Financial mathematics, MATH /ECO 3900

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1 Practice Questions

1. If $dS = \mu S dt + \sigma S dW$, where W is Wiener process. Find the stochastic differential equation satisfied by
   - $f(S) = As$
   - $f(S) = S^n$
2. If there are two assets satisfying the following stochastic differential equations:

   $$dS_i = \mu_i S_i + \sigma_i S_i dW_i \text{ for } i = 1, 2.$$

   The wiener process dW_i satisfy $\mathcal{E}(dW_i^2) = dt$. The asset price changes are co related with each other

   $$\mathcal{E}(dW_i dW_j) = \rho_{ij} dt,$$

   where $-1 \leq \rho_{ij} \leq 1$. Derive the Ito's lemma for a function $f(t, S_1, S_i)$. Generalize the result to n assets.
3. Consider

   $$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

   $$d_1 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

   Also $N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx$. Show that

   $$N^{'}(d_2) = N^{'}(d_1) \frac{S}{E} e^{r(T-t)},$$

   where

   $$N^{'}(d) = \frac{1}{2\pi} e^{-d^2/2}.$$
4. Show that

   $$SN^{'}(d_1) - Ee^{-r(T-t)} N^{'}(d_2) = 0.$$

5. Show that
$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}$$
for European call option and
$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)$$
satisfy
$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$
6. Show that $C(S,t)$ and $P(S,t)$ also satisfy the put-call parity. Calculate the following Greeks for both European call and Put options
7. $\Delta_{call} = \frac{\partial C}{\partial S}$
8. $\Delta_{put} = \frac{\partial P}{\partial S}$
9. $\Gamma_{call} = \frac{\partial^2 C}{\partial S^2}$
10. $\Gamma_{put} = \frac{\partial^2 P}{\partial S^2}$
11. Assume $r = 0$, calculate $\Theta = -\frac{\partial V}{\partial t}$
12. Calculate $\Theta_{call} = -\frac{\partial C}{\partial t}$
13. Show that $\Theta_{put} = \Theta_{call} - rEe^{-r(T-t)}$
14. Calculate $Vega_{call} = \frac{\partial C}{\partial \sigma}$
15. Calculate $Vega_{put} = \frac{\partial P}{\partial \sigma}$
16. Calculate $\rho_{call} = \frac{\partial C}{\partial r}$
17. Calculate $\rho_{put} = \frac{\partial P}{\partial r}$
18. What is the random walk followed by European call option.
19. Suppose that European calls of all exercise prices are available. regarding S as fixed and E as variable, show that their price $C(E,t)$ satisfy the partial differential equations
$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial E^2} + rS\frac{\partial C}{\partial E} - rC = 0.$$
20. Transform the Black-Scholes partial differential equation with continuous divided yield D_0
$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0,$$
into the diffusion equation
$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$
Use the transformation $\tau = T - t$, $x = \log(S/E)$, and $v(x,t) = \frac{V(S,t)}{E}$.

21. Expand $U(x, \tau + \delta\tau)$ and $U(x, \tau - \delta\tau)$ as Taylor series about (x, τ). Deduce the central difference approximation.

22. Develop implicit finite difference Scheme for Black-Scholes partial differential equation.

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24. Derive the partial differential equation for the Asian option, where

$$\text{payoff} = \max(S(T) - \frac{1}{T}\int_0^T f(S,t)dt, 0)$$

and

$$I = \int_0^T f(S,t)dt.$$