Differentiability of Functions of Two Variables

So far, we have an informal definition of differentiability for functions $f : \mathbb{R}^2 \to \mathbb{R}$: if the graph of f "looks like" a plane near a point, then f is differentiable at that point.

Definition 1. *(Informal Definition) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose for some point (a, b) in $\mathbb{R}^2$, if we zoom in on the graph of f near the point (a, b), the graph of f looks like a plane. Then f is differentiable at (a, b).*

In the case where a function is differentiable at a point, we defined the tangent plane at that point.

Definition 2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then the* tangent plane *to the graph of f at (a, b) is defined by the equation*

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We would like a formal, precise definition of differentiability. The key idea behind this definition is that a function should be differentiable if the plane above is a "good" linear approximation. To see what this means, let's revisit the single variable case.

In single variable calculus, a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = a$ if the following limit exists:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This limit exists if and only if

$$\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right) = 0.$$

In turn, this is true if and only if

$$\lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = 0.$$

If we let $L(x) = f(a) + f'(a)(x - a)$, this is equivalent to

$$\lim_{x \to a} \frac{f(x) - L(x)}{x - a} = 0.$$

Learning outcomes:
Author(s):

Recall that $L(x)$, as defined above, is the linear approximation to f at $x = a$. This is also a function whose graph is the tangent line to f at $x = a$. So, roughly speaking, we have shown that a single variable function is differentiable if and only the difference between $f(x)$ and its linear approximation goes to 0 quickly as x approaches a.

This idea will inform our definition for differentiability of multivariable functions: a function will be differentiable at a point if it has a good linear approximation, which will mean that the difference between the function and the linear approximation gets small quickly as we approach the point.

Formal definition of differentiability

We are now in position to give our formal definition of differentiability for a function $f : \mathbb{R}^2 \to \mathbb{R}$. We'll make our definition so that a function is differentiable at a point if the difference between the function and the linear approximation

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

gets small "quickly". Here, "quickly" is relative to how $\vec{x}$ is approaching $\vec{a}$, so relative to the distance $\|\vec{x} - \vec{a}\|$ between these points.

Notice that the function $h(x, y)$ matches the equation for the tangent plane, when the function f is differentiable.

Definition 3. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, and suppose that the partial derivatives f_x and f_y are defined at the point $(x, y) = (a, b)$. Define the linear function*

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We say that f is differentiable *at $(x, y) = (a, b)$ if*

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0.$$

If either of the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ do not exist, or the above limit does not exist or is not 0, then f is not differentiable at (a, b).

We had previously used our informal definition of differentiability to determine that the function $f(x, y) = xy + 2x + y$ is differentiable at $(0, 0)$. Let's verify this using our new, formal definition of differentiability.

Example 1. *We'll show that the function $f(x, y) = xy + 2x + y$ is differentiable at $(0, 0)$. In order to do this, we first need to find the function $h(x, y)$. This repeats earlier work, where we found the tangent plane to $f(x, y) = xy + 2x + y$ at $(0, 0)$.*

We begin by finding the partial derivatives with respect to x and y.

$$\begin{aligned} f_x(x,y) &= \boxed{y+2} \\ f_y(x,y) &= \boxed{x+1} \end{aligned}$$

At $(0,0)$, we have

$$\begin{aligned} f_x(0,0) &= \boxed{2}, \\ f_y(0,0) &= \boxed{1}. \end{aligned}$$

Finding the value of f at $(0,0)$, we have

$$f(0,0) = \boxed{0}.$$

Putting all of this together, we obtain an equation for the function $h(x,y)$.

$$\begin{aligned} h(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= \boxed{2x+y} \end{aligned}$$

Now, we show that f is differentiable at $(a,b) = (0,0)$, by evaluating the limit

$$\begin{aligned} \lim_{(x,y)\to(a,b)} \frac{f(x,y)-h(x,y)}{\|(x,y)-(a,b)\|} &= \lim_{(x,y)\to(0,0)} \frac{(xy+2x+y)-(2x+y)}{\sqrt{(x-0)^2+(y-0)^2}} \\ &= \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}. \end{aligned}$$

Switching to polar coordinates, we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{r\cos\theta \cdot r\sin\theta}{\sqrt{r^2}} \\ &= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{|r|}. \end{aligned}$$

Since $-1 \le \cos\theta\sin\theta \le 1$, we have

$$-|r| \le \frac{r^2\cos\theta\sin\theta}{|r|} \le |r|.$$

Since $\lim\limits_{r\to 0} -|r| = \lim\limits_{r\to 0} |r| = 0$, by the squeeze theorem, we have

$$\lim_{r\to} \frac{r^2\cos\theta\sin\theta}{|r|} = 0.$$

Thus, we have shown that $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)-h(x,y)}{\|(x,y)-(0,0)\|} = 0$, showing that f is differentiable at $(0,0)$.