n\,(-k)\,.\end{aligned}$$

Since $-k \in \mathbb{Z}$, this last equation implies that $n \mid (b - a)$ and hence that $b \equiv a \pmod{n}$. We have thus proven that if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$. This means that the relation of congruence modulo n is a symmetric relation. ■
