Cylindrical Coordinates

In this activity, we introduce cylindrical coordinates, a new coordinate system on $\mathbb{R}^3$. We also discuss how to convert between cylindrical and Cartesian coordinates.

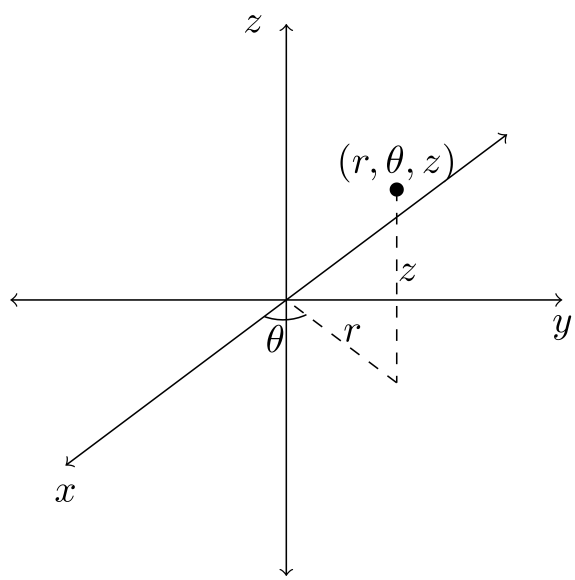
Cylindrical Coordinates

We've seen how points in $\mathbb{R}^2$ can be written using polar coordinates. Polar coordinates can be useful for describing shapes that are difficult to describe in Cartesian coordinates.

We'd now like to extend this idea to $\mathbb{R}^3$, using a coordinate system called *cylindrical coordinates*. Like polar coordinates, cylindrical coordinates will be useful for describing shapes in $\mathbb{R}^3$ that are difficult to describe using Cartesian coordinates. Later in the course, we will also see how cylindrical coordinates can be useful in multivariable Calculus, when differentiating or integrating in Cartesian coordinates is difficult or impossible.

Cylindrical coordinates are really just a simple extension of polar coordinates. For points in the xy-plane, we describe them using r and θ, where r is the distance from the origin and θ is the angle with the positive x-axis. We then tack on a z-coordinate, the exact same as the z-coordinate in Cartesian coordinates, which tells us the vertical displacement of the point.

Learning outcomes:
Author(s):



Example 1. *We'll convert the point $(x, y, z) = (1, 1, 1)$ to cylindrical coordinates.*

We can figure out r and θ by just considering the x- and y-coordinates of the point, $(1, 1)$. Then this becomes equivalent to representing the point in polar coordinates, so we have

$$(r, \theta) = \boxed{(\sqrt{2}, \pi/4)}.$$

For last coordinate, z, notice that this is telling us the height of the point, which is the exact same as the z-coordinate of the point written in Cartesian coordinates! So, our z coordinate is $\boxed{1}$, and the point $(x, y, z) = (1, 1, 1)$ can be written in cylinderical coordinates as

$$(r, \theta, z) = \boxed{(\sqrt{2}, \pi/4, 1)}.$$

You may use the applet below to experiment with how changing the different coordinates changes the point given in cylindrical coordinates.

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Uniqueness

When we studied polar coordinates, we saw that there were many different ways to represent a point. For example, the point $(x, y) = (0, 1)$ could be written as $(r, \theta) = (1, \pi/2)$, $(1, 5\pi/2)$, or even $(-1, 3\pi/2)$. And the origin was especially devious, it could be written as $(0, \theta)$ for any angle θ.

Because of this and the relationship between polar and cylindrical coordinates, it's not surprisingly that cylindrical coordinates have similar issues with uniqueness. For example, the point $(0, 1, 1)$ can be written as $(r, \theta, z) = (1, \pi/2, 1)$, $(1, 5\pi/2, 1)$, $(-1, 3\pi/w, 1)$, and so on. Any point on the z-axis can be written as $(0, \theta, z)$, where z is its z-coordinate, and θ is any angle.

Problem 1 *Which of the following, written in cylindrical coordinates, is equivalent to the point $(x, y, z) = (1, 1, 1)$? Select all that apply.*

Select All Correct Answers:

(a) $(1, 1, 1)$

(b) $(1, \pi/4, 1)$

(c) $\sqrt{2}, \pi/4, 1)$ ✓

(d) $(-1, 3\pi/4, 1)$

(e) $-\sqrt{2}, \pi/4, 1)$

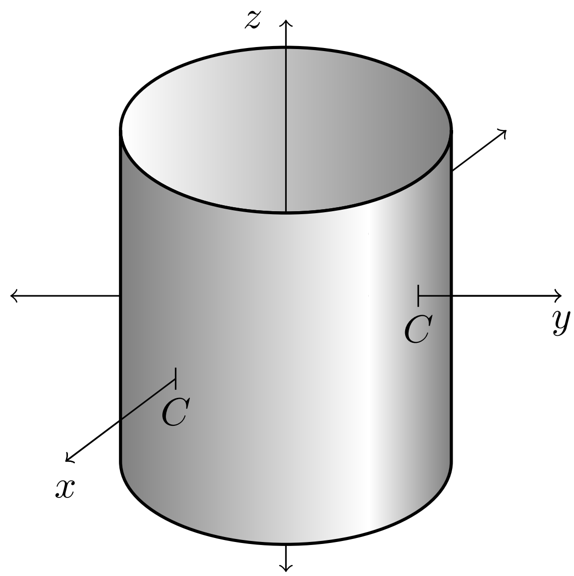
(f) $-\sqrt{2}, -3\pi/4, 1)$ ✓

As with polar coordinates, in situations where uniqueness is important, we will often make the restrictions $r \geq 0$ and $0 \leq \theta < 2\pi$.

Constant-Coordinate Surfaces

Let's look at what happens in cylindrical coordinates when we set each of the coordinates r, θ, z to be constant, with the standard restrictions that $0 \leq r$ and $0 \leq \theta \leq \pi/2$. This can give us insight to how cylindrical coordinates behave.

We'll begin by examining the set of points (r, θ, z), where $r = C$ is a constant. We have that $r = C$ is constant, which means that the distance between any such point and the z axis is constant, C. Also, θ and z can be anything. This will give us the cylinder of radius C, centered at the z-axis.



Next, we'll investigate the set of points (r, θ, z), where $\theta = C$ is constant. Let's consider the projection of this point onto the xy-plane. The projection will make an angle C with the positive x-axis, and have distance $r \geq 0$ from the origin. The height of the point can be any real number. From these observations, we conclude that the set of such points is the following half plane in $\mathbb{R}^3$.

Note that if we didn't have the restriction $r \geq 0$, we would get an entire plane rather than a half plane.

Finally, we'll consider the set of points (r, θ, z), where $z = C$ is constant. Since $z = C$, we will only have points at height C. Varying r and θ will then give us all points in the plane at height C parallel to the xy-plane, as below.

Converting between Cartesian and cylindrical coordinates

Perhaps not surprisingly, converting between Cartesian coordinates and cylindrical coordinates is very similar to how we converted between Cartesian coordinates and polar coordinates. That is, we can use the equations:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z, \\ r^2 &= x^2 + y^2, \\ \tan\theta &= \frac{y}{x}. \end{aligned}$$

Example 2. *We'll convert $z = \sqrt{1 - r^2}$ to Cartesian coordinates.*

Using $r^2 = x^2 + y^2$, we have

$$z = \sqrt{1 - x^2 - y^2}.$$

You may recognize this as the top half of the sphere of radius 1 centered at the origin. You could also rewrite this as

$$x^2 + y^2 + z^2 = 1,$$

keeping in mind that $z \geq 0$.

Example 3. *We'll convert $(x-2)^2 + y^2 = 1$ (where z can be anything) to cylindrical coordinates. Note that this is the cylinder of radius 1, centered at the vertical line through $(2, 0, 0)$.*

Expanding the expression, we have

$$x^2 - 4x + 1 + y^2 = 1.$$

Substituting $r^2 = x^2 + y^2$ and subtracting 1 from each side, we obtain

$$r^2 - 4x = 0.$$

We then substitute $x = r\cos\theta$.

$$r^2 - 4r\cos\theta = 0.$$

Dividing both sides by r, we have

$$r - 4\cos\theta = 0,$$

or

$$r = 4\cos\theta,$$

and z can be anything.

When we divided by r, we implicitly assumed that r was not 0. This means that we might accidentally be omitting the origin, but if we take $\theta = \pi/2$, we have

$$r = 4\cos(0) = 0,$$

so the origin is already included in the surface $r = 4\cos\theta$.

Conclusion

We introduced cylindrical coordinates and how to convert between cylindrical coordinates and Cartesian coordinates, and we discussed the uniqueness of cylindrical coordinates.