Written Homework

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Problem 1 *Prove that, for C^1 vector fields $\mathbf{F}$ and $\mathbf{G}$ in $\mathbf{R}^3$,*

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

(This shows that the curl is an additive operator*, so the curl of a sum is the sum of the curls.)*

Problem 2 (a) *Compute the curl of the vector field $\mathbf{F} = (x, y, z)$. Explain why your answer makes sense geometrically.*

(b) *Suppose we have a C^1 vector field $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$. Compute the curl of $\mathbf{F}$, and explain why your answer makes sense geometrically.*

Problem 3 (a) *Consider the function $f(x, y, z) = e^x \sin(y) + z^2 \cos(y)$. Compute ∇f, and verify that $\nabla \times (\nabla f) = \mathbf{0}$.*

(b) *Prove that for any C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$, the curl of the gradient of f is zero. That is, prove that*

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Professional Problem

Problem 4 *Consider a wheel W centered at a point on the z-axis, rotating about the z-axis. In this problem, we will investigate how the rotation of this wheel relates to the curl of a vector field describing its motion.*

Learning outcomes:
Author(s):

Let P be a point on the wheel W, of distance d from the center. The rotation of the wheel can be describe by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of the wheel W. You may assume that ω is constant. Let $\mathbf{x}(x, y, z)$ give the position of P at time t.

(a) *Carefully explain why $\mathbf{x}'(t)$ is orthogonal to both $\mathbf{w}$ and $\mathbf{x}(t)$. Then use the angle θ in the figure to show that $\mathbf{x}'(t) = \mathbf{w} \times \mathbf{x}(t)$. We can therefore define a "velocity field $\mathbf{F}(x, y, z) = \mathbf{w} \times (x, y, z)$, which describes the motion of the wheel W. In other words, every point on W has velocity given by $\mathbf{F}(x, y, z)$.*

(b) *Show that $\mathbf{x}'(t) = (-\omega y, \omega x)$.*

(c) *Show that $\nabla \times \mathbf{F} = 2\mathbf{w}$. Hence if the motion of an object is described by the velocity field $\mathbf{F}$, the curl vector points in the direction of the axis of (positive) rotation, and its length is proportional to the angular speed of the rotation.*

ADD IMAGE

Hint: *What is the linear speed of P in terms of $\mathbf{x}$?*

Recall that angular speed equals linear speed divided by radius. How you can write this in terms of ω, d, and $\mathbf{x}$?

Be sure your solution addresses why $\mathbf{x}'(t)$ is $\mathbf{w} \times \mathbf{x}(t)$, and not $\mathbf{x}(t) \times \mathbf{w}$.