Beginning Activities for Section 3.3

Mathematical Reasoning: Writing and Proof, Version 3

Beginning Activity 1 (Proof by Contradiction)

The main purpose of this beginning activity is to provide a logical justification for the method of proof know as "proof by contradiction." A different justification for this method of proof is given in Exercise 1 of this section.

1. The truth table for $(P \vee \neg P)$ shows that this statement is always true, and the truth table for $(P \wedge \neg P)$ shows that this statement is always false.

2.

P	Q	$P \to Q$	$\neg(P \to Q)$	$\neg Q$	$P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

This shows that $\neg(P \to Q)$ is logically equivalent to $P \wedge \neg Q$.

3. If $x = 2$, then

$$\frac{1}{x(1-x)} = \frac{1}{2(-1)} = -\frac{1}{2},$$

and $-\frac{1}{2} < 4$. So $x = -2$ is a counterexample that shows the statement is false.

4. The negation of the statement is:

$$\text{There exists a real number } x \text{ such that } 0 < x < 1 \text{ and } \frac{1}{x(1-x)} < 4.$$

So for a proof by contradiction, we would assume that there exists a real number x such that $0 < x < 1$ and $\frac{1}{x(1-x)} < 4$.

Beginning Activities for Section 3.3

Mathematical Reasoning: Writing and Proof, Version 3

Beginning Activity 2 (Constructing a Proof by Contradiction)

1. Since we have assumed that $x > 0$ and $y > 0$, we know that $xy > 0$. So if we multiply both sides of the inequality $\dfrac{x}{y} + \dfrac{y}{x} \leq 2$, we obtain

$$\begin{aligned} \left(\frac{x}{y} + \frac{y}{x}\right) xy &\leq 2xy \\ \left(\frac{x}{y}\right) \cdot xy + \left(\frac{y}{x}\right) \cdot xy &\leq 2xy \\ x^2 + y^2 &\leq 2xy \end{aligned}$$

If we now subtract $2xy$ from both sides of the last inequality, and then factor the left side, we obtain

$$\begin{aligned} x^2 - 2xy + y^2 &\leq 0 \\ (x - y)^2 &\leq 0 \end{aligned}$$

2. Another assumption that was made was that $x \neq y$. This implies that $x - y \neq 0$ and hence, that $(x - y)^2 > 0$. This contradicts the last inequality in part (1).
