Geometric Interpretation of Partial Derivatives

We've defined the partial derivatives of a function as follows.

Definition 1. *Consider a function* $f : \mathbb{R}^n \to \mathbb{R}$*. For* $1 \leq i \leq n$*, we define the* partial derivative of f with respect to x_i *to be*

$$f_{x_i}(x_1, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, ..., x_{i-1}, x_i + h, x_{i+1}, ..., x_n) - f(x_1, ..., x_n)}{h},$$

provided this limit exists.

In other words, if we treat all variables except for x_i as constants, and differentiate with respect to x_i, we get the partial derivative with respect to x_i.

When computing a partial derivative with respect to x_i, we're looking at the instantaneous rate of change of f with respect to x_i, if we keep the rest of the variables constant. Roughly speaking, we're asking: how does increasing x_i a tiny bit affect the value of f?

We can see the partial derivatives reflected in the shape of the graph of f. So that we can visualize the graph of f, we'll focus on a function $f : \mathbb{R}^2 \to \mathbb{R}$, so we're considering the partial derivative of f with respect to x, and with respect to y.

Suppose at the point $(1, 2)$, we have that $f_x(1, 2) > 0$ and $f_y(1, 2) > 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will increase. If we move a tiny amount in the positive y direction, the value of f will increase as well.

YouTube link: `https://www.youtube.com/watch?v=bRB4HpWPdNc`

Similarly, suppose at the point $(1, 2)$, we have that $f_x(1, 2) < 0$ and $f_y(1, 2) < 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will decrease. If we move a tiny amount in the positive y direction, the value of f will decrease as well.

YouTube link: `https://www.youtube.com/watch?v=17cKrUCjG9U`

Now, let's consider the case where $f_x(1, 2) > 0$ and $f_y(1, 2) < 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will increase. But, if we move a tiny amount in the positive y direction, the value of f will decrease.

Learning outcomes: Understand the geometric significance of partial derivatives.
Author(s): Melissa Lynn

YouTube link: https://www.youtube.com/watch?v=IEWHvz46pXw

Next, let's suppose that $f_x(1,2) > 0$ and $f_y(1,2) = 0$. As expected, f increases as we move a tiny amount in the positive x direction. On the other hand, the graph of f has flattened out as we move in the y direction. However, this doesn't mean that it's constant! It's just the instantaneous rate of change that's 0 at that one point.

YouTube link: https://www.youtube.com/watch?v=ukDx0LVk24Y

Now, let's look at a case where $f_x(1,2) = 0$ and $f_y(1,2) = 0$. As before, this does not mean that f is constant. This just means that the rates of change are both instantaneously 0. Points with this property will be important later in the course, when we study optimization.

YouTube link: https://www.youtube.com/watch?v=i6dsWWlQjNQ