Chapter 1 Solutions

1.1 Problem Solutions

1.1.1 (a) $2\begin{bmatrix}2\\1\\-1\end{bmatrix}+3\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix}$

(b) $\frac{1}{3}\begin{bmatrix}3\\1\\-2\end{bmatrix}-\begin{bmatrix}4\\1/2\\-2/3\end{bmatrix}=\begin{bmatrix}-3\\-1/6\\0\end{bmatrix}$

(c) $\sqrt{2}\begin{bmatrix}\sqrt{2}\\0\\\sqrt{2}\end{bmatrix}+\frac{2}{3}\begin{bmatrix}6\\0\\0\end{bmatrix}=\begin{bmatrix}6\\0\\2\end{bmatrix}$

(d) $3\begin{bmatrix}1\\0\\1\end{bmatrix}-2\begin{bmatrix}-2\\3\\1\end{bmatrix}+\begin{bmatrix}-7\\6\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(e) $x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$

(f) $-3\begin{bmatrix}1\\0\\1\end{bmatrix}+\pi\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-3\\\pi\\-3\end{bmatrix}$

(g) $(a+b)\begin{bmatrix}1\\1\\-1\end{bmatrix}+(-a+b)\begin{bmatrix}1\\2\\-3\end{bmatrix}+(-a+2b)\begin{bmatrix}-1\\-1\\2\end{bmatrix}=\begin{bmatrix}a\\b\\0\end{bmatrix}$

1.1.2 (a) Since $\begin{bmatrix}-2\\6\\-2\end{bmatrix}=-2\begin{bmatrix}1\\-3\\1\end{bmatrix}$ we get by Theorem 1.1.2 that $\operatorname{Span}\left\{\begin{bmatrix}1\\-3\\1\end{bmatrix},\begin{bmatrix}-2\\6\\-2\end{bmatrix}\right\}=\operatorname{Span}\left\{\begin{bmatrix}1\\-3\\1\end{bmatrix}\right\}$. Hence, the set represents the line in $\mathbb{R}^3$ with vector equation $\vec{x}=t\begin{bmatrix}1\\-3\\1\end{bmatrix}$, $t\in\mathbb{R}$.

(b) The set only contains two vectors, so the set represents the two points $(1,-3,1)$, $(-2,6,-2)$ in $\mathbb{R}^3$.

(c) Since $\left\{\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}\right\}$ is linearly independent (neither vector is a scalar multiple of the other), we get that the set represents the plane in $\mathbb{R}^3$ with vector equation $\vec{x} = s\begin{bmatrix}1\\0\\-2\end{bmatrix} + t\begin{bmatrix}2\\1\\-1\end{bmatrix}$, $s, t \in \mathbb{R}$.

(d) The set only contains the zero vector, so it represents the origin in $\mathbb{R}^3$. A vector equation is $\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

(e) The set is linearly independent (verify this) and so the set represents a hyperplane in $\mathbb{R}^4$ with vector equation

$$\vec{x} = t_1\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + t_2\begin{bmatrix}1\\0\\2\\1\end{bmatrix} + t_3\begin{bmatrix}3\\1\\0\\0\end{bmatrix}, \quad t_1, t_2, t_3 \in \mathbb{R}$$

(f) The set represents the line in $\mathbb{R}^4$ with vector equation $\vec{x} = t\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$, $t \in \mathbb{R}$.

1.1.3 (a) Observe that $(-1)\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. Hence, the set is linearly dependent. Solving for the first vector gives $\begin{bmatrix}1\\2\end{bmatrix} = 2\begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}1\\4\end{bmatrix}$.

(b) Since neither vector is a scalar multiple of the other, the set is linearly independent.

(c) Since neither vector is a scalar multiple of the other, the set is linearly independent.

(d) Observe that $2\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}-4\\-6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. Hence, the set is linearly dependent. Solving for the second vector gives $\begin{bmatrix}-4\\-6\end{bmatrix} = (-2)2\begin{bmatrix}2\\3\end{bmatrix}$.

(e) The only solution to $c\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is $c = 0$, so the set is linearly independent.

(f) Since the set contains the zero vector, it is linearly dependent by Theorem 1.1.4. Solving for the zero vector gives $\begin{bmatrix}0\\0\\0\end{bmatrix} = 0\begin{bmatrix}1\\-3\\-2\end{bmatrix} + 0\begin{bmatrix}4\\6\\1\end{bmatrix}$.

(g) Observe that $0\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\2\\-1\end{bmatrix} - \begin{bmatrix}-2\\-4\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$. Hence, the set is linearly dependent. Solving for the second vector gives $\begin{bmatrix}1\\2\\-1\end{bmatrix} = 0\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-2\\-4\\2\end{bmatrix}$.

(h) Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\-2\\1\end{bmatrix} + c_2\begin{bmatrix}2\\3\\4\end{bmatrix} + c_3\begin{bmatrix}0\\-1\\-2\end{bmatrix}$$

Performing operations on vectors on the right hand side, we get

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}c_1 + 2c_2\\-2c_1 + 3c_2 - c_3\\c_1 + 4c_2 - 2c_3\end{bmatrix}$$

Since vectors are equal if and only if there corresponding entries are equal we get the three equations in three unknowns

$$\begin{aligned} c_1 + 2c_2 &= 0\\ -2c_1 + 3c_2 - c_3 &= 0\\ c_1 + 4c_2 - 2c_3 &= 0 \end{aligned}$$

The first equation implies that $c_1 = -2c_2$. Substituting this into the second equation we get $7c_2 - c_3 = 0$. Thus, $c_3 = 7c_2$. Substituting these both into the third equation gives $-12c_2 = 0$. Therefore, the only solution is $c_1 = c_2 = c_3 = 0$. Hence, the set is linearly independent.

(i) Observe that

$$(-2)\begin{bmatrix}1\\1\\2\\1\end{bmatrix} + \begin{bmatrix}2\\2\\4\\2\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + 0\begin{bmatrix}2\\1\\3\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Hence, the set is linearly dependent. Solving for the second vector gives

$$\begin{bmatrix}2\\2\\4\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\\2\\1\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + 0\begin{bmatrix}2\\1\\3\\1\end{bmatrix}$$

(j) Since neither vector is a scalar multiple of the other, the set is linearly independent.

1.1.4 (a) Clearly $\begin{bmatrix}1\\0\end{bmatrix} \notin \operatorname{Span}\left\{\begin{bmatrix}3\\2\end{bmatrix}\right\}$. Thus, $\left\{\begin{bmatrix}3\\2\end{bmatrix}\right\}$ does not span $\mathbb{R}^2$ and so is not a basis for $\mathbb{R}^2$.

(b) Since neither vector is a scalar multiple of the other, the set is linearly independent. Let $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \in \mathbb{R}^2$ and consider

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2c_1 + c_2\\3c_1\end{bmatrix}$$

Comparing entries we get

$$\begin{aligned} x_1 &= 2c_1 + c_2\\ x_2 &= 3c_1 \end{aligned}$$

Thus, $c_1 = x_2/3$ and $c_2 = x_1 - 2x_2/3$. Since there is a solution for all $\vec{x} \in \mathbb{R}^2$, we have that $\left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ spans $\mathbb{R}^2$.

Since the set spans $\mathbb{R}^2$ and is linearly independent, it is a basis for $\mathbb{R}^2$.

(c) Since neither vector is a scalar multiple of the other, the set is linearly independent. Let $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \in \mathbb{R}^2$ and consider

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-c_1 + c_2\\c_1 + 3c_2\end{bmatrix}$$

Comparing entries we get

$$\begin{aligned} x_1 &= -c_1 + c_2 \\ x_2 &= c_1 + 3c_2 \end{aligned}$$

Solving, we get $c_2 = (x_1 + x_2)/4$ and $c_1 = (-3x_1 + x_2)/4$. Since there is a solution for all $\vec{x} \in \mathbb{R}^2$, we have that $\left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right\}$ spans $\mathbb{R}^2$.

Since the set spans $\mathbb{R}^2$ and is linearly independent, it is a basis for $\mathbb{R}^2$.

(d) Observe from our work in (c) that

$$\frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

Thus, the set is linearly dependent and hence is not a basis.

(e) Since neither vector is a scalar multiple of the other, the set is linearly independent. Let $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \in \mathbb{R}^2$ and consider

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}-3\\5\end{bmatrix} = \begin{bmatrix}c_1 - 3c_2\\5c_2\end{bmatrix}$$

Comparing entries we get

$$\begin{aligned} x_1 &= c_1 - 3c_2 \\ x_2 &= 5c_2 \end{aligned}$$

Solving, we get $c_2 = x_2/5$ and $c_1 = x_1 + 3x_2/5$. Since there is a solution for all $\vec{x} \in \mathbb{R}^2$, we have that $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-3\\5\end{bmatrix}\right\}$ spans $\mathbb{R}^2$.

Since the set spans $\mathbb{R}^2$ and is linearly independent, it is a basis for $\mathbb{R}^2$.

(f) The set contains the zero vector and so is linearly dependent by Theorem 1.1.4. Thus, the set cannot be a basis.

1.1.5 (a) The set contains the zero vector and so is linearly dependent by Theorem 1.1.4. Thus, the set cannot be a basis.

(b) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in $\mathbb{R}^3$. Consider the equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 \\ 2c_1 + c_2 \\ -c_1 + 2c_2 \end{bmatrix}$$

This gives the 3 equations in 2 unknowns

$$\begin{aligned} -c_1 + c_2 &= x_1 \\ 2c_1 + c_2 &= x_2 \\ -c_1 + 2c_2 &= x_3 \end{aligned}$$

Subtracting the first equation from the third equation gives $c_2 = x_3 - x_1$. Substituting this into the first equation we get $c_1 = -2x_1 + x_3$. Substituting both of these into the second equation gives

$$-5x_1 + 3x_3 = x_2$$

Thus, $\vec{x}$ is in the span of $\left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ if and only if $-5x_1 + 3x_3 = x_2$. Thus, the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in the span, so $\left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ does not span $\mathbb{R}^3$ and hence is not a basis.

(c) Clearly $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Thus, the set does not span $\mathbb{R}^3$ and hence is not a basis.

(d) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in $\mathbb{R}^3$. Consider the equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_3 \\ c_2 + c_3 \\ c_1 + c_2 \end{bmatrix} \tag{1.1}$$

This gives the 3 equations in 3 unknowns

$$\begin{aligned} c_1 + c_3 &= x_1 \\ c_2 + c_3 &= x_2 \\ c_1 + c_2 &= x_3 \end{aligned}$$

Solving using substitution and elimination we get

$$\begin{aligned} c_1 &= \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ c_2 &= -\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ c_3 &= \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \end{aligned}$$

Thus, $\vec{x}$ is in the span of $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$. Hence, it spans $\mathbb{R}^3$.

Moreover, if we let $x_1 = x_2 = x_3 = 0$ in equation (1.1), we get that the only solution is $c_1 = c_2 = c_3 = 0$, so the set is also linearly independent. Hence, it is a basis for $\mathbb{R}^3$.

(e) Observe that

$$\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}0\\0\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

so the set is linearly dependent and hence not a basis.

(f) Let $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be any vector in $\mathbb{R}^3$. Consider the equation

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = c_1\begin{bmatrix}1\\1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\2\\-3\end{bmatrix} + c_3\begin{bmatrix}-1\\-1\\2\end{bmatrix} = \begin{bmatrix}c_1 + c_2 - c_3\\ c_1 + 2c_2 - c_3\\ -c_1 - 3c_2 + 2c_3\end{bmatrix} \tag{1.2}$$

This gives the 3 equations in 3 unknowns

$$\begin{aligned} c_1 + c_2 - c_3 &= x_1 \\ c_1 + 2c_2 - c_3 &= x_2 \\ -c_1 - 3c_2 + 2c_3 &= x_3 \end{aligned}$$

Solving using substitution and elimination we get

$$\begin{aligned} c_1 &= x_1 + x_2 + x_3 \\ c_2 &= -x_1 + x_2 \\ c_3 &= -x_1 + 2x_2 + x_3 \end{aligned}$$

Thus, $\vec{x}$ is in the span of $\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\-3\end{bmatrix}, \begin{bmatrix}-1\\-1\\2\end{bmatrix}\right\}$. Hence, it spans $\mathbb{R}^3$.

Moreover, if we let $x_1 = x_2 = x_3 = 0$ in equation (1.2), we get that the only solution is $c_1 = c_2 = c_3 = 0$, so the set is also linearly independent. Hence, it is a basis for $\mathbb{R}^3$.

1.1.6 There are infinitely many correct answers. One simple choice is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. Since the vectors in the set are standard basis vectors for $\mathbb{R}^3$, we know they are linearly independent and hence the set forms a basis for a hyperplane in $\mathbb{R}^3$.

1.1.7 Assume that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. For a contradiction, assume without loss of generality that $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$. Then $\vec{v}_1 = t\vec{v}_2$ and hence $\vec{v}_1 - t\vec{v}_2 = \vec{0}$. This contradicts the fact that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent since the coefficient of $\vec{v}_1$ is non-zero.

On the other hand, assume that $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent. Then there exists $c_1, c_2 \in \mathbb{R}$ not both zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Without loss of generality assume that $c_1 \neq 0$. Then $\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2$ and hence $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$.

1.1.8 If $\vec{v}_1 \in \operatorname{Span}\{\vec{v}_2, \vec{v}_3\}$, then there exists $c_1, c_2 \in \mathbb{R}$ such that $\vec{v}_1 = c_1\vec{v}_2 + c_2\vec{v}_3$. Thus,

$$\vec{v}_1 - c_1\vec{v}_1 - c_2\vec{v}_3 = \vec{0}$$

where the coefficient of $\vec{v}_1$ is non-zero, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

1.1.9 Assume for a contradiction that $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent. Then there exists $c_1, c_2 \in \mathbb{R}$ with c_1, c_2 not both zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Hence, we have

$$c_1\vec{v}_1 + c_2\vec{v}_2 + 0\vec{v}_3 = \vec{0}$$

with not all coefficients equal to zero which contradicts the fact that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

1.1.10 To prove this, we will prove that both sets are a subset of the other.

Let $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Then there exists $c_1, c_2 \in \mathbb{R}$ such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Since $t \neq 0$ we get

$$\vec{x} = c_1\vec{v}_1 + \frac{c_2}{t}(t\vec{v}_2)$$

so $\vec{x} \in \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$. Thus, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$.

If $\vec{y} \in \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$, then there exists $d_1, d_2 \in \mathbb{R}$ such that

$$\vec{y} = d_1\vec{v}_1 + d_2(t\vec{v}_2) = d_1\vec{v}_1 + (d_2t)\vec{v}_2 \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$$

Hence, we also have $\operatorname{Span}\{\vec{v}_1, t\vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Therefore, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$.

1.1.11 Assume that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{v}_3 \notin \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Consider

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

If $c_3 \neq 0$, then we have $\vec{v}_3 = -\frac{c_1}{c_3}\vec{v}_1 - \frac{c_2}{c_3}\vec{v}_2$ which contradicts that fact that $\vec{v}_3 \notin \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Hence, $c_3 = 0$.

Thus, we have $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ implies $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Since $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent, the only solution to this is $c_1 = c_2 = 0$. Therefore, we have shown the only solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ is $c_1 = c_2 = c_3 = 0$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

1.1.12 Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in $\mathbb{R}^3$. Consider the equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_1 + c_2 \\ -c_1 + 2c_2 \end{bmatrix}$$

This gives the 3 equations in 2 unknowns

$$\begin{aligned} c_1 &= x_1 \\ 2c_1 + c_2 &= x_2 \\ -c_1 + 2c_2 &= x_3 \end{aligned}$$

Substituting $c_1 = x_1$ into the second equation we get $c_2 = -2x_1 + x_2$. Substituting both of these into the third equation gives

$$-5x_1 + 2x_2 = x_3$$

Thus, $\vec{x}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ if and only if $-5x_1 + 2x_2 = x_3$. Thus, the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the span, so $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ does not span $\mathbb{R}^3$ and hence is not a basis.

Then, by Problem 11, the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_1 + c_2 \\ -c_1 + 2c_2 + c_3 \end{bmatrix}$$

Thus, we have

$$\begin{aligned} c_1 &= x_1 \\ 2c_1 + c_2 &= x_2 \\ -c_1 + 2c_2 + c_3 &= x_3 \end{aligned}$$

Solving gives $c_1 = x_1$, $c_2 = -2x_1 + x_2$, and $c_3 = 3x_1 - 2x_2 + x_3$. Therefore, $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ also spans $\mathbb{R}^3$ and hence is a basis for $\mathbb{R}^3$.

1.1.13 We have

$$\begin{aligned}
\vec{x} + 2\vec{v} &= \vec{v} + (-\vec{x}) \\
\vec{x} + 2\vec{v} + \vec{x} &= \vec{v} + (-\vec{x}) + \vec{x} && \text{add } \vec{x} \text{ on the right to both sides} \\
\vec{x} + \vec{x} + 2\vec{v} &= \vec{v} + \vec{0} && \text{by V3 and V5} \\
1\vec{x} + 1\vec{x} + (1+1)\vec{v} &= \vec{v} && \text{by V4, V10, and normal addition in } \mathbb{R} \\
(1+1)\vec{x} + 1\vec{v} + 1\vec{v} &= \vec{v} && \text{by V8} \\
2\vec{x} + \vec{v} + \vec{v} &= \vec{v} && \text{by V10, and normal addition in } \mathbb{R} \\
2\vec{x} + \vec{v} + \vec{v} + (-\vec{v}) &= \vec{v} + (-\vec{v}) && \text{by V5, add } (-\vec{v}) \text{ on the right to both sides} \\
2\vec{x} + \vec{v} + \vec{0} &= \vec{0} && \text{by V5} \\
2\vec{x} + \vec{v} &= \vec{0} && \text{by V4} \\
2\vec{x} + \vec{v} + (-\vec{v}) &= \vec{0} + (-\vec{v}) && \text{by V5, add } (-\vec{v}) \text{ on the right to both sides} \\
2\vec{x} + \vec{0} &= \vec{0} + (-\vec{v}) && \text{by V5} \\
2\vec{x} &= (-\vec{v}) && \text{by V4} \\
\frac{1}{2}(2\vec{x}) &= \frac{1}{2}(-\vec{v}) && \text{multiply both sides by } \frac{1}{2} \\
\left(\frac{1}{2}2\right)\vec{x} &= \frac{1}{2}(-\vec{v}) && \text{by V7} \\
1\vec{x} &= \frac{1}{2}(-\vec{v}) && \text{by normal multiplication in } \mathbb{R} \\
\vec{x} &= \frac{1}{2}(-\vec{v}) && \text{by V10}
\end{aligned}$$

1.1.14 (a) The statement is true. If $\vec{v}_1 \neq \vec{0}$, then the only solution to $c\vec{v}_1 = \vec{0}$ is $c = 0$, so $\{\vec{v}_1\}$ is linearly independent. On the other hand, if $\vec{v}_1 = \vec{0}$, then $\{\vec{v}_1\}$ is linearly dependent by Theorem 1.1.4.

(b) The statement is false. For example Span $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}\right\}$ is a line in $\mathbb{R}^3$.

(c) The statement is true. Let $\vec{x} \in \mathbb{R}^2$. Since $\{\vec{v}_1, \vec{v}_2\}$ spans $\mathbb{R}^2$ there exists $c_1, c_2 \in \mathbb{R}$ such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Then we get

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_2\vec{v}_1 - c_2\vec{v}_1 = (c_1 - c_2)\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2)$$

Thus, $\mathbb{R}^2 \subseteq \{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$.

On the other hand, since $\vec{v}_1, \vec{v}_1 + \vec{v}_2 \in \mathbb{R}^2$ by Theorem 1.1.1, we have that $\operatorname{Span}\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\} \subseteq \mathbb{R}^2$ by Theorem 1.1.1.

Thus, $\operatorname{Span}\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\} = \mathbb{R}^2$.

(d) The statement is false. The set $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is linearly dependent, but $\begin{bmatrix}0\\1\end{bmatrix} \notin$ Span $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}$.

(e) The statement is false. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$ but $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent by Theorem 1.1.4.

(f) The statement is false. If $\vec{b} = \vec{d} = \vec{0}$, then $\vec{x} = t\vec{d} + \vec{b}$ is just the origin.

(g) The statement is true. By definition of Span we have $\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_k \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$.

1.2 Problem Solutions

1.2.1 (a) By Theorem 1.2.2, $\mathbb{S}_1$ is a subspace of $\mathbb{R}^3$. Since neither $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ nor $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ is a scalar multiple of the other, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$ is linearly independent. Also, $\operatorname{Span}\mathcal{B} = \mathbb{S}_1$, so $\mathcal{B}$ is a basis for $\mathbb{S}_1$. Thus, geometrically, $\mathbb{S}_1$ is a plane in $\mathbb{R}^3$.

(b) By definition $\mathbb{S}_2$ is a subset of $\mathbb{R}^2$. Observe that $\vec{0} \in \mathbb{S}_2$ since $0 = 0$. So, $\mathbb{S}_2$ is non-empty.

Let $\vec{x}, \vec{y} \in \mathbb{S}_2$, then $x_1 = x_2$ and $y_1 = y_2$. Hence,

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

where $x_1 + y_1 = x_2 + y_2$. So $\vec{x} + \vec{y} \in \mathbb{S}_2$.

For any $t \in \mathbb{R}$ we have

$$t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} \in \mathbb{S}_2$$

since $tx_1 = tx_2$.

Thus, $\mathbb{S}_2$ is a subspace of $\mathbb{R}^2$ by the Subspace Test.

To find a basis for $\mathbb{S}_2$ we need to find a linearly independent spanning set for $\mathbb{S}_2$. We first find a spanning set. Every vector $\vec{x} \in \mathbb{S}_2$ satisfies $x_1 = x_2$ and so has the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_1 \in \mathbb{R}$$

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{S}_2$. Moreover, it contains one non-zero vector so it also linearly independent. Thus, $\mathcal{B}$ is a basis for $\mathbb{S}_2$ and so $\mathbb{S}_2$ is a line in $\mathbb{R}^2$.

(c) By definition $\mathbb{S}_3$ is a subset of $\mathbb{R}^3$. But, observe that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin \mathbb{S}_3$ since it does not satisfy the condition on $\mathbb{S}_3$. Thus, $\mathbb{S}_3$ is not a subspace.

(d) By definition $\mathbb{S}_4$ is a subset of $\mathbb{R}^3$. Also, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{S}_4$ since $0 + 0 = 0$. Thus, $\mathbb{S}_4$ is non-empty.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{S}_4$. Then, $x_1 + x_2 = x_3$ and $y_1 + y_2 = y_3$. This gives

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

and

$$(x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = x_3 + y_3$$

So, $\vec{x} + \vec{y}$ satisfies the condition of $\mathbb{S}_4$, so $\vec{x} + \vec{y} \in \mathbb{S}_4$.

Similarly, for any $c \in \mathbb{R}$, $c\vec{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$ and

$$cx_1 + cx_2 = c(x_1 + x_2) = cx_3$$

so $c\vec{x} \in \mathbb{S}_4$.

Thus, by the Subspace Test, $\mathbb{S}_4$ is a subspace of $\mathbb{R}^3$.

To find a basis for $\mathbb{S}_4$ we need to find a linearly independent spanning set for $\mathbb{S}_4$. We first find a spanning set. Every vector $\vec{x} \in \mathbb{S}_4$ satisfies $x_1 + x_2 = x_3$ and so has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad x_1, x_2 \in \mathbb{R}$$

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{S}_4$. Moreover, neither vector is a scalar multiple of the other so the set is linearly independent. Thus, $\mathcal{B}$ is a basis for $\mathbb{S}_4$ and so $\mathbb{S}_4$ is a plane in $\mathbb{R}^3$.

(e) By definition $\mathbb{S}_5$ is non-empty subset of $\mathbb{R}^4$.

Let $\vec{x}, \vec{y} \in \mathbb{S}_5$. Then, $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Hence

$$\vec{x} + \vec{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{S}_5$$

and for any $c \in \mathbb{R}$

$$c\vec{x} = c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{S}_5$$

Therefore, $\mathbb{S}_5$ is a subspace of $\mathbb{R}^4$ by the Subspace Test.

By definition, the empty set is a basis for $\mathbb{S}_5$. Geometrically, $\mathbb{S}_5$ is the origin in $\mathbb{R}^4$.

(f) $\mathbb{S}_6$ is a subset of $\mathbb{R}^3$, but clearly $\begin{bmatrix}0\\0\\0\end{bmatrix} \notin \mathbb{S}_6$, so $\mathbb{S}_6$ is not a subspace.

(g) By definition $\mathbb{S}_7$ is a subset of $\mathbb{R}^4$. Also, $\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \in \mathbb{S}_7$ since $0 = 0 - 0 + 0$. Thus, $\mathbb{S}_7$ is non-empty.

Let $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}, \vec{y} = \begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix} \in \mathbb{S}_7$. Then, $x_2 = x_1 - x_3 + x_4$ and $y_2 = y_1 - y_3 + y_4$. This gives

$$\vec{x} + \vec{y} = \begin{bmatrix}x_1 + y_1\\x_2 + y_2\\x_3 + y_3\\x_4 + y_4\end{bmatrix}$$

and

$$x_2 + y_2 = (x_1 - x_3 + x_4) + (y_1 - y_3 + y_4) = (x_1 + y_1) - (x_3 + y_3) + (x_4 + y_4)$$

So, $\vec{x} + \vec{y}$ satisfies the condition of $\mathbb{S}_7$, so $\vec{x} + \vec{y} \in \mathbb{S}_7$.

Similarly, for any $t \in \mathbb{R}$, $t\vec{x} = \begin{bmatrix}tx_1\\tx_2\\tx_3\\tx_4\end{bmatrix} \in \mathbb{S}_7$ since

$$tx_2 = t(x_1 - x_3 + x_4) = (tx_1) - (tx_3) + (tx_4)$$

Thus, by the Subspace Test, $\mathbb{S}_7$ is a subspace of $\mathbb{R}^4$.

To find a basis for $\mathbb{S}_7$ we need to find a linearly independent spanning set for $\mathbb{S}_7$. We first find a spanning set. Every vector $\vec{x} \in \mathbb{S}_7$ satisfies $x_2 = x_1 - x_3 + x_4$ and so has the form

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\x_1 - x_3 + x_4\\x_3\\x_4\end{bmatrix} = x_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\-1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \quad x_1, x_3, x_4 \in \mathbb{R}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$ spans $\mathbb{S}_7$.

Consider

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}c_1\\c_1 - c_2 + c_3\\c_2\\c_3\end{bmatrix}$$

Comparing entries gives $c_1 = c_2 = c_3 = 0$. Thus, $\mathcal{B}$ is also linearly independent, and so is a basis for $\mathbb{S}_7$. Therefore, $\mathbb{S}_7$ is a hyperplane in $\mathbb{R}^4$.

(h) By definition $\mathbb{S}_8$ is a subset of $\mathbb{R}^4$. Also, take $a = b = c = d = 0$ gives $\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \in \mathbb{S}_8$. Thus, $\mathbb{S}_8$ is non-empty.

Let $\vec{x} = \begin{bmatrix} a_1 - b_1 \\ b_1 - d_1 \\ a_1 + b_1 - 2d_1 \\ c_1 - d_1 \end{bmatrix}, \vec{y} = \begin{bmatrix} a_2 - b_2 \\ b_2 - d_2 \\ a_2 + b_2 - 2d_2 \\ c_2 - d_2 \end{bmatrix} \in \mathbb{S}_8$. Then,

$$\vec{x} + \vec{y} = \begin{bmatrix} (a_1 + a_2) - (b_1 + b_2) \\ (b_1 + b_2) - (d_1 + d_2) \\ (a_1 + a_2) + (b_1 + b_2) - 2(d_1 + d_2) \\ (c_1 + c_2) - (d_1 + d_2) \end{bmatrix}$$

So, $\vec{x} + \vec{y}$ satisfies the condition of $\mathbb{S}_8$, so $\vec{x} + \vec{y} \in \mathbb{S}_8$.

For any $t \in \mathbb{R}$, we get

$$t\vec{x} = \begin{bmatrix} t(a_1 - b_1) \\ t(b_1 - d_1) \\ t(a_1 + b_1 - 2d_1) \\ t(c_1 - d_1) \end{bmatrix} = \begin{bmatrix} (ta_1) - (tb_1) \\ (tb_1) - (td_1) \\ (ta_1) + (tb_1) - 2(td_1) \\ (tc_1) - (td_1) \end{bmatrix}$$

So, $t\vec{x}$ satisfies the condition of $\mathbb{S}_8$, so $t\vec{x} \in \mathbb{S}_8$.

Thus, by the Subspace Test, $\mathbb{S}_8$ is a subspace of $\mathbb{R}^4$.

To find a basis for $\mathbb{S}_8$ we need to find a linearly independent spanning set for $\mathbb{S}_8$. We first find a spanning set. Every vector $\vec{x} \in \mathbb{S}_8$ has the form

$$\begin{bmatrix} a - b \\ b - d \\ a + b - 2d \\ c - d \end{bmatrix} = a\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + b\begin{bmatrix}-1\\1\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\0\\1\end{bmatrix} + d\begin{bmatrix}0\\-1\\-2\\-1\end{bmatrix}, \quad a, b, c, d \in \mathbb{R}$$

Thus, $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\-2\\-1\end{bmatrix}\right\}$ spans $\mathbb{S}_8$.

However, observe that

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \begin{bmatrix}-1\\1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\-1\\-2\\-1\end{bmatrix}$$

So, spanning set is linearly dependent. Using Theorem 1.1.2, we get

$$\operatorname{Span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}=\mathbb{S}_8$$

Now, consider

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}=c_1\begin{bmatrix}1\\0\\1\\0\end{bmatrix}+c_2\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}c_1-c_2\\c_2\\c_1+c_2\\c_3\end{bmatrix}$$

Comparing entries gives $c_1 - c_2 = 0$, $c_2 = 0$, $c_1 + c_2 = 0$, and $c_3 = 0$. Thus, $c_1 = c_2 = c_3 = 0$ is the only solution so, $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ is also linearly independent, and so is a basis for $\mathbb{S}_8$. Therefore, $\mathbb{S}_8$ is a hyperplane in $\mathbb{R}^4$.

1.2.2 By definition, a plane P in $\mathbb{R}^3$ has vector equation $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \vec{b}$, $c_1, c_2 \in \mathbb{R}$ where $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

If P is a subspace of $\mathbb{R}^3$, then $\vec{0} \in P$ and hence P passes through the origin. On the other hand, if P passes through the origin, then there exists $d_1, d_2 \in \mathbb{R}$ such that

$$\vec{0} = d_1\vec{v}_1 + d_2\vec{v}_2 + \vec{b}$$

Thus, $\vec{b} = -d_1\vec{v}_1 - d_2\vec{v}_2$. Hence, we can write the vector equation of P as

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 - d_1\vec{v}_1 - d_2\vec{v}_2 = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2$$

Since c_1 and c_2 can be any real numbers, $(c_1 - d_1)$ and $(c_2 - d_2)$ can be any real numbers, so $P = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Consequently, P is a subspace of $\mathbb{R}^2$ by Theorem 1.2.2.

1.2.3 Since $\mathbb{S}$ does not contain the zero vector it cannot be a subspace.

1.2.4 (a) Let $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be any vector in $\mathbb{R}^3$. Consider the equation

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=c_1\begin{bmatrix}1\\3\\1\end{bmatrix}+c_2\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}c_1+2c_2\\3c_1\\c_1+c_2\end{bmatrix}$$

This gives the 3 equations in 2 unknowns

$$\begin{aligned}c_1 + 2c_2 &= x_1\\ 3c_1 &= x_2\\ c_1 + c_2 &= x_3\end{aligned}$$

Subtracting the third equation from the first equation gives $c_2 = x_1 - x_3$. Substituting this and the second equation into the third equation we get

$$\frac{1}{3}x_2 + x_1 = 2x_3$$

Thus, $\vec{x}$ is in the span of $\left\{\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$ if and only if $\frac{1}{3}x_2 + x_1 = 2x_3$. Thus, the vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not in the span, so $\mathcal{B}$ does not span $\mathbb{R}^3$ and hence is not a basis for $\mathbb{R}^3$.

(b) From our work in (a), we have that $\mathcal{B}$ is a basis for the subspace

$$\mathbb{S} = \left\{\vec{x} \in \mathbb{R}^3 \mid \frac{1}{3}x_2 + x_1 = 2x_3\right\} = \operatorname{Span}\left\{\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$$

1.2.5 (a) If $d \neq 0$, then $\begin{bmatrix}0\\0\\0\end{bmatrix} \notin \mathbb{P}$ since $a(0) + b(0) + c(0) \neq d$. Hence, $\mathbb{P}$ cannot be a subspace.

(b) If $\mathbb{P}$ had a basis, then there would be a spanning set for $\mathbb{P}$. But, this would contradict Theorem 1.2.2, since $\mathbb{P}$ is not a subspace.

(c) If $a = b = c = 0$, then $\mathbb{P}$ is the empty set. If $a \neq 0$, then every vector in $\vec{x} \in \mathbb{P}$ has the form

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}\frac{d}{a} - \frac{b}{a}x_2 - \frac{c}{a}x_3\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}d/a\\0\\0\end{bmatrix} + x_2\begin{bmatrix}-b/a\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-c/a\\0\\1\end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

The remaining cases are similar.

(d) If $a = b = c = 0$, then $\mathbb{P}$ is the empty set. If $a \neq 0$, then $\mathbb{P}$ is a plane in $\mathbb{R}^3$ passing through $(d/a, 0, 0)$ since $\left\{\begin{bmatrix}-b/a\\1\\0\end{bmatrix}, \begin{bmatrix}-c/a\\0\\1\end{bmatrix}\right\}$ is linearly independent. The remaining cases are similar.

1.2.6 A set S is a subset of $\mathbb{R}^n$ if every element of S is in $\mathbb{R}^n$. For S to be a subspace, it not only has to be a subset, but it also must be closed under addition and scalar multiplication of vectors (by the Subspace Test).

1.3 Problem Solutions

1.3.1 Evaluate the following:

(a) $\begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 5(3) + 3(2) + (-6)(4) = -3$

(b) $\begin{bmatrix} 1 \\ -2 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1/2 \\ 1/2 \\ -1 \end{bmatrix} = 1(2) + (-2)(1/2) + (-2)(1/2) + 3(-1) = -3$

(c) $\left\| \begin{bmatrix} \sqrt{2} \\ 1 \\ \sqrt{2} \end{bmatrix} \right\| = \sqrt{(\sqrt{2})^2 + 1^2 + (-\sqrt{2})^2} = \sqrt{5}$

(d) $\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = 1(3) + 4(-1) + (-1)(-1) = 0$

(e) $\left\| \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2 + (-1)^2 + 3^2} = \sqrt{15}$

(f) $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

(g) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

(h) $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3a \\ 2a \\ 3a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

1.3.2 (a) Since the plane passes through $(0, 0, 0)$ we get that a scalar equation is

$$x_1 - 3x_2 + 3x_3 = 1(0) + (-3)(0) + 3(0) = 0$$

(b) For two planes to be parallel, their normal vectors must be scalar multiples of each other. Hence, a normal vector for the required plane is $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. Thus, a scalar equation of the plane is

$$3x_1 + 2x_2 - x_3 = 3(1) + 2(2) - 1(0) = 7$$

(c) A normal vector for the required plane is

$$\vec{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Since the plane passes through the origin, we get that a scalar equation is

$$2x_1 - x_2 + x_3 = 0$$

(d) We have $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \\ -3 \end{bmatrix}$. Any non-zero scalar multiple of this will be a normal vector for the plane. We pick $\vec{n} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$. Since the plane passes through the origin, we get that a scalar equation is

$$-2x_1 + 3x_2 - x_3 = 0$$

(e) A normal vector for the required plane is

$$\vec{n} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}$$

Since the plane passes through the origin, we get that a scalar equation is

$$5x_1 - 7x_2 + 4x_3 = 0$$

1.3.3 Determine which of the following sets are orthogonal.

(a) We have $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = -4$, so the set is not orthogonal.

(b) We have

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 0, \quad \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 0$$

Hence, the set is orthogonal.

(c) We have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

Hence, the set is orthogonal.

(d) We have

$$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{3}\\1/\sqrt{6}\end{bmatrix}\cdot\begin{bmatrix}0\\-1/\sqrt{3}\\2/\sqrt{6}\end{bmatrix}=0,\quad \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{3}\\1/\sqrt{6}\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{3}\\-1/\sqrt{6}\end{bmatrix}=0,\quad \begin{bmatrix}0\\-1/\sqrt{3}\\2/\sqrt{6}\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{3}\\-1/\sqrt{6}\end{bmatrix}=0$$

Hence, the set is orthogonal.

1.3.4 We have

$$\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}=0,\quad \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}=0,\quad \begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}=0$$

Hence, the set is orthogonal. Also,

$$\left\|\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\right\| = \sqrt{1/3+1/3+1/3}=1$$

$$\left\|\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}\right\| = \sqrt{1/2+0+1/2}=1$$

$$\left\|\begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}\right\| = \sqrt{1/6+4/6+1/6}=1$$

1.3.5 Let $\vec{x}=\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$. Then

$$\|c\vec{x}\|^2=\left\|\begin{bmatrix}cx_1\\\vdots\\cx_n\end{bmatrix}\right\|^2=(cx_1)^2+\cdots+(cx_n)^2=c^2x_1^2+\cdots+c^2x_n^2=c^2(x_1^2+\cdots+x_n^2)=c^2\|\vec{x}\|^2$$

Taking square roots of both sides gives $\|c\vec{x}\|=\sqrt{c^2}\|\vec{x}\|=|c|\,\|\vec{x}\|$.

1.3.6 Using Problem 1.3.5 we get

$$\left\|\frac{1}{\|\vec{x}\|}\vec{x}\right\|=\left|\frac{1}{\|\vec{x}\|}\right|\,\|\vec{x}\|=1$$

1.3.7 Consider $c_1\vec{v}_1+c_2\vec{v}_2=\vec{0}$. Then we have

$$0=\vec{v}_1\cdot\vec{0}=\vec{v}_1\cdot(c_1\vec{v}_1+c_2\vec{v}_2)=c_1\vec{v}_1\cdot\vec{v}_1+c_2\vec{v}_1\cdot\vec{v}_2=c_1\|\vec{v}_1\|^2+0=c_1$$

Similarly,

$$0=\vec{v}_2\cdot\vec{0}=\vec{v}_2\cdot(c_1\vec{v}_1+c_2\vec{v}_2)=c_1\vec{v}_2\cdot\vec{v}_1+c_2\vec{v}_2\cdot\vec{v}_2=0+c_2\|\vec{v}_2\|^2=c_2$$

Thus, $c_1\vec{v}_1+c_2\vec{v}_2=\vec{0}$ implies $c_1=c_2=0$, so $\{\vec{v}_1,\vec{v}_2\}$ is linearly independent.

1.3.8 We have

$$\begin{aligned}\|\vec{v}_1 + \vec{v}_2\|^2 &= (\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 + \vec{v}_2) \\ &= \vec{v}_1 \cdot \vec{v}_1 + \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_2 \cdot \vec{v}_1 + \vec{v}_2 \cdot \vec{v}_2 \\ &= \|\vec{v}_1\|^2 + 0 + 0 + \|\vec{v}_2\|^2 \\ &= \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2\end{aligned}$$

1.3.9 By definition of the cross product, we have that $\vec{n} \cdot \vec{v} = 0$ and $\vec{n} \cdot \vec{w} = 0$. If $\vec{y} \in \operatorname{Span}\{\vec{v}, \vec{w}\}$, then there exists $c_1, c_2 \in \mathbb{R}$ such that $\vec{y} = c_1\vec{v} + c_2\vec{w}$. Hence,

$$\vec{y} \cdot \vec{n} = (c_1\vec{v} + c_2\vec{w}) \cdot \vec{n} = c_1\vec{v} \cdot \vec{n} + c_2\vec{w} \cdot \vec{n} = 0 + 0 = 0$$

1.3.10 By definition, the set $\mathbb{S}$ of all vectors orthogonal to $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a subset of $\mathbb{R}^3$. Moreover, since $\vec{0} \cdot \vec{x} = 0$ we have that $\vec{0} \in \mathbb{S}$. Thus, $\mathbb{S}$ is non-empty.
Let $\vec{y}, \vec{z} \in \mathbb{S}$. Then $\vec{x} \cdot \vec{y} = 0$ and $\vec{x} \cdot \vec{z} = 0$. Thus, we have

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z} = 0 + 0 = 0$$

and

$$\vec{x} \cdot (t\vec{y}) = t(\vec{x} \cdot \vec{y}) = 0$$

for any $t \in \mathbb{R}$. Thus, $\vec{y} + \vec{z} \in \mathbb{S}$ and $t\vec{y} \in \mathbb{S}$, so $\mathbb{S}$ is a subspace of $\mathbb{R}^3$.

1.3.11 Let $\vec{x} \in \mathbb{R}^n$. Then,

$$\vec{0} \cdot \vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0x_1 + \cdots + 0x_n = 0$$

as required.

1.3.12 (a) The statement is false. If $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, then $\vec{x} \cdot \vec{y} = 0$, $\vec{y} \cdot \vec{z} = 0$, but $\vec{x} \cdot \vec{z} = 1$.

(b) The statement if false. If $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\vec{x} \cdot \vec{y} = 0$, but $\{\vec{x}, \vec{y}\}$ is linearly dependent since it contains the zero vector.

(c) The statement is true. We have

$$(s\vec{x}) \cdot (t\vec{y}) = (st)(\vec{x} \cdot \vec{y}) = 0$$

Thus, $\{s\vec{x}, t\vec{y}\}$ is an orthogonal set for any $s, t \in \mathbb{R}$.

(d) The statement is true. We have

$$\vec{z}\cdot\vec{v} = \vec{z}\cdot(s\vec{x}+t\vec{y}) = s(\vec{z}\cdot\vec{x}) + t(\vec{z}\cdot\vec{y}) = 0+0=0$$

Hence, $\vec{z}$ is orthogonal to $\vec{v} = s\vec{x}+t\vec{y}$ for any $s,t\in\mathbb{R}$.

(e) The statement is false. Take $\vec{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{y} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{z} = \begin{bmatrix}0\\1\\0\end{bmatrix}$. Then

$$\vec{x}\times(\vec{y}\times\vec{z}) = \begin{bmatrix}1\\0\\0\end{bmatrix}\times\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

But,

$$(\vec{x}\times\vec{y})\times\vec{z} = \begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

(f) The statement is true. If $\vec{x}\cdot\vec{n} = \vec{b}\cdot\vec{n}$ is a scalar equation of the plane, then multiplying both sides by $t\neq 0$ gives another scalar equation for the plane $\vec{x}\cdot(t\vec{n}) = \vec{b}\cdot(t\vec{n})$. Thus, $t\vec{n}$ is also a normal vector for the plane.

1.4 Problem Solutions

1.4.1 (a) We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-6}{13}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-12/13\\-18/13\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-3\\0\end{bmatrix} - \begin{bmatrix}-12/13\\-18/13\end{bmatrix} = \begin{bmatrix}-27/13\\18/13\end{bmatrix}$$

(b) We have

$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{9}{10}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}9/10\\27/10\end{bmatrix}$$

$$\text{perp}_{\vec{v}}\,\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}9/10\\27/10\end{bmatrix} = \begin{bmatrix}21/10\\-7/10\end{bmatrix}$$

(c) We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{0}{13}\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}6\\4\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix}$$

(d) We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{9}{13}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}27/13\\18/13\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}27/13\\18/13\end{bmatrix} = \begin{bmatrix}-14/13\\21/13\end{bmatrix}$$

(e) We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{3}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3\\2\\4\end{bmatrix} - \begin{bmatrix}3\\0\\0\end{bmatrix} = \begin{bmatrix}0\\2\\4\end{bmatrix}$$

(f) We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{9}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\3\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3\\2\\4\end{bmatrix} - \begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

(g) We have

$$\operatorname{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{8}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}4\\0\\0\\4\end{bmatrix}$$

$$\operatorname{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \operatorname{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}2\\5\\-6\\6\end{bmatrix} - \begin{bmatrix}4\\0\\0\\4\end{bmatrix} = \begin{bmatrix}-2\\5\\-6\\2\end{bmatrix}$$

(h) We have

$$\operatorname{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{8}{34}\begin{bmatrix}1\\4\\4\\-1\end{bmatrix} = \begin{bmatrix}4/17\\16/17\\16/17\\-4/17\end{bmatrix}$$

$$\operatorname{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \operatorname{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \begin{bmatrix}4/17\\16/17\\16/17\\-4/17\end{bmatrix} = \begin{bmatrix}13/17\\1/17\\1/17\\21/17\end{bmatrix}$$

(i) We have

$$\operatorname{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{4}{6}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} = \begin{bmatrix}2/3\\4/3\\0\\2/3\end{bmatrix}$$

$$\operatorname{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \operatorname{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3\\1\\1\\-1\end{bmatrix} - \begin{bmatrix}2/3\\4/3\\0\\2/3\end{bmatrix} = \begin{bmatrix}7/3\\-1/3\\1\\-5/3\end{bmatrix}$$

(j) We have

$$\operatorname{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-8}{24}\begin{bmatrix}-2\\-4\\0\\-2\end{bmatrix} = \begin{bmatrix}2/3\\4/3\\0\\2/3\end{bmatrix}$$

$$\operatorname{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \operatorname{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3\\1\\1\\-1\end{bmatrix} - \begin{bmatrix}2/3\\4/3\\0\\2/3\end{bmatrix} = \begin{bmatrix}7/3\\-1/3\\1\\-5/3\end{bmatrix}$$

1.4.2 (a) A normal vector for the plane P is $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$. Thus,

$$\operatorname{proj}_P(\vec{v}) = \operatorname{perp}_{\vec{n}}(\vec{v}) = \vec{v} - \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{0}{22}\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

This makes sense since $\vec{v} \in P$.

(b) A normal vector for the plane P is $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Thus,

$$\operatorname{proj}_P(\vec{v}) = \operatorname{perp}_{\vec{n}}(\vec{v}) = \vec{v} - \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \frac{5}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -14/9 \\ 8/9 \\ -1/9 \end{bmatrix}$$

(c) Observe that $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$. Hence, a normal vector for the plane P is $\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Thus,

$$\operatorname{proj}_P(\vec{v}) = \operatorname{perp}_{\vec{n}}(\vec{v}) = \vec{v} - \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} - \frac{3}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}$$

(d) A normal vector for the plane P is $\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$. Hence,

$$\operatorname{proj}_P(\vec{v}) = \operatorname{perp}_{\vec{n}}(\vec{v}) = \vec{v} - \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \frac{3}{18}\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ 3/2 \end{bmatrix}$$

1.4.3 A normal vector for the plane P is $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.

(a) The projection of $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto P is

$$\operatorname{proj}_P(\vec{x}) = \operatorname{perp}_{\vec{n}}(\vec{x}) = \vec{x} - \frac{\vec{x} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{0}{6}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(b) The projection of $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ onto P is

$$\operatorname{proj}_P(\vec{x}) = \operatorname{perp}_{\vec{n}}(\vec{x}) = \vec{x} - \frac{\vec{x}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{6}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(c) The projection of $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ onto P is

$$\operatorname{proj}_P(\vec{x}) = \operatorname{perp}_{\vec{n}}(\vec{x}) = \vec{x} - \frac{\vec{x}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \frac{-6}{6}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) The projection of $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ onto P is

$$\operatorname{proj}_P(\vec{x}) = \operatorname{perp}_{\vec{n}}(\vec{x}) = \vec{x} - \frac{\vec{x}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 13/6 \\ 8/3 \\ 19/6 \end{bmatrix}$$

1.4.4 (a) We have

$$\begin{aligned}
\operatorname{proj}_{\vec{v}}(\vec{x}+\vec{y}) &= \frac{(\vec{x}+\vec{y})\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} \\
&= \frac{\vec{x}\cdot\vec{v}+\vec{y}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} \\
&= \frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} + \frac{\vec{y}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} \\
&= \operatorname{proj}_{\vec{v}}(\vec{x}) + \operatorname{proj}_{\vec{v}}(\vec{y})
\end{aligned}$$

(b) We have

$$\begin{aligned}
\operatorname{proj}_{\vec{v}}(s\vec{x}) &= \frac{(s\vec{x})\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} \\
&= s\frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} \\
&= s\operatorname{proj}_{\vec{v}}(\vec{x})
\end{aligned}$$

(c) We have

$$\begin{aligned}
\operatorname{proj}_{\vec v}(\operatorname{proj}_{\vec v}(\vec x)) &= \frac{(\operatorname{proj}_{\vec v}(\vec x))\cdot\vec v}{\|\vec v\|^2}\vec v \\
&= \frac{\left(\frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v\right)\cdot\vec v}{\|\vec v\|^2}\vec v \\
&= \frac{\frac{\vec x\cdot\vec v}{\vec v\cdot\vec v}(\vec v\cdot\vec v)}{\|\vec v\|^2}\vec v \\
&= \frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v \\
&= \operatorname{proj}_{\vec v}(\vec x)
\end{aligned}$$

1.4.5 (a) The statement is true. We have

$$\operatorname{proj}_{-\vec v}(\vec x) = \frac{\vec x\cdot(-\vec v)}{\|-\vec v\|^2}(-\vec v) = \frac{(-1)(\vec x\cdot\vec v)}{\|\vec v\|^2}[(-1)\vec v] = (-1)(-1)\frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v = \operatorname{proj}_{\vec v}(\vec x)$$

(b) The statement is true. We have

$$\operatorname{proj}_{\vec v}(-\vec x) = \frac{(-\vec x)\cdot\vec v}{\|\vec v\|^2}\vec v = \frac{(-1)(\vec x\cdot\vec v)}{\|\vec v\|^2}\vec v = -\operatorname{proj}_{\vec v}(\vec x)$$

(c) The statement is false. If $\vec x$ is orthogonal to $\vec v$, then $\operatorname{proj}_{\vec v}(\vec x) = \vec 0$ and hence $\{\operatorname{proj}_{\vec v}(\vec x), \operatorname{perp}_{\vec v}(\vec x)\}$ is linearly dependent.

(d) The statement is true. We have

$$\begin{aligned}
\operatorname{proj}_{\vec v}(\operatorname{perp}_{\vec v}(\vec x)) &= \frac{\left(\vec x - \frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v\right)\cdot\vec v}{\|\vec v\|^2}\vec v \\
&= \frac{\vec x\cdot\vec v - \frac{\vec x\cdot\vec v}{\vec v\cdot\vec v}(\vec v\cdot\vec v)}{\|\vec v\|^2}\vec v \\
&= \frac{\vec x\cdot\vec v - \vec x\cdot\vec v}{\|\vec v\|^2}\vec v = \vec 0 \\
\operatorname{perp}_{\vec v}(\operatorname{proj}_{\vec v}(\vec x)) &= \frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v - \frac{\left(\frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v\right)\cdot\vec v}{\|\vec v\|^2}\vec v \\
&= \frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v - \frac{\frac{\vec x\cdot\vec v}{\vec v\cdot\vec v}(\vec v\cdot\vec v)}{\|\vec v\|^2}\vec v \\
&= \frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v - \frac{\vec x\cdot\vec v}{\|\vec v\|^2}\vec v \\
&= \vec 0 = \operatorname{proj}_{\vec v}(\operatorname{perp}_{\vec v}(\vec x))
\end{aligned}$$

1.4.6 Since P is a plane in $\mathbb{R}^3$, we must have $\{\vec v_1, \vec v_2\}$ is linearly independent. So, by Problem 1.1.11, we only need to prove that $\operatorname{proj}_{\vec n}(\vec x) \notin P$.

For a contradiction, assume that $\text{proj}_{\vec{n}}(\vec{x}) \in P$. Then, there exists $c_1, c_2 \in \mathbb{R}$ such that

$$\frac{\vec{x} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = c_1\vec{v}_1 + c_2\vec{v}_2$$

Then,

$$\vec{x} \cdot \vec{n} = \frac{\vec{x} \cdot \vec{n}}{\|\vec{n}\|^2}(\vec{n} \cdot \vec{n}) = \left(\frac{\vec{x} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n}\right) \cdot \vec{n} = (c_1\vec{v}_1 + c_2\vec{v}_2) \cdot \vec{n} = 0$$

since $\vec{n}$ is the normal vector for P and $\vec{v}_1, \vec{v}_2 \in P$. Hence, $\vec{x}$ is orthogonal to $\vec{n}$ which implies that $\vec{x} \in P$ which is a contradiction. Therefore, $\text{proj}_{\vec{n}}(\vec{x}) \notin P$ and the result follows from Problem 1.1.10.

Chapter 2 Solutions

2.1 Problem Solutions

2.1.1 (a) $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

(b) $x_1 = 1$, $x_1 = 2$, $x_2 + x_3 = 1$.

(c) $x_1 + x_2 + x_3 = 1$, $2x_1 + 2x_2 + 2x_3 = 2$, $3x_1 + 3x_2 + 3x_3 = 3$.

(d) If $\vec{x} = s\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix}$, $s \in \mathbb{R}$ is the solution, then we have

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}s+1\\2s\\s+1\end{bmatrix}$$

Thus, $x_3 = x_1 = s + 1 = \frac{1}{2}(2s) + 1 = \frac{1}{2}x_2 + 1$. Hence, we get

$$x_1 - x_3 = 0, \qquad -\frac{1}{2}x_2 + x_3 = 1, \qquad x_1 - \frac{1}{2}x_2 = 1$$

2.1.2 (a) We have

$$\begin{aligned}
6\left(\frac{3}{7}\right) + 2\left(-\frac{10}{7}\right) + 3\left(\frac{3}{7}\right) &= 1\\
2\left(\frac{3}{7}\right) + \left(-\frac{10}{7}\right) + 6\left(\frac{3}{7}\right) &= 2\\
4\left(\frac{3}{7}\right) + 5\left(-\frac{10}{7}\right) - 6\left(\frac{3}{7}\right) &= -8
\end{aligned}$$

Since the third equation is not satisfied, $\vec{x}$ is not a solution of the system.

(b) We have

$$\begin{aligned}
3(-2) + 6(-1) + 7(3) &= 9\\
-4(-2) + 3(-1) - 3(3) &= -4\\
(-2) - 13(-1) - 2(3) &= 5
\end{aligned}$$

Since the third equation is not satisfied, $\vec{x}$ is not a solution of the system.

(c) We have

$$\begin{aligned} 4 - 2(3) + (-1) - 3(-1) &= 0 \\ -4(4) + 5(3) + 5(-1) - 6(-1) &= 0 \\ 5(4) - 3(3) + 3(-1) + 8(-1) &= 0 \end{aligned}$$

Hence, $\vec{x}$ is a solution of the system.

(d) We have

$$\begin{aligned} (7 + 3t) + 2(-2 - 2t) + (t) &= 3 \\ -(7 + 3t) + (-2 - 2t) + 5t &= -9 \\ (7 + 3t) + (-2 - 2t) - t &= 5 \end{aligned}$$

Hence, $\vec{x}$ is a solution of the system.

2.1.3 By Theorem 2.1.1, we have that $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$ is a solution for any $c \in \mathbb{R}$. Thus, 3 other solutions are: $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

2.1.4 (a) TRUE. Since we can write the equation as $x_1 + 3x_2 + x_3 = 0$, it is linear.

(b) FALSE. Since it contains the square of a variable, it is not a linear equation.

(c) FALSE. The system $x_1 = 1$, $x_2 = 1$, and $x_3 = 1$ has solution $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, but clearly $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ is not a solution.

(d) FALSE. The systems $x_1 + x_2 = 0$ has more unknowns than equations, but it has a solution $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

2.1.5 (a) Since the right hand side of all the equations is 0, the system consists of hyperplanes which all pass through the origin. Since, the origin lies on all of the hyperplanes, it is a solution. Therefore, the system is consistent.

(b) The i-th equation of the system has the form

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0$$

Substituting in $\vec{x} = \vec{0}$, we get

$$a_{i1}(0) + \cdots + a_{in}(0) = 0$$

Hence, all of the equations are satisfied, so $\vec{x} = \vec{0}$ is a solution of the system. Therefore, the system is consistent.

2.1.6 (a) Observe that

$$\begin{aligned} 2(-1) + (2) + 4(1) + 2 &= 6 \\ (-1) + 1 + 2(2) &= 4 \\ (-1)(-1) + (-4)(2) - 9(1) + 10(2) &= 4 \end{aligned}$$

So, $P(-1, 2, 1, 2)$ lies on all three hyperplanes.

(b) Taking $t = s = 0$, gives $Q_1(4, -2, 0, 0)$ lies on all three hyperplanes.
Taking $t = 1$, $s = 0$, gives $Q_2(3, -4, 1, 0)$ lies on all three hyperplanes.
Taking $t = 0$, $s = 1$, gives $Q_3(2, 1, 0, 1)$ lies on all three hyperplanes.

2.2 Problem Solutions

2.2.1 1

(a) i. $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right]$

ii. Row reducing the augmented matrix we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right] R_2 - R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -1 & -1 & 1 \end{array}\right] (-1)R_2 \sim$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] R_1 - 2R_2 \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

Hence, the rank of the coefficient matrix and the rank of the augmented matrix are 2.

iii. x_3 is a free variable, so let $x_3 = t \in \mathbb{R}$. Then $x_1 = 3$ and $x_2 + x_3 = -1$, so $x_2 = -1 - t$. Therefore, all solutions have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1-t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

iv. The system is a pair of planes in $\mathbb{R}^3$ which intersect in a line passing through $(3, -1, 0)$.

(b) i. $\begin{bmatrix} 2 & 2 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \left[\begin{array}{ccc|c} 2 & 2 & -3 & 3 \\ 1 & 1 & 1 & 9 \end{array}\right]$

ii.

$$\left[\begin{array}{ccc|c} 2 & 2 & -3 & 3 \\ 1 & 1 & 1 & 9 \end{array}\right] R_1 \leftrightarrow R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 2 & -3 & 3 \end{array}\right] R_2 - 2R_1 \sim$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 0 & -5 & -15 \end{array}\right] -\tfrac{1}{5}R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right] R_1 - R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Hence, the rank of the coefficient matrix and the rank of the augmented matrix are 2.

iii. x_2 is a free variable, so let $x_2 = t \in \mathbb{R}$. Then $x_1 + x_2 = 6$, so $x_1 = 6 - t$, and $x_3 = 3$. Therefore, all solutions have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6-t \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

iv. The system is a pair of planes in $\mathbb{R}^3$ which intersect in a line passing through $(6, 0, 3)$.

(c) i. $\begin{bmatrix} 2 & 1 & -1 \\ 2 & -1 & -3 \\ 3 & 2 & -1 \end{bmatrix}, \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 2 & -1 & -3 & -1 \\ 3 & 2 & -1 & 8 \end{array}\right]$

ii.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 2 & -1 & -3 & -1 \\ 3 & 2 & -1 & 8 \end{array}\right] \begin{array}{l} \\ R_1 \leftrightarrow R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 3 & 2 & -1 & 8 \\ 2 & -1 & -3 & -1 \\ 2 & 1 & -1 & 4 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 2 & -1 & -3 & -1 \\ 2 & 1 & -1 & 4 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & -3 & -3 & -9 \\ 0 & -1 & -1 & -4 \end{array}\right] \begin{array}{l} \\ -\frac{1}{3}R_2 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -4 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 + 3R_3 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} \\ \\ (-1)R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, the rank of the coefficient matrix is 2, and the rank of the augmented matrix is 3.

iii. Since the rank of the augmented matrix is greater than the rank of the coefficient matrix, the system is inconsistent.

iv. The system is a set of three planes in $\mathbb{R}^3$ which have no common point of intersection.

(d) i. $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & -1 \\ -3 & -8 & 2 \end{bmatrix}, \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 3 & -1 & 4 \\ -3 & -8 & 2 & -11 \end{array}\right]$

ii.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 3 & -1 & 4 \\ -3 & -8 & 2 & -11 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + 3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 2 & -2 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 2R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, the rank of the coefficient matrix and the rank of the augmented matrix are 2.

iii. x_3 is a free variable, so let $x_3 = t \in \mathbb{R}$. Then $x_1 + 2x_3 = 1$ and $x_2 - x_3 = 1$, so $x_1 = 1 - 2t$, and $x_2 = 1 + t$. Therefore, all solutions have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 2t \\ 1 + t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

iv. The system is three planes in $\mathbb{R}^3$ which intersect in a line passing through $(1, 1, 0)$.

(e) i. $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & -2 \end{bmatrix}, \left[\begin{array}{ccc|c} 1 & 3 & -1 & 9 \\ 2 & 1 & -2 & -2 \end{array}\right]$

ii.

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 9 \\ 2 & 1 & -2 & -2 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & 9 \\ 0 & -5 & 0 & -20 \end{array}\right] \begin{array}{l} \\ -\frac{1}{5}R_2 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 9 \\ 0 & 1 & 0 & 4 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 0 & 4 \end{array}\right]$$

Hence, the rank of the coefficient matrix and the rank of the augmented matrix are 2.

iii. x_3 is a free variable, so let $x_3 = t \in \mathbb{R}$. Then $x_2 = 4$, and $x_1 - x_3 = -3$, so $x_1 = -3 + t$. Therefore, all solutions have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3+t \\ 4 \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

iv. The system is a pair of planes in $\mathbb{R}^3$ which intersect in a line passing through $(-3, 4, 0)$.

(f) i. $\begin{bmatrix} 2 & -2 & 2 & 1 \\ 3 & -3 & 3 & 4 \end{bmatrix}, \left[\begin{array}{cccc|c} 2 & -2 & 2 & 1 & -4 \\ 3 & -3 & 3 & 4 & 9 \end{array}\right]$

ii.

$$\left[\begin{array}{cccc|c} 2 & -2 & 2 & 1 & -4 \\ 3 & -3 & 3 & 4 & 9 \end{array}\right] \begin{array}{c} \\ R_2 - \frac{3}{2}R_1 \end{array} \sim \left[\begin{array}{cccc|c} 2 & -2 & 2 & 1 & -4 \\ 0 & 0 & 0 & 5/2 & 15 \end{array}\right] \begin{array}{c} \frac{1}{2}R_1 \\ \frac{2}{5}R_2 \end{array} \sim$$
$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1/2 & -2 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{c} R_1 - \frac{1}{2}R_2 \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Hence, the rank of the coefficient matrix and the rank of the augmented matrix are 2.

iii. x_2 and x_3 are free variables, so let $x_2 = s \in \mathbb{R}$ and $x_3 = t \in \mathbb{R}$. Then $x_4 = 6$, and $x_1 - x_2 + x_3 = -5$, so $x_1 = -5 + s - t$. Therefore, all solutions have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5+s-t \\ s \\ t \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 6 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

iv. The system is a pair of hyperplanes in $\mathbb{R}^4$ which intersect in a plane passing through $(-5, 0, 0, 6)$.

(g) i. $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -4 & 4 \\ 2 & 0 & 4 & 5 \\ -1 & 2 & -3 & 4 \end{bmatrix}, \left[\begin{array}{cccc|c} 1 & 1 & 1 & 6 & 5 \\ 0 & 2 & -4 & 4 & -4 \\ 2 & 0 & 4 & 5 & 4 \\ -1 & 2 & -3 & 4 & 9 \end{array}\right]$

ii.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 6 & 5 \\ 0 & 2 & -4 & 4 & -4 \\ 2 & 0 & 4 & 5 & 4 \\ -1 & 2 & -3 & 4 & 9 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 2R_1 \\ R_4 + R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 6 & 5 \\ 0 & 2 & -4 & 4 & -4 \\ 0 & -2 & 2 & -7 & -6 \\ 0 & 3 & -2 & 10 & 14 \end{array}\right] \begin{array}{c} \\ \frac{1}{2}R_2 \\ \\ \\ \end{array} \sim$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 6 & 5 \\ 0 & 1 & -2 & 2 & -2 \\ 0 & -2 & 2 & -7 & -6 \\ 0 & 3 & -2 & 10 & 14 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ R_3 + 2R_2 \\ R_4 - 3R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 7 \\ 0 & 1 & -2 & 2 & -2 \\ 0 & 0 & -2 & -3 & -10 \\ 0 & 0 & 4 & 4 & 20 \end{array}\right] \begin{array}{c} \\ \\ (-1)R_3 \\ \frac{1}{4}R_4 \end{array} \sim$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 7 \\ 0 & 1 & -2 & 2 & -2 \\ 0 & 0 & 2 & 3 & 10 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right] \begin{array}{c} \\ \\ \\ R_3 \leftrightarrow R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 7 \\ 0 & 1 & -2 & 2 & -2 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 3 & 10 \end{array}\right] \begin{array}{c} R_1 - 3R_3 \\ R_2 + 2R_3 \\ \\ R_4 - 2R_3 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -8 \\ 0 & 1 & 0 & 4 & 8 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} R_1 - R_4 \\ R_2 - 4R_4 \\ R_3 - R_4 \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -8 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Hence, the rank of the coefficient matrix and the rank of the augmented matrix are 4.

iii. The solution is $x_1 = -8$, $x_2 = 8$, $x_3 = 5$, $x_4 = 0$.

iv. The system is four hyperplanes in $\mathbb{R}^4$ which intersect only at the point $(-8, 8, 5, 0)$.

(h) i. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & -2 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 6 \end{array}\right]$

ii.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & -2 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 6 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & -2 \\ 0 & -3 & 0 & -3 & 6 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & -1 & 1 & -2 & 8 \end{array}\right] \quad -\tfrac{1}{3}R_2 \quad \sim$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & -2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & -1 & 1 & -2 & 8 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -1 & 6 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ \\ \\ R_4 - R_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, the rank of the coefficient matrix and the rank of the augmented matrix are 3.

iii. x_4 is a free variables, so let $x_4 = t \in \mathbb{R}$. Then $x_1 + x_4 = -4$, $x_2 + x_4 = -2$, and $x_3 - x_4 = 6$. Thus, $x_1 = -4 - t$, $x_2 = -2 - t$, and $x_3 = 6 + t$. Therefore, all solutions have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 - t \\ -2 - t \\ 6 + t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

iv. The system is four hyperplanes in $\mathbb{R}^4$ which intersect in a line passing through $(-4, -2, 6, 0)$.

(i) i. $\begin{bmatrix} 0 & 1 & -2 \\ 2 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 1 & -1 \end{bmatrix}, \left[\begin{array}{cccc|} 0 & 1 & -2 & 3 \\ 2 & 2 & 3 & 1 \\ 1 & 2 & 4 & -1 \\ 2 & 1 & -1 & 1 \end{array}\right]$

ii.

$$\begin{bmatrix} 0 & 1 & -2 & 3 \\ 2 & 2 & 3 & 1 \\ 1 & 2 & 4 & -1 \\ 2 & 1 & -1 & 1 \end{bmatrix} R_1 \leftrightarrow R_3 \sim \begin{bmatrix} 1 & 2 & 4 & -1 \\ 2 & 2 & 3 & 1 \\ 0 & 1 & -2 & 3 \\ 2 & 1 & -1 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array} \sim$$

$$\begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & -2 & -5 & 3 \\ 0 & 1 & -2 & 3 \\ 0 & -3 & -9 & 3 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_2 \leftrightarrow R_3 \end{array} \sim \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & -2 & -5 & 3 \\ 0 & -3 & -9 & 3 \end{bmatrix} \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 2R_2 \\ R_4 + 3R_2 \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & 8 & -7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -9 & 9 \\ 0 & 0 & -15 & 12 \end{bmatrix} \begin{array}{l} \\ \\ \\ -\frac{1}{9}R_3 \end{array} \sim \begin{bmatrix} 1 & 0 & 8 & -7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -15 & 12 \end{bmatrix} \begin{array}{l} R_1 - 8R_3 \\ R_2 + 2R_3 \\ \\ R_4 + 15R_3 \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{array}{l} \\ \\ \\ -\frac{1}{3}R_4 \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 + 15R_4 \\ R_2 - R_4 \\ R_3 + R_4 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, the rank of the coefficient matrix is 3, and the rank of the augmented matrix is 4.

iii. Since the rank of the augmented matrix is greater than the rank of the coefficient matrix, the system is inconsistent.

iv. The system is a set of four planes in $\mathbb{R}^3$ which have no common point of intersection.

2.2.2 (a) We need to determine if there exists $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 \\ 2c_1 - 4c_2 \\ -c_1 + 2c_2 \end{bmatrix}$$

Hence, we need to solve the system of equations

$$\begin{aligned} c_1 + 3c_2 &= 1 \\ 2c_1 - 4c_2 &= 3 \\ -c_1 + 2c_2 &= 1 \end{aligned}$$

We row reduce the corresponding augmented matrix to get

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & -4 & 3 \\ -1 & 2 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -10 & 1 \\ 0 & 5 & 2 \end{array}\right] R_2 + 2R_3 \sim \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 0 & 5 \\ 0 & 5 & 2 \end{array}\right]$$

The second row corresponds to the equation $0 = 5$. Thus, the system is inconsistent, so $\vec{x} \notin \operatorname{Span} \mathcal{B}$.

(b) We need to determine if there exists $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\2\\-1\end{bmatrix} + c_2\begin{bmatrix}3\\-4\\2\end{bmatrix} = \begin{bmatrix}c_1+3c_2\\2c_1-4c_2\\-c_1+2c_2\end{bmatrix}$$

Hence, we need to solve the system of equations

$$\begin{aligned} c_1 + 3c_2 &= 1\\ 2c_1 - 4c_2 &= 0\\ -c_1 + 2c_2 &= 0 \end{aligned}$$

We row reduce the corresponding augmented matrix to get

$$\left[\begin{array}{rr|r} 1 & 3 & 1\\ 2 & -4 & 0\\ -1 & 2 & 0 \end{array}\right] \begin{array}{c} \\ R_2-2R_1\\ R_3+R_1 \end{array} \sim \left[\begin{array}{rr|r} 1 & 3 & 1\\ 0 & -10 & -2\\ 0 & 5 & 1 \end{array}\right] -\tfrac{1}{10}R_2 \sim \left[\begin{array}{rr|r} 1 & 3 & 1\\ 0 & 1 & 1/5\\ 0 & 5 & 1 \end{array}\right] \begin{array}{c} R_1-3R_2\\ \\ R_3-5R_2 \end{array} \sim \left[\begin{array}{rr|r} 1 & 0 & 2/5\\ 0 & 1 & 1/5\\ 0 & 0 & 0 \end{array}\right]$$

Hence, $c_1 = \frac{2}{5}$ and $c_2 = \frac{1}{5}$, and so $\vec{x} \in \operatorname{Span}\mathcal{B}$. We can verify this answer by checking that

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{2}{5}\begin{bmatrix}1\\2\\-1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}3\\-4\\2\end{bmatrix}$$

(c) We need to determine if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\begin{bmatrix}1\\-3\\-7\\-9\end{bmatrix} = c_1\begin{bmatrix}3\\1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}2\\1\\3\\-4\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\-4\\3\end{bmatrix} = \begin{bmatrix}3c_1+2c_2-c_3\\c_1+c_2\\2c_1+3c_2-4c_3\\3c_1-4c_2+3c_3\end{bmatrix}$$

Hence, we need to solve the system of equations

$$\begin{aligned} 3c_1 + 2c_2 - c_3 &= 1\\ c_1 + c_2 &= -3\\ 2c_1 + 3c_2 - 4c_3 &= -7\\ 3c_1 - 4c_2 + 3c_3 &= -9 \end{aligned}$$

We row reduce the corresponding augmented matrix to get

$$\left[\begin{array}{rrr|r} 3 & 2 & -1 & 1\\ 1 & 1 & 0 & -3\\ 2 & 3 & -4 & -7\\ 3 & -4 & 3 & -9 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2\\ \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|r} 1 & 1 & 0 & -3\\ 3 & 2 & -1 & 1\\ 2 & 3 & -4 & -7\\ 3 & -4 & 3 & -9 \end{array}\right] \begin{array}{c} \\ R_2-3R_1\\ R_3-2R_1\\ R_4-3R_1 \end{array} \sim$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & -3\\ 0 & -1 & -1 & 10\\ 0 & 1 & -4 & -1\\ 0 & -7 & 3 & 0 \end{array}\right] \begin{array}{c} R_1+R_2\\ \\ R_3+R_2\\ R_4-7R_2 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 0 & -1 & 7\\ 0 & -1 & -1 & 10\\ 0 & 0 & -5 & 9\\ 0 & 0 & 10 & -70 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4+2R_3 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 0 & -1 & 7\\ 0 & -1 & -1 & 10\\ 0 & 0 & -5 & 9\\ 0 & 0 & 0 & -52 \end{array}\right]$$

The fourth row corresponds to the equation $0 = -52$. Thus, the system is inconsistent, so $\vec{x} \notin \operatorname{Span}\mathcal{B}$.

(d) We need to determine if there exists $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ -7 \\ -9 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ 3 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 5c_1 - 6c_2 + 6c_3 \\ -3c_1 + 3c_2 - 6c_3 \\ 4c_1 - 4c_2 + 8c_3 \end{bmatrix}$$

Hence, we need to solve the system of equations

$$\begin{aligned} 5c_1 - 6c_2 + 6c_3 &= 1 \\ -3c_1 + 3c_2 - 6c_3 &= -7 \\ 4c_1 - 4c_2 + 8c_3 &= -9 \end{aligned}$$

We row reduce the corresponding augmented matrix to get

$$\left[\begin{array}{ccc|c} 5 & -6 & 6 & 1 \\ -3 & 3 & -6 & -7 \\ 4 & -4 & 8 & -9 \end{array}\right] \begin{array}{c} \\ \\ R_3 + \frac{4}{3}R_2 \end{array} \sim \left[\begin{array}{ccc|c} 5 & -6 & 6 & 1 \\ -3 & 3 & -6 & -7 \\ 0 & 0 & 0 & -\frac{55}{3} \end{array}\right]$$

The third row corresponds to the equation $0 = -55/3$. Thus, the system is inconsistent, so $\vec{x} \notin \operatorname{Span} \mathcal{B}$.

2.2.3 (a) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Observe that this will give 3 equations in 4 unknowns. Hence, the rank of the matrix is at most 3, so by the System Rank Theorem, the system must have at least $4 - 3 = 1$ free variable. Therefore, the set is linearly dependent.

(b) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 + 3c_3 \\ c_2 + 3c_3 \\ -c_1 - 2c_2 - c_3 \end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ c_2 + 3c_3 &= 0 \\ -c_1 - 2c_2 - c_3 &= 0 \end{aligned}$$

To find the solution space of the system, we row reduce the corresponding coefficient matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ -1 & -2 & -1 \end{bmatrix} \begin{array}{c} \\ \\ R_3 + R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{array}{c} R_1 - 2R_2 \\ \\ \frac{1}{2}R_3 \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{c} R_1 + 3R_3 \\ R_2 - 3R_3 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the only solution to the system is $c_1 = c_2 = c_3 = 0$. Therefore, the set is linearly independent.

(c) Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}4\\5\\6\end{bmatrix} + c_3\begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}c_1 + 4c_2 + 7c_3\\2c_1 + 5c_2 + 8c_3\\3c_1 + 6c_2 + 9c_3\end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{aligned} c_1 + 4c_2 + 7c_3 &= 0\\ 2c_1 + 5c_2 + 8c_3 &= 0\\ 3c_1 + 6c_2 + 9c_3 &= 0 \end{aligned}$$

To find the solution space of the system, we row reduce the corresponding coefficient matrix.

$$\begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix} \begin{matrix}\\ R_2 - 2R_1\\ R_3 - 3R_1\end{matrix} \sim \begin{bmatrix}1 & 4 & 7\\0 & -3 & -6\\0 & -6 & -12\end{bmatrix} \begin{matrix}\\ \\ R_3 - 2R_2\end{matrix} \sim \begin{bmatrix}1 & 4 & 7\\0 & -3 & -6\\0 & 0 & 0\end{bmatrix}$$

Hence, we see that there will be a free-variable. Consequently, the homogeneous system has infinitely many solutions, and so the set is linearly dependent.

(d) Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}5\\-3\\4\end{bmatrix} + c_2\begin{bmatrix}-6\\3\\-4\end{bmatrix} + c_3\begin{bmatrix}6\\-6\\8\end{bmatrix} = \begin{bmatrix}5c_1 - 6c_2 + 6c_3\\-3c_1 + 3c_2 - 6c_3\\4c_1 - 4c_2 + 8c_3\end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{aligned} 5c_1 - 6c_2 + 6c_3 &= 0\\ -3c_1 + 3c_2 - 6c_3 &= 0\\ 4c_1 - 4c_2 + 8c_3 &= 0 \end{aligned}$$

From our work in Problem 2.2.2 (d), we know the rank of the coefficient matrix is 2. Thus, there is one free variable and so the homogeneous system has infinitely many solutions. Thus, the set is linearly dependent.

(e) Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}3\\1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\1\\3\end{bmatrix} + c_3\begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}3c_1 + 2c_2 + c_3\\c_1 + c_2 + 3c_3\\2c_1 + 3c_2 + 2c_3\end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{aligned} 3c_1 + 2c_2 + c_3 &= 0\\ c_1 + c_2 + 3c_3 &= 0\\ 2c_1 + 3c_2 + 2c_3 &= 0 \end{aligned}$$

We row reduce the corresponding coefficient matrix to get

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 3 \\ 2 & 3 & 2 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -8 \\ 0 & 1 & -4 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & -1 & -8 \\ 0 & 0 & -12 \end{bmatrix} \begin{matrix} \\ (-1)R_2 \\ -\frac{1}{12}R_3 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + 5R_3 \\ R_2 - 8R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we see the rank of the coefficient matrix is 3 and hence the solution is unique. Consequently, the set is linearly independent.

(f) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \\ 3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 4 \\ -3 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 3c_1 + 6c_2 \\ c_1 + 4c_2 + 2c_3 \\ 3c_1 - 3c_2 - c_3 \\ 5c_1 + 4c_2 + 8c_3 \end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{aligned} 3c_1 + 6c_2 &= 0 \\ c_1 + 4c_2 + 2c_3 &= 0 \\ 3c_1 - 3c_2 - c_3 &= 0 \\ 5c_1 + 4c_2 + 8c_3 &= 0 \end{aligned}$$

We row reduce the corresponding coefficient matrix to get

$$\begin{bmatrix} 3 & 6 & 0 \\ 1 & 4 & 2 \\ 3 & -3 & -1 \\ 5 & 4 & 8 \end{bmatrix} \begin{matrix} \frac{1}{3}R_1 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & 2 \\ 3 & -3 & -1 \\ 5 & 4 & 8 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 - 5R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & -9 & -1 \\ 0 & -6 & 8 \end{bmatrix} \frac{1}{2}R_2 \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -9 & -1 \\ 0 & -6 & 8 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 9R_2 \\ R_4 + 6R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & 14 \end{bmatrix} \begin{matrix} \\ \\ \\ \frac{1}{8}R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 14 \end{bmatrix} \begin{matrix} R_1 + 2R_3 \\ R_2 - R_3 \\ \\ R_4 - 14R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, we see the rank of the coefficient matrix is 3 and hence the solution is unique. Consequently, the set is linearly independent.

2.2.4 (a) Consider

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = c_1\begin{bmatrix}5\\-3\\7\end{bmatrix} + c_2\begin{bmatrix}-4\\1\\6\end{bmatrix}$$

This corresponds to a system of 3 equations in 2 unknowns. Therefore, the rank of the system is less than 3. Hence, by the System Rank Theorem, the system cannot be consistent for all $\vec{x} \in \mathbb{R}^3$. Thus, the set does not span $\mathbb{R}^3$, and so it is not a basis for $\mathbb{R}^3$.

(b) Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\-1\\3\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}6\\-2\\5\end{bmatrix}, +c_4\begin{bmatrix}2\\-1\\4\end{bmatrix}$$

Observe that this will give 3 equations in 4 unknowns. Hence, the rank of the matrix is at most 3. So, by the System Rank Theorem, the system must have at least $4-3 = 1$ free variable. Therefore, the set is linearly dependent, and so it is not a basis for $\mathbb{R}^3$.

(c) Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\3\\-2\end{bmatrix} + c_2\begin{bmatrix}2\\-5\\3\end{bmatrix} + c_3\begin{bmatrix}4\\1\\0\end{bmatrix} = \begin{bmatrix}c_1 + 2c_2 + 4c_3\\3c_1 - 5c_2 + c_3\\-2c_1 + 3c_2\end{bmatrix}$$

This gives the homogeneous system

$$\begin{aligned}c_1 + 2c_2 + 4c_3 &= 0\\3c_1 - 5c_2 + c_3 &= 0\\-2c_1 + 3c_2 &= 0\end{aligned}$$

Row reducing the coefficient matrix we get

$$\begin{bmatrix}1 & 2 & 4\\3 & -5 & 1\\-2 & 3 & 0\end{bmatrix}\begin{matrix}\\R_2 - 3R_1\\R_3 + 2R_1\end{matrix} \sim \begin{bmatrix}1 & 2 & 4\\0 & -11 & -11\\0 & 7 & 8\end{bmatrix}\begin{matrix}\\-\frac{1}{11}R_2\\\\\end{matrix} \sim \begin{bmatrix}1 & 2 & 4\\0 & 1 & 1\\0 & 7 & 8\end{bmatrix}\begin{matrix}R_1 - 2R_2\\\\R_3 - 7R_2\end{matrix} \sim$$

$$\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\begin{matrix}R_1 - 2R_3\\R_2 - R_3\\\\\end{matrix} \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Hence, the rank of the coefficient matrix is 3. Therefore, the only solution to the homogeneous system is $c_1 = c_2 = c_3 = 0$, so the set is linearly independent. Moreover, since the rank equals the number of rows, by the System Rank Theorem (3), the system $\begin{bmatrix}A \mid \vec{b}\end{bmatrix}$ is consistent for every $\vec{b} \in \mathbb{R}^3$, so the set also spans $\mathbb{R}^3$. Therefore, it is a basis for $\mathbb{R}^3$.

(d) Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\-3\end{bmatrix} = \begin{bmatrix}c_1 + 2c_2 + c_3\\c_1 + c_2\\c_1 - c_2 - 3c_3\end{bmatrix}$$

This gives the homogeneous system

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_1 - c_2 - 3c_3 &= 0 \end{aligned}$$

Row reducing the coefficient matrix we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & -3 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -3 & -4 \end{bmatrix} \begin{matrix} \\ (-1)R_2 \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ (-1)R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ R_2 - R_3 \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the rank of the coefficient matrix is 3. Therefore, the only solution to the homogeneous system is $c_1 = c_2 = c_3 = 0$, so the set is linearly independent. Moreover, since the rank equals the number of rows, by the System Rank Theorem (3), the system $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^3$, so the set also spans $\mathbb{R}^3$. Therefore, it is a basis for $\mathbb{R}^3$.

(e) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_2 + 3c_3 \\ 3c_2 + 6c_3 \\ -4c_1 - 5c_2 + 2c_3 \end{bmatrix}$$

This gives the homogeneous system

$$\begin{aligned} c_1 + 3c_2 + 3c_3 &= 0 \\ 3c_2 + 6c_3 &= 0 \\ -4c_1 - 5c_2 + 2c_3 &= 0 \end{aligned}$$

Row reducing the coefficient matrix we get

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 3 & 6 \\ -4 & -5 & 2 \end{bmatrix} \begin{matrix} \\ \frac{1}{3}R_2 \\ R_3 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 7 & 14 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ \\ R_3 - 7R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the rank of the coefficient matrix is 2. Therefore, the homogeneous system has infinitely many solution and so the set is linearly dependent. Consequently, it is not a basis.

(f) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3c_1 + 4c_2 + 2c_3 \\ 2c_1 + 4c_2 + c_3 \\ -3c_1 - 2c_3 \end{bmatrix}$$

This gives the homogeneous system

$$\begin{aligned} 3c_1 + 4c_2 + 2c_3 &= 0 \\ 2c_1 + 4c_2 + c_3 &= 0 \\ -3c_1 - 2c_3 &= 0 \end{aligned}$$

Row reducing the coefficient matrix we get

$$\begin{bmatrix} 3 & 4 & 2 \\ 2 & 4 & 1 \\ -3 & 0 & -2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ -3 & 0 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ R_2 + R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \frac{1}{4}R_2 \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the rank of the coefficient matrix is 3. Therefore, the only solution to the homogeneous system is $c_1 = c_2 = c_3 = 0$, so the set is linearly independent. Moreover, since the rank equals the number of rows, by the System Rank Theorem (3), the system $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^3$, so the set also spans $\mathbb{R}^3$. Therefore, it is a basis for $\mathbb{R}^3$.

2.2.5 Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a set of n-vectors in $\mathbb{R}^n$ and let $A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$.

Assume that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent. Then, the system $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ has a unique solution. But, this system can be represented by $\left[A \mid \vec{0}\right]$. Thus, by the System Rank Theorem (2), $\operatorname{rank}(A) = n$. Hence, $\left[A \mid \vec{b}\right]$ is consistent for all $\vec{b} \in \mathbb{R}^n$ by Theorem 2.2.5 (3). Therefore, for every $\vec{b} \in \mathbb{R}^n$ there exists a $\vec{x} \in \mathbb{R}^n$ such that $\vec{b} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$. Consequently, $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \mathbb{R}^n$.

On the other hand, if $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \mathbb{R}^n$, then $\left[A \mid \vec{b}\right]$ is consistent for all $\vec{b} \in \mathbb{R}^n$, so $\operatorname{rank}(A) = n$ by Theorem 2.2.5 (3). Thus, $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent by the System Rank Theorem (2).

2.2.6 Assume that $\mathcal{B} = \left\{\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$. Then, $\mathcal{B}$ must be linear independent. Hence, the only solution to

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} a \\ c \end{bmatrix} + t_2 \begin{bmatrix} b \\ d \end{bmatrix}$$

is $t_1 = t_2 = 0$. Thus, the coefficient matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of the corresponding system must have rank 2. So, we cannot have $a = 0 = c$. Assume that $a \neq 0$. Row reducing we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} \\ R_2 - \frac{c}{a}R_1 \end{matrix} \sim \begin{bmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{bmatrix} \begin{matrix} \\ aR_2 \end{matrix} \sim \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

Since we need rank $A = 2$, we must have $ad - bc \neq 0$ as required. The case of $c \neq 0$ is similar.

On the other hand, if $ad - bc \neq 0$, then at least one of a and c are non-zero. Hence, we can row reduce as in the first case to find that the rank of the coefficient matrix is 2. Then, the System Rank Theorem implies that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

2.2.7 Since there is no point in common between the three planes, we know that $\operatorname{rank} A < \operatorname{rank}[A \mid \vec{b}]$. But the planes are not parallel, so $\operatorname{rank} A > 1$, and there are only three rows of $[A \mid \vec{b}]$, so $\operatorname{rank}[A \mid \vec{b}] \leq 3$. It follows that $\operatorname{rank} A = 2$.

2.2.8 The intersection of two planes in $\mathbb{R}^3$ is represented by a system of 2 linear equations in 3 unknowns. This will correspond to a system whose coefficient matrix has 2 rows and 3 columns. Hence, the rank of the coefficient matrix of the system is either 1 or 2. Since we are given that the system is consistent, Theorem 2.2.5(2) tells us that the solution set has either 1 or 2 free variables. Theorem 2.2.6 tells us that the vectors corresponding to the free variables must be linearly independent. Thus, the solution set is either a line or a plane in $\mathbb{R}^3$.

2.2.9 (a) The statement is false. The solution set of $x_1 + x_2 = 1$ is not a subspace of $\mathbb{R}^2$, since it does not contain the zero vector.

(b) The statement is false. The system $x_1 + x_2 + x_3 + x_4 + x_5 = 1$, $x_1 + x_2 + x_3 + x_4 + x_5 = 2$, $x_1 + x_2 + x_3 + x_4 + x_5 = 3$, does not have infinitely many solutions. It is not consistent.

(c) The statement is false. The system $x_1 + x_2 + x_3 = 1$, $2x_1 + 2x_2 + 2x_3 = 2$, $3x_1 + 3x_2 + 3x_3 = 3$, $4x_1 + 4x_2 + 4x_3 = 4$, $5x_1 + 5x_2 + 5x_3 = 5$ has infinitely many solutions.

(d) The statement is true. We know that a homogeneous system is always consistent and the maximum rank is the minimum of the number of equation and the number of variables. Hence, the rank of the coefficient matrix is at most 3. Therefore, there are at least $5 - 3 = 2$ free variables in the system. Thus, there must be infinitely many solutions.

(e) The statement is false. The system $x_1 + x_2 + x_3 + x_4 = 1$, $x_1 + x_2 + x_3 + x_4 = 2$, $x_2 + x_3 + x_4 = 0$, has rank 2, but it is inconsistent, so the solution set is not a plane.

(f) The statement is true. Since the rank equals the number of equations, the system must be consistent by Theorem 2.2.5. Then, by Theorem 2.2.6, a vector equation for the solution set must have the form

$$\vec{x} = \vec{c} + t_1\vec{v}_1 + t_2\vec{v}_2, \quad t_1, t_2 \in \mathbb{R}$$

where $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set in $\mathbb{R}^5$. Therefore, the solution set is a plane in $\mathbb{R}^5$.

(g) The statement is true. If the solution space is a line, then it must have the form $\vec{x} = t\vec{v}_1$, $t \in \mathbb{R}$, where $\vec{v}_1 \neq \vec{0}$. Therefore, the system must have had 1 free variable. Hence, by Theorem 2.2.5, we get the rank of the coefficient matrix equals the number of variables - the number of free variables. Hence, the rank is $3 - 2 = 1$.

(h) The statement is false. Consider the system $x_1 + x_2 + x_3 = 1$, $2x_1 + 2x_2 + 2x_3 = 2$. The system has $m = 2$ equations and $n = 3$ variables, but the rank of the corresponding coefficient matrix A is $1 \neq m$.

(i) If $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^m$, then $\operatorname{rank} A = m$ by Theorem 2.2.5(3). Since the solution is unique, we have that $\operatorname{rank} A = n$ by Theorem 2.2.5(2). Thus, $m = \operatorname{rank} A = n$.

2.2.10 Assume that $\mathcal{B}$ is linearly independent. Then, the only solution to the system $\left[\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_k\end{bmatrix} \mid \vec{0}\right]$ is the trivial solution. Hence, by Theorem 2.2.5, the rank of the coefficient matrix must equal the number of columns which is k.

On the other hand, if the rank of $\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_k\end{bmatrix}$ is k, then $\left[\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_k\end{bmatrix} \mid \vec{0}\right]$ has a unique solution by Theorem 2.2.5. Thus, $\mathcal{B}$ is linearly independent.

2.2.11 (a) We have that $\{\vec{v}_1, \dots, \vec{v}_k\}$ spans $\mathbb{R}^n$ if and only if the equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^n$. The equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^n$ if and only if the corresponding system of n equation in k unknowns $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^n$. Finally, by Theorem 2.2.5(3) the corresponding system of n equation in k unknowns $\left[A \mid \vec{b}\right]$ is consistent for every $\vec{b} \in \mathbb{R}^n$ if and only if $\operatorname{rank} A = n$.

(b) If $k < n$, then the coefficient matrix A of the system $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{b}$ must have $\operatorname{rank} A < n$ since the rank is the number of leading ones and there cannot be more leading ones than rows. But then, we have $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{R}^n$ and $\operatorname{rank} A < n$ which contradicts part (a). Hence, $k \geq n$.

2.2.12 (a) Since $\vec{u}_i \in \mathbb{S}$ and $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_\ell\} = \mathbb{S}$, we can write each $\vec{u}_i$ as a linear combination of $\vec{v}_1, \dots, \vec{v}_\ell$. Say,

$$\begin{aligned}
\vec{u}_1 &= a_{11}\vec{v}_1 + \cdots + a_{1\ell}\vec{v}_\ell \\
&\vdots \\
\vec{u}_k &= a_{k1}\vec{v}_1 + \cdots + a_{k\ell}\vec{v}_\ell
\end{aligned}$$

Consider

$$c_1\vec{u}_1 + \cdots + c_k\vec{u}_k = \vec{0}$$

Substituting, we get

$$\begin{aligned}
\vec{0} &= c_1(a_{11}\vec{v}_1 + \cdots + a_{1\ell}\vec{v}_\ell) + \cdots + c_k(a_{k1}\vec{v}_1 + \cdots + a_{k\ell}\vec{v}_\ell) \\
0\vec{v}_1 + \cdots + 0\vec{v}_\ell &= (c_1a_{11} + \cdots + c_ka_{k1})\vec{v}_1 + \cdots + (c_1a_{1\ell} + \cdots + c_ka_{k\ell})\vec{v}_\ell
\end{aligned}$$

Comparing coefficients of $\vec{v}_i$ on both sides we get the system of equations

$$\begin{aligned}
a_{11}c_1 + \cdots + a_{k1}c_k &= 0 \\
&\vdots \\
a_{1\ell}c_1 + \cdots + a_{k\ell}c_k &= 0
\end{aligned}$$

This homogeneous system of ℓ equations in k unknowns $(c_1, \dots, c_k)$ must have a unique solution as otherwise, we would contradict the fact that $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent. By Theorem 2.2.5 (2), the rank of the matrix must equal the number of variables k. Therefore, $\ell \geq k$ as otherwise there would not be enough rows to contain the k leadings ones in the RREF of the coefficient matrix.

(b) If $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ is a basis, then $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_\ell\} = \mathbb{S}$. If $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a basis, then it is linearly independent. Hence, by part (a), $k \leq \ell$. Similarly, we also have that $\{\vec{u}_1, \dots, \vec{u}_k\}$ spans $\mathbb{S}$ and $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ is linearly independent, so $\ell \leq k$. Therefore, $k = \ell$.

2.2.13 The i-th equation of the system $\left[A \mid \vec{b}\right]$ has the form

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i$$

If $\vec{y}$ is a solution of $\left[A \mid \vec{0}\right]$, then we have

$$a_{i1}y_1 + \cdots + a_{in}y_n = 0$$

If $\vec{z}$ is a solution of $\left[A \mid \vec{b}\right]$, then we have

$$a_{i1}z_1 + \cdots + a_{in}z_n = b_i$$

So, if we take $\vec{x} = \vec{z} + c\vec{y}$ we get

$$\begin{aligned} a_{i1}(z_1 + cy_1) + \cdots + a_{in}(z_n + cy_n) &= a_{i1}z_1 + \cdots + a_{in}z_n + c(a_{i1}y_1 + \cdots + a_{in}y_n) \\ &= b_i + c(0) \\ &= b_i \end{aligned}$$

Thus, $\vec{x} = \vec{z} + c\vec{y}$ is a solution for each $c \in \mathbb{R}$.

2.2.14 (a) We have

$$\begin{aligned} f_1 + f_2 &= 50 \\ f_2 + f_3 + f_5 &= 60 \\ f_4 + f_5 &= 50 \\ f_1 - f_3 + f_4 &= 40 \end{aligned}$$

(b) We row reduce the augmented matrix corresponding to the system in part (a).

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 50 \\ 1 & 0 & -1 & 1 & 0 & 40 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 50 \\ 0 & -1 & -1 & 1 & 0 & -10 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ \\ R_4 + R_2 \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & -10 \\ 0 & 1 & 1 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 1 & 1 & 50 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & -10 \\ 0 & 1 & 1 & 0 & 1 & 60 \\ 0 & 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Rewriting this into equation form gives

$$\begin{aligned} f_1 - f_3 - f_5 &= -10 \\ f_2 + f_3 + f_5 &= 60 \\ f_4 + f_5 &= 50 \end{aligned}$$

We see that f_3 and f_5 are free variables. Thus, we get that the general solution is

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} -10 + f_3 + f_5 \\ 60 - f_3 - f_5 \\ f_3 \\ 50 - f_5 \\ f_5 \end{bmatrix} = \begin{bmatrix} -10 \\ 60 \\ 0 \\ 50 \\ 0 \end{bmatrix} + f_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + f_5 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad f_3, f_5 \in \mathbb{R}$$

(c) For f_1 to be non-negative, we require that $f_3 + f_5 \geq 10$. For f_2 to be non-negative, we require $f_3 + f_5 \leq 60$. For f_4 to be non-negative we require $f_5 \leq 50$.

Chapter 3 Solutions

3.1 Problem Solutions

3.1.1 (a) $2\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -6 & 10 \end{bmatrix}$

(b) $\frac{1}{3}\begin{bmatrix} 1 & 3 \\ 1 & -6 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 4 & 8 \\ 1/2 & 3 \end{bmatrix} = \begin{bmatrix} -2/3 & -1 \\ 5/24 & -11/4 \end{bmatrix}$

(c) $\frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} + \frac{1}{\sqrt{3}}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} \sqrt{2} + 2\sqrt{3} & 2\sqrt{2} \\ 0 & \sqrt{2} + 2\sqrt{3} \end{bmatrix}$

(d) $-2\begin{bmatrix} 3 & 3 \\ 0 & -1 \end{bmatrix} - 3\begin{bmatrix} -2 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$

(e) $x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$

(f) $0\begin{bmatrix} -4 & 3 \\ 3 & 5 \end{bmatrix} + 0\begin{bmatrix} 9 & -8 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3.1.2 (a) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 4 & -3 \end{bmatrix}$

(b) The product doesn't exist since the number of rows of the second matrix does not equal the number of columns of the first matrix.

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \\ 8 & 9 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 11 & 8 \\ 7 & 4 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & -1 \\ 0 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 8 & 2 & -2 \\ 15 & 4 & -2 \end{bmatrix}$

(g) The product doesn't exist since the number of rows of the second matrix does not equal the number of columns of the first matrix.

(h) $\begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ 2 & 7 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 17 \\ 11 & -2 \end{bmatrix}$

(i) $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 4$

(j) $\begin{bmatrix} 5 & -3 \\ 2 & 7 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 7 & -3 \\ 12 & 11 & 7 \\ -5 & -5 & -3 \end{bmatrix}$

(k) $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$

(l) $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 5x_3$

3.1.3 (a) We have

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n}(A+B)_{ii} = \sum_{i=1}^{n}[(A)_{ii} + (B)_{ii}] = \sum_{i=1}^{n}(A)_{ii} + \sum_{i=1}^{n}(B)_{ii} = \operatorname{tr} A + \operatorname{tr} B$$

(b) We have

$$\operatorname{tr}(A^T B) = \sum_{i=1}^{n}(A^T B)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n}(A^T)_{ij}(B)_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n}(A)_{ji}(B)_{ji}$$

3.1.4 We take $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

3.1.5 Simplifying the left side we get

$$\begin{bmatrix} 3c_1 + 2c_2 + c_3 & c_1 + c_2 + c_3 \\ 3c_1 + 2c_2 + c_3 & c_1 - c_2 - 2c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Comparing entires we get the homogeneous system of linear equations

$$\begin{aligned} 3c_1 + 2c_2 + c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \\ c_1 - c_2 - 2c_3 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the only solution is $c_1 = c_2 = c_3 = 0$.

3.1.6 Simplifying the left side we get

$$\begin{bmatrix} t_1 + t_2 + 2t_3 & 2t_1 + t_2 + 3t_3 \\ -2t_1 + t_3 & -t_1 - t_2 + t_3 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & -2 \end{bmatrix}$$

Comparing entires we get the homogeneous system of linear equations

$$\begin{aligned} t_1 + t_2 + 2t_3 &= -1 \\ 2t_1 + t_2 + 3t_3 &= -4 \\ -2t_1 + t_3 &= 3 \\ -t_1 - t_2 + t_3 &= -2 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & -4 \\ -2 & 0 & 1 & 3 \\ -1 & -1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, $t_1 = -2$, $t_2 = 3$, and $t_3 = -1$.

3.1.7 (a) The statement is false. If $A\vec{x}$ is defined, then $\vec{x}$ must have the same number of entries as the number of columns of A. Therefore, we would need $\vec{x} \in \mathbb{R}^2$.

(b) The statement is false. $A\vec{x}$ gives a linear combination of the columns of A. Since A has 2 rows, the columns of A are in $\mathbb{R}^2$. Thus, $A\vec{x} \in \mathbb{R}^2$.

(c) The statement is false. Take $B = I_3$. Then, by Theorem 3.1.5 we have $AB = BA$.

(d) The statement is false. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(e) The statement is false. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = B$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but neither A nor B is the zero matrix.

3.1.8 We have $(A + B)_{ij} = (A)_{ij} + (B)_{ij} = (B)_{ij} + (A)_{ij} = (B + A)_{ij}$.

3.1.9 Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$. We have

$$AI_n = A\begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} A\vec{e}_1 & \cdots & A\vec{e}_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} = A$$

Using this result for the $n \times m$ matrix A^T, we get

$$A^T = A^T I_m = A^T I_m^T = (I_m A)^T$$

Taking transposes of both sides gives $A = I_m A$.

3.1.10 By definition of matrix-vector multiplication we get that

$$A\vec{e}_i = 0\vec{a}_1 + \cdots + 0\vec{a}_{i-1} + 1\vec{a}_i + 0\vec{a}_{i+1} + \cdots + 0\vec{a}_n = \vec{a}_i$$

as required.

3.1.11 (a) Since A has a row of zeros, we can write A in block form as $A = \begin{bmatrix} A_1 \\ \vec{0}^T \\ A_2 \end{bmatrix}$. Thus,

$$AB = \begin{bmatrix} A_1 \\ \vec{0}^T \\ A_2 \end{bmatrix} B = \begin{bmatrix} A_1 B \\ \vec{0}^T B \\ A_2 B \end{bmatrix} = \begin{bmatrix} A_1 B \\ \vec{0}^T \\ A_2 B \end{bmatrix}$$

Hence, AB has a row of zeros.

(b) B could be the zero matrix.

3.1.12 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + bd & bc + d^2 \end{bmatrix}$$

Comparing entries gives

$$a^2 + bc = 1 \qquad (3.3)$$
$$ab + bd = 0 \qquad (3.4)$$
$$ac + cd = 0 \qquad (3.5)$$
$$bc + d^2 = 1 \qquad (3.6)$$

(1)-(4) gives $a^2 - d^2 = 0$, so $a = \pm d$. If $a = -d$, then all equations are satisfied when $a^2 + bc = 1$.

If $a = d$, then (2) and (3) give $2ab = 0$ and $2ac = 0$. If $a = 0$, then $a = -d$ and we have the case above.

If $a \neq 0$, then $b = c = 0$. So, (1) and (4) give $a^2 = 1$ and $d^2 = 1$. So, we have matrices of the form $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$.

3.1.13 (a) $\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$.

(b) We have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} 1000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 1300 \\ 1700 \end{bmatrix}$$
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} 1300 \\ 1700 \end{bmatrix} = \begin{bmatrix} 1510 \\ 1490 \end{bmatrix}$$
$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} 1510 \\ 1490 \end{bmatrix} = \begin{bmatrix} 1657 \\ 1343 \end{bmatrix}$$

The population will always stay at 3000 since we are not losing or gaining population. In particular, each year-to-year we will have $9/10 + 1/10$ of the population of X and $1/5 + 4/5$ of the population of Y.

(c) Observe that for any t, we have

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 9/10 & 1/5 \\ 1/10 & 4/5 \end{bmatrix} \begin{bmatrix} 2000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \end{bmatrix}$$

Therefore, by Induction, the population will always be $x_t = 2000$, $y_t = 1000$.

3.1.14 (a) Let x_t be the number of RealWest's customers, let y_t be the number of Newservices customers at month t. We get

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

(b) We have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.34 \\ 0.66 \end{bmatrix}$$

(c) Observe that

$$\begin{bmatrix} x_6 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}^6 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Using a computer to calculate $\begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}^6$ we get

$$\begin{bmatrix} x_6 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}^6 \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \approx \begin{bmatrix} 0.26 \\ 0.74 \end{bmatrix}$$

3.1.15 Let x_t be the number of cars at the airport, let y_t be the number of cars at the train station, and let z_t be the number of cars in the city centre. Then, we get

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 8/10 & 3/10 & 3/10 \\ 1/10 & 6/10 & 1/10 \\ 1/10 & 1/10 & 6/10 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

3.1.16 (a) Observe that

$$x_{n+2} = y_{n+1} = x_n + y_n = x_n + x_{n+1}$$

Thus, x_n is the Fibonacci sequence. Also, we have that

$$y_{n+2} = x_{n+1} + y_{n+1} = y_n + y_{n+1}$$

So, y_n is also the Fibonacci sequence.
We see that

$$x_0 = 1, x_1 = y_0 = 1, x_2 = 2, x_3 = 3, \ldots$$

while

$$y_0 = 1, y_1 = x_0 + y_0 = 2, y_2 = 3, \ldots$$

So, y_n is one step ahead of x_n.

(b) NOTE: by Problem XYZ in section WXY it means Problem 6.4.11 We have

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

3.2 Problem Solutions

3.2.1 (a) Since A has 2 columns for $A\vec{x}$ to be defined we must have $\vec{x} \in \mathbb{R}^2$. Thus, the domain of L is $\mathbb{R}^2$. Moreover, since A has 4 rows, we get that $A\vec{x} \in \mathbb{R}^4$. Thus, the codomain of L is $\mathbb{R}^4$.

(b) We have

$$L(2,-5) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -19 \\ 6 \\ -23 \\ 38 \end{bmatrix}$$

$$L(-3,4) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ -9 \\ 17 \\ -36 \end{bmatrix}$$

(c) We have

$$L(1,0) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

$$L(0,1) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -6 \end{bmatrix}$$

Thus, $[L] = A$.

3.2.2 (a) Since B has 4 columns for $B\vec{x}$ to be defined we must have $\vec{x} \in \mathbb{R}^4$. Thus, the domain of f is $\mathbb{R}^4$. Moreover, since B has 3 rows, we get that $B\vec{x} \in \mathbb{R}^3$. Thus, the codomain of f is $\mathbb{R}^3$.

(b) We have

$$f(2,-2,3,1) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \\ 7 \end{bmatrix}$$

$$f(-3,1,4,2) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -13 \\ -1 \\ 3 \end{bmatrix}$$

(c) We have

$$f(\vec{e}_1) = B\vec{e}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$f(\vec{e}_2) = B\vec{e}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$f(\vec{e}_3) = B\vec{e}_3 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$$f(\vec{e}_4) = B\vec{e}_4 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

Thus, $[f] = B$.

3.2.3 (a) Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$. Then,

$$\begin{aligned} L(s\vec{x} + t\vec{y}) &= L(sx_1 + ty_1, sx_2 + ty_2, sx_3 + ty_3) = ([sx_1 + ty_1] + [sx_2 + ty_2], 0) \\ &= s(x_1 + x_2, 0) + t(y_1 + y_2, 0) \\ &= sL(\vec{x}) + tL(\vec{y}) \end{aligned}$$

Hence, L is linear. We have

$$L(1,0,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad L(0,1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad L(0,0,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence,

$$[L] = \begin{bmatrix} L(1,0,0) & L(0,1,0) & L(0,0,1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$. Then,

$$\begin{aligned} L(s\vec{x} + t\vec{y}) &= L(sx_1 + ty_1, sx_2 + ty_2, sx_3 + ty_3) = ([sx_1 + ty_1] - [sx_2 + ty_2], [sx_2 + ty_2] + [sx_3 + ty_3]) \\ &= s(x_1 - x_2, x_2 + x_3) + t(y_1 - y_2, y_2 + y_3) \\ &= sL(\vec{x}) + tL(\vec{y}) \end{aligned}$$

Hence, L is linear. We have

$$L(1,0,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad L(0,1,0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad L(0,0,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence,

$$[L] = \begin{bmatrix} L(1,0,0) & L(0,1,0) & L(0,0,1) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(c) Observe that $L(0,0,0) = (1,0,0)$ and $L(1,0,0) = (1,1,0)$, so $L(0,0,0) + L(1,1,0) = (2,1,0)$. But,

$$L[(0,0,0) + (1,0,0)] = L(1,0,0) = (1,1,0) \neq L(0,0,0) + L(1,0,0)$$

So, L does not preserve addition, so it is not linear.

(d) Observe that $L(1,1) = (1,1)$, so $2L(1,1) = (2,2)$. But, $L[2(1,1)] = L(2,2) = (4,4) \neq 2L(1,1)$. Hence, L does not preserve scalar multiplication, so L is not linear.

(e) Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$. Then,

$$\begin{aligned} L(s\vec{x} + t\vec{y}) &= L(sx_1 + ty_1, sx_2 + ty_2, sx_3 + ty_3) = (0,0,0) \\ &= s(0,0,0) + t(0,0,0) = sL(\vec{x}) + tL(\vec{y}) \end{aligned}$$

Hence, L is linear. We have

$$L(1,0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad L(0,1,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad L(0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence,

$$[L] = \begin{bmatrix} L(1,0,0) & L(0,1,0) & L(0,0,1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(f) Let $\vec{x}, \vec{y} \in \mathbb{R}^4$ and $s, t \in \mathbb{R}$. Then,

$$\begin{aligned} L(s\vec{x} + t\vec{y}) &= L(sx_1 + ty_1, sx_2 + ty_2, sx_3 + ty_3, sx_4 + ty_4) \\ &= ([sx_4 + ty_4] - [sx_1 + ty_1], 2[sx_2 + ty_2] + 3[sx_3 + ty_3]) \\ &= s(x_4 - x_1, 2x_2 + 3x_3) + t(y_4 - y_1, 2y_2 + 3y_3) \\ &= sL(\vec{x}) + tL(\vec{y}) \end{aligned}$$

Hence, L is linear. We have

$$L(1,0,0,0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad L(0,1,0,0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad L(0,0,1,0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad L(0,0,0,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence,

$$[L] = \begin{bmatrix} L(1,0,0,0) & L(0,1,0,0) & L(0,0,1,0) & L(0,0,0,1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

3.2.4 Let $\vec{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

(a) We have

$$\text{proj}_{\vec{a}}(\vec{e}_1) = \frac{\vec{e}_1 \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = \frac{2}{8}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$
$$\text{proj}_{\vec{a}}(\vec{e}_2) = \frac{\vec{e}_2 \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = \frac{2}{8}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Hence,

$$[\text{proj}_{\vec{a}}] = \begin{bmatrix} \text{proj}_{\vec{a}}(\vec{e}_1) & \text{proj}_{\vec{a}}(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

(b) We have

$$\text{proj}_{\vec{a}}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(c) We have

$$\text{proj}_{\vec{a}}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \frac{4}{8}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3.2.5 We have

$$\text{refl}_P(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{6}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7/11 \\ -6/11 \\ 6/11 \end{bmatrix}$$
$$\text{refl}_P(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6/11 \\ 9/11 \\ 2/11 \end{bmatrix}$$
$$\text{refl}_P(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{2}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6/11 \\ 2/11 \\ 9/11 \end{bmatrix}$$

Hence,

$$[\text{refl}_P] = \begin{bmatrix} -7/11 & -6/11 & 6/11 \\ -6/11 & 9/11 & 2/11 \\ 6/11 & 2/11 & 9/11 \end{bmatrix}$$

3.2.6 Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$. Then

$$\begin{aligned}
\operatorname{perp}_{\vec{d}}(s\vec{x} + t\vec{y}) &= (s\vec{x} + t\vec{y}) - \operatorname{proj}_{\vec{d}}(s\vec{x} + t\vec{y}) \\
&= s\vec{x} + t\vec{y} - [s \operatorname{proj}_{\vec{d}}(\vec{x}) + t \operatorname{proj}_{\vec{d}}(\vec{y})] \qquad \text{since } \operatorname{proj}_{\vec{d}} \text{ is linear} \\
&= s[\vec{x} - \operatorname{proj}_{\vec{d}}(\vec{x})] + t[\vec{y} - \operatorname{proj}_{\vec{d}}(\vec{y})] \\
&= s \operatorname{perp}_{\vec{d}}(\vec{x}) + t \operatorname{perp}_{\vec{d}}(\vec{y})
\end{aligned}$$

Therefore, $\operatorname{perp}_{\vec{d}}$ is a linear mapping.

3.2.7 Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$. Then

$$\begin{aligned}
\operatorname{DOT}_{\vec{v}}(s\vec{x} + t\vec{y}) &= (s\vec{x} + t\vec{y}) \cdot \vec{v} \\
&= s(\vec{x} \cdot \vec{v}) + t(\vec{y} \cdot \vec{v}) \\
&= s \operatorname{DOT}_{\vec{v}}(\vec{x}) + t \operatorname{DOT}_{\vec{v}}(\vec{y})
\end{aligned}$$

Therefore, $\operatorname{DOT}_{\vec{v}}$ is a linear mapping.

3.2.8 $L(\vec{0}) = L(0\vec{x}) = 0L(\vec{x}) = \vec{0}$.

3.2.9 (a) Since L has 4 columns, the domain of L is $\mathbb{R}^4$.

(b) Since L has 2 rows, the codomain of L is $\mathbb{R}^2$.

(c) We have

$$L(\vec{x}) = [L]\vec{x} = \begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 2x_3 + x_4 \\ -x_1 + 3x_2 + 5x_4 \end{bmatrix}$$

3.2.10 Consider the rotation $R_{\pi/3} : \mathbb{R}^2 \to \mathbb{R}^2$.

(a) We have $[R_{\pi/3}] = \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

(b) We have

$$R_{\pi/3}(1/\sqrt{2}, 1/\sqrt{2}) = [R_{\pi/3}] \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} (1-\sqrt{3})/2\sqrt{2} \\ (\sqrt{3}+1)/2\sqrt{2} \end{bmatrix}$$

3.2.11 (a) The standard matrix of a such a linear mapping would be $[L] = \begin{bmatrix} L(1,0) & L(0,1) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -5 \\ 4 & 9 \end{bmatrix}$.

Hence, a linear mapping would be

$$L(\vec{x}) = [L]\vec{x} = \begin{bmatrix} 3x_1 + x_2 \\ -x_1 - 5x_2 \\ 4x_1 + 9x_2 \end{bmatrix}$$

(b) We could take $L(x_1,x_2) = \begin{bmatrix} 3x_1^2 + x_2 \\ -x_1 - 5x_2 \\ 4x_1 + 9x_2 \end{bmatrix}$. Then, $L(1,0) = (3,-1,4)$, $L(0,1) = (1,-5,9)$, but L is not linear since $L(2,0) = (12,-2,8) \neq 2L(1,0)$.

3.2.12 If L is linear, then we would get

$$L(1,0) = L\left(\frac{1}{2}\left[(1,1)+(1,-1)\right]\right) = \frac{1}{2}L(1,1) + \frac{1}{2}L(1,-1) = \frac{1}{2}(2,3) + \frac{1}{2}(3,1) = (5/2,2)$$

and

$$L(0,1) = L\left(\frac{1}{2}\left[(1,1)-(1,-1)\right]\right) = \frac{1}{2}L(1,1) - \frac{1}{2}L(1,-1) = \frac{1}{2}(2,3) - \frac{1}{2}(3,1) = (-1/2,1)$$

Hence, we can define the linear mapping by

$$L(\vec{x}) = [L]\vec{x} = \begin{bmatrix} 5/2 & -1/2 \\ 2 & 1 \end{bmatrix}\vec{x} = \begin{bmatrix} \frac{5}{2}x_1 - \frac{1}{2}x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

3.2.13 (a) Consider

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

Then,

$$\begin{aligned} L(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) &= L(\vec{0}) \\ c_1L(\vec{v}_1) + \cdots + c_kL(\vec{v}_k) &= \vec{0} \end{aligned}$$

Thus, $c_1 = \cdots = c_k = 0$ since $\{L(\vec{v}_1),\dots,L(\vec{v}_k)\}$ is linearly independent. Thus, $\{\vec{v}_1,\dots,\vec{v}_k\}$ is linearly independent.

(b) Take $L(\vec{x}) = \vec{0}$. Then, $\{L(\vec{v}_1),\dots,L(\vec{v}_k)\}$ contains the zero vector, so it is linearly dependent.

3.2.14 Let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$. By definition, the i-th column of $[\operatorname{proj}_{\vec{u}}]$ is

$$\operatorname{proj}_{\vec{u}}(\vec{e}_i) = \frac{\vec{e}_i \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} = u_i\vec{u}$$

since $\vec{u}$ is a unit vector. Thus,

$$\begin{aligned} [\operatorname{proj}_{\vec{u}}] &= \begin{bmatrix} u_1\vec{u} & \cdots & u_n\vec{u} \end{bmatrix} \\ &= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \\ &= \vec{u}\vec{u}^T \end{aligned}$$

3.3 Problem Solutions

3.3.1 (a) i. If $\vec{x} \in \ker(L)$, then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(x_1, x_2, x_3) = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$$

Hence, $x_2 = -x_1$. Thus, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_1, x_3 \in \mathbb{R}$$

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $\ker(L)$ and is clearly linearly independent. Hence, $\mathcal{B}$ is a basis for $\ker(L)$.

Every vector $\vec{y} \in \text{Range}(L)$ has the form

$$L(\vec{x}) = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix} = (x_1 + x_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, (x_1 + x_2) \in \mathbb{R}$$

Hence, $C = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ spans Range(L) and is linearly independent, so it is a basis for Range(L).

ii. We have

$$L(1, 0, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad L(0, 1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad L(0, 0, 1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $[L] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

iii. By Theorem 3.3.4, $\mathcal{B}$ is a basis for Null$([L])$. By Theorem 3.3.5, C is a basis for Col$([L])$.

By definition, Row$([L])$ = Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Thus, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ spans Row$([L])$ and is clearly linearly independent, so it is a basis for Row$([L])$.

To find a basis for the left nullspace of $[L]$ we solve the homogeneous system $[L]^T \vec{x} = \vec{0}$. Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, we have $x_1 = 0$, so every $\vec{x} \in \text{Null}([L]^T)$ has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence, $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ spans Null$([L]^T)$ and is clearly linearly independent, so it is a basis for Null$([L]^T)$.

(b) i. If $\vec{x} \in \ker(L)$, then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(x_1, x_2, x_3) = \begin{bmatrix} x_1 - x_2 \\ x_2 + x_3 \end{bmatrix}$$

Hence, $x_2 = x_1$ and $x_3 = -x_2 = -x_1$. Thus, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad x_1 \in \mathbb{R}$$

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ spans $\ker(L)$ and is clearly linearly independent. Hence, $\mathcal{B}$ is a basis for $\ker(L)$.

Every vector $\vec{y} \in \operatorname{Range}(L)$ has the form

$$L(\vec{x}) = \begin{bmatrix} x_1 - x_2 \\ x_2 + x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R}$$

Hence, $C = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ spans $\operatorname{Range}(L)$ and is linearly independent, so it is a basis for $\operatorname{Range}(L)$.

ii. We have

$$L(1,0,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad L(0,1,0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad L(0,0,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, $[L] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

iii. By Theorem 3.3.4 $\mathcal{B}$ is a basis for $\operatorname{Null}([L])$. By Theorem 3.3.5 C is a basis for $\operatorname{Col}([L])$. By definition, $\operatorname{Row}([L]) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Thus, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans $\operatorname{Row}([L])$ and is clearly linearly independent, so it is a basis for $\operatorname{Row}([L])$.

To find a basis for the left nullspace of $[L]$ we solve the homogeneous system $[L]^T\vec{x} = \vec{0}$. Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, we have $x_1 = x_2 = 0$, so $\operatorname{Null}([L]^T) = \{\vec{0}\}$. Thus, a basis for $\operatorname{Null}([L]^T)$ is the empty set.

(c) i. Observe that for any $\vec{x} \in \mathbb{R}^3$, we have

$$L(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, $\ker(L) = \mathbb{R}^3$. Thus, a basis for $\ker(L)$ is the standard basis for $\mathbb{R}^3$.
Every vector $\vec{y} \in \text{Range}(L)$ has the form

$$L(\vec{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, $\text{Range}(L) = \{\vec{0}\}$ and so a basis for $\text{Range}(L)$ is the empty set.

ii. We have

$$L(1,0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad L(0,1,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad L(0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, $[L] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

iii. By Theorem 3.3.4 the standard basis for $\mathbb{R}^3$ is a basis for $\text{Null}([L])$. By Theorem 3.3.5 the empty set is a basis for $\text{Col}([L])$.
By definition, $\text{Row}([L]) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Thus, $\text{Row}([L]) = \{\vec{0}\}$ and so a basis for $\text{Row}([L])$ is the empty set.
To find a basis for the left nullspace of $[L]$ we solve the homogeneous system $[L]^T\vec{x} = \vec{0}$. Clearly, the solution space is $\mathbb{R}^3$. Hence, the standard basis for $\mathbb{R}^3$ is a basis for $\text{Null}([L]^T)$.

(d) i. If $\vec{x} \in \ker(L)$, then we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 \\ x_2 + x_4 \end{bmatrix}$$

Hence, $x_1 = 0$ and $x_4 = -x_2$. Thus, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ spans $\ker(L)$ and is clearly linearly independent. Hence, $\mathcal{B}$ is a basis for $\ker(L)$.
Every vector $\vec{y} \in \text{Range}(L)$ has the form

$$L(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 + x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (x_2 + x_4) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_1, (x_2 + x_4) \in \mathbb{R}$$

Hence, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ spans $\text{Range}(L)$ and is linearly independent, so it is a basis for $\text{Range}(L)$.

ii. We have

$$L(1,0,0,0) = \begin{bmatrix}1\\0\end{bmatrix} \quad L(0,1,0,0) = \begin{bmatrix}0\\1\end{bmatrix} \quad L(0,0,1,0) = \begin{bmatrix}0\\0\end{bmatrix} \quad L(0,0,0,1) = \begin{bmatrix}0\\1\end{bmatrix}$$

Thus, $[L] = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 1\end{bmatrix}$.

iii. By Theorem 3.3.4 $\mathcal{B}$ is a basis for Null($[L]$). By Theorem 3.3.5 C is a basis for Col($[L]$).
By definition, Row($[L]$) = Span $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$. Thus, $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$ spans Row($[L]$) and is clearly linearly independent. Therefore, it is a basis for Row($[L]$).
To find a basis for the left nullspace of $[L]$ we solve the homogeneous system $[L]^T\vec{x} = \vec{0}$. Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{bmatrix}$$

Thus, we have $x_1 = x_2 = 0$, so Null($[L]^T$) $= \{\vec{0}\}$. Thus, a basis for Null($[L]^T$) is the empty set.

(e) i. If $\vec{x} \in \ker(L)$, then we have

$$\begin{bmatrix}0\\0\end{bmatrix} = L(x_1, x_2) = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Hence, $x_1 = x_2 = 0$. So, $\ker(L) = \{\vec{0}\}$ and so a basis for $\ker(L)$ is the empty set.
Every vector $\vec{y} \in$ Range(L) has the form

$$L(\vec{x}) = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}, \quad x_1, x_2 \in \mathbb{R}$$

Hence, $C = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ spans Range(L) and is linearly independent, so it is a basis for Range(L).

ii. We have

$$L(1,0) = \begin{bmatrix}1\\0\end{bmatrix} \quad L(0,1) = \begin{bmatrix}0\\1\end{bmatrix}$$

Thus, $[L] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

iii. By Theorem 3.3.4 the empty set is a basis for Null($[L]$). By Theorem 3.3.5 C is a basis for Col($[L]$).
By definition, Row($[L]$) = Span $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$. Thus, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis for Row($[L]$).

To find a basis for the left nullspace of $[L]$ we solve the homogeneous system $[L]^T\vec{x} = \vec{0}$. We see that the only solution is $\vec{x} = \vec{0}$, so $\text{Null}([L]^T) = \{\vec{0}\}$. Thus, a basis for $\text{Null}([L]^T)$ is the empty set.

(f) $L(x_1, x_2, x_3, x_4) = (x_4 - x_1, 2x_2 + 3x_3)$

i. If $\vec{x} \in \ker(L)$, then we have

$$\begin{bmatrix}0\\0\end{bmatrix} = L(x_1, x_2, x_3, x_4) = \begin{bmatrix}x_4 - x_1\\2x_2 + 3x_3\end{bmatrix}$$

Hence, $x_4 = x_1$ and $x_3 = -\frac{2}{3}x_2$. Thus, we have

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\-2x_2/3\\x_1\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + x_2\begin{bmatrix}0\\1\\-2/3\\0\end{bmatrix}, \quad x_1, x_2 \in \mathbb{R}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-2/3\\0\end{bmatrix}\right\}$ spans $\ker(L)$ and is clearly linearly independent. Hence, $\mathcal{B}$ is a basis for $\ker(L)$.

Every vector $\vec{y} \in \text{Range}(L)$ has the form

$$L(\vec{x}) = \begin{bmatrix}x_4 - x_1\\2x_2 + 3x_3\end{bmatrix} = (x_4 - x_1)\begin{bmatrix}1\\0\end{bmatrix} + (2x_2 + 3x_3)\begin{bmatrix}0\\1\end{bmatrix}$$

Hence, $\mathcal{C} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ spans $\text{Range}(L)$ and is linearly independent, so it is a basis for $\text{Range}(L)$.

ii. We have

$$L(1,0,0,0) = \begin{bmatrix}-1\\0\end{bmatrix} \quad L(0,1,0,0) = \begin{bmatrix}0\\2\end{bmatrix} L(0,0,1,0) = \begin{bmatrix}0\\3\end{bmatrix} \quad L(0,0,0,1) = \begin{bmatrix}1\\0\end{bmatrix}$$

Thus, $[L] = \begin{bmatrix}-1 & 0 & 0 & 1\\0 & 2 & 3 & 0\end{bmatrix}$.

iii. By Theorem 3.3.4 $\mathcal{B}$ is a basis for $\text{Null}([L])$. By Theorem 3.3.5 $\mathcal{C}$ is a basis for $\text{Col}([L])$.

By definition, $\text{Row}([L]) = \text{Span}\left\{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\2\\3\\0\end{bmatrix}\right\}$. Thus, $\left\{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\2\\3\\0\end{bmatrix}\right\}$ spans $\text{Row}([L])$ and is clearly linearly independent. Therefore, it is a basis for $\text{Row}([L])$.

To find a basis for the left nullspace of $[L]$ we solve the homogeneous system $[L]^T\vec{x} = \vec{0}$. Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}-1 & 0\\0 & 2\\0 & 3\\1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{bmatrix}$$

Thus, we have $x_1 = x_2 = 0$, so $\text{Null}([L]^T) = \{\vec{0}\}$. Thus, a basis for $\text{Null}([L]^T)$ is the empty set.

(g) i. If $\vec{x} \in \ker(L)$, then we have

$$0 = L(\vec{x}) = (3x_1)$$

Hence, $x_1 = 0$. Therefore, $\ker(L) = \{0\}$, so a basis for $\ker(L)$ is the empty set.
Every vector in the range of L has the form $L(\vec{x}) = (3x_1)$. Hence, $\{1\}$ spans $\text{Range}(L)$ and is linearly independent, so it is a basis for $\text{Range}(L)$.

ii. We have $L(1) = 3$. Thus, $[L] = [3]$.

iii. By Theorem 3.3.4, the empty set is a basis for $([L])$. By Theorem 3.3.5, $\{1\}$ is a basis for $\text{Col}([L])$. Now notice that $[L]^T = [L]$. Hence, the empty set is also a basis for $([L]^T)$ and $\{1\}$ is also a basis for $\text{Row}([L])$.

i. If $\vec{x} \in \ker(L)$, then

$$(0, 0, 0) = L(x_1, x_2, x_3) = (2x_1 + x_3, x_1 - x_3, x_1 + x_3)$$

Hence, $2x_1 + x_3 = 0$, $x_1 - x_3 = 0$, and $x_1 + x_3 = 0$. This implies that $x_1 = x_3 = 0$. Therefore, every vector in $\ker(L)$ has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 \in \mathbb{R}$$

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ spans $\ker(L)$ and is linearly independent. Hence, it is a basis for $\ker(L)$.
Every vector $\vec{y} \in \text{Range}(L)$ has the form

$$L(\vec{x}) = \begin{bmatrix} 2x_1 + x_3 \\ x_1 - x_3 \\ x_1 + x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad x_1, x_3 \in \mathbb{R}$$

Thus, $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ spans $\text{Range}(L)$ and is clearly linearly independent, so it is a basis for $\text{Range}(L)$.

ii. We have

$$L(1, 0, 0) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad L(0, 1, 0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad L(0, 0, 1) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, $[L] = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$.

iii. By Theorem 3.3.4 $\mathcal{B}$ is a basis for Null($[L]$). By Theorem 3.3.5 $\mathcal{C}$ is a basis for Col($[L]$). By definition, Row($[L]$) = Span $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$. However, observe that

$$\frac{1}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix}$$

Hence, Row($[L]$) = Span $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$. Since $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ is also clearly linearly independent, it is a basis for Row($[L]$).

To find a basis for the left nullspace of $[L]$, we solve $[L]^T\vec{x} = \vec{0}$. Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}2 & 1 & 1\\0 & 0 & 0\\1 & -1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2/3\\0 & 1 & -1/3\\0 & 0 & 0\end{bmatrix}$$

Hence, every vector in Null($[L]^T$) has the form

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}x_3\\ \frac{1}{3}x_3\\ x_3\end{bmatrix} = x_3\begin{bmatrix}-2/3\\1/3\\1\end{bmatrix}, \quad x_3 \in \mathbb{R}$$

Hence, $\left\{\begin{bmatrix}-2/3\\1/3\\1\end{bmatrix}\right\}$ spans the left nullspace of $[L]$ and is clearly linearly independent, so it is a basis for the left nullspace of $[L]$.

(h) i. If $\vec{x} \in \ker(L)$, then we have

$$0 = L(\vec{x}) = (x_1, 2x_1, -x_1)$$

Hence, $x_1 = 0$. Therefore, $\ker(L) = \{0\}$, so a basis for $\ker(L)$ is the empty set.

Every vector in the range of L has the form

$$L(\vec{x}) = \begin{bmatrix}x_1\\2x_1\\-x_1\end{bmatrix} = x_1\begin{bmatrix}1\\2\\-1\end{bmatrix}, \quad x_1 \in \mathbb{R}$$

Hence, $\mathcal{C} = \left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$ spans Range(L) and is linearly independent, so it is a basis for Range(L).

ii. We have $L(1) = \begin{bmatrix}1\\2\\-1\end{bmatrix}$. Thus, $[L] = \begin{bmatrix}1\\2\\-1\end{bmatrix}$.

iii. By Theorem 3.3.4, the empty set is a basis for $([L])$. By Theorem 3.3.5, C is a basis for $\mathrm{Col}([L])$.
We have $\mathrm{Row}([L]) = \mathrm{Span}\{1, 2, -1\} = \mathrm{Span}\{1\}$. Thus, $\{1\}$ is a basis for $\mathrm{Row}([L])$.
To find a basis for the left nullspace of $[L]$, we solve

$$0 = [L]^T\vec{x} = x_1 + 2x_2 - x_3$$

Thus, every vector in the left nullspace of $[L]$ has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

Hence, $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for $\mathrm{Null}([L]^T)$.

i. If $\vec{x} \in \ker(L)$, then

$$(0, 0) = L(x_1, x_2) = (x_1 + x_2, 2x_1 + 4x_2)$$

Hence, $x_1 + x_2 = 0$ and $2x_1 + 4x_2 = 0$. This implies that $x_1 = x_2 = 0$. Therefore, $\ker(L) = \{\vec{0}\}$ and so a basis for $\ker(L)$ is the empty set.
Every vector $\vec{y} \in \mathrm{Range}(L)$ has the form

$$L(\vec{x}) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 4x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad x_1, x_2 \in \mathbb{R}$$

Thus, $C = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$ spans $\mathrm{Range}(L)$ and is clearly linearly independent, so it is a basis for $\mathrm{Range}(L)$.

ii. We have

$$L(1, 0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad L(0, 1) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Thus, $[L] = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$.

iii. By Theorem 3.3.4 the empty set is a basis for $\mathrm{Null}([L])$. By Theorem 3.3.5 C is a basis for $\mathrm{Col}([L])$.
By definition, $\mathrm{Row}([L]) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$. Hence, $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ spans $\mathrm{Row}([L])$ and is also clearly linearly independent, so it is a basis for $\mathrm{Row}([L])$.
To find a basis for the left nullspace of $[L]$, we solve $[L]^T\vec{x} = \vec{0}$. Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, the only vector in $\mathrm{Null}([L]^T)$ is the zero vector. So, a basis for the left nullspace of $[L]$ is the empty set.

3.3.2 (a) By definition, $\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$ spans Col(A). Since it is also linearly independent, it is a basis for Col(A).

By definition, $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$ spans Row(A). Since it is also linearly independent, it is a basis for Row(A).

To find Null(A), we solve $A\vec{x} = \vec{0}$. Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Thus, the only solutions is $\vec{x} = \vec{0}$. Hence, Null(A) $= \{\vec{0}\}$ and so a basis is the empty set.

To find Null(A^T), we solve $A^T\vec{x} = \vec{0}$. Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Thus, the only solutions is $\vec{x} = \vec{0}$. Hence, Null(A^T) $= \{\vec{0}\}$ and so a basis is the empty set.

(b) By definition, Col(B) = Span $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$. Since $\begin{bmatrix}1\\-1\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\end{bmatrix}$, we get that

$$\mathrm{Col}(B) = \mathrm{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathrm{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

by Theorem 1.1.2. Thus, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a spanning set for Col(B) and is also linearly independent, so it is a basis for Col(B).

By definition, $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$ spans Row(B). Since it is also linearly independent, it is a basis for Row(B).

To find Null(B), we solve $B\vec{x} = \vec{0}$. The coefficient matrix is already in RREF, so we get $x_1 + x_3 = 0$ and $x_2 - x_3 = 0$. Let $x_3 = t \in \mathbb{R}$. Then, every vector $\vec{x} \in$ Null(B) has the form

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-x_3\\x_3\\x_3\end{bmatrix} = x_3\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

So, $\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ spans Null(B) and is clearly linearly independent and hence is a basis for Null(B).

To find Null(B^T), we solve $B^T\vec{x} = \vec{0}$. Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix}1 & 0\\0 & 1\\1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

Thus, the only solutions is $\vec{x} = \vec{0}$. Hence, $\operatorname{Null}(B^T) = \{\vec{0}\}$ and so a basis is the empty set.

(c) By definition, $\operatorname{Col}(C) = \operatorname{Span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}\right\}$ spans $\operatorname{Col}(C)$. Observe that

$$2\begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}2\\-2\\3\end{bmatrix}$$

Hence, $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}\right\}$ also spans $\operatorname{Col}(C)$. Since it is also linearly independent, it is a basis for $\operatorname{Col}(C)$.

By definition,

$$\operatorname{Row}(C) = \operatorname{Span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-1\end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-1\end{bmatrix}\right\}$$

Hence, $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-1\end{bmatrix}\right\}$ is a basis for $\operatorname{Row}(C)$ since it spans $\operatorname{Row}(C)$ and is linearly independent.

To find $\operatorname{Null}(C)$, we solve $C\vec{x} = \vec{0}$. We row reduce the corresponding coefficient matrix to get

$$\begin{bmatrix}1 & 2 & 0\\-1 & -2 & 0\\1 & 3 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

$x_1 + 2x_3 = 0$ and $x_2 - x_3 = 0$. Then, every vector $\vec{x} \in \operatorname{Null}(C)$ has the form

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2x_3\\x_3\\x_3\end{bmatrix} = x_3\begin{bmatrix}-2\\1\\1\end{bmatrix}, \quad x_3 \in \mathbb{R}$$

So, $\left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$ spans $\operatorname{Null}(C)$ and is clearly linearly independent and hence is a basis for $\operatorname{Null}(C)$.

To find $\operatorname{Null}(C^T)$, we solve $C^T\vec{x} = \vec{0}$. Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix}1 & -1 & 1\\2 & -2 & 3\\0 & 0 & -1\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

Hence, $x_1 - x_2 = 0$ and $x_3 = 0$. Then, every vector $\vec{x} \in \operatorname{Null}(C^T)$ has the form

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_2\\x_2\\0\end{bmatrix} = x_2\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad x_2 \in \mathbb{R}$$

So, $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ spans Null(C^T) and is clearly linearly independent and hence is a basis for Null(C^T).

(d) To find a basis for Col(A), we need to determine which columns of A can be written as linear combinations of other columns. We consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\-2\\-1\end{bmatrix} + c_2\begin{bmatrix}2\\-5\\-3\end{bmatrix} + c_3\begin{bmatrix}1\\-2\\-1\end{bmatrix} + c_4\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}c_1+2c_2+c_3+c_4\\-2c_1-5c_2-2c_3+c_4\\-c_1-3c_2-c_3+2c_4\end{bmatrix}$$

Row reducing the coefficient matrix of the homogeneous system gives

$$\begin{bmatrix}1&2&1&1\\-2&-5&-2&1\\-1&-3&-1&2\end{bmatrix} \sim \begin{bmatrix}1&0&1&7\\0&1&0&-3\\0&0&0&0\end{bmatrix}$$

If we ignore the last column and pretend the third column is an augmented part, this shows that the third column equals 1 times the first column (which is obvious). If we ignore the third column and pretend the fourth column is an augmented part, we see that 7 times the first column plus -3 times the second column gives the fourth column. Hence, we have that $\left\{\begin{bmatrix}1\\-2\\-1\end{bmatrix},\begin{bmatrix}2\\-5\\-3\end{bmatrix}\right\}$ spans Col(A). Since it is also linearly independent (just consider the first two columns in the system above) and hence it is a basis for Col(A).

To find a basis for Row(A), we need to determine which rows of A can be written as linear combinations of other columns. We consider

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\2\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\-5\\-2\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\-3\\-1\\2\end{bmatrix} = \begin{bmatrix}c_1-2c_2-c_3\\2c_1-5c_2-3c_3\\c_1-2c_2-c_3\\c_1+c_2+2c_3\end{bmatrix}$$

Row reducing the coefficient matrix of the homogeneous system gives

$$\begin{bmatrix}1&-2&-1\\2&-5&-3\\1&-2&-1\\1&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

If we ignore the last column and pretend the third column is an augmented part, this shows that the third column equals the sum of the first two columns. That is, the third row of A equals the sum of the first two rows. Hence, we have that $\left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix},\begin{bmatrix}-2\\-5\\-2\\1\end{bmatrix}\right\}$ spans Row(A). Since it is also linearly independent (just consider the first two columns in the system above) and hence it is a basis for Row(A).

To find Null(A), we solve $A\vec{x} = \vec{0}$. Observe that the coefficient matrix is A which we already row reduced above. Hence, we get $x_1 + x_3 + 7x_4 = 0$ and $x_2 - 3x_4 = 0$. Then, every vector $\vec{x} \in \text{Null}(A^T)$ has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 7x_4 \\ 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

So, $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans Null(A^T) and is clearly linearly independent and hence is a basis for Null(A).

To find Null(A^T), we solve $A^T\vec{x} = \vec{0}$. We observe that we row reduced A^T above. We get $x_1 + x_3 = 0$ and $x_2 + x_3 = 0$. Then, every vector $\vec{x} \in \text{Null}(A^T)$ has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

So, $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ spans Null(A^T) and is clearly linearly independent and hence is a basis for Null(A^T).

3.3.3 If $A\vec{x} = \vec{b}$ has a unique solution, then $A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = \vec{0}$. Thus, Null(A) = $\{\vec{0}\}$.

On the other hand, if Null(A) = $\{\vec{0}\}$, then $A\vec{x} = \vec{0}$ has a unique solution. Thus, by Theorem 2.2.4, rank(A) = n and so $A\vec{x} = \vec{b}$ has a unique solution.

3.3.4 By definition Range(L) is a subset of $\mathbb{R}^m$ and $\vec{0} \in$ Range(L) by Lemma 3.3.1.

Let $\vec{y}, \vec{z} \in$ Range(L). Then, there exists $\vec{x}, \vec{w} \in \mathbb{R}^n$ such that $L(\vec{x}) = \vec{y}$ and $L(\vec{w}) = \vec{z}$. We now see that

$$L(\vec{x} + \vec{w}) = L(\vec{x}) + L(\vec{w}) = \vec{y} + \vec{z}$$

and

$$L(t\vec{x}) = tL(\vec{x}) = t\vec{y}$$

So, $\vec{y} + \vec{z} \in$ Range(L) and $t\vec{y} \in$ Range(L) for all $t \in \mathbb{R}$. Thus, Range(L) is a subspace of $\mathbb{R}^m$ by the Subspace Test.

3.3.5 Consider

$$c_1L(\vec{v}_1) + \cdots + c_kL(\vec{v}_k) = \vec{0}$$

Then,

$$L(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = \vec{0}$$

Therefore, $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \in \ker(L)$ and hence $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ since $\ker(L) = \{\vec{0}\}$. Thus, $c_1 = \cdots = c_k = 0$ since $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent. Hence, $\{L(\vec{v}_1), \ldots, L(\vec{v}_k)\}$ is linearly independent.

3.3.6 Let $[L]$ be the standard matrix of L. By definition of the standard matrix, we have $[L]$ is an $n \times n$ matrix. Thus, we get that $\ker(L) = \{\vec{0}\}$ if and only if

$$\vec{0} = L(\vec{x}) = [L]\vec{x}$$

has a unique solution. By Theorem 2.2.4(2), this is true if and only if $\operatorname{rank}[L] = n$. Moreover, by Theorem 2.2.4(3), we have that $\vec{b} = [L]\vec{x} = L(\vec{x})$ is consistent for all $\vec{b} \in \mathbb{R}^n$ if and only if $\operatorname{rank}[L] = n$. Hence, $\ker(L) = \{\vec{0}\}$ if and only if $\operatorname{rank}[L] = n$ if and only if $\operatorname{Range}(L) = \mathbb{R}^n$.

3.3.7 Let $\vec{y} \in \operatorname{Col}(A)$. Then, by definition there exists $\vec{x} \in \mathbb{R}^n$ such that $\vec{y} = A\vec{x}$. Thus,

$$A\vec{y} = A(A\vec{x}) = A^2\vec{x} = O_{n,n}\vec{x} = \vec{0}$$

Thus, $\vec{y} \in \operatorname{Null}(A)$, so $\operatorname{Null}(A) \subseteq \operatorname{Col}(A)$.

3.3.8 (a) By definition $\operatorname{Null}(A)$ is a subset of $\mathbb{R}^n$. Also, we have $A\vec{0} = \vec{0}$, so $\vec{0} \in \operatorname{Null}(A)$.
Let $\vec{x}, \vec{y} \in \operatorname{Null}(A)$. Then, $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$ and hence

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$$

Similarly, for any $s \in \mathbb{R}$ we have

$$A(s\vec{x}) = sA\vec{x} = s\vec{0} = \vec{0}$$

Thus, by the Subspace Test, $\operatorname{Null}(A)$ is a subset of $\mathbb{R}^n$.

(b) Observe that every $\vec{x} \in \operatorname{Null}(A)$ satisfies

$$\vec{0} = A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

Thus, $\operatorname{Null}(A)$ is the solution space of this homogeneous system which we know is a subspace by Theorem 2.2.3.

3.3.9 We just need to prove that if B is obtained from A by applying any one of the three elementary row operations, then $\operatorname{Row}(A) = \operatorname{Row}(B)$.

Let $A = \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix}$. Then, $\operatorname{Row}(A) = \operatorname{Span}\{\vec{a}_1, \dots, \vec{a}_m\}$.

If B is obtained from A by swapping row i and row j, then B still has the same rows as A (just in a different order), so

$$\operatorname{Row}(B) = \operatorname{Span}\{\vec{a}_1, \dots, \vec{a}_m\} = \operatorname{Row}(A)$$

If B is obtained from A by multiplying row i by a non-zero constant c, then we have

$$\begin{aligned}
\operatorname{Row}(B) &= \operatorname{Span}\{\vec{a}_1, \dots, \vec{a}_{i-1}, c\vec{a}_i, \vec{a}_{i+1}, \dots, \vec{a}_m\} \\
&= \{t_1\vec{a}_1 + \cdots + t_{i-1}\vec{a}_{i-1} + t_i(c\vec{a}_i) + t_{i+1}\vec{a}_{i+1} + \cdots + t_m\vec{a}_m \mid t_1, \dots, t_m \in \mathbb{R}\} \\
&= \{t_1\vec{a}_1 + \cdots + t_{i-1}\vec{a}_{i-1} + ct_i(\vec{a}_i) + t_{i+1}\vec{a}_{i+1} + \cdots + t_m\vec{a}_m \mid t_1, \dots, t_m \in \mathbb{R}\} \\
&= \operatorname{Span}\{\vec{a}_1, \dots, \vec{a}_m\} = \operatorname{Row}(A)
\end{aligned}$$

If B is obtained from A by applying the elementary row operation $R_i + cR_j$, then we have

$$\begin{aligned}
\operatorname{Row}(B) &= \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_{i-1}, \vec{a}_i + c\vec{a}_j, \vec{a}_{i+1}, \ldots, \vec{a}_m\} \\
&= \{t_1\vec{a}_1 + \cdots + t_{i-1}\vec{a}_{i-1} + t_i(\vec{a}_i + c\vec{a}_j) + t_{i+1}\vec{a}_{i+1} + \cdots + t_m\vec{a}_m \mid t_1, \ldots, t_m \in \mathbb{R}\} \\
&= \{t_1\vec{a}_1 + \cdots + t_{i-1}\vec{a}_{i-1} + t_i\vec{a}_i + t_{i+1}\vec{a}_{i+1} + \cdots + (t_j + ct_i)\vec{a}_j + \cdots + t_m\vec{a}_m \mid t_1, \ldots, t_m \in \mathbb{R}\} \\
&= \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_m\} = \operatorname{Row}(A)
\end{aligned}$$

3.3.10 (a) If $\vec{b} = A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$, then by definition $\vec{b} \in \operatorname{Col}(A)$. On the other hand, if $\vec{b} \in \operatorname{Col}(A)$, then there exists $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$, so the system is consistent. Hence, we have proven the statement.

(b) We disprove the statement with a counter example. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L(\vec{x}) = \vec{0}$. Then, $L(1, 1) = \vec{0}$ and so $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \ker(L)$, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \vec{0}$.

(c) We disprove the statement with a counter example. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L(\vec{x}) = A\vec{x}$. Then, observe that $\operatorname{Range}(L) = \operatorname{Span}\left\{\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}\right\}$, but $[L] \neq \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$.

3.4 Problem Solutions

3.4.1 (a) We have

$$(L + M)(x_1, x_2) = L(x_1, x_2) + M(x_1, x_2) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 3x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ 2x_1 + x_2 \end{bmatrix}$$

Hence,

$$[L + M] = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$$

(b) We have

$$(L + M)(x_1, x_2, x_3) = L(x_1, x_2, x_3) + M(x_1, x_2, x_3) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_3 \end{bmatrix} + \begin{bmatrix} -x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + x_2 + 2x_3 \end{bmatrix}$$

Hence,

$$[L + M] = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

(c) We have

$$(L + M)(x_1, x_2, x_3) = L(x_1, x_2, x_3) + M(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 - x_2 \\ x_2 + 3x_3 \end{bmatrix} + \begin{bmatrix} -2x_1 + x_2 \\ -x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence,

$$[L + M] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.4.2 (a) We have

$$(M \circ L)(x_1, x_2) = M(L(x_1, x_2)) = M(2x_1 - x_2, x_1) = \begin{bmatrix} 3(2x_1 - x_2) + x_1 \\ 2x_1 - x_2 + x_1 \end{bmatrix} = \begin{bmatrix} 7x_1 - 3x_2 \\ 3x_1 - x_2 \end{bmatrix}$$
$$(L \circ M)(x_1, x_2) = L(M(x_1, x_2)) = L(3x_1 + x_2, x_1 + x_2) = \begin{bmatrix} 2(3x_1 + x_2) - (x_1 + x_2) \\ 3x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + x_2 \\ 3x_1 + x_2 \end{bmatrix}$$

Hence, $[M \circ L] = \begin{bmatrix} 7 & -3 \\ 3 & -1 \end{bmatrix}$ and $[L \circ M] = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix}$.

(b) We have

$$\begin{aligned}(M \circ L)(x_1, x_2, x_3) &= M(L(x_1, x_2, x_3)) = M(x_1 - 2x_2, x_1 + x_2 + x_3) \\ &= \begin{bmatrix} 0 \\ (x_1 - 2x_2) + (x_1 + x_2 + x_3) \\ x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_1 - x_2 + x_3 \\ x_1 - 2x_2 \end{bmatrix}\end{aligned}$$
$$(L \circ M)(x_1, x_2) = L(M(x_1, x_2)) = L(0, x_1 + x_2, x_1) = \begin{bmatrix} 0 - 2(x_1 + x_2) \\ 0 + (x_1 + x_2) + x_1 \end{bmatrix} = \begin{bmatrix} -2x_1 - 2x_2 \\ 2x_1 + x_2 \end{bmatrix}$$

Hence, $[M \circ L] = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$ and $[L \circ M] = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix}$.

(c) We have

$$\begin{aligned}(M \circ L)(x_1, x_2) &= M(L(x_1, x_2)) = M(3x_1 + x_2, 5x_1 + 2x_2) \\ &= \begin{bmatrix} 2(3x_1 + x_2) - (5x_1 + 2x_2) \\ -5(3x_1 + x_2) - 3(5x_1 + 2x_2) \end{bmatrix} = \begin{bmatrix} x_1 \\ -30x_1 - 11x_2 \end{bmatrix} \\ (L \circ M)(x_1, x_2) &= L(M(x_1, x_2)) = L(2x_1 - x_2, -5x_1 - 3x_2) \\ &= \begin{bmatrix} 3(2x_1 - x_2) + (-5x_1 - 3x_2) \\ 5(2x_1 - x_2) + 2(-5x_1 - 3x_2) \end{bmatrix} = \begin{bmatrix} x_1 - 6x_2 \\ -11x_2 \end{bmatrix}\end{aligned}$$

Hence, $[M \circ L] = \begin{bmatrix} 1 & 0 \\ -30 & -11 \end{bmatrix}$ and $[L \circ M] = \begin{bmatrix} 1 & -6 \\ 0 & -11 \end{bmatrix}$.

3.4.3 Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ be linear mappings.

(a) Let $\vec{z} \in \mathbb{R}^p$. Since M is onto, there exists $\vec{y} \in \mathbb{R}^m$ such that $L(\vec{y}) = \vec{z}$. Since L is onto, there exists $\vec{x} \in \mathbb{R}^n$ such that $L(\vec{x}) = \vec{y}$. Hence, we have that $(M \circ L)(\vec{x}) = M(L(\vec{x})) = M(\vec{y}) = \vec{z}$. Hence, Range$(M \circ L) = \mathbb{R}^p$.

(b) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $L(x_1, x_2) = (x_1, x_2, 0)$. Clearly Range$(L) \neq \mathbb{R}^3$. However, if we define $M : \mathbb{R}^3 \to \mathbb{R}^2$ by $M(x_1, x_2, x_3) = (x_1, x_2)$, then we get that

$$(M \circ L)(x_1, x_2) = M(L(x_1, x_2)) = M(x_1, x_2, 0) = (x_1, x_2)$$

So, Range$(M \circ L) = \mathbb{R}^2$.

(c) No, it isn't. If $M \circ L$ is onto, then for any $\vec{z} \in \mathbb{R}^p$, there exists $\vec{x} \in \mathbb{R}^n$ such that $\vec{z} = (M \circ L)(\vec{x}) = M(L(\vec{x}))$. Hence, $\vec{z} \in \text{Range}(M)$. So, $\text{Range}(M) = \mathbb{R}^p$.

3.4.4 Let $L, M \in \mathbb{L}$ and let $c, d \in \mathbb{R}$.

V4 Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$. Hence,

$$O(s\vec{x} + t\vec{y}) = \vec{0} = sO(\vec{x}) + tO(\vec{y})$$

so, O is linear and hence $O \in \mathbb{L}$. For any $\vec{x} \in \mathbb{R}^n$ we have

$$(L + O)(\vec{x}) = L(\vec{x}) + O(\vec{x}) = L(\vec{x}) + \vec{0} = L(\vec{x})$$

Hence, $L + O = L$.

V5 Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$. Then,

$$(-L)(s\vec{x} + t\vec{y}) = -L(s\vec{x} + t\vec{y}) = -[sL(\vec{x}) + tL(\vec{y})] = s(-L)(\vec{x}) + t(-L)(\vec{y})$$

Hence, $(-L)$ is linear and so $(-L) \in \mathbb{L}$. For any $\vec{x} \in \mathbb{R}^n$ we have

$$(L + (-L))(\vec{x}) = L(\vec{x}) + (-L)(\vec{x}) = L(\vec{x}) - L(\vec{x}) = \vec{0} = O(\vec{x})$$

Thus, $L + (-L) = O$

V8 For any $\vec{x} \in \mathbb{R}^n$ we have

$$[(c + d)L](\vec{x}) = (c + d)L(\vec{x}) = cL(\vec{x}) + dL(\vec{x}) = [cL + dL](\vec{x})$$

Hence, $(c + d)L = cL + dL$.

3.4.5 We have

$$\begin{aligned} \begin{bmatrix} (M \circ L)(\vec{e}_1) & \cdots & (M \circ L)(\vec{e}_n) \end{bmatrix} &= \begin{bmatrix} M(L(\vec{e}_1)) & \cdots & M(L(\vec{e}_n)) \end{bmatrix} \\ &= \begin{bmatrix} [M](L(\vec{e}_1)) & \cdots & [M](L(\vec{e}_n)) \end{bmatrix} \\ &= [M] \begin{bmatrix} L(\vec{e}_1) & \cdots & L(\vec{e}_n) \end{bmatrix} \\ &= [M][L] \end{aligned}$$

3.4.6 For any $\vec{x} \in \mathbb{R}^3$ we have

$$\begin{aligned} O(\vec{x}) &= (t_1L_1 + t_2L_2)(\vec{x}) = t_1L_1(\vec{x}) + t_2L_2(\vec{x}) \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= t_1 \begin{bmatrix} 2x_1 \\ x_2 - x_3 \end{bmatrix} + t_2 \begin{bmatrix} x_1 + x_2 + x_3 \\ x_3 \end{bmatrix} \\ \begin{bmatrix} 0x_1 + 0x_2 + 0x_3 \\ 0x_1 + 0x_2 + 0x_3 \end{bmatrix} &= \begin{bmatrix} (2t_1 + t_2)x_1 + t_2x_2 + t_2x_3 \\ t_1x_2 + (-t_1 + t_2)x_3 \end{bmatrix} \end{aligned}$$

Hence, we have the system of equations

$$\begin{aligned} 2t_1 + t_2 &= 0 \\ t_2 &= 0 \\ t_2 &= 0 \\ t_1 &= 0 \\ -t_1 + t_2 &= 0 \end{aligned}$$

Therefore, the only solution is $t_1 = t_2 = 0$.

3.4.7 For any $\vec{x} \in \mathbb{R}^3$ we have

$$\begin{aligned} O(\vec{x}) &= (t_1L_1 + t_2L_2 + t_3L_3)(\vec{x}) = t_1L_1(\vec{x}) + t_2L_2(\vec{x}) + t_3L_3(\vec{x}) \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} t_1(x_1 + 3x_2) + t_2(2x_1 + 5x_2) + t_3(x_1 + x_2) \\ t_1(6x_1 + x_2) + t_2(8x_1 + 3x_2) + t_3(-2x_1 + 3x_2) \end{bmatrix} \\ \begin{bmatrix} 0x_1 + 0x_2 \\ 0x_1 + 0x_2 \end{bmatrix} &= \begin{bmatrix} (t_1 + 2t_2 + t_3)x_1 + (3t_1 + 5t_2 + t_3)x_2 \\ (6t_1 + 8t_2 - 2t_3)x_1 + (t_1 + 3t_2 + 3t_3)x_2 \end{bmatrix} \end{aligned}$$

Hence, we have the system of equations

$$\begin{aligned} t_1 + 2t_2 + t_3 &= 0 \\ 3t_1 + 5t_2 + t_3 &= 0 \\ 6t_1 + 8t_2 - 2t_3 &= 0 \\ t_1 + 3t_2 + 3t_3 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 6 & 8 & -2 \\ 1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, we get

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 3t_1 \\ -2t_3 \\ t_3 \end{bmatrix} = t_3 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \quad t_3 \in \mathbb{R}$$

3.4.8 Define $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$ by $\mathrm{Id}(\vec{x}) = \vec{x}$. Then, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$, we have

$$\mathrm{Id}(s\vec{x} + t\vec{y}) = s\vec{x} + t\vec{y} = s\,\mathrm{Id}(\vec{x}) + t\,\mathrm{Id}(\vec{y})$$

Hence, Id is linear.

For any $\vec{x} \in \mathbb{R}^n$ we have

$$(L \circ \mathrm{Id})(\vec{x}) = L(\mathrm{Id}(\vec{x})) = L(\vec{x})$$

and

$$(\mathrm{Id} \circ L)(\vec{x}) = \mathrm{Id}(L(\vec{x})) = L(\vec{x})$$

as required.

Chapter 4 Solutions

4.1 Problem Solutions

4.1.1 (a) Every $q(x) \in \mathbb{S}_1$ has the form $x^2(a+bx+cx^2) = ax^2+bx^3+cx^4 \in P_4(\mathbb{R})$. Additionally, $0 \in \mathbb{S}_1$, so $\mathbb{S}_1$ is a non-empty subset of $P_4(\mathbb{R})$.

Let $q_1(x), q_2(x) \in \mathbb{S}_1$. Then there exists $p_1(x), p_2(x) \in P_2(\mathbb{R})$ such that $q_1(x) = x^2 p_1(x)$ and $q_2(x) = x^2 p_2(x)$. Then

$$q_1(x) + q_2(x) = x^2 p_1(x) + x^2 p_2(x) = x^2[p_1(x) + p_2(x)] \in \mathbb{S}_1$$

since $p_1(x) + p_2(x) \in P_2(\mathbb{R})$. Also, for any $c \in \mathbb{R}$ we have

$$cq_1(x) = c[x^2 p_1(x)] = x^2[cp_1(x)] \in \mathbb{S}_1$$

since $cp_1(x) \in P_2(\mathbb{R})$. Therefore, by the Subspace Test, $\mathbb{S}_1$ is a subspace of $P_4(\mathbb{R})$.

(b) By definition $\mathbb{S}_2$ is a subset of $M_{2\times 2}(\mathbb{R})$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}_2$ since $0+0=0$. Thus, we can apply the Subspace Test.

Let $\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} \in \mathbb{S}_2$ and $t \in \mathbb{R}$. Then $a_1 + a_3 = 0$ and $b_1 + b_3 = 0$. Hence,

$$\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ 0 & a_3+b_3 \end{bmatrix} \in \mathbb{S}_2$$

since $(a_1+b_1)+(a_3+b_3) = a_1+a_3+b_1+b_3 = 0+0 = 0$. Similarly,

$$t\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} = \begin{bmatrix} ta_1 & ta_2 \\ 0 & ta_3 \end{bmatrix} \in \mathbb{S}_2$$

since $ta_1 + ta_3 = t(a_1+a_3) = t(0) = 0$. Hence, $\mathbb{S}_2$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

(c) By definition $\mathbb{S}_3$ is a subset of $P_2(\mathbb{R})$. Also, the zero polynomial z satisfies $z(x) = 0$ for all x, hence $z(1) = 0$ so $z \in \mathbb{S}_3$. Therefore, we can apply the Subspace Test.

Let $p, q \in \mathbb{S}_3$ and $t \in \mathbb{R}$. Then, $p(1) = 0$ and $q(1) = 0$. Hence,

$$(p+q)(1) = p(1) + q(1) = 0 + 0 = 0$$

Hence, $p + q \in \mathbb{S}_2$. Similarly,

$$(tp)(1) = tp(1) = t(0) = 0$$

So, $tp \in \mathbb{S}_3$. Consequently, $\mathbb{S}_3$ is a subspace of $P_2(\mathbb{R})$.

(d) Clearly $\mathbb{S}_4$ is a non-empty subset of $M_{2\times 2}(\mathbb{R})$.

Let $A, B \in \mathbb{S}_4$. Then, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}_4$. For any $t \in \mathbb{R}$ we have $tA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}_4$. So, $\mathbb{S}_4$ is a subspace of $M_{2\times 2}(\mathbb{R})$ by the Subspace Test.

NOTE: $\mathbb{S}_4$ is called the trivial subspace of $M_{2\times 2}(\mathbb{R})$.

(e) By definition $\mathbb{S}_5$ is a subset of $P_2(\mathbb{R})$ and the zero polynomial $z(x) = 0 + 0x + 0x^2 \in \mathbb{S}_5$ since $0 = 0$. Let $a + bx + cx^2, d + ex + fx^2 \in \mathbb{S}_5$. Then, $a = c$ and $d = f$. Hence,

$$a + bx + cx^2 + d + ex + fx^2 = (a + d) + (b + e)x + (c + f)x^2 \in \mathbb{S}_5$$

since $a + d = c + f$ and for any $t \in \mathbb{R}$

$$t(a + bx + cx^2) = ta + tbx + tcx^2 \in \mathbb{S}_5$$

because $ta = tc$. Consequently, by the Subspace Test, $\mathbb{S}_5$ is a subspace of $P_2(\mathbb{R})$.

(f) By definition $\mathbb{S}_6$ is a subset of $M_{n\times n}(\mathbb{R})$. Also, the $n\times n$ zero matrix $O_{n,n}$ clearly satisfies $O_{n,n}^T = O_{n,n}$. Hence, we can apply the Subspace Test.

Let $A, B \in \mathbb{S}_6$ and $t \in \mathbb{R}$. Then, $A^T = A$ and $B^T = B$. Using properties of the transpose gives

$$(A + B)^T = A^T + B^T = A + B \quad \text{ and } (tA)^T = tA^T = tA$$

so, $A + B \in \mathbb{S}_6$ and $tA \in \mathbb{S}_6$, so $\mathbb{S}_6$ is a subspace of $M_{n\times n}(\mathbb{R})$.

(g) The set does not contain the zero polynomial and hence cannot be a subspace of $P_2(\mathbb{R})$.

(h) By Theorem 4.1.3, this is a subspace of $P_3(\mathbb{R})$.

(i) By definition $\mathbb{S}_9$ is a subset of $M_{2\times 2}(\mathbb{R})$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}_2$ since $0 + 0 = 0$ and $0 = 2(0)$. Thus, we can apply the Subspace Test.

Let $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbb{S}_9$ and $t \in \mathbb{R}$. Then $a_1 + a_3 = 0$, $a_2 = 2a_4$, $b_1 + b_3 = 0$, and $b_2 = 2b_4$. Hence,

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} \in \mathbb{S}_9$$

since $(a_1 + b_1) + (a_3 + b_3) = a_1 + a_3 + b_1 + b_3 = 0 + 0 = 0$ and $a_2 + b_2 = 2a_4 + 2b_4 = 2(a_4 + b_4)$. Similarly,

$$t\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} ta_1 & ta_2 \\ ta_3 & ta_4 \end{bmatrix} \in \mathbb{S}_9$$

since $ta_1 + ta_3 = t(a_1 + a_3) = t(0) = 0$ and $ta_2 = t(2a_4) = 2(ta_4)$. Hence, $\mathbb{S}_9$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

4.1.2 (a) Consider

$$\begin{aligned}1 + x^2 &= c_1(1 + x + x^2) + c_2(-1 + 2x + 2x^2) + c_3(5 + x + x^2)\\ &= c_1 - c_2 + 5c_3 + (c_1 + 2c_2 + c_3)x + (c_1 + 2c_2 + c_3)x^2\end{aligned}$$

Comparing coefficients we get the system

$$\begin{aligned}c_1 - c_2 + 5c_3 &= 1\\ c_1 + 2c_2 + c_3 &= 0\\ c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

Row reducing the augmented matrix gives

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & 1\\ 1 & 2 & 1 & 0\\ 1 & 2 & 1 & 1\end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 11/3 & 0\\ 0 & 1 & -4/3 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$$

Hence, the system is inconsistent, so $\vec{v} \notin \operatorname{Span}\mathcal{B}$.

(b) Consider

$$\begin{bmatrix}2 & 5\\ 4 & 4\end{bmatrix} = c_1\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 1\\ 2 & 2\end{bmatrix} + c_3\begin{bmatrix}3 & 1\\ -3 & 1\end{bmatrix} = \begin{bmatrix}c_1 + 3c_3 & c_2 + c_3\\ -c_1 + 2c_2 - 3c_3 & c_1 + 2c_2 + c_3\end{bmatrix}$$

Comparing entries we get the system

$$\begin{aligned}c_1 + 3c_3 &= 2\\ c_2 + c_3 &= 5\\ -c_1 + 2c_2 - 3c_3 &= 4\\ c_1 + 2c_2 + c_3 &= 4\end{aligned}$$

Row reducing the augmented matrix gives

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 2\\ 0 & 1 & 1 & 5\\ -1 & 2 & -3 & 4\\ 1 & 2 & 1 & 4\end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & -4\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{array}\right]$$

Hence, the system is consistent, so $\vec{v} \in \operatorname{Span}\mathcal{B}$.

(c) Consider

$$\begin{bmatrix}0\\ -3\\ 3\\ -6\end{bmatrix} = c_1\begin{bmatrix}1\\ 2\\ 0\\ -1\end{bmatrix} + c_2\begin{bmatrix}1\\ 3\\ 1\\ -1\end{bmatrix} + c_3\begin{bmatrix}2\\ 4\\ 3\\ -5\end{bmatrix} = \begin{bmatrix}c_1 + c_2 + 2c_3\\ 2c_1 + 3c_2 + 4c_3\\ c_2 + 3c_3\\ -c_1 - c_2 - 5c_3\end{bmatrix}$$

Comparing entries we get the system

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 3c_2 + 4c_3 &= -3 \\ c_2 + 3c_3 &= 3 \\ -c_1 - c_2 - 5c_3 &= -6 \end{aligned}$$

Row reducing the augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 3 & 4 & -3 \\ 0 & 1 & 3 & 3 \\ -1 & -1 & -5 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, the system is consistent, so $\vec{v} \in \operatorname{Span} \mathcal{B}$.

(d) Consider

$$\begin{aligned} -3x + 3x^2 - 6x^3 &= c_1(1 + 2x - x^3) + c_2(1 + 3x + x^2 - x^3) + c_3(2 + 4x + 3x^2 - 5x^3) \\ &= c_1 + c_2 + 2c_3 + (2c_1 + 3c_2 + 4c_3)x + (c_2 + c_3)x^2 + (-c_1 - c_2 - 5c_3)x^3 \end{aligned}$$

Comparing coefficients we get the system

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 3c_2 + 4c_3 &= -3 \\ c_2 + 3c_3 &= 3 \\ -c_1 - c_2 - 5c_3 &= -6 \end{aligned}$$

Observe this is the same system as in (c). Hence, $\vec{v} \in \operatorname{Span} \mathcal{B}$.

(e) Consider

$$\begin{bmatrix} 0 & -3 \\ 3 & -6 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + 2c_3 & 2c_1 + 3c_2 + 4c_3 \\ c_2 + 3c_3 & -c_1 - c_2 - 5c_3 \end{bmatrix}$$

Comparing entries we get the system

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 3c_2 + 4c_3 &= -3 \\ c_2 + 3c_3 &= 3 \\ -c_1 - c_2 - 5c_3 &= -6 \end{aligned}$$

Observe this is the same system as in (c). Hence, $\vec{v} \in \operatorname{Span} \mathcal{B}$.

4.1.3 (a) Consider

$$\begin{aligned} 0 + 0x + 0x^2 &= c_1(3 - 2x + x^2) + c_2(2 - 5x + 2x^2) + c_3(6 + 7x - 2x^2) \\ &= 3c_1 + 2c_2 + 6c_3 + (-2c_1 - 5c_2 + 7c_3)x + (c_1 + 2c_2 - 2c_3)x^2 \end{aligned}$$

This gives the homogeneous system of equations

$$\begin{aligned} 3c_1 + 2c_2 + 6c_3 &= 0 \\ -2c_1 - 5c_2 + 7c_3 &= 0 \\ c_1 + 2c_2 - 2c_3 &= 0 \end{aligned}$$

Row reducing the coefficient matrix gives

$$\begin{bmatrix} 3 & 2 & 6 \\ -2 & -5 & 7 \\ 1 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the system has infinitely many solutions, so $\mathcal{B}$ is linearly dependent.

(b) By Theorem 4.1.6 the set is linearly dependent, since it contains the zero vector.

(c) Consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 + c_3 & c_1 + 2c_2 - 2c_3 \\ -2c_1 + c_2 + 3c_3 & c_1 + 4c_3 \end{bmatrix}$$

This gives the homogeneous system

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ c_1 + 2c_2 - 2c_3 &= 0 \\ -2c_1 + c_2 + 3c_3 &= 0 \\ c_1 + 4c_3 &= 0 \end{aligned}$$

Row reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & -2 \\ -2 & 1 & 3 \\ 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the only solution is $c_1 = c_2 = c_3 = 0$, so $\mathcal{B}$ is linearly independent.

(d) Consider

$$\begin{aligned} 0 &= c_1(1 + x - 2x^2 + x^3) + c_2(2 + 2x + x^2) + c_3(1 - 2x + 3x^2 + 4x^3) \\ &= c_1 + 2c_2 + c_3 + (c_1 + 2c_2 - 2c_3)x + (-2c_1 + c_2 + 3c_3)x^2 + (c_1 + 4c_3)x^3 \end{aligned}$$

Hence, we have the same homogeneous system as in part (c) so this set is also linearly independent.

(e) Consider

$$\begin{bmatrix}0&0\\0&0\end{bmatrix} = c_1\begin{bmatrix}1&-1\\2&1\end{bmatrix}+c_2\begin{bmatrix}3&1\\2&1\end{bmatrix}+c_3\begin{bmatrix}-1&5\\0&1\end{bmatrix}+c_4\begin{bmatrix}3&1\\2&2\end{bmatrix} = \begin{bmatrix}c_1+3c_2-c_3+3c_4 & -1c_1+c_2+5c_3+c_4\\ 2c_1+2c_2+2c_4 & c_1+c_2+c_3+2c_4\end{bmatrix}$$

This gives the homogeneous system

$$\begin{aligned} c_1+3c_2-c_3+3c_4 &= 0\\ -c_1+c_2+5c_3+c_4 &= 0\\ 2c_1+2c_2+2c_4 &= 0\\ c_1+c_2+c_3+2c_4 &= 0 \end{aligned}$$

Row reducing the coefficient matrix gives

$$\begin{bmatrix}1&3&-1&3\\-1&1&5&1\\2&2&0&2\\1&1&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Hence, the only solution is $c_1 = c_2 = c_3 = c_4 = 0$. Therefore, the set is linearly independent.

4.1.4 For any $\vec{x} \in \mathbb{R}^3$ we have

$$\begin{aligned} O(\vec{x}) &= (t_1L_1+t_2L_2+t_3L_3)(\vec{x}) = t_1L_1(\vec{x})+t_2L_2(\vec{x})+t_3L_3(\vec{x})\\ \begin{bmatrix}0\\0\end{bmatrix} &= \begin{bmatrix}t_1(x_1+3x_2)+t_2(2x_1+5x_2)+t_3(x_1+x_2)\\ t_1(6x_1+x_2)+t_2(8x_1+3x_2)+t_3(-2x_1+3x_2)\end{bmatrix}\\ \begin{bmatrix}0x_1+0x_2\\0x_1+0x_2\end{bmatrix} &= \begin{bmatrix}(t_1+2t_2+t_3)x_1+(3t_1+5t_2+t_3)x_2\\ (6t_1+8t_2-2t_3)x_1+(t_1+3t_2+3t_3)x_2\end{bmatrix} \end{aligned}$$

Hence, we have the system of equations

$$\begin{aligned} t_1+2t_2+t_3 &= 0\\ 3t_1+5t_2+t_3 &= 0\\ 6t_1+8t_2-2t_3 &= 0\\ t_1+3t_2+3t_3 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}1&2&1\\3&5&1\\6&8&-2\\1&3&3\end{bmatrix} \sim \begin{bmatrix}1&0&-3\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}$$

Since there are infinitely many solutions, the set is linearly dependent.

4.1.5 (a) The statement is true. Assume that $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ is any subset of $\mathcal{B}$ that is linearly dependent. Then there exists coefficients $c_1, \dots, c_\ell$ not all zero such that

$$c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell = \vec{0}$$

But then we have

$$c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell + 0\vec{v}_{\ell+1} + \cdots + 0\vec{v}_k = \vec{0}$$

with not all coefficients 0 which contradicts $\mathcal{B}$ is linearly independent. Hence, any subset of $\mathcal{B}$ is linearly independent.

(b) The statement is false. Take $\vec{v}_1 = \vec{0}$, $\vec{v}_2 \neq \vec{0}$, and $c = 0$. Then, $c\vec{v}_1 = c\vec{v}_2$, but $\vec{v}_1 \neq \vec{v}_2$.

(c) The statement is true. We have $0\vec{x} = \vec{0} \in \mathbb{S}$ by Theorem 4.1.1. If $c \neq 0$, then for any $\vec{z} \in \mathbb{V}$ we can let $\vec{w} = \frac{1}{c}\vec{z}$. Then, we get

$$\vec{z} + c\vec{0} = c\vec{w} + c\vec{0} = c(\vec{w} + \vec{0}) = c\vec{w} = \vec{z}$$

Hence, $c\vec{0} = \vec{0}$ since the zero vector is unique.

4.1.6 By definition $\{\vec{0}_\mathbb{V}\}$ is a non-empty subset of $\mathbb{V}$. Hence, we can apply the Subspace Test. Let $\vec{x}, \vec{y} \in \mathbb{S}$. Then, $\vec{x} = \vec{0}_\mathbb{V} = \vec{y}$. Hence,

$$\vec{x} + \vec{y} = \vec{0}_\mathbb{V} + \vec{0}_\mathbb{V} = \vec{0}_\mathbb{V} \qquad \text{by V4 since } \mathbb{V} \text{ is a vector space}$$

Hence, $\vec{x} + \vec{y} \in \{\vec{0}_\mathbb{V}\}$. We need to show that $s\vec{x} = \vec{0}$ for all $s \in \mathbb{R}$. We have $0\vec{x} = \vec{0} \in \mathbb{S}$ by Theorem 4.1.1. If $s \neq 0$, then for any $\vec{z} \in \mathbb{V}$ we can let $\vec{w} = \frac{1}{s}\vec{z}$. Then, we get

$$\vec{z} + s\vec{0} = s\vec{w} + s\vec{0} = s(\vec{w} + \vec{0}) = s\vec{w} = \vec{z}$$

Hence, $s\vec{0} = \vec{0} \in \{\vec{0}_\mathbb{V}\}$. Therefore, $\{\vec{0}_\mathbb{V}\}$ is a subspace of $\mathbb{V}$. Consequently, by definition, it is a vector space under the same operations as $\mathbb{V}$.

4.1.7 It is not a vector space since we know that matrix multiplication is not commutative. That is, in general $A \oplus B = AB \neq AB = B \oplus A$.

4.1.8 Denote the set of all sequences by $\mathbb{S}$. Let $\vec{s} = \{s_1, s_2, s_3, \dots\})$, $\vec{t} = \{t_1, t_2, t_3, \dots\}$, $\vec{u} = \{u_1, u_2, u_3, \dots\}$, and let $a, b \in \mathbb{R}$.

V1 By definition $\vec{s} + \vec{t} = \{s_1 + t_1, s_2 + t_2, \dots\} \in \mathbb{S}$.

V2 We have

$$\begin{aligned}\vec{s} + (\vec{t} + \vec{u}) &= \vec{s} + \{t_1 + u_1, t_2 + u_2, \dots\} = \{s_1 + (t_1 + u_1), s_2 + (t_2 + u_2), \dots\} \\ &= \{(s_1 + t_1) + u_1, (s_2 + t_2) + u_2, \dots\} = \{s_1 + t_1, s_2 + t_2, \dots\} + \vec{u} = (\vec{s} + \vec{t}) + \vec{u}\end{aligned}$$

V3 We have

$$\vec{s} + \vec{t} = \{s_1 + t_1, s_2 + t_2, \dots\} = \{t_1 + s_1, t_2 + s_2, \dots\} = \vec{t} + \vec{s}$$

V4 Let $\vec{0} = \{0, 0, \dots\}$. Then, $\vec{0} \in \mathbb{S}$ and

$$\vec{s} + \vec{0} = \{s_1 + 0, s_2 + 0, \dots\} = \{s_1, s_2, \dots\} = \vec{s}$$

V5 Let $-\vec{s} = \{-s_1, -s_2, \ldots\}$. Then, $-\vec{s} \in \mathbb{S}$ and

$$\vec{s} + (-\vec{s}) = \{s_1 + (-s_1), s_2 + (-s_2), \ldots\} = \{0, 0, \ldots\} = \vec{0}$$

V6 By definition $a\vec{s} = \{as_1, as_2, \ldots\} \in \mathbb{S}$

V7 We have

$$a(b\vec{s}) = a\{bs_1, bs_2, \ldots\} = \{a(bs_1), a(bs_2), \ldots\} = \{(ab)s_1, (ab)s_2, \ldots\} = (ab)\{s_1, s_2, \ldots\} = (ab)\vec{s}$$

V8 We have

$$(a+b)\vec{s} = \{(a+b)s_1, (a+b)s_2, \ldots\} = \{as_1+bs_1, as_2+bs_2), \ldots\} = \{as_1, as_2, \ldots\}+\{bs_1, bs_2, \ldots\} = a\{s_1, s_2, \ldots\}+b\{s_1$$

V9 We have

$$a(\vec{s}+\vec{t}) = a\{s_1+t_1, s_2+t_2, \ldots\} = \{a(s_1+t_1), a(s_2+t_2), \ldots\} = \{as_1+at_1, as_2+at_2, \ldots\} = \{as_1, as_2, \ldots\}+\{at_1, at_2, \ldots\} =$$

V10 We have

$$1\vec{s} = \{1s_1, 1s_2, \ldots\} = \{s_1, s_2, \ldots\} = \vec{s}$$

Thus, $\mathbb{S}$ is a vector space under these operations.

4.1.9 Let $\mathbb{V}$ be a vector space. Prove that $(-\vec{x}) = (-1)\vec{x}$ for all $\vec{x} \in \mathbb{V}$. We have

$$\begin{aligned}
\vec{0} &= \vec{x} + (-\vec{x}) && \text{by V5}\\
0\vec{x} &= \vec{x} + (-\vec{x}) && \text{by Theorem 4.1.1(1)}\\
(1-1)\vec{x} &= \vec{x} + (-\vec{x}) && \text{operations on reals}\\
1\vec{x} + (-1)\vec{x} &= \vec{x} + (-\vec{x}) && \text{by V8}\\
\vec{x} + (-1)\vec{x} &= \vec{x} + (-\vec{x}) && \text{by V10}\\
(-\vec{x}) + \vec{x} + (-1)\vec{x} &= (-\vec{x}) + \vec{x} + (-\vec{x}) && \text{by V5}\\
\vec{0} + (-1)\vec{x} &= \vec{0} + (-\vec{x}) && \text{by V5}\\
(-1)\vec{x} &= (-\vec{x}) && \text{by V4}
\end{aligned}$$

4.1.10 We are assuming that there exist $c_1, \ldots, c_{k-1} \in \mathbb{R}$ such that

$$c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_i\vec{v}_{i+1} + \cdots + c_{k-1}\vec{v}_k = \vec{v}_i$$

Let $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. Then, there exist $d_1, \ldots, d_k \in \mathbb{R}$ such that

$$\begin{aligned}
\vec{x} &= d_1\vec{v}_1 + \cdots + d_{i-1}\vec{v}_{i-1} + d_i\vec{v}_i + d_{i+1}\vec{v}_{i+1} + \cdots + d_k\vec{v}_k\\
&= d_1\vec{v}_1 + \cdots + d_{k-1}\vec{v}_{k-1} + d_i(c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_i\vec{v}_{i+1} + \cdots + c_{k-1}\vec{v}_k) + d_{i+1}\vec{v}_{i+1} + \cdots + d_k\vec{v}_k\\
&= (d_1 + d_ic_1)\vec{v}_1 + \cdots + (d_{i-1} + d_ic_{i-1})\vec{v}_{i-1} + (d_{i+1} + d_ic_i)\vec{v}_{i+1} + \cdots + (d_k + d_ic_{k-1})\vec{v}_k
\end{aligned}$$

Thus, $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$. Hence, $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$. Clearly, we have $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\} \subseteq \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ and so

$$\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$$

as required.

4.1.11 If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent, then there exists $c_1, \dots, c_k \in \mathbb{R}$ not all zero such that

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

Assume that $c_i \neq 0$. Then, we get

$$\vec{v}_i = \frac{-c_1}{c_i}\vec{v}_1 + \cdots + \frac{-c_{i-1}}{c_i}\vec{v}_{i-1} + \frac{-c_{i+1}}{c_i}\vec{v}_{i+1} + \cdots + \frac{-c_k}{c_i}\vec{v}_k$$

Thus, $\vec{v}_i \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$.

On the other hand, if $\vec{v}_i \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$, then there exists, $c_1, \dots, c_{k-1}$ such that

$$c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_i\vec{v}_{i+1} + \cdots + c_{k-1}\vec{v}_k = \vec{v}_i$$

Thus,

$$c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} - \vec{v}_i + c_i\vec{v}_{i+1} + \cdots + c_{k-1}\vec{v}_k = \vec{0}$$

Hence, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent.

4.2 Problem Solutions

4.2.1 (a) Consider

$$\begin{aligned}0 &= c_1(1+x^2)+c_2(1-x+x^3)+c_3(2x+x^2-3x^3)+c_4(1+4x+x^2)+c_5(1-x-x^2+x^3)\\ &= (c_1+c_2+c_4+c_5)+(-c_2+2c_3+4c_4-c_5)x+(c_1+c_3+c_4-c_5)x^2+(c_2-3c_3+c_5)x^3\end{aligned}$$

This gives us a homogeneous system of 4 equations in 5 unknowns. Hence, by Theorem 2.2.4, there must be at least one parameter in the general solution, so there are infinitely many solutions. Thus, $\mathcal{B}$ is linearly dependent and hence not a basis.

(b) We need to show that for any polynomial $p(x) = a + bx + cx^2$ we can find c_1, c_2, c_3 such that

$$\begin{aligned}a+bx+cx^2 &= c_1(1+2x+x^2)+c_2(2+9x)+c_3(3+3x+4x^2)\\ &= (c_1+2c_2+3c_3)+(2c_1+9c_2+3c_3)x+(c_1+4c_3)x^2\end{aligned}$$

ow reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix}1&2&3\\2&9&3\\1&0&4\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Hence, there is a leading one in each row, so the system is consistent for every right hand side $\begin{bmatrix}a\\b\\c\end{bmatrix}$ and $\mathcal{B}$ spans $P_2(\mathbb{R})$ by Theorem 2.2.4. Moreover, if we let $a = b = c = 0$, we see that we get the unique solution $c_1 = c_2 = c_3 = 0$, and hence $\mathcal{B}$ is also linearly independent. Therefore, $\mathcal{B}$ is a basis for $P_2(\mathbb{R})$.

(c) Consider

$$\begin{bmatrix}0&0\\0&0\end{bmatrix} = c_1\begin{bmatrix}1&1\\0&1\end{bmatrix}+c_2\begin{bmatrix}1&2\\0&-1\end{bmatrix}+c_3\begin{bmatrix}1&3\\0&2\end{bmatrix} = \begin{bmatrix}c_1+c_2+c_3 & c_1+2c_2+3c_3\\0 & c_1-c_2+2c_3\end{bmatrix}$$

This gives us the homogeneous system of linear equations

$$\begin{aligned}0 &= c_1+c_2+c_3\\0 &= c_1+2c_2+3c_3\\0 &= c_1-c_2+2c_3\end{aligned}$$

Row reducing the coefficient matrix of the homogeneous system we get

$$\begin{bmatrix}1&1&1\\1&2&3\\1&-1&2\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Hence, the only solution is $c_1 = c_2 = c_3 = 0$, so $\mathcal{B}$ is linearly independent.

To show it is also a spanning set we need to show that any matrix $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ can be written as a linear combination of of the elements of the set. We consider

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

This gives a system of linear equations with the same coefficient matrix as above. So, using the same row-operations we did above we see that the matrix has a leading one in each row of its RREF, so by Theorem 2.2.4, the system is consistent for all matrices $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, so $\mathcal{B}$ spans $\mathbb{U}$.

Thus, $\mathcal{B}$ is a basis for $\mathbb{U}$.

(d) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 1 & 2 \end{bmatrix} \right\}$ of $M_{2\times 2}(\mathbb{R})$ Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ and consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = c_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} + c_3 \begin{bmatrix} 9 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} c_1 + 5c_2 + 9c_3 & 2c_1 + 6c_2 \\ 3c_1 + 7c_2 + c_3 & 4c_1 + 8c_2 + 2c_3 \end{bmatrix}$$

This gives a system of 4 equations in 3 unknowns. Since the rank of the coefficient matrix is at most 3 we get by Theorem 2.2.4 that the equation cannot be consistent for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ and hence $\mathcal{B}$ does not span $M_{2\times 2}(\mathbb{R})$. Thus, $\mathcal{B}$ is not a basis.

4.2.2 (a) Observe that every vector in $\mathbb{S}_1$ has the form $xp(x) = x(a + bx + cx^2) = ax + bx^2 + cx^3$. Hence, $\mathbb{S}_1 = \operatorname{Span}\{x, x^2, x^3\}$. Clearly $\{x, x^2, x^3\}$ is also linearly independent, and hence it is a basis for $\mathbb{S}_1$ and $\dim \mathbb{S}_1 = 3$.

(b) Observe that every vector in $\mathbb{S}_2$ has the form

$$\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 0 & -a_1 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ spans $\mathbb{S}_2$ and is clearly linearly independent. Consequently, $\mathcal{B}$ is a basis for $\mathbb{S}_2$ and $\dim \mathbb{S}_2 = 2$.

(c) Let $p(x) = ax^2 + bx + c$ be any vector in $\mathbb{S}_3$. Then $p(2) = 0$, so $a(2)^2 + b(2) + c = 0$. Thus, $c = -4a - 2b$. Hence, every $p(x) \in \mathbb{S}_3$ has the form

$$ax^2 + bx + (-4a - 2b) = a(x^2 - 4) + b(x - 2)$$

Thus, $\mathbb{S}_3 = \operatorname{Span}\{x^2 - 4, x - 2\}$. Since $\{x^2 - 4, x - 2\}$ is also clearly linearly independent, it is a basis for $\mathbb{S}_3$. Thus, $\dim \mathbb{S}_3 = 2$.

(d) By definition a basis is the empty set and the dimension is 0.

(e) Let $p(x) = a + bx + cx^2$ be any vector in $\mathbb{S}_5$. Then $a = c$, so every $p(x) \in \mathbb{S}_5$ has the form

$$a + bx + ax^2 = a(1 + x^2) + bx$$

Thus, $\mathbb{S}_5 = \operatorname{Span}\{1 + x^2, x\}$. Since $\{1 + x^2, x\}$ is also clearly linearly independent, it is a basis for $\mathbb{S}_5$. Thus, $\dim \mathbb{S}_5 = 2$.

(f) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{S}_6$. Then A satisfies $A^T = A$ which implies that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Hence, $b = c$. Thus, every $A \in \mathbb{S}_6$ has the form

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ spans $\mathbb{S}_6$ and is clearly linearly independent, so $\mathcal{B}$ is a basis for $\mathbb{S}_6$. Thus, $\dim \mathbb{S}_6 = 3$.

(g) Every $A \in \mathbb{S}_7$ has the form

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} a_1 & 2a_4 \\ -a_1 & a_4 \end{bmatrix} = a_1\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + a_4\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\right\}$ spans $\mathbb{S}_7$ and is clearly linearly independent, so $\mathcal{B}$ is a basis for $\mathbb{S}_7$. Thus, $\dim \mathbb{S}_7 = 2$.

(h) Consider

$$\begin{aligned} 0 &= c_1(1+x) + c_2(1+x+x^2) + c_3(x+2x^2) + c_4(x+x^2+x^3) + c_5(1+x^2+x^3) \\ &= c_1 + c_2 + c_5 + (c_1+c_2+c_3+c_4)x + (c_2+2c_3+c_4+c_5)x^2 + (c_4+c_5)x^3 \end{aligned}$$

Row reducing the coefficient matrix of the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus, treating this an augmented matrix we get that $1+x^2+x^3$ can be written as a linear combination of the other vectors. Ignoring the last column shows that $\{1+x, 1+x+x^2, x+2x^2, x+x^2+x^3\}$ is linearly independent, and hence a basis for the set it spans. Thus, $\dim \mathbb{S}_8 = 4$.

(i) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + c_4\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + c_5\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 + c_3 \\ -3c_1 + 4c_2 - c_3 + 2c_4 + c_5 \\ 2c_1 - c_2 + 4c_3 + 2c_4 + c_5 \end{bmatrix}$$

Row reducing the coefficient matrix of the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -3 & 4 & -1 & 2 & 1 \\ 2 & -1 & 4 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we see that the last three vectors can be written as a linear combination of the first two vectors. Moreover, the first two columns shows that the set $\left\{\begin{bmatrix}1\\-3\\2\end{bmatrix}, \begin{bmatrix}-1\\4\\-1\end{bmatrix}\right\}$ is linearly independent, and hence a basis for the set it spans. Thus, $\dim \mathbb{S}_9 = 2$.

4.2.3 $\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$

4.2.4 (1) This is the contrapositive of Theorem 4.2.1.

(2) Assume $\{\vec{v}_1, \ldots, \vec{v}_k\}$ spans $\mathbb{V}$ where $k < n$. Then, we can use Theorem 4.1.1 to remove linearly dependent vectors (if any) to get a basis for $\mathbb{V}$. Thus, we can find a basis for $\mathbb{V}$ with fewer than n vectors which contradicts Theorem 4.2.4.

(3) Assume that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent, but does not span $\mathbb{V}$. Then, there exists $\vec{v} \in \mathbb{V}$ such that $\vec{v} \notin \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$. Hence, by Theorem 4.2.2 we have that $\{\vec{v}_1, \ldots, \vec{v}_n, \vec{v}\}$ is a linearly independent set of $n + 1$ vectors in $\mathbb{V}$. But, this contradicts (1). Therefore, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ also spans $\mathbb{V}$.

Assume $\{\vec{v}_1, \ldots, \vec{v}_n\}$ spans $\mathbb{V}$, but is linearly dependent. Then, there exists some vector $\vec{v}_i \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\}$. Hence, by Theorem 4.2.1 we have that

$$\mathbb{V} = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\}$$

So, $\mathbb{V}$ is spanned by $n - 1$ vectors which contradicts (2). Thus, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is also linearly independent.

4.2.5 Since we know that $\dim P_3(\mathbb{R}) = 4$, we need to add two vectors to the set $\{1 + x + x^3, 1 + x^2\}$ so that the set is still linearly independent. There are a variety of ways of picking such vectors. We observe that neither x^2 nor x can be written as a linear combination of $1 + x + x^3$ and $1 + x^2$, so we will try to prove that $\mathcal{B} = \{1 + x + x^3, 1 + x^2, x^2, x\}$ is a basis for P_3. Consider

$$0 = c_1(1 + x + x^3) + c_2(1 + x^2) + c_3(x^2) + c_4x = (c_1 + c_2) + (c_1 + c_4)x + (c_2 + c_3)x^2 + c_1x^3$$

Solving the corresponding homogeneous system we get $c_1 = c_2 = c_3 = c_4 = 0$, so $\mathcal{B}$ is a linearly independent set of 4 vectors in $P_3(\mathbb{R})$ and hence is a basis for $P_3(\mathbb{R})$ which includes $1 + x + x^3$ and $1 + x^2$.

4.2.6 Let $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ denote the standard basis vectors for $M_{2\times2}(\mathbb{R})$. Then, clearly $\{\vec{v}_1, \vec{v}_2, \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ spans $M_{2\times2}(\mathbb{R})$. Consider

$$\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{e}_1 + c_4\vec{e}_2 + c_5\vec{e}_3 + c_6\vec{e}_4$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}1&1&1&0&0&0\\2&2&0&1&0&0\\2&3&0&0&1&0\\1&4&0&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0&0&4/5&-3/5\\0&1&0&0&-1/5&2/5\\0&0&1&0&-3/5&1/5\\0&0&0&1&-6/5&2/5\end{bmatrix}$$

If we ignore the last two columns, we see that this implies that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{e}_1, \vec{e}_2\}$ is a linearly independent set of 4 vectors in $M_{2\times2}(\mathbb{R})$. Thus, since $\dim M_{2\times2}(\mathbb{R}) = 4$ we get that $\mathcal{B}$ is linearly independent by Theorem 4.2.3.

4.2.7 Three vectors in the hyperplane are $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$. Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} c_1 \\ -c_1 + 2c_2 \\ -c_2 + c_3 \\ -2c_3 \end{bmatrix}$$

The only solution is $c_1 = c_2 = c_3 = 0$ so the vectors are linearly independent. Since a hyperplane in $\mathbb{R}^4$ has dimension 3 and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of three vectors in the hyperplane, it forms a basis for the hyperplane.

We need to add a vector $\vec{v}_4$ so that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$. Since $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ does not satisfy the equation of the hyperplane, it is not in $\operatorname{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ and so is not a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Thus, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent set with four elements in $\mathbb{R}^4$ and hence is a basis for $\mathbb{R}^4$ since $\mathbb{R}^4$ has dimension 4.

4.2.8 Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for $\mathbb{S}$. Then, $\mathcal{B}$ is a linearly independent set of k vectors in $\mathbb{V}$. But, $\dim \mathbb{V} = \dim \mathbb{S} = k$, so $\mathcal{B}$ is also a basis for $\mathbb{V}$. Hence, $\mathbb{V} = \mathbb{S}$.

4.2.9 Since $k < n$, we have that $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \neq \mathbb{V}$. Thus, there exists a vector $\vec{v}_{k+1} \notin \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$. Thus, by Theorem 4.1.5, $\{\vec{v}_1, \dots, \vec{v}_{k+1}\}$ is linearly independent.

If $k + 1 = n$, then $\{\vec{v}_1, \dots, \vec{w}_{k+1}\}$ is a basis for $\mathbb{V}$. If not, we can keep repeating this procedure until we get $\{\vec{v}_1, \dots, \vec{w}_{k+1}, \vec{w}_{k+2}, \dots, \vec{w}_n\}$.

4.2.10 (a) This statement is true as it is the contrapositive of Theorem 4.2.3(1).

(b) This is false. Since $\dim P_2(\mathbb{R}) = 3$, we have that every basis for $P_2(\mathbb{R})$ has 3 vectors in it.

(c) This is false. Taking $a = b = 1$ and $c = d = 2$, gives $\{\vec{v}_1 + \vec{v}_2, 2\vec{v}_1 + 2\vec{v}_2\}$ which is clearly linearly dependent.

(d) This is true. If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, then it is a basis. If it is linearly dependent, then by Theorem 4.1.5, there is some vector $\vec{v}_i$ such that $\vec{v}_i \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$. Then, by Theorem 4.1.4, we have that $\mathbb{V} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}$. We can continue to repeat this process until we have a linearly indpendent spanning set.

(e) This is true. If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a linearly independent set in $\mathbb{V}$, then either $\dim \mathbb{V} = k$ or, by Theorem 4.2.3, there exist $\vec{w}_{k+1}, \dots, \vec{w}_n$ such that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_{k+1}, \dots, \vec{w}_n\}$ is a basis for $\mathbb{V}$. Thus, $\dim \mathbb{V} = n \geq k$.

(f) This is false. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ is clearly linearly dependent and hence not a basis for $M_{2\times 2}(\mathbb{R})$.

4.2.11 Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for $\mathbb{S}$ and $\mathcal{C} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be a basis for $\mathbb{T}$. Assume for a contradiction that $\mathbb{S} \cup \mathbb{T} = \{\vec{0}\}$. Then, $\vec{v}_i \notin \operatorname{Span} \mathcal{C}$ for $i = 1, 2, 3$. Therefore, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a linearly independent set of 6 vectors in $\mathbb{V}$, but this contradicts the fact that $\dim \mathbb{V} = 5$.

4.3 Problem Solutions

4.3.1 (a) We have $\vec{x} = \sqrt{2}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} + (-4)\begin{bmatrix} 3 & 6 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} - 12 & 3\sqrt{2} - 24 \\ -\sqrt{2} + 16 & 2\sqrt{2} - 12 \end{bmatrix}$

(b) $\vec{x} = 1(1 + x) + 0(1 + x + x^2) + 3(1 + x^2) = 4 + x + 3x^2$

4.3.2 We need to find c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}$$

$$d_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix} + d_3 \begin{bmatrix} -2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -3 \end{bmatrix}$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{ccc|cc} 1 & 3 & -2 & -2 & 4 \\ 2 & 7 & -3 & -5 & 8 \\ 0 & 3 & 5 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 11 & -7/2 \\ 0 & 1 & 0 & -3 & 3/2 \\ 0 & 0 & 1 & 2 & -3/2 \end{array}\right]$$

Thus, $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -3 \\ 2 \end{bmatrix}$ and $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -7/2 \\ 3/2 \\ -3/2 \end{bmatrix}$.

4.3.3 (a) We need to find c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$c_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$$
$$d_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + d_2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 3 & 7 \end{bmatrix}$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 3 \\ -1 & 1 & 2 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, for the first system we have $c_1 = -2$, $c_2 = 3$, and $c_3 = -1$, and for the second system we have $d_1 = -2$, $d_2 = 1$, and $d_3 = 2$. Thus,

$$[A]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad [B]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

(b) We need to find c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$c_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$
$$d_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + d_3 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 1 & 4 \end{bmatrix}$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & -4 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, for the first system we have $c_1 = -2$, $c_2 = 3$, and $c_3 = 1$, and for the second system we have $d_1 = -2$, $d_2 = 3$, and $d_3 = -1$. Thus,

$$[A]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad [B]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

4.3.4 (a) We need to find c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$c_1(1 + x + x^2) + c_2(1 + 3x + 2x^2) + c_3(4 + x^2) = -2 + 8x + 5x^2$$
$$d_1(1 + x + x^2) + d_2(1 + 3x + 2x^2) + d_3(4 + x^2) = -4 + 8x + 4x^2$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{ccc|cc} 1 & 1 & 4 & -2 & -4 \\ 1 & 3 & 0 & 8 & 8 \\ 1 & 2 & 1 & 5 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 5 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & -2 \end{array}\right]$$

Hence, $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$ and $[q(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$.

(b) We need to find c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$c_1(1 + x - 4x^2) + c_2(2 + 3x - 3x^2) + c_3(3 + 6x + 4x^2) = -3 - 5x - 3x^2$$
$$d_1(1 + x - 4x^2) + d_2(2 + 3x - 3x^2) + d_3(3 + 6x + 4x^2) = x$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & -3 & 0 \\ 1 & 3 & 6 & -5 & 1 \\ -4 & -3 & 4 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -14 & -17 \\ 0 & 1 & 0 & 13 & 16 \\ 0 & 0 & 1 & -5 & -5 \end{array}\right]$$

Hence, $[p(x)]_{\mathcal{B}} = \begin{bmatrix} -14 \\ 13 \\ -5 \end{bmatrix}$ and $[q(x)]_{\mathcal{B}} = \begin{bmatrix} -17 \\ 16 \\ -5 \end{bmatrix}$.

4.3.5 To find the change of coordinates matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates, we need to determine the $\mathcal{C}$-coordinates of the vectors in $\mathcal{B}$. That is, we need to find c_1, c_2, d_1, d_2 such that

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad d_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

We row reduce the corresponding doubly augmented matrix to get

$$\left[\begin{array}{cc|cc} 2 & 5 & 3 & 5 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 5 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

Thus, ${}_{\mathcal{C}}P_{\mathcal{B}} = \begin{bmatrix} -1 & 5 \\ 1 & -1 \end{bmatrix}$.

To find the change of coordinates matrix from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates, we need to determine the $\mathcal{B}$-coordinates of the vectors in $\mathcal{C}$. That is, we need to find c_1, c_2, d_1, d_2 such that

$$c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad d_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

We row reduce the corresponding doubly augmented matrix to get

$$\left[\begin{array}{cc|cc} 3 & 5 & 2 & 5 \\ 1 & 3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/4 & 5/4 \\ 0 & 1 & 1/4 & 1/4 \end{array}\right]$$

Thus, ${}_{\mathcal{B}}P_{\mathcal{C}} = \begin{bmatrix} 1/4 & 5/4 \\ 1/4 & 1/4 \end{bmatrix}$.

4.3.6 To find ${}_{\mathcal{B}}P_{\mathcal{C}}$ we need to find the $\mathcal{B}$-coordinates of the vectors in $\mathcal{C}$. Thus, we need to solve the systems

$$\begin{aligned} b_1(1) + b_2(-1+x) + b_3(1-2x+x^2) &= 1+x+x^2 \\ c_1(1) + c_2(-1+x) + c_3(1-2x+x^2) &= 1+3x-x^2 \\ d_1(1) + d_2(-1+x) + d_3(1-2x+x^2) &= 1-x-x^2 \end{aligned}$$

Row reducing the corresponding triply augmented matrix gives

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & 3 & 1 & -3 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

Thus, ${}_{\mathcal{B}}P_{\mathcal{C}} = \begin{bmatrix} 3 & 3 & -1 \\ 3 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$.

Similarly, to find ${}_{\mathcal{C}}P_{\mathcal{B}}$ we find the $\mathcal{C}$-coordinates of the vectors in $\mathcal{B}$ by row reducing the corresponding triply augmented matrix to get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 3 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1 \\ 0 & 1 & 0 & 0 & 1/4 & -3/4 \\ 0 & 0 & 1 & 1/2 & -3/4 & 3/4 \end{array}\right]$$

Thus, ${}_{\mathcal{C}}P_{\mathcal{B}} = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 0 & 1/4 & -3/4 \\ 1/2 & -3/4 & 3/4 \end{bmatrix}$.

4.3.7 (a) To find ${}_{\mathcal{S}}P_{\mathcal{B}}$ we need to find the $\mathcal{S}$-coordinates of the vectors in $\mathcal{B}$. We have

$$[1-x+x^2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad [1-2x+3x^2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad [1-2x+4x^2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Hence, ${}_{\mathcal{S}}P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 1 & 3 & 4 \end{bmatrix}$.

To find ${}_{\mathcal{B}}P_{\mathcal{S}}$ we need to find the $\mathcal{B}$-coordinates of the vectors in $\mathcal{S}$. Thus, we need to solve the systems

$$\begin{aligned} b_1(1-x+x^2) + b_2(1-2x+3x^2) + b_3(1-2x+4x^2) &= 1 \\ c_1(1-x+x^2) + c_2(1-2x+3x^2) + c_3(1-2x+4x^2) &= x \\ d_1(1-x+x^2) + d_2(1-2x+3x^2) + d_3(1-2x+4x^2) &= x^2 \end{aligned}$$

Row reducing the corresponding triply augmented matrix gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & -2 & -2 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 & -3 & -1 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$$

Hence, ${}_{\mathcal{B}}P_{\mathcal{S}} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

(b) We have

$$[p(x)]_{\mathcal{S}} = {}_{\mathcal{S}}P_{\mathcal{B}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Hence, $p(x) = 2 - x - 2x^2$.

(c) We have

$$[q(x)]_{\mathcal{B}} = {}_{\mathcal{B}}P_{\mathcal{S}}[q(x)]_{\mathcal{S}} = \begin{bmatrix} 2 & 1 & 0 \\ -2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

(d) We have

$$[r(x)]_{\mathcal{S}} = {}_{\mathcal{S}}P_{\mathcal{B}}[r(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $r(x) = 1 + x^2$, as expected.

4.3.8 To find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ we need to find the coordinates of the vectors in $\mathcal{B}$ with respect to $\mathcal{S}$. We have

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} = -1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,

$${}_{\mathcal{S}}P_{\mathcal{B}} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 3 & -3 \\ 3 & 0 & 1 \end{bmatrix}$$

To find the change of coordinates matrix from $\mathcal{S}$ to $\mathcal{B}$ we need to find the coordinates of the vectors in $\mathcal{S}$ with respect to $\mathcal{B}$. We need to find $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a_1\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + a_2\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} + a_3\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = b_1\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + b_2\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} + b_3\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = c_1\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + c_2\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

Row reducing the augmented matrix to the corresponding system of equations gives

$$\left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -3 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/9 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1/3 & 1 \end{array}\right]$$

Hence,

$$_{\mathcal{B}}P_{\mathcal{S}} = \begin{bmatrix} 1/3 & 1/9 & 0 \\ -1 & 0 & 1 \\ -1 & -1/3 & 1 \end{bmatrix}$$

It is easy to check that $_{\mathcal{B}}P_{\mathcal{S}}\,{}_{\mathcal{S}}P_{\mathcal{B}} = I$.

4.3.9 Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & 8 & 0 \end{bmatrix}$. Find a matrix B such that $AB = I$. We can think of A as being the change of coordinates matrix $_{\mathcal{S}}P_{\mathcal{B}}$ from the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ to the standard basis $\mathcal{S}$ for $\mathbb{R}^3$. Hence, Theorem 4.3.3 tells us that we can take $B = {}_{\mathcal{B}}P_{\mathcal{S}}$. So, we need find the $\mathcal{B}$-coordinates of the vectors in $\mathcal{S}$. That is, we need to solve

$$b_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b_2 \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} + b_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$d_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d_2 \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} + d_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Row reducing the corresponding triply augmented matrix gives

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/3 & 8/3 & -1 \\ 0 & 1 & 0 & 1/2 & -1 & 1/2 \\ 0 & 0 & 1 & 2/3 & -1/3 & 0 \end{array}\right]$$

Thus, we take

$$B = {}_{\mathcal{B}}P_{\mathcal{S}} = \begin{bmatrix} -4/3 & 8/3 & -1 \\ 1/2 & -1 & 1/2 \\ 2/3 & -1/3 & 0 \end{bmatrix}$$

It is easy to verify that $AB = I$.

4.3.10 If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{V}$, show that $\{[\vec{v}_1]_{\mathcal{B}}, \dots, [\vec{v}_n]_{\mathcal{B}}\}$ is a basis for $\mathbb{R}^n$. Consider

$$\vec{0} = c_1[\vec{v}_1]_{\mathcal{B}} + \cdots + c_n[\vec{v}_n]_{\mathcal{B}}$$

Then, by Theorem 4.3.2, we get

$$\vec{0} = [c_1\vec{v}_1 + \cdots + c_n\vec{v}_n]_{\mathcal{B}}$$

Hence, by definition of $\mathcal{B}$-coordinates we get that

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0\vec{v}_1 + \cdots + 0\vec{v}_n = \vec{0}$$

Thus, since $\mathcal{B}$ is linearly independent, this implies that $c_1 = \cdots = c_n = 0$. Consequently, $\{[\vec{v}_1]_{\mathcal{B}}, \dots, [\vec{v}_n]_{\mathcal{B}}\}$ is a linearly independent set of n vectors in $\mathbb{R}^n$. Hence, by Theorem 4.2.3, it is a basis for $\mathbb{R}^n$.

4.3.11 Observe that $[\vec{v}_i]_{\mathcal{B}} = \vec{e}_i$. Thus, we have that $[\vec{v}_i]_{\mathcal{C}} = \vec{e}_i$ and so

$$\vec{v}_i = 0\vec{w}_1 + \cdots + 0\vec{w}_{i-1} + 1\vec{w}_i + 0\vec{w}_{i+1} + \cdots + 0\vec{w}_n = \vec{w}_i$$

Thus, $\vec{v}_i = \vec{w}_i$ for all i.

Chapter 5 Solutions

5.1 Problem Solutions

5.1.1 (a) Using the result of Example 5.1.4 we get

$$\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}^{-1} = \frac{1}{2(4)-1(-3)}\begin{bmatrix} 4 & -1 \\ -(-3) & 2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$$

(b) Using the result of Example 5.1.4 we get

$$\begin{bmatrix} 1 & 3 \\ 1 & -6 \end{bmatrix}^{-1} = \frac{1}{1(-6)-3(1)}\begin{bmatrix} -6 & -3 \\ -1 & 1 \end{bmatrix} = -\frac{1}{9}\begin{bmatrix} -6 & -3 \\ -1 & 1 \end{bmatrix}$$

(c) Using the result of Example 5.1.4 we get $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ is not invertible since $2(6) - 4(3) = 0$.

(d) Row reducing the multiple augmented system $[A \mid I]$ we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

Thus, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$.

(e) Row reducing the multiple augmented system $[A \mid I]$ we get

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -8/3 & 5/3 \\ 0 & 1 & 2 & 0 & 7/3 & -4/3 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

Therefore, A is not invertible since the RREF of A is not I.

(f) Row reducing the multiple augmented system $[A \mid I]$ we get

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 0 \\ 3 & 4 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -2/3 & 5/3 \\ 0 & 1 & 0 & 2 & 1/6 & -2/3 \\ 0 & 0 & 1 & 1 & 1/3 & -1/3 \end{array}\right]$$

Thus, $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 3 & 4 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & -2/3 & 5/3 \\ 2 & 1/6 & -2/3 \\ 1 & 1/2 & -1/3 \end{bmatrix}$.

(g) Row reducing the multiple augmented system $[A \mid I]$ we get

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/10 & -3/10 & 1/2 \\ 0 & 1 & 0 & 1/10 & 7/10 & -1/2 \\ 0 & 0 & 1 & -3/10 & -1/10 & 1/2 \end{array}\right]$$

Thus, $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/10 & -3/10 & 1/2 \\ 1/10 & 7/10 & -1/2 \\ -3/10 & -1/10 & 1/2 \end{bmatrix}$.

(h) Row reducing the multiple augmented system $[A \mid I]$ we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

Thus, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 \end{bmatrix}$.

(i) Row reducing the multiple augmented system $[A \mid I]$ we get

$$\left[\begin{array}{cccc|cccc} -1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 1 & -1 & -2 & 4 & 0 & 1 & 0 & 0 \\ -3 & -1 & 5 & 2 & 0 & 0 & 1 & 0 \\ -1 & -2 & 4 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} -1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & -3 & 6 & 1 & 1 & 0 & 0 \\ 0 & 0 & 11 & -10 & -4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

Therefore, A is not invertible since the RREF of A is not I.

(j) Row reducing the multiple augmented system $[A \mid I]$ we get

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -3 & 1 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & -5 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & 3 & 6 & 7 \\ 0 & 1 & 0 & 0 & -5 & -2 & -7 & -8 \\ 0 & 0 & 1 & 0 & 5 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & -4 & -1 & -5 & -6 \end{array}\right]$$

Thus, $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 \\ -1 & -3 & 1 & 4 \\ 0 & 1 & -3 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 & 6 & 7 \\ -5 & -2 & -7 & -8 \\ 5 & 1 & 6 & 7 \\ -4 & -1 & -5 & -6 \end{bmatrix}$.

5.1.2 Row reducing the multiple augmented system $[A \mid I]$ we get

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -10 & -2 \\ 0 & 1 & 0 & -3 & -9 & -2 \\ 0 & 0 & 1 & -1 & -4 & -1 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} -3 & -10 & -2 \\ -3 & -9 & -2 \\ -1 & -4 & -1 \end{bmatrix}$.

(a) The solution to $A\vec{x} = \vec{d}$ is

$$\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} 11 \\ 9 \\ 4 \end{bmatrix}$$

(b) The solution to $A\vec{x} = \vec{d}$ is

$$\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} -3 \\ -3 \\ -1 \end{bmatrix}$$

(c) The solution to $A\vec{x} = \vec{d}$ is

$$\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} \frac{11}{6} - 2\sqrt{2} \\ \frac{3}{2} - 2\sqrt{2} \\ \frac{5}{6} - \sqrt{2} \end{bmatrix}$$

5.1.3 Using the result of Example 5.1.4 we get

$$\begin{bmatrix} \sqrt{2} & 3/2 \\ 1/2 & 0 \end{bmatrix}^{-1} = \frac{1}{\sqrt{2}(0) - \frac{3}{2}\left(\frac{1}{2}\right)} \begin{bmatrix} 0 & -\frac{3}{2} \\ -\frac{1}{2} & \sqrt{2} \end{bmatrix} = -\frac{4}{3} \begin{bmatrix} 0 & 2 \\ 2/3 & -4\sqrt{2}/3 \end{bmatrix}$$

(a) The solution to $A\vec{x} = \vec{d}$ is

$$\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}$$

(b) The solution to $A\vec{x} = \vec{d}$ is

$$\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} 2 \\ -2\sqrt{2}/3 \end{bmatrix}$$

(c) The solution to $A\vec{x} = \vec{d}$ is

$$\vec{x} = A^{-1}\vec{d} = \begin{bmatrix} 2\pi \\ \frac{2}{3}\sqrt{3} - \frac{4}{3}\sqrt{2}\pi \end{bmatrix}$$

5.1.4 We need to find all 3×2 matrices $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$ such that

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix}$$

This gives us two systems $A\vec{b}_1 = \vec{e}_1$ and $A\vec{b}_2 = \vec{e}_2$. We solve both by row reducing a double augmented matrix. We get

$$\left[\begin{array}{ccc|c|c} 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 2/5 & -1/5 \\ 0 & 1 & 0 & -1/5 & 3/5 \end{array}\right]$$

Thus, the general solution of the first system is $\begin{bmatrix} 2/5 \\ -1/5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and the general solution of the second system is $s\begin{bmatrix} -1/5 \\ 3/5 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore, every right inverse of A has the form

$$B = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \\ t & s \end{bmatrix}$$

5.1.5 We need to find all 3×2 matrices $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$ such that

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix}$$

This gives us two systems $A\vec{b}_1 = \vec{e}_1$ and $A\vec{b}_2 = \vec{e}_2$. We solve both by row reducing a double augmented matrix. We get

$$\left[\begin{array}{ccc|c|c} 1 & -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Thus, the general solution of the first system is $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and the general solution of the second system is $s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Therefore, every right inverse of A has the form

$$B = \begin{bmatrix} -1-t & 2-s \\ -1 & 1 \\ t & s \end{bmatrix}$$

5.1.6 Observe that $B = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix}^T$. Also, observe that if $AB = I_2$, then $(AB)^T = I_2$ and so $B^T A^T = I_2$. Thus, from our work in Problem 5, we have that all left inverses of B are $\begin{bmatrix} -1-t & -1 & t \\ 2-s & 1 & s \end{bmatrix}$.

5.1.7 To find all left inverses of B, we use the same trick we did in Problem 6. We will find all right inverses of $B^T = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 3 \end{bmatrix}$. We get

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 3 & 1 & 0 \\ 2 & -1 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 \end{array}\right]$$

Thus, the general solution of the first system is $\begin{bmatrix}1\\2\\0\end{bmatrix}+t\begin{bmatrix}-3\\-3\\1\end{bmatrix}$ and the general solution of the second system is $s\begin{bmatrix}0\\-1\\0\end{bmatrix}+s\begin{bmatrix}-3\\-3\\1\end{bmatrix}$. Therefore, every left inverse of B has the form

$$B=\begin{bmatrix}1-3t & 2-3t & t\\ -3s & -1-3s & s\end{bmatrix}$$

5.1.8 (a) Using our work in Example 4, we get that

$$A^{-1}=\frac{1}{2(1)-1(3)}\begin{bmatrix}1 & -1\\ -3 & 2\end{bmatrix}=\begin{bmatrix}-1 & 1\\ 3 & -2\end{bmatrix}$$
$$B^{-1}=\frac{1}{1(5)-2(3)}\begin{bmatrix}5 & -2\\ -3 & 1\end{bmatrix}=\begin{bmatrix}-5 & 2\\ 3 & -1\end{bmatrix}$$

(b) We have $AB=\begin{bmatrix}5 & 9\\ 6 & 11\end{bmatrix}$ and hence

$$(AB)^{-1}=\frac{1}{5(11)-9(6)}\begin{bmatrix}11 & -9\\ -6 & 5\end{bmatrix}=\begin{bmatrix}11 & -9\\ -6 & 5\end{bmatrix}=B^{-1}A^{-1}$$

(c) We have $2A=\begin{bmatrix}4 & 2\\ 6 & 2\end{bmatrix}$, so

$$(2A)^{-1}=\frac{1}{4(2)-2(6)}\begin{bmatrix}2 & -2\\ -6 & 4\end{bmatrix}=\begin{bmatrix}-1/2 & 1/2\\ 3/2 & -1\end{bmatrix}=\frac{1}{2}A^{-1}$$

(d) We have $A^T=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}$, so $(A^T)^{-1}=\frac{1}{2(1)-3(1)}\begin{bmatrix}1 & -3\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-1 & 3\\ 1 & -2\end{bmatrix}$ and

$$A^T(A^T)^{-1}=\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 3\\ 1 & -2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

5.1.9 We have

$$A^T(A^{-1})^T=[A^{-1}A]^T=I^T=I$$

Hence, $(A^T)^{-1}=(A^{-1})^T$ by Theorem 5.1.4.

5.1.10 Consider

$$c_1A\vec{v}_1+\cdots+c_kA\vec{v}_k=\vec{0}$$

Then, we have

$$A(c_1\vec{v}_1+\cdots+c_k\vec{v}_k)=\vec{0}$$

Since A is invertible, the only solution to this equation is

$$c_1\vec{v}_1+\cdots+c_k\vec{v}_k=A^{-1}\vec{0}=\vec{0}$$

Hence, we get $c_1=\cdots=c_k=0$, because $\{\vec{v}_1,\dots,\vec{v}_k\}$ is linearly independent.

5.1.11 (a) Consider $B\vec{x} = \vec{0}$. Multiplying on the left by A gives

$$AB\vec{x} = A\vec{0} = \vec{0}$$

Since AB is invertible, this has the unique solution $\vec{x} = (AB)^{-1}\vec{0} = \vec{0}$. Thus, B is invertible. Since AB and B^{-1} are invertible, we have that $A = (AB)B^{-1}$ is invertible by Theorem 5.1.5(3).

(b) Let $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $D = C^T$. Then C and D are both not invertible since they are not square, but $CD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible.

5.1.12 Assume that A has a right inverse $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_m \end{bmatrix}$. Then, we have

$$\begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_m \end{bmatrix} = I_m = AB = A\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_m \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_m \end{bmatrix}$$

Thus, $A\vec{b}_i = \vec{e}_i$. Then, for any $\vec{y} = y_1\vec{e}_1 + \cdots + y_m\vec{e}_m \in \mathbb{R}^m$, we have

$$\vec{y} = y_1\vec{e}_1 + \cdots + y_m\vec{e}_m = y_1A\vec{b}_1 + \cdots + y_mA\vec{b}_m = A(y_1\vec{b}_1 + \cdots + y_m\vec{b}_m)$$

Hence, $A\vec{x} = \vec{y}$ is consistent for all $\vec{y} \in \mathbb{R}^m$. Therefore, rank $A = m$ by Theorem 2.2.4(3). But then we must have $n \geq m$ which is a contradiction.

5.1.13 (a) The statement if false. The 3×2 zero matrix cannot have a left inverse.

(b) The statement is true. By the Invertible Matrix Theorem, we get that if $\operatorname{Null}(A^T) = \{\vec{0}\}$, then A^T is invertible. Hence, A is invertible by Theorem 5.1.5.

(c) The statement is false. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are both invertible, but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

(d) The statement is false. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ satisfies $AA = A$, but A is not invertible.

(e) Let $\vec{b} \in \mathbb{R}^n$. Consider the system of equations $A\vec{x} = \vec{b}$. Then, we have

$$\begin{aligned} AA\vec{x} &= A\vec{b} \\ \vec{x} &= (AA)^{-1}A\vec{b} \end{aligned}$$

Thus, $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$ and hence A is invertible by the Invertible Matrix Theorem.

(f) The statement is false. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $AB = O_{2,2}$ and B is invertible.

(g) The statement is true. One such basis is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$.

(h) If A has a column of zeroes, then A^T has a row of zeroes and hence $\operatorname{rank}(A^T) < n$. Thus, A^T is not invertible and so A is also not invertible by the Invertible Matrix Theorem.

(i) If $A\vec{x} = \vec{0}$ has a unique solution, then $\operatorname{rank}(A) = n$ by Theorem 2.2.5 and so A is invertible and hence the columns of A form a basis for $\mathbb{R}^n$ by the Invertible Matrix Theorem. Thus, $\operatorname{Col}(A) = \mathbb{R}^n$, so the statement is true.

(j) If $\operatorname{rank}(A^T) = n$, then A^T is invertible by the Invertible Matrix theorem, and hence A is also invertible by the Invertible Matrix Theorem. Thus, $\operatorname{rank}(A) = n$ by Theorem 5.1.6.

(k) If A is invertible, then A^T is also invertible. Thus, the columns of A^T form a basis for $\mathbb{R}^n$ and hence must be linearly independent.

(l) We have that $I = -A^2 + 2A = A(-A + 2I)$. Thus, A is invertible by Theorem 5.1.4. In particular $A^{-1} = -A + 2I$.

5.1.14 (a) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. Then, $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$. Then, $AB = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$ and $(AB)^{-1} = \begin{bmatrix} 5 & -3 \\ -8 & 5 \end{bmatrix}$, but $A^{-1}B^{-1} = \begin{bmatrix} 5 & -4 \\ -5 & 3 \end{bmatrix}$.

(b) If $(AB)^{-1} = A^{-1}B^{-1} = (BA)^{-1}$, then we must have $AB = BA$.

5.2 Problem Solutions

5.2.1 (a) The matrix is elementary. The corresponding row operation is $R_1 + 3R_2$.

(b) The matrix is not elementary.

(c) The matrix is elementary. The corresponding row operation is $R_2 - 4R_1$.

(d) The matrix is elementary. The corresponding row operation is $R_2 \leftrightarrow R_3$.

(e) The matrix is not elementary.

(f) The matrix is elementary. The corresponding row operation is $(-4)R_3$.

(g) The matrix is not elementary.

(h) The matrix is elementary. The corresponding row operation is $R_1 \leftrightarrow R_4$.

(i) The matrix is not elementary.

(j) The matrix is not elementary.

5.2.2 (a) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

5.2.3 (a) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & -3 \end{bmatrix} \tfrac{1}{3}R_2 \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -3 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

and $E_5E_4E_3E_2E_1A = R$. Then

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}R \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

(b) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix} R_2 - 2R_1 \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \end{bmatrix} \tfrac{1}{3}R_2 \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} R_1 + R_2 \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = R$$

Hence, $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $E_3E_2E_1A = R$. Then

$$A = E_1^{-1}E_2^{-1}E_3^{-1}R = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

(c) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & -1 & -2 \\ 2 & 2 & 4 \end{bmatrix} R_1 \leftrightarrow R_3 \sim \begin{bmatrix} 2 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \tfrac{1}{2}R_1 \\ (-1)R_2 \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} R_2 - 2R_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R$$

Hence,

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

and $E_6E_5E_4E_3E_2E_1A = R$. Then

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}R \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(d) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ -\frac{1}{3}R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Hence,

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $E_5E_4E_3E_2E_1A = R$. Then

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}R \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(e) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \\ -2 & 2 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_1 \\ R_4 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & 9 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} \\ \\ \\ \frac{1}{3}R_4 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ R_3 \leftrightarrow R_4 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad E_7 = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $E_8E_7E_6E_5E_4E_3E_2E_1A = R$. Then

$$\begin{aligned} A =& E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1}R \\ =& \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \\ & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(f) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ -1 & -1 & 3 & 6 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 4 & 8 \end{bmatrix} \begin{matrix} \\ \\ R_2 \leftrightarrow R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -7 & -14 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 + 7R_3 \\ R_2 - 4R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = R$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

and $E_5E_4E_3E_2E_1A = R$. Then

$$\begin{aligned} A =& E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}R \\ =& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{aligned}$$

5.2.4 (a) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix} \begin{matrix} \\ \frac{1}{3}R_2 \end{matrix} \sim \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E_3E_2E_1A = I$. Then, $A^{-1} = E_3E_2E_1$ and

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(b) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ -\frac{1}{2}R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 4R_3 \\ R_2 + R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and $E_6E_5E_4E_3E_2E_1A = I$. Hence,

$$A^{-1} = E_6E_5E_4E_3E_2E_1$$

and

$$\begin{aligned} A =& E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1} \\ =& \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(c) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \begin{matrix} \\ \frac{1}{3}R_2 \end{matrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} R_1 + 2R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

and $E_3E_2E_1A = I$. Hence,

$$A^{-1} = E_3E_2E_1$$

and

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

(d) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 0 & 2 & 6 \\ 1 & 4 & 4 \\ -1 & 2 & 8 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 4 & 4 \\ 0 & 2 & 6 \\ -1 & 2 & 8 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 3 \\ 0 & 6 & 12 \end{bmatrix} \begin{matrix} R_1 - 4R_2 \\ \\ R_3 - 6R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{bmatrix} \begin{matrix} \\ \\ -\frac{1}{6}R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + 8R_3 \\ R_2 - 3R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/6 \end{bmatrix} \quad E_7 = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

and $E_8E_7E_6E_5E_4E_3E_2E_1A = I$. Hence,

$$A^{-1} = E_8E_7E_6E_5E_4E_3E_2E_1$$

and

$$\begin{aligned} A =& E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1} \\ =& \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(e) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} R_1 - R_2 \sim \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

and $E_4E_3E_2E_1A = I$. Hence,

$$A^{-1} = E_4E_3E_2E_1$$

and

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(f) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & -2 \\ -4 & 1 & 4 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -4 \\ 0 & 1 & 0 \end{bmatrix} R_2 \leftrightarrow R_3 \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} -\tfrac{1}{4}R_3 \sim$$
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 + R_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/4 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $E_5E_4E_3E_2E_1A = I$. Hence,

$$A^{-1} = E_5E_4E_3E_2E_1$$

and

$$\begin{aligned} A =& E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} \\ =& \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(g) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} R_2 + 3R_1 \sim \begin{bmatrix} 1 & 3 \\ 0 & 10 \end{bmatrix} \tfrac{1}{10}R_2 \sim$$
$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} R_1 - 3R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/10 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

and $E_3E_2E_1A = I$. Hence,

$$A^{-1} = E_3E_2E_1$$

and

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(h) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ -1 & 3 & -4 & -1 \\ 0 & 1 & 2 & 0 \\ -2 & 4 & -8 & -1 \end{bmatrix} \begin{array}{l} \\ R_2 + R_1 \\ \\ R_4 + 2R_1 \end{array} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - R_2 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 - 2R_3 \\ \\ \\ \\ \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 - R_4 \\ \\ \frac{1}{2}R_3 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $E_7E_6E_5E_4E_3E_2E_1A = I$. Hence,

$$A^{-1} = E_7E_6E_5E_4E_3E_2E_1$$

and

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.2.5 Since multiplying on the left is the same as applying the corresponding elementary row operations we get

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3/2 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 2 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

5.2.6 (a) We row reduce A to I keeping track of our elementary row-operations.

$$\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} R_2 - 2R_1 \sim \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} R_1 - R_2 \sim$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tfrac{1}{2}R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,

$$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

and $E_3E_2E_1A = I$.

(b) Since multiplying on the left is the same as applying the corresponding elementary row operations we get

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -1/2 \end{bmatrix} \end{aligned}$$

(c) We have

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 6 & 5 \end{array}\right] R_2 - 2R_1 \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 2 & -1 \end{array}\right] R_1 - R_2 \sim$$
$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 2 & -1 \end{array}\right] \tfrac{1}{2}R_2 \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1/2 \end{array}\right]$$

5.2.7 If A is invertible, then by Corollary 5.2.7, there exists a sequence of elementary matrices $E_1, \ldots, E_k$ such that $A = E_1^{-1} \cdots E_k^{-1}$. If B is row equivalent to A, then there exists a sequence of elementary matrices $F_1, \ldots, F_k$ such that $F_k \cdots F_1 A = B$. Thus,

$$B = F_k \cdots F_1 E_1^{-1} \cdots E_k^{-1}$$

Therefore, B is a product of invertible matrices and hence is invertible by Theorem 5.1.5(3).

5.2.8 Let $B = AE$. Taking transposes of both sides gives $B^T = E^T A^T$. We observe that the transpose of an elementary matrix is still an elementary matrix. Thus, B^T is the matrix obtained from A by performing the elementary row operation associated with E^T on A^T. But, the rows of A^T are just the columns of B. Thus, multiplying on the right by an elementary performs an elementary column operation on A.

5.3 Problem Solutions

5.3.1 The cofactor matrix is

(a) $\begin{bmatrix} 8 & -5 \\ 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & -3 & 3 \\ 3 & 6 & -1 \\ -6 & 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 11 & 6 & -8 \\ -3 & 8 & 7 \\ 10 & -9 & 12 \end{bmatrix}$

(d) $\begin{bmatrix} -29 & -13 & -4 \\ -19 & 7 & 16 \\ -11 & 23 & 14 \end{bmatrix}$

5.3.2 (a) $\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = 2(4) - 1(-3) = 11$

(b) $\begin{vmatrix} 1 & 3 \\ 1 & -6 \end{vmatrix} = 1(-6) - 3(1) = -9$

(c) $\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 4(3) - 2(6) = 0$

(d) $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{vmatrix} = 1(2)(3) = 6$

(e) $\begin{vmatrix} 3 & 0 & -4 \\ -2 & 0 & 5 \\ 3 & -1 & 8 \end{vmatrix} = (-1)(-1)^{3+2} \begin{vmatrix} 3 & -4 \\ -2 & 5 \end{vmatrix} = 3(5) - (-4)(-2) = 7$

(f) $\begin{vmatrix} 1 & 2 & 1 \\ 5 & 2 & 0 \\ 3 & 0 & 0 \end{vmatrix} = 1(-1)^{1+3} \begin{vmatrix} 5 & 2 \\ 3 & 0 \end{vmatrix} = 5(0) - 2(3) = -6$

(g) Using row operations to simplify the determinant we get

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & -8 \\ 2 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -5 \\ 0 & -9 & -4 \end{vmatrix}$$

Expanding along the first column we get

$$= 1(-1)^{1+1} \begin{vmatrix} 5 & -5 \\ -9 & -4 \end{vmatrix} = 5(-4) - (-5)(-9) = -65$$

(h) Using a column operation we get

$$\begin{vmatrix} 2 & 3 & 1 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 5 & 3 & 1 \\ 0 & 2 & 0 \\ 4 & 4 & 4 \end{vmatrix}$$

Expanding along the second column gives

$$= 2(-1)^{2+2} \begin{vmatrix} 5 & 1 \\ 4 & 4 \end{vmatrix} = 2[5(4) - 1(4)] = 32$$

(i) Using row operations we get

$$\begin{vmatrix} 6 & 8 & -8 \\ 7 & 5 & 8 \\ -2 & -4 & 8 \end{vmatrix} = \begin{vmatrix} 6 & 8 & -8 \\ 13 & 13 & 0 \\ 4 & 4 & 0 \end{vmatrix} = 13(4) \begin{vmatrix} 6 & 8 & -8 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

(j) Using row operations we get

$$\begin{vmatrix} 1 & 10 & 7 & -9 \\ 7 & -7 & 7 & 7 \\ 2 & -2 & 6 & 2 \\ -3 & -3 & 4 & 1 \end{vmatrix} = (7) \begin{vmatrix} 1 & 10 & 7 & -9 \\ 1 & -1 & 1 & 1 \\ 2 & -2 & 6 & 2 \\ -3 & -3 & 4 & 1 \end{vmatrix} = 7 \begin{vmatrix} 0 & 11 & 6 & -10 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & -6 & 7 & 4 \end{vmatrix}$$

Expanding along the third row we get

$$\det B = (7)(4) \begin{vmatrix} 0 & 11 & -10 \\ 1 & -1 & 1 \\ 0 & -6 & 4 \end{vmatrix}$$

Now expanding along the first column gives

$$\det B = 28(-1) \begin{vmatrix} 11 & -10 \\ -6 & 4 \end{vmatrix} = (-28)(44 - 60) = 448$$

(k) Using row operations we get

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix} \\ &= \begin{vmatrix} b-a & b^2-a^2 & b^3-a^3 \\ c-a & c^2-a^2 & c^3-a^3 \\ d-a & d^2-a^2 & d^3-a^3 \end{vmatrix} \\ &= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & b+a & b^2+ba+a^2 \\ 1 & c+a & c^2+ca+a^2 \\ 1 & d+a & d^2+da+a^2 \end{vmatrix} \end{aligned}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b+a & b^2+ba+a^2 \\ 0 & c-b & c^2+ca-b^2-ba \\ 0 & d-b & d^2+da-b^2-ba \end{vmatrix}$$
$$= (b-a)(c-a)(d-a)\begin{vmatrix} c-b & c^2+ca-b^2-ba \\ d-b & d^2+da-b^2-ba \end{vmatrix}$$
$$= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & a+b+c \\ 1 & a+b+d \end{vmatrix}$$
$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

(l) Using row operations we get

$$\begin{vmatrix} -1 & 2 & 6 & 4 \\ 0 & 3 & 5 & 6 \\ 1 & -1 & 4 & -2 \\ 1 & 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 6 & 4 \\ 0 & 3 & 5 & 6 \\ 0 & 1 & 10 & 2 \\ 0 & 4 & 7 & 6 \end{vmatrix} = (-1)(-1)^{1+1}\begin{vmatrix} 3 & 5 & 6 \\ 1 & 10 & 2 \\ 4 & 7 & 6 \end{vmatrix}$$
$$= (-1)\begin{vmatrix} 0 & -25 & 0 \\ 1 & 10 & 2 \\ 0 & -33 & -2 \end{vmatrix} = (-1)(-25)(-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -25(-2) = 50$$

5.3.3 If A is invertible, then $AA^{-1} = I$. By Theorem 5.3.9 we get

$$1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}$$

Since A is invertible, $\det A \neq 0$ and so we get

$$\det A^{-1} = \frac{1}{\det A}$$

5.3.4 By Theorem 5.3.9 and Corollary 5.3.10 we get

$$\det B = \det(P^{-1}AP) = \det P^{-1} \det A \det P = \frac{1}{\det P} \det A \det P = \det A$$

5.3.5 (a) We have $AA^T = AA^{-1} = I$. Hence, by Theorem 5.3.9 and Corollary 5.3.10 we get

$$1 = \det I = \det(AA^T) = \det A \det A^T = \det A \det A = (\det A)^2$$

Hence, $\det A = \pm 1$.

(b) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A = A^T$ and $\det A = -1$.

5.3.6 (a) The statement is true. If the columns of A are linearly independent, then A is invertible and hence, $\det A \neq 0$ by the Invertible Matrix Theorem.

(b) The statement is false. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\det B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $\det(A+B) = \det\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$, but $\det A + \det B = 1 + 1 = 1$.

(c) The statement is false. Take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then, $A\vec{x} = \vec{b}$ is consistent, but $\det A = 0$.

(d) The statement is true. By Theorem 5.3.9 we get $0 = \det(AB) = \det A \det B$. Hence, $\det A = 0$ or $\det B = 0$.

5.3.7 We will use induction on n.
Base Case: $n = 2$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$. Hence,

$$\det A = ad - bc = -(bc - ad) = -\det B$$

Inductive Hypothesis: Assume that the result holds for any $(n-1) \times (n-1)$ matrix.
Inductive Step: Suppose that B is an $n \times n$ matrix obtained from A by swapping two rows. If the i-th row of A was not swapped, then the cofactors of the i-th row of B are $(n-1) \times (n-1)$ matrices which can be obtained from the cofactors of the i-th row of A by swapping the same two rows. Hence, by the inductive hypothesis, the cofactors C_{ij}^* of B and C_{ij} of A satisfy $C_{ij}^* = -C_{ij}$. Hence,

$$\begin{aligned} \det B &= a_{i1}C_{i1}^* + \cdots + a_{in}C_{in}^* = a_{i1}(-C_{i1}) + \cdots + a_{in}(-C_{in}) \\ &= -(a_{i1}C_{i1} + \cdots + a_{in}C_{in}) = -\det A \end{aligned}$$

5.4 Problem Solutions

5.4.1 (a) We have $\operatorname{adj} A = \begin{bmatrix} 5 & -3 \\ 2 & 1 \end{bmatrix}$. Hence,

$$A(\operatorname{adj} A) = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = (\det A)I$$

(b) We have $\operatorname{adj} A = \begin{bmatrix} 3 & -5 \\ 1 & 4 \end{bmatrix}$. Hence,

$$A(\operatorname{adj} A) = \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix} = (\det A)I$$

(c) We have $\operatorname{adj} A = \begin{bmatrix} 10 & -3 \\ -4 & 1 \end{bmatrix}$. Hence,

$$A(\operatorname{adj} A) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = (\det A)I$$

(d) We have $\operatorname{adj} A = \begin{bmatrix} 6 & -6 & 16 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}$. Hence,

$$A(\operatorname{adj} A) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = (\det A)I$$

(e) We have $\operatorname{adj} A = \begin{bmatrix} -1 & 4 & -2 \\ 2 & -9 & 5 \\ 3 & -11 & 6 \end{bmatrix}$. Hence,

$$A(\operatorname{adj} A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (\det A)I$$

(f) We have $\operatorname{adj} A = \begin{bmatrix} -1 & -8 & -4 \\ -2 & -12 & -4 \\ -2 & -8 & -4 \end{bmatrix}$. Hence,

$$A(\operatorname{adj} A) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = (\det A)I$$

5.4.2 (a) We have $\det A = ad - bc$ and $\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, thus

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(b) We have $\det A = -17 - 2t$ and $\operatorname{adj} A = \begin{bmatrix} -1 & t+3 & -3 \\ 5 & 2-3t & -2t-2 \\ -2 & -11 & -6 \end{bmatrix}$. Hence,

$$A^{-1} = \frac{1}{-17-2t}\begin{bmatrix} -1 & t+3 & -3 \\ 5 & 2-3t & -2t-2 \\ -2 & -11 & -6 \end{bmatrix}$$

(c) We have $\det A = 14t + 1$ and $\operatorname{adj} A = \begin{bmatrix} 4t+1 & -1 & -3t-1 \\ -10 & 14 & 11 \\ 3+2t & -4 & 2t-3 \end{bmatrix}$. Hence,

$$A^{-1} = \frac{1}{14t+1}\begin{bmatrix} 4t+1 & -1 & -3t-1 \\ -10 & 14 & 11 \\ 3+2t & -4 & 2t-3 \end{bmatrix}$$

(d) We have $\det A = acd - b^2c$ and $\operatorname{adj} A = \begin{bmatrix} cd & 0 & -cb \\ 0 & ad-b^2 & 0 \\ -cb & 0 & ac \end{bmatrix}$. Hence,

$$A^{-1} = \frac{1}{acd-b^2c}\begin{bmatrix} cd & 0 & -cb \\ 0 & ad-b^2 & 0 \\ -cb & 0 & ac \end{bmatrix}$$

5.4.3 Solve the following systems of linear equations using Cramer's Rule.

(a) The coefficient matrix is $A = \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix}$, so $\det A = 21 + 4 = 25$. Hence,

$$x_1 = \frac{1}{25}\begin{vmatrix} 2 & -1 \\ 5 & 7 \end{vmatrix} = \frac{19}{25}$$
$$x_2 = \frac{1}{25}\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = \frac{7}{25}$$

Thus, the solution is $\vec{x} = \begin{bmatrix} 19/25 \\ 7/25 \end{bmatrix}$.

(b) The coefficient matrix is $A = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$, so $\det A = 14 - 3 = 11$. Hence,

$$x_1 = \frac{1}{11}\begin{vmatrix} 1 & 1 \\ -2 & 7 \end{vmatrix} = \frac{9}{11}$$
$$x_2 = \frac{1}{11}\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -\frac{7}{11}$$

Thus, the solution is $\vec{x} = \begin{bmatrix} 9/11 \\ -7/11 \end{bmatrix}$.

(c) The coefficient matrix is $A = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$, so $\det A = 14 - 3 = 11$. Hence,

$$x_1 = \frac{1}{11}\begin{vmatrix} 3 & 1 \\ 5 & 7 \end{vmatrix} = \frac{16}{11}$$
$$x_2 = \frac{1}{11}\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = \frac{1}{11}$$

Thus, the solution is $\vec{x} = \begin{bmatrix} 16/11 \\ 1/11 \end{bmatrix}$.

(d) The coefficient matrix is $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & -1 \\ -2 & 0 & 2 \end{bmatrix}$, so $\det A = 6$. Hence,

$$x_1 = \frac{1}{6}\begin{vmatrix} 1 & 3 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} = \frac{4}{6}$$
$$x_2 = \frac{1}{6}\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{vmatrix} = \frac{-3}{6}$$
$$x_3 = \frac{1}{6}\begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = \frac{7}{6}$$

Hence, the solution is $\vec{x} = \begin{bmatrix} 2/3 \\ -1/2 \\ 7/6 \end{bmatrix}$.

(e) The coefficient matrix is $A = \begin{bmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{bmatrix}$, so $\det A = -16$. Hence,

$$x_1 = \frac{1}{-16}\begin{vmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{vmatrix} = \frac{12}{-16}$$

$$x_2 = \frac{1}{-16}\begin{vmatrix} 5 & 4 & -1 \\ 9 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} = \frac{-166}{-16}$$

$$x_3 = \frac{1}{-16}\begin{vmatrix} 5 & 1 & 4 \\ 9 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \frac{-42}{-16}$$

Thus, the solution is $\vec{x} = \begin{bmatrix} -3/4 \\ 83/8 \\ 21/8 \end{bmatrix}$.

5.5 Problem Solutions

5.5.1 (a) $A = \left|\det \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\right| = |2(2) - 3(-3)| = 13$

(b) $A = \left|\det \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}\right| = |4(1) - (-1)(1)| = 5$

(c) $A = \left|\det \begin{bmatrix} 4 & -3 \\ 1 & 3 \end{bmatrix}\right| = |4(3) - (-3)(1)| = 15$

(d) $A = \left|\det \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}\right| = |2(2) - (-2)(3)| = 10$

5.5.2 (a) $V = \left|\det \begin{bmatrix} 5 & 1 & 1 \\ -1 & 4 & 2 \\ 1 & -1 & -6 \end{bmatrix}\right| = \left|\det \begin{bmatrix} 0 & 6 & 31 \\ 0 & 3 & -4 \\ 1 & -1 & 6 \end{bmatrix}\right| = 117$

(b) $V = \left|\det \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & 1 \\ 4 & 4 & -5 \end{bmatrix}\right| = \left|\det \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}\right| = 6$

(c) $V = \left|\det \begin{bmatrix} 2 & 2 & 1 \\ 3 & -1 & 5 \\ 4 & -5 & 2 \end{bmatrix}\right| = \left|\det \begin{bmatrix} 0 & 0 & 1 \\ -7 & -11 & 5 \\ 0 & -9 & 2 \end{bmatrix}\right| = 63$

5.5.3 $V = \left|\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 5 \\ 1 & 3 & 0 & 7 \end{bmatrix}\right| = \left|\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 2 & -1 & 6 \end{bmatrix}\right| = \left|\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 12 \end{bmatrix}\right| = 13$

5.5.4 Since adding a multiple of one row to another does not change the determinant, we get

$$V = |\det\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}| = |\det\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n + t\vec{v}_1\end{bmatrix}|$$

5.5.5 (a) We have

$$\begin{aligned}\text{area}(L(\vec{u}), L(\vec{v})) &= \left|\det\begin{bmatrix}L(\vec{u}) & L(\vec{v})\end{bmatrix}\right| = \left|\det\begin{bmatrix}[L]\vec{u} & [L]\vec{v}\end{bmatrix}\right| \\ &= \left|\det([L]\begin{bmatrix}\vec{u} & \vec{v}\end{bmatrix}\right| = \left|\det[L]\det\begin{bmatrix}\vec{u} & \vec{v}\end{bmatrix}\right| \\ &= |\det[L]|\,\text{area}(\vec{u}, \vec{v})\end{aligned}$$

(b) The area scaling factor is

$$|\det[L]| = \left|\det\begin{bmatrix}2 & 8 \\ -1 & 5\end{bmatrix}\right| = 18$$

(c) The area scaling factor is

$$|\det[M \circ L]| = |\det[M][L]|$$

Chapter 6 Solutions

6.1 Problem Solutions

6.1.1 (a) We have

$$L(1,-2) = \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$L(2,1) = \begin{bmatrix} 8 \\ 14 \end{bmatrix} = (-4)\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(1,-2)]_{\mathcal{B}} & [L(2,1)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -1 & 6 \end{bmatrix}$$

(b) We have

$$L(1,1) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 10\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$L(1,2) = \begin{bmatrix} 10 \\ 1 \end{bmatrix} = 19\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 9\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(1,1)]_{\mathcal{B}} & [L(1,2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 10 & 19 \\ -4 & -9 \end{bmatrix}$$

(c) We have

$$L(-3,5) = \begin{bmatrix} 6 \\ -10 \end{bmatrix} = -2\begin{bmatrix} -3 \\ 5 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$L(1,-2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0\begin{bmatrix} -3 \\ 5 \end{bmatrix} - 1\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(-3,5)]_{\mathcal{B}} & [L(1,-2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

(d) We have

$$L(1,1,1)=\begin{bmatrix}4\\0\\5\end{bmatrix}=(-1)\begin{bmatrix}1\\1\\1\end{bmatrix}+5\begin{bmatrix}1\\-1\\0\end{bmatrix}-6\begin{bmatrix}0\\-1\\-1\end{bmatrix}$$
$$L(1,-1,0)=\begin{bmatrix}1\\-1\\-1\end{bmatrix}=1\begin{bmatrix}1\\1\\1\end{bmatrix}+0\begin{bmatrix}1\\-1\\0\end{bmatrix}+2\begin{bmatrix}0\\-1\\-1\end{bmatrix}$$
$$L(0,-1,-1)=\begin{bmatrix}-3\\0\\-3\end{bmatrix}=0\begin{bmatrix}1\\1\\1\end{bmatrix}-3\begin{bmatrix}1\\-1\\0\end{bmatrix}+3\begin{bmatrix}0\\-1\\-1\end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}}=\begin{bmatrix}[L(1,1,1)]_{\mathcal{B}} & [L(1,-1,0)]_{\mathcal{B}} & [L(0,-1,-1)]_{\mathcal{B}}\end{bmatrix}=\begin{bmatrix}-1&1&0\\5&0&-3\\-6&2&3\end{bmatrix}$$

(e) We have

$$L(2,-1,-1)=\begin{bmatrix}16\\-2\\10\end{bmatrix}=6\begin{bmatrix}2\\-1\\-1\end{bmatrix}+4\begin{bmatrix}1\\1\\1\end{bmatrix}-12\begin{bmatrix}0\\0\\-1\end{bmatrix}$$
$$L(1,1,1)=\begin{bmatrix}-1\\2\\-1\end{bmatrix}=(-1)\begin{bmatrix}2\\-1\\-1\end{bmatrix}+1\begin{bmatrix}1\\1\\1\end{bmatrix}+3\begin{bmatrix}0\\0\\-1\end{bmatrix}$$
$$L(0,0,-1)=\begin{bmatrix}6\\0\\4\end{bmatrix}=2\begin{bmatrix}2\\-1\\-1\end{bmatrix}+2\begin{bmatrix}1\\1\\1\end{bmatrix}-4\begin{bmatrix}0\\0\\-1\end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}}=\begin{bmatrix}[L(2,-1,-1)]_{\mathcal{B}} & [L(1,1,1)]_{\mathcal{B}} & [L(0,0,-1)]_{\mathcal{B}}\end{bmatrix}=\begin{bmatrix}6&-1&2\\4&1&2\\-12&3&-4\end{bmatrix}$$

(f) We have

$$L(0,1,0)=\begin{bmatrix}0\\2\\0\end{bmatrix}=2\begin{bmatrix}0\\1\\0\end{bmatrix}+0\begin{bmatrix}2\\0\\1\end{bmatrix}+0\begin{bmatrix}1\\0\\1\end{bmatrix}$$
$$L(2,0,1)=\begin{bmatrix}4\\0\\2\end{bmatrix}=0\begin{bmatrix}0\\1\\0\end{bmatrix}+2\begin{bmatrix}2\\0\\1\end{bmatrix}+0\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$L(1,0,1) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(0,1,0)]_{\mathcal{B}} & [L(2,0,1)]_{\mathcal{B}} & [L(1,0,1)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(g) We have

$$L(0,1,0) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$L(-1,0,1) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$L(1,1,1) = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(0,1,0)]_{\mathcal{B}} & [L(-1,0,1)]_{\mathcal{B}} & [L(1,1,1)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

6.1.2 (a) We have

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 22 \\ 15 \end{bmatrix}$$

Thus,

$$L(\vec{x}) = 4\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 22\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 15\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 19 \\ 22 \end{bmatrix}$$

(b) We have

$$[L(\vec{y})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

Thus,

$$L(\vec{y}) = 0\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

(c) From our work on similar matrices, we have that $[L]_{\mathcal{B}} = P^{-1}[L]P$ where P is the change of coordinates matrix from $\mathcal{B}$-coordinates to standard coordinates. Thus,

$$[L] = P[L]_{\mathcal{B}}P^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 4 & 2 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -7 & 6 \\ 5 & -3 & 7 \\ -2 & 4 & 2 \end{bmatrix}$$

6.1.3 (a) We have

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 13 \\ 2 \end{bmatrix}$$

Thus,

$$L(\vec{x}) = 6\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 13\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \\ 15 \end{bmatrix}$$

(b) We have

$$[L(\vec{y})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix}$$

Thus,

$$L(\vec{y}) = (-3)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$$

(c) From our work on similar matrices, we have that $[L]_{\mathcal{B}} = P^{-1}[L]P$ where P is the change of coordinates matrix from $\mathcal{B}$-coordinates to standard coordinates. Thus,

$$[L] = P[L]_{\mathcal{B}}P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 3 \\ 5 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 3/2 & -3/2 & 7/2 \\ 7/2 & 3/2 & -3/2 \\ 2 & 3 & -1 \end{bmatrix}$$

6.1.4 (a) Our geometrically natural basis should include $\vec{a}$ and a vector orthogonal to $\vec{a}$. We pick $\mathcal{B} = \left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$. Then we have

$$\text{proj}_{\vec{a}}(\vec{a}) = \vec{a} = 1\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\text{proj}_{\vec{a}}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Therefore,

$$[\text{proj}_{\vec{a}}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) For our geometrically natural basis, we include $\vec{b}$ and a vector orthogonal to $\vec{b}$. We pick $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$. Then we have

$$\operatorname{proj}_{\vec{b}}(\vec{b}) = \vec{b} = 1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix}$$
$$\operatorname{proj}_{\vec{b}}\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \vec{0} = 0\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix}$$

Hence,

$$[\operatorname{perp}_{\vec{b}}]_{\mathcal{B}} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

(c) For our geometrically natural basis, we include $\vec{b}$ and a vector orthogonal to $\vec{b}$. We pick $\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$. Then we have

$$\operatorname{perp}_{\vec{b}}(\vec{b}) = \vec{0} = 0\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix}$$
$$\operatorname{perp}_{\vec{b}}\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix} = 0\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$$

Hence,

$$[\operatorname{perp}_{\vec{b}}]_{\mathcal{B}} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

(d) For our geometrically natural basis, we include the normal vector $\vec{n}$ for the plane of reflection and a basis for the plane. To pick a basis for the plane, we need to pick a set of two linearly independent vectors which are orthogonal to $\vec{n}$. Thus, we pick

$$\mathcal{B} = \left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

Then we get

$$\operatorname{refl}_{\vec{n}}(\vec{n}) = -\vec{n} = -\begin{bmatrix}1\\0\\-1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\end{bmatrix}$$
$$\operatorname{refl}_{\vec{n}}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} = 0\begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\end{bmatrix}$$
$$\operatorname{refl}_{\vec{n}}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\\-1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Hence,

$$[\text{refl}_{\vec{n}}]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.1.5 Let A, B, and C be $n \times n$ matrices.

(a) Taking $P = I$ we get $I^{-1}AI = A$, so by definition A is similar to A.

(b) If A is similar to B, then there exists an invertible matrix P such that $P^{-1}AP = B$. If B is similar to C, then there exists an invertible matrix Q such that $Q^{-1}BQ = C$. Hence,

$$Q^{-1}P^{-1}APQ = Q^{-1}BQ = C$$

Thus,

$$(PQ)^{-1}A(PQ) = C$$

So, A is similar to C.

6.1.6 If A^TA is similar to I, then there exists an invertible matrix P such that

$$P^{-1}A^TAP = I$$

Hence, $A^TA = PP^{-1} = I$, so $A^T = A^{-1}$.

If $A^T = A^{-1}$, then $A^TA = I$. Taking $P = I$ we get $I^{-1}A^TAI = I$, so A^TA is similar to I.

6.1.7 We prove the result by induction.

Base Case: $n = 1$: If A is similar to B, then there exists an invertible matrix P such that $P^{-1}AP = B$, so A^1 is similar to B^1.

Inductive Hypothesis: Assume that for some positive integer k we have $P^{-1}A^kP = B^k$.

Inductive Step: We have

$$B^{k+1} = B^nB = (P^{-1}A^kP)(P^{-1}AP) = P^{-1}A^kPP^{-1}AP = P^{-1}A^kAP = P^{-1}A^{k+1}P$$

Thus, by induction, A^n is similar to B^n for all positive integers n.

6.1.8 If A and B are similar, then there exists an invertible matrix P such that $P^{-1}AP = B$ which we can rewrite as $P^{-1}A = BP^{-1}$. Since P is invertbile, there exists elementary matrices $E_1, \dots, E_k$ such that $P^{-1} = E_k \cdots E_1$. Thus, we have

$$\begin{aligned} \text{rank}(E_k \cdots E_1A) &= \text{rank}(BE_k \cdots E_1) \\ \text{rank}(A) &= \text{rank}(BE_k \cdots E_1) \quad \text{by Corollary 5.2.5} \\ \text{rank}(A^T) &= \text{rank}([BE_k \cdots E_1]^T) \\ \text{rank}(A^T) &= \text{rank}(E_1^T \cdots E_k^T B^T) \\ \text{rank}(A^T) &= \text{rank}(B^T) \quad \text{by Corollary 5.2.5} \\ \text{rank}(A) &= \text{rank}(B) \end{aligned}$$

as required.

6.1.9 (a) The statement is false. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, then $\operatorname{rank} A = \operatorname{rank} B$, but $\operatorname{tr} A \neq \operatorname{tr} B$, so A and B cannot be similar by Theorem 6.1.1.

(b) The statement is true. If A and B are similar, then there exists an invertible matrix P such that $P^{-1}AP = B$. Thus, B is a product of invertible matrices, so B is invertible by Theorem 5.1.5.

(c) The statement is false. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Its RREF is $R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We have $\det A \neq \det R$, so A and R cannot be similar by Theorem 6.1.1.

(d) The statement is true. If A is similar to the diagonal matrix $D = \operatorname{diag}(d_{11}, d_{22}, \ldots, d_{nn})$, then there exists an invertible matrix P such that $P^{-1}AP = D$ and so $A = PDP^{-1}$. By Theorem 5.3.2, Theorem 5.3.9, and Corollary 5.3.10 we get

$$\det A = \det(PDP^{-1}) = \det P \det D \det P^{-1} = \det D \det P \frac{1}{\det P} = \det D = d_{11}d_{22} \cdots d_{nn}$$

6.2 Problem Solutions

6.2.1 (a) We have

$$A\vec{v}_1 = \begin{bmatrix} -1 & 1 & -4 \\ 5 & -5 & -4 \\ -5 & 11 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 16 \end{bmatrix}$$

is not a scalar multiple of $\vec{v}_1$. Thus, $\vec{v}_1$ is not an eigenvector.

(b) We have

$$A\vec{v}_2 = \begin{bmatrix} -1 & 1 & -4 \\ 5 & -5 & -4 \\ -5 & 11 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ -18 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$$

Thus, $\vec{v}_2$ is an eigenvector.

(c) By definition, the zero vector cannot be an eigenvector.

6.2.2 (a) We have

$$A\vec{v}_1 = \begin{bmatrix} 1 & 3 & 3 \\ 6 & 7 & 12 \\ -3 & -3 & -5 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, $\vec{v}_1$ is an eigenvector.

(b) We have

$$A\vec{v}_2 = \begin{bmatrix} 1 & 3 & 3 \\ 6 & 7 & 12 \\ -3 & -3 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \\ -6 \end{bmatrix} = -2\begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

Thus, $\vec{v}_2$ is an eigenvector.

(c) We have

$$A\vec{v}_3 = \begin{bmatrix} 1 & 3 & 3 \\ 6 & 7 & 12 \\ -3 & -3 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \\ -5 \end{bmatrix}$$

is not a scalar multiple of $\vec{v}_3$. Thus, $\vec{v}_3$ is not an eigenvector.

6.2.3 (a) We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 4 & 5-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 7 = (\lambda - 7)(\lambda - 1)$$

Hence, the eigenvalues are $\lambda_1 = 7$ and $\lambda_2 = 1$ with $a_{\lambda_1} = 1 = a_{\lambda_2}$. Therefore, by Theorem 6.2.4, we have that $g_{\lambda_1} = 1 = g_{\lambda_2}$.

(b) We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 0 \\ 5 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$$

Thus, $\lambda_1 = -2$ is the only eigenvalue and $a_{\lambda_1} = 2$. We have

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. So, $g_{\lambda_1} = 1$.

(c) We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ 3 & -4-\lambda \end{vmatrix} = \lambda^2 - 7 = (\lambda - \sqrt{7})(\lambda + \sqrt{7})$$

Hence, the eigenvalues are $\lambda_1 = \sqrt{7}$ and $\lambda_2 = -\sqrt{7}$. We have $a_{\lambda_1} = 1 = a_{\lambda_2}$. Therefore, by Theorem 6.2.4, $g_{\lambda_1} = 1 = g_{\lambda_2}$.

(d) We have

$$\begin{aligned} C(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} -3-\lambda & -2 & -2 \\ 2 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -3-\lambda & -2 \\ 2 & 2-\lambda \end{vmatrix} \\ &= -(\lambda - 1)(\lambda^2 + \lambda - 2) = -(\lambda - 1)^2(\lambda + 2) \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = 1$ with $a_{\lambda_1} = 2$, and $\lambda_2 = -2$ with $a_{\lambda_2} = 1$. Since λ_2 has algebraic multiplicity 1 it also has geometric multiplicity 1 by Theorem 6.2.4. For λ_1 we get

$$A - \lambda_1 I = \begin{bmatrix} -4 & -2 & -2 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for E_{λ_1} is $\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right\}$. Thus, λ_1 has geometric multiplicity 2.

(e) We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 6 & -2 \\ -1 & 2-\lambda & -1 \\ 1 & -3 & -\lambda \end{vmatrix}$$
$$= \begin{vmatrix} -3-\lambda & 6 & -2 \\ 0 & -1-\lambda & -1-\lambda \\ 1 & -3 & -\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 8 & -2 \\ 0 & 0 & -1-\lambda \\ 1 & -3+\lambda & -\lambda \end{vmatrix}$$
$$= -(\lambda+1)(\lambda^2-1) = -(\lambda+1)^2(\lambda-1)$$

Hence, the eigenvalues are $\lambda_1 = -1$ with algebraic multiplicity 2 and $\lambda_2 = 1$ with algebraic multiplicity 1. Since λ_2 has algebraic multiplicity 1 it also has geometric multiplicity 1 by Theorem 6.2.4. For λ_1 we get

$$A - \lambda_1 I = \begin{bmatrix} -2 & 6 & -2 \\ -1 & 3 & -1 \\ 1 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\operatorname{rank}(A - \lambda_1 I) = 1$, so $g_{\lambda_1} = 3 - 1 = 2$ by Theorem 2.2.6.

(f) We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix}$$
$$= \begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & -2+\lambda & 4-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 4 & 0 & 6-\lambda \end{vmatrix}$$
$$= -(\lambda-1)(\lambda^2-10\lambda+16) = -(\lambda-2)^2(\lambda-8)$$

Hence, the eigenvalues are $\lambda_1 = 2$ with $a_{\lambda_1} = 2$, and $\lambda_2 = 8$ with $a_{\lambda_2} = 1$. Since λ_2 has algebraic multiplicity 1 it also has geometric multiplicity 1 by Theorem 6.2.4. For λ_1 we get

$$A - \lambda_1 I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\operatorname{rank}(A - \lambda 1 I) = 1$, so $g_{\lambda_1} = 3 - 1 = 2$ by Theorem 2.2.6.

(g) We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$
$$= -(\lambda-1)(\lambda^2-2\lambda-3) = -(\lambda-1)(\lambda-3)(\lambda+1)$$

Hence, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = -1$ all with algebraic multiplicity 1. Thus, they all have geometric multiplicity 1 by Theorem 6.2.4.

(h) We have

$$
\begin{aligned}
C(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 2 & -1 \\ -2 & 1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 2 \\ 1-\lambda & 2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} \\
&= -\lambda(\lambda-1)^2
\end{aligned}
$$

Hence, the eigenvalues are $\lambda_1 = 0$ with algebraic multiplicity 1 and $\lambda_2 = 1$ with algebraic multiplicity 2. Since λ_1 has algebraic multiplicity 1 it also has geometric multiplicity 1 by Theorem 6.2.4. For λ_2 we get

$$
A - \lambda_2 I = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 2 \\ 2 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Thus, $\operatorname{rank}(A - \lambda_2 I) = 2$, and so $g_{\lambda_2} = 3 - 2 = 1$ by Theorem 2.2.6.

(i) We have

$$
\begin{aligned}
C(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 6 & -4 \\ 5 & -3-\lambda & 1 \\ -5 & 6 & 2-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda & 6 & -4 \\ 5 & -3-\lambda & 1 \\ 0 & 3-\lambda & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 10 & -4 \\ 5 & -4-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)(\lambda^2 + 3\lambda - 54) = -(\lambda-3)(\lambda+9)(\lambda-6)
\end{aligned}
$$

Hence, the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -9$, and $\lambda_3 = 6$ all with algebraic multiplicity 1. Thus, they all have geometric multiplicity 1 by Theorem 6.2.4.

(j) We have

$$
\begin{aligned}
C(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)(2-\lambda)^2
\end{aligned}
$$

Hence, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. The algebraic multiplicity of λ_1 is 1, so the algebraic multiplicity is also 1 by Theorem 6.2.4. For $\lambda_2 = 2$, we have

$$
A - \lambda_2 I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

Thus, $\operatorname{rank}(A - \lambda_2 I) = 2$, and so $g_{\lambda_2} = 3 - 2 = 1$ by Theorem 2.2.6.

6.2.4 Geometrically, we are looking for vectors that are mapped to a scalar multiple of themselves. For a rotation by $\pi/3$, the only vector that will be mapped to a scalar multiple of itself is the zero vector. Thus, $R_{\pi/3}$ has no real eigenvalues and eigenvectors. Since the projection onto $\vec{v}$, is always a scalar multiple of $\vec{v}$, the only eigenvectors can be vectors that a parallel to $\vec{v}$ or orthogonal to $\vec{v}$. Indeed, by definition, we have $\operatorname{proj}_{\vec{v}} \vec{x} = \frac{\vec{x}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$. Thus, if $\vec{x} = t\vec{v}$, for $t \neq 0$, then we get

$$\operatorname{proj}_{\vec{v}}(t\vec{v}) = \frac{(t\vec{v})\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = t\vec{v} = 1(t\vec{v})$$

Thus, $\lambda_1 = 1$ is an eigenvalue with $E_{\lambda_1} = \operatorname{Span}\{\vec{v}\}$.

If $\vec{y}$ is any non-zero vector orthogonal to $\vec{v}$, then we get

$$\operatorname{proj}_{\vec{v}} \vec{y} = \vec{0} = 0\vec{v}$$

Hence, $\lambda_2 = 0$ is an eigenvalue with $E_{\lambda_2} = \operatorname{Span}\{\vec{y}\}$.

6.2.5 If A is invertible and $\vec{v}$ is an eigenvector of A, then there exists $\lambda \in \mathbb{R}$ such that $A\vec{v} = \lambda\vec{v}$. Multiplying both sides on the left by A^{-1} gives

$$\vec{v} = A^{-1}(\lambda\vec{v}) = \lambda A^{-1}\vec{v}$$

Since $\vec{v} \neq \vec{0}$, this implies that $\lambda \neq 0$. Thus, $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$. Hence, $\vec{v}$ is also an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

6.2.6 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We have that

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

This gives

$$\begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a+3b \\ c+3d \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

Solving $a + 2b = 2$ and $a + 3b = 3$ gives $b = 1$ and $a = 0$. Solving $c + 2d = 4$ and $c + 3d = 9$ gives $d = 5$ and $c = -6$. Thus, $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

6.2.7 Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then, observe that $C(\lambda) = \lambda^4$, so $\lambda = 0$ is an eigenvalue with $a_\lambda = 4$. However, since $\operatorname{rank}(A) = 1$, we have by Theorem 6.2.4, that $g_\lambda = 3$. Thus, $g_\lambda < a_\lambda$ as required.

6.2.8 Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Then, observe that

$$C(\lambda) = (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)$$

Thus, A has 4 distinct eigenvalues $1, 2, 3, 4$.

6.2.9 (a) We have

$$(AB)(5\vec{u}+3\vec{v}) = A(5B\vec{u}+3B\vec{v}) = A(25\vec{u}+9\vec{v}) = 25A\vec{u}+9A\vec{v} = 150\vec{u}+90\vec{v} = 30(5\vec{u}+3\vec{v})$$

So, $5\vec{u}+3\vec{v}$ is an eigenvector of C with eigenvalue 30.

(b) Since $\vec{u}$ and $\vec{v}$ correspond to different eigenvalues, $\{\vec{u},\vec{v}\}$ is linearly independent and hence a basis for $\mathbb{R}^2$. Thus, there exists coefficients c_1, c_2 such that $\vec{w} = c_1\vec{u}+c_2\vec{v}$. Therefore

$$AB\vec{w} = AB(c_1\vec{u}+c_2\vec{v}) = c_1(AB)\vec{u}+c_2(AB)\vec{v} = c_1(30\vec{u})+c_2(30\vec{v}) = 30(c_1\vec{u}+c_2\vec{v}) = 30\vec{w} = \begin{bmatrix} 6 \\ 42 \end{bmatrix}$$

6.2.10 It does not imply that $\lambda\mu$ is an eigenvalue of AB. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$. Then $\lambda = 2$ is an eigenvalue of A and $\mu = 1$ is an eigenvalue of B, but $\lambda\mu = 2$ is not an eigenvalue of $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

6.2.11 By definition of matrix-vector multiplication, we have that

$$A\vec{v} = \begin{bmatrix} c \\ \vdots \\ c \end{bmatrix} = c\vec{v}$$

Thus, $\vec{v}$ is an eigenvector with eigenvalue c.

6.2.12 Observe that if A is upper or lower triangular, then $A - \lambda I$ is also upper or lower triangular. Thus, by Theorem 5.3.2, we have that

$$\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

Therefore, the eigenvalues of A are $a_{11}, a_{22}, \dots, a_{nn}$ as required.

6.2.13 If A and B are similar, then there exists an invertible matrix P such that $P^{-1}AP = B$. We get that the characteristic polynomial for B is

$$\begin{aligned} C_B(\lambda) &= \det(B-\lambda I) = \det(P^{-1}AP - \lambda P^{-1}P) \\ &= \det(P^{-1}(A-\lambda I)P) \\ &= \det P^{-1}\det(A-\lambda I)\det P \\ &= \det(A-\lambda I) \end{aligned}$$

which is the characteristic polynomial for A. Thus, the result follows.

6.2.14 If A is not invertible, then $\det A = 0$ and hence $0 = \det(A - 0I)$ and so 0 is an eigenvalue of A.
On the other hand, if 0 is an eigenvalue of A, then $0 = \det(A - 0I) = \det A$ and so A is not invertible.

6.3 Problem Solutions

6.3.1 (a) We have

$$C(\lambda) = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4)$$

Thus, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$. Since all the eigenvalues have algebraic multiplicity 1, we know that A is diagonalizable.

For $\lambda_1 = 2$ we have $A - \lambda_1 I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 4$ we have $A - \lambda_2 I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The resulting diagonal matrix D has the eigenvalues of A as its diagonal entries, so $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$$

Thus, the only eigenvalue of A is $\lambda_1 = 1$ which has algebraic multiplicity 2.

For $\lambda_1 = 1$ we have $A - \lambda_1 I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

A basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Hence, the geometric multiplicity of λ_1 is 1. Since the geometric multiplicity is less than the algebraic multiplicity the matrix is not diagonalizable.

(c) We have

$$C(\lambda) = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 7-\lambda \end{vmatrix} = \lambda^2 - 11\lambda + 24 = (\lambda-3)(\lambda-8)$$

Thus, the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 8$. Since all the eigenvalues have algebraic multiplicity 1, we know that A is diagonalizable.

For $\lambda_1 = 3$ we have $A - \lambda_1 I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 8$ we have $A - \lambda_2 I = \begin{bmatrix} 4 & -2 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. The resulting diagonal matrix D has the eigenvalues of A as its diagonal entries, so $D = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$.

(d) We have

$$C(\lambda) = \begin{vmatrix} 5-\lambda & -1 & 1 \\ -1 & 5-\lambda & 1 \\ 1 & 1 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 1 \\ 0 & 6-\lambda & 6-\lambda \\ 1 & 1 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 2 \\ 0 & 6-\lambda & 0 \\ 1 & 1 & 4-\lambda \end{vmatrix}$$
$$= (6-\lambda)(\lambda^2 - 9\lambda + 18) = -(\lambda-6)^2(\lambda-3)$$

Thus, the eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = 3$.

For $\lambda_1 = 6$ we have $A - \lambda_1 I = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 3$ we have $A - \lambda_2 I = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. The resulting diagonal matrix D has the eigenvalues of A as its diagonal entries, so $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(e) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 2 & 2-\lambda & 1 \\ 4 & 0 & 5-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 6\lambda + 9) = -(\lambda-2)(\lambda-3)^2$$

Thus, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 3$.

For $\lambda_2 = 3$ we have $A - \lambda_2 I = \begin{bmatrix} -2 & 0 & -1 \\ 2 & -1 & 1 \\ 4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Consequently, the matrix is not diagonalizable since $g_{\lambda_2} < a_{\lambda_2}$.

(f) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 2 & -\lambda & 1 \\ 1 & -1 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 2-\lambda & -\lambda & 1 \\ 0 & -1 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & -1 \\ 0 & -1 & -1-\lambda \end{vmatrix}$$
$$= (2-\lambda)(\lambda^2 + 2\lambda) = -\lambda(\lambda - 2)(\lambda + 2)$$

Thus, the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = -2$.

For $\lambda_1 = 0$ we have $A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$.

For $\lambda_2 = 2$ we have $A - \lambda_2 I = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -2 & 1 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda_3 = -2$ we have $A - \lambda_3 I = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/4 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_3} is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} -1 & 1 & -3 \\ -3 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix}$. The resulting diagonal matrix D has the eigenvalues of A as its diagonal entries, so $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

(g) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & -2 & 3 \\ 2 & 6-\lambda & -6 \\ 1 & 2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 2-\lambda \\ 2 & 6-\lambda & -6 \\ 1 & 2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 2 & 6-\lambda & -8 \\ 1 & 2 & -2-\lambda \end{vmatrix}$$
$$= (2-\lambda)(\lambda^2 - 4\lambda + 4) = -(\lambda - 2)^3$$

Thus, the only eigenvalues of A is $\lambda_1 = 2$ with algebraic multiplicity 3.

We have $A - \lambda_1 I = \begin{bmatrix} -1 & -2 & 3 \\ 2 & 4 & -6 \\ 1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$. Thus, $g_{\lambda_1} = 2 < a_{\lambda_1}$, so A is not diagonalizable.

(h) We have

$$C(\lambda) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ -4 & -2-\lambda & -5 \\ 2 & 2 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & 0 \\ -4 & -2-\lambda & -3+\lambda \\ 2 & 2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & 0 \\ -2 & -\lambda & 0 \\ 2 & 2 & 3-\lambda \end{vmatrix}$$
$$= (3-\lambda)(\lambda^2 - 3\lambda + 2) = -(\lambda - 3)(\lambda - 2)(\lambda - 1)$$

Thus, the eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 1$.

For $\lambda_1 = 3$ we have $A - \lambda_1 I = \begin{bmatrix} 0 & 1 & 1 \\ -4 & -5 & -5 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 2$ we have $A - \lambda_2 I = \begin{bmatrix} 1 & 1 & 1 \\ -4 & -4 & -5 \\ 2 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda_3 = 1$ we have $A - \lambda_3 I = \begin{bmatrix} 2 & 1 & 1 \\ -4 & -3 & -5 \\ 2 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_3} is $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$. The resulting diagonal matrix D has the eigenvalues of A as its diagonal entries, so $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(i) We have

$$C(\lambda) = \begin{vmatrix} -3-\lambda & -3 & 5 \\ 13 & 10-\lambda & -13 \\ 3 & 2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & -3 & 5 \\ 0 & 10-\lambda & -13 \\ 2-\lambda & 2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & -3 & 5 \\ 0 & 10-\lambda & -13 \\ 0 & 5 & -6-\lambda \end{vmatrix}$$
$$= (2-\lambda)(\lambda^2 - 4\lambda + 5)$$

The roots of $\lambda^2 - 4\lambda + 5$ are not real, so A is not diagonalizable over $\mathbb{R}$.

(j) We have

$$C(\lambda) = \begin{vmatrix} 8-\lambda & 6 & -10 \\ 5 & 1-\lambda & -5 \\ 5 & 3 & -7-\lambda \end{vmatrix} = \begin{vmatrix} 8-\lambda & 6 & -10 \\ 0 & -2-\lambda & 2+\lambda \\ 5 & 3 & -7-\lambda \end{vmatrix} = \begin{vmatrix} 8-\lambda & -4 & -10 \\ 0 & 0 & 2+\lambda \\ 5 & -4-\lambda & -7-\lambda \end{vmatrix}$$
$$= -(2+\lambda)(\lambda^2 - 4\lambda - 12) = -(\lambda+2)^2(\lambda - 6)$$

Thus, the eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 6$.

For $\lambda_1 = -2$ we have $A - \lambda_1 I = \begin{bmatrix} 10 & 6 & -10 \\ 5 & 3 & -5 \\ 5 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/5 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -3/5 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 6$ we have $A - \lambda_2 I = \begin{bmatrix} 2 & 6 & -10 \\ 5 & -5 & -5 \\ 5 & 3 & -13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} -3/5 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. The resulting diagonal matrix D has the eigenvalues of A as its diagonal entries, so $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

(k) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ -1+\lambda & 1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ -3+2\lambda & 0 & 4-\lambda \end{vmatrix}$$
$$= (2-\lambda)(3-\lambda)(4-\lambda) - 3 + 2\lambda = \lambda^3 - 9\lambda^2 + 24\lambda - 21$$

We find that $C(\lambda)$ has non-real eigenvalues and so is not diagonalizable over $\mathbb{R}$.

(l) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 1 & \lambda & \lambda & 1-\lambda \end{vmatrix}$$
$$= \begin{vmatrix} 1-\lambda & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 2 & 0 & \lambda & 2-\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 1 \\ 2 & \lambda & 2-\lambda \end{vmatrix}$$
$$= (-\lambda)\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 1 \\ 3 & 0 & 3-\lambda \end{vmatrix} = \lambda^2(\lambda^2 - 4\lambda) = \lambda^3(\lambda - 4)$$

Thus, the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 4$.

For $\lambda_1 = 0$ we have $A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 4$ we have $A - \lambda_2 I = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The resulting diagonal matrix D has the eigenvalues of A as its diagonal entries, so $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

6.3.2 (a) We have $[L] = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. Thus, from our work in problem 6.3.1(a), we get that if we take $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, then $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) We have $[L] = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda = 10 = (\lambda - 5)(\lambda + 2)$$

Thus, the eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -2$. Since all the eigenvalues have algebraic multiplicity 1, we know that $[L]$ is diagonalizable.

For $\lambda_1 = 5$ we have $A - \lambda_1 I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$.

For $\lambda_2 = -2$ we have $A - \lambda_2 I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Therefore, we take $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and our we get $[L]_{\mathcal{B}} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$.

(c) We have $[L] = \begin{bmatrix} -2 & 2 & 2 \\ -3 & 3 & 2 \\ -2 & 2 & 2 \end{bmatrix}$.

We have

$$C(\lambda) = \begin{vmatrix} -2-\lambda & 2 & 2 \\ -3 & 3-\lambda & 2 \\ -2 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 2 & 2 \\ -3 & 3-\lambda & 2 \\ \lambda & 0 & -\lambda \end{vmatrix}$$
$$= \begin{vmatrix} -2-\lambda & 2 & -\lambda \\ -3 & 3-\lambda & -1 \\ \lambda & 0 & 0 \end{vmatrix} = \lambda(-2+3\lambda-\lambda^2)$$
$$= -\lambda(\lambda-2)(\lambda-1)$$

Thus, the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 1$. Since all the eigenvalues have algebraic multiplicity 1, we know that $[L]$ is diagonalizable.

For $\lambda_1 = 0$ we have $A - \lambda_1 I = \begin{bmatrix} -2 & 2 & 2 \\ -3 & 3 & 2 \\ -2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda_2 = 2$ we have $A - \lambda_2 I = \begin{bmatrix} -4 & 2 & 2 \\ -3 & 1 & 2 \\ -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_3 = 1$ we have $A - \lambda_3 I = \begin{bmatrix} -3 & 2 & 2 \\ -3 & 2 & 2 \\ -2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, a basis for E_{λ_3} is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$.

Therefore, we take $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ and our we get $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

6.3.3 (a) By definition A is similar to $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Hence, by Theorem 6.1.1, we have that

$$\operatorname{tr} A = \operatorname{tr} \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \lambda_1 + \cdots + \lambda_n$$

(b) By definition A is similar to $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Hence, by Theorem 6.1.1, we have that

$$\det A = \det \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdots \lambda_n$$

6.3.4 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible, but not diagonalizable.

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible and diagonalizable.

$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible and not diagonalizable.

$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible, but diagonalizable.

This shows that there is no connection between a matrix being invertible and being diagonalizable.

6.3.5 If A is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$, then there exists an invertible matrix P such that

$$P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

Then, observe that

$$P^{-1}(A - \lambda_1 I)P = P^{-1}AP - \lambda_1 P^{-1}P = \operatorname{diag}(\lambda_1, \dots, \lambda_n) - \lambda_1 I = \operatorname{diag}(0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1)$$

as required.

6.3.6 If A is diagonalizable, then there exists an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. Thus,

$$D = D^T = (P^{-1}AP)^T = P^T A^T (P^{-1})^T$$

Let $Q = (P^{-1})^T$, then $Q^{-1} = P^T$ and so we have that

$$Q^{-1}A^T Q = D$$

and so A^T is also diagonalizable.

6.3.7 (a) Observe that

$$C(\lambda) = \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc$$

Thus, the eigenvalues of A are

$$\lambda = \frac{-(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}$$

This will only have real roots when

$$0 \leq (a + d)^2 - 4(ad - bc) = a^2 + 2ad + d^2 - 4ad + 4bc = a^2 - 2ad + d^2 - 4bc = (a - d)^2 + 4bc$$

as required.

(b) If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $(a-d)^2 + 4bc = (0-0)^2 + 4(1)(0) = 0$, but A is not diagonalizble since $a_0 = 2$, but $g_0 = 1 < a_0$.

If A is the zero matrix, then $(a-d)^2 + 4bc = 0$, but A is diagonalizable since it is already diagonal.

(c) If $(a-d)^2+4bc = (2x)^2$ where $x \in \mathbb{Z}$, then observe that $a-d$ must be even since $(a-d)^2 = (2x)^2-4bc$ is even. Let $a - d = 2y$ where $y \in \mathbb{Z}$, then we have that the eigenvalues of A are

$$\lambda = \frac{-2y \pm \sqrt{(2x)^2}}{2} = \frac{-2y \pm 2|x|}{2} = -y \pm |x|$$

which are both integers.

6.4 Problem Solutions

6.4.1 We have

$$C(\lambda) = \begin{vmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{vmatrix} = \lambda^2 - 9\lambda + 18 = (\lambda-3)(\lambda-6)$$

Thus, the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 6$.

For $\lambda_1 = 3$ we get

$$A - \lambda_1 I = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = 6$ we get

$$A - \lambda_2 I = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ to $D = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$. Thus, we have $A = PDP^{-1}$ and so

$$\begin{aligned} A^3 = PD^3P^{-1} &= \begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 27 & 0 \\ 0 & 216 \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 90 & -63 \\ -126 & 153 \end{bmatrix} \end{aligned}$$

It is easy to verify that this does equal A^3.

6.4.2 We have

$$C(\lambda) = \begin{vmatrix} -6-\lambda & -10 \\ 4 & 7-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

Thus, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$.

For $\lambda_1 = 2$ we get

$$A - \lambda_1 I = \begin{bmatrix} -8 & -10 \\ 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5/4 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} -5 \\ 4 \end{bmatrix}\right\}$.

For $\lambda_2 = -1$ we get

$$A - \lambda_2 I = \begin{bmatrix} -5 & -10 \\ 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} -5 & -2 \\ 4 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. Thus, we have $A = PDP^{-1}$ and so

$$\begin{aligned} A^{100} = PD^{100}P^{-1} &= \begin{bmatrix} -5 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 4 & -5 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} -5 \cdot 2^{100} + 8 & -5 \cdot 2^{101} + 10 \\ 2^{102} - 4 & 2^{103} - 5 \end{bmatrix} \end{aligned}$$

6.4.3 We have

$$C(\lambda) = \begin{vmatrix} 2 - \lambda & 2 \\ -3 & -5 - \lambda \end{vmatrix} = \lambda^2 + 3\lambda - 4 = (\lambda - 1)(\lambda + 4)$$

Thus, the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -4$.

For $\lambda_1 = 1$ we get

$$A - \lambda_1 I = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = -4$ we get

$$A - \lambda_2 I = \begin{bmatrix} 6 & 2 \\ -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} -2 & -1 \\ 1 & 3 \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$. Thus, we have $A = PDP^{-1}$ and so

$$\begin{aligned} A^{100} = PD^{100}P^{-1} &= \begin{bmatrix} -2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4^{100} \end{bmatrix}\frac{1}{-5}\begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix} \\ &= -\frac{1}{5}\begin{bmatrix} -6 + 4^{100} & -2 + 2 \cdot 4^{100} \\ 3 - 3 \cdot 4^{100} & 1 - 6 \cdot 4^{100} \end{bmatrix} \end{aligned}$$

6.4.4 We have

$$C(\lambda) = \begin{vmatrix} -2 - \lambda & 2 \\ -3 & 5 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4)$$

Thus, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

For $\lambda_1 = -1$ we have

$$A - \lambda_1 I = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = 4$ we have

$$A - \lambda_2 I = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ to $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$. Thus, we have $A = PDP^{-1}$ and so

$$\begin{aligned} A^{200} = PD^{200}P^{-1} &= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4^{200} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 6 - 4^{200} & -2 + 2 \cdot 4^{200} \\ 3 - 3 \cdot 4^{20} & -1 + 6 \cdot 4^{200} \end{bmatrix} \end{aligned}$$

6.4.5 We have

$$\begin{aligned} C(\lambda) &= \begin{vmatrix} -2-\lambda & 1 & 1 \\ -1 & -\lambda & 1 \\ -2 & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -1-\lambda & 1 & 1 \\ -1-\lambda & -\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix} \\ &= \begin{vmatrix} -1-\lambda & 1 & 1 \\ 0 & -1-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-1) \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$.

For $\lambda_1 = -1$ we have

$$A - \lambda_1 I = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 1$ we have

$$A - \lambda_2 I = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -1 & 1 \\ -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ to $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus, we have $A = PDP^{-1}$ and so

$$A^{100} = PD^{100}P^{-1} = PIP^{-1} = I$$

6.4.6 If λ is an eigenvalue of A, then there exists a non-zero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. We will prove by induction that $A^n\vec{v} = \lambda^n\vec{v}$.

Base Case: $n = 1$. We have $A^1\vec{v} = \lambda^1\vec{v}$.

Inductive Hypothesis: Assume that $A^k\vec{v} = \lambda^k\vec{v}$.

Inductive Step: We have

$$A^{k+1}\vec{v} = A^k(A\vec{v}) = A^k(\lambda\vec{v}) = \lambda(A^k\vec{v}) = \lambda(\lambda^k\vec{v}) = \lambda^{k+1}\vec{v}$$

as required.

6.4.7 Let $S = T - 1I$. Then, we have

$$\sum_{i=1}^{n} s_{ij} = \left(\sum_{i=1}^{n} t_{ij}\right) - 1 = 0$$

Thus, the column sums of S equal 0. Therefore, $\det S = 0$ since if we add each of the first $n-1$ rows of S to the n-th row using Theorem 5.3.6, we will get a row of zeros in the new matrix.
Hence, $0 = \det S = \det(T - 1I)$, so $\lambda = 1$ is an eigenvalue of T.

6.4.8 (a) By question 7 we know that $\lambda_1 = 1$ is an eigenvalue of A. So, by Theorem 6.3.6, the other eigenvalue is $\lambda_2 = \left(\frac{9}{10} + \frac{4}{5}\right) - 1 = \frac{7}{10}$.

For $\lambda_1 = 1$ we have

$$A - \lambda_1 I = \begin{bmatrix} -\frac{1}{10} & \frac{1}{5} \\ \frac{1}{10} & -\frac{1}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = \frac{7}{10}$ we have

$$A - \lambda_2 I = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{1}{10} & \frac{1}{10} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{10} \end{bmatrix}$. Thus, we have $A = PDP^{-1}$ and so

$$\begin{aligned} A^t = PD^tP^{-1} &= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{7}{10}\right)^t \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 + \left(\frac{7}{10}\right)^t & 2 - \left(\frac{7}{10}\right)^t \\ 1 - \left(\frac{7}{10}\right)^t & 1 + 2\left(\frac{7}{10}\right)^t \end{bmatrix} \end{aligned}$$

(b) We have

$$\lim_{t\to\infty}\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \lim_{t\to\infty} A^t \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \lim_{t\to\infty} \frac{1}{3}\begin{bmatrix} 2+\left(\frac{7}{10}\right)^t & 2-2\left(\frac{7}{10}\right)^t \\ 1-\left(\frac{7}{10}\right)^t & 1+2\left(\frac{7}{10}\right)^t \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 2x_0+2y_0 \\ x_0+y_0 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 2T \\ T \end{bmatrix}$$

as required.

6.4.9 We have $A = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$. By question 7 we know that $\lambda_1 = 1$ is an eigenvalue of A. So, by Theorem 6.3.6, the other eigenvalue is $\lambda_2 = \left(\frac{7}{10}+\frac{9}{10}\right) - 1 = \frac{3}{5}$.

For $\lambda_1 = 1$ we have

$$A - \lambda_1 I = \begin{bmatrix} -\frac{3}{10} & \frac{1}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.

For $\lambda_2 = \frac{3}{5}$ we have

$$A - \lambda_2 I = \begin{bmatrix} \frac{1}{10} & \frac{1}{10} \\ \frac{3}{10} & \frac{3}{10} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

It follows that A is diagonalized by $P = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{bmatrix}$. Thus, we have $A = PDP^{-1}$ and so

$$A^t = PD^tP^{-1} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{3}{5}\right)^t \end{bmatrix}\frac{1}{4}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 1+3\left(\frac{3}{5}\right)^t & 1-\left(\frac{3}{5}\right)^t \\ 3-3\left(\frac{3}{5}\right)^t & 3+\left(\frac{3}{5}\right)^t \end{bmatrix}$$

Therefore, in 6 months, the market share will be

$$\begin{bmatrix} x_6 \\ y_6 \end{bmatrix} = A^6\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 1+3\left(\frac{3}{5}\right)^6 & 1-\left(\frac{3}{5}\right)^6 \\ 3-3\left(\frac{3}{5}\right)^6 & 3+\left(\frac{3}{5}\right)^6 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$
$$\approx \begin{bmatrix} 0.26 \\ 0.74 \end{bmatrix}$$

In the long run, we have

$$\lim_{t\to\infty}\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \lim_{t\to\infty} A^t \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$

6.4.10 Let $A = \begin{bmatrix} 8/10 & 3/10 & 3/10 \\ 1/10 & 6/10 & 1/10 \\ 1/10 & 1/10 & 6/10 \end{bmatrix}$. We can find that A is diagonalized by $P = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. Thus, as $t \to \infty$ we get $D^t \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore, in the long run we have

$$\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = PD^tP^{-1}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3(x_0 + y_0 + z_0) \\ x_0 + y_0 + z_0 \\ x_0 + y_0 + z_0 \end{bmatrix}$$

Thus, if $T = x_0 + y_0 + z_0$ is the total number of cars, then in the long run $\frac{3}{5}$ of the cars will be at the airport, $\frac{1}{5}$ will be at the train station, and $\frac{1}{5}$ will be at the city centre.

6.4.11 First, recall that 2e have $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Now, observe that we have

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^{n-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6.7}$$

Thus, we need to calculate A^{n-1}.
We have

$$C(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$$

By the quadratic formula we get

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

For $\lambda_1 = \frac{1+\sqrt{5}}{2}$ we get

$$A - \lambda_1 I = \begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1+\sqrt{5}}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix}$$

Thus, a corresponding eigenvector is $\vec{v}_1 = \begin{bmatrix} 1-\sqrt{5} \\ -2 \end{bmatrix}$.

For $\lambda_2 = \frac{1-\sqrt{5}}{2}$ we get

$$A - \lambda_2 I = \begin{bmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1-\sqrt{5}}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix}$$

Thus, a corresponding eigenvector is $\vec{v}_2 = \begin{bmatrix} 1+\sqrt{5} \\ -2 \end{bmatrix}$.

Therefore,

$$A^{n-1} = PD^{n-1}P^{-1} = \begin{bmatrix} 1-\sqrt{5} & 1+\sqrt{5} \\ -2 & -2 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{bmatrix} \frac{1}{4\sqrt{5}} \begin{bmatrix} -2 & -1-\sqrt{5} \\ 2 & 1-\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{(-1+\sqrt{5})}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \frac{(1+\sqrt{5})}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & \frac{(1+\sqrt{5})}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{(1-\sqrt{5})}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{bmatrix}$$

Using equation (6.7) we get

$$a_n = x_n = \frac{1}{\sqrt{5}}\left(\frac{(-1+\sqrt{5})}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \frac{(1+\sqrt{5})}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right)$$

Chapter 7 Solutions

7.1 Problem Solutions

7.1.1 (a) Row reducing we get

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$$

Hence, a basis for the row space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$, a basis for the column space is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. To find a basis for the nullspace we solve

$$\begin{aligned} x_1 - 3x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

x_3 is a free variable, so we let $x_3 = t \in \mathbb{R}$. Then we have any vector $\vec{x}$ in the nullspace satisfies

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

Thus, a basis for the nullspace is $\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$.

To find a basis for the left nullspace, we row reduce the transpose of the matrix. We get

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, the left nullspace is $\{\vec{0}\}$. Therefore, a basis for the left nullspace is the empty set.

(b) Row reducing we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, a basis for the row space is $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, a basis for the column space is $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right\}$. To find a basis for the nullspace we solve

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

x_2 is a free variable, so we let $x_2 = t \in \mathbb{R}$. Then we have any vector $\vec{x}$ in the nullspace satisfies

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2t\\t\\0\end{bmatrix} = t\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

Thus, a basis for the nullspace is $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}$.

To find a basis for the left nullspace, we row reduce the transpose of the matrix. We get

$$\begin{bmatrix}1 & 2\\2 & 4\\-1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

Thus, the left nullspace is $\{\vec{0}\}$. Therefore, a basis for the left nullspace is the empty set.

(c) Row reducing we get

$$\begin{bmatrix}2 & -1 & 5\\3 & 4 & 2\\1 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

Hence, a basis for the row space is $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$, a basis for the column space is $\left\{\begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}-1\\4\\1\end{bmatrix}\right\}$. To find a basis for the nullspace we solve

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

x_3 is a free variable, so we let $x_3 = t \in \mathbb{R}$. Then we have any vector $\vec{x}$ in the nullspace satisfies

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2t\\t\\t\end{bmatrix} = t\begin{bmatrix}-2\\1\\1\end{bmatrix}$$

Thus, a basis for the nullspace is $\left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$.

To find a basis for the left nullspace, we row reduce the transpose of the matrix. We get

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 1 \\ 5 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/11 \\ 0 & 1 & 3/11 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we need to solve the system

$$\begin{aligned} x_1 + \frac{1}{11}x_3 &= 0 \\ x_2 + \frac{3}{11}x_3 &= 0 \end{aligned}$$

Let $x_3 = t \in \mathbb{R}$. Then every vector $\vec{x}$ in the left nullspace satisfies

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t/11 \\ -3t/11 \\ t \end{bmatrix} = t\begin{bmatrix} -1/11 \\ -3/11 \\ 1 \end{bmatrix}$$

Hence, a basis for the left nullspace is $\left\{\begin{bmatrix} -1/11 \\ -3/11 \\ 1 \end{bmatrix}\right\}$.

(d) Row reducing we get

$$\begin{bmatrix} 1 & -2 & 3 & 5 \\ -2 & 4 & 0 & -4 \\ 3 & -6 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the row space is $\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$, a basis for the column space is $\left\{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}\right\}$. To find a basis for the nullspace we solve

$$\begin{aligned} x_1 - 2x_2 + 2x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

x_2 and x_4 are free variables, so we let $x_2 = s \in \mathbb{R}$ and $x_4 = t \in \mathbb{R}$. Then we have any vector $\vec{x}$ in the nullspace satisfies

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s - 2t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus, a basis for the nullspace is $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$.

To find a basis for the left nullspace, we row reduce the transpose of the matrix. We get

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 6 \\ 3 & 0 & -5 \\ 5 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & -7/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we need to solve the system

$$\begin{aligned} x_1 - \frac{5}{3}x_3 &= 0 \\ x_2 - \frac{7}{3}x_3 &= 0 \end{aligned}$$

Let $x_3 = t \in \mathbb{R}$. Then every vector $\vec{x}$ in the left nullspace satisfies

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5t/3 \\ 7t/3 \\ t \end{bmatrix} = t \begin{bmatrix} 5/3 \\ 7/3 \\ 1 \end{bmatrix}$$

Hence, a basis for the left nullspace is $\left\{ \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix} \right\}$.

(e) Row reducing we get

$$\begin{bmatrix} 3 & 2 & 5 & 3 \\ -1 & 0 & -3 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the row space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, a basis for the column space is $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

To find a basis for the nullspace we solve

$$\begin{aligned} x_1 + 3x_3 &= 0 \\ x_2 - 2x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

x_3 is a free variable, so we let $x_3 = t \in \mathbb{R}$. Then we have any vector $\vec{x}$ in the nullspace satisfies

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3t \\ 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Thus, a basis for the nullspace is $\left\{\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right\}$.

To find a basis for the left nullspace, we row reduce the transpose of the matrix. We get

$$\begin{bmatrix} 3 & -1 & 1 & 1 \\ 2 & 0 & 1 & 4 \\ 5 & -3 & 1 & -5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we need to solve the system

$$\begin{aligned} x_1 - 3x_4 &= 0 \\ x_2 &= 0 \\ x_3 + 10x_4 &= 0 \end{aligned}$$

Let $x_4 = t \in \mathbb{R}$. Then every vector $\vec{x}$ in the left nullspace satisfies

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3t \\ 0 \\ -10t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ -10 \\ 1 \end{bmatrix}$$

Hence, a basis for the left nullspace is $\left\{\begin{bmatrix} 3 \\ 0 \\ -10 \\ 1 \end{bmatrix}\right\}$.

7.1.2 We have that $\dim \operatorname{Null}(A) = 2$. Thus, by the Dimension Theorem

$$\dim \operatorname{Col}(A) = \operatorname{rank}(A) = 3 - \dim \operatorname{Null}(A) = 3 - 2 = 1$$

We are also given that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{a}_1$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{a}_2$$

Also, $\vec{a}_3 \neq \vec{0}$, since $\vec{e}_3$ is not in the nullspace of A. Thus, a basis for $\operatorname{Col}(A)$ is $\{\vec{a}_3\}$.

7.1.3 Pick $\vec{z} \in \{RE\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$. Then $\vec{z} = RE\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$. By definition of matrix-vector multiplication we have that $E\vec{x} = \vec{y} \in \mathbb{R}^n$ for some $\vec{y} \in \mathbb{R}^n$. Thus, $\vec{z} = R\vec{y} \in \{R\vec{y} \mid \vec{y} \in \mathbb{R}^n\}$.
On the other hand, pick $\vec{w} \in \{R\vec{y} \mid \vec{y} \in \mathbb{R}^n\}$. Since E is an invertible matrix, for any $\vec{y} \in \mathbb{R}^n$ we have that there exists a unique vector $\vec{x} \in \mathbb{R}^n$ such that $E\vec{x} = \vec{y}$. Thus, $\vec{z} = RE\vec{x} \in \{RE\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$.
Therefore, the sets are subsets of each other and hence equal.

7.1.4 The result follows immediately from the Invertible Matrix Theorem.

7.1.5 (a) If $\vec{x} \in \operatorname{Null}(B^T)$, then by definition $\vec{0} = B^T\vec{x}$. Taking transposes of both sides gives

$$\vec{0}^T = (B^T\vec{x})^T = \vec{x}^T(B^T)^T = \vec{x}^T B$$

(b) Let $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}^n \end{bmatrix}$. Then, $B^T = \begin{bmatrix} \vec{b}_1^T \\ \vdots \\ \vec{b}_n^T \end{bmatrix}$. If $\vec{x} \in \operatorname{Null}(B^T)$, then we have

$$\vec{0} = B^T\vec{x} = \begin{bmatrix} \vec{b}_1 \cdot \vec{x} \\ \vdots \\ \vec{b}_n \cdot \vec{x} \end{bmatrix}$$

Thus, $\vec{b}_i \cdot \vec{x} = 0$ for $1 \leq i \leq n$. Hence, if $\vec{y} \in \operatorname{Col}(B)$, then

$$\vec{x} \cdot \vec{y} = \vec{x} \cdot (c_1\vec{b}_1 + \cdots + c_n\vec{b}_n) = c_1(\vec{x} \cdot \vec{b}_1) + \cdots + c_n(\vec{x} \cdot \vec{b}_n) = 0$$

7.1.6 We need to find a matrix A such that $A\vec{x} = \vec{0}$ if and only if $\vec{x}$ is a linear combination of the columns of A. Observe that we need $\dim \operatorname{Null}(A) = \dim \operatorname{Col}(A) = \operatorname{rank}(A)$. So, by the Dimension Theorem, we have $2 = \operatorname{rank}(A) + \dim \operatorname{Null}(A) = 2\operatorname{rank}(A)$. Thus, we need $\operatorname{rank}(A) = 1$. Hence, we can pick A to be of the form $A = \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix}$. It is easy to see that taking $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ gives

$$\operatorname{Col}(A) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \operatorname{Null}(A)$$

7.1.7 First observe by the Dimension Theorem that $\dim \operatorname{Null}(A^T) = m - r$.

Let $E = \begin{bmatrix} \vec{e}_1^T \\ \vdots \\ \vec{e}_m^T \end{bmatrix}$ and observe that the bottom $m - r$ rows of R are all zeros since $\operatorname{rank}(A) = r$.

Thus, $EA = R$ implies that $\vec{e}_i^T A = \vec{0}$ for $r + 1 \leq i \leq m$. Hence, $\vec{e}_{r+1}, \dots, \vec{e}_m \in \operatorname{Null}(A^T)$. Moreover, since E is invertible, the columns of E are linearly independent by the Invertible Matrix Theorem. Thus, $\{\vec{e}_{r+1}, \dots, \vec{e}_m\}$ is a linearly independent set of $m-r$ vectors in $\operatorname{Null}(A^T)$ as hence forms a basis for $\operatorname{Null}(A^T)$ as required.

7.1.8 If A and B are similar, then there exists an invertible matrix P such that $P^{-1}AP = B$. Multiply on the left by P to get $AP = PB$. Then, since P is invertible, it can be written as a product of elementary matrices, say $P = E_1 \cdots E_k$. Hence, by Corollary 5.2.4, we get

$$\operatorname{rank}(PB) = \operatorname{rank}(E_1 \cdots E_k B) = \operatorname{rank}(B)$$

Similarly, since the rank of a matrix equals the rank of the transpose of a the matrix, we get

$$\operatorname{rank}(AP) = \operatorname{rank}((AP)^T) = \operatorname{rank}(E_k^T \cdots E_1^T A) = \operatorname{rank}(A^T) = \operatorname{rank} A$$

Thus, $\operatorname{rank}(A) = \operatorname{rank}(AP) = \operatorname{rank}(PB) = \operatorname{rank}(B)$ as required.

7.1.9 (a) If $A\vec{x} = \vec{0}$, then $A^TA\vec{x} = A^T\vec{0} = \vec{0}$. Hence, the nullspace of A is a subset of the nullspace of A^TA. On the other hand, consider $A^TA\vec{x} = \vec{0}$. Then,

$$\|A\vec{x}\|^2 = (A\vec{x}) \cdot (A\vec{x}) = (A\vec{x})^T(A\vec{x}) = \vec{x}^TA^TA\vec{x} = \vec{x}^T\vec{0} = 0$$

Thus, $A\vec{x} = \vec{0}$. Hence, the nullspace of A^TA is also a subset of the nullspace of A, and the result follows.

(b) Using part (a), we get that $\dim(\operatorname{Null}(A^TA)) = \dim(\operatorname{Null}(A))$. Thus, the Dimension Theorem gives

$$\operatorname{rank}(A^TA) = n - \dim(\operatorname{Null}(A^TA)) = n - \dim(\operatorname{Null}(A)) = \operatorname{rank}(A)$$

as required.

7.1.10 Let $\vec{x} \in \operatorname{Col}(B)$. Then, there exists $\vec{y}$ such that $B\vec{y} = \vec{x}$. Now, observe that

$$A\vec{x} = A(B\vec{y}) = (AB)\vec{y} = O_{m,n}\vec{y} = \vec{0}$$

Thus, $\vec{x} \in \operatorname{Null}(A)$. Hence, $\operatorname{Col}(B)$ is a subset of $\operatorname{Null}(A)$ and thus, since $\operatorname{Col}(B)$ is a vector space, $\operatorname{Col}(B)$ is a subspace of $\operatorname{Null}(A)$.

7.1.11 (a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

(c) We first observe that C must have 3 columns. So, lets make C a 1×3 matrix. We require that

$$\begin{aligned} c_{11} + c_{12} + c_{13} &= 0 \\ c_{11} + 2c_{13} &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Thus, we get that $\vec{c} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Hence, we can take

$$C = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix}$$

(d) Using our work in part (c), we see that we can take $D = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

(e) No, there cannot be such a matrix F since we would have $\dim(\operatorname{Col}(F)) = 2$, $\dim(\operatorname{Null}(F)) = 2$, and F would have to have 3 columns, but this would contradict the Dimension Theorem.

Chapter 8 Solutions

8.1 Problem Solutions

8.1.1 (a) Let $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ and $s, t \in \mathbb{R}$. Then we have

$$\begin{aligned} L\left(s\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + t\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} sa_1 + ta_2, sb_1 + tb_2 \end{bmatrix}\right) = \begin{bmatrix} sa_1 + ta_2 & 0 \\ 0 & sb_1 + tb_2 \end{bmatrix} \\ &= s\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + t\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = sL\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right) + tL\left(\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) \end{aligned}$$

Hence, L is linear.

(b) We have $L(x + x^2) = -1 + x^2$ and $L(2x + 2x^2) = -2 + 4x^2$, so $L(x + x^2) + L(2x + 2x^2) = -3 + 5x^2$. But

$$L[(x + x^2) + (2x + 2x^2)] = L(3x + 3x^2) = -3 + 9x^2$$

Hence, L is not linear.

(c) Since $L(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \vec{0}_{M_{2\times 2}(\mathbb{R})}$ we have that L is not linear by Theorem 8.2.1.

(d) Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ and $s, t \in \mathbb{R}$. Then we have

$$\begin{aligned} T\left(s\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + t\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} sa_1 + ta_2 & sb_1 + tb_2 \\ sc_1 + tc_2 & sd_1 + td_2 \end{bmatrix}\right) = \begin{bmatrix} sa_1 + ta_2 + sb_1 + tb_2 \\ sb_1 + tb_2 + sc_1 + tc_2 \\ sc_1 + tc_2 - (sa_1 + ta_2) \end{bmatrix} \\ &= s\begin{bmatrix} a_1 + b_1 \\ b_1 + c_1 \\ c_1 - a_1 \end{bmatrix} + t\begin{bmatrix} a_2 + b_2 \\ b_2 + c_2 \\ c_2 - a_2 \end{bmatrix} = sT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + tT\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \end{aligned}$$

Thus, T is linear.

(e) Let $a_1 + b_1x + c_1x^2 + d_1x^3, a_2 + b_2x + c_2x^3 + d_2x^3 \in P_3(\mathbb{R})$ and $s, t \in \mathbb{R}$. Then,

$$\begin{aligned} D[s(a_1 + b_1x + c_1x^2 + d_1x^3) + t(a_2 + b_2x + c_2x^3 + d_2x^3)] &= D(sa_1 + ta_2 + (sb_1 + tb_2)x + (sc_1 + tc_2)x^2 + (sd_1 + \\ &= (sb_1 + tb_2) + 2(sc_1 + tc_2)x + 3(sd_1 + td_2)x^2 \\ &= s(b_1 + 2c_1x + 3d_1x^2) + t(b_2 + 2c_2x + 3d_2x^2) \\ &= sD(a_1 + b_1x + c_1x^2 + d_1x^3) + tD(a_2 + b_2x + c_2x^3 + \end{aligned}$$

Thus, D is linear.

(f) We have

$$L\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)+L\left(\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\right)=1+1=2$$

but

$$L\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\right)=L\left(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\right)=0$$

So, L is not linear.

8.1.2 (a) If L is linear, then we have

$$\begin{aligned}L(a_1,a_2,a_3)&=a_1L(1,0,0)+a_2L(0,1,0)+a_3L(0,0,1)\\&=a_1(1+x)+a_2(1-x^2)+a_3(1+x+x^2)\\&=(a_1+a_2+a_3)1+(a_1+a_3)x+(-a_2+a_3)x^2\end{aligned}$$

(b) If L is linear, then we have

$$L(a+bx+cx^2)=aL(1)+bL(x)+cL(x^2)$$

Observe that

$$L(x^2)=L(1+x+x^2)-L(1)-L(x)=\begin{bmatrix}2 & 2\\2 & 3\end{bmatrix}-\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

Thus,

$$L(a+bx+cx^2)=a\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}+b\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}=\begin{bmatrix}a+b & 2a\\a+c & 2a+b\end{bmatrix}$$

8.1.3 Let $\vec{x},\vec{y}\in\mathbb{V}$, $s,t\in\mathbb{R}$ and let $\mathcal{B}=\{\vec{v}_1,\dots,\vec{v}_n\}$. Since $\vec{x},\vec{y}\in\mathbb{V}$ there exists $b_1,\dots,b_n,c_1,\dots,c_n\in\mathbb{R}$ such that

$$\vec{x}=b_1\vec{v}_1+\cdots+b_n\vec{v}_n\quad\text{and}\quad\vec{y}=c_1\vec{v}_1+\cdots+c_n\vec{v}_n$$

Hence,

$$\begin{aligned}L(s\vec{x}+t\vec{y})&=L\left[s(b_1\vec{v}_1+\cdots+b_n\vec{v}_n)+t(c_1\vec{v}_1+\cdots+c_n\vec{v}_n)\right]\\&=L[(sb_1+tc_1)\vec{v}_1+\cdots+(sb_n+tc_n)\vec{v}_n]\\&=\begin{bmatrix}sb_1+tc_1\\\vdots\\sb_n+tc_n\end{bmatrix}\\&=s\begin{bmatrix}b_1\\\vdots\\b_n\end{bmatrix}+t\begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix}\\&=sL(b_1\vec{v}_1+\cdots+b_n\vec{v}_n)+tL(c_1\vec{v}_1+\cdots+c_n\vec{v}_n)\\&=sL(\vec{x})+tL(\vec{y})\end{aligned}$$

hence, L is linear.

8.1.4 $\vec{0} = 0L(\vec{x}) = L(0\vec{x}) = L(\vec{0})$.

8.1.5 By definition the domain of $M \circ L$ is $\mathbb{V}$. Also, for any $\vec{v} \in \mathbb{V}$, we have

$$(M \circ L)(\vec{v}) = M(L(\vec{v}))$$

Observe that $L(\vec{v}) \in \mathbb{W}$ which is in the domain of M. Hence, we get that $M(L(\vec{v})) \in \text{Range}(M)$ which is in $\mathbb{U}$. Thus, the codomain of $M \circ L$ is $\mathbb{U}$.
Let $\vec{x}, \vec{y} \in \mathbb{V}$ and $s, t \in \mathbb{R}$. Then,

$$\begin{aligned}(M \circ L)(s\vec{x} + t\vec{y}) &= M(L(s\vec{x} + t\vec{y})) = M(sL(\vec{x}) + tL(\vec{y})) = sM(L(\vec{x})) + tM(L(\vec{y})) \\ &= s(M \circ L)(\vec{x}) + t(M \circ L)(\vec{y})\end{aligned}$$

Hence, $M \circ L$ is linear.

8.1.6 (a) Consider $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$. Then, we have

$$\begin{aligned}L(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) &= L(\vec{0}) \\ c_1L(\vec{v}_1) + \cdots + c_kL(\vec{v}_k) &= \vec{0} \qquad \text{by Problem 4}\end{aligned}$$

Thus, $c_1 = \cdots = c_k = 0$ since $\{L(\vec{v}_1), \dots, L(\vec{v}_k)\}$ is linearly independent. Thus, $\{\vec{v}_1, \dots, \vec{v}_k\}$ must also be linearly independent.

(b) If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear mapping defined by $L(\vec{x}) = \vec{0}$ and $\{\vec{v}_1, \vec{v}_2\} = \{\vec{e}_1, \vec{e}_2\}$ is the standard basis for $\mathbb{R}^2$, then $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent, but $\{L(\vec{v}_1), L(\vec{v}_2)\}$ is linearly dependent since it contains the zero vector.

8.1.7 Let $L, M \in \mathbb{L}$ and $t \in \mathbb{R}$.

(a) By definition tL is a mapping with domain $\mathbb{V}$. Also, since $\mathbb{W}$ is closed under scalar multiplication, we have that $(tL)(\vec{v}) = tL(\vec{v}) \in \mathbb{W}$. Thus, the codomain of tL is $\mathbb{W}$. For any $\vec{x}, \vec{y} \in \mathbb{V}$ and $c, d \in \mathbb{R}$ we have

$$\begin{aligned}(tL)(c\vec{x} + d\vec{y}) &= tL(c\vec{x} + d\vec{y}) = t[cL(\vec{x}) + dL(\vec{y})] \\ &= ctL(\vec{x}) + dtL(\vec{y}) = c(tL)(\vec{x}) + d(tL)(\vec{y})\end{aligned}$$

Hence, $tL \in \mathbb{L}$.

(b) For any $\vec{v} \in \mathbb{V}$ we have

$$[t(L + M)](\vec{v}) = t[(L + M)(\vec{v})] = t[L(\vec{v}) + M(\vec{v})] = tL(\vec{v}) + tM(\vec{v}) = [tL + tM](\vec{v})$$

Thus, $t(L + M) = tL + tM$.

8.1.8 Assume that L is invertible. Then, there exists a linear mapping $L^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $(L \circ L^{-1})(\vec{x}) = \vec{x}$. Hence, by definition of the standard matrix, we get

$$\vec{x} = L(L^{-1})(\vec{x}) = [L][L^{-1}]\vec{x}$$

Thus, $[L][L^{-1}] = I$ by Theorem 3.1.4. So, $[L]$ is invertible.

If $[L]$ is invertible, then there exists a matrix A such that $A[L] = I = [L]A$. Define $M : \mathbb{R}^n \to \mathbb{R}^n$ by $M(\vec{x}) = A\vec{x}$. Then, for any $\vec{x} \in \mathbb{R}^n$ we have

$$(M \circ L)(\vec{x}) = M(L(\vec{x})) = A[L]\vec{x} = I\vec{x} = \vec{x}$$

and

$$(L \circ M)(\vec{x}) = L(M(\vec{x})) = [L]A\vec{x} = I\vec{x} = \vec{x}$$

Thus, $M = L^{-1}$.

8.1.9 Using our work from problem 7, we know that the standard matrix of L^{-1} is the inverse of $[L] = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$. We have that $[L]^{-1} = \begin{bmatrix} 4/11 & 1/11 \\ 3/11 & -2/11 \end{bmatrix}$. Consequently,

$$L^{-1}(x_1, x_2) = \left(\frac{4}{11}x_1 + \frac{1}{11}x_2, \frac{3}{11}x_1 - \frac{2}{11}x_2\right)$$

8.1.10 Observe that

$$\begin{aligned} (L \circ R)(\vec{x}) &= L(R(x_1, x_2, x_3, \dots)) \\ &= L(0, x_1, x_2, x_3, \dots) \\ &= (x_1, x_2, x_3)(R \circ L)(\vec{x}) \qquad\qquad = R(L(x_1, x_2, x_3, \dots)) \\ &= R(x_2, x_3, \dots) \\ &= (0, x_2, x_3, \dots) \end{aligned}$$

Thus, $(L \circ R)(\vec{x}) = \vec{x}$, but $(R \circ L)(\vec{x}) \neq \vec{x}$.

8.2 Problem Solutions

8.2.1 (a) Every vector in Range(L) has the form

$$L(a+bx+cx^2) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, a basis for Range(L) is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. Thus, rank($L$) = 2 and so the Rank-Nullity Theorem gives

$$\text{nullity}(L) = \dim P_2(\mathbb{R}) - \text{rank}(L) = 3 - 2 = 1$$

(b) If $a+bx+cx^2 \in \ker(L)$, then

$$0 + 0x + 0x^2 = L(a+bx+cx^2) = (a-b) + (b+c)x^2$$

Thus, $a-b=0$ and $b+c=0$. So, $a=b=-c$. So, every vector $a+bx+cx^2 \in \ker(L)$ has the form $-c-cx+cx^2 = c(-1-x+x^2)$. Thus, a basis for $\ker(L)$ is $\{-1-x+x^2\}$. Consequently, nullity(L) = 1 and the Rank-Nullity Theorem gives

$$\text{rank}(L) = \dim P_2(\mathbb{R}) - \text{nullity}(L) = 3 - 1 = 2$$

(c) Every vector in Range(T) has the form

$$T(a+bx+cx^2) = \begin{bmatrix} 0 & 0 \\ a+c & b+c \end{bmatrix} = (a+c)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (b+c)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ spans Range(T) and is clearly linearly independent. Thus, it is a basis for Range(T). Hence, rank(T) = dim Range(T) = 2. By the Rank-Nullity Theorem we get

$$\text{nullity}(L) = \dim P_2(\mathbb{R}) - \text{rank}(T) = 3 - 2 = 1$$

(d) If $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \ker(L)$, then

$$0 = L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a+b) + (a+b+c)x$$

Hence, $a+b=0$ and $a+b+c=0$. Thus, $c=0$ and $b=-a$. So, every vector in $\ker(L)$ has the form

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Thus, a basis for $\ker(L)$ is $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$. Thus, $\text{nullity}(L) = 1$ and

$$\text{rank}(L) = \dim \mathbb{R}^3 - \text{nullity}(L) = 3 - 1 = 2$$

by the Rank-Nullity Theorem.

(e) If $A = \begin{bmatrix}a & b\\c & d\end{bmatrix} \in \ker(T)$, then

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = T(A) = \begin{bmatrix}a & c\\b & d\end{bmatrix}$$

Thus, $a = b = c = d = 0$ and so $A = O_{2,2}$. Hence, $\ker(T) = \{O_{2,2}\}$ and so $\text{nullity}(T) = 0$. Therefore, by the Rank-Nullity Theorem we have that $\text{rank}(T) = \dim M_{2\times 2}(\mathbb{R}) - 0 = 4$.

(f) If $a + bx + cx^2 \in \ker(M)$, then we have

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = M(a + bx + cx^2) = \begin{bmatrix}-a-2c & 2b-c\\-2a+2c & -2b-c\end{bmatrix}$$

Thus, we have $-a - 2c = 0$, $-2a + 2c = 0$, $2b - c = 0$, and $-2b - c = 0$. These imply that $a = b = c = 0$. Thus, $\text{nullity}(M) = 0$ and so $\text{rank}(M) = \dim P_2(\mathbb{R}) - 0 = 3$ by the Rank-Nullity Theorem.

8.2.2 If $\{L(\vec{v}_1), \dots, L(\vec{v}_k)\}$ spans $\mathbb{W}$, then $\text{Range}(L) = \mathbb{W}$. Thus, $\text{rank}(L) = \dim \text{Range}(L) = \dim \mathbb{W}$. So, by the Rank-Nullity Theorem, we have

$$\dim \mathbb{V} = \text{rank}(L) + \text{nullity}(L) = \dim \mathbb{W} + \text{nullity}(L)$$

Since $\text{nullity}(L) \geq 0$, we have that $\dim \mathbb{V} \geq \dim \mathbb{W}$.

8.2.3 One possible example is to take $L : P_3(\mathbb{R}) \to \mathbb{R}^2$ defined by $L(a + bx + cx^2 + dx^3) = \begin{bmatrix}0\\0\end{bmatrix}$. Then, we have that $\dim P_3(\mathbb{R}) \geq \dim \mathbb{R}^2$, and $\text{Range}(L) = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} \neq \mathbb{R}^2$ as required.

8.2.4 Using our work in Problem 7.1.6, we see that we can take $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(x_1, x_2) = \begin{bmatrix}x_2\\0\end{bmatrix}$. Then, we have

$$\text{Range}(L) = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} = \ker(L)$$

8.2.5 If $\text{Range}(L) = \mathbb{W}$, then $\text{rank}(L) = \dim \text{Range}(L) = \dim \mathbb{W} = n$. Hence, by the Rank-Nullity Theorem we get

$$\text{nullity}(L) = \dim \mathbb{V} - \text{rank}(L) = n - n = 0$$

Thus, $\ker(L) = \{\vec{0}\}$.
Similarly, if $\ker(L) = \{\vec{0}\}$, then $\text{nullity}(L) = 0$, so the Rank-Nullity Theorem gives

$$\dim(\text{Range}(L)) = \text{rank}(L) = \dim \mathbb{V} - \text{nullity}(L) = n - 0 = n$$

Thus, $\text{Range}(L) = \mathbb{W}$.

8.2.6 (a) Since the rank of a linear mapping is equal to the dimension of the range, we consider the range of both mappings. Let $\vec{x} \in \text{Range}(M \circ L)$. Then there exists $\vec{v} \in \mathbb{V}$ such that $\vec{x} = (M \circ L)(\vec{v}) = M(L(\vec{v})) \in \text{Range}(M)$.

Hence, $\text{Range}(M \circ L)$ is a subset, and hence a subspace, of $\text{Range}(M)$. Therefore, $\dim \text{Range}(M \circ L) \leq \dim \text{Range}(M)$ which implies $\text{rank}\, M \circ L \leq \text{rank}\, M$.

(b) The kernel of L is a subspace of the kernel of $M \circ L$, because if $L(\vec{x}) = \vec{0}$, then

$$(M \circ L)(\vec{x}) = M(L(\vec{x})) = M(\vec{0}) = \vec{0}$$

Therefore

$$\text{nullity}(L) \leq \text{nullity}(M \circ L)$$

so

$$n - \text{nullity}(L) \geq n - \text{nullity}(M \circ L)$$

Hence $\text{rank}(L) \geq \text{rank}(M \circ L)$, by the Rank-Nullity Theorem.

8.3 Problem Solutions

8.3.1 (a) We have

$$\begin{aligned} D(1) &= 0 = 0(1) + 0(x) \\ D(x) &= 1 = 1(1) + 0(x) \\ D(x^2) &= 2x = 0(1) + 2(x) \end{aligned}$$

Thus ${}_{\mathcal{C}}[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) We have

$$\begin{aligned} L(1,-1) &= x^2 = 0(1+x^2) + 1(1+x) + 1(-1-x+x^2) \\ L(1,2) &= 3 + x^2 = 3(1+x^2) - 2(1+x) - 2(-1-x+x^2) \end{aligned}$$

Thus ${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 3 \\ 1 & -2 \\ 1 & -2 \end{bmatrix}$.

(c) We have

$$\begin{aligned} T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ T\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

Hence, ${}_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 3/2 & 1/2 & 1 \\ 0 & 1/2 & -1/2 & 0 \end{bmatrix}$.

(d) We have

$$\begin{aligned} L(1+x^2) &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ L(x-x^2) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ L(x^2) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ L(x^3) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

Hence, ${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.

(e) We have

$$\begin{aligned} L(1) = 1 &= 1(1) + 0(x+1) + 0(x+1)^2 \\ L(x-1) = x-1 &= (-2)(1) + 1(x+1) + 0(x+1)^2 \\ L(x^2-2x+1) = x^2-2x+1 &= 4(1) - 4(x+1) + 1(x+1)^2 \end{aligned}$$

Hence, ${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

(f) We have

$$\begin{aligned} L(1,-1,0) &= 1 - x = 1(1-x) + 0(x) \\ L(1,0,-1) &= 0 + x = 0(1-x) + 1(x) \\ L(0,1,1) &= 1 + 2x = 1(1-x) + 3(x) \end{aligned}$$

Hence, ${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

8.3.2 (a) We have

$$\begin{aligned} L(1) &= 0 = 0 + 0x + 0x^2 \\ L(x) &= 1 = 1 + 0x + 0x^2 \\ L(x^2) &= 2x = 0 + 2x + 0x^2 \end{aligned}$$

Hence, $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) For ease, name the vectors in $\mathcal{B}$ as $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

$$\begin{aligned}
L\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 \\
L\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4 \\
L\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 0\vec{v}_1 + 0\vec{v}_2 + 1\vec{v}_3 + 0\vec{v}_4 \\
L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0\vec{v}_1 + -1\vec{v}_2 + 1\vec{v}_3 + 0\vec{v}_4
\end{aligned}$$

Thus, $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(c) We have

$$\begin{aligned}
T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\
T\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) &= \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}
\end{aligned}$$

Thus, $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

(d) We have

$$\begin{aligned}
L(1 + x^2) &= 1 + x^2 = 1(1 + x^2) + 0(-1 + x) + 0(1 - x + x^2) \\
L(-1 + x) &= -1 + x^2 = (-1)(1 + x^2) + 2(-1 + x) + 2(1 - x + x^2) \\
L(1 - x + x^2) &= 1 = (1)(1 + x^2) + (-1)(-1 + x) + (-1)(1 - x + x^2)
\end{aligned}$$

Thus, $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$.

8.3.3 Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$. Then for $1 \leq i \leq n$ we have

$$L(\vec{v}_i) = \vec{v}_i = 0\vec{v}_1 + \cdots + 0\vec{v}_{i-1} + \vec{v}_i + 0\vec{v}_{i+1} + \cdots + 0\vec{v}_n$$

Thus,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & \cdots & [L(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix} = I$$

8.3.4 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{x}) = A\vec{x}$. Clearly we have $\operatorname{Range}(L) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. Let $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. Then, we get

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Clearly, $\operatorname{Col}([L]_{\mathcal{B}}) \neq \operatorname{Range}(L)$.

8.4 Problem Solutions

8.4.1 (a) We define $L : P_1(\mathbb{R}) \to \mathbb{R}^2$ by $L(a_0 + a_1x) = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$.

Linear: Let any two elements of $P_1(\mathbb{R})$ be $\vec{a} = a_0 + a_1x$ and $\vec{b} = b_0 + b_1x$ and let $s, t \in \mathbb{R}$ then

$$\begin{aligned} L(s\vec{a} + t\vec{b}) &= L(s(a_0 + a_1x) + t(b_0 + b_1x)) \\ &= L((sa_0 + tb_0) + (sa_1 + tb_1)x) \\ &= \begin{bmatrix} sa_0 + tb_0 \\ sa_1 + tb_1 \end{bmatrix} = s\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + t\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = sL(\vec{a}) + tL(\vec{b}) \end{aligned}$$

Therefore, L is linear.

One-to-one: If $a_0 + a_1x \in \ker(L)$, then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(a_0 + a_1x) = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Hence, $a_0 = a_1 = 0$ and so $\ker(L) = \{\vec{0}\}$. Consequently, L is one-to-one.

Onto: For any $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \in \mathbb{R}^2$ we have $L(a_0 + a_1x) = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$. Thus, L is onto.

Thus, L is an isomorphism from $P_1(\mathbb{R})$ to $\mathbb{R}^2$.

(b) We define $L : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $L(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$.

Linear: Let any two elements of $P_3(\mathbb{R})$ be $\vec{a} = a_0 + a_1x + a_2x^2 + a_3x^3$ and $\vec{b} = b_0 + b_1x + b_2x^2 + b_3x^3$ and let $s, t \in \mathbb{R}$ then

$$\begin{aligned} L(s\vec{a} + t\vec{b}) &= L(s(a_0 + a_1x + a_2x^2 + a_3x^3) + t(b_0 + b_1x + b_2x^2 + b_3x^3)) \\ &= L((sa_0 + tb_0) + (sa_1 + tb_1)x + (sa_2 + tb_2)x^2 + (sa_3 + tb_3)x^3) \\ &= \begin{bmatrix} sa_0 + tb_0 & sa_1 + tb_1 \\ sa_2 + tb_2 & sa_3 + tb_3 \end{bmatrix} = s\begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix} + t\begin{bmatrix} b_0 & b_1 \\ b_2 & b_3 \end{bmatrix} = sL(\vec{a}) + tL(\vec{b}) \end{aligned}$$

Therefore, L is linear.

<u>One-to-one:</u> If $a_0 + a_1x + a_2x^2 + a_3x^3 \in \ker(L)$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = L(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$$

Hence, $a_0 = a_1 = a_2 = a_3 = 0$ and so $\ker(L) = \{\vec{0}\}$. Therefore, L is one-to-one.

<u>Onto:</u> For any $\begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ we have $L(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$. Hence, L is onto.

Thus, L is an isomorphism from P_3 to $M_{2\times 2}(\mathbb{R})$.

(c) We know that a general vector in $\mathbb{P}$ has the form $(x-1)(a_2x^2+a_1x+a_0)$. Thus, we define $L : \mathbb{P} \to \mathbb{U}$ by $L((x-1)(a_2x^2 + a_1x + a_0)) = \begin{bmatrix} a_2 & a_1 \\ 0 & a_0 \end{bmatrix}$.

<u>Linear:</u> Let any two elements of $\mathbb{P}$ be $\vec{a} = (x-1)(a_2x^2 + a_1x + a_0)$ and $\vec{b} = (x-1)(b_2x^2 + b_1x + b_0)$ and let $s, t \in \mathbb{R}$ then

$$\begin{aligned} L(s\vec{a} + t\vec{b}) &= L((x-1)(sa_2 + tb_2)x^2 + (sa_1 + tb_1)x + (sa_0 + tb_0)) \\ &= \begin{bmatrix} sa_2 + tb_2 & sa_1 + tb_1 \\ 0 & sa_0 + tb_0 \end{bmatrix} \\ &= s\begin{bmatrix} a_2 & a_1 \\ 0 & a_0 \end{bmatrix} + t\begin{bmatrix} b_2 & b_1 \\ 0 & b_0 \end{bmatrix} = sL(\vec{a}) + t(\vec{b}) \end{aligned}$$

Therefore L is linear.

<u>One-to-one:</u> Assume $\vec{a} \in \ker(L)$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = L((x-1)(a_2x^2 + a_1x + a_0)) = \begin{bmatrix} a_2 & a_1 \\ 0 & a_0 \end{bmatrix}$$

This gives $a_2 = a_1a_0 = 0$. Hence $\ker(L) = \{\vec{0}\}$ so L is one-to-one.

<u>Onto:</u> Since $\dim \mathbb{P} = \dim \mathbb{U}$ and L is one-to-one, by Theorem 8.4.5 L is also onto.

Thus, L is an isomorphism from $\mathbb{P}$ to $\mathbb{U}$.

(d) Observe that every vector in $\mathbb{S}$ has the form $x_2\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and that a every vector in $\mathbb{U}$ has the form $a(1+x)+cx^2$. In particular, $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for $\mathbb{S}$ and $\{1+x, x^2\}$ is a basis for $\mathbb{U}$. Thus, we define $L : \mathbb{S} \to \mathbb{U}$ by

$$L\left(a\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right) = a(1+x) + bx^2$$

<u>Linear:</u> Let any two elements of $\mathbb{S}$ be $\vec{a} = a_1\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + b_1\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ and $\vec{b} = a_2\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + b_2\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ and let $s, t \in \mathbb{R}$ then

$$\begin{aligned}
L(s\vec{a} + t\vec{b}) &= L\left((sa_1 + ta_2)\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + (sb_1 + tb_2)\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right)\\
&= (sa_1 + ta_2)(1+x) + (sb_1 + tb_2)x^2\\
&= sa_1(1+x) + ta_2(1+x) + sb_1x^2 + tb_2x^2\\
&= s[a_1(1+x) + b_1x^2] + t[a_2(1+x) + b_2x^2] = sL(\vec{a}) + tL(\vec{b})
\end{aligned}$$

Therefore L is linear.

<u>One-to-one:</u> Assume $L(\vec{a}) = L(\vec{b})$. Then $a_1(1+x) + b_1x^2 = a_2(1+x) + b_2x^2$. This gives $a_1 = b_1$ and $a_2 = b_2$ hence $\vec{a} = \vec{b}$ so L is one-to-one.

<u>Onto:</u> For any $a(1+x) + bx^2 \in \mathbb{U}$ we can pick $\vec{a} = a\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \in \mathbb{S}$ so that we have $L(\vec{a}) = a(1+x) + bx^2$ hence L is onto.

Thus, L is an isomorphism from $\mathbb{S}$ to $\mathbb{U}$.

(e) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{V}$. Define $L : \mathbb{V} \to \mathbb{R}^n$ by

$$L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) = b_1\vec{e}_1 + \cdots + b_n\vec{e}_n$$

<u>Linear:</u> Let $b_1\vec{v}_1 + \cdots + b_n\vec{v}_n, c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in \mathbb{V}$ and let $s, t \in \mathbb{R}$ then

$$\begin{aligned}
L[s(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) + t(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)] &= L[(sb_1 + tc_1)\vec{v}_1 + \cdots + (sb_n + tc_n)\vec{v}_n]\\
&= (sb_1 + tc_1)\vec{e}_1 + \cdots + (sb_n + tc_n)\vec{e}_n\\
&= s(b_1\vec{e}_1 + \cdots + b_n\vec{e}_n) + t(c_1\vec{e}_1 + \cdots + c_n\vec{e}_n)\\
&= sL(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) + tL(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)
\end{aligned}$$

Therefore, L is linear.

<u>One-to-one:</u> If $L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) = L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)$, then

$$b_1\vec{e}_1 + \cdots + b_n\vec{e}_n = c_1\vec{e}_1 + \cdots + c_n\vec{e}_n$$

Hence, $b_i = c_i$ for $1 \le i \le n$ and so L is one-to-one.

<u>Onto:</u> For any $\vec{x} = x_1\vec{e}_1 + + \cdots + x_n\vec{e}_n \in \mathbb{R}^n$ we have $L(x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) = x_1\vec{e}_1 + + \cdots + x_n\vec{e}_n = \vec{x}$. So, L is onto.

Thus, L is an isomorphism from $\mathbb{V}$ to $\mathbb{R}^n$.

(f) Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{V}$. Define $L : \mathbb{V} \to P_{n-1}(\mathbb{R})$ by

$$L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) = b_1 + b_2x + b_3x^2 + \cdots + b_nx^{n-1}$$

<u>Linear:</u> Let $b_1\vec{v}_1 + \cdots + b_n\vec{v}_n, c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in \mathbb{V}$ and let $s, t \in \mathbb{R}$ then

$$\begin{aligned} L[s(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) + t(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)] &= L[(sb_1 + tc_1)\vec{v}_1 + \cdots + (sb_n + tc_n)\vec{v}_n] \\ &= (sb_1 + tc_1) + (sb_2 + tc_2)x + \cdots + (sb_n + tc_n)x^{n-1} \\ &= s(b_1 + b_2x + \cdots + b_nx^{n-1}) + t(c_1 + c_2x + \cdots + c_nx^{n-1}) \\ &= sL(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) + tL(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \end{aligned}$$

Therefore, L is linear.

<u>One-to-one:</u> If $L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) = L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)$, then

$$b_1 + b_2x + b_3x^2 + \cdots + b_nx^{n-1} = c_1 + c_2x + c_3x^2 + \cdots + c_nx^{n-1}$$

Hence, $b_i = c_i$ for $1 \le i \le n$ and so L is one-to-one.

<u>Onto:</u> For any $b_1 + b_2x + b_3x^2 + \cdots + b_nx^{n-1} \in P_{n-1}(\mathbb{R})$ we have $L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) = b_1 + b_2x + b_3x^2 + \cdots + b_nx^{n-1}$. So, L is onto.

Thus, L is an isomorphism from $\mathbb{V}$ to $P_{n-1}(\mathbb{R})$.

8.4.2 If L is one-to-one, then $\ker(L) = \{\vec{0}\}$ by Lemma 8.4.1. Thus, $\operatorname{nullity}(L) = 0$ and so by the Rank-Nullity Theorem we get

$$\dim(\operatorname{Range}(L)) = \operatorname{rank}(L) = \dim\mathbb{V} - \operatorname{nullity}(L) = n - 0 = n$$

Since $\dim(\operatorname{Range}(L)) = \dim\mathbb{W}$, we have $\operatorname{Range}(L) = \mathbb{W}$, so L is onto.
Similarly, if L is onto, then $\operatorname{rank}(L) = \dim(\operatorname{Range}(L)) = \dim\mathbb{W} = n$. Hence, by the Rank-Nullity Theorem,

$$\operatorname{nullity}(L) = \dim\mathbb{V} - \operatorname{rank}(L) = n - n = 0$$

Thus, L is one-to-one by Lemma 8.4.1.

8.4.3 Consider $c_1L(\vec{v}_1) + \cdots + c_nL(\vec{v}_n) = \vec{0}$. Then

$$L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \vec{0}$$

Hence, $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in \ker(L)$. Since L is an isomorphism, it is one-to-one and thus $\ker(L) = \{\vec{0}\}$ by Lemma 8.4.1. Thus, we get $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$. This implies that $c_1 = \cdots = c_n = 0$ since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent. Consequently, $\{L(\vec{v}_1), \dots, L(\vec{v}_n)\}$ is a linearly independent set of n vectors in an n-dimensional vector space, and hence is a basis.

8.4.4 (a) Let $\vec{w} \in \mathbb{W}$. Since M is onto, there exist a $\vec{u} \in \mathbb{U}$ such that $M(\vec{u}) = \vec{w}$. Then, since L is onto, there exists a $\vec{v} \in \mathbb{V}$ such that $L(\vec{v}) = \vec{u}$. Hence,

$$(M \circ L)(\vec{v}) = M(L(\vec{v})) = M(\vec{u}) = \vec{w}$$

as required.

(b) Let $L : \mathbb{R} \to \mathbb{R}^2$ be defined by $L(x_1) = (x_1, 0)$. Clearly, L is not onto. Now, define $M : \mathbb{R}^2 \to \mathbb{R}$ by $M(x_1, x_2) = x_1$. Then, we get

$$(M \circ L)(x_1) = M(L(x_1)) = M(x_1, 0) = x_1$$

Hence, $\text{Range}(M \circ L) = \mathbb{R}$ so $M \circ L$ is onto.

(c) It is not possible. Observe that if $M \circ L$ is onto, then for any $\vec{w} \in \mathbb{W}$, there exists a $\vec{v} \in \mathbb{V}$ such that

$$\vec{w} = (M \circ L)(\vec{v}) = M(L(\vec{v}))$$

Hence, $\vec{w} \in \text{Range}(M)$, and so M is onto.

8.4.5 (a) By definition $[L(\vec{x})]_{\mathcal{C}} = A[\vec{x}]_{\mathcal{B}}$. Hence $[L(\vec{x})]_{\mathcal{C}} \in \text{Col}(A)$. Thus, we define $F : \text{Range}(L) \to \text{Col}(A)$ by $F(\vec{w}) = [\vec{w}]_{\mathcal{C}}$.

We know that taking coordinates is a linear operation, so F is linear.

F is one-to-one since $F(\vec{w}) = \vec{0}$ gives $[\vec{w}]_{\mathcal{C}} = \vec{0}$, so $\vec{w} = \vec{0}$.

F is onto: Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ and let $\vec{y} \in \text{Col}(A)$. Then, $\vec{y} = A\vec{x}$ for some $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Let $\vec{v} = x_1\vec{v}_1 + \cdots x_n\vec{v}_n$ so that $[\vec{v}]_{\mathcal{B}} = \vec{x}$. Then,

$$F(L(\vec{v})) = [L(\vec{v})]_{\mathcal{C}} = A[\vec{v}]_{\mathcal{B}} = A\vec{x} = \vec{y}$$

(b) Since $\text{Range}(L)$ is isomorphic to $\text{Col}(A)$, they have the same dimension. Hence,

$$\text{rank}(L) = \dim(\text{Range}(L)) = \dim(\text{Col}(A)) = \text{rank}(A)$$

8.4.6 Since L and M are one-to-one, we have $\text{nullity}(L) = 0 = \text{nullity}(M)$ by Lemma 8.4.1. Since the range of a linear mapping is a subspace of its codomain, we have that $\dim \mathbb{V} \geq \dim \text{Range}(L)$ and $\dim \mathbb{U} \geq \dim \text{Range}(M)$. Thus, by the Rank-Nullity Theorem we get

$$\dim \mathbb{V} \geq \dim \text{Range}(L) = \text{rank}(L) = \dim \mathbb{U}$$

and

$$\dim \mathbb{U} \geq \dim \text{Range}(M) = \text{rank}(M) = \dim \mathbb{V}$$

Hence, $\dim \mathbb{U} = \dim \mathbb{V}$ and so $\mathbb{U}$ and $\mathbb{V}$ are isomorphic by Theorem 8.4.2.

8.4.7 (a) We disprove the statement with a counter example. Let $\mathbb{U} = \mathbb{R}^2$ and $\mathbb{V} = \mathbb{R}^2$ and $L : \mathbb{U} \to \mathbb{V}$ be the linear mapping defined by $L(\vec{x}) = \vec{0}$. Then, clearly $\dim \mathbb{U} = \dim \mathbb{V}$, but L is not an isomorphism since it isn't one-to-one nor onto.

(b) By definition, we have that $\dim(\text{Range}(L)) = \text{rank}(L)$. Thus, the Rank-Nullity Theorem gives us that

$$\dim(\text{Range}(L)) = \dim \mathbb{V} - \text{nullity}(L)$$

If $\dim \mathbb{V} < \dim \mathbb{W}$, then $\dim V - \text{nullity}(L) < \dim \mathbb{W}$ and hence the range of L cannot be equal to $\mathbb{W}$. Thus, we have proven the statement is true.

(c) We disprove the statement with a counter example. Let $\mathbb{V} = \mathbb{R}^3$ and $\mathbb{W} = \mathbb{R}^2$ and $L : \mathbb{V} \to \mathbb{W}$ be the linear mapping defined by $L(\vec{x}) = \vec{0}$. Then, clearly $\dim \mathbb{V} > \dim \mathbb{W}$, but L is not onto.

(d) The Rank-Nullity Theorem tells us that

$$\text{nullity}(L) = \dim \mathbb{V} - \text{rank}(L) \geq \dim \mathbb{V} - \dim \mathbb{W} > 0$$

since $\text{rank}(L) \leq \dim \mathbb{W}$. Thus, $\text{nullity}(L) \neq \{\vec{0}\}$ and hence L is not one-to-one by Lemma 8.4.1. Therefore, we have proven the statement is true.

Chapter 9 Solutions

9.1 Problem Solutions

9.1.1 (a) $\left\langle \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \right\rangle = 1(1) + 2(2) + 0(0) + (-1)(-1) = 6$

(b) $\left\langle \begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix} \right\rangle = 3(4) + (-4)(1) + 2(2) + 5(-1) = 7$

(c) $\left\langle \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}, \begin{bmatrix} -5 & 3 \\ 1 & 5 \end{bmatrix} \right\rangle = 2(-5) + 3(3) + (-4)(1) + 1(5) = 0$

(d) $\left\langle \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 7 \\ -4 & 3 \end{bmatrix} \right\rangle = 0(3) + 0(7) + 0(-4) + 0(3) = 0$

9.1.2 (a) $\langle x, 1 + x + x^2 \rangle = 0(1) + 1(3) + 2(7) = 17$

(b) $\langle 1 + x^2, 1 + x^2 \rangle = 1(1) + 2(2) + 5(5) = 30$

(c) $\langle x + x^2, 1 + x + x^2 \rangle = 0(1) + 2(3) + 6(7) = 48$

(d) $\langle x, x \rangle = 0(0) + 1(1) + 2(2) = 5$

9.1.3 (a) For any $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ and $s, t \in \mathbb{R}$, we have

$$\langle \vec{x}, \vec{x} \rangle = 2x_1^2 + 3x_2^2 + 4x_3^2 \geq 0$$

and $0 = \langle \vec{x}, \vec{x} \rangle = 2x_1^2 + 3x_2^2 + 4x_3^2$ if and only if $x_1 = x_2 = x_3 = 0$. Thus, $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$. So, it is positive definite.

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 3x_2y_2 + 4x_3y_3 = 2y_1x_1 + 3y_2x_2 + 4y_3x_3 = \langle \vec{y}, \vec{x} \rangle$$

Thus, it is symmetric.

$$\begin{aligned} \langle s\vec{x} + t\vec{y}, \vec{z} \rangle &= 2(sx_1 + ty_1)z_1 + 3(sx_2 + ty_2)z_2 + 4(sx_3 + ty_3)z_3 \\ &= s(2x_1z_1 + 3x_2z_2 + 4x_3z_3) + t(2y_1z_1 + 3y_2z_2 + 4y_3z_3) \qquad = s\langle \vec{x}, \vec{x} \rangle + t\langle \vec{y}, \vec{z} \rangle \end{aligned}$$

Thus, it is also bilinear. Hence, it is an inner product.

(b) For any $\vec{x},\vec{y},\vec{z} \in \mathbb{R}^3$ and $s,t \in \mathbb{R}$, we have

$$\begin{aligned}\langle \vec{x},\vec{x}\rangle &= 2x_1^2 - x_1x_2 - x_2x_1 + 2x_2^2 + x_3^2 = 2x_1^2 - 2x_1x_2 + 2x_2^2 + x_3^2\\ &= x_1^2 + x_1^2 - 2x_1x_2 + x_2^2 + x_2^2 + x_3^2\\ &= x_1^2 + (x_1 - x_2)^2 + x_2^2 + x_3^2\end{aligned}$$

Thus, $\langle \vec{x},\vec{x}\rangle \geq 0$ and $\langle \vec{x},\vec{x}\rangle = 0$ if and only if $\vec{x} = \vec{0}$. So, it is positive definite.

$$\langle \vec{x},\vec{y}\rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 + x_3y_3 = 2y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 + y_3x_3 = \langle \vec{y},\vec{x}\rangle$$

Thus, it is symmetric.

$$\begin{aligned}\langle s\vec{x}+t\vec{y},\vec{z}\rangle &= 2(sx_1+ty_1)z_1 - (sx_1+ty_1)z_2 - (sx_2+ty_2)z_1 + 2(sx_2+ty_2)z_2 + (sx_3+ty_3)z_3\\ &= s(2x_1z_1 - x_1z_2 - x_2z_1 + 2x_2z_2 + x_3z_3) + t(2y_1z_1 - y_1z_2 - y_2z_1 + 2y_2z_2 + y_3z_3)\\ &= s\langle \vec{x},\vec{x}\rangle + t\langle \vec{y},\vec{z}\rangle\end{aligned}$$

Thus, it is also bilinear. Hence, it is an inner product.

9.1.4 (a) Observe that if $\langle 1-x^2, 1-x^2\rangle = 0(0)+0(0) = 0$, but $1-x^2$ is not the zero vector in $P_2(\mathbb{R})$, so the function is not positive definite and hence not an inner product.

(b) Let $p,q,r \in P_2(\mathbb{R})$ and $s,t \in \mathbb{R}$. We have

$$\langle p,p\rangle = 2[p(-2)]^2 + [p(1)]^2 + [p(2)]^2 \geq 0$$

and $\langle p,p\rangle = 0$ if and only if $p(-2) = p(1) = p(2) = 0$. Since p is a polynomial of degree at most 2 with three roots we get that $p = 0$. Hence, $\langle p,p\rangle = 0$ if and only if $p = 0$.

$$\begin{aligned}\langle p,q\rangle &= p(-2)q(-2) + p(1)q(1) + p(2)q(2)\\ &= q(-2)p(-2) + q(1)p(1) + q(2)p(2) = \langle q,p,\rangle\\ \langle sp+tq,r\rangle &= (sp+tq)(-2)[r(-2)] + (sp+tq)(1)[r(1)] + (sp+tq)(2)[r(2)]\\ &= [sp(-2)+tq(-2)]r(-2) + [sp(1)+tq(1)]r(1) + [sp(2)+tq(2)]r(2)\\ &= s[p(-2)r(-2) + p(1)r(1) + p(2)r(2)] + t[q(-2)r(-2) + q(1)r(1) + q(2)r(2)] \quad = s\langle p,r\rangle + t\langle q,r\rangle\end{aligned}$$

Thus, it is an inner product.

(c) Let $p,q,r \in P_2(\mathbb{R})$ and $a,b \in \mathbb{R}$. We have

$$\begin{aligned}\langle p,p\rangle &= 2[p(-1)]^2 + 2[p(0)]^2 + 2[p(1)]^2 - 2p(-1)p(1)\\ &= [p(-1)]^2 + 2[p(0)]^2 + [p(1)]^2 + [p(-1)]^2 - 2p(-1)p(1) + [p(1)]^2\\ &= [p(-1)]^2 + 2[p(0)]^2 + [p(1)]^2 + [p(-1)-p(1)]^2 \geq 0\end{aligned}$$

and $\langle p, p\rangle = 0$ if and only if $p(-1) = p(0) = p(1) = 0$. Thus, $\langle p, p\rangle = 0$ if and only if $p = 0$.

$$\begin{aligned}
\langle p, q\rangle =& 2p(-1)q(-1) + 2p(0)q(0) + 2p(1)q(1) - p(-1)q(1) - p(1)q(-1)\\
=& 2q(-1)p(-1) + 2q(0)p(0) + 2q(1)p(1) - q(-1)p(1) - q(1)p(-1) = \langle q, p, \rangle\\
\langle ap + bq, r\rangle =& 2[ap(-1) + bq(-1)]r(-1) + 2[ap(0) + bq(0)]r(0) + 2[ap(1) + bq(1)]r(1)\\
&- [ap(-1) + bq(-1)]r(1) - [ap(1) + bq(1)]r(-1)\\
=& a[2p(-1)r(-1) + 2p(0)r(0) + 2p(1)r(1) - p(-1)r(1) - p(1)r(-1)]\\
&+ b[2q(-1)r(-1) + 2q(0)r(0) + 2q(1)r(1) - q(-1)r(1) - q(1)r(-1)]\\
&= a\langle p, r\rangle + b\langle q, r\rangle
\end{aligned}$$

Thus, it is an inner product.

9.1.5 (a) Observe that

$$\left\langle \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}\right\rangle = 2(0)^2 + 0^2 + 2(0)^2 = 0$$

but, $\begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix} \neq \vec{0}$. Hence, the function is not positive definite and hence not an inner product.

(b) Observe that

$$\left\langle \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}\right\rangle = 0^2 + 0^2 - 0^2 - 1^2 = -1$$

Hence, the function is not positive definite and hence not an inner product.

9.1.6 We have

$$\langle \vec{v}, \vec{0}\rangle = \langle \vec{v}, 0\vec{v}\rangle = 0\langle \vec{v}, \vec{v}\rangle = 0$$

as required.

9.1.7 We have

$$\langle A\vec{x}, \vec{y}\rangle = (A\vec{x})\cdot\vec{y} = (A\vec{x})^T\vec{y} = \vec{x}^T A^T\vec{y} = \vec{x}\cdot(A^T\vec{y}) = \langle \vec{x}, A^T\vec{y}\rangle$$

9.1.8 We have

$$\begin{aligned}
\langle \vec{x}, \vec{y}\rangle &= \langle x_1\vec{e}_1 + x_2\vec{e}_2, y_1\vec{e}_1 + y_2\vec{e}_2\rangle\\
&= \langle x_1\vec{e}_1, y_1\vec{e}_1 + y_2\vec{e}_2\rangle + \langle x_2\vec{e}_2, y_1\vec{e}_1 + y_2\vec{e}_2\rangle\\
&= \langle x_1\vec{e}_1, y_1\vec{e}_1\rangle + \langle x_1\vec{e}_1, y_2\vec{e}_2\rangle + \langle x_2\vec{e}_2, y_1\vec{e}_1\rangle + \langle x_2\vec{e}_2, y_2\vec{e}_2\rangle\\
&= x_1y_1\langle \vec{e}_1, \vec{e}_1\rangle + x_1y_2\langle \vec{e}_1, \vec{e}_2\rangle + x_2y_1\langle \vec{e}_2, \vec{e}_1\rangle + x_2y_2\langle \vec{e}_2, \vec{e}_2\rangle
\end{aligned}$$

as required.

9.1.9 Let $f, g, h \in C[-\pi, \pi]$ and $a, b \in \mathbb{R}$. We have

$$\langle f, f\rangle = \int_{-\pi}^{\pi} (f(x))^2\, dx \geq 0$$

since $(f(x))^2 \geq 0$. Moreover, we have $\langle f, f\rangle = 0$ if and only if $f(x) = 0$ for all $x \in [-\pi, \pi]$.

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx = \int_{-\pi}^{\pi} g(x)f(x)\,dx = \langle g, f, \rangle$$

$$\begin{aligned}\langle af + bg, b\rangle &= \int_{-\pi}^{\pi} (2f(x) + bg(x))h(x)\,dx \\ &= a\int_{-\pi}^{\pi} f(x)h(x)\,dx + b\int_{-\pi}^{\pi} g(x)h(x)\,dx \qquad = a\langle f, h\rangle + b\langle g, h\rangle\end{aligned}$$

Thus, it is an inner product.

9.1.10 (a) The statement is false. Consider the standard inner product on $\mathbb{R}^2$ and take $\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}-1\\0\end{bmatrix}$. Then, we get $\vec{x}\cdot\vec{y} = -1$.

(b) The statement is false. We have

$$\begin{aligned}\langle a\vec{x} + b\vec{y}, c\vec{x} + d\vec{y}\rangle &= \langle a\vec{x} + b\vec{y}, c\vec{x}\rangle + \langle a\vec{x} + b\vec{y}, d\vec{y}\rangle \\ &= \langle a\vec{x}, c\vec{x}\rangle + \langle b\vec{y}, c\vec{x}\rangle + \langle a\vec{x}, d\vec{y}\rangle + \langle b\vec{y}, d\vec{y}\rangle \\ &= ab\langle\vec{x}, \vec{x}\rangle + bc\langle\vec{y}, \vec{x}\rangle + ad\langle\vec{x}, \vec{y}\rangle + bd\langle\vec{y}, \vec{y}\rangle \\ &= ab\langle\vec{x}, \vec{x}\rangle + (bc + ad)\langle\vec{x}, \vec{y}\rangle + bd\langle\vec{y}, \vec{y}\rangle\end{aligned}$$

9.2 Problem Solutions

9.2.1 (a) We have

$$\begin{aligned}\langle 1 + x, -2 + 3x\rangle &= 0(-5) + 1(-2) + 2(1) = 0 \\ \langle 1 + x, 2 - 3x^2\rangle &= 0(-1) + 1(2) + 2(-1) = 0 \\ \langle -2 + 3x, 2 - 3x^2\rangle &= (-5)(-1) + (-2)(2) + 1(-1) = 0\end{aligned}$$

Therefore, it is an orthogonal set. Since it does not contain the zero vector, we get that it is a linearly independent by Theorem 9.2.3. Thus, it is a linearly independent set of 3 vectors in $P_2(\mathbb{R})$ and hence it is an orthogonal basis for $P_2(\mathbb{R})$.

(b) We have

$$\begin{aligned}\frac{\langle 1 + x, 1\rangle}{\|1 + x\|^2} &= \frac{3}{5} \\ \frac{\langle -2 + 3x, 1\rangle}{\| -2 + 3x\|^2} &= \frac{-6}{30} = -\frac{1}{5} \\ \frac{\langle 2 - 3x^2, 1\rangle}{\|2 - 3x^2\|^2} &= \frac{0}{6} = 0\end{aligned}$$

Hence, $[1]_{\mathcal{B}} = \begin{bmatrix}3/5\\-1/5\\0\end{bmatrix}$.

(c) We have

$$\begin{aligned}
\frac{\langle 1+x, x\rangle}{\|1+x\|^2} &= \frac{2}{5} \\
\frac{\langle -2+3x, x\rangle}{\|-2+3x\|^2} &= \frac{6}{30} = \frac{1}{5} \\
\frac{\langle 2-3x^2, x\rangle}{\|2-3x^2\|^2} &= \frac{0}{6} = 0
\end{aligned}$$

Hence, $[x]_{\mathcal{B}} = \begin{bmatrix} 2/5 \\ 1/5 \\ 0 \end{bmatrix}$.

(d) We have

$$\begin{aligned}
\frac{\langle 1+x, x^2\rangle}{\|1+x\|^2} &= \frac{2}{5} \\
\frac{\langle -2+3x, x^2\rangle}{\|-2+3x\|^2} &= \frac{-4}{30} = -\frac{2}{15} \\
\frac{\langle 2-3x^2, x^2\rangle}{\|2-3x^2\|^2} &= \frac{-2}{6} = -\frac{1}{3}
\end{aligned}$$

Hence, $[x^2]_{\mathcal{B}} = \begin{bmatrix} 2/5 \\ -2/15 \\ -1/3 \end{bmatrix}$.

(e) i. Consider

$$3 - 7x_x^2 = c_1(1+x) + c_2(-2+3x) + c_3(2-3x^2) = (c_1 - 2c_2 + 2c_3) + (c_1 - 3c_2)x + (-3c_3)x^2$$

Row reducing the corresponding augmemanted matrix gives

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 1 & -3 & 0 & -7 \\ 0 & 0 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/5 \\ 0 & 1 & 0 & -32/15 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

ii. We have

$$\begin{aligned}
[3 - 7x + x^2]_{\mathcal{B}} &= 3[1]_{\mathcal{B}} - 7[x]_{\mathcal{B}} + 1[x_2]_{\mathcal{B}} \\
&= 3\begin{bmatrix} 3/5 \\ -1/5 \\ 0 \end{bmatrix} - 7\begin{bmatrix} 2/5 \\ 1/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/5 \\ -2/15 \\ -1/3 \end{bmatrix} \\
&= \begin{bmatrix} -3/5 \\ -32/15 \\ -1/3 \end{bmatrix}
\end{aligned}$$

iii. We have

$$\frac{\langle 1+x, 3-7x+x^2\rangle}{\|1+x\|^2} = \frac{-3}{5}$$
$$\frac{\langle -2+3x, 3-7x+x^2\rangle}{\|-2+3x\|^2} = \frac{-64}{30} = -\frac{32}{15}$$
$$\frac{\langle 2-3x^2, 3-7x+x^2\rangle}{\|2-3x^2\|^2} = \frac{-2}{6} = -\frac{1}{3}$$

Hence, $[3-7x+x^2]_{\mathcal{B}} = \begin{bmatrix} -3/5 \\ -32/15 \\ -1/3 \end{bmatrix}$.

9.2.2 (a) We have

$$\left\langle \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\rangle = 1(2) + (-2)(2) + 2(1) = 0$$
$$\left\langle \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\rangle = 1(-2) + (-2)(1) + 2(2) = 0$$
$$\left\langle \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\rangle = 2(-2) + 2(1) + 1(2) = 0$$

Therefore, $\mathcal{B}$ is an orthogonal set in $\mathbb{R}^3$. Since it does not contain the zero vector, we get that it is a linearly independent by Theorem 9.2.3. Thus, it is a linearly independent set of 3 vectors in $\mathbb{R}^3$ and hence it is an orthogonal basis for $\mathbb{R}^3$.

(b) We have

$$\left\langle \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\rangle = 1^2 + (-2)^2 + 2^2 = 9$$
$$\left\langle \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\rangle = 2^2 + 2^2 + 1^2 = 9$$
$$\left\langle \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\rangle = (-2)^2 + 1^2 + 2^2 = 9$$

Thus, an orthonormal basis for $\mathbb{S}$ is

$$\mathcal{C} = \left\{ \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \right\}$$

(c) We have

$$\left\langle \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \right\rangle = 1(4) + (-2)(3) + 2(5) = 8$$

$$\left\langle \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \right\rangle = 2(4) + 2(3) + 1(5) = 19$$

$$\left\langle \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \right\rangle = (-2)(4) + 1(3) + 2(5) = 5$$

Thus, using Theorem 9.2.4 we get

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 8/9 \\ 19/9 \\ 5/9 \end{bmatrix}$$

(d) We have

$$\left\langle \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \right\rangle = \frac{1}{3}(4) + \left(-\frac{2}{3}\right)(3) + \frac{2}{3}(5) = 8/3$$

$$\left\langle \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \right\rangle = \frac{2}{3}(4) + \frac{2}{3}(3) + \frac{1}{3}(5) = 19/3$$

$$\left\langle \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \right\rangle = -\frac{2}{3}(4) + \frac{1}{3}(3) + \frac{2}{3}(5) = 5/3$$

Thus, using Corollary 9.2.5 we get

$$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 8/3 \\ 19/3 \\ 5/3 \end{bmatrix}$$

9.2.3 (a) We have

$$\left\langle \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\rangle = 1(3) + 2(2)(-1) + (-1)(-1) = 0$$

$$\left\langle \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\rangle = 1(3) + 2(2)(1) + (-1)(7) = 0$$

$$\left\langle \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\rangle = 3(3) + 2(-1)(1) + (-1)(7) = 0$$

Therefore, $\mathcal{B}$ is an orthogonal set in $\mathbb{R}^3$ under the given inner product. Since it does not contain the zero vector, we get that it is a linearly independent by Theorem 9.2.3. Thus, it is a linearly independent set of 3 vectors in $\mathbb{R}^3$ and hence it is an orthogonal basis for $\mathbb{R}^3$.

(b) We have

$$\left\langle \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix} \right\rangle = 1^2 + 2(2)^2 + (-1)^2 = 10$$

$$\left\langle \begin{bmatrix}3\\-1\\-1\end{bmatrix}, \begin{bmatrix}3\\-1\\-1\end{bmatrix} \right\rangle = 3^2 + 2(-1)^2 + (-1)^2 = 12$$

$$\left\langle \begin{bmatrix}3\\1\\7\end{bmatrix}, \begin{bmatrix}3\\1\\7\end{bmatrix} \right\rangle = 3^2 + 2(1)^2 + 7^2 = 60$$

Hence, an orthonormal basis C for $\mathbb{R}^3$ under this inner product is

$$C = \left\{ \frac{1}{\sqrt{10}}\begin{bmatrix}1\\2\\-1\end{bmatrix}, \frac{1}{\sqrt{12}}\begin{bmatrix}3\\-1\\-1\end{bmatrix}, \frac{1}{\sqrt{60}}\begin{bmatrix}3\\1\\7\end{bmatrix} \right\}$$

(c) We have

$$\left\langle \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle = 1(1) + 2(2)(1) + (-1)(1) = 4$$

$$\left\langle \left[\begin{bmatrix}3\\-1\\-1\end{bmatrix}\right], \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle 3(1) + 2(-1)(1) + (-1)(1) = 0$$

$$\left\langle \begin{bmatrix}3\\1\\7\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle = 3(1) + 2(1)(1) + 7(1) = 12$$

Therefore, by Theorem 9.2.4 we have $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}4/10\\0/12\\12/60\end{bmatrix} = \begin{bmatrix}2/5\\0\\1/5\end{bmatrix}$.

(d) We have

$$\left\langle \frac{1}{\sqrt{10}}\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle = \frac{4}{\sqrt{10}}$$

$$\left\langle \left[\tfrac{1}{\sqrt{12}}\begin{bmatrix}3\\-1\\-1\end{bmatrix}\right], \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle = 0$$

$$\left\langle \frac{1}{\sqrt{60}}\begin{bmatrix}3\\1\\7\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\rangle = \frac{12}{\sqrt{60}}$$

Therefore, by Corollary 9.2.5 we have $[\vec{x}]_C = \begin{bmatrix}4/\sqrt{10}\\0\\12/\sqrt{60}\end{bmatrix}$.

9.2.4 (a) We have

$$\left\langle \begin{bmatrix}1&1\\-1&-1\end{bmatrix}, \begin{bmatrix}1&-1\\1&-1\end{bmatrix}\right\rangle = 1(1)+1(-1)+(-1)(1)+(-1)(-1) = 0$$
$$\left\langle \begin{bmatrix}1&1\\-1&-1\end{bmatrix}, \begin{bmatrix}-1&1\\1&-1\end{bmatrix}\right\rangle = 1(-1)+1(1)+(-1)(1)+(-1)(-1) = 0$$
$$\left\langle \begin{bmatrix}1&-1\\1&-1\end{bmatrix}, \begin{bmatrix}-1&1\\1&-1\end{bmatrix}\right\rangle = 1(-1)+(-1)(1)+1(1)+(-1)(-1) = 0$$

Therefore, it is an orthogonal set. Since it does not contain the zero vector, we get that it is a linearly independent by Theorem 9.2.3. By definition, it is also a spanning set, and hence it is an orthogonal basis for $\mathbb{S}$.

(b) We have

$$\left\langle \begin{bmatrix}1&1\\-1&-1\end{bmatrix}, \begin{bmatrix}1&1\\-1&-1\end{bmatrix}\right\rangle = 1^2+1^2+(-1)^2+(-1)^2 = 4$$
$$\left\langle \begin{bmatrix}1&-1\\1&-1\end{bmatrix}, \begin{bmatrix}1&-1\\1&-1\end{bmatrix}\right\rangle = 1^2+(-1)^2+1^2+(-1)^2 = 4$$
$$\left\langle \begin{bmatrix}-1&1\\1&-1\end{bmatrix}, \begin{bmatrix}-1&1\\1&-1\end{bmatrix}\right\rangle = (-1)^2+1^2+1^2+(-1)^2 = 4$$

Hence, an orthonormal basis C for $\mathbb{S}$ is

$$C = \left\{\begin{bmatrix}1/2&1/2\\-1/2&-1/2\end{bmatrix}, \begin{bmatrix}1/2&-1/2\\1/2&-1/2\end{bmatrix}, \begin{bmatrix}-1/2&1/2\\1/2&-1/2\end{bmatrix}\right\}$$

(c) We have

$$\left\langle \begin{bmatrix}1/2&1/2\\-1/2&-1/2\end{bmatrix}, \begin{bmatrix}-3&6\\-2&-1\end{bmatrix}\right\rangle = 3$$
$$\left\langle \begin{bmatrix}1/2&-1/2\\1/2&-1/2\end{bmatrix}, \begin{bmatrix}-3&6\\-2&-1\end{bmatrix}\right\rangle = -5$$
$$\left\langle \begin{bmatrix}-1/2&1/2\\1/2&-1/2\end{bmatrix}, \begin{bmatrix}-3&6\\-2&-1\end{bmatrix}\right\rangle = 4$$

Therefore, $[\vec{x}]_C = \begin{bmatrix}3\\-5\\4\end{bmatrix}$.

9.2.5 (a) The matrix satisfies $AA^T = I$, so it is orthogonal.

(b) The matrix satisfies $AA^T = I$, so it is orthogonal.

(c) The first column is not a unit vector, so the matrix is not orthogonal.

(d) The matrix satisfies $AA^T = I$, so it is orthogonal.

(e) The first column is not a unit vector, so the matrix is not orthogonal.

(f) The columns are standard basis vectors and so form an orthonormal basis for $\mathbb{R}^3$. Hence, the matrix is orthogonal.

(g) The first column is not a unit vector, so the matrix is not orthogonal.

9.2.6 We first look for a polynomial $a + bx + cx^2 \in P_2(\mathbb{R})$ that is orthogonal to both $1 - x^2$ and $x - x^2$.

$$\langle a + bx + cx^2, 1 - x^2\rangle = (a - b + c)(0) + (a)(1) + (a + b + c)(0) = a$$
$$\langle a + bx + cx^2, x - x^2\rangle = (a - b + c)(-2) + (a)(0) + (a + b + c)(0) = -2a + 2b - 2c$$

This implies that $a = 0$ and $-2a + 2b - 2c = 0$, so any polynomial of the form $bx + cx^2$ is orthogonal to each polynomial in $\mathcal{B}$. We use $x + x^2$, and normalize it to $\frac{1}{2}(x + x^2)$. This polynomial extends $\mathcal{B}$ to an orthonormal basis of $P_2(\mathbb{R})$.

9.2.7 We have

$$\|t\vec{v}\| = \sqrt{\langle t\vec{v}, t\vec{v}\rangle} = \sqrt{t^2\langle \vec{v}, \vec{v}\rangle} = \sqrt{t^2}\sqrt{\langle \vec{v}, \vec{v}\rangle} = |t|\|\vec{v}\|$$

9.2.8 (a) Since $PP^T = I$ we get

$$1 = \det I = \det PP^T = \det P \det P^T = (\det P)^2$$

Thus, $\det P = \pm 1$.

(b) Let λ be a real eigenvalue of P with corresponding eigenvector $\vec{x}$. Then, using (a) we get

$$\|\vec{x}\| = \|P\vec{x}\| = \|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$$

Hence, we get $|\lambda| = 1$ and so $\lambda = \pm 1$ since λ is real.

(c) Take $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then P is clearly orthogonal and the eigenvalues of P are $\pm i$.

(d) If $P^T = P^{-1}$ and $Q^T = Q^{-1}$, then

$$(PQ)^T = Q^T P^T = Q^{-1}P^{-1} = (PQ)^{-1}$$

Thus, $PQ$is orthogonal.

9.2.9 The matrix whose columns are the vectors in $\mathcal{B}$ is orthogonal. Thus, the rows also form an orthonormal basis for $\mathbb{R}^3$ by Theorem 9.2.6. Thus, such an orthonormal basis is

$$\left\{ \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

9.2.10 We have

$$Q^TQ = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \cdots & \vec{v}_2 \cdot \vec{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \vec{v}_n \cdot \vec{v}_2 & \cdots & \vec{v}_n \cdot A\vec{v}_n \end{bmatrix}$$

But, $\{\vec{v}_1, \dots, \vec{v}_m\}$ is orthonormal, so $\vec{v}_i \cdot \vec{v}_i = 1$ and $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. Hence we have $Q^TQ = I$ as required.

9.2.11 (a) We have

$$\begin{aligned} < \vec{x}, \vec{y} >= &< c_1\vec{v}_1 + \cdots + c_n\vec{v}_n, d_1\vec{v}_1 + \cdots + d_n\vec{v}_n > \\ =&c_1 < \vec{v}_1, d_1\vec{v}_1 + \cdots + d_n\vec{v}_n > + \cdots + c_n < \vec{v}_n, d_1\vec{v}_1 + \cdots + d_n\vec{v}_n > \\ =&c_1d_1 < \vec{v}_1, \vec{v}_1 > + \cdots + c_1d_n < \vec{v}_1, \vec{v}_n > +c_2d_1 < \vec{v}_2, \vec{v}_1 > +c_2d_2 < \vec{v}_2, \vec{v}_2 > + \\ &c_2d_2 < \vec{v}_2, \vec{v}_2 > + \cdots + c_2d_n < \vec{v}_2, \vec{v}_n > + \cdots + c_nd_1 < \vec{v}_n, \vec{v}_1 > + \cdots + c_nd_n < \vec{v}_n, \vec{v}_n > \end{aligned}$$

But, since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is orthonormal we have $< \vec{v}_i, \vec{v}_i >= 1$ and $< \vec{v}_i, \vec{v}_j >= 0$ for $i \neq j$, hence we get

$$< \vec{x}, \vec{y} >= c_1d_1 + \cdots + c_nd_n$$

(b) Taking $\vec{y} = \vec{x}$, the result of part (a) gives $||\vec{x}|| =< \vec{x}, \vec{x} >= c_1^2 + \cdots + c_n^2$.

9.2.12 (a) We have $d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}|| \geq 0$ by Theorem 9.2.1(1).

(b) We have $0 = d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||$ But, by Theorem 9.2.1(1), we have that $||\vec{x} - \vec{y}|| = 0$ if and only if $\vec{x} - \vec{y} = \vec{0}$ as required.

(c) We have

$$d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}|| = \sqrt{\langle \vec{x} - \vec{y}, \vec{x} - \vec{y} \rangle} = \sqrt{\langle (-1)(\vec{y} - \vec{x}), (-1)(\vec{y} - \vec{x}) \rangle} = \sqrt{(-1)(-1)\langle \vec{y} - \vec{x}, \vec{y} - \vec{x} \rangle} = ||\vec{y} - \vec{x}|| = d(\vec{y}, \vec{x})$$

(d) Using Theorem 9.2.1(4) we get

$$d(\vec{x}, \vec{z}) = ||\vec{x} - \vec{z}|| = ||\vec{x} - \vec{y} + \vec{y} - \vec{z}|| \leq |\vec{x} - \vec{y}| + ||\vec{y} - \vec{z}|| = d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$$

(e) We have

$$d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}|| = ||\vec{x} + \vec{z} - \vec{z} - \vec{y}|| = ||\vec{x} + \vec{z} - (\vec{y} + \vec{z})|| = d(\vec{x} + \vec{z}, \vec{y} + \vec{z})$$

(f) Using Theorem 9.2.1(2) we get

$$d(c\vec{x}, c\vec{y}) = ||c\vec{x} - c\vec{y}|| = ||c(\vec{x} - \vec{y})|| = |c|\,||\vec{x} - \vec{y}|| = |c|\,d(\vec{x}, \vec{y})$$

9.3 Problem Solutions

9.3.1 Let the vectors in $\mathcal{B}$ be denoted $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ respectively. We first take $\vec{v}_1 = \vec{w}_1$. Next, we get

$$\vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 2 \\ 5/2 \end{bmatrix}$$

So, we take $\vec{v}_2 = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$. Finally, we get

$$\begin{aligned} \vec{v}_3 &= \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{w}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\ &= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{22}{66}\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 5/3 \\ -2/3 \end{bmatrix} \end{aligned}$$

Consequently, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

9.3.2 Let the vectors in $\mathcal{B}$ be denoted $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ respectively. We first take $\vec{v}_1 = \vec{w}_1$. Next, we get

$$\vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} - \frac{-10}{11}\begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -8/11 \\ -12/11 \\ 12/11 \end{bmatrix}$$

So, we take $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$. Finally, we get

$$\begin{aligned} \vec{v}_3 &= \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{w}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\ &= \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \frac{-2}{11}\begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} - \frac{16}{22}\begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Consequently, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

9.3.3 Take $p_1(x) = 1$. Then

$$\begin{aligned} p_2(x) &= x - \frac{\langle x, 1 \rangle}{\|1\|^2}1 = x - \frac{0}{3}1 = x \\ p_3(x) &= x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2}1 - \frac{\langle x^2, x \rangle}{\|x\|^2}x \\ &= x^2 - \frac{2}{3}1 - \frac{0}{2}x = x^2 - \frac{2}{3} \end{aligned}$$

Instead, we take $p_3(x) = -2 + 3x^2$. Then, $\{1, x, -2 + 3x^2\}$ is an orthogonal basis for $P_2(\mathbb{R})$. To make it an orthonormal basis, we divide each vector by its length to get $\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{6}}(-2 + 3x^2)\}$.

9.3.4 Take $p_1(x) = 1$. Then

$$\begin{aligned} p_2(x) &= x - \frac{\langle x, 1\rangle}{\|1\|^2}1 = x - \frac{3}{3}1 = x - 1 \\ p_3(x) &= x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2}1 - \frac{\langle x^2, x-1\rangle}{\|x-1\|^2}(x-1) \\ &= x^2 - \frac{5}{3}1 - \frac{4}{2}(x-1) = x^2 - 2x + \frac{1}{3} \end{aligned}$$

Instead, we take $p_3(x) = 1 - 6x + 3x^2$. Then, $\{1, x-1, 1-6x+3x^2\}$ is an orthogonal basis for $P_2(\mathbb{R})$. To make it an orthonormal basis, we divide each vector by its length to get $\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}(x-1), \frac{1}{\sqrt{18}}(1-6x+3x^2)\}$.

9.3.5 To find an orthogonal basis for $\mathbb{S}$, we apply the Gram-Schmidt Procedure. Let the vectors in the spanning set be denoted $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ respectively. Then, we first take $\vec{v}_1 = \vec{w}_1$. Next, we get

$$\vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{12}{16}\begin{bmatrix} -2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

So, we take $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix}$. Finally, we get

$$\begin{aligned} \vec{v}_3 &= \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{w}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\ &= \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix} - \frac{-2}{16}\begin{bmatrix} -2 \\ 2 \\ 2 \\ 2 \end{bmatrix} - \frac{9}{12}\begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

Consequently, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{S}$.

9.3.6 Let the vectors in $\mathcal{B}$ be denoted $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$, and $\vec{w}_4$ respectively. Then, we first take $\vec{v}_1 = \vec{w}_1$. Next, we get

$$\vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} + \frac{5}{10}\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$$

So, we take $\vec{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore, we see that $\vec{w}_3 \in \operatorname{Span}\{\vec{w}_1, \vec{w}_2\} = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$, so we can ignore it. Finally, we get

$$\begin{aligned} \vec{v}_3 &= \vec{w}_4 - \frac{\langle \vec{w}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{w}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3/5 & -3/5 \\ -1/5 & 4/5 \end{bmatrix} \end{aligned}$$

Consequently, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{S}$.

9.3.7 We let $w_1 = x$. Then

$$w_2 = 2x^2 + x - \frac{\langle 2x^2 + x, x\rangle}{\langle x, x\rangle}x = 2x^2 + x - \frac{2\langle x^2, x\rangle + \langle x, x\rangle}{4}x = 2x^2 + x - \frac{2(-2) + 4}{4}x = 2x^2 + x$$

$$\begin{aligned}
w_3 &= 2 - \frac{\langle 2, x\rangle}{\langle x, x\rangle}x - \frac{\langle 2, 2x^2 + x\rangle}{\langle 2x^2 + x, 2x^2 + x\rangle}(2x^2 + x) \\
&= 2 - \frac{2\cdot 2}{4}x - \frac{2\cdot 2\langle 1, x^2\rangle + 2\langle 1, x\rangle}{4\langle x^2, x^2\rangle + 2\langle x^2, x\rangle + 2\langle x, x^2\rangle + \langle x, x\rangle}(2x^2 + x) \\
&= 2 - x - \frac{4(-2) + 2\cdot 2}{4\cdot 3 + 2(-2) + 2(-2) + 4}(2x^2 + x) = 2 - x + \frac{1}{2}(2x^2 + x) = 2 + x^2 - \frac{1}{2}x
\end{aligned}$$

So an orthogonal basis is $\mathcal{B}_2 = \{x, 2x^2 + x, 2 + x^2 - \frac{1}{2}x\}$.

9.3.8 (a) Consider

$$\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{e}_1 + c_4\vec{e}_2 + c_5\vec{e}_3 + c_6\vec{e}_4$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 3 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/4 & 0 & 0 & 3/4 \\ 0 & 1 & 1/4 & 0 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus, $\{\vec{v}_1, \vec{v}_2, \vec{e}_2, \vec{e}_3\}$ is a basis for $\mathbb{R}^4$.

(b) Observe that $\vec{v}_1$ and $\vec{v}_2$ are already orthogonal. Hence, the next step in the Gram-Schmidt procedure gives

$$\vec{e}_2 - \frac{\langle \vec{e}_2, \vec{v}_1\rangle}{||\vec{v}_1||^2}\vec{v}_1 - \frac{\langle \vec{e}_2, \vec{v}_2\rangle}{||\vec{v}_2||^2}\vec{v}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{-1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{12}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

Hence, we take $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$. Next, we get

$$\begin{aligned}
\vec{v}_4 &= \vec{e}_3 - \frac{\langle \vec{e}_3, \vec{v}_1\rangle}{||\vec{v}_1||^2}\vec{v}_1 - \frac{\langle \vec{e}_3, \vec{v}_2\rangle}{||\vec{v}_2||^2}\vec{v}_2 - \frac{\langle \vec{e}_3, \vec{v}_3\rangle}{||\vec{v}_3||^2}\vec{v}_2 \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{-1}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{12}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{-1}{6}\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}
\end{aligned}$$

So, we have extended $\{\vec{v}_1, \vec{v}_2\}$ to an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ for $\mathbb{R}^4$.

9.4 Problem Solutions

9.4.1 (a) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{S}_1^\perp$. Then, we have

$$0 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3x_1 + 2x_2 + x_3$$

$$0 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 - 2x_2 + 3x_3$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Thus, we get $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{S}_1^\perp$ and $\mathcal{B}$ is clearly linearly independent, so it is a basis for $\mathbb{S}_1^\perp$.

(b) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{S}_2^\perp$. Then, we have

$$0 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 - 2x_2 + x_4$$

The general solution to this linear equation is

$$\vec{x} = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad a, b, c \in \mathbb{R}$$

Therefore, we have

$$\mathbb{S}_2^\perp = \operatorname{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent by Theorem 2.2.6., we have that $\mathcal{B}$ is a basis for $\mathbb{S}_2^\perp$.

(c) We first observe that a basis for $\mathbb{S}_3$ is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$. Let $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{S}_3^\perp$. Then

$$0 = \begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1 + x_3$$

The general solution to this linear equaiton is

$$\vec{x} = a\begin{bmatrix}-1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix}, \quad a, b \in \mathbb{R}$$

Consequently, we have that $\mathcal{B} = \left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ spans $\mathbb{S}_3^\perp$ and is clearly linearly independent, so $\mathcal{B}$ is a basis for $\mathbb{S}_3^\perp$.

(d) Let $A = \begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix} \in \mathbb{S}_4^\perp$. Then

$$0 = \left\langle \begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} \right\rangle = a_1 + a_3$$
$$0 = \left\langle \begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}, \begin{bmatrix}2 & -1\\1 & 3\end{bmatrix} \right\rangle = 2a_1 - a_2 + a_3 + 3a_4$$

Row reducing the coefficient matrix of the homogeneous system gives

$$\begin{bmatrix}1 & 0 & 1 & 0\\2 & -1 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & -3\end{bmatrix}$$

Hence, we get the general solution is

$$\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix} = s\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\3\\0\\1\end{bmatrix}, \quad s, t \in \mathbb{R}$$

Therefore, the orthogonal complement of $\mathbb{S}_4$ is $\mathbb{S}_4^\perp = \operatorname{Span}\left\{\begin{bmatrix}-1 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 3\\0 & 1\end{bmatrix}\right\}$. Since $\left\{\begin{bmatrix}-1 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 3\\0 & 1\end{bmatrix}\right\}$ is also clearly linearly independent, it is a basis for $\mathbb{S}_4^\perp$.

(e) Let $A = \begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix} \in \mathbb{S}_5^\perp$. Then

$$0 = \left\langle \begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}, \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} \right\rangle = 2a_1 + a_2 + a_3 + a_4$$
$$0 = \left\langle \begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}, \begin{bmatrix}-1 & 1\\3 & 1\end{bmatrix} \right\rangle = -a_1 + a_2 + 3a_3 + a_4$$

Row reducing the coefficient matrix of the homogeneous system gives

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & 1 & 7/3 & 1 \end{bmatrix}$$

Hence, we get the general solution is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = s \begin{bmatrix} 2/3 \\ -7/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Therefore, the orthogonal complement of $\mathbb{S}_5$ is $\mathbb{S}_5^\perp = \operatorname{Span}\left\{\begin{bmatrix} 2/3 & -7/3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right\}$. Since $\left\{\begin{bmatrix} 2/3 & -7/3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right\}$ is also clearly linearly independent, it is a basis for $\mathbb{S}_5^\perp$.

(f) We first need to find a basis for $\mathbb{S}_6$. Every vector in $\mathbb{S}_6$ has the form

$$\begin{bmatrix} -b-d & b \\ c & d \end{bmatrix} = b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ spans $\mathbb{S}_6$ and is clearly linearly independent, so it is a basis for $\mathbb{S}_6$.

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathbb{S}_6^\perp$. Then

$$0 = \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle = -a_1 + a_2$$

$$0 = \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle = a_3$$

$$0 = \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle = -a_1 + a_4$$

We get that the general solution of this system is $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $s \in \mathbb{R}$.

Therefore, the orthogonal complement of $\mathbb{S}_6$ is $\mathbb{S}_6^\perp = \operatorname{Span}\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}$. Since $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}$ is also clearly linearly independent, it is a basis for $\mathbb{S}_6^\perp$.

9.4.2 (a) We need to find the general form of any vector orthogonal to $x^2 + 1$. If $a + bx + cx^2 \in \mathbb{S}^\perp$, then we have

$$0 = \langle a + bx + cx^2, x^2 + 1 \rangle = (a - b + c)(2) + a(1) + (a + b + c)(2) = 5a + 4c$$

Hence, $c = -\frac{5}{4}a$. Thus, we have

$$a + bx + cx^2 = a + bx - \frac{5}{4}ax^2 = a\left(1 - \frac{5}{4}x^2\right) + bx$$

Hence, $\mathbb{S}^\perp$ is spanned by $\mathcal{B} = \{4 - 5x^2, x\}$. Observe that

$$\langle 4 - 5x^2, x\rangle = (-1)(-1) + 4(0) + (-1)(1) = 0$$

So, $\mathcal{B} = \{4 - 5x^2, x\}$ is an orthogonal basis for $\mathbb{S}^\perp$.

(b) We need to find the general form of any vector orthogonal to x^2. If $a + bx + cx^2 \in \mathbb{S}^\perp$, then we have

$$0 = \langle a + bx + cx^2, x^2\rangle = (a - b + c)(1) + a(0) + (a + b + c)(1) = 2a + 2c$$

Hence, $c = -a$. Thus, we have

$$a + bx + cx^2 = a + bx - ax^2 = a\left(1 - x^2\right) + bx$$

Hence, $\mathbb{S}^\perp$ is spanned by $\mathcal{B} = \{1 - x^2, x\}$. Observe that

$$\langle 1 - x^2, x\rangle = (0)(-1) + 1(0) + (0)(1) = 0$$

So, $\mathcal{B} = \{1 - x^2, x\}$ is an orthogonal basis for $\mathbb{S}^\perp$.

9.4.3 (a) We apply the Gram-Schdmit Procedure. Let $\vec{v}_1 = \vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

$$\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1\rangle}{||\vec{v}_1||^2}\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \frac{6}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_1\rangle}{||\vec{v}_1||^2}\vec{v}_1 - \frac{\langle \vec{w}_3, \vec{v}_2\rangle}{||\vec{v}_2||^2}\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix} - \frac{6}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{6}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

Then, normalizing the vectors we get the orthonormal basis

$$\left\{ \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 3 \\ -1 \end{bmatrix} \right\}$$

(b) Denote the vectors in the orthonormal basis from part (a) by $\vec{c}_1$, $\vec{c}_2$, and $\vec{c}_3$ respectively. Then we get

$$\begin{aligned}\operatorname{proj}_{\mathbb{S}}(\vec{y}) &= \langle \vec{y}, \vec{c}_1\rangle \vec{c}_1 + \langle \vec{y}, \vec{c}_2\rangle \vec{c}_2 + \langle \vec{y}, \vec{c}_3\rangle \vec{c}_3 \\ &= \frac{12}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} + \frac{12}{2}\frac{1}{2}\begin{bmatrix}1\\-1\\1\\1\end{bmatrix} + \frac{-12}{12}\begin{bmatrix}-1\\1\\3\\-1\end{bmatrix} = \begin{bmatrix}6\\0\\0\\6\end{bmatrix}\end{aligned}$$

(c) We have

$$\begin{aligned}\operatorname{perp}_{\mathbb{S}}(\vec{z}) &= \vec{z} - \operatorname{proj}_{\mathbb{S}}(\vec{z}) = \vec{z} - \langle \vec{z}, \vec{c}_1\rangle \vec{c}_1 + \langle \vec{y}, \vec{c}_2\rangle \vec{c}_2 + \langle \vec{y}, \vec{c}_3\rangle \vec{c}_3 \\ &= \begin{bmatrix}1\\0\\0\\2\end{bmatrix} - \frac{3}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} + \frac{3}{2}\frac{1}{2}\begin{bmatrix}1\\-1\\1\\1\end{bmatrix} + \frac{-3}{12}\begin{bmatrix}-1\\1\\3\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\0\\2\end{bmatrix} - \begin{bmatrix}3/2\\0\\0\\3/2\end{bmatrix} \\ &= \begin{bmatrix}-1/2\\0\\0\\1/2\end{bmatrix}\end{aligned}$$

9.4.4 (a) Denote the given basis by $\vec{z}_1 = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$, $\vec{z}_2 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$, $\vec{z}_3 = \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$. Let $\vec{w}_1 = \vec{z}_1$. Then, we get $\vec{w}_2 = \vec{z}_2 - \operatorname{proj}_{\vec{w}_1}(\vec{z}_2) = \vec{z}_2 - \frac{\vec{z}_2\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix} = \begin{bmatrix}2/3 & 1\\4/3 & 2/3\end{bmatrix}$ To simplify calculations we use $\vec{w}_2 = \begin{bmatrix}2 & 3\\4 & 2\end{bmatrix}$ instead. Then, we get

$$\vec{w}_3 = z_3 - \frac{\vec{z}_3\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 - \frac{(\vec{z}_3\cdot\vec{w}_2)}{\|\vec{w}_2\|^2}\vec{w}_2 = \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix} - \frac{10}{33}\begin{bmatrix}2 & 3\\4 & 2\end{bmatrix} = \begin{bmatrix}8/11 & -10/11\\5/11 & -3/11\end{bmatrix}$$

We pick $\vec{w}_3 = \begin{bmatrix}8 & -10\\5 & -3\end{bmatrix}$. Then the set $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthogonal basis for $\mathbb{S}$.

(b) From our work in (a)

$$\operatorname{proj}_{\mathbb{S}}(A) = \frac{\vec{x}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 + \frac{\vec{x}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\vec{w}_2 + \frac{\vec{x}\cdot\vec{w}_3}{\|\vec{w}_3\|^2}\vec{w}_3 = \frac{11}{3}\vec{w}_1 + \frac{19}{33}\vec{w}_2 + \frac{-23}{198}\vec{w}_3 = \begin{bmatrix}35/9 & 13/9\\-35/18 & 93/18\end{bmatrix}$$

(c) From our work in (a)

$$\operatorname{proj}_{\mathbb{S}}(B) = \frac{\vec{x}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\vec{w}_1 + \frac{\vec{x}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\vec{w}_2 + \frac{\vec{x}\cdot\vec{w}_3}{\|\vec{w}_3\|^2}\vec{w}_3 = \frac{0}{3}\vec{w}_1 + \frac{12}{33}\vec{w}_2 + \frac{-18}{198}\vec{w}_3 = \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}$$

(d) Since $\operatorname{proj}_{\mathbb{S}}(B) = B$ we have that $B \in \mathbb{S}$. Since $\operatorname{proj}_{\mathbb{S}}(A) \neq A$, we have $A \notin \mathbb{S}$.

9.4.5 (a) We let $\vec{v}_1 = 1$. Then

$$\vec{v}_2 = x - x^2 - \frac{\langle x - x^2, 1\rangle}{||1||^2}1 = x - x^2 - \frac{-2}{3}1 = \frac{2}{3} + x - x^2$$

Thus, an orthogonal basis for $\mathbb{S}$ is $\{1, 2 + 3x - 3x^2\}$.

(b) We have

$$\begin{aligned}
\mathrm{proj}_{\mathbb{S}}(1 + x + x^2) &= \frac{\langle 1 + x + x^2, 1\rangle}{||1||^2}1 + \frac{\langle 1 + x + x^2, 2 + 3x - 3x^2\rangle}{||2 + 3x - 3x^2||} \\
&= \frac{5}{3}1 + \frac{4}{24}(2 + 3x - 3x^2) \\
&= 2 + \frac{1}{2}x - \frac{1}{2}x^2 \\
\mathrm{perp}_{\mathbb{S}}(1 + x + x^2) &= 1 + x + x^2 - \mathrm{proj}_{\mathbb{S}}(1 + x + x^2) = -1 + \frac{1}{2}x + \frac{3}{2}x^2
\end{aligned}$$

(c) Since $\mathrm{proj}_{\mathbb{S}}$ and $\mathrm{perp}_{\mathbb{S}}$ are linear mappings, from our work in (b) we get

$$\mathrm{proj}_{\mathbb{S}}(2 + 2x + 2x^2) = 2\,\mathrm{proj}_{\mathbb{S}}(1 + x + x^2) = 4 + x - x^2$$

and

$$\mathrm{perp}_{\mathbb{S}}(2 + 2x + 2x^2) = 2\,\mathrm{perp}_{\mathbb{S}}(1 + x + x^2) = -2 + x + 3x^2$$

9.4.6 We first extend the set to a basis for $\mathbb{R}^3$. We take

$$\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

We then apply the Gram-Schmidt procedure. Take $\vec{v}_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$. Then

$$\vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1\rangle}{||\vec{v}_1||^2}\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{2}{5}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}3/5\\0\\2/5\end{bmatrix}$$

Hence, we take $\vec{v}_2 = \begin{bmatrix}3\\0\\2\end{bmatrix}$.

$$\vec{v}_3 = \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_1\rangle}{||\vec{v}_1||^2}\vec{v}_1 - \frac{\langle \vec{w}_3, \vec{v}_2\rangle}{||\vec{v}_2||^2}\vec{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} - 0\begin{bmatrix}1\\0\\-1\end{bmatrix} - 0\begin{bmatrix}3\\0\\2\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Hence, an orthogonal basis is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

9.4.7 Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthonormal basis for $\mathbb{W}$. Then, for any $\vec{u}, \vec{v} \in \mathbb{V}$ and $s, t \in \mathbb{R}$, we have

$$\begin{aligned}\operatorname{proj}_{\mathbb{W}}(s\vec{u} + t\vec{v}) &= \langle s\vec{u} + t\vec{v}, \vec{v}_1\rangle\vec{v}_1 + \cdots + \langle s\vec{u} + t\vec{v}, \vec{v}_k\rangle\vec{v}_k \\ &= s(\langle \vec{u}, \vec{v}_1\rangle\vec{v}_1 + \cdots + \langle \vec{u}, \vec{v}_k\rangle\vec{v}_k) + t(\langle \vec{v}, \vec{v}_1\rangle\vec{v}_1 + \cdots + \langle \vec{v}, \vec{v}_k\rangle\vec{v}_k) \\ &= s \operatorname{proj}_{\mathbb{W}} \vec{u} + t \operatorname{proj}_{\mathbb{W}} \vec{v}\end{aligned}$$

Hence, $\operatorname{proj}_{\mathbb{W}}$ is linear.

Let $\vec{w} \in \mathbb{W}^{\perp}$. Then $\langle \vec{w}, \vec{v}_i\rangle = 0$ for $1 \leq i \leq k$. Hence $\operatorname{proj}_{\mathbb{W}} \vec{w} = \langle \vec{w}, \vec{v}_1\rangle\vec{v}_1 + \cdots + \langle \vec{w}, \vec{v}_k\rangle\vec{v}_k = \vec{0}$, so $\vec{w} \in \ker(\operatorname{proj}_{\mathbb{W}})$. Therefore, $\mathbb{W}^{\perp} \subseteq \ker(\operatorname{proj}_{\mathbb{W}})$.

Let $\vec{x} \in \ker(\operatorname{proj}_{\mathbb{W}})$, then $\vec{0} = \operatorname{proj}_{\mathbb{W}} \vec{x} = \langle \vec{x}, \vec{v}_1\rangle\vec{v}_1 + \cdots + \langle \vec{x}, \vec{v}_k\rangle\vec{v}_k$
Since $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, we get that $\langle \vec{x}, \vec{v}_i\rangle = 0$ for $1 \leq i \leq k$. Therefore, $\vec{x} \in \mathbb{W}^{\perp}$ by theorem. Hence, $\ker(\operatorname{proj}_{\mathbb{W}}) \subseteq \mathbb{W}^{\perp}$. Consequently, $\mathbb{W}^{\perp} = \ker(\operatorname{proj}_{\mathbb{W}})$ as required.

9.4.8 A basis for $\mathbb{S}^{\perp}$ is $\{p_4(x)\} = \{1\}$. Then

$$\operatorname{proj}_{\mathbb{S}^{\perp}}(-1 + x + x^2 - x^3) = \frac{\langle 1, -1 + x + x^2 - x^3\rangle}{||1||^2} 1 = \frac{-4}{4} = -1$$

Therefore,

$$\operatorname{proj}_{\mathbb{S}}(-1 + x + x^2 - x^3) = (-1 + x + x^2 - x^3) - \operatorname{proj}_{\mathbb{S}^{\perp}}(-1 + x + x^2 - x^3) = x + x^2 - x^3$$

9.5 Problem Solutions

9.5.1 By the Fundamental Theorem of Linear Algebra the orthogonal complement of the rowspace is the nullspace of A. We have

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 2 & 6 & 2 \\ -1 & 0 & -2 & -3 \\ 3 & 1 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, a basis for $\operatorname{Row}(A)^{\perp}$ is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

9.5.2 We will prove that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_\ell\}$ is a linearly independent spanning set for $\mathbb{U} \oplus \mathbb{W}$. Consider

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{w}_1 + \cdots + c_{k+\ell}\vec{w}_\ell = \vec{0}$$

Then, we have

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = -c_{k+1}\vec{w}_1 - \cdots - c_{k+\ell}\vec{w}_\ell$$

The vector on the left is in $\mathbb{U}$ and the vector on the right is in $\mathbb{W}$. But, the only vector that is both in $\mathbb{U}$ and $\mathbb{W}$ is the zero vector. Therefore, each $c_i = 0$ and hence $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_\ell\}$ is linearly independent.
For any $\vec{v} \in \mathbb{U} \oplus \mathbb{W}$ we have that $\vec{v} = \vec{u} + \vec{w}$ by definition of $\mathbb{U} \oplus \mathbb{W}$. We can write $\vec{u} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ and $\vec{w} = d_1\vec{w}_1 + \cdots + d_\ell\vec{w}_\ell$ and hence

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + d_1\vec{w}_1 + \cdots + d_\ell\vec{w}_\ell$$

and so $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_\ell\}$ also spans $\mathbb{U} \oplus \mathbb{W}$.

9.6 Problem Solutions

9.6.1 (a) We have $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$. Hence, the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ is

$$\begin{aligned}
\vec{x} &= (A^T A)^{-1} A^T \vec{b} \\
&= \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 25/7 \\ -19/14 \end{bmatrix}
\end{aligned}$$

(b) We have $A = \begin{bmatrix} 1 & 5 \\ -2 & -7 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Hence, the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ is

$$\begin{aligned}
\vec{x} &= (A^T A)^{-1} A^T \vec{b} \\
&= \begin{bmatrix} 6 & 21 \\ 21 & 78 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 & 1 \\ 5 & -7 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{27} \begin{bmatrix} 78 & -21 \\ -21 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} -4/3 \\ -1/3 \end{bmatrix}
\end{aligned}$$

(c) We have $A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$. Hence, the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ is

$$\begin{aligned}
\vec{x} &= (A^T A)^{-1} A^T \vec{b} \\
&= \begin{bmatrix} 17 & 6 \\ 6 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} \\
&= \frac{1}{66} \begin{bmatrix} 6 & -6 \\ -6 & 17 \end{bmatrix} \begin{bmatrix} 30 \\ 21 \end{bmatrix} \\
&= \begin{bmatrix} 9/11 \\ 59/22 \end{bmatrix}
\end{aligned}$$

(d) We have $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}$. Hence, the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ is

$$\begin{aligned} \vec{x} &= (A^T A)^{-1} A^T \vec{b} \\ &= \begin{bmatrix} 4 & 10 \\ 10 & 26 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 26 & -10 \\ -10 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 25 \end{bmatrix} \\ &= \begin{bmatrix} 5/2 \\ 0 \end{bmatrix} \end{aligned}$$

(e) We have $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ -2 \\ 3 \\ -2 \end{bmatrix}$. Hence, the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ is

$$\begin{aligned} \vec{x} &= (A^T A)^{-1} A^T \vec{b} \\ &= \begin{bmatrix} 10 & 0 & 4 \\ 0 & 7 & 1 \\ 4 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 2 & 1 \\ -1 & -1 & 2 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 5/3 \\ 5/3 \\ -11/3 \end{bmatrix} \end{aligned}$$

(f) We have $A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 3 & 1 \\ 2 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$. Hence, the vector $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ is

$$\begin{aligned} \vec{x} &= (A^T A)^{-1} A^T \vec{b} \\ &= \begin{bmatrix} 18 & -4 \\ -4 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 3 & 2 \\ -2 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 9/22 \\ 1/11 \end{bmatrix} \end{aligned}$$

9.6.2 (a) By the Approximation Theorem, the vector in P that is closet to $\vec{x}$ is

$$\text{proj}_P(\vec{x}) = \text{perp}_{\vec{n}}(\vec{x}) = \vec{x} - \frac{\vec{x}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 0 \\ 3/2 \\ 3/2 \end{bmatrix}$$

(b) By the Approximation Theorem, the vector in P that is closet to $\vec{y}$ is

$$\text{proj}_P(\vec{y}) = \text{perp}_{\vec{n}}(\vec{y}) = \vec{y} - \frac{\vec{y}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

(c) By the Approximation Theorem, the vector in P that is closet to $\vec{x}$ is

$$\text{proj}_P(\vec{z}) = \text{perp}_{\vec{n}}(\vec{z}) = \vec{z} - \frac{\vec{z}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(d) In our calculations in (a), (b), and (c), we see that $\vec{y}\cdot\vec{n} > \vec{x}\cdot\vec{n}$ and $\vec{z} \in P$, so $\vec{y}$ is the furthest from P.

9.6.3 (a) Let $X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$. Then we get

$$\vec{a} = (X^TX)^{-1}X^T\vec{y} = \begin{bmatrix} 1/3 \\ 5/2 \end{bmatrix}$$

Hence, the best fitting line is $y = \frac{1}{3} + \frac{5}{2}x$.

(b) Let $X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Then we get

$$\vec{a} = (X^TX)^{-1}X^T\vec{y} = \begin{bmatrix} -7/5 \\ 13/10 \end{bmatrix}$$

Hence, the best fitting line is $y = -\frac{7}{5} + \frac{13}{10}x$.

9.6.4 (a) Let $X = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$. Then we get

$$\vec{a} = (X^TX)^{-1}X^T\vec{y} = \begin{bmatrix} -3/2 \\ 2/5 \\ 1/2 \end{bmatrix}$$

Hence, the best fitting quadratic is $y = -\frac{3}{2} + \frac{2}{5}x + \frac{1}{2}x^2$.

(b) Let $X = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}$. Then we get

$$\vec{a} = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} 4/5 \\ 2/5 \\ 0 \end{bmatrix}$$

Hence, the best fitting quadratic is $y = \frac{4}{5} + \frac{2}{5}x + 0x^2$.

9.6.5 Let $\vec{y} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. Since the desired polynomial has the form $bx + cx^2$, the design matrix is $X = \begin{bmatrix} -1 & (-1)^2 \\ 0 & 0^2 \\ 1 & 1^2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

We have

$$\vec{a} = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 5/2 \end{bmatrix}$$

So, $y = -\frac{3}{2}x + \frac{5}{2}x^2$.

Chapter 10 Solutions

10.1 Problem Solutions

10.1.1 (a) We have $C(\lambda) = \det(A - \lambda I) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$. We pick $\lambda_1 = -2$. We have

$$A + 2I = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, a unit eigenvector for $\lambda_1 = -2$ is $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We extend $\{\vec{v}_1\}$ to the orthonormal basis $\{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$ with $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Taking $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ gives $Q^TAQ = \begin{bmatrix} -2 & -8 \\ 0 & -2 \end{bmatrix} = T$.

(b) We have $C(\lambda) = \det(A - \lambda I) = \lambda^2 - 5\lambda - 24 = (\lambda - 8)(\lambda + 3)$. We pick $\lambda_1 = 8$. We have

$$A - 8I = \begin{bmatrix} -7 & 4 \\ 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/7 \\ 0 & 0 \end{bmatrix}$$

Hence, a unit eigenvector for $\lambda_1 = 8$ is $\vec{v}_1 = \frac{1}{\sqrt{65}}\begin{bmatrix} 4 \\ 7 \end{bmatrix}$. We extend $\{\vec{v}_1\}$ to the orthonormal basis $\{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$ with $\vec{v}_2 = \frac{1}{\sqrt{65}}\begin{bmatrix} -7 \\ 4 \end{bmatrix}$. Taking $Q = \frac{1}{\sqrt{65}}\begin{bmatrix} 4 & -7 \\ 7 & 4 \end{bmatrix}$ gives $Q^TAQ = \begin{bmatrix} 8 & -3 \\ 0 & -3 \end{bmatrix} = T$.

(c) Observe that $\lambda = -3$ is an eigenvalue of A with corresponding eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We extend $\{\vec{v}_1\}$ to the basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ for $\mathbb{R}^3$. Thus, we pick $P_1 = I$. Thus, we have

$$P_1^TAP_1 = \begin{bmatrix} -3 & 4 & 5 \\ 0 & 1 & 4 \\ 0 & 7 & 4 \end{bmatrix} = \begin{bmatrix} -3 & \vec{b}^T \\ \vec{0} & A_1 \end{bmatrix}$$

where $A_1 = \begin{bmatrix} 1 & 4 \\ 7 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Our work in part (b) tells us that we can take $Q = \frac{1}{\sqrt{65}}\begin{bmatrix} 4 & -7 \\ 7 & 4 \end{bmatrix}$ and get $T_1 = \begin{bmatrix} 8 & -3 \\ 0 & -3 \end{bmatrix}$.

We now take $P_2 = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/\sqrt{65} & -7/\sqrt{65} \\ 0 & 7/\sqrt{65} & 4/\sqrt{65} \end{bmatrix}$. Let $P = P_1P_2 = P_2$, then

$$P^TAP = \begin{bmatrix} -3 & \vec{b}^TQ \\ \vec{0} & T_1 \end{bmatrix} = \begin{bmatrix} -3 & 51/\sqrt{65} & -8/\sqrt{65} \\ 0 & 8 & -3 \\ 0 & 0 & -3 \end{bmatrix}$$

(d) Observe that $\lambda = 7$ is an eigenvalue of A with corresponding eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We extend $\{\vec{v}_1\}$ to the basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ for $\mathbb{R}^3$. Thus, we pick $P_1 = I$. Thus, we have

$$P_1^TAP_1 = \begin{bmatrix} 7 & -6 & 5 \\ 0 & 1 & 4 \\ 0 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 7 & \vec{b}^T \\ \vec{0} & A_1 \end{bmatrix}$$

where $A_1 = \begin{bmatrix} 1 & 4 \\ 6 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$.

We have $C(\lambda) = \det(A - \lambda I) = \lambda^2 - 4\lambda - 21 = (\lambda + 3)(\lambda - 7)$. So, $\lambda_1 = -3$ and

$$A + 3I = \begin{bmatrix} 4 & 4 \\ 6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, a unit eigenvector for $\lambda_1 = -3$ is $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We extend $\{\vec{v}_1\}$ to the orthonormal basis $\{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$ with $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Taking $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ gives $Q^TA_1Q = \begin{bmatrix} -3 & 2 \\ 0 & 7 \end{bmatrix} = T_1$.

We now take $P_2 = \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Let $P = P_1P_2 = P_2$, then

$$P^TAP = \begin{bmatrix} 7 & \vec{b}^TQ \\ \vec{0} & T_1 \end{bmatrix} = \begin{bmatrix} 7 & 11/\sqrt{2} & -1/\sqrt{2} \\ 0 & -3 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

(e) Observe that $\lambda = 2$ is an eigenvalue of A with corresponding eigenvector $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. We extend $\{\vec{v}_1\}$

to the basis $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ for $\mathbb{R}^3$. Thus, we pick $P_1 = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$. Thus, we have

$$P_1^TAP_1 = \begin{bmatrix}2 & 2 & 1\\0 & 1 & -1\\0 & 4 & 5\end{bmatrix} = \begin{bmatrix}2 & \vec{b}^T\\ \vec{0} & A_1\end{bmatrix}$$

where $A_1 = \begin{bmatrix}1 & -1\\4 & 5\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}2\\1\end{bmatrix}$.

We have $C(\lambda) = \det(A - \lambda I) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$. So, $\lambda_1 = 3$ and

$$A - 3I = \begin{bmatrix}-2 & -1\\4 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 1/2\\0 & 0\end{bmatrix}$$

Hence, a unit eigenvector for $\lambda_1 = 3$ is $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}$. We extend $\{\vec{v}_1\}$ to the orthonormal basis $\{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$ with $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}$. Taking $Q = \frac{1}{\sqrt{5}}\begin{bmatrix}-1 & 2\\2 & 1\end{bmatrix}$ gives $Q^TA_1Q = \begin{bmatrix}3 & 5\\0 & 3\end{bmatrix} = T_1$.

We now take $P_2 = \begin{bmatrix}1 & \vec{0}^T\\ \vec{0} & Q\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & -1/\sqrt{5} & 2/\sqrt{5}\\0 & 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$. Let $P = P_1P_2$, then

$$P^TAP = \begin{bmatrix}2 & \vec{b}^TQ\\ \vec{0} & T_1\end{bmatrix} = \begin{bmatrix}2 & 0 & \sqrt{5}\\0 & 3 & 5\\0 & 0 & 3\end{bmatrix}$$

(f) We first need to find a real eigenvalue of A. We have

$$C(\lambda) = \begin{vmatrix}-\lambda & 1 & 1\\2 & 3-\lambda & 6\\-1 & -1 & -2-\lambda\end{vmatrix} = \begin{vmatrix}-\lambda & 1 & 1\\0 & 1-\lambda & 2-2\lambda\\-1 & -1 & -2-\lambda\end{vmatrix} = \begin{vmatrix}-\lambda & 1 & -1\\0 & 1-\lambda & 0\\-1 & -1 & -\lambda\end{vmatrix}$$

Thus, $\lambda_1 = 1$ is an eigenvalue. Next, we need to pick a unit eigenvector corresponding to λ_1. We have

$$A - I = \begin{bmatrix}-1 & 1 & 1\\2 & 2 & 6\\-1 & -1 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

Thus, we take $\vec{v}_1 = \begin{bmatrix}-1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$.

Next, we need to extend this to an orthonormal basis for $\mathbb{R}^3$. We take $\vec{w}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ and $\vec{w}_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. Then we define $P_1 = \begin{bmatrix} \vec{v}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix}$.

We find that

$$P_1^T A P_1 = \begin{bmatrix} 1 & -10/\sqrt{3} & -3\sqrt{2} \\ 0 & -1 & 0 \\ 0 & 4/\sqrt{(6)} & 1 \end{bmatrix}$$

Notice that this is almost upper triangular. We can make it upper triangular, by swapping the order of $\vec{w}_2$ and $\vec{w}_3$. In particular, if we take $P_2 = \begin{bmatrix} \vec{v}_1 & \vec{w}_3 & \vec{w}_2 \end{bmatrix}$, then

$$P_2^T A P_2 = \begin{bmatrix} 1 & -10/\sqrt{3} & -3\sqrt{2} \\ 0 & 1 & 4/\sqrt{(6)} \\ 0 & 0 & -1 \end{bmatrix} = T$$

is upper triangular.

(g) Observe that $\lambda = 2$ is an eigenvalue of A with corresponding eigenvector $\vec{v}_1 = \vec{e}_3$. We extend $\{\vec{v}_1\}$ to the basis $\{\vec{e}_3, \vec{e}_1, \vec{e}_2, \vec{e}_4\}$ for $\mathbb{R}^4$. Thus, we pick $P_1 = \begin{bmatrix} \vec{e}_3 & \vec{e}_1 & \vec{e}_2 & \vec{e}_4 \end{bmatrix}$. Thus, we have

$$P_1^T A P_1 = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

is upper triangular.

10.1.2 If A is orthogonally similar to B, then there exists an orthogonal matrix P such that $P^T AP = B$. If B is orthogonally similar to C, then there exists an orthogonal matrix Q such that $Q^T BQ = C$. Hence,

$$C = Q^T BQ = Q^T P^T APQ = (PQ)^T A(PQ) = R^T AR$$

where $R = PQ$ is orthogonal since a product of orthogonal matrices is orthogonal. Thus, A is orthogonal similar to C.

10.1.3 (a) If A and B are orthogonally similar, then there exists an orthogonal matrix P such that $P^T AP = B$. Since P is orthogonal, we get that P is invertible, and hence

$$B^{-1} = (P^T AP)^{-1} = P^{-1}A^{-1}(P^T)^{-1}$$

Let $Q = (P^T)^{-1}$. Then, $Q^T = ((P^T)^{-1})^T = P^{-1}$. So, we have

$$B^{-1} = Q^T A^{-1} Q$$

and thus A^{-1} and B^{-1} are orthogonally similar.

(b) If A and B are orthogonally similar, then there exists an orthogonal matrix P such that $P^TAP = B$. Thus, we have
$$B^2 = (P^TAP)(P^TAP) = P^TA(PP^T)AP = P^TA^2P$$
Thus, A^2 and B^2 are also orthogonally similar.

(c) If A and B are orthogonally similar, then there exists an orthogonal matrix P such that $P^TAP = B$. Thus,
$$B^T = (P^TAP)^T = P^TA^T(P^T)^T = P^TAP = B$$

(d) If A and B are orthogonally similar, then there exists an orthogonal matrix P such that $P^TAP = B$. Assume that there exists an orthogonal matrix Q such that $Q^TBQ = T$ is upper triangular. Then,
$$T = Q^T(P^TAP)Q = (PQ)^TA(PQ)$$
where PQ is orthogonal since both P and Q are orthogonal. Thus, A is also orthogonally similar to T.

10.2 Problem Solutions

10.2.1 (a) We have $C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda = \lambda(\lambda - 4)$

Hence, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 4$.

For $\lambda_1 = 0$ we get $A - \lambda_1 I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 4$ we get $A - \lambda_2 I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^2$. Hence, $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal and $P^TAP = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) We have
$$C(\lambda) = \begin{vmatrix} -\lambda & 2 & -2 \\ 2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & -2 \\ 2 & 1-\lambda & 0 \\ -2 & 1-\lambda & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & -2 \\ 2 & 1-\lambda & 0 \\ -4 & 0 & 1-\lambda \end{vmatrix}$$
$$= -(\lambda - 1)(\lambda^2 - \lambda - 8)$$

Hence, $\lambda_1 = 1$ is an eigenvalue, and by the quadratic formula, we get the other eigenvalues are $\lambda_2 = (1 + \sqrt{33})/2$ and $\lambda_3 = (1 - \sqrt{33})/2$.

For $\lambda_1 = 1$ we get $A - \lambda_1 I = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

For $\lambda_2 = \frac{1+\sqrt{33}}{2}$ we get $A - \lambda_2 I = \begin{bmatrix} -\frac{1+\sqrt{33}}{2} & 2 & -2 \\ 2 & \frac{1-\sqrt{33}}{2} & 0 \\ -2 & 0 & \frac{1-\sqrt{33}}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-1+\sqrt{33}}{4} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1-\sqrt{33} \\ -4 \\ 4 \end{bmatrix} \right\}$.

For $\lambda_3 = \frac{1-\sqrt{33}}{2}$ we get $A - \lambda_3 I = \begin{bmatrix} -\frac{1-\sqrt{33}}{2} & 2 & -2 \\ 2 & \frac{1+\sqrt{33}}{2} & 0 \\ -2 & 0 & \frac{1+\sqrt{33}}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-1-\sqrt{33}}{4} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_3 is $\left\{ \begin{bmatrix} 1+\sqrt{33} \\ -4 \\ 4 \end{bmatrix} \right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^3$. Hence, $P = \begin{bmatrix} 0 & (1-\sqrt{33})/(66-2\sqrt{33}) & (1+\sqrt{33})/(66+2\sqrt{33}) \\ 1/\sqrt{2} & -1/(66-2\sqrt{33}) & -1/(66+2\sqrt{33}) \\ 1/\sqrt{2} & 1/(66-2\sqrt{33}) & 1/(66+2\sqrt{33}) \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+\sqrt{33})/2 & 0 \\ 0 & 0 & (1-\sqrt{33})/2 \end{bmatrix}$.

(c) We have $C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 7-\lambda \end{vmatrix} = \lambda^2 - 11\lambda - 24 = (\lambda - 3)(\lambda - 8)$

Hence, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 8$.

For $\lambda_1 = 3$ we get $A - \lambda_1 I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 8$ we get $A - \lambda_2 I = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^2$. Hence, $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$ is orthogonal and $P^TAP = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$.

(d) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & -2 & -5 \\ -2 & -5-\lambda & -2 \\ -5 & -2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 7-\lambda & 0 & -7+\lambda \\ -2 & -5-\lambda & -2 \\ -5 & -2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 7-\lambda & 0 & 0 \\ -2 & -5-\lambda & -4 \\ -5 & -2 & -3-\lambda \end{vmatrix}$$
$$= -(\lambda - 7)(\lambda^2 + 8\lambda + 7) = -(\lambda - 7)(\lambda + 1)(\lambda + 7)$$

Hence, the eigenvalues are $\lambda_1 = 7$, $\lambda_2 = -1$, and $\lambda_3 = -7$.

For $\lambda_1 = 7$ we get $A - \lambda_1 I = \begin{bmatrix} -5 & -2 & -5 \\ -2 & -12 & -2 \\ -5 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$

For $\lambda_2 = -1$ we get $A - \lambda_2 I = \begin{bmatrix}3 & -2 & -5\\-2 & -4 & -2\\-5 & -2 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$.

For $\lambda_3 = -7$ we get $A - \lambda_3 I = \begin{bmatrix}9 & -2 & -5\\-2 & 2 & -2\\-5 & -2 & 9\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$

A basis for the eigenspace of λ_3 is $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^3$. Hence, $P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\\0 & -1/\sqrt{3} & 2/\sqrt{6}\\1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix}7 & 0 & 0\\0 & -1 & 0\\0 & 0 & -7\end{bmatrix}$.

(e) We have $C(\lambda) = \det(A - \lambda I) = \begin{vmatrix}1-\lambda & -2\\-2 & -2-\lambda\end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$
Hence, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$.

For $\lambda_1 = -3$ we get $A - \lambda_1 I = \begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -1/2\\0 & 0\end{bmatrix}$. Thus, a basis for E_{λ_1} is $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

For $\lambda_2 = 2$ we get $A - \lambda_2 I = \begin{bmatrix}-1 & -2\\-2 & -4\end{bmatrix} \sim \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$. Thus, a basis for E_{λ_2} is $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^2$. Hence, $P = \begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$ is orthogonal and $P^TAP = \begin{bmatrix}-3 & 0\\0 & 2\end{bmatrix}$.

(f) We have

$$C(\lambda) = \begin{vmatrix}1-\lambda & 0 & 2\\0 & 1-\lambda & 4\\2 & 4 & 2-\lambda\end{vmatrix} = (1-\lambda)(\lambda^2 - 3\lambda - 14) + 2(-1)(1-\lambda)(2) \quad = -(\lambda - 1)(\lambda^2 - 3\lambda - 18) = -(\lambda - 1$$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 6$, and $\lambda_3 = -3$.

For $\lambda_1 = 2$ we get $A - 2I = A = \begin{bmatrix}0 & 0 & 2\\0 & 0 & 4\\2 & 4 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}$.

For $\lambda_2 = 6$ we get $A - 6I = \begin{bmatrix} -5 & 0 & 2 \\ 0 & -5 & 4 \\ 2 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2/5 \\ 0 & 1 & -4/5 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \right\}$.

For $\lambda_3 = -3$ we get $A + 3I = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 4 \\ 2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_3 is $\left\{ \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^3$. Hence, $P = \begin{bmatrix} -2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

(g) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix}$$
$$= -\lambda(\lambda^2 - 3\lambda) = -\lambda^2(\lambda - 3)$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$.

For $\lambda_1 = 0$ we get $A - 0I = A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{w}_1, \vec{w}_2\}$.

However, we need an orthogonal basis for each eigenspace, so we apply the Gram-Schmidt procedure to this basis.

Pick $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Then $\vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

Thus, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis for the eigenspace of λ_1.

For $\lambda_2 = 3$ we get $A - 3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^3$. Hence, $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(h) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 2 & 4 \\ 2 & 4-\lambda & 2 \\ 4 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & 2+\lambda \\ 2 & 4-\lambda & 0 \\ 4 & 2 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & 2+\lambda \\ 2 & 4-\lambda & 0 \\ 6-\lambda & 4 & 0 \end{vmatrix} = -(\lambda+2)(\lambda^2-10\lambda+16)$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 8$, and $\lambda_3 = 2$.

For $\lambda_1 = -2$ we get $A + 2I = A = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 6 & 2 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 8$ we get $A - 8I = \begin{bmatrix} -6 & 2 & 4 \\ 2 & -4 & 2 \\ 4 & 2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_3 = 2$ we get $A - 2I = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_3 is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^3$. Hence, $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(i) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & -4 & -4 \\ -4 & 2-\lambda & -4 \\ -4 & -4 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & -4 & -4 \\ -4 & 2-\lambda & -4 \\ 0 & -6+\lambda & 6-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & -8 & -4 \\ -4 & -2-\lambda & -4 \\ 0 & 0 & 6-\lambda \end{vmatrix}$$
$$= -(\lambda-6)(\lambda^2-36) = -(\lambda-6)^2(\lambda+6)$$

The eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -6$.

For $\lambda_1 = 6$ we get $A - 6I = \begin{bmatrix} -4 & -4 & -4 \\ -4 & -4 & -4 \\ -4 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{w}_1, \vec{w}_2\}$.

However, we need an orthogonal basis for each eigenspace, so we apply the Gram-Schmidt procedure to this basis.

Pick $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Then $\vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

Thus, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal basis for the eigenspace of λ_1.

For $\lambda_2 = -6$ we get $A + 6I = \begin{bmatrix} 8 & -4 & -4 \\ -4 & 8 & -4 \\ -4 & -4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^3$. Hence, $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$.

(j) We have

$$C(\lambda) = \begin{vmatrix} 5-\lambda & -1 & 1 \\ -1 & 5-\lambda & 1 \\ 1 & 1 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & -1 & 1 \\ -1 & 5-\lambda & 1 \\ 0 & 6-\lambda & 6-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & -2 & 1 \\ -1 & 4-\lambda & 1 \\ 0 & 0 & 6-\lambda \end{vmatrix}$$
$$= -(\lambda - 6)(\lambda^2 - 9\lambda + 18) = -(\lambda - 6)^2(\lambda - 3)$$

The eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 3$.

For $\lambda_1 = 6$ we get $A - 6I = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{w}_1, \vec{w}_2\}$.

However, we need an orthogonal basis for each eigenspace, so we apply the Gram-Schmidt procedure to this basis.

Pick $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Then $\vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$.

Thus, $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$ is an orthogonal basis for the eigenspace of λ_1.

For $\lambda_2 = 3$ we get $A - 3I = \begin{bmatrix}2 & -1 & 1\\-1 & 2 & 1\\1 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^3$. Hence, $P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix}6 & 0 & 0\\0 & 6 & 0\\0 & 0 & 3\end{bmatrix}$.

(k) We have

$$\begin{aligned}C(\lambda) &= \begin{vmatrix}-\lambda & 1 & -1\\1 & -\lambda & 1\\-1 & 1 & -\lambda\end{vmatrix} = \begin{vmatrix}-\lambda & 1 & -1\\1 & -\lambda & 1\\0 & 1-\lambda & 1-\lambda\end{vmatrix}\\ &= \begin{vmatrix}-\lambda & 2 & -1\\1 & -1-\lambda & 1\\0 & 0 & 1-\lambda\end{vmatrix} = (1-\lambda)(\lambda^2+\lambda-2) = -(\lambda-1)^2(\lambda+2)\end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$.

We have

$$A - I = \begin{bmatrix}-1 & 1 & -1\\1 & -1 & 1\\-1 & 1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Thus, a basis for E_{λ_1} is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$. Since, this isn't an orthogonal basis, we need to apply the Gram-Schimdt procedure. We take $\vec{w}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and get

$$\begin{bmatrix}-1\\0\\1\end{bmatrix} - \frac{-1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}$$

Remembering that we need an orthonormal basis, we take $\vec{v}_1 = \begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}-1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}$.

Next, we have

$$A + 2I = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\vec{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

Thus, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$ and hence, taking $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ gives

$$P^T AP = \text{diag}(1, 1, -2)$$

(l) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 1 & \lambda & \lambda & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 1 & \lambda & \lambda & 1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 3 & 0 & 0 & 3-\lambda \end{vmatrix} = (-\lambda)^2(\lambda^2 - 4\lambda) = -\lambda^3(\lambda - 4)$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 4$.

For $\lambda_1 = 0$ we get $A - 0I = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

However, we need an orthogonal basis for each eigenspace, so we apply the Gram-Schmidt procedure to this basis.

Pick $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$.

So, we pick $\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$. Next,

$$\vec{w}_3 - \frac{\vec{w}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{w}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{bmatrix}$$

Thus, we take $\vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$ and we get that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for the eigenspace of λ_1.

For $\lambda_2 = 4$ we get $A - 4I = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

After normalizing, the basis vectors for the eigenspaces form an orthonormal basis for $\mathbb{R}^3$. Hence, $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 0 & 2/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 0 & 0 & 3/\sqrt{12} & 1/2 \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

10.2.2 Observe that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set, but we need an orthogonal matrix. So, we normalize the vectors and take $P = \begin{bmatrix} -\frac{1}{3} & -2/\sqrt{5} & 2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$.

We want to find A such that $P^TAP = \operatorname{diag}(-3, 1, 6)$. Thus,

$$A = P \operatorname{diag}(-3, 1, 6)P^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

10.2.3 If A is orthogonally diagonalizable, then there exists an orthogonal matrix P and diagonal matrix D such that $D = P^TAP = P^{-1}AP$. Since A and P are invertible, we get that D is invertible and hence,

$$D^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^TA^{-1}P$$

Hence, A^{-1} is orthogonally diagonalizable

10.2.4 (a) We have $(A^TA)^T = A^T(A^T)^T = A^TA$, so A^TA is symmetric.

(b) We have $(AA^T)^T = (A^T)^TA^T = AA^T$, so AA^T is symmetric.

(c) We have $(AB)^T = B^TA^T = BA$, so it seems that AB does not have to be symmetric. To demonstrate it might not be symmetric, we give a counter example. Take $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then, A and B are symmetric, but $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not symmetric.

(d) We have $(AA)^T = A^TA^T = AA$, so A^2 is symmetric.

10.2.5 Assume that there exists a symmetric matrix B such that $A = B^2$. Since B is symmetric, there exists an orthogonal matrix P such that $P^TBP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Hence, we have $B = PDP^T$ and so

$$A = B^2 = (PDP^T)(PDP^T) = PD^2P^T$$

Thus, $P^TAP = D^2 = \operatorname{diag}(\lambda_1^2, \dots, \lambda_n^2)$, so the eigenvalues of A are all non-negative.

On the other hand, assume that all eigenvalues $\mu_1, \dots, \mu_n$ of A are non-negative. Define $\lambda_i = \sqrt{\mu_i}$ which exists since μ_i is non-negative. Then, since A is symmetric, there exists an orthogonal matrix Q such that

$$Q^TAQ = \operatorname{diag}(\mu_1, \dots, \mu_n) = \operatorname{diag}(\lambda_1^2, \dots, \lambda_n^2)$$

So,

$$A = Q\operatorname{diag}(\lambda_1^2, \dots, \lambda_n^2)Q^T = [Q\operatorname{diag}(\lambda_1, \dots, \lambda_n)Q^T][Q\operatorname{diag}(\lambda_1, \dots, \lambda_n)Q^T]$$

Thus, we define $B = Q\operatorname{diag}(\lambda_1, \dots, \lambda_n)Q^T$. We have that $Q^TBQ = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and so B is symmetric since it is orthogonally diagonalizable.

10.2.6 (a) The statement is false. The matrix $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal, but it is not symmetric, so it is not orthogonally diagonalizable.

(b) The statement is false by the result of 4(c).

(c) The statement is true. If A is orthogonally similar to B, then there exists an orthogonal matrix Q such that $Q^TAQ = B$. If B is symmetric, then there exists an orthogonal matrix P such that $P^TBP = D$. Hence,

$$D = P^T(Q^TAQ)P = P^TQ^TAQP = (QP)^TA(QP)$$

where QP is orthogonal since a product of orthogonal matrices is orthogonal. Hence, A is orthogonally diagonalizable.

(d) If A is symmetric, then it is orthogonally diagonalizable by the Principal Axis Theorem. Thus, A is diagonalizable and hence $g_\lambda = a_\lambda$ by Corollary 6.3.4.

10.3 Problem Solutions

10.3.1 (a) i. $A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 1 \end{bmatrix}$

ii. We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 3/2 \\ 3/2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 5/4 = (\lambda - \frac{5}{2})(\lambda + \frac{1}{2})$$

Hence, the eigenvalues are $\lambda_1 = \frac{5}{2}$ and $\lambda_2 = -\frac{1}{2}$, so the quadratic form and symmetric matrix are indefinite.

iii. The corresponding diagonal form is $Q = \frac{5}{2}y_1^2 - \frac{1}{2}y_2^2$

For $\lambda_1 = -5/2$ we get $A - \lambda_1 I = \begin{bmatrix} -3/2 & 3/2 \\ 3/2 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = -1/2$ we get $A - \lambda_2 I = \begin{bmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

Hence, $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ diagonalizes A. So, computing $\vec{x} = P\vec{y}$ we get that the required change of variables is $x_1 = \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}}$, $x_2 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}$.

(b) i. $A = \begin{bmatrix} 8 & 2 \\ 2 & 11 \end{bmatrix}$

ii. We have

$$C(\lambda) = \begin{vmatrix} 8-\lambda & 2 \\ 2 & 11-\lambda \end{vmatrix} = (\lambda - 12)(\lambda - 7)$$

Hence, the eigenvalues are $\lambda_1 = 12$ and $\lambda_2 = 7$, so the quadratic form and symmetric matrix are positive definite.

iii. The corresponding diagonal form is $Q = 12y_1^2 + 7y_2^2$

For $\lambda_1 = 12$ we get $A - \lambda_1 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

For $\lambda_2 = 7$ we get $A - \lambda_2 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

Hence, $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ diagonalizes A. So, computing $\vec{x} = P\vec{y}$ we get that the required change of variables is $x_1 = \frac{1}{\sqrt{5}}y_1 - \frac{2}{\sqrt{5}}y_2$, $x_2 = \frac{2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2$.

(c) i. $A = \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix}$

ii. We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 4 \\ 4 & -5-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 7)$$

Hence, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -7$, so the quadratic form and symmetric matrix are indefinite.

iii. The corresponding diagonal form is $Q = 3y_1^2 - 7y_2^2$

For $\lambda_1 = 3$ we get $A - \lambda_1 I = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = -7$ we get $A - \lambda_2 I = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$.

Hence, $P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ diagonalizes A. So, computing $\vec{x} = P\vec{y}$ we get that the required change of variables is $x_1 = \frac{2}{\sqrt{5}}y_1 - \frac{1}{\sqrt{5}}y_2$, $x_2 = \frac{1}{\sqrt{5}}y_1 + \frac{2}{\sqrt{5}}y_2$.

(d) i. $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$

ii. We have

$$C(\lambda) = \begin{vmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 2)$$

Hence, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, so the quadratic form and symmetric matrix are indefinite.

iii. The corresponding diagonal form is $Q = 3y_1^2 - 2y_2^2$

For $\lambda_1 = 3$ we get $A - \lambda_1 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

For $\lambda_2 = -2$ we get $A - \lambda_2 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

Hence, $P = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ diagonalizes A. So, computing $\vec{x} = P\vec{y}$ we get that the required change of variables is $x_1 = \frac{1}{\sqrt{5}}y_1 - \frac{2}{\sqrt{5}}y_2$, $x_2 = \frac{2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2$.

(e) i. $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

ii. We have

$$C(\lambda) = -(\lambda - 2)^2(\lambda - 8)$$

Hence, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 8$, so the quadratic form and symmetric matrix are positive definite.

iii. The corresponding diagonal form is $Q = 2y_1^2 + 2y_2^2 + 8y_3^2$

We find that a matrix which orthogonally diagonalizes A is $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

diagonalizes A. So, computing $\vec{x} = P\vec{y}$ we get that the required change of variables is

$$\begin{aligned} x_1 &= -\frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \\ x_2 &= \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \\ x_3 &= \frac{2}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \end{aligned}$$

(f) i. $A = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$

ii. We have

$$C(\lambda) = -\lambda(\lambda - 3)(\lambda + 3)$$

Hence, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = -3$ so the quadratic form and symmetric matrix are indefinite.

iii. The corresponding diagonal form is $Q = 3y_2^2 - 3y_3^2$

We find that a matrix which orthogonally diagonalizes A is $P = \begin{bmatrix} -1/3 & -2/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$ diagonalizes A. So, computing $\vec{x} = P\vec{y}$ we get that the required change of variables is

$$\begin{aligned} x_1 &= -\frac{1}{3}y_1 - \frac{2}{3}y_2 + \frac{2}{3}y_3 \\ x_2 &= \frac{2}{3}y_1 - \frac{2}{3}y_2 - \frac{1}{3}y_3 \\ x_3 &= \frac{2}{3}y_1 + \frac{1}{3}y_2 + \frac{2}{3}y_3 \end{aligned}$$

(g) i. $A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

ii. We have

$$C(\lambda) = -(\lambda - 5)^2(\lambda - 2)$$

Hence, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$ so the quadratic form and symmetric matrix are positive definite.

iii. The corresponding diagonal form is $Q = 5y_1^2 + 5y_2^2 + 2y_3^2$

We find that a matrix which orthogonally diagonalizes A is $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ diagonalizes A. So, computing $\vec{x} = P\vec{y}$ we get that the required change of variables is

$$\begin{aligned} x_1 &= -\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3 \\ x_2 &= \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3 \\ x_3 &= \frac{2}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \end{aligned}$$

(h) i. $A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & -1 \\ 1 & -1 & -4 \end{bmatrix}$

ii. We have

$$C(\lambda) = -(\lambda + 3)^2(\lambda + 6)$$

Hence, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -6$ so the quadratic form and symmetric matrix are negative definite.

iii. The corresponding diagonal form is $Q = -3y_1^2 - 3y_2^2 - 6y_3^2$

We find that a matrix which orthogonally diagonalizes A is $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ diagonalizes A. So, computing $\vec{x} = P\vec{y}$ we get that the required change of variables is

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3 \\ x_2 &= \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \\ x_3 &= \frac{2}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \end{aligned}$$

10.3.2 (a) Let λ_1, λ_2 be the eigenvalues of A. If $\det A > 0$ then $ac - b^2 > 0$ so a and c must both have the same sign. Thus, $c > 0$. We know that $\det A = \lambda_1\lambda_2$ and $\lambda_1 + \lambda_2 = a + c$ and so λ_1 and λ_2 must have the same sign since $\det A > 0$ and we have $\lambda_1 + \lambda_2 = a + c > 0$ so we must have λ_1 and λ_2 both positive so Q is positive definite.

(b) Let λ_1, λ_2 be the eigenvalues of A. If $\det A > 0$ then $ac - b^2 > 0$ so a and c must both have the same sign. Thus, $c < 0$. We know that $\det A = \lambda_1\lambda_2$ and $\lambda_1 + \lambda_2 = a + c$ and so λ_1 and λ_2 must have the same sign since $\det A > 0$ and we have $\lambda_1 + \lambda_2 = a + c < 0$ so we must have λ_1 and λ_2 both negative so Q is negative definite.

(c) Let λ_1, λ_2 be the eigenvalues of A. If $\det A < 0$ then $\det A = \lambda_1\lambda_2 < 0$ so λ_1 and λ_2 must have different signs thus Q is indefinite.

10.3.3 If $\langle\, , \rangle$ is an inner product on $\mathbb{R}^n$, then it is symmetric and positive definite. Observe that $\vec{y}^T A\vec{x}$ is a 1×1 matrix, and hence it is symmetric. Thus,

$$\vec{x}^T A\vec{y} = \langle \vec{x}, \vec{y}\rangle = \langle \vec{y}, \vec{x}\rangle = \vec{y}^T A\vec{x} = (\vec{y}^T A\vec{x})^T = \vec{x}^T A^T\vec{y}$$

Hence, $\vec{x}^T A\vec{y} = \vec{x}^T A^T\vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Therefore, $A = A^T$ by Theorem 3.1.4. Observe that

$$\vec{x}^T A\vec{x} = \langle \vec{x}, \vec{x}\rangle > 0$$

whenever $\vec{x} \neq \vec{0}$, so A is positive definite.

Assume that A is a positive definite, symmetric matrix. Let $\vec{x}, \vec{y}, \vec{w} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$. Then,

$$\langle \vec{x}, \vec{x}\rangle = \vec{x}^T A\vec{x} \geq 0$$

for all $\vec{x} \neq \vec{0}$ and $\langle \vec{x}, \vec{x}\rangle = 0$ if and only if $\vec{x} = \vec{0}$. Thus, $\langle\, , \rangle$ is positive definite.
We have

$$\langle \vec{x}, \vec{y}\rangle = \vec{x}^T A\vec{y} = (\vec{x}^T A\vec{y})^T = \vec{y}^T A^T\vec{x} = \vec{y}^T A\vec{x} = \langle \vec{y}, \vec{x}\rangle$$

So, $\langle\, , \rangle$ is symmetric.
We have

$$\begin{aligned} \langle s\vec{x} + t\vec{y}, \vec{w}\rangle = (s\vec{x} + t\vec{y})^T A\vec{w} &= (s\vec{x}^T + t\vec{y}^T)A\vec{w} = s\vec{x}^T A\vec{w} + t\vec{y}^T A\vec{w} \\ &= s\langle \vec{x}, \vec{w}\rangle + t\langle \vec{y}, \vec{w}\rangle \end{aligned}$$

So, $\langle\, , \rangle$ is also bilinear. Therefore, it is an inner product.

10.3.4 (a) If A is a positive definite symmetric matrix, then $\vec{x}^T A\vec{x} > 0$ for all $\vec{x} \neq \vec{0}$. Hence, for $1 \leq i \leq n$, we have

$$a_{ii} = \vec{e}_i^T A\vec{e}_i > 0$$

as required.

(b) If A is a positive definite symmetric matrix, then all the eigenvalues of A are positive. Thus, since the determinant of A is the product of the eigenvalues, we have that $\det A > 0$ and hence A is invertible.

10.3.5 We first must observe that $A + B$ is symmetric. Since the eigenvalues of A and B are all positive, the quadratic forms $\vec{x}^T A\vec{x}$ and $\vec{x}^T B\vec{x}$ are positive definite. Let $\vec{x} \neq \vec{0}$. Then $\vec{x}^T A\vec{x} > 0$ and $\vec{x}^T B\vec{x} > 0$, so $\vec{x}^T(A + B)\vec{x} = \vec{x}^T A\vec{x} + \vec{x}^T B\vec{x} > 0$, and the quadratic form $\vec{x}^T(A + B)\vec{x}$ is positive definite. Thus the eigenvalues of $A + B$ must be positive.

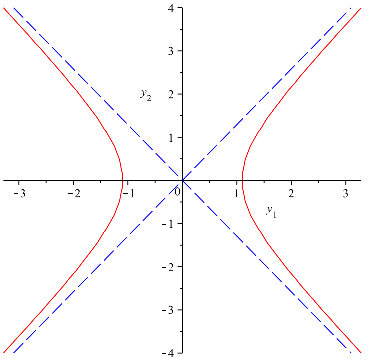
10.4 Problem Solutions

10.4.1 (a) The corresponding symmetric matrix is $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$. We find that the characteristic polynomial is $C(\lambda) = (\lambda - 5)(\lambda + 3)$. So, we have eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -3$.

For $\lambda_1 = 5$ we get $A - \lambda_1 I = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

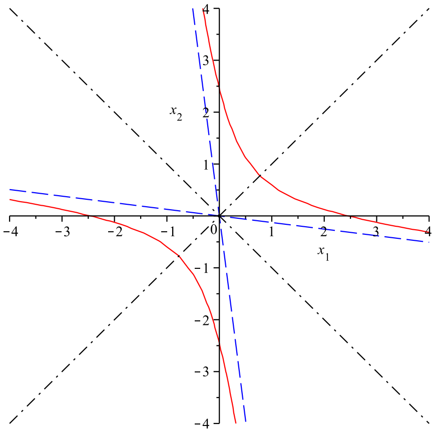
For λ_2 we get $A - \lambda_2 I = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Thus, we have the hyperbola $5y_1^2 - 3y_2^2 = 6$ with principal axis $\vec{v}_1$ for x_1 and $\vec{v}_2$ for y_1. The asymptotes of the hyperbola are when $0 = 5y_1^2 - 3y_2^2 \Rightarrow y_2 = \pm\sqrt{\frac{5}{3}}y_1$. Plotting this we get the graph in the y_1y_2-plane is



To sketch $x_1^2 + 8x_1x_2 + x_2^2 = 6$ in the x_1x_2-plane we first draw the y_1-axis in the direction of $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the y_2-axis in the direction of $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Next, to convert the asymptotes into the x_1x_2-plane, we use the change of variables $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\vec{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix}$. So, the asymptotes in the x_1x_2-plane are

$$
\begin{aligned}
y_2 &= \pm\sqrt{\frac{5}{3}}y_1 \\
\frac{1}{\sqrt{2}}(-x_1 + x_2) &= \pm\sqrt{\frac{5}{3}}\left(\frac{1}{\sqrt{2}}(x_1 + x_2)\right) \\
\sqrt{3}(-x_1 + x_2) &= \pm\sqrt{5}(x_1 + x_2) \\
(\sqrt{3} \mp \sqrt{5})x_2 &= (\sqrt{3} \pm \sqrt{5})x_1 \\
x_2 &= \frac{\sqrt{3} \pm \sqrt{5}}{\sqrt{3} \mp \sqrt{5}}x_1
\end{aligned}
$$



Plotting gives the graph to the right.

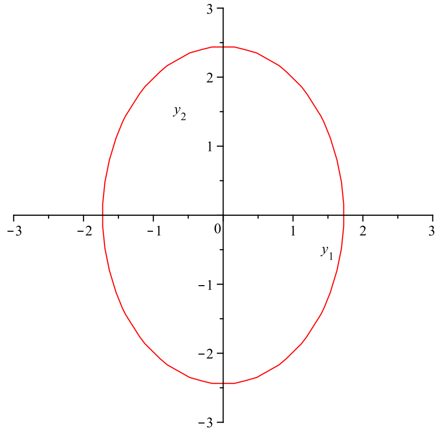
(b) The corresponding symmetric matrix is $\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. We find that the characteristic polynomial is $C(\lambda) = (\lambda - 4)(\lambda - 2)$. So, we have eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 2$.

For λ_1 we get $A - \lambda_1 I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Thus, a corresponding eigenvector is $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

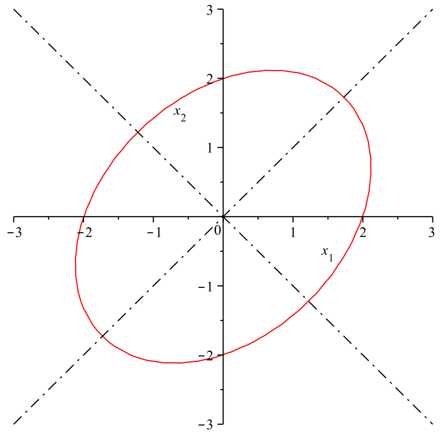
For λ_2 we get $A - \lambda_2 I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, we have the ellipse $4y_1^2 + y_2^2 = 12$ with principal axis $\vec{v}_1$ for x_1 and $\vec{v}_2$ for y_1.

Plotting this we get the graph in the y_1y_2-plane is



To sketch $3x_1^2 - 2x_1x_2 + 3x_2^2 = 12$ in the x_1x_2-plane we first draw the y_1-axis in the direction of $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and the y_2-axis in the direction of $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Plotting gives



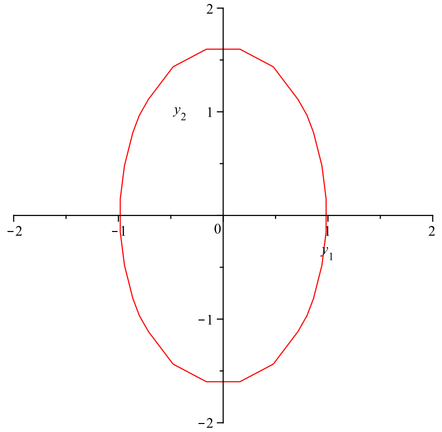
(c) The corresponding symmetric matrix is $\begin{bmatrix} -4 & 2 \\ 2 & -7 \end{bmatrix}$. We find that the characteristic polynomial is $C(\lambda) = (\lambda + 8)(\lambda + 3)$. So, we have eigenvalues $\lambda_1 = -8$ and $\lambda_2 = -3$.

For λ_1 we get $A - \lambda_1 I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

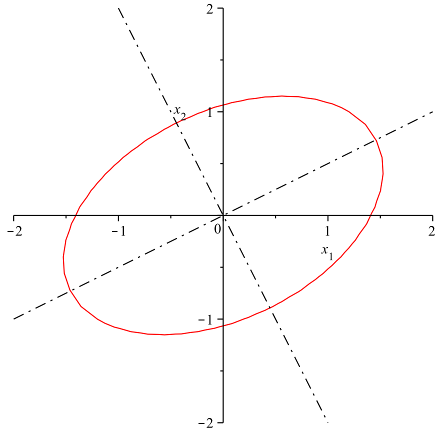
For λ_2 we get $A - \lambda_2 I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Thus, we have the ellipse $-8y_1^2 - 3y_2^2 = -8$ with principal axis $\vec{v}_1$ for x_1 and $\vec{v}_2$ for y_1.

Plotting this we get the graph in the y_1y_2-plane is



To sketch $-4x_1^2 + 4x_1x_2 - 7x_2^2 = -8$ in the x_1x_2-plane we first draw the y_1-axis in the direction of $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and the y_2-axis in the direction of $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Plotting gives

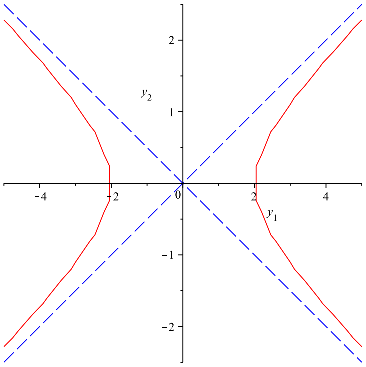


(d) The corresponding symmetric matrix is $\begin{bmatrix} -3 & -2 \\ -2 & 0 \end{bmatrix}$. We find that the characteristic polynomial is $C(\lambda) = (\lambda + 4)(\lambda - 1)$. So, we have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -4$.

For λ_1 we get $A - \lambda_1 I = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

For λ_2 we get $A - \lambda_2 I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

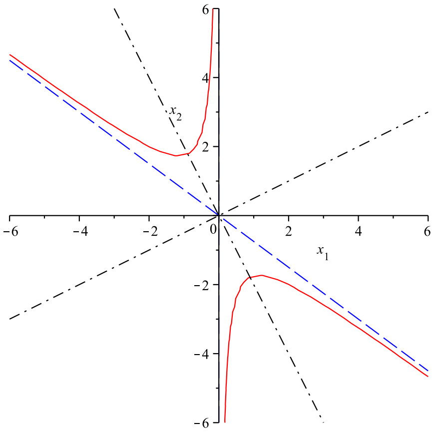
Thus, we have the hyperbola $y_1^2 - 4y_2^2 = 4$ with principal axis $\vec{v}_1$ for x_1 and $\vec{v}_2$ for y_1. The asymptotes of the hyperbola are when $0 = y_1^2 - 4y_2^2 \Rightarrow y_2 = \pm\frac{1}{2}y_1$. Plotting this we get the graph in the y_1y_2-plane is



To sketch $-3x_1^2 - 4x_1x_2 = 4$ in the x_1x_2-plane we first draw the y_1-axis in the direction of $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and the y_2-axis in the direction of $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Next, to convert the asymptotes into the x_1x_2-plane, we use the change of variables $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\vec{x} = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix}$. So, the asymptotes in the x_1x_2-plane are

$$\begin{aligned} y_2 &= \pm\frac{1}{2}y_1 \\ \frac{1}{\sqrt{5}}(2x_1 + x_2) &= \pm\frac{1}{2}\left(\frac{1}{\sqrt{5}}(-x_1 + 2x_2)\right) \\ 2(2x_1 + x_2) &= \pm(-x_1 + 2x_2) \\ (2 \mp 2)x_2 &= (-4 \mp 1)x_1 \end{aligned}$$

Thus, $x_1 = 0$ or $x_2 = -\frac{3}{4}x_1$. Plotting gives the graph to the right.

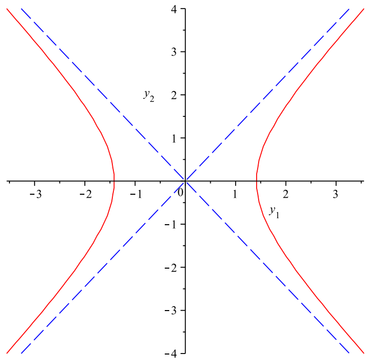


(e) The corresponding symmetric matrix is $\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$. We find that the characteristic polynomial is $C(\lambda) = (\lambda - 3)(\lambda + 2)$. So, we have eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$.

For λ_1 we get $A - \lambda_1 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

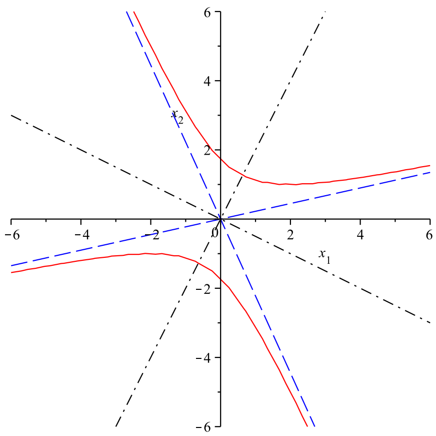
For λ_2 we get $A - \lambda_2 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Thus, we have the hyperbola $3y_1^2 - 2y_2^2 = 6$ with principal axis $\vec{v}_1$ for x_1 and $\vec{v}_2$ for y_1. The asymptotes of the hyperbola are when $0 = 3y_1^2 - 2y_2^2 \Rightarrow y_2 = \pm\sqrt{\frac{3}{2}}y_1$. Plotting this we get the graph in the y_1y_2-plane is



To sketch $-x_1^2 + 4x_1x_2 + 2x_2^2 = 6$ in the x_1x_2-plane we first draw the y_1-axis in the direction of $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the y_2-axis in the direction of $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Next, to convert the asymptotes into the x_1x_2-plane, we use the change of variables $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\vec{x} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + x_2 \end{bmatrix}$. So, the asymptotes in the x_1x_2-plane are

$$\begin{aligned}
y_2 &= \pm\sqrt{\frac{3}{2}}y_1 \\
\frac{1}{\sqrt{5}}(-2x_1 + x_2) &= \pm\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{5}}(x_1 + 2x_2)\right) \\
\sqrt{2}(-2x_1 + x_2) &= \pm\sqrt{3}(x_1 + 2x_2) \\
(\sqrt{2} \mp 2\sqrt{3})x_2 &= (2\sqrt{2} \pm \sqrt{3})x_1 \\
x_2 &= \frac{2\sqrt{2} \pm \sqrt{3}}{\sqrt{2} \mp 2\sqrt{3}}x_1
\end{aligned}$$



Plotting gives the graph to the right.

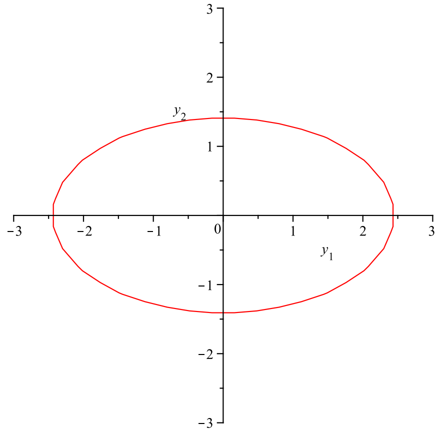
(f) The corresponding symmetric matrix is $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$. We find that the characteristic polynomial is $C(\lambda) = (\lambda - 2)(\lambda - 6)$. So, we have eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 6$.

For λ_1 we get $A - \lambda_1 I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

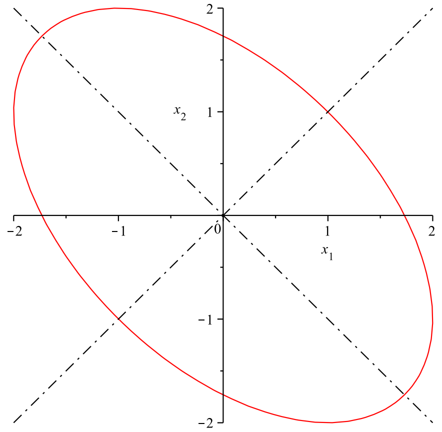
For λ_2 we get $A - \lambda_2 I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. So, a corresponding eigenvector is $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, we have the ellipse $2y_1^2 + 6y_2^2 = 12$ with principal axis $\vec{v}_1$ for x_1 and $\vec{v}_2$ for y_1.

Plotting this we get the graph in the y_1y_2-plane is



To sketch $4x_1^2 + 4x_1x_2 + 4x_2^2 = 12$ in the x_1x_2-plane we first draw the y_1-axis in the direction of $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and the y_2-axis in the direction of $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Plotting gives



(g) The corresponding symmetric matrix is $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus, A is diagonalized by $P =$

$\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$ to $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Hence, we have the hyperbola $3y_1^2 - 2y_2^2 = 6$ with principal axis $\vec{v}_1$, and $\vec{v}_2$. Since this is a hyperbola we need to graph the asymptotes. They are when $0 = 3y_1^2 - 2y_2^2$.

Hence, the equations of the asymptotes are $y_2 = \pm\sqrt{\frac{3}{2}}y_1$. Graphing give the diagram below to the left. We have

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\vec{x} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$$

So, the asymptotes in the x_1x_2-plane are given by

$$\frac{1}{\sqrt{5}}(x_1 + 2x_2) = \pm\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{5}}(-2x_1 + x_2)\right)$$

Solving for x_2 gives

$$x_2 = \frac{-1 \pm 2\sqrt{3/2}}{2 \pm \sqrt{3/2}}x_1$$

which is $x_2 \approx 0.449x_1$ and $x_2 \approx -4.449x_1$.

Alternately, we can find that the direction vectors of the asymptotes are

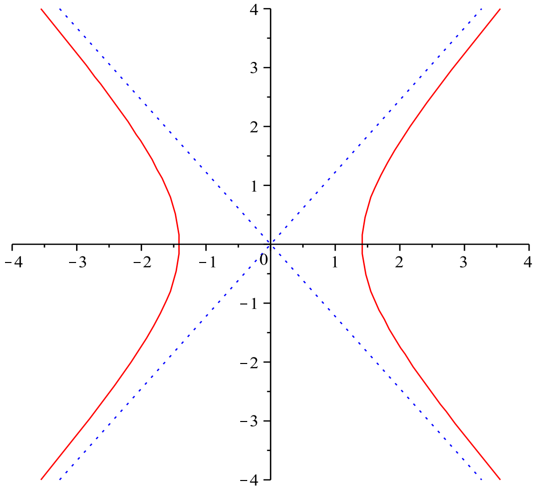
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P\vec{x} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ \sqrt{3/2} \end{bmatrix} = \begin{bmatrix} (-2 + \sqrt{3/2})/\sqrt{5} \\ (1 + 2\sqrt{3/2})/\sqrt{5} \end{bmatrix}$$

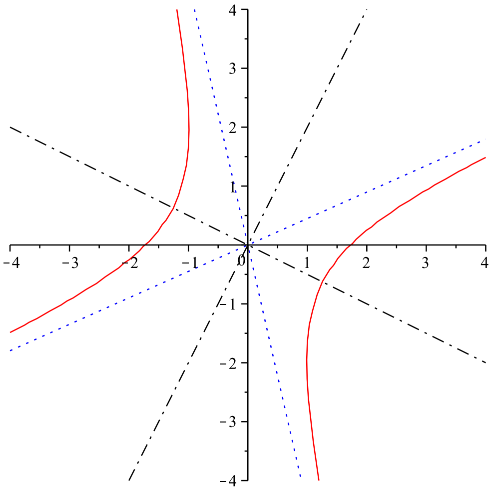
and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P\vec{x} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -\sqrt{3/2} \end{bmatrix} = \begin{bmatrix} (-2 - \sqrt{3/2})/\sqrt{5} \\ (1 - 2\sqrt{3/2})/\sqrt{5} \end{bmatrix}$$

These are the direction vectors of the lines $x_2 \approx -4.449x_1$ and $x_2 \approx 0.449x_1$.

Also, the principal axes are $\vec{v}_1$ and $\vec{v}_2$, so graphing this gives the diagram below to the right.



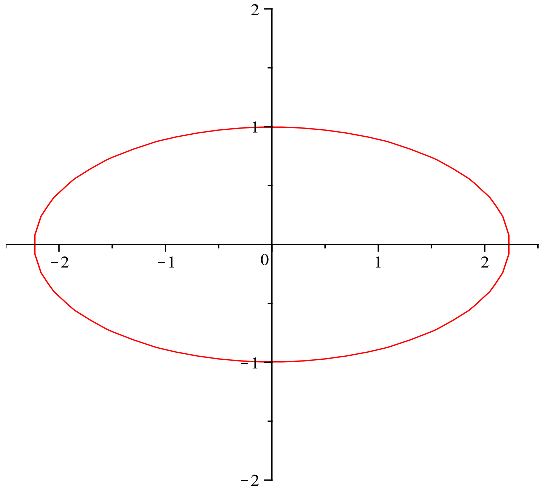


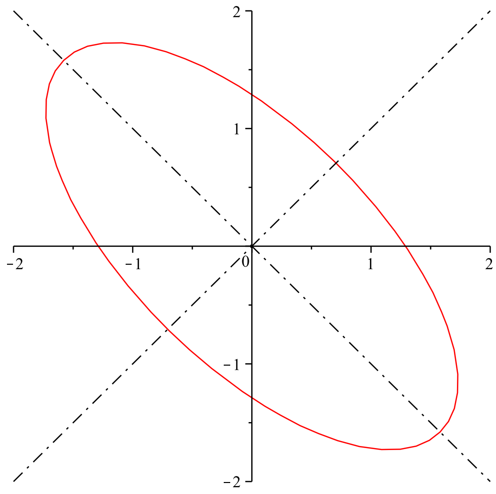
(h) The corresponding symmetric matrix is $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, A is orthogonally diagonalized by $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$. Thus, we have the ellipse

$$y_1^2 + 5y_2^2 = 5$$

Graphing this in the y_1y_2-plane gives the diagram below to the left.

The principal axes are $\vec{v}_1$ and $\vec{v}_2$. So, graphing it in the x_1x_2-plane gives the diagram below to the right.





10.5 Problem Solutions

10.5.1 (a) The corresponding symmetric matrix is $A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$. Hence, we have

$$C(\lambda) = (\lambda - 5)(\lambda - 3)$$

Hence, the eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 3$. Therefore, the maximum is 5 and the minimum is 3.

(b) The corresponding symmetric matrix is $A = \begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$. Hence, we have

$$C(\lambda) = (\lambda - \sqrt{34})(\lambda + \sqrt{34})$$

Hence, the eigenvalues of A are $\lambda_1 = \sqrt{34}$ and $\lambda_2 = -\sqrt{34}$. Therefore, the maximum is $\sqrt{34}$ and the minimum is $-\sqrt{34}$.

(c) The corresponding symmetric matrix is $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 8 & 2 \\ 0 & 2 & 11 \end{bmatrix}$. Hence, we have

$$C(\lambda) = (\lambda - 7)(\lambda + 1)(\lambda - 12)$$

Hence, the eigenvalues of A are $\lambda_1 = 7$, $\lambda_2 = -1$, and $\lambda_3 = 12$. Therefore, the maximum is 12 and the minimum is -1.

(d) The corresponding symmetric matrix is $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. Hence, we have

$$C(\lambda) = (\lambda - 7)^2(\lambda + 2)$$

Hence, the eigenvalues of A are $\lambda_1 = 7$ and $\lambda_2 = -2$. Therefore, the maximum is 7 and the minimum is -2.

(e) The corresponding symmetric matrix is $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Hence, we have

$$C(\lambda) = (\lambda - 1)^2(\lambda - 4)$$

Hence, the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 4$. Therefore, the maximum is 4 and the minimum is 1.

10.6 Problem Solutions

10.6.1 (a) We have $A^TA = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$. The eigenvalue of A^TA are (ordered from greatest to least) $\lambda_1 = 9$ and $\lambda_2 = 4$. Corresponding normalized eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ for λ_1 and $\vec{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ for λ_2. Hence, A^TA is orthogonally diagonalized by

$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. The singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 2$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Next compute

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

Thus we have $U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. Then $A = U\Sigma V^T$ as required.

(b) We have $A^TA = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}$. The eigenvalue of A^TA are (ordered from greatest to least) $\lambda_1 = 100$ and $\lambda_2 = 25$. Corresponding normalized eigenvectors are $\vec{v}_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$ for λ_1 and $\vec{v}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ for λ_2. Hence, A^TA is orthogonally diagonalized by

$V = \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$. The singular values of A are $\sigma_1 = 10$ and $\sigma_2 = 5$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$. Next compute

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{10}\begin{bmatrix} 3 & -4 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus we have $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A = U\Sigma V^T$ as required.

(c) We have $A^TA = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$. The eigenvalue of A^TA are (ordered from greatest to least) $\lambda_1 = 16$ and $\lambda_2 = 1$. Corresponding normalized eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ for λ_1 and $\vec{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ for λ_2. Hence, A^TA is orthogonally diagonalized by

$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. The singular values of A are $\sigma_1 = 4$ and $\sigma_2 = 1$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$. Next compute

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{4}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{1}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

Thus we have $U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. Then $A = U\Sigma V^T$ as required.

(d) We have $A^TA = \begin{bmatrix} 6 & 6 \\ 6 & 11 \end{bmatrix}$. The eigenvalues of A^TA are (ordered from greatest to least) $\lambda_1 = 15$ and $\lambda_2 = 2$.

Normalized eigenvectors are $\vec{v}_1 = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}$ for λ_1 and $\vec{v}_2 = \begin{bmatrix} -3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix}$ for λ_2.

Hence, A^TA is orthogonally diagonalized by $V = \begin{bmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$.

The singular values of A are $\sigma_1 = \sqrt{15}$ and $\sigma_2 = \sqrt{2}$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} \sqrt{15} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$.

Next compute

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{15}}\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix} = \begin{bmatrix} 11/\sqrt{195} \\ 7/\sqrt{195} \\ 5/\sqrt{195} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3/\sqrt{13} \\ 2/\sqrt{13} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{26} \\ -4/\sqrt{26} \\ -1/\sqrt{26} \end{bmatrix}$$

We then need to extend $\{\vec{u}_1, \vec{u}_2\}$ to an orthonormal basis for $\mathbb{R}^3$. Since we are in $\mathbb{R}^3$, we can use the cross product. We have

$$\begin{bmatrix} 11 \\ 7 \\ 5 \end{bmatrix} \times \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ 26 \\ -65 \end{bmatrix}$$

So, we can take $\vec{u}_3 = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}$. Thus, we have $U = \begin{bmatrix} 11/\sqrt{95} & 3/\sqrt{26} & 1/\sqrt{30} \\ 7/\sqrt{195} & -4/\sqrt{26} & 2/\sqrt{30} \\ 5/\sqrt{195} & -1/\sqrt{26} & -5/\sqrt{30} \end{bmatrix}$. Then, $A = U\Sigma V^T$ as required.

(e) We have $A^TA = \begin{bmatrix} 12 & -6 \\ -6 & 3 \end{bmatrix}$. The eigenvalues of A^TA are (ordered from greatest to least) $\lambda_1 = 15$ and $\lambda_2 = 0$.

Corresponding normalized eigenvectors are $\vec{v}_1 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ for λ_1 and $\vec{v}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ for λ_2.

Hence, A^TA is orthogonally diagonalized by $V = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$.

The singular values of A are $\sigma_1 = \sqrt{15}$ and $\sigma_2 = 0$. Thus, the matrix Σ is $\Sigma = \begin{bmatrix} \sqrt{15} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Next compute

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{15}} \begin{bmatrix} 2 & -1 \\ 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

We then need to extend $\{\vec{u}_1\}$ to an orthonormal basis for $\mathbb{R}^3$. One way of doing this is to find an orthonormal basis for Null(A^T). A basis for Null(A^T) is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. To make this an orthogonal basis, we apply the Gram-Schmidt procedure. We take $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and then we get

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Thus, we pick $\vec{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\vec{u}_3 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2\sqrt{6} \end{bmatrix}$. Consequently, we have
$U = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$ and $A = U\Sigma V^T$ as required.

(f) We have $A^TA = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. The only eigenvalue of A^TA is $\lambda_1 = 3$ with multiplicity 3.

Since A^TA is diagonal, we can take $V = I$.

The singular values of A are $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{3}$, and $\sigma_3 = \sqrt{3}$. So, we get that $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$.

Next compute

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sigma_3}A\vec{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

We then need to extend $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ to an orthonormal basis for $\mathbb{R}^4$. We find that a basis for Null(A^T) is $\left\{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$. After normalizing this vector, we take
$U = \begin{bmatrix} -1/\sqrt{3} & 0 & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$ and $A = U\Sigma V^T$ as required.

(g) We have $A^TA = \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix}$. The eigenvalue of A^TA are (ordered from greatest to least) $\lambda_1 = 20$ and $\lambda_2 = 0$. Corresponding normalized eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ for λ_1 and $\vec{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ for λ_2. Hence, A^TA is orthogonally diagonalized by
$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{20}$ and $\sigma_2 = 1$. Thus the matrix Σ is
$\Sigma = \begin{bmatrix} \sqrt{20} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Next compute

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{20}}\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

We then need to extend $\{\vec{u}_1\}$ to an orthonormal basis for $\mathbb{R}^4$. We pick

$$\vec{u}_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Thus, we have $U = \begin{bmatrix} 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}$. Then $A = U\Sigma V^T$ as required.

(h) Observe that the matrix is the transpose of the matrix in part (d). Hence, we have

$$A = \left(\begin{bmatrix} 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \sqrt{20} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

Thus, we take

$$U = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}$$

and get a singular value decomposition $A = U\Sigma V^T$.

NOTE: Make sure that you have the correct matrix V.

(i) It would be difficult to find the SVD of A directly. Therefore, as we did in part (e), we will first find the SVD of A^T.

We have $(A^T)^T A^T = AA^T = \begin{bmatrix} 36 & 12 \\ 12 & 29 \end{bmatrix}$. The eigenvalue of A^TA are (ordered from greatest to least) $\lambda_1 = 45$ and $\lambda_2 = 20$. Corresponding normalized eigenvectors are $\vec{v}_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$ for λ_1 and $\vec{v}_2 = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ for λ_2. Hence, A^TA is orthogonally diagonalized by $V = \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{45}$ and $\sigma_2 = \sqrt{20}$. Thus the matrix Σ is

$\Sigma = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{20} \\ 0 & 0 \end{bmatrix}$. Next compute

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{45}}\begin{bmatrix} 4 & 3 \\ 4 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{45} \\ 22/\sqrt{1125} \\ 4/\sqrt{45} \end{bmatrix}$$
$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{\sqrt{20}}\begin{bmatrix} 4 & 3 \\ 4 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/\sqrt{125} \\ 11/\sqrt{125} \end{bmatrix}$$

We then need to extend $\{\vec{u}_1, \vec{u}_2\}$ to an orthonormal basis for $\mathbb{R}^3$. We pick

$$\vec{u}_3 = \vec{u}_1 \times \vec{u}_2 = \begin{bmatrix} 2/3 \\ -11/15 \\ 2/15 \end{bmatrix}$$

Thus, we have $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$. Then, $A = (A^T)^T = (U\Sigma V^T)^T = V\Sigma^T U^T$.

10.6.2 (a) We have $A^TA = \begin{bmatrix} 40 & 0 \\ 0 & 10 \end{bmatrix}$. Thus, the eigenvalues of A^TA are $\lambda_1 = 40$ and $\lambda_2 = 10$. Hence, the maximum of $\|A\vec{x}\|$ subject to $\|\vec{x}\| = 1$ is $\sqrt{40}$.

(b) We have $A^TA = \begin{bmatrix} 6 & -3 \\ -3 & 14 \end{bmatrix}$. Thus,

$$C(\lambda) = \lambda^2 - 20\lambda + 75 = (\lambda - 15)(\lambda - 5)$$

Hence, the eigenvalues of A^TA are $\lambda_1 = 15$ and $\lambda_2 = 5$. So, the maximum of $\|A\vec{x}\|$ subject to $\|\vec{x}\| = 1$ is $\sqrt{15}$.

(c) We have seen that A^TA and AA^T have the same non-zero eigenvalues. Hence, we find that $AA^T = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix}$, and see by inspection that the eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 19$. Hence, the maximum of $\|A\vec{x}\|$ subject to $\|\vec{x}\| = 1$ is $\sqrt{19}$.

10.6.3 By definition, the singular values of PA are the eigenvalues of $(PA)^T(PA) = A^TP^TPA = A^TA$. But, the eigenvalues of A^TA are the singular values of A as required.

10.6.4 If $\vec{u}$ is a left singular vector of A, then $A^T\vec{u} = \vec{0}$, or $A^T\vec{u} = \sigma\vec{v}$ and $A\vec{v} = \sigma\vec{u}$.

If $A^T\vec{u} = \vec{0}$, then $AA^T\vec{u} = A\vec{0} = \vec{0} = 0\vec{u}$, so $\vec{u}$ is an eigenvector of AA^T.

If $A^T\vec{u} = \sigma\vec{v}$, then
$$AA^T\vec{u} = A(\sigma\vec{v}) = \sigma(A\vec{v}) = \sigma(\sigma\vec{u}) = \sigma^2\vec{u}$$

Hence, $\vec{u}$ is an eigenvector of AA^T.

On the other hand, assume that $\vec{u}$ is an eigenvector of AA^T with corresponding eigenvalue σ^2.

If $\sigma \neq 0$, then define $\vec{v} = \frac{1}{\sigma}A^T\vec{u}$. Thus, we $A^T\vec{u} = \sigma\vec{v}$ and

$$\begin{aligned} AA^T\vec{u} &= \sigma^2\vec{u} \\ A(\sigma\vec{v}) &= \sigma^2\vec{u} \\ A\vec{v} &= \sigma\vec{u} \end{aligned}$$

Thus, $\vec{u}$ is a left singular vector of A.

If $\sigma = 0$, then we have $AA^T\vec{u} = \vec{0}$ and hence

$$\|A^T\vec{u}\|^2 = \langle A^T\vec{u}, A^T\vec{u}\rangle = (A^T\vec{u})^T(A^T\vec{u}) = \vec{u}^TAA^T\vec{u} = \vec{u}^T\vec{0} = 0$$

Thus, $A^T\vec{u} = \vec{0}$ and so $\vec{u}$ is a left singular vector of A.

Replacing A by A^T in our result above gives that $\vec{v}$ is a right singular vector if and only if $\vec{v}$ is an eigenvector of A^TA.

10.6.5 Let $U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$ and $V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$.

By Lemma 10.6.7, $\{\vec{u}_1, \dots, \vec{u}_r\}$ forms an orthonormal basis for Col(A).

By the Fundamental Theorem of Linear Algebra, the nullspace of A^T is the orthogonal complement of Col(A). We are given that $\{\vec{u}_1, \dots, \vec{u}_m\}$ forms an orthonormal basis for $\mathbb{R}^n$ and hence a basis for the orthogonal compliment of Span$\{\vec{u}_1, \dots, \vec{u}_r\}$ is $B = \{\vec{u}_{r+1}, \dots, \vec{u}_m\}$. Thus B is a basis for Null(A^T).

By Theorem 10.6.1, we have that $\|A\vec{v}_i\| = \sqrt{\lambda_i}$. But, since $U\Sigma V^T$ is a singular value decomposition of A, we have that $\lambda_{r+1}, \dots, \lambda_n$ are the zero eigenvalues of A. Hence, $\{\vec{v}_{r+1}, \dots, \vec{v}_n\}$ is an orthonormal set of $n - r$ vectors in the nullspace of A. But, since A has rank r, we know by the Rank-Nullity Theorem, that $\dim \text{Null}(A) = n - r$. So $\{\vec{v}_{r+1}, \dots, \vec{v}_n\}$ is a basis for Null(A).

Hence, by the Fundamental Theorem of Linear Algebra we have that the orthogonal complement of Null(A) is Row(A). Hence $\{\vec{v}_1, \dots, \vec{v}_r\}$ forms a basis for Row(A).

10.6.6 (a) We get $A^TA = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix}$ which has eigenvalues $\lambda_1 = 8$, $\lambda_2 = 4$, and $\lambda_3 = 2$ and corresponding unit eigenvectors $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$. Thus, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(b) We have

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 4/\sqrt{24} \\ 2/\sqrt{24} \\ 2/\sqrt{24} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sigma_3}A\vec{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -2/\sqrt{12} \\ 2/\sqrt{12} \\ 2/\sqrt{12} \end{bmatrix}$$

Hence, $\mathcal{C} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

(c) We have

$$L(\vec{v}_1) = A\vec{v}_1 = \begin{bmatrix} 4/\sqrt{3} \\ 2/\sqrt{3} \\ 2/\sqrt{3} \end{bmatrix} = \sqrt{8}\begin{bmatrix} 4/\sqrt{24} \\ 2/\sqrt{24} \\ 2/\sqrt{24} \end{bmatrix} + 0\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + 0\begin{bmatrix} -2/\sqrt{12} \\ 2/\sqrt{12} \\ 2/\sqrt{12} \end{bmatrix}$$

$$L(\vec{v}_2) = A\vec{v}_2 = \begin{bmatrix} 0 \\ -2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = 0\begin{bmatrix} 4/\sqrt{24} \\ 2/\sqrt{24} \\ 2/\sqrt{24} \end{bmatrix} + 2\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + 0\begin{bmatrix} -2/\sqrt{12} \\ 2/\sqrt{12} \\ 2/\sqrt{12} \end{bmatrix}$$

$$L(\vec{v}_3) = A\vec{v}_3 = \begin{bmatrix} -2/\sqrt{6} \\ 2/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = 0\begin{bmatrix} 4/\sqrt{24} \\ 2/\sqrt{24} \\ 2/\sqrt{24} \end{bmatrix} + 0\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \sqrt{2}\begin{bmatrix} -2/\sqrt{12} \\ 2/\sqrt{12} \\ 2/\sqrt{12} \end{bmatrix}$$

Hence,

$$_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

NOTE: Observe that A is not even diagonalizable.

10.6.7 Using block multiplication, we get

$$\begin{aligned} A &= U\Sigma V^T \\ &= \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & \sigma_r & \ddots & & \vdots \\ \vdots & & \ddots & 0 & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1\vec{u}_1 & \cdots & \sigma_r\vec{u}_r & \vec{0} & \cdots & \vec{0} \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix} \\ &= \sigma_1\vec{u}_1\vec{v}_1^T + \cdots + \sigma_r\vec{u}_r\vec{v}_r^T \end{aligned}$$

Chapter 11 Solutions

11.1 Problem Solutions

11.1.1 (a) $(3+4i)-(-2+6i) = 5-2i$

(b) $(2-3i)+(1+3i) = 3$

(c) $(-1+2i)(3+2i) = -7+4i$

(d) $-3i(-2+3i) = 9+6i$

(e) $\overline{3-7i} = 3+7i$

(f) $\overline{(1+i)(1-i)} = \overline{2} = 2$

(g) $|(1+i)(1-2i)(3+4i)| = |1+i|\,|1-2i|\,|3+4i| = \sqrt{2}(\sqrt{5})(5) = 5\sqrt{10}$

(h) $|1+6i| = \sqrt{37}$

(i) $|\frac{2}{3} - \sqrt{2}i| = \frac{1}{3}\sqrt{22}$

(j) $\frac{2}{1-i} = \frac{2}{1-i}\frac{1+i}{1+i} = \frac{2(1+i)}{2} = 1+i$

(k) $\frac{4-3i}{3-4i} = \frac{4-3i}{3-4i}\frac{3+4i}{3+4i} = \frac{24}{25} + \frac{7i}{25}$

(l) $\frac{2+5i}{-3-6i} = \frac{2+5i}{-3-6i}\frac{-3+6i}{-3+6i} = -\frac{4}{5} - \frac{1}{15}i$

11.1.2 (a) We have

$$\begin{aligned} 2iz &= 4 \\ z &= -2i \end{aligned}$$

(b) We have

$$\begin{aligned} 1-z &= \frac{1}{1-5i} \\ -z &= -1 + \frac{1+5i}{26} \\ z &= \frac{25}{26} - \frac{5}{26}i \end{aligned}$$

(c) We have

$$\begin{aligned}(1+i)z+(2+i)z&=2\\(3+2i)z&=2\\z&=\frac{2}{3+2i}\\z&=\frac{6}{13}-\frac{4}{13}i\end{aligned}$$

11.1.3 (a) Row reducing gives

$$\left[\begin{array}{cccc|c}1&2+i&i&0&1+i\\i&-1+2i&0&2i&-i\\1&2+i&1+i&2i&2-i\end{array}\right]\begin{array}{l}\\R_2-iR_1\\R_3-R_1\end{array}\sim\left[\begin{array}{cccc|c}1&2+i&i&0&1+i\\0&0&1&2i&1-2i\\0&0&1&2i&1-2i\end{array}\right]\begin{array}{l}R_1-iR_2\\\\R_3-R_2\end{array}\sim$$

$$\left[\begin{array}{cccc|c}1&2+i&0&2&-1\\0&0&1&2i&1-2i\\0&0&0&0&0\end{array}\right]$$

Hence, the system is consistent with two parameters. Let $z_2=s\in\mathbb{C}$ and $z_4=t\in\mathbb{C}$. Then, the general solution is

$$\vec{z}=\begin{bmatrix}-1\\0\\1-2i\\0\end{bmatrix}+s\begin{bmatrix}-2-i\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-2\\0\\-2i\\1\end{bmatrix}$$

(b) Row reducing gives

$$\left[\begin{array}{ccc|c}i&2&-3-i&1\\1+i&2-2i&-4&i\\i&2&-3-3i&1+2i\end{array}\right]\begin{array}{l}\\R_2-R_1\\R_3-R_1\end{array}\sim\left[\begin{array}{ccc|c}i&2&-3-i&1\\1&-2i&-1+i&-1+i\\0&0&-2i&2i\end{array}\right]\begin{array}{l}R_1-iR_2\\\\\frac{1}{2}iR_3\end{array}\sim$$

$$\left[\begin{array}{ccc|c}0&0&-2&2+i\\1&-2i&-1+i&-1+i\\0&0&1&-1\end{array}\right]\begin{array}{l}R_1+2R_3\\\\\\end{array}\sim\left[\begin{array}{ccc|c}0&0&0&i\\1&-2i&-1+i&-1+i\\0&0&1&-1\end{array}\right]$$

Hence, the system is inconsistent.

(c) Row reducing gives

$$\left[\begin{array}{ccc|c}i&1+i&1&2i\\1-i&1-2i&-2+i&-2+i\\2i&2i&2&4+2i\end{array}\right]\begin{array}{l}-iR_3\\\\-\frac{1}{2}iR_3\end{array}\sim\left[\begin{array}{ccc|c}1&1-i&-i&2\\1-i&1-2i&-2+i&-2+i\\1&1&-i&1-2i\end{array}\right]\begin{array}{l}\\R_2+(1-i)R_1\\R_3-R_1\end{array}\sim$$

$$\left[\begin{array}{ccc|c}1&1-i&-i&2\\0&1&-1+2i&-4+3i\\0&i&0&-1-2i\end{array}\right]\begin{array}{l}R_1-(1-i)R_2\\\\R_3-iR_2\end{array}\sim\left[\begin{array}{ccc|c}1&0&-1-4i&3-7i\\0&1&-1+2i&-4+3i\\0&0&2+i&2+2i\end{array}\right]\begin{array}{l}\\\\\frac{1}{2+i}R_3\end{array}\sim$$

$$\left[\begin{array}{ccc|c}1&0&-1-4i&3-7i\\0&1&-1+2i&-4+3i\\0&0&1&\frac{6}{5}+\frac{2}{5}i\end{array}\right]\begin{array}{l}R_1+(1+4i)R_3\\R_2+(1-2i)R_2\\\\\end{array}\sim\left[\begin{array}{ccc|c}1&0&0&\frac{13}{5}-\frac{9}{5}i\\0&1&0&-2+i\\0&0&1&\frac{6}{5}+\frac{2}{5}i\end{array}\right]$$

Hence, the solution is $z_1=\frac{13}{5}-\frac{9}{5}i$, $z_2=-2+i$, and $z_3=\frac{6}{5}+\frac{2}{5}i$.

(d) Row reducing gives

$$\left[\begin{array}{ccc|c} 1 & -1 & i & 2i \\ 1+i & -i & i & -2+i \\ 1-i & -1+2i & 1+2i & 3+2i \end{array}\right] \begin{array}{l} \\ R_2-(1+i)R_1 \\ R_3-(1-i)R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & i & 2i \\ 0 & 1 & 1 & -i \\ 0 & i & i & 1 \end{array}\right] \begin{array}{l} R_1-R_2 \\ \\ R_3-iR_2 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1+i & i \\ 0 & 1 & 1 & -i \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, the system is consistent with one parameter. Let $z_3 = t \in \mathbb{C}$. Then, the general solution is

$$\vec{z} = \begin{bmatrix} i \\ -i \\ 0 \end{bmatrix} + t \begin{bmatrix} -1-i \\ -1 \\ 1 \end{bmatrix}$$

11.1.4 (a) Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, and $z_3 = a_3 + b_3 i$. We have

$$\begin{aligned} z_1(z_2+z_3) &= (a_1+b_1i)(a_2+b_2i+a_3+b_3i) \\ &= (a_1+b_1i)[(a_2+a_3)+(b_2+b_3)i] \\ &= a_1(a_2+a_3) - b_1(b_2+b_3) + b_1(a_2+a_3)i + a_1(b_2+b_3)i \\ &= a_1a_2 - b_1b_2 + b_1a_2i + a_1b_2i + a_1a_3 - b_1b_3 + b_1a_3i + a_1b_3i \\ &= (a_1+b_1i)(a_2+b_2i) + (a_1+b_1i)(a_3+b_3i) \\ &= z_1z_2 + z_1z_3 \end{aligned}$$

(b) Let $z = a+bi$. Define $\frac{1}{z} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. Then $\frac{1}{z} \in \mathbb{C}$ since $\frac{a}{a^2+b^2} \in \mathbb{R}$ and $\frac{-b}{a^2+b^2} \in \mathbb{R}$. Also, we have

$$\begin{aligned} z(1/z) &= (a+bi)\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i\right) \\ &= a\left(\frac{a}{a^2+b^2}\right) - b\left(\frac{-b}{a^2+b^2}\right) + \left[a\frac{-b}{a^2+b^2} + b\frac{a}{a^2+b^2}\right]i \\ &= \frac{a^2+b^2}{a^2+b^2} + 0i \\ &= 1 \end{aligned}$$

(c) If $z_1 = a+bi \in \mathbb{R}$, then $b = 0$. Hence, $\overline{z_1} = \overline{a+0i} = a - 0i = a + 0i = z_1$

On the other hand, assume that $\overline{z_1} = z_1$. Then, we have $a - bi = a + bi$ which implies that $b = 0$. Thus, $z_1 \in \mathbb{R}$.

(d) Let $z_1 = a+bi$ and $z_2 = c+di$. Then,

$$\overline{z_1z_2} = \overline{ac-bd+(ad+bc)i} = ac-bd-(ad+bc)i = ac-bd+(-ad-bc)i = (a-bi)(c-di) = \overline{z_1}\,\overline{z_2}$$

(e) We have

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2 \\
&\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

11.1.5 Let $z_1 = a + bi$ and $z_2 = c + di$. If $z_1 + z_2$ is a negative real number, then $b = -d$ and $a + c < 0$. Also, we have $z_1z_2 = (ac - bd) + (ad + bc)i$ is a negative real number, so $ad + bc = 0$ and $ac - bd < 0$. Combining these we get

$$0 = ad + bc = ad - dc = d(a - c)$$

so either $d = 0$ or $a = c$. If $a = c$, then $ac - bd < 0$ implies that $a^2 + b^2 < 0$ which is impossible. Hence, we must have $d = 0$. But then $b = -d = 0$ so z_1 and z_2 are real numbers.

11.1.6 Let $z = a + bi$. Then $a^2 + b^2 = 1$. Hence

$$\begin{aligned}
\frac{1}{1-z} &= \frac{1}{1-a-bi} = \frac{1}{1-a-bi}\,\frac{1-a+bi}{1-a+bi} = \frac{1-a+bi}{(1-a)^2+b^2} \\
&= \frac{1-a+bi}{1-2a+a^2+b^2} = \frac{1-a+bi}{1-2a+1} \\
&= \frac{1-a}{2-2a} + \frac{b}{2-2a}i
\end{aligned}$$

Hence, $\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}$.

11.1.7 Let $z = a + bi$. We will prove this by induction on n. If $n = 1$, then $\overline{z} = \overline{z}$. Assume that $\overline{z^k} = (\overline{z})^k$ for some integer k. Then, by Theorem 5.1.3 part 5,

$$\overline{z^{k+1}} = \overline{z^k z} = \overline{z^k}\,\overline{z} = (\overline{z})^k\overline{z} = (\overline{z})^{k+1}$$

as required.

11.2 Problem Solutions

11.2.1 (a) By definition $\mathbb{S}_1$ is a subset of $\mathbb{C}^3$ and $\vec{0} \in \mathbb{S}_1$ since $i(0) = 0$. Let $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{S}_1$. Then, $iz_1 = z_3$ and $iy_1 = y_3$. Hence, $\vec{z} + \vec{y} = \begin{bmatrix} z_1 + y_1 \\ z_2 + y_2 \\ z_3 + y_3 \end{bmatrix} \in \mathbb{S}_1$ since $i(z_1 + y_1) = iz_1 + iy_1 = z_3 + y_3$. Similarly,

$\alpha\vec{z} = \begin{bmatrix}\alpha z_1\\ \alpha z_2\\ \alpha z_3\end{bmatrix} \in \mathbb{S}_1$ since $i(\alpha z_1) = \alpha(iz_1) = \alpha z_3$. Therefore, by the Subspace Test, $\mathbb{S}_1$ is a subspace of $\mathbb{C}^3$.

Every $\vec{z} \in \mathbb{S}_1$ has the form

$$\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}z_1\\ z_2\\ iz_1\end{bmatrix} = z_1\begin{bmatrix}1\\0\\i\end{bmatrix} + z_2\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\i\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ spans $\mathbb{S}_1$ and is clearly linearly independent, so it is a basis for $\mathbb{S}_1$. Consequently, $\dim \mathbb{S}_1 = 2$.

(b) Observe that $\begin{bmatrix}1\\0\\0\end{bmatrix} \in \mathbb{S}_2$ since $1(0) = 0$ and $\begin{bmatrix}0\\1\\0\end{bmatrix} \in \mathbb{S}_2$ since $0(1) = 0$. However, $\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} \notin \mathbb{S}_2$ since $1(1) \neq 0$. Consequently, $\mathbb{S}_2$ is not closed under addition and hence is not a subspace.

(c) By definition $\mathbb{S}_3$ is a subset of $\mathbb{C}^3$ and $\vec{0} \in \mathbb{S}_3$ since $0 + 0 + 0 = 0$. Let $\vec{z} = \begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}, \vec{y} = \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} \in \mathbb{S}_3$.

Then, $z_1 + z_2 + z_3 = 0$ and $y_1 + y_2 + y_3 = 0$. Hence, $\vec{z} + \vec{y} = \begin{bmatrix}z_1 + y_1\\ z_2 + y_2\\ z_3 + y_3\end{bmatrix} \in \mathbb{S}_1$ since $(z_1 + y_1) + (z_2 + y_2) + (z_3 + y_3) = z_1 + z_2 + z_3 + y_1 + y_2 + y_3 = 0 + 0 = 0$. Similarly, $\alpha\vec{z} = \begin{bmatrix}\alpha z_1\\ \alpha z_2\\ \alpha z_3\end{bmatrix} \in \mathbb{S}_1$ since $\alpha z_1 + \alpha z_2 + \alpha z_3 = \alpha(z_1 + z_2 + z_3) = \alpha(0) = 0$. Therefore, by the Subspace Test, $\mathbb{S}_3$ is a subspace of $\mathbb{C}^3$.

Every $\vec{z} \in \mathbb{S}_1$ has the form

$$\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}z_1\\ z_2\\ -z_1 - z_2\end{bmatrix} = z_1\begin{bmatrix}1\\0\\1\end{bmatrix} + z_2\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Thus, $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$ spans $\mathbb{S}_3$ and is clearly linearly independent, so it is a basis for $\mathbb{S}_3$. Consequently, $\dim \mathbb{S}_3 = 2$.

11.2.2 (a) Row reducing the matrix A to its reduced row echelon form R gives

$$\begin{bmatrix}1 & i\\ 1+i & -1+i\\ -1 & i\end{bmatrix} \sim \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

The non-zero rows of R form a basis for the rowspace of A. Hence, a basis for Row(A) is

$$\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

The columns from A which correspond to columns in R which have leadings ones form a basis for the columnspace of A. So, a basis for Col(A) is

$$\left\{\begin{bmatrix}1\\1+i\\-1\end{bmatrix},\begin{bmatrix}i\\-1+i\\i\end{bmatrix}\right\}$$

Solve the homogeneous system $A\vec{z}=\vec{0}$, we find that a basis for Null(A) is the empty set.

To find a basis for the left nullspace, we row reduce A^T. We get

$$\begin{bmatrix}1 & 1+i & -1\\ i & -1+i & i\end{bmatrix}\sim\begin{bmatrix}1 & 1+i & 0\\ 0 & 0 & 1\end{bmatrix}$$

Solving $A^T\vec{z}=\vec{0}$, we find that a basis for Null(A^T) is

$$\left\{\begin{bmatrix}-1-i\\1\\0\end{bmatrix}\right\}$$

(b) Row reducing the matrix B to its reduced row echelon form R gives

$$\begin{bmatrix}1 & 1+i & -1\\ i & -1+i & i\end{bmatrix}\sim\begin{bmatrix}1 & 1+i & 0\\ 0 & 0 & 1\end{bmatrix}$$

The non-zero rows of R form a basis for the rowspace of B. Hence, a basis for Row(B) is

$$\left\{\begin{bmatrix}1\\1+i\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

The columns from B which correspond to columns in R which have leadings ones form a basis for the columnspace of B. So, a basis for Col(B) is

$$\left\{\begin{bmatrix}1\\i\end{bmatrix},\begin{bmatrix}-1\\i\end{bmatrix}\right\}$$

Solve the homogeneous system $B\vec{z}=\vec{0}$, we find that a basis for Null(B) is $\left\{\begin{bmatrix}-1-i\\1\\0\end{bmatrix}\right\}$.

To find a basis for the left nullspace, we row reduce B^T. We get

$$\begin{bmatrix}1 & i\\ 1+i & -1+i\\ -1 & i\end{bmatrix}\sim\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

Therefore, a basis for Null(B^T) is the empty set.

(c) Row reducing the matrix A to its reduced row echelon form R gives

$$\begin{bmatrix} 1 & 1+i & 3 \\ 0 & 2 & 2-2i \\ i & 1-i & -i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1-i \\ 0 & 0 & 0 \end{bmatrix}$$

The non-zero rows of R form a basis for the rowspace of A. Hence, a basis for Row(A) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The columns from A which correspond to columns in R which have leadings ones form a basis for the columnspace of A. So, a basis for Col(A) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}, \begin{bmatrix} 1+i \\ 2 \\ 1-i \end{bmatrix} \right\}$$

Solve the homogeneous system $A\vec{z} = \vec{0}$, we find that a basis for Null(A) is $\left\{ \begin{bmatrix} -1 \\ -1+i \\ 1 \end{bmatrix} \right\}$.

To find a basis for the left nullspace, we row reduce A^T. We get

$$\begin{bmatrix} 1 & 0 & i \\ 1+i & 2 & 1-i \\ 3 & 2-2i & -i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1-i \\ 0 & 0 & 0 \end{bmatrix}$$

Solving $A^T\vec{z} = \vec{0}$, we find that a basis for Null(A^T) is

$$\left\{ \begin{bmatrix} -i \\ -1+i \\ 1 \end{bmatrix} \right\}$$

11.2.3 If $\vec{z} \in \mathbb{C}^n$, then there exists $z_1, \ldots, z_n \in \mathbb{C}$ such that

$$\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + i \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{x} + i\vec{y}$$

where $\vec{x}, \vec{y} \in \mathbb{R}^n$.

11.2.4 (a) Let $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\alpha, \beta \in \mathbb{C}$. Then

$$L(\alpha\vec{z} + \beta\vec{y}) = L\left(\begin{bmatrix} \alpha z_1 + \beta y_1 \\ \alpha z_2 + \beta y_2 \\ \alpha z_2 + \beta y_2 \end{bmatrix} \right) = \begin{bmatrix} -i(\alpha z_1 + \beta y_1) + (1+i)(\alpha z_2 + \beta y_2 + (1+2i)(\alpha z_3 + \beta y_3) \\ (-1+i)(\alpha z_1 + \beta y_1) - 2i(\alpha z_2 + \beta y_2) - 3i(\alpha z_3 + \beta y_3) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} -iz_1 + (1+i)z_2 + (1+2i)z_3 \\ (-1+i)z_1 - 2iz_2 - 3iz_3 \end{bmatrix} + \beta \begin{bmatrix} -iy_1 + (1+i)y_2 + (1+2i)y_3 \\ (-1+i)y_1 - 2iy_2 - 3iy_3 \end{bmatrix} = \alpha L(\vec{z}) + \beta L(\vec{y})$$

Thus L is linear.

We have $L(1,0,0) = \begin{bmatrix} -i \\ -1+i \end{bmatrix}$, $L(0,1,0) = \begin{bmatrix} 1+i \\ -2i \end{bmatrix}$, and $L(0,0,1) = \begin{bmatrix} 1+2i \\ -3i \end{bmatrix}$. Hence $[L] = \begin{bmatrix} -i & 1+i & 1+2i \\ -1+i & -2i & -3i \end{bmatrix}$.

(b) Row reducing $[L]$ we get

$$\begin{bmatrix} -i & 1+i & 1+2i \\ -1+i & -2i & -3i \end{bmatrix} \sim \begin{bmatrix} 1 & -1+i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A basis for $\ker(L)$ is the general solution of $[L]\vec{x} = \vec{0}$, hence a basis is $\left\{\begin{bmatrix} 1-i \\ 1 \\ 0 \end{bmatrix}\right\}$.

The range of L is equal to the columnspace of $[L]$. Thus, a basis for the range of L is $\left\{\begin{bmatrix} -i \\ -1+i \end{bmatrix}, \begin{bmatrix} 1+2i \\ -3i \end{bmatrix}\right\}$.

11.2.5 (a) Let $\vec{w}, \vec{z} \in \mathbb{C}^2$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned} L(\alpha\vec{z} + \beta\vec{w}) = L(\alpha z_1 + \beta w_1, \alpha z_2 + \beta w_2) &= \begin{bmatrix} \alpha z_1 + \beta w_1 + (1+i)(\alpha z_2 + \beta w_2) \\ (1+i)(\alpha z_1 + \beta w_1) + 2i(\alpha z_2 + \beta w_2) \end{bmatrix} \\ &= \alpha \begin{bmatrix} z_1 + (1+i)z_2 \\ (1+i)z_1 + 2iz_2 \end{bmatrix} + \beta \begin{bmatrix} w_1 + (1+i)w_2 \\ (1+i)w_1 + 2iw_2 \end{bmatrix} \\ &= \alpha L(\vec{z}) + \beta L(\vec{w}) \end{aligned}$$

Hence, L is linear.

(b) If $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \ker(L)$, then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(\vec{z}) = \begin{bmatrix} z_1 + (1+i)z_2 \\ (1+i)z_1 + 2iz_2 \end{bmatrix}$$

Hence, we need to solve the homogeneous system of equations

$$\begin{aligned} z_1 + (1+i)z_2 &= 0 \\ (1+i)z_1 + 2iz_2 &= 0 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1+i \\ 1+i & 2i \end{bmatrix} \sim \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for $\ker(L)$ is $\left\{\begin{bmatrix} -1-i \\ 1 \end{bmatrix}\right\}$.

Every vector in the range of L has the form

$$\begin{aligned}\begin{bmatrix} z_1 + (1+i)z_2 \\ (1+i)z_1 + 2iz_2 \end{bmatrix} &= z_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} + z_2 \begin{bmatrix} 1+i \\ 2i \end{bmatrix} \\ &= z_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} + (1+i)z_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \\ &= \Big(z_1 + (1+i)z_2\Big) \begin{bmatrix} 1 \\ 1+i \end{bmatrix}\end{aligned}$$

Therefore, a basis for Range(L) is $\left\{\begin{bmatrix} 1 \\ 1+i \end{bmatrix}\right\}$.

(c) If $\vec{z} \in \ker(L)$, then $L(\vec{z}) = \vec{0} = 0\vec{z}$. Consequently, we pick $\vec{z}_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$. In part (b) we showed that $L(\vec{z}) = \Big(z_1 + (1+i)z_2\Big)\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$. Hence, if we take$\vec{z}_2 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$, we get

$$L(\vec{z}_2) = \Big(1 + (1+i)(1+i)\Big)\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = (1+2i)\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

Thus, if we take $\mathcal{B} = \{\vec{z}_1, \vec{z}_2\}$ we get

$$\begin{aligned} L(\vec{z}_1) &= \vec{0} = 0\vec{z}_1 + 0\vec{z}_2 \\ L(\vec{z}_2) &= (1+2i)\vec{z}_2 = 0\vec{z}_1 + (1+2i)\vec{z}_2 \end{aligned}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 1+2i \end{bmatrix}$$

11.2.6 We have

$$\begin{vmatrix} 1 & -1 & i \\ 1+i & -i & i \\ 1-i & -1+2i & 1+2i \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1+i & 1 & 1 \\ 1-i & i & i \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ i & i \end{vmatrix} = 0$$

11.2.7 We have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1+i & 1 & 0 & 1 & 0 \\ i & 1+i & 1+2i & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2+2i & -i & -2+i \\ 0 & 1 & 0 & 1 & -i & i \\ 0 & 0 & 1 & -1-i & i & 1-i \end{array}\right]$$

Hence,

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1+i & 1 \\ i & 1+i & 1+2i \end{bmatrix}^{-1} = \begin{bmatrix} 2+2i & -i & -2+i \\ 1 & -i & i \\ -1-i & i & 1-i \end{bmatrix}$$

11.2.8 We prove this by induction. If $A = [a]$ is a 1×1 matrix, then

$$\overline{\det A} = \overline{\det[a]} = \overline{a} = \det[\overline{a}] = \det \overline{A}$$

Assume the result holds for $n-1 \times n-1$ matrices and consider an $n \times n$ matrix A. If we expand $\det \overline{A}$ along the first row, we get by definition of the determinant

$$\det \overline{A} = \sum_{i=1}^{n} \overline{a_{1i}} C_{1i}(\overline{A})$$

where $C_{1i}(\overline{A})$ represents the cofactors of $\overline{A}$. But, each of these cofactors is the determinant of an $n-1 \times n-1$ matrix, so we have by our inductive hypothesis that $C_{1i}(\overline{A}) = \overline{C_{1i}(A)}$. Hence,

$$\det \overline{A} = \sum_{i=1}^{n} \overline{a_{1i}} C_{1i}(\overline{A}) = \sum_{i=1}^{n} \overline{a_{1i} C_{1i}(A)} = \overline{\det A}$$

11.3 Problem Solutions

11.3.1 (a) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 1+i \\ 1-i & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda = \lambda(\lambda - 3)$$

Hence, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$. For $\lambda_1 = 0$ we get

$$A - 0I = \begin{bmatrix} 2 & 1+i \\ 1-i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & (1+i)/2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} 1+i \\ -2 \end{bmatrix} \right\}$. For $\lambda_2 = 3$ we get

$$A - 3I = \begin{bmatrix} -1 & 1+i \\ 1-i & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1-i \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\}$.

Therefore, A is diagonalized by $P = \begin{bmatrix} 1+i & 1+i \\ -2 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$.

(b) We have

$$C(\lambda) = \begin{vmatrix} 3-\lambda & 5 \\ -5 & -3-\lambda \end{vmatrix} = \lambda^2 + 16$$

Solving $\lambda^2 + 16 = 0$ we find that the eigenvalues are $\lambda_1 = 4i$ and $\lambda_2 = -4i$. For $\lambda_1 = 4i$ we get

$$A - 4iI = \begin{bmatrix} 3-4i & 5 \\ -5 & -3-4i \end{bmatrix} \sim \begin{bmatrix} 5 & 3+4i \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 3+4i \\ -5 \end{bmatrix}\right\}$. For $\lambda_2 = -4i$ we get

$$A + 4iI = \begin{bmatrix} 3+4i & 5 \\ -5 & -3+4i \end{bmatrix} \sim \begin{bmatrix} 5 & 3-4i \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix} 3-4i \\ -5 \end{bmatrix}\right\}$.

Therefore, A is diagonalized by $P = \begin{bmatrix} 3+4i & 3-4i \\ -5 & -5 \end{bmatrix}$ to $D = \begin{bmatrix} 4i & 0 \\ 0 & -4i \end{bmatrix}$.

(c) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 2 & -1 \\ -4 & 1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 0 & \lambda \\ -4 & 1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & \lambda \\ -2 & 1-\lambda & 2 \\ 1-\lambda & 2 & -1-\lambda \end{vmatrix} = -\lambda(\lambda^2 - 2\lambda + 5)$$

Hence, the eigenvalues are $\lambda_1 = 0$ and the roots of $\lambda^2 - 2\lambda + 5$. By the quadratic formula, we get that $\lambda_2 = 1 + 2i$ and $\lambda_3 = 1 - 2i$. For $\lambda_1 = 0$ we get

$$A - 0I = \begin{bmatrix} 2 & 2 & -1 \\ -4 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right\}$. For $\lambda_2 = 1 + 2i$ we get

$$A - \lambda_2 I = \begin{bmatrix} 1-2i & 2 & -1 \\ -4 & -2i & 2 \\ 2 & 2 & -2-2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}\right\}$. For $\lambda_3 = 1 - 2i$ we get

$$A - \lambda_3 I = \begin{bmatrix} 1+2i & 2 & -1 \\ -4 & 2i & 2 \\ 2 & 2 & -2+2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_3} is $\left\{\begin{bmatrix}1\\-i\\1\end{bmatrix}\right\}$.

Therefore, A is diagonalized by $P = \begin{bmatrix}1 & 1 & 1\\0 & i & -i\\1 & 1 & 1\end{bmatrix}$ to $D = \begin{bmatrix}0 & 0 & 0\\0 & 1+2i & 0\\0 & 0 & 1-2i\end{bmatrix}$.

(d) We have

$$C(\lambda) = \begin{vmatrix}1-\lambda & i\\ i & -1-\lambda\end{vmatrix} = \lambda^2$$

Hence, the only eigenvalue is $\lambda_1 = 0$ with algebraic multiplicity 2. We get

$$A - 0I = \begin{bmatrix}1 & i\\ i & -1\end{bmatrix} \sim \begin{bmatrix}1 & i\\ 0 & 0\end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix}-i\\1\end{bmatrix}\right\}$. Hence, the geometric multiplicity is 1 and so the matrix is not diagonalizable.

(e) We have

$$C(\lambda) = \begin{vmatrix}2-\lambda & 1 & -1\\ 2 & 1-\lambda & 0\\ 3 & -1 & 2-\lambda\end{vmatrix} = \begin{vmatrix}2-\lambda & 0 & -1\\ 2 & 1-\lambda & 0\\ 3 & 1-\lambda & 2-\lambda\end{vmatrix}$$
$$= \begin{vmatrix}2-\lambda & 0 & -1\\ 2 & 1-\lambda & 0\\ 1 & 0 & 2-\lambda\end{vmatrix} = -(\lambda-1)(\lambda^2-4\lambda+5)$$

Hence, the eigenvalues are $\lambda_1 = 1$ and the roots of $\lambda^2 - 4\lambda + 5$. By the quadratic formula, we get that $\lambda_2 = 2 + i$ and $\lambda_3 = 2 - i$. For $\lambda_1 = 1$ we get

$$A - I = \begin{bmatrix}1 & 1 & -1\\ 2 & 0 & 0\\ 3 & -1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. For $\lambda_2 = 2 + i$ we get

$$A - \lambda_2 I = \begin{bmatrix}-i & 1 & -1\\ 2 & -1-i & 0\\ 3 & -1 & -i\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -(1+2i)/5\\ 0 & 1 & -(3-i)/5\\ 0 & 0 & 0\end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix}1+2i\\3+i\\5\end{bmatrix}\right\}$. For $\lambda_3 = 2 - i$ we get

$$A - \lambda_3 I = \begin{bmatrix} i & 1 & -1 \\ 2 & -1+i & 0 \\ 3 & -1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -(1-2i)/5 \\ 0 & 1 & -(3+i)/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_3} is $\left\{\begin{bmatrix}1-2i\\3-i\\5\end{bmatrix}\right\}$.

Therefore, A is diagonalized by $P = \begin{bmatrix} 0 & 1+2i & 1-2i \\ 1 & 3+i & 3-i \\ 1 & 5 & 5 \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}$.

(f) We have

$$C(\lambda) = \begin{vmatrix} 1+i-\lambda & 1 & 0 \\ 1 & 1-\lambda & -i \\ 1 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1+i-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1-i+\lambda \\ 1 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1+i-\lambda)(1-\lambda)^2 + (-1-i+\lambda) = -(\lambda-(1+i))[\lambda^2 - 2\lambda + 1 - 1]$$

$$= -(\lambda-(1+i))(\lambda-2)\lambda$$

Hence, the eigenvalues are $\lambda_1 = 1+i$, $\lambda_2 = 2$, and $\lambda_3 = 0$. For $\lambda_1 = 1+i$ we get

$$A - \lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -i & -i \\ 1 & 0 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix}i\\0\\1\end{bmatrix}\right\}$. For $\lambda_2 = 2$ we get

$$A - \lambda_2 I = \begin{bmatrix} -1+i & 1 & 0 \\ 1 & -1 & -i \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1+i \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix}1\\1-i\\1\end{bmatrix}\right\}$. For $\lambda_3 = 0$ we get

$$A - \lambda_3 I = \begin{bmatrix} -1+i & 1 & 0 \\ 1 & -1 & -i \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1-i \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_3} is $\left\{\begin{bmatrix} -1 \\ 1+i \\ 1 \end{bmatrix}\right\}$.

Therefore, A is diagonalized by $P = \begin{bmatrix} i & 1 & -1 \\ 0 & 1-i & 1+i \\ 1 & 1 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(g) We have

$$\begin{aligned} C(\lambda) &= \begin{vmatrix} 5-\lambda & -1+i & 2i \\ -2-2i & 2-\lambda & 1-i \\ 4i & -1-i & -1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & i-i\lambda \\ -2-2i & 2-\lambda & 1-i \\ 4i & -1-i & -1-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ -2-2i & 2-\lambda & -1+i \\ 4i & -1-i & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)[\lambda^2 - 5\lambda + 4] = -(\lambda-1)^2(\lambda-4) \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$. For $\lambda_1 = 1$ we get

$$A - \lambda_1 I = \begin{bmatrix} 4 & -1+i & 2i \\ -2-2i & 1 & 1-i \\ 4i & -1-i & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & (-1+i)/4 & i/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{\begin{bmatrix} 1-i \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -i \\ 0 \\ 2 \end{bmatrix}\right\}$. For $\lambda_2 = 4$ we get

$$A - \lambda_2 I = \begin{bmatrix} 1 & -1+i & 2i \\ -2-2i & -2 & 1-i \\ 4i & -1-i & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -i \\ 0 & 1 & (1-i)/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_2} is $\left\{\begin{bmatrix} -2i \\ -1+i \\ 2 \end{bmatrix}\right\}$.

Therefore, A is diagonalized by $P = \begin{bmatrix} 1-i & -i & -2i \\ 4 & 0 & -1+i \\ 0 & 2 & 2 \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(h) We have

$$\begin{aligned} C(\lambda) &= \begin{vmatrix} -6-3i-\lambda & -2 & -3-2i \\ 10 & 2-\lambda & 5 \\ 8+6i & 3 & 4+4i-\lambda \end{vmatrix} = \begin{vmatrix} i-\lambda & -2 & -3-2i \\ 0 & 2-\lambda & 5 \\ -2i+2\lambda & 3 & 4+4i-\lambda \end{vmatrix} \\ &= \begin{vmatrix} i-\lambda & -2 & -3-2i \\ 0 & 2-\lambda & 5 \\ 0 & -1 & -2-\lambda \end{vmatrix} \\ &= (i-\lambda)(\lambda^2-1) = -(\lambda-i)^2(\lambda+i) \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. For $\lambda_1 = i$ we get

$$A - \lambda_1 I = \begin{bmatrix} -6-4i & -2 & -3-2i \\ 10 & 2-i & 5 \\ 8+6i & 3 & 4+3i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

Thus, A is not diagonalizable since $g_{\lambda_1} < a_{\lambda_1}$.

11.3.2 (a) The characteristic polynomial is

$$C(\lambda) = \det(R_\theta - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = \lambda^2 - 2\cos\theta + 1$$

Hence, by the quadratic formula, we have

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm i\sin\theta$$

Observe, that if $\sin\theta = 0$, then R_θ is diagonal, so we could just take $P = I$. So, we now assume that $\sin\theta \neq 0$.

For $\lambda_1 = \cos\theta + i\sin\theta$, we get $R_\theta - \lambda_1 I = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

We get that an eigenvector of λ_1 is $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

Similarly, for $\lambda_2 = \cos\theta - i\sin\theta$, we get $R_\theta - \lambda_2 I = \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$

We get that an eigenvector of λ_1 is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Thus, $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}$.

(b) We have $R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is already diagonal, so R_0 is diagonalized by $P = I$ as we stated in (a).

$R_{\pi/4} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. From our work in a) $R_{\pi/4}$ is diagonalized by $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ to $D = \begin{bmatrix} (1+i)/\sqrt{2} & 0 \\ 0 & (1-i)/\sqrt{2} \end{bmatrix}$. Indeed, we find that

$$P^{-1}R_{\pi/4}P = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} (1+i)/\sqrt{2} & 0 \\ 0 & (1-i)/\sqrt{2} \end{bmatrix}$$

11.3.3 If $A\vec{z} = \lambda\vec{z}$, then

$$\overline{A}\,\overline{\vec{z}} = \overline{A\vec{z}} = \overline{\lambda\vec{z}} = \overline{\lambda}\,\overline{\vec{z}}$$

Hence, $\overline{\vec{z}}$ is an eigenvector of $\overline{A}$ with eigenvalue $\overline{\lambda}$.

11.3.4 By Theorem 10.3.1, since A is real we know that the other eigenvalue of A is $2-i$ with corresponding eigenvector $\begin{bmatrix} 1-i \\ -i \end{bmatrix}$. Hence, A is diagonalized by $P = \begin{bmatrix} 1+i & 1-i \\ i & -i \end{bmatrix}$ to $D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$. Then $P^{-1}AP = D$, so

$$A = PDP^{-1} = \begin{bmatrix} 1+i & 1-i \\ i & -i \end{bmatrix}\begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}\frac{1}{-2i}\begin{bmatrix} -i & -1+i \\ -i & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

11.3.5 If A is a 3×3 real matrix with a non-real eigenvalue λ, then by Theorem 11.3.1, $\overline{\lambda}$ is another eigenvalues of A. Also, by Corollary 11.3.2, we have that A must have a real eigenvalue. Therefore, A has 3 distinct eigenvalues and hence is diagonalizable.

11.4 Problem Solutions

11.4.1 (a) $\|\vec{u}\| = \sqrt{\langle\vec{u},\vec{u}\rangle} = \sqrt{\vec{u}\cdot\overline{\vec{u}}} = \sqrt{1(1)+2i(-2i)+(1-i)(1+i)} = \sqrt{1+4+2} = \sqrt{7}.$

(b) $\|\vec{w}\| = \sqrt{\langle\vec{w},\vec{w}\rangle} = \sqrt{\vec{w}\cdot\overline{\vec{w}}} = \sqrt{(1+i)(1-i)+(1-2i)(1+2i)+i(-i)} = \sqrt{2+5+1} = \sqrt{8}.$

(c) $\langle\vec{u},\vec{z}\rangle = \vec{u}\cdot\overline{\vec{z}} = 1(2)+2i(2i)+(1-i)(1+i) = 2-4+2 = 0$

(d) $\langle\vec{z},\vec{u}\rangle = \overline{\langle\vec{u},\vec{z}\rangle} = 0$

(e) $\langle\vec{u},(2+i)\vec{z}\rangle = \overline{(2+i)}\langle\vec{u},\vec{z}\rangle = 0$

(f) $\langle\vec{z},\vec{w}\rangle = \vec{z}\cdot\overline{\vec{w}} = 2(1-i)+(-2i)(1+2i)+(1-i)(-i) = 2-2i-2i+4-i-1 = 5-5i$

(g) $\langle\overline{\vec{w}},\overline{\vec{z}}\rangle = \overline{\vec{w}}\cdot\overline{\overline{\vec{z}}} = \overline{\vec{w}}\cdot\vec{z} = \vec{z}\cdot\overline{\vec{w}} = \langle\vec{z},\vec{w}\rangle = 5-5i$

(h) $\langle\vec{u}+\vec{z},2i\vec{w}-i\vec{z}\rangle = \left\langle\begin{bmatrix} 3 \\ 0 \\ 2-2i \end{bmatrix},\begin{bmatrix} -2 \\ 2+2i \\ -3-i \end{bmatrix}\right\rangle = 3(-2)+0(2-2i)+(2-2i)(-3+i) = -10+8i$

11.4.2 (a) We have

$$\langle A,B\rangle = 1(-i)+2i(2)+(-i)(1+i)+(1+i)(3) = 4+5i$$

(b) We have

$$\langle B,A\rangle = i(1)+2(-2i)+(1-i)(i)+3(1-i) = 4-5i$$

(c) We have

$$\|A\| = \sqrt{\langle A,A\rangle} = \sqrt{1(1)+2i(-2i)+(-i)(i)+(1+i)(1-i)} = \sqrt{8}$$

(d) We have

$$\|B\| = \sqrt{\langle B,B\rangle} = \sqrt{i(-i)+2(2)+(1-i)(1+i)+3(3)} = 4$$

11.4.3 (a) Observe that

$$\left\langle \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} \right\rangle = \frac{i}{\sqrt{2}} \cdot \overline{\frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}} \cdot \overline{\frac{-i}{\sqrt{2}}} = \frac{i}{2} + \frac{i}{2} = i$$

Hence, the columns are not orthogonal, so the matrix is not unitary.

(b) Let $U = \begin{bmatrix} (1+i)/2 & (1-i)/\sqrt{6} & (1-i)/\sqrt{12} \\ 1/2 & 0 & 3i/\sqrt{12} \\ 1/2 & 2i/\sqrt{6} & -i/\sqrt{12} \end{bmatrix}$. Then we find that $UU^* = I$, so U is unitary.

(c) The matrix is a real orthogonal matrix, and hence is unitary.

(d) Observe that

$$\left\| \begin{bmatrix} 0 \\ 0 \\ (1+i)/2 \end{bmatrix} \right\| = \sqrt{0 + 0 + \frac{2}{4}} = \sqrt{\frac{1}{2}} \neq 0$$

Thus, the matrix is not unitary.

11.4.4 Let $U = \begin{bmatrix} \vec{u}_1 & \cdots \vec{u}_n \end{bmatrix}$. Then, we have

$$\begin{aligned} U^*U &= \begin{bmatrix} \vec{u}_1^* \\ \vdots \\ \vec{u}_n^* \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix} \\ &= \begin{bmatrix} \vec{u}_1^*\vec{u}_1 & \cdots & \vec{u}_1^*\vec{u}_n \\ \vdots & \ddots & \vdots \\ \vec{u}_n^*\vec{u}_1 & \cdots & \vec{u}_n^*\vec{u}_n \end{bmatrix} \end{aligned}$$

Since $\{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthonormal basis for $\mathbb{C}^n$ we get

$$\vec{u}_i^*\vec{u}_i = \overline{\vec{u}_i^T\overline{\vec{u}_i}} = \overline{\langle \vec{u}_i, \vec{u}_i \rangle} = 1$$

and

$$\vec{u}_i^*\vec{u}_j = \overline{\vec{u}_i^T\overline{\vec{u}_j}} = \overline{\langle \vec{u}_i, \vec{u}_j \rangle} = 0$$

whenever $i \neq j$. Thus, $U^*U = I$ as required.

11.4.5 (a) We have

$$\begin{aligned} \langle \vec{z}, \vec{w} \rangle &= (1+i)(1+i) + (2-i)(-2+3i) + (-1+i)(-1) = 2i - 1 + 8i + 1 - i = 9i \\ \langle \vec{w}, 2i\vec{z} \rangle &= \overline{2i}\overline{\langle \vec{z}, \vec{w} \rangle} = -2i(-9i) = -18 \end{aligned}$$

(b) A vector orthogonal to $\vec{z}$ in $\mathrm{Span}\{\vec{z}, \vec{w}\}$ is

$$\vec{v} = \mathrm{perp}_{\vec{z}}\,\vec{w} = \vec{w} - \frac{\langle \vec{w}, \vec{z} \rangle}{\|\vec{z}\|^2}\vec{z} = \begin{bmatrix} 1-i \\ -2-3i \\ -1 \end{bmatrix} + \frac{9i}{9}\begin{bmatrix} 1+i \\ 2-i \\ -1+i \end{bmatrix} = \begin{bmatrix} 0 \\ 1-i \\ -2-i \end{bmatrix}$$

(c) Note that $\text{Span}\{\vec z, \vec w\} = \text{Span}\{\vec z, \vec v\}$ where $\vec v$ is the vector from part (b) and $\{\vec z, \vec v\}$ is orthogonal over $\mathbb{C}$ so we have

$$\text{proj}_S\ \vec u = \frac{<\vec u, \vec z>}{\|\vec z\|^2}\vec z + \frac{<\vec u, \vec v>}{\|\vec v\|^2}\vec v$$

11.4.6 Denote the vectors in the spanning set by A_1, A_2, and A_3 respectively.
Let $B_1 = A_1$. Next, we get

$$A_2 - \frac{\langle A_2, B_1\rangle}{\|B_1\|^2}B_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} - \frac{i}{2}\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} i/2 & 1/2 \\ 0 & i \end{bmatrix}$$

So, we let $B_2 = \begin{bmatrix} i & 1 \\ 0 & 2i \end{bmatrix}$. Next, we get

$$B_3 = A_3 - \frac{\langle A_3, B_1\rangle}{\|B_1\|^2}B_1 - \frac{\langle A_3, B_2\rangle}{\|B_2\|^2}B_2 = \begin{bmatrix} 0 & 2 \\ 0 & i \end{bmatrix} - \frac{-2i}{2}\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} - \frac{4}{6}\begin{bmatrix} i & 1 \\ 0 & 2i \end{bmatrix} = \begin{bmatrix} i/3 & 1/3 \\ 0 & -i/3 \end{bmatrix}$$

So, an orthogonal basis for $\mathbb{S}$ is $\{B_1, B_2, B_3\}$.

11.4.7 (a) We have

$$(A^*)^* = \left[\overline{A}^T\right]^* = \overline{\overline{A^T}}^T = (A^T)^T = A$$

(b) We have

$$(\alpha A)^* = \overline{(\alpha A)}^T = \overline{\alpha A^T} = \overline{\alpha}\,\overline{A^T} = \overline{\alpha}A^*$$

(c) We have

$$(AB)^* = \overline{(AB)}^T = \overline{B^T A^T} = \overline{B^T}\,\overline{A^T} = B^*A^*$$

(d) For any $\vec z, \vec w \in \mathbb{C}^n$ we have

$$\langle A\vec z, \vec w\rangle = (A\vec z)\cdot\overline{\vec w} = (A\vec z)^T\overline{\vec w} = \vec z^T A^T\overline{\vec w} = \vec z^T\overline{\overline{A^T}}\,\overline{\vec w} = \vec z^T\overline{A^*}\,\overline{\vec w} = \vec z^T\overline{A^*\vec w} = \vec z\cdot\overline{A^*\vec w} = \langle \vec z, A^*\vec w\rangle$$

11.4.8 (a) For any $\vec z, \vec w \in \mathbb{C}^n$ we have

$$\langle U\vec z, U\vec w\rangle = (U\vec z)^T\overline{U\vec w} = \vec z^T U^T\overline{U}\,\overline{\vec w} = \vec z^T\overline{U^*}\,\overline{U}\,\overline{\vec w} = \vec z^T\overline{U^*U}\,\overline{\vec w} = \vec z^T\overline{\vec w} = \langle \vec z, \vec w\rangle$$

(b) Suppose λ is an eigenvalue of U with eigenvector $\vec v$. We get $\|U\vec v\|^2 = \langle U\vec v, U\vec v\rangle = \langle \vec v, \vec v\rangle = \|\vec v\|^2$ by part (a). But we also have $\|U\vec v\|^2 = \|\lambda\vec v\|^2 = \langle \lambda\vec v, \lambda\vec v\rangle = \lambda\overline{\lambda}\langle \vec v, \vec v\rangle = |\lambda|^2\|\vec v\|^2$, so since $\|\vec v\| \neq 0$, we must have $|\lambda| = 1$.

(c) The matrix $U = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ is unitary since $U^*U = I$ and the only eigenvalue is i with multiplicity 2.

11.4.9 If $\langle \vec u, \vec v\rangle = 0$, then we have

$$\begin{aligned} \|\vec u + \vec v\|^2 &= \langle \vec u + \vec v, \vec u + \vec v\rangle = \langle \vec u, \vec u + \vec v\rangle + \langle \vec v, \vec u + \vec v\rangle = \langle \vec u, \vec u\rangle + \langle \vec u, \vec v\rangle + \langle \vec v, \vec u\rangle + \langle \vec v, \vec v\rangle \\ &= \|\vec u\|^2 + 0 + \overline{0} + \|\vec v\|^2 = \|\vec u\|^2 + \|\vec v\|^2 \end{aligned}$$

The converse is not true. One counter example is: Consider $V = \mathbb{C}$ with its standard inner product and let $\vec u = 1 + i$ and $\vec v = 1 - i$. Then $\|\vec u + \vec v\|^2 = \|2\|^2 = 4$ and $\|\vec u\|^2 + \|\vec v\|^2 = 2 + 2 = 4$, but $\langle \vec u, \vec v\rangle = (1 + i)(1 + i) = 2i \neq 0$.

11.5 Problem Solutions

11.5.1 (a) If A is skew-Hermitian, then $A^* = -A$. By Schur's Theorem, there exists a unitary matrix U such that $U^*AU = T$ is upper triangular. Now, observe that

$$T^* = (U^*AU)^* = U^*A * U = U^*(-A)U = (-1)U^*AU = -T$$

Thus, T is also skew-Hermitian. Since T is upper triangular, we have that T^* is lower triangluar. Consequently, T is both upper and lower triangular and hence diagonal.

So, $U^*AU = T$ is diagonal as required.

(b) If A is skew-Hermitian, then $A^* = -A$ and hence

$$AA^* = A[(-1)A] = (-1)AA = A^*A$$

Therefore, A is normal and hence unitarily diagonalizable.

11.5.2 Let λ be an eigenvalue of A with eigenvector $\vec{z}$. Since A is skew-Hermitian we have $A^* = -A$ and hence

$$\lambda\langle\vec{z},\vec{z}\rangle = \langle\lambda\vec{z},\vec{z}\rangle = \langle A\vec{z},\vec{z}\rangle = \langle\vec{z},A^*\vec{z}\rangle = \langle\vec{z},-A\vec{z}\rangle = \langle\vec{z},-\lambda\vec{z}\rangle = -\overline{\lambda}\langle\vec{z},\vec{z}\rangle$$

Since $\langle\vec{z},\vec{z}\rangle \neq 0$ as $\vec{z} \neq \vec{0}$, we get that $\lambda = -\overline{\lambda}$ so $\operatorname{Re}(\lambda) = \frac{1}{2}(\lambda + \overline{\lambda}) = 0$.

11.5.3 (a) If A is unitary, then $A^* = A^{-1}$. By Schur's Theorem, there exists a unitary matrix U such that $U^*AU = T$ is upper triangular. Now, observe that T is a product of unitary matrices and thus, T is also unitary by Theorem 11.4.8.

Let $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix} = \begin{bmatrix} \vec{t}_1 & \vec{t}_2 & \cdots & \vec{t}_n \end{bmatrix}$.

Since T is unitary, the columns of T must form an orthonormal basis for $\mathbb{C}^n$. Thus,

$$1 = \|\vec{t}_1\| = |t_{11}|$$

Also, we must have

$$0 = \langle\vec{t}_1,\vec{t}_2\rangle = t_{11}t_{12}$$

Since $t_{11} \neq 0$, we must have $t_{12} = 0$. Then, we get $1 = \|\vec{t}_2\| = |t_{22}|$. Next, we have

$$0 = \langle\vec{t}_1,\vec{t}_3\rangle = t_{11}t_{13}$$
$$0 = \langle\vec{t}_2,\vec{t}_3\rangle = t_{22}t_{23}$$

Therefore, $t_{13} = t_{23} = 0$. Continuing in this way we get that T is diagonal and all of the diagonal entries satisfy $|t_{ii}| = 1$.

So, $U^*AU = T$ is diagonal as required.

(b) If A is unitary, then $A^* = A^{-1}$ and so

$$AA^* = I = A^*A$$

Thus, A is normal and hence unitarily diagonalizable.

11.5.4 (a) Since $A^* = A$, we have that A is Hermitian and hence it is also normal.

(b) We have that $A^* = \begin{bmatrix} 2 & -i \\ i & 1-i \end{bmatrix}$, so A is not Hermitian, nor skew-Hermitian. We find that $AA^* = \begin{bmatrix} 5 & -1-3i \\ -1+3i & 3 \end{bmatrix} \neq A^*A$, so A is not normal either.

(c) We have that $A^* = \begin{bmatrix} 0 & -1+i \\ 1+i & 0 \end{bmatrix} = -A$, so A is skew-Hermitian and hence it is also normal.

(d) We have that $A^* = \begin{bmatrix} 1+i & 2 \\ -2i & 3 \end{bmatrix}$, so A is not Hermitian, nor skew-Hermitian. We find that $AA^* = \begin{bmatrix} 6 & 2+4i \\ 2-4i & 13 \end{bmatrix} \neq A^*A$, so A is not normal either.

(e) We have that $A^* = \begin{bmatrix} -i & 2 \\ -2i & 5 \end{bmatrix}$, so A is not Hermitian, nor skew-Hermitian. We find that $AA^* = \begin{bmatrix} 5 & 2+2i \\ 2-2i & 5 \end{bmatrix} = A^*A$, so A is normal .

(f) Since $A^* = A$, we have that A is Hermitian and hence it is also normal.

11.5.5 (a) We have $C(\lambda) = \lambda^2 - 2a\lambda + a^2 + b^2$ so by the quadratic formula we get eigenvalues $\lambda = a \pm bi$. For $\lambda = a + bi$ we get

$$A - \lambda I = \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} \Rightarrow \vec{z}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

For $\lambda = a - bi$ we get

$$A - \lambda I = \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix} \Rightarrow \vec{z}_2 = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

Hence we have $D = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$ and $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$.

(b) We have $C(\lambda) = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$. Thus, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 1$. For $\lambda_1 = 4$ we get

$$A - \lambda_1 I = \begin{bmatrix} -2 & 1+i \\ 1-i & -1 \end{bmatrix} \Rightarrow \vec{z}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

For $\lambda_2 = 1$ we get

$$A - \lambda I = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \Rightarrow \vec{z}_2 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

Hence we have $D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{-1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$.

(c) We have $C(\lambda) = \lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6)$. Thus, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 6$. For $\lambda_1 = 2$ we get

$$A - \lambda_1 I = \begin{bmatrix} 3 & \sqrt{2} - i \\ \sqrt{2} + i & 1 \end{bmatrix} \Rightarrow \vec{z}_1 = \begin{bmatrix} -\sqrt{2} + i \\ 3 \end{bmatrix}$$

For $\lambda_2 = 6$ we get

$$A - \lambda I = \begin{bmatrix} -1 & \sqrt{2} - i \\ \sqrt{2} + i & -3 \end{bmatrix} \Rightarrow \vec{z}_2 = \begin{bmatrix} \sqrt{2} - i \\ 1 \end{bmatrix}$$

Hence we have $D = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$ and $U = \begin{bmatrix} \frac{-\sqrt{2}+i}{\sqrt{12}} & \frac{\sqrt{2}-i}{2} \\ \frac{3}{\sqrt{12}} & \frac{1}{2} \end{bmatrix}$.

(d) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + 1 = -\lambda^3 + 3\lambda^2 - 3\lambda + 2$$

Since A is a 3×3 real matrix, we know that A must have at least one real eigenvalue. We try the Rational Roots Theorem to see if the real root is rational. The possible rational roots are ± 1 and ± 2. We find that $\lambda_1 = 2$ is a root. Then, by the Factor Theorem and polynomial division, we get

$$-\lambda^3 + 3\lambda^2 - 3\lambda + 2 = -(\lambda - 2)(\lambda^2 - \lambda + 1)$$

Using the quadratic formula, we find that the other eigenvalues are $\lambda_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\lambda_3 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

For λ_1 we get $A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Thus, a corresponding unit eigenvector is $z_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

For λ_2 we get

$$A - \lambda_2 I = \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 & 1 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ 0 & 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a corresponding unit eigenvector is $z_2 = \begin{bmatrix} (-1 - \sqrt{3}i)/12 \\ (-1 + \sqrt{3}i)/12 \\ 1/6 \end{bmatrix}$.

By Theorem 11.3.1, we have that a unit eigenvector corresponding to $\lambda_3 = \overline{\lambda_2}$ is

$$\vec{z}_3 = \overline{\vec{z}_2} = \begin{bmatrix} (-1 + \sqrt{3}i)/12 \\ (-1 - \sqrt{3}i)/12 \\ 1/6 \end{bmatrix}$$

Hence, taking $U = \begin{bmatrix} \vec{z}_1 & \vec{z}_2 & \vec{z}_3 \end{bmatrix}$ gives $U^*AU = \operatorname{diag}(2, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i)$.

(e) We have $C(\lambda) = -(\lambda-2)(\lambda^2-\lambda+2) = -(\lambda-2)^2(\lambda+1)$ so we get eigenvalues $\lambda = 2$ and $\lambda = -1$.
For $\lambda = 2$ we get

$$C-\lambda I = \begin{bmatrix} -1 & 0 & 1+i \\ 0 & 0 & 0 \\ 1-i & 0 & -2 \end{bmatrix} \Rightarrow \vec{z}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{z}_2 = \begin{bmatrix} 1+i \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = -1$ we get

$$C-\lambda I = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 3 & 0 \\ 1-i & 0 & 1 \end{bmatrix} \Rightarrow \vec{z}_3 = \begin{bmatrix} 1+i \\ 0 \\ -2 \end{bmatrix}$$

Hence we have $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $U = \begin{bmatrix} 0 & \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \end{bmatrix}$.

(f) We have

$$\begin{aligned} C(\lambda) &= \begin{vmatrix} 1-\lambda & i & -i \\ -i & -1-\lambda & i \\ i & -i & -\lambda \end{vmatrix} = \begin{vmatrix} 1+i-\lambda & i & -i \\ -1-i-\lambda & -1-\lambda & i \\ 0 & -i & -\lambda \end{vmatrix} \\ &= \begin{vmatrix} 1+i-\lambda & 0 & -i-\lambda \\ -1-i-\lambda & -1-\lambda & i \\ 0 & -i & -\lambda \end{vmatrix} = \begin{vmatrix} 1+i-\lambda & 0 & -i-\lambda \\ -2\lambda & -1-\lambda & -\lambda \\ 0 & -i & -\lambda \end{vmatrix} \\ &= \begin{vmatrix} 1+i-\lambda & 0 & -i-\lambda \\ -2\lambda & -1+i-\lambda & 0 \\ 0 & -i & -\lambda \end{vmatrix} = i(i+\lambda)(-2\lambda) - \lambda(1+i-\lambda)(-1+i-\lambda) \\ &= -\lambda^3 + 4\lambda = -\lambda(\lambda-2)(\lambda+2) \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = -2$.

For λ_1 we get $A - 0I = \begin{bmatrix} 1 & i & -i \\ -i & -1 & i \\ i & -i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & (-1-i)/2 \\ 0 & 1 & (-1-i)/2 \\ 0 & 0 & 0 \end{bmatrix}$

Thus, a corresponding unit eigenvector is $z_1 = \begin{bmatrix} (1+i)/\sqrt{8} \\ (1+i)/\sqrt{8} \\ 1/\sqrt{2} \end{bmatrix}$.

For λ_2 we get $A - 2I = \begin{bmatrix} -1 & i & -i \\ -i & -3 & i \\ i & -i & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & (1+3i)/2 \\ 0 & 1 & (1-i)/2 \\ 0 & 0 & 0 \end{bmatrix}$

Thus, a corresponding unit eigenvector is $z_2 = \begin{bmatrix} (-1-3i)/4 \\ (-1+i)/4 \\ 1/2 \end{bmatrix}$.

For λ_3 we get $A + 2I = \begin{bmatrix} 3 & i & -i \\ -i & 1 & i \\ i & -i & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & (1-i)/2 \\ 0 & 1 & (1+3i)/2 \\ 0 & 0 & 0 \end{bmatrix}$

Thus, a corresponding unit eigenvector is $z_3 = \begin{bmatrix} (-1+i)/4 \\ (-1-3i)/4 \\ 1/2 \end{bmatrix}$.

Hence, taking $U = \begin{bmatrix} \vec{z}_1 & \vec{z}_2 & \vec{z}_3 \end{bmatrix}$ gives $U^*AU = \text{diag}(0, 2, -2)$.

11.5.6 We have $A^*A = AA^*$. So

$$B^*B = (A^*A^{-1})^*(A^*A^{-1}) = (A^{-1})^*AA^*A^{-1} = (A^{-1})^*A^*AA^{-1} = (AA^{-1})^*(AA^{-1}) = I$$

11.5.7 If AB is Hermitian, then $(AB) = (AB)^* = B^*A^* = BA$, since A and B are Hermitian.

If $AB = BA$, then $(AB)^* = B^*A^* = BA = AB$.

11.5.8 (a) For $\lambda = 2 + i$ we have

$$A - \lambda I = \begin{bmatrix} -2+i & -2+i \\ 1-i & 1-i \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus $\det(A - \lambda I) = 0$ as so λ is an eigenvalue of A.

(b) For our work in (a) we get that a unit eigenvector corresponding to $\lambda = 2 + i$ is $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We extend $\{\vec{u}_1\}$ to an orthonormal basis for by choosing $\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We can take

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$T = U^*AU = \begin{bmatrix} 2+i & 3-2i \\ 0 & 1+i \end{bmatrix}$$

11.5.9 (a) We have

$$AA^* = A(iA) = iAA = (iA)A = A^*A$$

so A is normal.

(b) Assume that $\vec{v}$ is an eigenvector of A corresponding to λ, we get

$$i\lambda\vec{v} = iA\vec{v} = A^*\vec{v} = \overline{\lambda}\vec{v}$$

Thus, $(i\lambda - \overline{\lambda})\vec{v} = \vec{0}$, so $i\lambda = \overline{\lambda}$ since $\vec{v} \neq \vec{0}$.

(c) By Schur's Theorem, A is unitarily similar to an upper triangular matrix T whose diagonal entries are the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. Thus, we have that $\det T = \lambda_1 \cdots \lambda_n$ and $\operatorname{tr} T = \lambda_1 + \cdots + \lambda_n$. The result now follows since similar matrices have the same trace and the same determinant.

11.6 Problem Solutions

11.6.1 By the Cayley-Hamilton Theorem, we have that

$$\begin{aligned} A^3 + 2A - 2I &= O \\ A^3 + 2A &= 2I \\ A\left(\frac{1}{2}A^2 + I\right) &= I \end{aligned}$$

Thus, $A^{-1} = \frac{1}{2}A^2 + I$.

11.6.2 By the Cayley-Hamilton Theorem, we have that

$$\begin{aligned} A^3 - 147A + 686I &= O \\ A^3 &= 147A - 686I \\ &= \begin{bmatrix} 441 & -1176 & -294 \\ -1176 & -1323 & -588 \\ -294 & -588 & 882 \end{bmatrix} - \begin{bmatrix} 686 & 0 & 0 \\ 0 & 686 & 0 \\ 0 & 0 & 686 \end{bmatrix} \\ &= \begin{bmatrix} -245 & -1176 & -294 \\ -1176 & -2009 & -588 \\ -294 & -588 & 196 \end{bmatrix} \end{aligned}$$

11.6.3 Observe that $\lambda^3 - 3\lambda + 2 = (\lambda + 2)(\lambda - 1)^2$. Hence, one choice is to take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$. This gives

$$A^3 - 3A + 2A = I^3 - 3I + 2I = I - 3I + 2I = O$$

NOTE: The characteristic polynomial of our choice of A is $(\lambda - 1)^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 1$. Thus, we have shown that the converse of the Cayley-Hamilton Theorem is not true.