

Parametric Curves

We've dealt with several ways to describe curves in \mathbb{R}^2 :

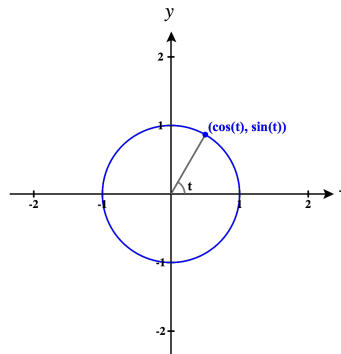
- As the graph of a function. For example, $f(x) = x^2$.
- As the set of points satisfying an equation. For example, the points (x, y) such that $x^2 + y^2 = 1$.
- As the set of points satisfying an equation in another coordinate system. For example, $r = \sin(\theta)$ in polar coordinates.

Another way that we can describe a curve is using *parametric equations*. When describing a curve using parametric equations, we define x and y in terms of a third variable, often t , called the *parameter*. We often think about t as representing time, and imagine the curve being drawn out as t increases. This gives us another way to describe curves in \mathbb{R}^2 , and potentially describe some new and strange curves.

We can describe the unit circle in \mathbb{R}^2 with the parametric equations

$$\begin{aligned}x &= \cos(t), \\y &= \sin(t),\end{aligned}$$

for $0 \leq t \leq 2\pi$.



Learning outcomes: Understand the definition of a parametric curves, and that there can be multiple parametrizations for a given curve. Be able to graph a parametric curve, and parametrize curves.

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We can think of t as giving the angle that a point makes with the positive axis. It can also be helpful to imagine t as representing time, and the parametric equations tracing out the circle as time passes.

YouTube link: <https://www.youtube.com/watch?v=1J6bT8RE3so>

Consider the parametric equations for the unit circle in \mathbb{R}^2 :

$$\begin{aligned}x &= \cos(t), \\y &= \sin(t),\end{aligned}$$

for $0 \leq t \leq 2\pi$.

We can combine these equations into a single vector,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

We can visualize the vectors $\vec{x}(t)$ tracing out the unit circle as t goes from 0 to 2π .

YouTube link: <https://www.youtube.com/watch?v=sYYekBSL4cg>

Notice that we are blurring the distinction between vectors and points. Although we can think of $\vec{x}(t)$ as a position vector, we would more commonly think of $\vec{x}(t)$ as a point on a curve. Although this might seem a bit sloppy, it will prove very useful throughout the course. Although intuitively we might prefer to use points, a lot of the computations tools that we'll require are more appropriately used with vectors.

We have defined a function \vec{x} from the interval $[0, 2\pi] \subset \mathbb{R}$ to \mathbb{R}^2 , and this idea provides the motivation behind our definition for paths.

Parametric Curves in \mathbb{R}^n

Definition 1. A path in \mathbb{R}^n is a continuous function

$$\vec{x} : I \subset \mathbb{R} \rightarrow \mathbb{R}^n,$$

where $I \subset \mathbb{R}$ is an interval.

This is also called a parametrized curve or parametric curve. When we wish to emphasize that $\vec{x}(t)$ is a vector, we'll call it the position vector.

We'll focus on the cases $n = 2$ and $n = 3$ in this course.

We defined a path as a continuous function, however, we haven't said what it means for a multivariable function to be continuous. We'll come back to this later, and we'll give a rigorous definition for continuity. For now, this should fit with your intuition: you can draw the path without lifting your pencil from the paper.

Sometimes we care more about the image of a path than how the path is drawn out, and then we refer to a curve.

Definition 2. A curve C in \mathbb{R}^n is the image of some path $\vec{x} : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$.

In this case, we say that \vec{x} is a parametrization for the curve C .

The difference between a curve and a path is largely a matter of perspective: when working with a curve, we pay attention to *what* is drawn; when working with a path, we care about *how* it is drawn.

Example 1. There are many different parametrizations for a given curve.

Consider again the unit circle C in \mathbb{R}^2 . Which of the following are parametrizations for C ?

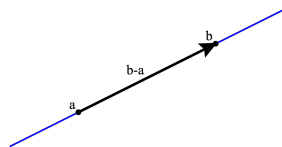
Select All Correct Answers:

- (a) $\vec{x}(t) = (\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$ ✓
- (b) $\vec{x}(t) = (\sin(t), \cos(t))$ for $0 \leq t \leq \pi$
- (c) $\vec{x}(t) = (t, \pm\sqrt{1-t^2})$ for $-1 \leq t \leq 1$
- (d) $\vec{x}(t) = (\sin(2\pi t), \cos(2\pi t))$ for $0 \leq t \leq 1$ ✓
- (e) $\vec{x}(t) = (\cos(t), \sin(t))$ for $-10 \leq t \leq 10$ ✓

Example 2. In this example, we review a parametrization for the line through points \vec{a} and \vec{b} in \mathbb{R}^n .

Given points \vec{a} and \vec{b} in \mathbb{R}^n , we obtain a vector starting at \vec{a} and ending at \vec{b} by taking $\vec{b} - \vec{a}$. This vector is parallel to the line through \vec{a} and \vec{b} . Then, taking scalar multiples $t(\vec{b} - \vec{a})$ for $t \in \mathbb{R}$, we have a line parallel to the line through \vec{a} and \vec{b} . Finally, we add one of the points, \vec{a} , to ensure that our line passes through these two points. Thus, we arrive at our parametrization,

$$\vec{l}(t) = \vec{a} + t(\vec{b} - \vec{a}) \text{ for } t \in \mathbb{R}.$$

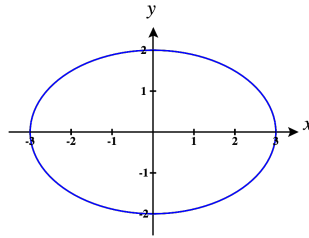


Example 3. In this example, we see how we can obtain new transformations from old ones, using linear algebra and simple transformations.

Recall the parametrization for the unit circle in \mathbb{R}^2 ,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Now, consider the ellipse below.



We can think of this ellipse as the result of stretching the unit circle horizontally by a factor of 3 and vertically by a factor of 2. That is, we are applying the linear transformation

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

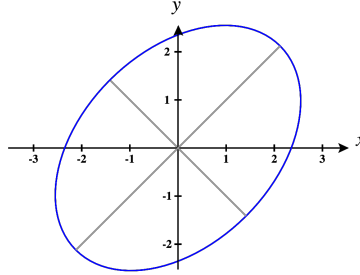
We can apply this to the parametrization for the unit circle, in order to parametrize the ellipse.

$$\begin{aligned} \vec{y}(t) &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \\ &= (3 \cos(t), 2 \sin(t)). \end{aligned}$$

Thus, we have a parametrization for the ellipse given by

$$\vec{y}(t) = (3 \cos(t), 2 \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Next, consider the following ellipse.



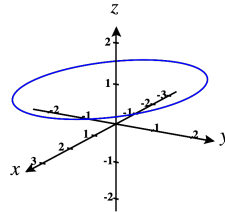
We can obtain this from our previous ellipse by counterclockwise rotation of $\pi/4$. The matrix for this linear transformation is

$$\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \boxed{1/\sqrt{2}} & \boxed{-1/\sqrt{2}} \\ \boxed{1/\sqrt{2}} & \boxed{1/\sqrt{2}} \end{pmatrix}.$$

Applying this rotation to our parametrization for the previous ellipse, we obtain a parametrization for our new ellipse.

$$\vec{z}(t) = \boxed{(3/\sqrt{2} \cos(t) - 2/\sqrt{2} \sin(t), 3/\sqrt{2} \cos(t) + 2/\sqrt{2} \sin(t))} \text{ for } 0 \leq t \leq 2\pi$$

Finally, we consider an ellipse in \mathbb{R}^3 , shown below.



This ellipse is parallel to the xy -plane, and will have constant z -coordinate of 1. Note the similarity to the first ellipse we considered. A parametrization for

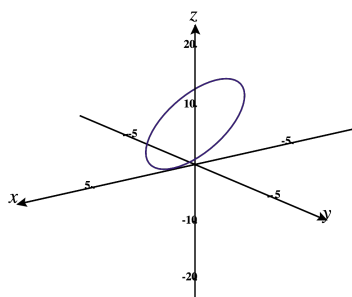
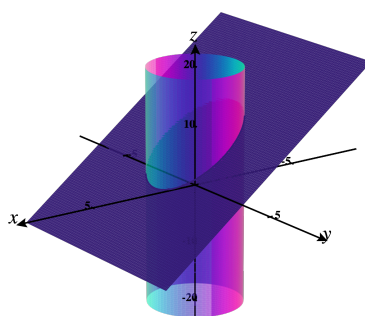
this ellipse can be obtained by taking the parametrization \vec{y} for our first ellipse in \mathbb{R}^2 , and appending the constant z -coordinate.

$$\vec{a}(t) = \boxed{(3 \cos(t), 2 \sin(t), 1)} \text{ for } 0 \leq t \leq 2\pi$$

Examples in \mathbb{R}^3

In this section, we give examples of parametrizations of a couple of more complicated curves in \mathbb{R}^3 , taking advantage of our previous experience with cylindrical coordinates.

Example 4. We'll parametrize the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = 7 - 3x$ in \mathbb{R}^3 , pictured below.



Our x and y coordinates must satisfy $x^2 + y^2 = 4$, which would define a circle, if we were in \mathbb{R}^2 . Recalling our parametrizations for circles, these coordinates

can be written as

$$\begin{aligned}x(t) &= 2 \cos(t) \\ y(t) &= 2 \sin(t)\end{aligned}$$

for $0 \leq t \leq 2\pi$.

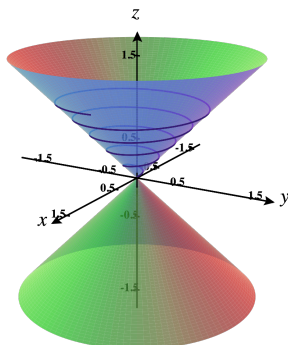
It remains to write the z -coordinate in terms of the parameter t . Turning our attention to the equation for the plane, $z = 7 - 3x$, we have z expressed in terms of x . Since we have expressed x in terms of t , we can make this substitution to describe z in terms of t ,

$$z(t) = 7 - 6 \cos(t).$$

Putting all of this together, we have a parametrization for this intersection given by

$$\vec{x}(t) = (2 \cos(t), 2 \sin(t), 7 - 6 \cos(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Example 5. Consider the curve below, which lies on the cone $z^2 = x^2 + y^2$, and makes five rotations around the z -axis as the height ranges from 0 to 1. We'll refer to this curve as a "tornado."



We'll parametrize this curve by thinking about it in cylindrical coordinates, using the height as the parameter.

First, let's consider what's happening with the z -coordinate. Since the height of the tornado ranges from 0 to 1, so will z . We'll set $z = t$, with $0 \leq t \leq 1$, and express x and y in terms of t as well.

Now, we turn our attention to the angle θ . As the height ranges from 0 to 1, the tornado makes five revolutions, so θ should range from 0 to 10π . Thus, expressing θ in terms of t , we let $\theta = 10\pi t$.

Next, we consider the radius r . Since we are on the cone $z^2 = x^2 + y^2$, we have $z^2 = r^2$. Since $z \geq 0$, we have $z = r$. Thus, we can write r in terms of t as $r = t$.

Finally, putting all of this together with $x = r \cos \theta$ and $y = r \sin \theta$, we have a parametrization for the tornado given by

$$\vec{x}(t) = \boxed{(t \cos(10\pi t), t \sin(10\pi t), t)} \text{ for } t \in [0, 1].$$

Images were generated using *CalcPlot3D*.