Geometry of Differentiability

In single variable calculus, derivatives were closely related to the slope of the tangent line to a graph at a point. We used this idea of the slope of the tangent line to define derivatives as a limit of slopes of secant lines,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

YouTube link: `https://www.youtube.com/watch?v=1FATfS-nTl8`

In the other direction, we were able to use differentiation rules to more easily find the equation for the tangent line to a graph at a point.

Example 1. *We'll find the equation for the tangent line to the graph of* $f(x) = x^3 + 2x + 1$ *at* $x = 2$.

We can find the slope of the tangent line by computing $f'(2)$*. Using differentiation rules, we have*

$$f'(x) = \boxed{3x^2 + 2}.$$

Plugging in $x = 2$*, we have* $f'(2) = \boxed{14}$.

Since the tangent line will have to pass through the point $(2, f(2))$*, we compute*

$$f(2) = \boxed{13}.$$

So, the tangent line will pass through the point $(2, 13)$*, and will have slope* 14*. Writing the equation of the line in point-slope form, we have*

$$y - 13 = \boxed{14(x - 2)}.$$

When a single variable function is differentiable, we can use the above method to find an equation for the tangent line. In addition, the tangent line provides us with a good linear approximation for the function.

We would like to do something analogous for multivariable functions, but this raises a few questions. What would be the equivalent of the tangent line? What does it mean for a function to be differentiable?

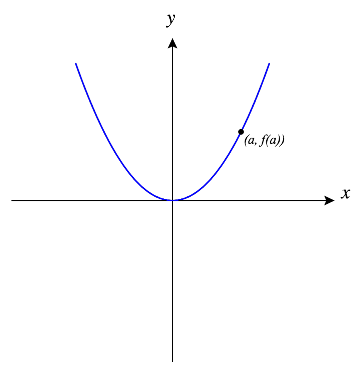
As we begin to explore these questions, we'll focus on functions $\mathbb{R}^2 \to \mathbb{R}$, so that we can visualize their graphs.

Learning outcomes:
Author(s):

Geometric Interpretation of Differentiability

In single variable calculus, we could get a good sense of whether a function was differentiable by looking at its graph.

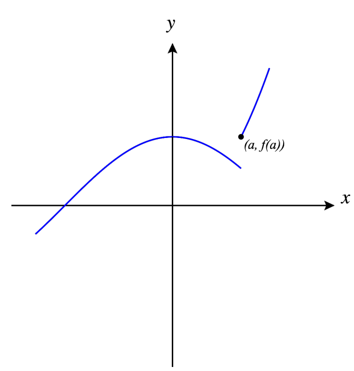
Problem 1 *For each of the graphs, determine whether the given function is differentiable at $x = a$.*



Multiple Choice:

(a) *differentiable* ✓

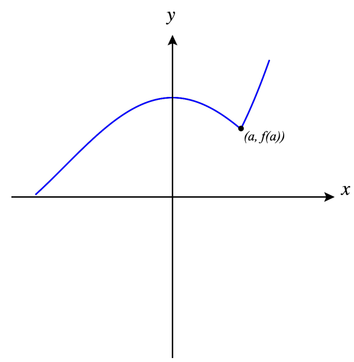
(b) *not differentiable*



Multiple Choice:

(a) *differentiable*

(b) *not differentiable* ✓



Multiple Choice:

(a) *differentiable*

(b) *not differentiable* ✓

If there is a discontinuity or some sort of corner or cusp in the graph at a point, then the function will not be differentiable at that point. Roughly speaking, if we "zoom in" on the graph of a function near a point, and the graph looks very close to a line, then the function will be differentiable at that point.

YouTube link: `https://www.youtube.com/watch?v=U_t2RFukDlA`

We'll extend this idea to make our first, informal definition of differentiability for a function from $\mathbb{R}^2$ to $\mathbb{R}$.

Definition 1. *(Informal Definition) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose for some point (a, b) in $\mathbb{R}^2$, if we zoom in on the graph of f near the point (a, b), the graph of f looks like a plane. Then f is differentiable at (a, b).*

Although this definition provides us with nice geometric intuition for determining if a function is differentiable, it's not at all precise or rigorous. Eventually, we'll need a more formal definition of differentiability, so we'll return to this concept later.

For now, let's use this informal definition to investigate differentiability for a couple of functions.

Example 2. *Consider the function $f(x, y) = xy + 2x + y$, graphed below.*

YouTube link: https://www.youtube.com/watch?v=AuaD1C1jLsA

Is f differentiable at $(0, 0)$?

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Example 3. *Consider the function*

$$f(x, y) = \begin{cases} 1 - |y| & \text{if } |y| \leq |x| \\ 1 - |x| & \text{if } |y| > |x| \end{cases},$$

graphed below.

YouTube link: https://www.youtube.com/watch?v=u5CZGY65wtc

Is f differentiable at $(0, 0)$?

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

The tangent plane

If a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, when we zoom in, the graph will look like a plane. Because of this, there will be a plane that's a good linear approximation for the function near that point. We can use the partial derivatives with respect to x and y to find an equation for this plane, which we call the tangent plane.

Definition 2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then the* tangent plane *to the graph of f at (a, b) is defined by the equation*

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Let's revisit our previous examples, and see what happens with the tangent plane.

Example 4. *Consider the function $f(x, y) = xy + 2x + y$, which we found is differentiable at the point $(0, 0)$. Let's find an equation for the tangent plane to the graph of f at $(0, 0)$.*

We begin by finding the partial derivatives with respect to x and y.

$$f_x(x,y) = \boxed{y+2}$$
$$f_y(x,y) = \boxed{x+1}$$

At $(0,0)$, we have

$$f_x(0,0) = \boxed{2},$$
$$f_y(0,0) = \boxed{1}.$$

Finding the value of f at $(0,0)$, we have

$$f(0,0) = \boxed{0}.$$

Putting all of this together, we obtain an equation for the tangent plane to the graph of f at $(0,0)$.

$$\begin{aligned} z &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= \boxed{2x+y} \end{aligned}$$

Example 5. *Consider the function*

$$f(x,y) = \begin{cases} 1-|y| & \text{if } |y| \le |x| \\ 1-|x| & \text{if } |y| > |x| \end{cases},$$

which we found is not differentiable $(0,0)$. Even though this function is not differentiable, let's see what happens when we try to find an equation for the tangent plane.

To compute the partial derivatives with respect to x and y, we'll need to use the limit definition.

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{(1-|0|)-(1-|0|)}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= 0 \\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{(1-|0|)-(1-|0|)}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$

We can also find $f(0,0)$*,*

$$f(0,0) = 1.$$

So, we have all of the necessary pieces to find an equation for the (nonexistent) "tangent plane":

$$\begin{aligned} z &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= 1. \end{aligned}$$

However, we decided that the function wasn't differentiable at $(0,0)$*, so the graph does not have a tangent plane at the point.*

This example brings up a couple of important points.

- It's possible for the partial derivatives of a function to all exist, and yet the function is not differentiable.
- It's possible that we can find the equation $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, and yet f has no tangent plane at the point (a,b).

For these reasons, differentiability is a much more subtle concept in multivariable calculus than it was in single variable calculus, and our next task will be to find a formal definition for differentiability.