

The Gradient

We've given a formal definition for differentiability of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

Definition 1. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, and suppose that the partial derivatives f_x and f_y are defined at the point $(x, y) = (a, b)$. Define the linear function

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We say that f is differentiable at $(x, y) = (a, b)$ if

$$\lim_{(x, y) \rightarrow (a, b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0.$$

If either of the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ do not exist, or the above limit does not exist or is not 0, then f is not differentiable at (a, b) .

The idea behind this definition is that $h(x, y)$ will be a “good” linear approximation to $f(x, y)$ near (a, b) if f is differentiable at (a, b) .

We would now like to define differentiability for scalar-valued functions of more than two variables, so functions from \mathbb{R}^n to \mathbb{R} . This definition will closely resemble our definition above, which handles the case $n = 2$. For example, in the case $n = 3$, we will use the linear function

$$h(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).$$

For larger n , we'll define a similar function h , but this notation will quickly become unwieldy! In order to simplify notation, we'll now introduce a new object to organize our partial derivatives: the gradient of a scalar-valued function.

The gradient

In order to organize our information about partial derivatives, and streamline our definition of differentiability for functions $\mathbb{R}^n \rightarrow \mathbb{R}$, we now define the gradient of a scalar-valued function.

Definition 2. Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. The gradient of f is the function $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Learning outcomes:
Author(s):

The gradient vector will be a useful computation tool in general, not only for defining differentiability.

Example 1. For $f(x, y, z) = x^2 + ye^z$, we can compute the partial derivatives

$$\begin{aligned}f_x(x, y, z) &= 2x, \\f_y(x, y, z) &= e^z, \\f_z(x, y, z) &= ye^z.\end{aligned}$$

Then the gradient of f is

$$\nabla f = (2x, e^z ye^z).$$

Problem 1 Find the gradient of each function.

$$f(x, y, z) = \sin(xyz)$$

$$\nabla f(x, y, z) = (yz \cos(xyz), xz \cos(xyz), xy \cos(xyz))$$

$$g(x, y) = x^2 e^y + y$$

$$\nabla g(x, y) = (2xe^y, x^2 e^y + 1)$$

$$h(x_1, x_2, x_3, x_4) = x_1^2 x_2 + x_1 x_3 + x_2 x_4$$

$$\nabla h(x_1, x_2, x_3, x_4) = (2x_1 x_2 + x_3, x_1^2 + x_4, x_1, x_2)$$

Differentiability

Now that we've defined the gradient, let's revisit our definition of differentiability for a function from \mathbb{R}^2 to \mathbb{R} . We used the function

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Looking at the terms $f_x(a, b)(x - a) + f_y(a, b)(y - b)$, we can rewrite this as a dot product of two vectors:

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = (f_x(a, b), f_y(a, b)) \cdot (x - a, y - b).$$

The first vector is the gradient of f evaluated at (a, b) , so we can rewrite this as

$$(f_x(a, b), f_y(a, b)) \cdot (x - a, y - b) = \nabla f(a, b) \cdot (x - a, y - b).$$

If we take $\vec{x} = (x, y)$ and $\vec{a} = (a, b)$, we can write this as

$$\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}).$$

With these notational changes in mind, we now define differentiability for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

Definition 3. Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. For a point $\vec{a} \in \mathbb{R}^n$, define

$$h(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}).$$

We say that f is differentiable at \vec{a} if

$$\lim_{\vec{x} \rightarrow \vec{a}} \frac{f(\vec{x}) - h(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0.$$

If f is differentiable, we say that $h(\vec{x})$ is the tangent hyperplane to f at \vec{a} .

If any of the partial derivatives of f do not exist, or the above limit does not exist or is not 0, then f is not differentiable at \vec{a} .

Example 2. We'll use this definition of differentiability to prove that the function $f(x, y, z) = xy + z$ is differentiable at $(1, 1, 1)$.

First, we find the gradient of f .

$$\nabla f(x, y, z) = \boxed{(y, x, 1)}$$

At the point $(1, 1, 1)$, we have

$$\nabla f(1, 1, 1) = \boxed{(1, 1, 1)}.$$

From this, we find the formula for $h(x, y, z)$.

$$\begin{aligned} h(x, y, z) &= f(1, 1, 1) + \nabla f(1, 1, 1) \cdot ((x, y, z) - (1, 1, 1)) \\ &= 2 + (1, 1, 1) \cdot (x - 1, y - 1, z - 1) \\ &= \boxed{x + y + z - 1} \end{aligned}$$

Next, we evaluate the limit

$$\begin{aligned} \lim_{(x, y, z) \rightarrow (1, 1, 1)} \frac{f(x, y, z) - h(x, y, z)}{\|(x, y, z) - (1, 1, 1)\|} &= \lim_{(x, y, z) \rightarrow (1, 1, 1)} \frac{(xy + z) - (x + y + z - 1)}{\sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2}} \\ &= \lim_{(x, y, z) \rightarrow (1, 1, 1)} \frac{xy - x - y + 1}{\sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2}}. \end{aligned}$$

To evaluate this limit, we switch to translated spherical coordinates

$$\begin{aligned} x &= 1 + \rho \cos \theta \sin \phi, \\ y &= 1 + \rho \sin \theta \sin \phi, \\ z &= 1 + \rho \cos \phi. \end{aligned}$$

Making this change, we obtain

$$\begin{aligned} \lim_{(x, y, z) \rightarrow (1, 1, 1)} \frac{xy - x - y + 1}{\sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2}} &= \lim_{\rho \rightarrow 0} \frac{(1 + \rho \cos \theta \sin \phi)(1 + \rho \sin \theta \sin \phi) - (1 + \rho \cos \theta \sin \phi) - (1 + \rho \sin \theta \sin \phi) - (1 + \rho \cos \phi)}{|\rho|} \\ &= \lim_{\rho \rightarrow 0} \frac{\rho^2 \cos \theta \sin \theta \sin^2 \phi}{|\rho|}. \end{aligned}$$

Since $-|\rho| \leq \frac{\rho^2 \cos \theta \sin \theta \sin^2 \phi}{|\rho|} \leq |\rho|$, we use the squeeze theorem to obtain

$$\lim_{\rho \rightarrow 0} \frac{\rho^2 \cos \theta \sin \theta \sin^2 \phi}{|\rho|} = 0.$$

Thus, we have shown that $f(x, y, z) = xy + z$ is differentiable at $(1, 1, 1)$.