Homework 11: Taylor's Theorem

Graded Problems

Problem 1 *Give a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a > 0$ and $\det(A) > 0$ for which the quadratic form $p(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is **NOT** positive definite. Does this mean the theorem in problem 6 is incorrect?*

Problem 2 *Prove the following theorem: Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a symmetric 2×2 matrix. If $\det(A) < 0$, then the quadratic form $p(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is indefinite, regardless of the value of a.*

(Hint: think about the cases $a > 0$, $a = 0$ and $a < 0$. In two cases you can apply Sylvester's Theorem. In the third case, you'll have to do some work by hand to show $p(x, y)$ can have both positive and negative values.)

Professional Problem

Problem 3 *Complete the online peer review form posted on moodle.*

Completion Packet

Problem 4 *Find the symmetric matrix that represents each quadratic form.*

(a) $r(x_1, x_2, x_3, x_4) = x_3^2 - x_2x_3 + x_1x_4$

Learning outcomes:
Author(s):

(b) $t(\mathbf{x}) = \mathbf{x}^T \begin{bmatrix} 6 & 1 & 8 & -2 \\ 0 & 5 & 1 & 9 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$

Problem 5 *Prove the following theorem* without *using Sylvester's theorem: Let* $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ *be a symmetric* 2×2 *matrix. If* $a > 0$ *and* $\det(A) > 0$*, then the quadratic form* $p(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ *is positive definite.*

(Hint: Write out p *in terms of the variables* x *and* y*, then complete the square with respect to* x *and collect the remaining terms.)*

Problem 6 (a) *Write the Taylor series for* $f(x) = \sin(x)$ *centered at* $x = 0$*.*

(b) *Find the second-order Taylor approximation for* $f(x, y) = \sin(xy)$ *centered at* $(0, 0)$*, using your answer to part (a).*

(c) *Verify your answer to part (b), by computing the second-order Taylor approximation for* $f(x, y) = \sin(xy)$ *directly.*

Problem 7 (a) *Write the Taylor series for* $f(x) = e^x$ *centered at* $x = 0$*.*

(b) *Find the second-order Taylor approximation for* $f(x, y) = e^{x^2+y^2}$ *centered at* $(0, 0)$*, using your answer to part (a).*

(c) *Verify your answer to part (b), by computing the second-order Taylor approximation for* $f(x, y) = e^{x^2+y^2}$ *directly.*

Problem 8 *Compute the Hessian matrix for each function at the given point.*

(a) $f(x, y) = \sqrt{xy}$ at $(1, 1)$

(b) $f(x, y) = \cos(x) + x^2 \sin(y)$ at $(0, \pi)$

(c) $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2 + 1}$ at $(0, 0, 0)$

Problem 9 *Consider the function* $f(x,y) = \dfrac{x}{x+y}$ *and the point* $\mathbf{a} = (1,2)$.

(a) *Find the first-order Taylor polynomial of* f *at* $\mathbf{a}$.

(b) *Find the second-order Taylor polynomial of* f *at* $\mathbf{a}$.

(c) *Express the second-order Taylor polynomial using the derivative matrix and the Hessian matrix, as in formula (10) of section 4.1 of the textbook.*

Problem 10 *Consider the function* $f(x,y) = x^2e^y$ *and the point* $\mathbf{a} = (1,0)$.

(a) *Find the first-order Taylor polynomial of* f *at* $\mathbf{a}$.

(b) *Find the second-order Taylor polynomial of* f *at* $\mathbf{a}$.

(c) *Express the second-order Taylor polynomial using the derivative matrix and the Hessian matrix, as in formula (10) of section 4.1 of the textbook.*