Chapter 1 Exercise Solutions

1.1 Exercise Solutions

1.1.1 Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$.

V2

$$\vec{x} + (\vec{y} + \vec{w}) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 + w_1 \\ \vdots \\ y_n + w_n \end{bmatrix} = \begin{bmatrix} x_1 + (y_1 + w_1) \\ \vdots \\ x_n + (y_n + w_n) \end{bmatrix}$$
$$= \begin{bmatrix} (x_1 + y_1) + w_1 \\ \vdots \\ (x_n + y_n) + w_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = (\vec{x} + \vec{y}) + \vec{w}$$

V4 Take $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$. Then, we have

$$\vec{x} + \vec{0} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$$

as required.

V5 Take $-\vec{x} = \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix} \in \mathbb{R}^n$. Then, $\vec{x} + (-\vec{x}) = \begin{bmatrix} x_1 + (-x_1) \\ \vdots \\ x_n + (-x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$ as required.

V6 $c\vec{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix} \in \mathbb{R}^n$, since $cx_i \in \mathbb{R}$ for $1 \leq i \leq n$.

V7 $c(d\vec{x}) = c\begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = \begin{bmatrix} c(dx_1) \\ \vdots \\ c(dx_n) \end{bmatrix} = \begin{bmatrix} (cd)x_1 \\ \vdots \\ (cd)x_n \end{bmatrix} = cd\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (cd)\vec{x}.$

V8

$$(c+d)\vec{x} = (c+d)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (c+d)x_1 \\ \vdots \\ (c+d)x_n \end{bmatrix} = \begin{bmatrix} cx_1 + dx_1 \\ \vdots \\ cx_n + dx_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix} + \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = c\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + d\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = c\vec{x} + d\vec{x}$$

V9

$$c(\vec{x}+\vec{y}) = c\begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} c(x_1 + y_1) \\ \vdots \\ c(x_n + y_n) \end{bmatrix} = \begin{bmatrix} cx_1 + cy_1 \\ \vdots \\ cx_n + cy_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix} + \begin{bmatrix} cy_1 \\ \vdots \\ cy_n \end{bmatrix} = c\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + c\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = c\vec{x} + c\vec{y}$$

V10 $1\vec{x} = \begin{bmatrix} 1x_1 \\ \vdots \\ 1x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}.$

1.1.2 Since neither $\vec{y}$ nor $\vec{z}$ is a scalar multiple of the other, the set is a plane in $\mathbb{R}^3$ passing through the origin.

1.1.3 To determine if the set is linearly independent, we use the definition. Consider

$$c_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using operations on vectors this gives

$$\begin{bmatrix} c_1 + c_2 + c_3 \\ 2c_1 - c_2 - c_3 \\ c_1 + 2c_2 - 3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since vectors are equal if and only if their corresponding entires are equal, we get a simultaneous system of equations

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 - c_2 - c_3 &= 0 \\ c_1 + 2c_2 - 3c_3 &= 0 \end{aligned}$$

Adding the first equation to the second, we get $3c_1 = 0$ and so $c_1 = 0$. Therefore, the first equation gives $c_2 = -c_3$. Substituting into the third equation, we get $-5c_3 = 0$. Therefore, $c_3 = 0 = c_2$. Hence, the only solution is $c_1 = c_2 = c_3 = 0$, so the set is linearly independent.

1.1.4 To determine if the set is linearly independent, we use the definition. Consider

$$c_1\begin{bmatrix}1\\-1\\3\end{bmatrix}+c_2\begin{bmatrix}2\\-4\\1\end{bmatrix}+c_3\begin{bmatrix}1\\0\\-1\end{bmatrix}+c_4\begin{bmatrix}3\\-4\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

We observe that a non-trivial solution to this equation is $c_1 = 0$, $c_2 = 1$, $c_3 = 1$, and $c_4 = -1$. That is,

$$0\begin{bmatrix}1\\-1\\3\end{bmatrix}+1\begin{bmatrix}2\\-4\\1\end{bmatrix}+1\begin{bmatrix}1\\0\\-1\end{bmatrix}+(-1)\begin{bmatrix}3\\-4\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1.1}$$

Hence, the set is linearly dependent.

As indicated in the proof of Theorem 1.1.3, to write one of the vectors as a linear combination of the others, we pick one of the vectors with a non-zero coefficient in equation (1.1). We pick $\vec{v}_4 = \begin{bmatrix}3\\-4\\0\end{bmatrix}$. Solving for this vector gives

$$\begin{bmatrix}3\\-4\\0\end{bmatrix}=0\begin{bmatrix}1\\-1\\3\end{bmatrix}+1\begin{bmatrix}2\\-4\\1\end{bmatrix}+1\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

NOTE: We could also have written the second or third vector as a linear combination of the others, but we could not have written the first vector as a linear combination of the others. It is important to understand that the definition of linear dependence implies that at least one of the vectors can be written as linear combination of the others. However, as in this exercise, there may be some vectors in the set which cannot be written as a linear combination of the others.

1.1.5 The standard basis for $\mathbb{R}^4$ is $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

1.1.6 We have

$$\begin{aligned}
c_1\vec{v}_1+\cdots+c_k\vec{v}_k-d_1\vec{v}_1-\cdots-d_k\vec{v}_k &= d_1\vec{v}_1+\cdots+d_k\vec{v}_k-d_1\vec{v}_1-\cdots-d_k\vec{v}_k \text{ by V5}\\
c_1\vec{v}_1-d_1\vec{v}_1+\cdots+c_k\vec{v}_k-d_k\vec{v}_k &= d_1\vec{v}_1-d_1\vec{v}_1+\cdots+d_k\vec{v}_k-d_k\vec{v}_k \text{ by V3}\\
(c_1-d_1)\vec{v}_1+\cdots+(c_k-d_k)\vec{v}_k &= \vec{0}+\cdots+\vec{0} \text{ by V8, V5}\\
(c_1-d_1)\vec{v}_1+\cdots+(c_k-d_k)\vec{v}_k &= \vec{0} \text{ by V4}
\end{aligned}$$

1.1.7 To write a vector equation for a hyperplane in $\mathbb{R}^5$ we need 4 linearly independent vectors in $\mathbb{R}^5$. The easiest way to do this is to pick four of the standard basis vectors for $\mathbb{R}^5$. One answer is

$$\vec{x}=c_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}+c_3\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}+c_4\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix},\quad c_1,c_2,c_3,c_4\in\mathbb{R}$$

1.2 Exercise Solutions

1.2.1 No, L cannot be a subspace of $\mathbb{R}^2$. If it does not pass through the origin, it does not contain $\vec{0}$.

1.2.2 If $\mathbb{S}$ is a subspace of $\mathbb{R}^n$, then $\vec{0} \in \mathbb{S}$. Hence, there exists $t \in \mathbb{R}$ such that $\vec{0} = t\vec{v} + \vec{b}$. Thus, $\vec{b} = -t\vec{v}$. Therefore, $\vec{b}$ is a scalar multiple of $\vec{v}$.

On the other hand, if $\vec{b}$ is a scalar multiple of $\vec{v}$, there exists $s \in \mathbb{R}$ such that $\vec{b} = s\vec{v}$. We need to show that $\mathbb{S}$ is a subspace of $\mathbb{R}^n$.

By definition $\mathbb{S}$ is a subset of $\mathbb{R}^n$. Also, $\mathbb{S}$ is non-empty since

$$\vec{0} = -\vec{b} + \vec{b} = -(s\vec{v}) + \vec{b} = (-s)\vec{v} + \vec{b} \in \mathbb{S}$$

Let $\vec{x}, \vec{y} \in \mathbb{S}$ and let $c \in \mathbb{R}$. Then there exists $t_1, t_2 \in \mathbb{R}$ such that $\vec{x} = t_1\vec{v} + \vec{b}$ and $\vec{y} = t_2\vec{v} + \vec{b}$. Observe that

$$\vec{x} + \vec{y} = t_1\vec{v} + \vec{b} + t_2\vec{v} + \vec{b} = t_1\vec{v} + t_2\vec{v} + s\vec{v} + \vec{b} = (t_1 + t_2 + s)\vec{v} + \vec{b}$$

Thus, $\vec{x} + \vec{y} \in \mathbb{S}$, so $\mathbb{S}$ is closed under addition. Similarly,

$$c\vec{x} = c(t_1\vec{v} + \vec{b}) = ct_1\vec{v} + (c-1)\vec{b} + \vec{b} = ct_1\vec{v} + (c-1)(s\vec{v}) + \vec{b} = (ct_1 + (c-1)s)\vec{v} + \vec{b}$$

so $c\vec{x} \in \mathbb{S}$. Therefore, $\mathbb{S}$ is also closed under scalar multiplication. Hence, $\mathbb{S}$ is a subspace of $\mathbb{R}^n$ by the subspace test.

1.3 Exercise Solutions

1.3.1 We have (a) $\begin{bmatrix}1\\0\\0\end{bmatrix} \times \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ (b) $\begin{bmatrix}3\\-1\\1\end{bmatrix} \times \begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}-1\\4\\7\end{bmatrix}$ (c) $\begin{bmatrix}1\\1\\3\end{bmatrix} \times \begin{bmatrix}c\\c\\3c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

1.3.2 There are infinitely many choices. Two choices are $\vec{n}_1 = \begin{bmatrix}-3\\-1\\1\end{bmatrix}$ and $\vec{n}_2 = \begin{bmatrix}6\\2\\-2\end{bmatrix}$.

1.3.3 We have

$$\begin{bmatrix}-1\\-1\\3\end{bmatrix} \times \begin{bmatrix}2\\1\\4\end{bmatrix} = \begin{bmatrix}-7\\10\\1\end{bmatrix}$$

Thus, a normal vector for the plane is $\vec{n} = \begin{bmatrix}-7\\10\\1\end{bmatrix}$. Taking $s = 1$ and $t = 0$, we get that a point on the plane is $(-1, -1, 3)$. Thus, a scalar equation for the plane is

$$-7x_1 + 10x_2 + x_3 = (-7)(-1) + 10(-1) + 1(3) = 0$$

1.4 Exercise Solutions

1.4.1 We have

$$\begin{aligned}
\operatorname{proj}_{\vec{v}}(\vec{u}) \cdot \operatorname{perp}_{\vec{v}}(\vec{u}) &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} \cdot \left(\vec{u} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}\right) = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}\right) \cdot \vec{u} - \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}\right) \cdot \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}\right) \\
&= \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)(\vec{v} \cdot \vec{u}) - \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)^2 (\vec{v} \cdot \vec{v}) = \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} - \left(\frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^4}\right)(\|\vec{v}\|^2) \\
&= \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} - \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} = 0
\end{aligned}$$

as required.

1.4.2 We have

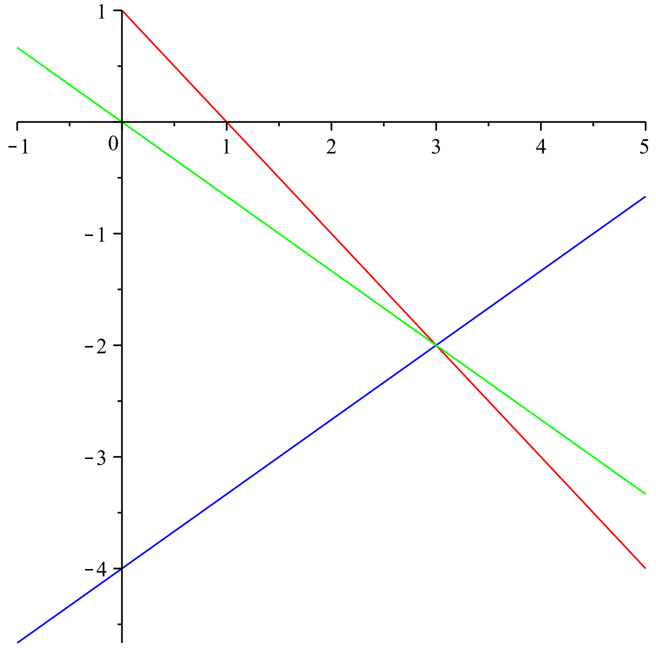
$$\operatorname{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = \frac{3}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\operatorname{perp}_{\vec{a}}(\vec{b}) = \vec{b} - \operatorname{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

Chapter 2 Exercise Solutions

2.1 Exercise Solutions

2.1.1 Graphing the three lines gives



2.1.2 We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7+2t \\ 2+t \\ -t \end{bmatrix}$$

so $x_1 = 7 + 2t$, $x_2 = 2 + t$, and $x_3 = -t$. Thus,

$$(7+2t) - 2(2+t) = 7 + 2t - 4 - 2t = 3$$
$$(7+2t) + (2+t) + 3(-t) = 9 + 3t - 3t = 9$$

as required.

2.2 Exercise Solutions

2.2.1 We first want to solve for one variable, say x_3. To do this we need to eliminate x_1 and x_2 from one of the equations. First, add -2 times the first equation to the second equation.

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= 0 \\ 5x_2 + 11x_3 &= -1 \\ -x_1 + x_2 - x_3 &= 2 \end{aligned}$$

Next add the first equation to the third equation

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= 0 \\ 5x_2 + 11x_3 &= -1 \\ -x_2 - 4x_3 &= 2 \end{aligned}$$

Add 5 times the third equation to the second equation to get

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= 0 \\ -9x_3 &= 9 \\ -x_2 - 4x_3 &= 2 \end{aligned}$$

Multiplying the second equation by $-\frac{1}{9}$ and the third equation by-1.

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= 0 \\ x_3 &= -1 \\ x_2 + 4x_3 &= -2 \end{aligned}$$

Add 2 times the third equation to first equation to get

$$\begin{aligned} x_1 + 5x_3 &= -4 \\ x_3 &= -1 \\ x_2 + 4x_3 &= -2 \end{aligned}$$

Add -5 times the second row to the first row to get

$$\begin{aligned} x_1 &= 1 \\ x_3 &= -1 \\ x_2 + 4x_3 &= -2 \end{aligned}$$

Adding -4 times the second equation to the third equation gives

$$\begin{aligned} x_1 &= 1 \\ x_3 &= -1 \\ x_2 &= 2 \end{aligned}$$

We can easily check our answer. We have

$$\begin{aligned} 1 - 2(2) - 3(-1) &= 0 \\ 2(1) + 2 + 5(-1) &= -1 \\ -(1) + 2 - (-1) &= 2 \end{aligned}$$

Since every equation is satisfied, our answer is correct.

2.2.2 The initial system was

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 2 & 1 & 5 & -1 \\ -1 & 1 & -1 & 2 \end{array}\right]$$

Add -2 times the first row to the second row to get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 5 & 11 & -1 \\ -1 & 1 & -1 & 2 \end{array}\right]$$

Add the first row to the third row to get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 5 & 11 & -1 \\ 0 & -1 & -4 & 2 \end{array}\right]$$

Add 5 times the third row to the second row to get

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 0 & -9 & 9 \\ 0 & -1 & -4 & 2 \end{array}\right]$$

Multiplying the second row by $-\frac{1}{9}$ and the third row by -1 gives

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 4 & -2 \end{array}\right]$$

Add 2 times the third row to first row to get

$$\left[\begin{array}{rrr|r} 1 & 0 & 5 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 4 & -2 \end{array}\right]$$

Add -5 times the second row to the first row to get

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 4 & -2 \end{array}\right]$$

Adding -4 times the second row to the third row gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

This is the augmented matrix for the system $x_1 = 1$, $x_3 = -1$, and $x_2 = 2$.

2.2.3 (a) This matrix is not in RREF since there are non-zero entries about the leading 1 in the third row.

(b) The matrix is in RREF

(c) The matrix is not in RREF since the first row (nor the second row) have leading 1s.

(d) The matrix is not in RREF since there is a non-zero row beneath a zero row.

2.2.4 The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & 2 & 4 & -3 \end{array}\right]$$

Row reducing we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & 2 & 4 & -3 \end{array}\right] \begin{array}{c} \\ R_2 + R_1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 4 & -3 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ \frac{1}{3}R_2 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 4 & -3 \end{array}\right] \begin{array}{c} \\ \\ R_2 \leftrightarrow R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 2 & 4 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} R_1 + 3R_3 \\ R_2 - 4R_3 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \frac{1}{2}R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system is

$$\begin{aligned} x_1 &= 3 \\ x_2 &= -3/2 \\ x_3 &= 0 \end{aligned}$$

We check the answer. We have

$$\begin{aligned} 3 + 2\left(-\frac{3}{2}\right) + 0 &= 0 \\ -3 - 2\left(-\frac{3}{2}\right) + 2(0) &= 0 \\ 0 + 2\left(-\frac{3}{2}\right) + 4(0) &= -3 \end{aligned}$$

Since every equation is satisfied, our answer is correct.

2.2.5 The augmented matrix for the system is

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & 0 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 0 & 8 & 0 & 0 \end{array}\right]$$

Row reducing we get

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & 0 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 0 & 8 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_3 \\ \sim \\ \end{array} \left[\begin{array}{cccc|c} -2 & 0 & 8 & 0 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{array}\right] \quad R_2 + R_1 \quad \sim$$

$$\left[\begin{array}{cccc|c} -2 & 0 & 8 & 0 & 0 \\ 0 & 3 & 9 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{array}\right] \begin{array}{c} -\frac{1}{2}R_1 \\ R_2 - 3R_3 \ \sim \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \sim \\ R_2 \leftrightarrow R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system is

$$\begin{aligned} x_1 - 4x_3 &= 0 \\ x_2 + 3x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

Since x_3 is a free variable, we let $x_3 = t \in \mathbb{R}$. Solving the first equation for x_1 gives $x_1 = 4x_3 = 4t$. Solving the second equation for x_2 gives $x_2 = -3x_3 = -3t$. Therefore, all solutions have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4t \\ -3t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

We check the answer. We have

$$\begin{aligned} -3t + 3t &= 0 \\ 2(4t) + 3(-3t) + t + 0 &= 0 \\ -2(4t) + 8t &= 0 \end{aligned}$$

Since every equation is satisfied, our answer is correct.

2.2.6 We need to prove that $\mathcal{B}$ spans $\mathbb{R}^3$ and is linearly independent. For spanning, we need to show that every vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors in $\mathcal{B}$. So, for any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$, we consider

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 + 3t_3 \\ 2t_1 + t_2 + 3t_3 \\ -t_1 + t_2 + t_3 \end{bmatrix}$$

Thus, we have the system of linear equations

$$\begin{aligned} t_1 + 2t_2 + 3t_3 &= x_1 \\ 2t_1 + t_2 + 3t_3 &= x_2 \\ -t_1 + t_2 + t_3 &= x_3 \end{aligned}$$

Next, we row reduce the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 2 & 1 & 3 & x_2 \\ -1 & 1 & 1 & x_3 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & -3 & -3 & x_2 - 2x_1 \\ 0 & 3 & 4 & x_3 + x_1 \end{array}\right] \begin{array}{c} \\ \\ R_3 + R_2 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & -3 & -3 & x_2 - 2x_1 \\ 0 & 0 & 1 & -x_1 + x_2 + x_3 \end{array}\right] -\tfrac{1}{3}R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 1 & \frac{2}{3}x_1 - \frac{1}{3}x_2 \\ 0 & 0 & 1 & -x_1 + x_2 + x_3 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{3}x_1 + \frac{2}{3}x_2 \\ 0 & 1 & 1 & \frac{2}{3}x_1 - \frac{1}{3}x_2 \\ 0 & 0 & 1 & -x_1 + x_2 + x_3 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{3}x_1 - \frac{1}{3}x_2 - x_3 \\ 0 & 1 & 0 & \frac{5}{3}x_1 - \frac{4}{3}x_2 - x_3 \\ 0 & 0 & 1 & -x_1 + x_2 + x_3 \end{array}\right]$$

Thus, the system is consistent for all x_1, x_2, and x_3 as required. Since $\mathcal{B} \subset \mathbb{R}^3$ we have that $\operatorname{Span}\mathcal{B} \subset \mathbb{R}^3$ by Theorem 1.1.1. So, $\mathcal{B}$ spans $\mathbb{R}^3$.

To determine if $\mathcal{B}$ is linearly independent, we use the definition of linear independence. Consider

$$t_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We see that this is going to give us exactly the same system as above, except that the right-hand side is now all zeros. Hence, performing the same elementary row operations will give

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 1 & 3 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Consequently, the only solution is $t_1 = t_2 = t_3 = 0$ and so $\mathcal{B}$ is also linearly independent. Therefore, it is a basis for $\mathbb{R}^3$.

Chapter 3 Exercise Solutions

3.1 Exercise Solutions

3.1.1 $\vec{a}_1 = \begin{bmatrix}1\\3\\1\end{bmatrix}, \vec{a}_2 = \begin{bmatrix}-4\\0\\2\end{bmatrix}, \vec{a}_3 = \begin{bmatrix}5\\9\\-3\end{bmatrix}$

3.1.2 We have

(a) $\begin{bmatrix}1 & 3 & 2\\-1 & 4 & 5\end{bmatrix}\begin{bmatrix}2\\-1\\6\end{bmatrix} = \begin{bmatrix}1(2)+3(-1)+2(6)\\-1(2)+4(-1)+5(6)\end{bmatrix} = \begin{bmatrix}11\\24\end{bmatrix}$

(b) $\begin{bmatrix}6 & -1 & 1\end{bmatrix}\begin{bmatrix}2\\3\\1\end{bmatrix} = [6(2)+(-1)(3)+1(1)] = [10]$

(c) $\begin{bmatrix}1 & -4\\3 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1(0)+(-4)(1)\\3(0)+1(1)\\1(0)+(-2)(1)\end{bmatrix} = \begin{bmatrix}-4\\1\\-2\end{bmatrix}$

3.1.3 We take $A = \begin{bmatrix}1 & 1\\2 & 0\\-2 & -3\end{bmatrix}$, $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$, and $\vec{b} = \begin{bmatrix}3\\15\\-5\end{bmatrix}$.

3.1.4 We have

(a) $\begin{bmatrix}1 & 3 & 2\\-1 & 4 & 5\end{bmatrix}\begin{bmatrix}2\\-1\\6\end{bmatrix} = 2\begin{bmatrix}1\\-1\end{bmatrix} + (-1)\begin{bmatrix}3\\4\end{bmatrix} + 6\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}11\\24\end{bmatrix}$

(b) $\begin{bmatrix}6 & -1 & 1\end{bmatrix}\begin{bmatrix}2\\3\\1\end{bmatrix} = 2[6] + 3[-1] + 1[1] = [10]$

(c) $\begin{bmatrix}1 & -4\\3 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0\begin{bmatrix}1\\3\\1\end{bmatrix} + 1\begin{bmatrix}-4\\1\\-2\end{bmatrix} = \begin{bmatrix}-4\\1\\-2\end{bmatrix}$

3.1.5 (a) $\begin{bmatrix}1 & 3\\2 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 3 & 0 & 1\\0 & 1 & -2 & 1\end{bmatrix} = \begin{bmatrix}2 & 6 & -6 & 4\\4 & 5 & 2 & 1\\0 & 1 & -2 & 1\end{bmatrix}$

(b) Since the number of columns of the first matrix does not equal the number of rows of the second matrix, the product does not exist.

(c) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = [6]$

(d) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

3.1.6 Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$. We have

$$AI = A\begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} A\vec{e}_1 & \cdots & A\vec{e}_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} = A$$

Observe that $I^T = I$. Hence, from our result above, we have that

$$A^T I^T = A^T I = A^T$$

Taking transposes of both sides gives

$$\begin{aligned} (A^T I^T)^T &= (A^T)^T \\ (I^T)^T (A^T)^T &= A \\ IA &= A \end{aligned}$$

3.2 Exercise Solutions

3.2.1 Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$. Then

$$\begin{aligned} L(s\vec{x} + t\vec{y}) &= L(sx_1 + ty_1, sx_2 + ty_2, sx_3 + ty_3) \\ &= \begin{bmatrix} sx_1 + ty_1 - (sx_2 + ty_2) \\ sx_1 + ty_1 - 2(sx_3 + ty_3) \end{bmatrix} \\ &= s\begin{bmatrix} x_1 - x_2 \\ x_1 - 2x_3 \end{bmatrix} + t\begin{bmatrix} y_1 - y_2 \\ y_1 - 2y_3 \end{bmatrix} \\ &= sL(\vec{x}) + tL(\vec{y}) \end{aligned}$$

Hence, L is linear.

3.2.2 The columns of the standard matrix of L are the images of the standard basis vectors under L. We have

$$\begin{aligned} L(1, 0, 0) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ L(0, 1, 0) &= \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ L(0, 0, 1) &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$

Therefore, the standard matrix of L is

$$[L] = \begin{bmatrix} L(1,0,0) & L(0,1,0) & L(0,0,1) \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

3.3 Exercise Solutions

3.3.1 Every vector in Range(L) has the form

$$L(\vec{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 - 2x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Thus, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}$ is a spanning set for Range(L). However, it is clearly linearly dependent since the first vector is a linear combination of the second and the third. By Theorem 1.1.2 we get that $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}$ also spans the range of L. Moreover, since neither vector in $\mathcal{B}$ is a scalar multiple of the other we have the $\mathcal{B}$ is also linearly independent. Thus, $\mathcal{B}$ is a basis for the range of L.

3.3.2 Let $\vec{x} \in \ker(L)$. Then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(\vec{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 - 2x_3 \end{bmatrix}$$

Thus, $x_1 - x_2 = 0$ and $x_1 - 2x_3 = 0$, and so $x_2 = x_1$ and $x_3 = \frac{1}{2}x_1$. Thus, every vector in $\ker(L)$ has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1/2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$$

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix} \right\}$ spans $\ker(L)$. Moreover, $\mathcal{B}$ is a set containing one non-zero vector and so it is linearly independent. Therefore, it is a basis for $\ker(L)$.

3.4 Exercise Solutions

3.4.1 For any $\vec{x} \in \mathbb{R}^n$ we have

$$\begin{aligned} c(dL)(\vec{x}) &= c[dL(\vec{x})] && \text{by definition of } dL \\ &= (cd)L(\vec{x}) && \text{by Theorem 1.1.1 V7} \\ &= ((cd)L)(\vec{x}) && \text{by definition of } (cd)L \end{aligned}$$

Thus, $c(dL) = (cd)L$.

Chapter 4 Exercise Solutions

4.1 Exercise Solutions

4.1.1 We need to show that all 10 vector space axioms hold. Let $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in \mathbb{V} \times \mathbb{W}$ and $s, t \in \mathbb{R}$. Then there exists $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{V}$ and $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in \mathbb{W}$ such that $\vec{x}_1 = (\vec{v}_1, \vec{w}_1)$, $\vec{x}_2 = (\vec{v}_2, \vec{w}_2)$, and $\vec{x}_3 = (\vec{v}_3, \vec{w}_3)$. Let $s, t \in \mathbb{R}$.

V1 We have $\vec{x}_1 \oplus \vec{x}_2 = (\vec{v}_1, \vec{w}_1) \oplus (\vec{v}_2, \vec{w}_2) = (\vec{v}_1 + \vec{v}_2, \vec{w}_1 + \vec{w}_2)$. Since $\mathbb{V}$ is a vector space it is closed under addition so $\vec{v}_1 + \vec{v}_2 \in \mathbb{V}$. Similarly, $\vec{w}_1 + \vec{w}_2 \in \mathbb{W}$. Thus, $\vec{x}_1 \oplus \vec{x}_2 \in \mathbb{V} \times \mathbb{W}$.

V2 We have

$$\begin{aligned}
(\vec{x}_1 \oplus \vec{x}_2) \oplus \vec{x}_3 &= [(\vec{v}_1, \vec{w}_1) \oplus (\vec{v}_2, \vec{w}_2)] \oplus (\vec{v}_3, \vec{w}_3) \\
&= (\vec{v}_1 + \vec{v}_2, \vec{w}_1 + \vec{w}_2) \oplus (\vec{v}_3, \vec{w}_3) \\
&= ([\vec{v}_1 + \vec{v}_2] + \vec{v}_3, [\vec{w}_1 + \vec{w}_2] + \vec{w}_3) \\
&= (\vec{v}_1 + [\vec{v}_2 + \vec{v}_3], \vec{w}_1 + [\vec{w}_2 + \vec{w}_3]) \quad \text{by V2 in the vector spaces } \mathbb{V} \text{ and } \mathbb{W} \\
&= (\vec{v}_1, \vec{w}_1) \oplus (\vec{v}_2 + \vec{v}_3, \vec{w}_2 + \vec{w}_3) \\
&= (\vec{v}_1, \vec{w}_1) \oplus [(\vec{v}_2, \vec{w}_2) \oplus (\vec{v}_3, \vec{w}_3)] \\
&= \vec{x} \oplus (\vec{x}_2 \oplus \vec{x}_3)
\end{aligned}$$

V3 We have

$$\begin{aligned}
\vec{x}_1 \oplus \vec{x}_2 &= (\vec{v}_1, \vec{w}_1) \oplus (\vec{v}_2, \vec{w}_2) \\
&= (\vec{v}_1 + \vec{v}_2, \vec{w}_1 + \vec{w}_2) \\
&= (\vec{v}_2 + \vec{v}_1, \vec{w}_2 + \vec{w}_1) \quad \text{by V3 in the vector spaces } \mathbb{V} \text{ and } \mathbb{W} \\
&= (\vec{v}_2, \vec{w}_2) \oplus (\vec{v}_1, \vec{w}_1) \\
&= \vec{x}_2 \oplus \vec{x}_1
\end{aligned}$$

V4 Since $\mathbb{V}$ and $\mathbb{W}$ are vector spaces there exists $\vec{0}_{\mathbb{V}} \in \mathbb{V}$ and $\vec{0}_{\mathbb{W}} \in \mathbb{W}$ such that $\vec{v} + \vec{0}_{\mathbb{V}} = \vec{v}$ for all $\vec{v} \in \mathbb{V}$ and $\vec{w} + \vec{0}_{\mathbb{W}} = \vec{w}$ for all $\vec{w} \in \mathbb{V}$. Observe that $(\vec{0}_{\mathbb{V}}, \vec{0}_{\mathbb{W}}) \in \mathbb{V} \times \mathbb{W}$ and

$$(\vec{v}_1, \vec{w}_1) \oplus (\vec{0}_{\mathbb{V}}, \vec{0}_{\mathbb{W}}) = (\vec{v}_1 + \vec{0}_{\mathbb{V}}, \vec{w}_1 + \vec{0}_{\mathbb{W}}) = (\vec{v}_1, \vec{w}_1)$$

V5 Since $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, we get that there exists $(-\vec{v}_1) \in \mathbb{V}$ and $(-\vec{w}_1) \in \mathbb{W}$ such that $\vec{v}_1+(-\vec{v}_1) = \vec{0}_{\mathbb{V}}$ and $\vec{w}_1 + (-\vec{w}_1) = \vec{0}_{\mathbb{W}}$. Let $(-\vec{x}) = (-\vec{v}_1, -\vec{w}_1)$. Then, by definition, $(-\vec{x}) \in \mathbb{V} \times \mathbb{W}$ and

$$\vec{x} \oplus (-\vec{x}) = (\vec{v}_1, \vec{w}_1) \oplus (-\vec{v}_1, -\vec{w}_1) = (\vec{v}_1 + (-\vec{v}_1), \vec{w}_1 + (-\vec{w}_1)) = (\vec{0}_{\mathbb{V}}, \vec{0}_{\mathbb{W}}) = \vec{0}_{\mathbb{V}\times\mathbb{W}}$$

as required.

V6 We have $s \odot \vec{x} = s \odot (\vec{v}_1, \vec{w}_1) = (s\vec{v}_1, s\vec{w}_1) \in \mathbb{V} \times \mathbb{W}$ since $s\vec{v}_1 \in \mathbb{V}$ and $s\vec{w}_1 \in \mathbb{W}$ since $\mathbb{V}$ and $\mathbb{W}$ are vector spaces.

V7 Since $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, we get

$$s \odot (t \odot \vec{x}) = s \odot (t\vec{v}_1, t\vec{v}_2) = \Big(s(t\vec{v}_1), s(t\vec{w}_1)\Big) = \Big((st)\vec{v}_1, (st)\vec{w}_1\Big) = (st) \odot (\vec{v}_1, \vec{w}_1) = (st)\vec{x}$$

V8 Since $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, we get

$$\begin{aligned}(c+d) \odot \vec{x} &= \Big((c+d)\vec{v}_1, (c+d)\vec{w}_1\Big) = (c\vec{v}_1 + d\vec{v}_1, c\vec{w}_1 + d\vec{w}_1)\\ &= (c\vec{v}_1, c\vec{w}_1) \oplus (d\vec{v}_1, d\vec{w}_1) = c \odot (\vec{v}_1, \vec{w}_1) \oplus d \odot (\vec{v}_1, \vec{w}_1) = c \odot \vec{x} \oplus d \odot \vec{x}\end{aligned}$$

V9 Since $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, we get

$$\begin{aligned}c \odot (\vec{x} \oplus \vec{y}) &= c \odot \Big[(\vec{v}_1, \vec{w}_1) + (\vec{v}_2, \vec{w}_2)\Big] = c \odot (\vec{v}_1 + \vec{v}_2, \vec{w}_1 + \vec{w}_2)\\ &= \Big(c(\vec{v}_1 + \vec{v}_2), c(\vec{w}_1 + \vec{w}_2)\Big) = (c\vec{v}_1 + c\vec{v}_2, c\vec{w}_1 + c\vec{w}_2)\\ &= (c\vec{v}_1, c\vec{w}_1) \oplus (c\vec{v}_2, c\vec{w}_2) = c \odot (\vec{v}_1, \vec{w}_1) \oplus c \odot (\vec{v}_2, \vec{w}_2) = c \odot \vec{x} \oplus c \odot \vec{y}\end{aligned}$$

V10 Since $\mathbb{V}$ and $\mathbb{W}$ are vector spaces, we get

$$1 \odot \vec{x} = 1 \odot (\vec{v}_1, \vec{w}_1) = (1\vec{v}_1, 1\vec{w}_1) = (\vec{v}_1, \vec{w}_1) = \vec{x}$$

Hence, $\mathbb{V} \times \mathbb{W}$ is a vector space.

4.1.2 By definition, $\mathbb{S}$ is a subset of $M_{2\times 2}(\mathbb{R})$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}$ since $0 + 0 = 0 - 0$. Thus, $\mathbb{S}$ is non-empty.

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbb{S}$. Hence, $a_1 + a_2 = a_3 - a_4$ and $b_1 + b_2 = b_3 - b_4$. Then,

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$

satisfies

$$(a_1 + b_1) + (a_2 + b_2) = a_1 + a_2 + b_1 + b_2 = a_3 - a_4 + b_3 - b_4 = (a_3 + b_3) - (a_4 + b_4)$$

So, $A + B \in \mathbb{S}$. Similarly, for any $t \in \mathbb{R}$ we get

$$tA = \begin{bmatrix} ta_1 & ta_2 \\ ta_3 & ta_4 \end{bmatrix}$$

satisfies

$$(ta_1) + (ta_2) = t(a_1 + a_2) = t(a_3 - a_4) = (ta_3) - (ta_4)$$

Thus, $tA \in \mathbb{S}$. Therefore, $\mathbb{S}$ is a subspace of $M_{2\times 2}(\mathbb{R})$ by the Subspace Test.

4.1.3 Let $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$. We need to show that $p(x)$ can be written as a linear combination of the vectors in $\mathcal{B}$. To do this, we can show that the equation

$$a + bx + cx^2 = t_1(1 + x + x^2) + t_2(1 + 2x - x^2) + t_3(1 + 3x + 2x^2)$$

is consistent for all $a, b, c \in \mathbb{R}$.

Simplifying the right hand side gives

$$a + bx + cx^2 = (t_1 + t_2 + t_3) + (t_1 + 2t_2 + 3t_3)x + (t_1 - t_2 + 2t_3)x^2$$

Comparing coefficients of like powers of x we get the system of linear equations

$$\begin{aligned} t_1 + t_2 + t_3 &= a \\ t_1 + 2t_2 + 3t_3 &= b \\ t_1 - t_2 + 2t_3 &= c \end{aligned}$$

Row reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, by Theorem 2.2.5, the system is consistent for all $a, b, c \in \mathbb{R}$ and so $\mathcal{B}$ spans $P_2(\mathbb{R})$.

4.1.4 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$, and let $\mathcal{B} = \{B_1, B_2, B_3\}$ be a set of three vectors in $M_{2\times 2}(\mathbb{R})$. Consider

$$A = c_1B_1 + c_2B_2 + c_3B_3$$

Calculating the linear combination on the right-hand side and comparing coefficients gives a system of 4 linear equations (one for each entry) in 3 unknowns (c_1, c_2, c_3). Observe that the coefficient matrix is 4×3, so the maximum rank of the coefficient matrix is 3. Hence, by Theorem 2.2.5, this system cannot be consistent for all $A \in M_{2\times 2}(\mathbb{R})$. Hence, $\mathcal{B}$ cannot span $M_{2\times 2}(\mathbb{R})$.

4.1.5 Let $\mathcal{B} = \{p_1(x), p_2(x), p_3(x), p_4(x)\}$ in $P_2(\mathbb{R})$ and consider

$$0 = c_1p_1(x) + c_2p_2(x) + c_3p_3(x) + c_4p_4(x) \tag{4.2}$$

Calculating the linear combination on the right-hand side and comparing coefficients gives a system of 3 linear equations (one for each power of x) in 4 unknowns (c_1, c_2, c_3, c_4). Observe that the coefficient matrix is 3×4, so the maximum rank of the coefficient matrix is 3. Hence, by Theorem 2.2.5, there is at least one free variable, so equation **??** has infinitely many solutions. Thus, $\mathcal{B}$ must be linearly dependent.

4.2 Exercise Solutions

4.2.1 Let $a + bx + cx^2 \in P_2(\mathbb{R})$ and consider

$$\begin{aligned} a + bx + cx^2 &= t_1(3 - 2x + x^2) + t_2(2 - 5x + 2x^2) + t_3(1 - x + x^2) \\ &= (3t_1 + 2t_2 + t_3) + (-2t_1 - 5t_2 - t_3)x + (t_1 + 2t_2 + t_3)x^2 \end{aligned}$$

Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix} 3 & 2 & 1 \\ -2 & -5 & -1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, by Theorem 2.2.5, the equation is consistent for all $a+bx+cx^2 \in P_2(\mathbb{R})$ and so Span $\mathcal{B} = P_2(\mathbb{R})$. Moreover, if we let $a = b = c = 0$, we get a unique solution. Therefore, $\mathcal{B}$ is also linearly independent and hence is a basis for $P_2(\mathbb{R})$.

4.2.2 Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$. Consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = c_1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & c_1 \\ c_3 & c_2 + c_3 \end{bmatrix}$$

Thus, we get a system of 4 equations in 3 unknowns. Since the rank of the corresponding coefficient matrix is at most 3, the system cannot be consistent for $a, b, c, d \in \mathbb{R}$ by Theorem 2.2.5. Thus, $\mathcal{B}$ does not span $M_{2\times 2}(\mathbb{R})$ and so it is not a basis for $M_{2\times 2}(\mathbb{R})$.

Consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & c_1 \\ c_3 & c_2 + c_3 \end{bmatrix}$$

This gives us the system of linear equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 &= 0 \\ c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

Since the only solution is $c_1 = c_2 = c_3 = c_4 = 0$, we have that $\mathcal{B}$ is linearly independent. Thus, $\mathcal{B}$ is a basis for the subspace $\mathbb{S} = \operatorname{Span}\mathcal{B}$.

4.3 Exercise Solutions

4.3.1 By definition of coordinates, we have

$$A = 2\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 5 \end{bmatrix}$$

$$B = -4\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 6 & 15 \end{bmatrix}$$

4.3.2 For $p(x)$, we need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$1 + x + x^2 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2)$$

By inspection, we see that $c_3 = 1$, $c_2 = 0$, and $c_1 = 0$. Hence, $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

For $q(x)$, we need to find $d_1, d_2, d_3 \in \mathbb{R}$ such that

$$1 - 2x + 3x^2 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2) = c_1 + c_2 + c_3 + (c_2 + c_3)x + c_3x^2$$

By inspection, we see that $c_3 = 3$, $c_2 = -5$, and $c_1 = 3$. Hence, $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix}$.

4.3.3 By definition, we have

$$A = 2\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 5 \end{bmatrix}$$

$$B = (-4)\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 6 & 15 \end{bmatrix}$$

Chapter 5 Exercise Solutions

5.1 Exercise Solutions

5.1.1 We have

$$\begin{bmatrix}1&1&2\\1&2&2\\1&-1&3\end{bmatrix}\begin{bmatrix}8&-5&-2\\-1&1&0\\-3&2&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Hence, A is the inverse of B (and B is the inverse of A).

5.1.2 We row reduce the multiple augmented system $[A \mid I]$. We get

$$\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&1&2&0&1&0\\-1&3&10&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1&0&0&-4&23&-5\\0&1&0&2&-9&2\\0&0&1&-1&5&-1\end{array}\right]$$

Thus, $A^{-1}=\begin{bmatrix}-4&23&-5\\2&-9&2\\-1&5&-1\end{bmatrix}$.

5.1.3 For A to be invertible, we require that $\operatorname{rank} A = 3$. Thus, we see that we must a, d, and f all non-zero (that is $adf \neq 0$). Given this, we row reduce the multiple augmented system $[A \mid I]$ to get

$$\left[\begin{array}{ccc|ccc}a&b&c&1&0&0\\0&d&e&0&1&0\\0&0&f&0&0&1\end{array}\right]\begin{array}{l}\frac{1}{a}R_1\\\frac{1}{d}R_2\\\frac{1}{f}R_3\end{array}\sim$$

$$\left[\begin{array}{ccc|ccc}1&b/a&c/a&1/a&0&0\\0&1&e/d&0&1/d&0\\0&0&1&0&0&1/f\end{array}\right]\begin{array}{l}R_1-\frac{c}{a}R_3\\R_2-\frac{e}{d}R_3\\\ \end{array}\sim$$

$$\left[\begin{array}{ccc|ccc}1&b/a&0&1/a&0&-c/af\\0&1&0&0&1/d&-e/df\\0&0&1&0&0&1/f\end{array}\right]\begin{array}{l}R_1-\frac{b}{a}R_2\\\ \\\ \end{array}\sim$$

$$\left[\begin{array}{ccc|ccc}1&0&0&1/a&-\frac{b}{ad}&(-cd+be)/adf\\0&1&0&0&1/d&-e/df\\0&0&1&0&0&1/f\end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 1/a & -\frac{b}{ad} & (-cd+be)/adf \\ 0 & 1/d & -e/df \\ 0 & 0 & 1/f \end{bmatrix}$.

5.2 Exercise Solutions

5.2.1 (a) E_1 is the elementary matrix associated with $R_1 + R_2$.

(b) E_2 is the elementary matrix associated with $(-1)R_1$.

(c) E_3 is not elementary. We would require the two elementary row operations $2R_1$ and then $R_1 + 3R_2$ to get E_3 from the identity matrix.

5.2.2 We row reduce A to I keeping track of our elementary row-operations.

$$\begin{aligned} &\begin{bmatrix} 0 & 1 & 2 & 4 \\ -2 & 3 & 4 & 4 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \end{matrix} \sim \begin{bmatrix} -2 & 3 & 4 & 4 \\ 0 & 1 & 2 & 4 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \end{matrix} \sim \\ &\begin{bmatrix} -2 & 0 & -2 & -8 \\ 0 & 1 & 2 & 4 \end{bmatrix} \begin{matrix} -\frac{1}{2}R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 4 \end{bmatrix} = R \end{aligned}$$

Hence,

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

and $E_3E_2E_1A = R$. Then

$$A = E_1^{-}E_2^{-1}E_3^{-1}R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

5.3 Exercise Solutions

5.3.1 Since A is lower triangular, we get that $\det A = 1(3)(5) = 15$. Since B is lower triangular, we get that $\det B = 2(2)(2) = 8$.
However, notice that C is not upper nor lower triangular. Performing a cofactor expansion along the third row gives

$$\det C = (-1)^{3+1}(1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1(1)[0-1] = -1$$

5.3.2 For A we get

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ -1 & 3 & -1 & 1 \\ 2 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & 2 \\ -1 & 3 & 0 & 1 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{vmatrix}$$

$$= (-1)^{3+3}(1) \begin{vmatrix} 1 & 2 & 2 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 0 & 4 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (-1)^{3+1}(1) \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= -2$$

For B we get

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ -1 & -4 & -5 & 4 \\ -1 & -3 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 0 & -3 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= (-1)^{1+1}(1) \begin{vmatrix} 0 & -3 & 7 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (-1)^{2+1}(1) \begin{vmatrix} -3 & 7 \\ 1 & 1 \end{vmatrix}$$

$$= (-1)(-10) = 10$$

5.3 Exercise Solutions

5.3.1 We have

$$c_{11} = (-1)^{1+1}[4] = 4 \qquad c_{12} = (-1)^{1+2}[3] = -3$$

$$c_{21} = (-1)^{2+1}[-2] = 2 \qquad c_{22} = (-1)^{2+2}[-1] = -1$$

5.3.2 By definition, we have

$$\begin{aligned}\det A &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}\\ &= (-1)(-1)^{1+1}\begin{vmatrix}4 & 0\\ 1 & 1\end{vmatrix} + (-2)(-1)^{2+1}\begin{vmatrix}-2 & 1\\ 1 & 1\end{vmatrix} + 0(-1)^{3+1}\begin{vmatrix}-2 & 1\\ 4 & 0\end{vmatrix}\\ &= (-1)\Big(4(1) - 0(1)\Big) + 2\Big((-2)(1) - 1(1)\Big) + 0\\ &= -10\end{aligned}$$

5.3.3 (a) Expanding along the third row gives

$$\begin{aligned}\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} &= (-3)(-1)^{3+2}\begin{vmatrix}2 & 1\\ -4 & -5\end{vmatrix}\\ &= 3\Big(2(-5) - 1(-4)\Big) = -18\end{aligned}$$

(b) We expand along the first row to get

$$\begin{aligned}\det B = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} &= 4(-1)^{1+1}\begin{vmatrix}3 & 0\\ 17 & 8\end{vmatrix}\\ &= 4\Big(3(8) - 0(17)\Big) = 4(3)(8) = 96\end{aligned}$$

5.3.4 A is lower triangular, hence by Theorem 5.3.2 we get that $\det A = 1(3)(5) = 15$.
B is upper triangular, hence by Theorem 5.3.2 we get that $\det B = 2(2)(2) = 8$.
C is neither lower or upper triangular, so we cannot use Theorem 5.3.2. Using a cofactor expansion along the third row gives

$$\det C = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = 1(-1)^{3+1}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + 0 = -1$$

5.3.5 Since adding a multiple of one row to another does not change the determinant, we have

$$\det A = \begin{vmatrix}1 & 2 & 1 & 2\\ 0 & 5 & 0 & 3\\ 0 & -1 & 1 & -2\\ 0 & -1 & 0 & -1\end{vmatrix}$$

Since swapping rows multiplies the determinant by (-1) we get

$$\begin{aligned}\det A = (-1)\begin{vmatrix}1 & 2 & 1 & 2\\ 0 & -1 & 0 & -1\\ 0 & -1 & 1 & -2\\ 0 & 5 & 0 & 3\end{vmatrix} &= (-1)\begin{vmatrix}1 & 2 & 1 & 2\\ 0 & -1 & 0 & -1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & -2\end{vmatrix}\\ &= (-1)(1)(-1)(1)(-2) = -2\end{aligned}$$

Similarly, we get

$$\det B = \begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 0 & -3 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -3 & 7 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= (-1)(-1)\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -7 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

$$= (-1)(-1)(1)(1)(1)(-4) = -4$$

5.4 Exercise Solutions

5.4.1 We have $\operatorname{cof} A = \begin{bmatrix} -5 & 4 & -7 \\ -4 & 3 & -5 \\ 8 & -6 & 11 \end{bmatrix}$. Hence, $\operatorname{adj} A = (\operatorname{cof} A)^T = \begin{bmatrix} -5 & -4 & 8 \\ 4 & 3 & -6 \\ -7 & -5 & 11 \end{bmatrix}$.

Observe that $A \operatorname{adj} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and that $\det A = 1$.

Chapter 6 Exercise Solutions

6.1 Exercise Solutions

6.1.1 We have

$$L(1,2) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \frac{10}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{14}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} L(1,-1) \qquad = \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \frac{4}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{7}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, $[L]_{\mathcal{B}} = \begin{bmatrix} [L(1,2)]_{\mathcal{B}} & [L(1,-1)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 10/3 & 4/3 \\ 14/3 & -7/3 \end{bmatrix}$.

Hence, $[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 10/3 & 4/3 \\ 14/3 & -7/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/3 \\ 7/3 \end{bmatrix}$.

6.2 Exercise Solutions

6.2.1 (a) We have

$$\begin{bmatrix} 1 & 4 & 0 \\ -1 & -5 & 1 \\ -3 & -4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -8 \\ -17 \end{bmatrix}$$

Since, $\begin{bmatrix} 9 \\ -8 \\ -17 \end{bmatrix}$ is not a scalar multiple of $\vec{v}_1$, we get that $\vec{v}_1$ is not an eigenvector of A.

(b) We have

$$\begin{bmatrix} 1 & 4 & 0 \\ -1 & -5 & 1 \\ -3 & -4 & -2 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ -10 \end{bmatrix} = (-2)\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$$

Thus, $\vec{v}_2$ is an eigenvector of A with eigenvalue $\lambda = -2$.

(c) We have

$$\begin{bmatrix} 1 & 4 & 0 \\ -1 & -5 & 1 \\ -3 & -4 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Thus, $\vec{v}_3$ is an eigenvector of A with eigenvalue $\lambda = -1$.

6.2.2 To find all of the eigenvalues, we find the roots of the characteristic polynomial. We have

$$0 = C(\lambda) = \det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} = (1-\lambda)^2(-2-\lambda) = -(\lambda+2)(\lambda-1)^2$$

Hence, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1$.
For $\lambda_1 = -2$ we have

$$A - \lambda_1 I = A + 2I = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/9 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1/9 \\ -2/3 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 1$ we have

$$A - \lambda_2 I = A - I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

6.2.3 From our work in Exercise 2, we see that $\lambda_1 = -2$ is a single root of $C(\lambda)$, so $a_{\lambda_1} = 1$. Since $\dim E_{\lambda_1} = 1$ we get that $g_{\lambda_1} = 1$. $\lambda_2 = 1$ is a double root of $C(\lambda)$, so $a_{\lambda_2} = 2$, and we have $g_{\lambda_2} = \dim E_{\lambda_2} = 1$.

6.3 Exercise Solutions

6.3.1 Another matrix which diagonalizes A is $P_1 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1/3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. We have $P_1^{-1}AP_1 = \text{diag}(2, 0, 2)$.

We could also choose $P_2 = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 3 \end{bmatrix}$. We get $P_2^{-1}AP_2 = \text{diag}(2, 2, 0)$.

Chapter 8 Exercise Solutions

8.2 Exercise Solutions

8.2.1 Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \ker(L)$. Then,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_1 + x_2 \\ 0 & x_1 + x_2 \end{bmatrix}$$

Hence, $x_1 = 0$ and $x_1 + x_2 = 0$. Thus, $\ker(L) = \{\vec{0}\}$ and so a basis for $\ker(L)$ is the empty set.

Every vector in Range(L) has the form

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_1 + x_2 \\ 0 & x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$ spans Range(L). Moreover, since neither vector in $\mathcal{B}$ is a scalar multiple of the other, $\mathcal{B}$ is linearly independent. Therefore, it is a basis for Range(L).

8.4 Exercise Solutions

8.4.1 Let $(a, b, c) \in \ker(L)$. Then,

$$0 = L(a, b, c) = a + bx + cx^2$$

Thus, $a = b = c = 0$, so $\ker(L) = \{\vec{0}\}$. Hence, L is one-to-one by Lemma 8.4.1.
Let $d + ex + fx^2 \in P_2(\mathbb{R})$. Then, $L(d, e, f) = d + ex + fx^2$ so Range$(L) = P_2(\mathbb{R})$ and thus L is onto.

8.4.2 Define $L : \mathbb{R}^3 \to P_2(\mathbb{R})$ by $L(a, b, c) = a + bx + cx^2$. Then, by Exercise 1, we have that L is linear, one-to-one, and onto. Thus, it is an isomorphism between $\mathbb{R}^3$ and $P_2(\mathbb{R})$, hence $\mathbb{R}^3$ and $P_2(\mathbb{R})$ are isomorphic.

8.4.3 Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{V}$ and define $T : \mathbb{L} \to M_{2\times 2}(\mathbb{R})$ by $T(L) = [L]_{\mathcal{B}}$ for any linear mapping $L : \mathbb{V} \to \mathbb{V}$.

1. We need to prove that T is linear, one-to-one, and onto.

   <u>Linear:</u> Let $L, M : \mathbb{V} \to \mathbb{V}$ be linear mappings, and let $s, t \in \mathbb{R}$. Then for all $\vec{x} \in \mathbb{V}$ we get

$$\begin{aligned} {}_{C}[sL + tM]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} &= [(sL + tM)(\vec{x})]_C = [sL(\vec{x}) + tM(\vec{x})]_C = s[L(\vec{x})]_C + t[M(\vec{x})]_C \\ &= s_C[L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} + t_C[M]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = (s_C[L]_{\mathcal{B}} + t_C[M]_{\mathcal{B}})[\vec{x}]_{\mathcal{B}} \end{aligned}$$

Hence $T(kL+M) = [kL+M]_B = k[L]_{\mathcal{B}} + [M]_{\mathcal{B}} = kT(L) + T(M)$ as required.

One-To-One: Let $L \in \ker(T)$. Then $T(L) = O_{2,2}$ and hence for all $\vec{x} \in \mathbb{V}$ we have $[L(\vec{x})]_{\mathcal{B}} = O_{2,2}[\vec{x}]_{\mathcal{B}} = \vec{0}$, hence $L(\vec{x}) = \vec{0}$ for all $\vec{x}$ and so L is the zero vector in $\mathbb{L}$. Thus, $\ker(T) = \{\vec{0}_{\mathbb{L}}\}$ so T is one-to-one.

Onto: Let $A \in M_{2\times 2}(\mathbb{R})$, then define a function $L : \mathbb{V} \to \mathbb{V}$ by $[L(\vec{x})]_{\mathcal{B}} = A[\vec{x}]_{\mathcal{B}}$. Then, by definition, L is linear and $[L]_{\mathcal{B}} = A$. Hence T is onto.

Thus, T is an isomorphism from the vector space of all linear operators on $\mathbb{V}$ to $M_{2\times 2}(\mathbb{R})$.

2. As in the onto proof above, we define $T^{-1} : M_{2\times 2}(\mathbb{R}) \to \mathbb{L}$ by $T^{-1}(A)$ is the linear mapping $L : \mathbb{V} \to \mathbb{V}$ such that $[L]_{\mathcal{B}} = A$.

   Then, we have $T^{-1}(\vec{e}_1)$ is the linear mapping such that $[L_1]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, if $\vec{x} = b_1\vec{v}_1 + b_2\vec{v}_2$, we get

$$[L_1(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

   So, $L_1(x_1, x_2) = b_1\vec{v}_1$.
   Similarly, $T^{-1}(\vec{e}_2)$ is the linear mapping

$$[L_2(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ 0 \end{bmatrix}$$

   So, $L_2(x_1, x_2) = b_2\vec{v}_1$.
   $T^{-1}(\vec{e}_3)$ is the linear mapping

$$[L_3(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \end{bmatrix}$$

   So, $L_3(x_1, x_2) = b_1\vec{v}_2$.
   $T^{-1}(\vec{e}_4)$ is the linear mapping

$$[L_4(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}$$

   So, $L_4(x_1, x_2) = b_2\vec{v}_2$.

3. Consider

$$c_1L_1 + c_2L_2 + c_3L_3 + c_4L_4 = Z$$

   where $Z : \mathbb{V} \to \mathbb{V}$ is the zero mapping: $Z(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{V}$. Then, for any $\vec{x} = b_1\vec{v}_1 + b_2\vec{v}_2$ we get

$$\begin{aligned} Z(x_1, x_2) &= (c_1L_1 + c_2L_2 + c_3L_3 + c_4L_4)(x_1, x_2) \\ \vec{0} &= c_1L_1(x_1, x_2) + c_2L_2(x_1, x_2) + c_3L_3(x_1, x_2) + c_4L_4(x_1, x_2) \\ &= c_1b_1\vec{v}_1 + c_2b_2\vec{v}_1 + c_3b_1\vec{v}_2 + c_4b_2\vec{v}_2 \end{aligned}$$

   Hence, $c_1b_1 + c_2b_2 = 0$ and $c_3b_1 + c_4b_2 = 0$. Since this is valid for every $b_1, b_2 \in \mathbb{R}$ we get that $c_1 = c_2 = c_3 = c_4 = 0$, so $\{L_1, L_2, L_3, L_4\}$ is linearly independent.

Let $L : \mathbb{V} \to \mathbb{V}$ be a linear mapping with $[L]_{\mathcal{B}} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. Then, for any $\vec{x} = b_1\vec{v}_1 + b_2\vec{v}_2$ we get

$$[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 + a_2b_2 \\ a_3b_1 + a_4b_2 \end{bmatrix}$$

Thus,

$$\begin{aligned} L(\vec{x}) &= (a_1b_1 + a_2b_2)\vec{v}_1 + (a_3b_1 + a_4b_2)\vec{v}_2 \\ &= a_1b_1\vec{v}_1 + a_2b_2\vec{v}_1 + a_3b_1\vec{v}_2 + a_4b_2\vec{v}_2 \\ &= a_1L_1(\vec{x}) + a_2L_2(\vec{x}) + a_3L_3(\vec{x}) + a_4L_4(\vec{x}) \end{aligned}$$

Thus, $\{L_1, L_2, L_3, L_4\}$ also spans $\mathbb{L}$. Consequently, it is a basis for $\mathbb{L}$.

Chapter 9 Exercise Solutions

9.2 Exercise Solutions

9.2.1 Observe that

$$\left\langle \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \right\rangle = 3(1) + 1(-3) + (-1)(2) + 2(1) = 0$$

$$\left\langle \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \right\rangle = 3(1) + 1(2) + (-1)(3) + 2(-1) = 0$$

$$\left\langle \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \right\rangle = 1(1) + (-3)(2) + 2(3) + 1(-1) = 0$$

Hence, the set is orthogonal.

9.2.2 Observe that

$$\langle 1 - x - 2x^2, 2 - x^2 \rangle = 0(1) + 1(2) + (-2)(1) = 0$$

$$\langle 1 - x - 2x^2, 2 + 3x - 4x^2 \rangle = 0(-5) + 1(2) + (-2)(1) = 0$$

$$\langle 2 - x^2, 2 + 3x - 4x^2 \rangle = 1(-5) + 2(2) + 1(1) = 0$$

Thus, $\mathcal{B}$ is an orthogonal set of non-zero vectors, so it is a linearly independent set of 3 vectors in $P_2(\mathbb{R})$ and hence is an orthogonal basis for $P_2(\mathbb{R})$.
We have

$$\langle 1 + x^2, 1 - x - 2x^2 \rangle = 2(0) + 1(1) + 2(-2) = -3$$

$$\langle 1 + x^2, 2 - x^2 \rangle = 2(1) + 1(2) + 2(1) = 6$$

$$\langle 1 + x^2, 2 + 3x - 4x^2 \rangle = 2(-5) + 1(2) + 2(1) = -6 \langle 1 - x - 2x^2, 1 - x - 2x^2 \rangle \quad = 0(0) + 1(1) + (-2)(-2) = 5$$

$$\langle 2 - x^2, 2 - x^2 \rangle = 1(1) + 2(2) + 1(1) = 6$$

$$\langle 2 + 3x - 4x^2, 2 + 3x - 4x^2 \rangle = (-5)(-5) + 2(2) + 1(1) = 30$$

Hence, $[1 + x^2]_{\mathcal{B}} = \begin{bmatrix} -3/5 \\ 1 \\ -1/5 \end{bmatrix}$.

9.2.3 We have

$$\left\langle \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \right\rangle = \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(0) + 0\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = 0$$

$$\left\langle \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \right\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right) + 0\left(\frac{1}{2}\right) + 0\left(-\frac{1}{2}\right) = 0$$

$$\left\langle \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \right\rangle = 0\left(\frac{1}{2}\right) + 0\left(-\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right) = 0$$

Therefore, the set is orthogonal. Also,

$$\left\langle \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} \right\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + 0^2 = 1$$

$$\left\langle \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \right\rangle = 0^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

$$\left\langle \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \right\rangle = \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = 1$$

Consequently, the set is orthonormal.

9.2.4 Observe that

$$\langle 1, x^2 \rangle = 1(4) + 1(0) + 1(4) = 8$$

Thus, the set is not even orthogonal.

9.3 Exercise Solutions

9.3.1 Since the algorithm is recursive, we use proof by induction!

Base Case: We have $\text{Span}\{\vec{w}_1\} = \text{Span}\{\vec{v}_1\}$, so $\{\vec{v}_1\}$ is an orthogonal basis for $\text{Span}\{\vec{w}_1\}$.

Inductive Hypothesis: Assume that $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ is an orthogonal basis for $\text{Span}\{\vec{w}_1, \dots, \vec{w}_{k-1}\}$.

Inductive Step: By definition in the statement of the theorem we have

$$\vec{v}_k = \vec{w}_k - \frac{\langle \vec{w}_k, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{w}_k, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \cdots - \frac{\langle \vec{w}_k, \vec{v}_{k-1} \rangle}{\|\vec{v}_{k-1}\|^2}\vec{v}_{k-1}$$

First observe that $\vec{v}_k \neq \vec{0}$ since $\vec{w}_k \notin \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\} = \text{Span}\{\vec{w}_1, \dots, \vec{w}_{k-1}\}$. Consequently, $\vec{v}_k \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{w}_k\} = \text{Span}\{\vec{w}_1, \dots, \vec{w}_k\}$.

For $1 \leq i \leq k-1$, we get

$$\langle \vec{v}_i, \vec{v}_k \rangle = \langle \vec{v}_i, \vec{w}_k - \frac{\langle \vec{w}_k, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{w}_k, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \cdots - \frac{\langle \vec{w}_k, \vec{v}_{k-1} \rangle}{\|\vec{v}_{k-1}\|^2}\vec{v}_{k-1} \rangle$$

Since the inner product is bilinear and using the fact that $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ is orthogonal, we get

$$\langle \vec{v}_i, \vec{v}_k \rangle = \langle \vec{v}_i, \vec{w}_k \rangle - \frac{\langle \vec{w}_k, \vec{v}_i \rangle}{||\vec{v}_i||^2} \langle \vec{v}_i, \vec{v}_i \rangle = \langle \vec{v}_i, \vec{w}_k \rangle - \langle \vec{w}_k, \vec{v}_i \rangle = 0$$

since the inner product is symmetric. Hence, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal set of k non-zero vectors in the k-dimensional subspace Span$\{\vec{w}_1, \dots, \vec{w}_k\}$ and so it is an orthogonal basis for Span$\{\vec{w}_1, \dots, \vec{w}_k\}$.

9.3.2 Take $\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Next, we get

$$\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{||\vec{v}_1||^2} \vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Then,

$$\vec{v}_3 = \vec{w}_1 - \frac{\langle \vec{w}_1, \vec{v}_1 \rangle}{||\vec{v}_1||^2} \vec{v}_1 - \frac{\langle \vec{w}_1, \vec{v}_2 \rangle}{||\vec{v}_2||^2} \vec{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{0}{1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Note: This demonstrates that applying the Gram-Schmidt procedure to vectors in a set in a different order can produce a different orthogonal basis for the set.

Chapter 10 Exercise Solutions

10.3 Exercise Solutions

10.3.1 $Q(x_1, x_2, x_3) = -3x_1^2 + x_1x_2 + 2x_1x_3 + \frac{3}{2}x_2^2 + 4x_2x_3$

10.3.2 $A = \begin{bmatrix} 3 & \frac{1}{4} & 0 \\ \frac{1}{4} & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

10.6 Exercise Solutions

10.6.1 We have $A^TA = \begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix}$. The characterstic polynomial is $c(\lambda) = \lambda^2 - 14\lambda - 24 = (\lambda - 2)(\lambda - 12)$. Thus, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 12$. Hence, the singular values are $\sigma_1 = \sqrt{12}$ and $\sigma_2 = \sqrt{2}$.

We have $B^TB = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 5 & 3 \\ 1 & 3 & 5 \end{bmatrix}$. We find that the characterstic polynomial is $C(\lambda) = \lambda(\lambda - 8)(\lambda - 3)$. Hence, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 8$, and $\lambda_3 = 3$. So, the singular values are $\sigma_1 = \sqrt{8}$, $\sigma_2 = \sqrt{3}$, and $\sigma_3 = 0$.

Chapter 11 Exercise Solutions

11.1 Exercise Solutions

11.1.1 Let $z_1 = a + bi$ and $z_2 = c + di$. Then,

$$\begin{aligned}
|z_1 z_2|^2 &= |(ac - bd) + (ad + bc)i|^2 \\
&= (ac - bd)^2 + (ad + bc)^2 \\
&= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \\
&= (a^2 + b^2)(c^2 + d^2) \\
&= |z_1|^2 |z_2|^2
\end{aligned}$$

11.1.2 $\dfrac{5}{3-4i} = \dfrac{5}{3-4i} \cdot \dfrac{3+4i}{3+4i} = \dfrac{15+20i}{3^2+4^2} = \dfrac{3}{5} + \dfrac{4}{5}i.$

11.2 Exercise Solutions

11.2.1 It is the same as the standard basis for $\mathbb{R}^n$.

11.2.2 Row reducing A gives $A \sim \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 2i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. A basis for Row$(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2i \end{bmatrix} \right\}$, a basis for Col$(A)$ is $\left\{ \begin{bmatrix} 1 \\ i \\ -i \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1+2i \\ 0 \end{bmatrix} \right\}$,

and a basis for Null(A) is $\left\{ \begin{bmatrix} -i \\ -2i \\ 1 \end{bmatrix} \right\}$.

Row reducing A^T gives $A^T \sim \begin{bmatrix} 1 & 0 & -2i & 1-i \\ 0 & 1 & 1 & 1-i \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Hence, a basis for Null(A^T) is $\left\{ \begin{bmatrix} 2i \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1+i \\ -1+i \\ 0 \\ 1 \end{bmatrix} \right\}$.

11.2.3 We have

$$\det A = \begin{vmatrix} 1+i & 2 & i \\ 2 & 3-2i & 1+i \\ 1-i & 1-3i & 1+i \end{vmatrix} = \begin{vmatrix} 1+i & 2 & i \\ 2 & 3-2i & 1+i \\ -1-i & -21-i & 0 \end{vmatrix} = \begin{vmatrix} 1+i & 2 & i \\ 0 & 1 & 0 \\ -1-i & -21-i & 0 \end{vmatrix} = i[0 - 1(-1-i)] = -1+i$$

Hence, $\det A^{-1} = \dfrac{1}{\det A} = \dfrac{1}{-1+i} = -\dfrac{1}{2} - \dfrac{i}{2}$.

11.3 Exercise Solutions

11.3.1 The characteristic polynomial is

$$\begin{aligned}
C(\lambda) &= \begin{vmatrix} -3-\lambda & -1 & 4 \\ -5 & 3-\lambda & 5 \\ -8 & -2 & 9-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda & -1 & 4 \\ 0 & 3-\lambda & 5 \\ 1-\lambda & -2 & 9-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda & -1 & 4 \\ 0 & 3-\lambda & 5 \\ 0 & -1 & 5-\lambda \end{vmatrix} \\
&= (1-\lambda)(\lambda^2 - 8\lambda + 20)
\end{aligned}$$

By the quadratic formula, we get that the roots of $\lambda^2 - 8\lambda + 20$ are

$$\lambda = \frac{8 \pm \sqrt{-16}}{2} = 4 \pm 2i$$

Hence, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2 + 2i$ and $\lambda_3 = 2 - 2i$.
For $\lambda_1 = 1$ we get

$$A - \lambda_1 I = \begin{bmatrix} -4 & -1 & 4 \\ -5 & 2 & 5 \\ -8 & -2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, an eigenvector for λ_1 is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 4 + 2i$ we get

$$A - \lambda_2 I = \begin{bmatrix} -7-2i & -1 & 4 \\ -5 & -1-2i & 5 \\ -8 & -2 & 5-2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -\frac{1}{2}+i \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, an eigenvector for λ_2 is $\vec{v}_2 = \begin{bmatrix} 1/2 \\ \frac{1}{2} - i \\ 1 \end{bmatrix}$.

For $\lambda_3 = 4 - 2i$ we get

$$A - \lambda_3 I = \begin{bmatrix} -7+2i & -1 & 4 \\ -5 & -1+2i & 5 \\ -8 & -2 & 5+2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -\frac{1}{2}-i \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, an eigenvector for λ_3 is $\vec{v}_3 = \begin{bmatrix} 1/2 \\ \frac{1}{2} + i \\ 1 \end{bmatrix}$.

11.4 Exercise Solutions

11.4.1 We have

$$\langle \vec{z}, \vec{w} \rangle = (1+i)(1+i) + (1-i)(-2i) + 2(-2) = 2i - 2i - 2 - 4 = -6$$
$$\langle \vec{w}, \vec{z} \rangle = (1-i)(1-i) + (2i)(1+i) + (-2)(2) = -2i + 2i - 2 - 4 = -6$$

11.4.2 We have

$$\langle A, B \rangle = 1(2-i) + (1-i)(1) + (2+2i)(-2i) + i(1-i) = 8 - 5i$$
$$\langle B, A \rangle = (2+i)(1) + (1)(1+i) + (2i)(2-2i) + (1+i)(-i) = 8 + 5i$$

11.4.3 We have

$$\|\alpha\vec{z}\|^2 = \langle \alpha\vec{z}, \alpha\vec{z} \rangle = \alpha^2 \|\vec{z}\|$$

Taking square roots, gives $\|\alpha\vec{z}\| = |\alpha| \|\vec{z}\|$.

11.4.4 If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{V}$, then for any $\vec{z} \in \mathbb{V}$, there exists $c_1, \dots, c_n \in \mathbb{C}$ such that

$$\vec{z} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

Taking the inner product of both sides with respect to $\vec{v}_i$ gives

$$\begin{aligned} \langle \vec{z}, \vec{v}_i \rangle &= \langle c_1\vec{v}_1 + \cdots + c_n\vec{v}_n, \vec{v}_i \rangle \\ &= c_1 \langle \vec{v}_1, \vec{v}_i \rangle + \cdots c_n \langle \vec{v}_n \vec{v}_i \rangle \\ &= c_i \langle \vec{v}_i, \vec{v}_i \rangle \end{aligned}$$

since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is orthogonal. Since $\vec{v}_i \neq \vec{0}$, we have $\|\vec{v}_i\| \neq 0$ and so we get that $c_i = \frac{\langle \vec{z}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}$ for all i. The result follows.

11.6 Exercise Solutions

11.6.1 Since A is upper triangular we have that its characteristic polynomial is

$$C(\lambda) = (\lambda - a)(\lambda - d)(\lambda - f) = \lambda^3 + (-a - d - f)\lambda^2 + (ad + af + df)x - adf$$

Then,

$$\begin{aligned} C(A) =& A^3 + (-a - d - f)A^2 \\ =& \begin{bmatrix} a^3 & a^2b + (ab + bd)d & c + (ab + bd)e + (ac + be + cf)f \\ 0 & d^3 & d^2e + (de + ef)f \\ 0 & 0 & f^3 \end{bmatrix} \\ &- (a + d + f) \begin{bmatrix} a^2 & ab + bd & ac + be + cf \\ 0 & d^2 & de + ef \\ 0 & 0 & f^2 \end{bmatrix} + (ad + af + df) \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} - adf \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ =& \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Alternately, we have

$$
\begin{aligned}
C(A) &= (A - aI)(A - dI)(A - fI) \\
&= \begin{bmatrix} 0 & b & c \\ 0 & d-a & e \\ 0 & 0 & f-a \end{bmatrix} \begin{bmatrix} a-d & b & c \\ 0 & 0 & e \\ 0 & 0 & f-d \end{bmatrix} \begin{bmatrix} a-f & b & c \\ 0 & d-f & e \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & be + c(f-d) \\ 0 & 0 & ef - ea \\ 0 & 0 & f^2 - (a+d)f + ad \end{bmatrix} \begin{bmatrix} a-f & b & c \\ 0 & d-f & e \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

as required.