

# Practice Problems

## Online Problems

**Problem 1** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (x, y, z)$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

**Problem 2** Compute the 2-dimensional curl of the vector field  $\mathbf{F}(x, y) = (xy^2, y \sin(x^2))$ . Your answer should be given as a vector.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 2xy(\cos(x^2) - 1))}$$

**Problem 3** Compute the 2-dimensional curl of the vector field  $\mathbf{F}(x, y) = (e^{xy}, \cos(y))$ . Your answer should be given as a vector.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, -xe^{xy})}$$

**Problem 4** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (y, z, x)$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(-1, -1, -1)}$$

Learning outcomes:  
Author(s):

**Problem 5** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (yz, xz, xy)$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

**Problem 6** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (\sin(x), \cos(y), xy)$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(x, -y, 0)}$$

**Problem 7** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (0, x^3, \sin(x+y))$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(\cos(x+y), -\cos(x+y), 3x^2)}$$

**Problem 8** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (e^{xy}, \cos(z), e^y)$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(e^y + \sin(z), 0, -xe^{xy})}$$

**Problem 9** Compute the 2-dimensional curl of the vector field  $\mathbf{F}(x, y) = (y, x)$ . Your answer should be given as a vector.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

**Problem 10** Compute the 2-dimensional curl of the vector field  $\mathbf{F}(x, y) = (y, -x)$ . Your answer should be given as a vector.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, -2)}$$

**Problem 11** Compute the 2-dimensional curl of the vector field  $\mathbf{F}(x, y) = (x^2y + 3xy, xy^4 - 2x^2y)$ . Your answer should be given as a vector.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, -x^2 - 3x - 4xy + y^4)}$$

**Problem 12** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (yz, 2xz, 3xy)$ .

$$\nabla \times \mathbf{F} = \boxed{(x, -2y, z)}$$

Find the curl of  $\mathbf{F}$  at the point  $(x, y, z) = (0, 0, 0)$ .

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}$$

Is  $\mathbf{F}$  irrotational?

**Multiple Choice:**

- (a) Yes.
- (b) No. ✓
- (c) Not enough information.

**Problem 13** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (x^2, y^3, z^4)$ .

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}.$$

Find the curl of  $\mathbf{F}$  at the point  $(x, y, z) = (1, 2, 3)$ .

$$(\nabla \times \mathbf{F})(1, 2, 3) = \boxed{(0, 0, 0)}.$$

Is  $\mathbf{F}$  irrotational?

**Multiple Choice:**

- (a) Yes. ✓
- (b) No.
- (c) Not enough information.

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**Problem 14** Compute the two-dimensional curl of the vector field  $\mathbf{F}(x, y) = (-xy, xy)$ .

$$\nabla \times \mathbf{F} = \left(0, 0, \boxed{x + y}\right)$$

Describe the local rotation of  $\mathbf{F}$  at the point  $(1, 1)$ .

**Multiple Choice:**

- (a) Counterclockwise. ✓
- (b) Clockwise.
- (c) No rotation.

Describe the local rotation of  $\mathbf{F}$  at the point  $(-1, 1)$ .

**Multiple Choice:**

- (a) Counterclockwise.
- (b) Clockwise.
- (c) No rotation. ✓

Describe the local rotation of  $\mathbf{F}$  at the point  $(-1, -1)$ .

**Multiple Choice:**

- (a) Counterclockwise.
  - (b) Clockwise. ✓
  - (c) No rotation.
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**Problem 15** Find the 2-dimensional curl of the vector field  $\mathbf{F}(x, y) = (-y, x)$ . Your answer should be given as a vector.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 2)}$$

Is this vector field irrotational?

**Multiple Choice:**

- (a) Yes
- (b) No ✓

Describe the direction of the local rotation of this vector field.

**Multiple Choice:**

- (a) Clockwise
- (b) Counterclockwise ✓
- (c) No rotation

Plot this vector field. *INCLUDE GRAPHING STUFF*

Find the 2-dimensional curl of the vector field  $\mathbf{G}(x, y) = \left( \frac{-y}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$ .

Your answer should be given as a vector.

$$\nabla \times \mathbf{G} = \boxed{(0, 0, 0)}$$

Is this vector field irrotational?

**Multiple Choice:**

- (a) Yes ✓
- (b) No

Describe the direction of the local rotation of this vector field.

**Multiple Choice:**

- (a) Clockwise
- (b) Counterclockwise
- (c) No rotation ✓

Plot this vector field. *INCLUDE GRAPHING STUFF*

Although the vector field  $\mathbf{F}$  and  $\mathbf{G}$  have the same flow lines, we see that one is irrotational and the other is not. Why does this happen?

**Multiple Choice:**

- (a) Math is broken.

- (b) Curl describes local rotation, not global rotation. ✓

**Problem 16** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (-2y \cos(3x), 3x \sin(-2y), 0)$ .

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 2 \cos(3x) - 3 \sin(2y))}$$

Find the curl of  $\mathbf{F}$  at the point  $(x, y, z) = (\pi, \pi, \pi)$ .

$$(\nabla \times \mathbf{F})(\pi, \pi, \pi) = \boxed{(0, 0, -2)}$$

Is  $\mathbf{F}$  a conservative vector field?

**Multiple Choice:**

- (a) Yes.
- (b) No. ✓
- (c) Not enough information.

Justify your answer.

**Free Response:**

**Problem 17** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (ze^{xz} + z \sin(x), xe^{xy}, -\cos(x))$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, x^2 e^{xz}, xye^{xy} + e^{xy})}$$

Is this vector field conservative?

**Multiple Choice:**

- (a) Yes
- (b) No ✓

**Problem 18** Compute the curl of the vector field  $\mathbf{F}(x, y) = (2x - y, -x + 4y)$ .

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is  $\mathbf{F}$  conservative?

**Multiple Choice:**

- (a) Yes. ✓  
 (b) No.

**Problem 18.1** Find a potential function  $f$  for  $\mathbf{F}$ , so that  $\nabla f = \mathbf{F}$ .

$$f(x, y) = \boxed{x^2 - xy + 2y^2}$$

**Problem 19** Compute the curl of the vector field  $\mathbf{F}(x, y) = (2y, 3x)$ .

$$\nabla \times \mathbf{F} = (0, 0, \boxed{1})$$

Is  $\mathbf{F}$  conservative?**Multiple Choice:**

- (a) Yes.  
 (b) No. ✓

**Problem 20** Compute the curl of the vector field  $\mathbf{F}(x, y) = (2x, 3y)$ .

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is  $\mathbf{F}$  conservative?**Multiple Choice:**

- (a) Yes. ✓  
 (b) No.

**Problem 20.1** Find a potential function  $f$  for  $\mathbf{F}$ , so that  $\nabla f = \mathbf{F}$ .

$$f(x, y) = \boxed{x^2 + \frac{3}{2}y^2}$$

**Problem 21** Compute the curl of the vector field  $\mathbf{F}(x, y) = (-4x + y \cos(x), \sin(x))$ .

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is  $\mathbf{F}$  conservative?

**Multiple Choice:**

- (a) Yes. ✓
- (b) No.

**Problem 21.1** Find a potential function  $f$  for  $\mathbf{F}$ , so that  $\nabla f = \mathbf{F}$ .

$$f(x, y) = \boxed{-2x^2 + y \sin(x)}$$

**Problem 22** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (\sin(x), y^2, e^z)$ .

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}$$

Is  $\mathbf{F}$  conservative?

**Multiple Choice:**

- (a) Yes. ✓
- (b) No.

**Problem 22.1** Find a potential function  $f$  for  $\mathbf{F}$ , so that  $\nabla f = \mathbf{F}$ .

$$f(x, y, z) = \boxed{-\cos(x) + \frac{1}{2}y^2 + e^z}$$



**Problem 23** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (x, y, z)$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

Is this vector field conservative?

**Multiple Choice:**

- (a) Yes ✓
- (b) No

**Problem 23.1** Find a potential function  $f$  for  $\mathbf{F}$ , so that  $\nabla f = \mathbf{F}$ .

$$f(x, y, z) = \boxed{\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2}$$

**Problem 24** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (z, 4yz, x^2 + 3y)$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(3 - 4y, 1 - 2x, 0)}$$

Is this vector field conservative?

**Multiple Choice:**

- (a) Yes
- (b) No ✓

**Problem 25** Compute the curl of the vector field  $\mathbf{F}(x, y, z) = (ye^{xy} + z \cos(x), xe^{xy}, \sin(x))$ .

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

Is this vector field conservative?

**Multiple Choice:**

- (a) Yes ✓

(b) No

**Problem 25.1** Find a potential function  $f$  for  $\mathbf{F}$ , so that  $\nabla f = \mathbf{F}$ .

$$f(x, y, z) = \boxed{e^{xy} + \sin(x)z}$$

## Written Problems

**Problem 26** Prove that, for  $C^1$  vector fields  $\mathbf{F}$  and  $\mathbf{G}$  in  $\mathbb{R}^3$ ,

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

(This shows that the curl is an additive operator, so the curl of a sum is the sum of the curls.)

**Problem 27** Let  $\mathbf{F}$  and  $\mathbf{G}$  be vector fields in  $\mathbb{R}^3$ , and let  $a$  and  $b$  be real numbers. Prove that

$$\nabla \times (a\mathbf{F} + b\mathbf{G}) = a(\nabla \times \mathbf{F}) + b(\nabla \times \mathbf{G}).$$

**Problem 28** Let  $\mathbf{F}$  be a vector field in  $\mathbb{R}^3$ , and let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a scalar valued function. Prove that

$$\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}.$$

**Problem 29** Let  $\mathbf{F}(x, y, z) = \frac{c\mathbf{r}}{\|\mathbf{r}\|^3}$ , where  $c$  is constant and  $\mathbf{r} = (x, y, z)$ . Prove that the curl of  $\mathbf{F}$ ,  $\nabla \times \mathbf{F}$ , is zero.

**Problem 30** Let  $\mathbf{F}(x, y, z) = (x, y, z)$ , and let  $\mathbf{a} \in \mathbb{R}^3$  be a vector with constant entries. Prove that

$$\nabla \times (\mathbf{a} \times \mathbf{F}) = 2\mathbf{a}.$$

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**Problem 31** (a) Compute the curl of the vector field  $\mathbf{F} = (x, y, z)$ . Explain why your answer makes sense geometrically.

(b) Suppose we have a  $C^1$  vector field  $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$ . Compute the curl of  $\mathbf{F}$ , and explain why your answer makes sense geometrically.

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**Problem 32** (a) Consider the function  $f(x, y, z) = e^x \sin(y) + z^2 \cos(y)$ . Compute  $\nabla f$ , and verify that  $\nabla \times (\nabla f) = \mathbf{0}$ .

(b) Prove that for any  $C^2$  function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , the curl of the gradient of  $f$  is zero. That is, prove that

$$\nabla \times (\nabla f) = \mathbf{0}.$$


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