Multivariable Calculus

October 25, 2018

Contents

Part I

Week 0: Review

Vectors

In this section, we review some basics about vectors. This includes the definition of a vector, basic vector operations, standard basis vectors, and notation.

Vectors

In linear algebra, we often worked with vectors. We begin by recalling the (algebraic) definition of a vector in $\mathbb{R}^n$.

Definition 1. *A* vector *in $\mathbb{R}^n$ is an ordered n-tuple of real numbers. That is, a vector $\vec{v}$ may be written as*

$$\vec{v} = (a_1, a_2, ..., a_n)$$

where $a_1, a_2, ..., a_n$ are real numbers.

We call the numbers a_i the components *or* entries *of the vector. We call n the* dimension *of the vector $\vec{v}$, and say that $\vec{v}$ is n*-dimensional*.*

We write the vector with an arrow above it, as $\vec{v}$, in order to make the distinction between vectors and *scalars*, which are just real numbers. Some other common notations for vectors are $\mathbf{v}$ and $\hat{v}$. It's important to make this distinction between vectors and scalars, so you should make use of one of these notations for vectors.

Example 1. $\vec{v} = (1, 3)$ *is a vector in* $\mathbb{R}^2$*.*

$\vec{w} = (-1, 5, 0)$ *is a vector in* $\mathbb{R}^3$*.*

$\vec{x} = (1, -2, 3)$ *is a vector in* $\mathbb{R}^3$*.*

$\vec{y} = (-6, \pi, 1/24, -0.5, 3)$ *is a vector in* $\mathbb{R}^5$*.*

It's sometimes convenient to write a vector as a column vector instead (particularly when working with linear transformations, which we'll review in a later

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section). We could write

$$\vec{v} = (a_1, a_2, ..., a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

or

$$\vec{v} = (a_1, a_2, ..., a_n) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

The choice between square brackets and parentheses is just a difference in notation, they mean the same thing, and you should feel free to use either.

Example 2. *We write the following vectors as column vectors.*

$\vec{v} = (1, 3) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$\vec{w} = (-1, 5, 0) = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$.

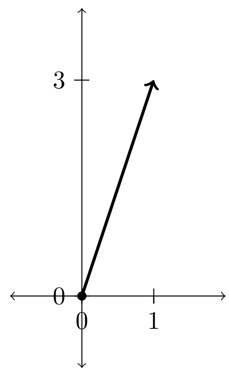
$\vec{x} = (1, -2, 3) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

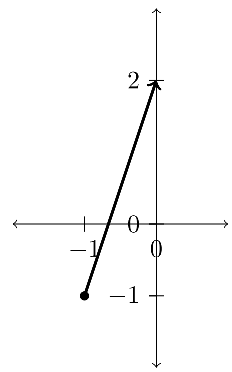
$\vec{y} = (-6, \pi, 1/24, -0.5, 3) = \begin{bmatrix} -6 \\ \pi \\ 1/24 \\ -0.5 \\ 3 \end{bmatrix}$.

The Geometric Perspective

We also can think of a vector geometrically, as giving a direction and magnitude, but without a fixed position.

In two or three dimensions, it is useful to visualize a vector as an arrow in $\mathbb{R}^n$. We might visualize a vector $\vec{v} = (1, 3)$ in $\mathbb{R}^2$ as the arrow starting at the origin and ending at the point $(1, 3)$, thus giving a direction and a magnitude. However, we typically don't think of a vector as having a set location. We could also visualize the vector $\vec{v}$ as starting at the point $(-1, -1)$ and ending at the point $(0, 2)$. Note that this arrow would have the same direction and magnitude as the one starting at the origin, thus they represent the same vector.





In four or higher dimensions, visualizing anything becomes very difficult. It can still be useful to think of a vector $(1,2,3,4,5)$ in $\mathbb{R}^5$ as starting at the origin and ending at the point $(1,2,3,4,5)$, but you probably won't be able to have a very clear picture of this in your head.

This concept will probably seem more useful once you think about a displacement vector.

Definition 2. *Given points $P_1 = (x_1, ..., x_n)$ and $P_2 = (y_1, ..., y_n)$ in $\mathbb{R}^n$, the* displacement vector *from P_1 to P_2 is*

$$\vec{P_1P_2} = (y_1 - x_1, ..., y_n - x_n).$$

This is the vector that starts at P_1 and ends at P_2.

Notice that the notation $(a_1, ..., a_n)$ that we use for a vector in $\mathbb{R}^n$ is identical to the notation that we'd use for a point in $\mathbb{R}^n$. Since both vectors and points in $\mathbb{R}^n$ are defined as n-tuples of points, they are, in some sense, the same thing. The difference between the two comes when we consider the context and geometric significance of the vector or point that we're working with. As we move into multivariable calculus, we'll often blur the distinction between a vector and a point, and sometimes think of a vector as a point and vice versa. This will be greatly simplify notation, and we promise that it won't be as confusing as it sounds!

Vector Operations

Before defining some basic vector operations, we define what it means for two vectors to be equal. This is done by comparing the components of the vectors.

Definition 3. *Two vectors $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ in $\mathbb{R}^n$ are* equal *if their corresponding components are equal, so $a_1 = b_1$, $a_2 = b_2$, ... , $a_n = b_n$.*

Notice that, in order to be equal, two vectors must have the same dimension and the same entries in the same order. Thus, the vectors $(1, 3)$ and $(1, 3, 0)$ are not equal.

We now define addition of two vectors of the same dimension, which is done componentwise.

Definition 4. *Let $\vec{a} = (a_1, a_2, ..., a_n)$ and $\vec{b} = (b_1, b_2, ..., b_n)$ be vectors in $\mathbb{R}^n$. We define $\vec{a} + \vec{b}$ to be the vector in $\mathbb{R}^n$ given by*

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n).$$

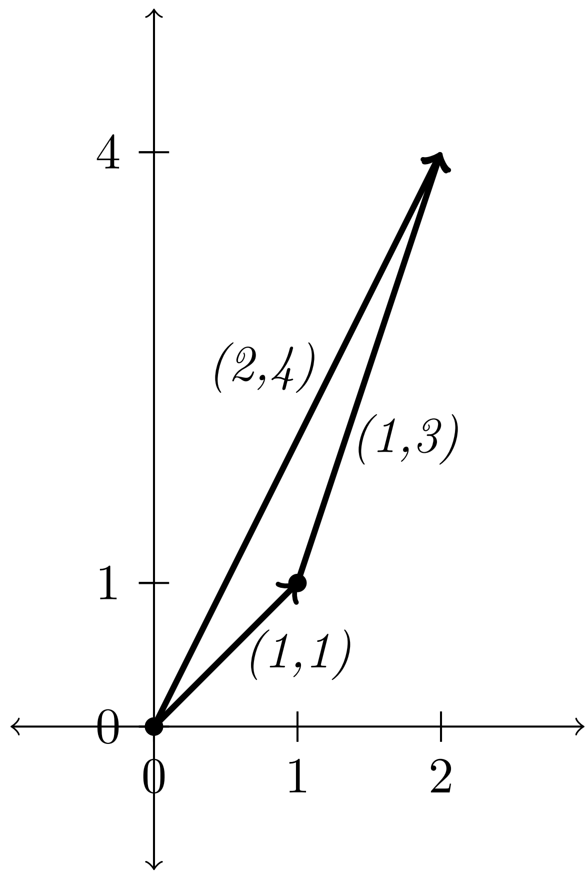
Note that we can only add two vectors if they have the same dimension.

Example 3. *Adding the vectors $\vec{a} = (1, -8, 2)$ and $\vec{b} = (3, -1, -2)$, we obtain*

$$\begin{aligned} \vec{a} + \vec{b} &= (1, -8, 2) + (3, -1, -2) \\ &= (1 + 3, -8 - 1, 2 - 2) \\ &= (4, -9, 0). \end{aligned}$$

Geometrically, we can add vectors by placing the start point of the second vector at the end point of the first vector, and drawing an arrow from the start point of the first vector to the end point of the second vector.

Example 4. *In this example, we add the vectors $(1, 1)$ and $(1, 3)$. Adding these vectors algebraically, we obtain $(2, 4)$. We can also see this geometrically by placing the start point of the vector $(1, 3)$ at the end of the vector $(1, 1)$ (so at the point $(1, 1)$), and drawing the vector from the origin to the end point of the vector $(1, 3)$, which is now at $(2, 4)$.*



Another vector operation is scalar multiplication. Here, we multiply a vector by a real number, possibly changing the length of the vector.

Definition 5. *Let $\vec{a} = (a_1, ..., a_n)$ be a vector in $\mathbb{R}^n$, and let r be a real number (also called a scalar). We define the* scalar product *$r\vec{a}$ to be*

$$r\vec{a} = (ra_1, ..., ra_n).$$

Thus, we see that scalar multiplication is defined by multiplying each component of the vector by the scalar r.

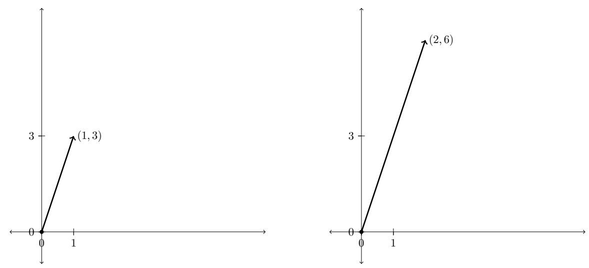
Example 5. $3(1, 5, -2) = (3, 15, -6)$

$-1(1, 1, 1) = (-1, -1, -1)$

$0(6, 2, 4) = (0, 0, 0)$

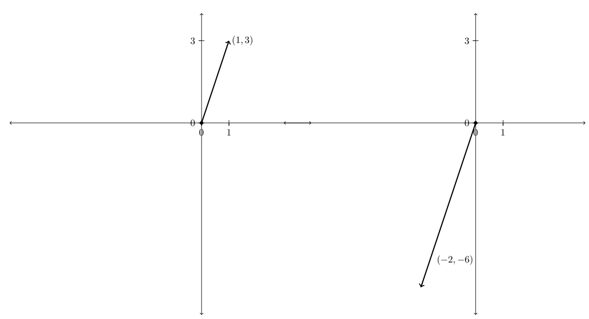
Now, let's look at what scalar multiplication does geometrically. Consider the vector $(1, 3)$.

When we multiply $(1, 3)$ by 2, we obtain $(2, 6)$, which is twice as long as $(1, 3)$ and goes in the same direction.



When we multiply $(1, 3)$ by $\dfrac{1}{2}$, we obtain $\left(\dfrac{1}{2}, \dfrac{3}{2}\right)$, which is half as long as $(1, 3)$ and goes in the same direction.

If we multiply $(1, 3)$ by -2, we obtain $(-2, -6)$, which is twice as long as $(1, 3)$ and goes in the exact opposite direction.



Thus, we have seen that multiplying a vector by a scalar changes the length of a vector, but not the direction (except for reversing it, if the scalar is negative).

Properties

Now, let's recall some useful properties of vector addition and scalar multiplication.

Proposition 1. *Suppose $\vec{a}, \vec{b}, \vec{c}$ are vectors in $\mathbb{R}^n$ and k, l are real numbers. Then*

(a) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ *(vector addition is* commutative*);*

(b) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$ *(vector addition is* associative*);*

(c) $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$*, where* $\vec{0}=(0,...,0)$ *is the* zero vector *in* $\mathbb{R}^n$*;*

(d) $(k+l)\vec{a}=k\vec{a}+l\vec{a}$*;*

(e) $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$ *(with the previous property, scalar multiplication is* distributive*);*

(f) $k(l\vec{a})=(kl)\vec{a}=l(k\vec{a})$*;*

(g) $1\vec{a}=\vec{a}$*.*

These properties tell us different kinds of algebraic manipulations that we can do with vectors.

Standard Basis Vectors

It's often useful to write things in terms of the standard basis vectors for $\mathbb{R}^n$.

Definition 6. *The vectors* $\vec{e}_1=(1,0,...0)$*,* $\vec{e}_2=(0,1,0,...,0)$*, ...,* $\vec{e}_n=(0,...,0,1)$ *in* $\mathbb{R}^n$ *are called the* standard basis vectors *for* $\mathbb{R}^n$*.*

Note that any vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of the standard unit vectors. For example, in $\mathbb{R}^4$,

$$\begin{aligned}(1,5,-3,6) &= 1(1,0,0,0)+5(0,1,0,0)-3(0,0,1,0)+6(0,0,0,1)\\ &= 1\vec{e}_1+5\vec{e}_2-3\vec{e}_3+6\vec{e}_4.\end{aligned}$$

In $\mathbb{R}^2$, we sometimes write the standard basis vectors as $\mathbf{i}=(1,0)$ and $\mathbf{j}=(0,1)$. This gives us a new notation for vectors, for example we could write

$$(3,4)=3\mathbf{i}+4\mathbf{j}.$$

Similarly, in $\mathbb{R}^3$, we sometimes write the standard basis vectors as $\mathbf{i}=(1,0,0)$, $\mathbf{j}=(0,1,0)$, and $\mathbf{k}=(0,0,1)$. We can then write

$$(2,3,4)=2\mathbf{i}+3\mathbf{j}+4\mathbf{k}.$$

Summary

In this section, we reviewed some basics about vectors, including the definition of a vector, basic vector operations, standard basis vectors, notation, and the geometric perspective.

The Dot Product

In this section we review the dot product on vectors. This also includes the angle between vectors and the projection of one vector onto another.

The Dot Product

We begin with the definition of the dot product.

Definition 7. *The* dot product *of two vectors* $\vec{v} = (v_1, v_2, ..., v_n)$ *and* $\vec{w} = (w_1, w_2, ..., w_n)$ *in* $\mathbb{R}^n$ *is*

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Notice that the dot product takes two vectors and outputs a scalar.

Example 6. $(1, 6) \cdot (-3, -6) = -3 - 36 = -39$

$(1, 2, 3) \cdot (7, -2, 4) = 7 - 4 + 12 = 15$

$(1, 7, -3) \cdot (3, 0, 1) = 3 + 0 - 3 = 0$

We can also compute the dot product using the magnitude (or length) of the vectors and the angle in between them.

Proposition 2. *If* $\vec{v}$ *and* $\vec{w}$ *are vectors in* $\mathbb{R}^n$*, then*

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta,$$

where $\|\vec{v}\|$ *and* $\|\vec{w}\|$ *are the lengths of the vectors* $\vec{v}$ *and* $\vec{w}$*, respectively, and* θ *is the angle between* $\vec{v}$ *and* $\vec{w}$*.*

This is illustrated in the picture below.

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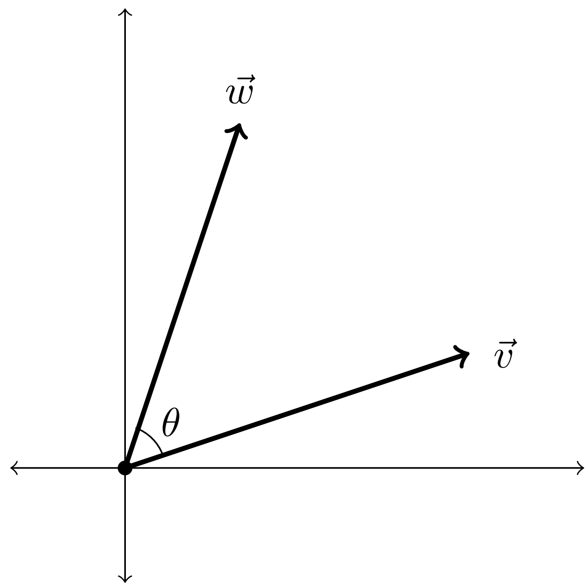

This provides us with a geometric interpretation of the dot product: it gives us a measure of "how much" in the same direction two vectors are (taking their lengths into account). This also gives us a useful way to compute the angle between two vectors.

Example 7. *Consider the vectors* $(1,4)$ *and* $(-2,2)$*. We have*

$$\begin{aligned}(1,4)\cdot(-2,2) &= -2+8 = 6,\\ \|(1,4)\| &= \sqrt{1^2+4^2} = \sqrt{17},\\ \|(-2,2)\| &= \sqrt{(-2)^2+2^2} = \sqrt{8}.\end{aligned}$$

From $\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$*, we then have*

$$6 = \sqrt{17}\sqrt{8}\cos\theta.$$

Solving for θ*, we obtain the angle between the vectors as*

$$\theta = \arccos\left(\frac{6}{\sqrt{17}\sqrt{8}}\right) \approx 59.04^\circ$$

Furthermore, note that for nonzero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, their dot product is 0 if and only if $\cos(\theta) = 0$. This means that θ would have to be 90° or 270°, meaning that $\vec{v}$ and $\vec{w}$ are perpendicular.

Proposition 3. *Two nonzero vectors* $\vec{v}$ *in* $\vec{w}$ *in* $\mathbb{R}^n$ *are perpendicular if and only if* $\vec{v}\cdot\vec{w} = 0$.

This provides us with a very useful algebraic method for determining if two vectors are perpendicular.

Example 8. *The vectors* $(1,7,-3)$ *and* $(3,0,1)$ *in* $\mathbb{R}^3$ *are perpendicular, since*

$$(1,7,-3)\cdot(3,0,1) = 3+0-3 = 0.$$

By taking the dot product of a vector with itself, we get an important relationship between the dot product and the length of a vector.

Proposition 4. *Let* $\vec{v}$ *be a vector in* $\mathbb{R}^n$*. Then*

$$\vec{v}\cdot\vec{v} = \|\vec{v}\|^2.$$

This can be shown directly, or using the fact that the angle between $\vec{v}$ and itself is 0.

Projection of one vector onto another

We can also use the dot product to define the projection of one vector onto another.

Definition 8. *For vectors* $\vec{a}$ *and* $\vec{b}$ *in* $\mathbb{R}^n$*, we define the* vector projection *of* $\vec{a}$ *onto* $\vec{b}$ *as*

$$proj_{\vec{b}}(\vec{a}) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}.$$

Example 9. *We can use this to find the projection of* $(2,4,3)$ *onto* $(1,-1,1)$*.*

$$\begin{aligned}
proj_{(1,-1,1)}(2,4,3) &= \frac{(2,4,3)\cdot(1,-1,1)}{(1,-1,1)\cdot(1,-1,1)}(1,-1,1)\\
&= \frac{2-4+3}{1+1+1}(1,-1,1)\\
&= \frac{1}{3}(1,-1,1)\\
&= \left(\frac{1}{3},-\frac{1}{3},\frac{1}{3}\right)
\end{aligned}$$

Summary

In this section we reviewed the dot product on vectors, the angle between vectors, and the projection of one vector onto another.

The Cross Product

In this section, we review the vector cross product, including the geometric perspective of the cross product, the area of a parallelogram, and the volume of parallelepiped.

The Cross Product

The cross product is fundamentally different from the dot product in a couple of ways. The cross product is defined only on vectors in $\mathbb{R}^3$, while the dot product is defined in $\mathbb{R}^n$ for any positive integer n. Furthermore, the cross product takes two vectors and produces another vector, while the dot product takes two vectors and produces a scalar.

We now give the algebraic definition of the cross product.

Definition 9. *Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$. The* cross product *of $\vec{a}$ and $\vec{b}$, denoted $\vec{a} \times \vec{b}$, is defined to be*

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Equivalently, we can compute the cross product as

$$\vec{a} \times \vec{b} = det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

where

$$\begin{aligned} \mathbf{i} &= (1, 0, 0), \\ \mathbf{j} &= (0, 1, 0), \\ \mathbf{k} &= (0, 0, 1). \end{aligned}$$

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Example 10.

$$\begin{aligned}
(3,2,-1)\times(9,0,2) &= det\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & -1 \\ 9 & 0 & 2 \end{pmatrix} \\
&= (2\cdot 2)\mathbf{i} - (0\cdot -1)\mathbf{i} + (-1\cdot 9)\mathbf{j} - (2\cdot 3)\mathbf{j} + (3\cdot 0)\mathbf{k} - (9\cdot -1)\mathbf{k} \\
&= 4\mathbf{i} - 15\mathbf{j} + 9\mathbf{k} \\
&= (4,-15,9)
\end{aligned}$$

The cross product has some nice algebraic properties, which can be very useful.

Proposition 5. *Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in $\mathbb{R}^3$, and let $k \in \mathbb{R}$ be a scalar. The cross product has the following properties:*

(a) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ *(the cross product is* anticommutative*);*

(b) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$*;*

(c) $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$ *(with the previous property, the cross product is* distributive *over vector addition);*

(d) $k(\vec{a}\times\vec{b}) = (k\vec{a})\times\vec{b} = \vec{a}\times(k\vec{b})$.

In particular, it's important to remember that the cross product is *not* commutative, so the order of the vectors matters!

Geometry of the Cross Product

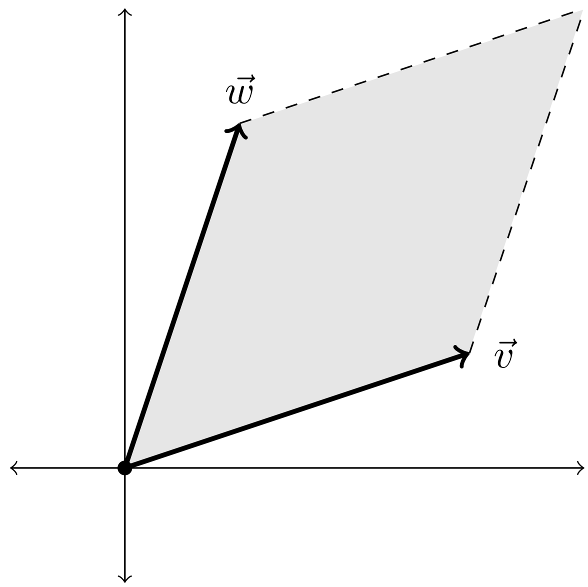
It's often easiest to compute cross products algebraically, but it's easier to understand their significance from a geometric perspective. We now discuss some of the geometric properties of the cross product.

Proposition 6. *Let $\vec{a}$ and $\vec{b}$ be vectors in $\mathbb{R}^3$, and consider their cross product $\vec{a}\times\vec{b}$.*

- *The* magnitude *of the vector $\vec{a}\times\vec{b}$ can be computed as*

  $$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin(\theta),$$

  where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$. Furthermore, this magnitude is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.



- *The vector $\vec{a} \times \vec{b}$ is always perpendicular to both $\vec{a}$ and $\vec{b}$, and follows the* right-hand rule. *That is, if you take your right hand and orient it so you can curl your fingers from the vector $\vec{a}$ to the $\vec{b}$, your thumb will be pointing in the same direction as the cross product $\vec{a} \times \vec{b}$.*

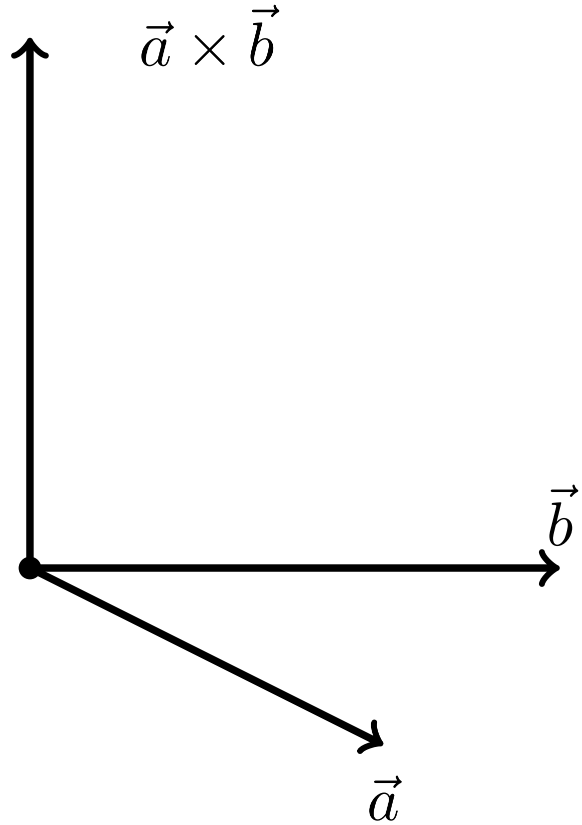
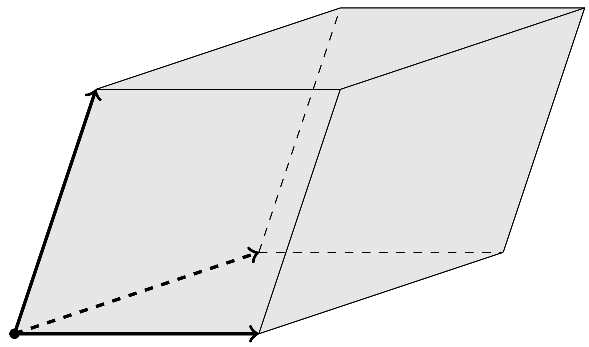


Image this image in $\mathbb{R}^3$, so that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Volume of a Parallelepiped

We can use the cross product and dot product together to compute the volume of a parallelepiped.



The volume of the parellelepiped can be computed as the area of the base times the height. We've seen that the area of the base can be computed as the magnitude of a cross product, $\|\vec{a} \times \vec{b}\|$. The height of the parallelepiped can be computed as $\|\vec{c}\| \, |\cos(\theta)|$, where θ is the angle between the vector $\vec{c}$ and a line perpendicular to the base. We then have that the volume is $\|\vec{a} \times \vec{b}\| \, \|\vec{c}\| \, |\cos(\theta)|$, which we can recognize as the absolute value of the dot product of the vectors $\vec{a} \times \vec{b}$ and $\vec{c}$. Thus we have the following proposition.

Proposition 7. *The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be computed as $|(\vec{a} \times \vec{b}) \cdot \vec{c}|$.*

Summary

We've reviewed the cross product, including its properties and geometric perspective, including its use in finding the area of parallelograms and volume of parallelepipeds.

Matrices

In this section, we review matrices, including the determinant and the linear transformation represented by a matrix.

Matrices

We begin with the definition of a matrix.

Definition 10. *An $m \times n$* matrix *A is a rectangular array of numbers a_{ij}, with m rows and n columns:*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where the a_{ij} are real numbers for i and j integers with $1 \leq i \leq m$ and $1 \leq j \leq n$.

The numbers a_{ij} are called the entries *of the matrix A.*

Note that for an entry a_{ij}, the subscript ij describes the location of a_{ij} in the matrix A: i gives the row, and j gives the column.

We can also think of a matrix as a "vector of vectors" in two different ways. If we imagine that the columns of A are vectors in $\mathbb{R}^n$, then the matrix of A can be viewed as a vector of column vectors. If we imagine that the rows of A are vectors in $\mathbb{R}^n$, then the matrix A can be viewed as a vector of row vectors.

Matrix Operations

Here, we'll define matrix addition and matrix multiplication.

In order to be able to add two matrices, they need to have the exact same dimensions. That is, they both need to be $m \times n$ matrices for some fixed values of m and n. When we have two matrices with the same dimensions, we define their sum component-wise or entry-wise.

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Definition 11. *Let A and B be two $m \times n$ matrices, with*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}.$$

Then we define the matrix sum $A+B$ *to be*

$$A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn}. \end{pmatrix},$$

That is, $A+B$ is the $m \times n$ matrix obtained by adding the corresponding entries of A and B.

Example 11. *We can add the matrices $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$ as follows:*

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} \\ &= \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix} \end{aligned}$$

Example 12. *We cannot add the matrices $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 \\ 10 & 11 \end{pmatrix}$, because their dimensions don't match.*

As you might expect, matrix addition has some nice properties which are inherited from addition of real numbers. We list some of them here.

Proposition 8. *Let A, B, and C be $m \times n$ matrices. Then we have*

(a) $A + B = B + A$ *(matrix addition is* commutative*);*

(b) $A + (B + C) = (A + B) + C$ *(matrix addition is* associative*).*

Furthermore, there is an $m \times n$ matrix O, called the zero matrix, *such that $A + O = A$ for any $m \times n$ matrix A. All of the entries of the zero matrix are the real number 0.*

We've seen that matrix addition works in a very natural way, and multiplying a matrix by a scalar (or real number) is similarly nice. We now define scalar multiplication for matrices.

Definition 12. *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

be an $m \times n$ matrix, and let $k \in \mathbb{R}$ be a scalar. Then the scalar product *of k and A, denoted kA, is*

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}.$$

That is, we obtain the scalar product by multiplying each entry in A by the scalar k.

Example 13. *We can compute the scalar product of 2 and the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ as follows:*

$$\begin{aligned} 2A &= 2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 2\cdot 1 & 2\cdot 2 & 2\cdot 3 \\ 2\cdot 4 & 2\cdot 5 & 2\cdot 6 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}. \end{aligned}$$

We now list some nice properties of scalar multiplication.

Proposition 9. *Let A and B be $m \times n$ matrices, and let k and l be scalars in $\mathbb{R}$. Then*

(a) $(k+l)A = kA + lA$ *(scalar multiplication is* distributive *over scalar addition);*

(b) $k(A+B) = kA + kB$ *(scalar multiplication is* distributive *over matrix addition);*

(c) $k(lA) = (kl)A = l(kA)$.

We'll now define matrix multiplication, which can be a bit trickier to work with than matrix addition or scalar multiplication. Here are some important things to remember about matrix multiplication:

- Not all matrices can be multiplied. In order to compute the product AB of two matrices A and B, the number of columns in A needs to be the same as the number of rows in B.
- Matrix multiplication is *not* commutative. In fact, its possible that the matrix product AB exists but the product BA does not.

Definition 13. *Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix, with*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$
$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}.$$

Note that we are assuming the number of columns in A is the same as the number of rows in B.

We define the matrix product *of A and B, denoted AB, to be the $m \times p$ matrix*

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + a_{12}b_{2p} + \cdots + \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + a_{22}b_{2p} + \cdots + \\ \vdots & \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + a_{m2}b_{2p} + \cdots + \end{pmatrix}$$

Equivalently, we could define the ijth entry of AB to be the dot product of the ith row of A with the jth column of B. This makes sense, since the number of columns in A is the same as the number of rows in B (both n), which ensures that the ith row of A and the jth column of B are both vectors in $\mathbb{R}^n$.

This definition can seem a bit convoluted, and it's easier to understand how matrix multiplication works by going through an example.

Example 14. *We can compute the product AB of the matrices* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

and $B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$ *as follows:*

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}, \\ &= \begin{pmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{pmatrix}, \\ &= \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}. \end{aligned}$$

Note that we multiplied a 2×3 *matrix by a* 3×2 *matrix, and we obtained a* 2×2 *matrix.*

We can also compute the product BA *for the same matrices as above.*

$$\begin{aligned} BA &= \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \\ &= \begin{pmatrix} 7 \cdot 1 + 8 \cdot 4 & 7 \cdot 2 + 8 \cdot 5 & 7 \cdot 3 + 8 \cdot 6 \\ 9 \cdot 1 + 10 \cdot 4 & 9 \cdot 2 + 10 \cdot 5 & 9 \cdot 3 + 10 \cdot 6 \\ 11 \cdot 1 + 12 \cdot 4 & 11 \cdot 2 + 12 \cdot 5 & 11 \cdot 3 + 12 \cdot 6 \end{pmatrix}, \\ &= \begin{pmatrix} 39 & 54 & 69 \\ 49 & 68 & 81 \\ 59 & 82 & 105 \end{pmatrix}. \end{aligned}$$

Note that we multiplied a 3×2 *matrix by a* 2×3 *matrix, and we obtained a* 3×3 *matrix.*

Note that in this case $AB \neq BA$*; matrix multiplication is not commutative, so the order of the matrices matters!*

Although matrix multiplication is not commutative, it still has some nice algebraic properties. We list some of them here.

Proposition 10. *Let* A*,* B*, and* C *be matrices of dimensions such that the following operations are defined, and let* k *be a scalar. Then*

(a) $A(BC) = (AB)C$ *(matrix multiplication is* associative*);*

(b) $k(AB) = (kA)B = A(kB)$*;*

(c) $A(B + C) = AB + AC$*;*

(d) $(A + B)C = AC + BC$ *(with the previous property, matrix multiplication is* distributive *over matrix addition).*

Determinants

When we have a square matrix (meaning an $n \times n$ matrix, where the number of rows and number of columns are the same), we can compute an important number, called the determinant of the matrix. It turns out that this single number can tell us some important things about the matrix!

We begin by defining the determinant of a 2×2 matrix.

Definition 14. *Consider the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define the* determinant *of the matrix A to be*

$$det(A) = ad - bc.$$

We also sometimes use the notation $|A|$ for the determinant of the matrix A.

Note that the determinate of a 2×2 matrix is just a number, not a matrix. We compute the determinant in a couple of examples.

Example 15. *We'll compute the determinant of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.*

$$\begin{aligned} det(A) &= 1 \cdot 4 - 2 \cdot 3 \\ &= -2. \end{aligned}$$

We've defined the determinant of 2×2 matrices, but we haven't defined the determinant of a larger square matrix yet. It turns out that the determinant is defined *inductively*. This means that the determinant of a 3×3 matrix is defined using determinants of 2×2 matrices, the determinant of a 4×4 matrix is defined using determinants of 3×3 matrices, the determinant of a 5×5 matrix is defined using determinants of 4×4 matrices, and so on. This means in order to compute the determinant of a large square matrix, we often need to compute the determinants of many smaller matrices.

We now give the definition of the determinant of an $n \times n$ matrix.

Definition 15. *Let A be an $n \times n$ matrix, with entries a_{ij}. We defined the* determinant *of A to be the number computed by*

$$det(A) = (-1)^{1+1} a_{11} det(A_{11}) + (-1)^{1+2} a_{12} det(A_{12}) + \cdots + (-1)^{1+n} a_{1n} det(A_{1n}),$$

where A_{ij} is the $(n-1) \times (n-1)$ submatrix of A which we obtain by deleting the ith row and $j$the column from A.

This definition is pretty confusing if you read through it without seeing an example, but this actually follows a nice pattern. This pattern is easier to see with an example.

Example 16. *We compute the determinant of the 4×4 matrix,*

$$A = \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that we begin by writing this in terms of determinants of 3×3 matrices. But in order to compute the determinant of each 3×3 matrix, we write it in terms of 2×2 matrices! This winds up being a lot of determinants to compute.

$$\begin{aligned} det(A) = (-1)^{1+1} 1 det \begin{pmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{1+2} 4 det \begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ + (-1)^{1+3} 2 det \begin{pmatrix} 0 & 0 & 1 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{1+4}(-1) det \begin{pmatrix} 0 & 0 & -2 \\ -3 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

We now compute the determinant of each of the 3×3 submatrices, which we *compute using determinants of* 2×2 *matrices.*

$$\begin{aligned}
det\begin{pmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (-1)^{1+1} 0 det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2}(-2) det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + (-1)^{1+3} 1 det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= 1 \cdot 0 \cdot (1 \cdot 1 - 0 \cdot 0) + -1 \cdot (-2) \cdot (0 \cdot 1 - 0 \cdot 0) + 1 \cdot 1 \cdot (0 \cdot 0 - 1 \cdot 0) \\
&= 0 \\
det\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (-1)^{1+1} 0 det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2}(-2) det\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + (-1)^{1+3} 1 det\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \\
&= 1 \cdot 0 \cdot (1 \cdot 1 - 0 \cdot 0) + -1 \cdot (-2) \cdot (-3 \cdot 1 - 0 \cdot 0) + 1 \cdot 1 \cdot (-3 \cdot 0 - 1 \cdot 0) \\
&= 6 \\
det\begin{pmatrix} 0 & 0 & 1 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (-1)^{1+1} 0 det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2} 0 det\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + (-1)^{1+3} 1 det\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \\
&= 1 \cdot 0 \cdot (0 \cdot 1 - 0 \cdot 0) + -1 \cdot 0 \cdot (-3 \cdot 1 - 0 \cdot 0) + 1 \cdot 1 \cdot (-3 \cdot 0 - 0 \cdot 0) \\
&= 0 \\
det\begin{pmatrix} 0 & 0 & -2 \\ -3 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} &= (-1)^{1+1} 0 det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-1)^{1+2} 0 det\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \\
&\quad + (-1)^{1+3}(-2) det\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \\
&= 1 \cdot 0 \cdot (0 \cdot 0 - 1 \cdot 0) + -1 \cdot 0 \cdot (-3 \cdot 0 - 1 \cdot 0) + 1 \cdot (-2) \cdot (-3 \cdot 0 - 0 \cdot 0) \\
&= 0
\end{aligned}$$

Substituting these in to our computation of the determinant of A*, we then obtain*

$$\begin{aligned}
det(A) &= 1 \cdot 1 \cdot 0 + (-1) \cdot 4 \cdot (6) + 1 \cdot 2 \cdot 0 + (-1) \cdot (-1) \cdot 0 \\
&= -24.
\end{aligned}$$

We sometimes call this method of computing a determinant as "expanding along the first row." This is because we can also compute the determinant of a matrix by similarly expanding along a different row, or even a column.

Proposition 11. *We can similarly compute the determinant of an* $n \times n$ *matrix*

A by expanding along any row or column. Expanding along the ith row, we have

$$det(A) = (-1)^{i+1} a_{i1}\, det(A_{i1}) + (-1)^{i+2} a_{i2}\, det(A_{i2}) + \cdots + (-1)^{i+n} a_{in}\, det(A_{in}).$$

Expanding along the jth column, we have

$$det(A) = (-1)^{1+j} a_{1j}\, det(A_{1j}) + (-1)^{2+j} a_{2j}\, det(A_{2j}) + \cdots + (-1)^{n+j} a_{nj}\, det(A_{nj}).$$

Once again, A_{ij} is the $(n-1)\times(n-1)$ submatrix obtained from A by deleting the ith row and jth column.

It can be useful to think about which row or column will be easiest to expand along. In particular, choosing a row or column with a lot of zeros greatly simplifies computation.

Example 17. *We'll once again compute the determinant of the 4×4 matrix*

$$A = \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

this time by expanding along the second column. Note that this column is a good choice, since there's only one nonzero element. We have

$$det(A) = (-1)^{1+2}(4)\, det\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{2+2}(0)\, det\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{3+2}(0)\, det\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We'll only compute the determinant of the submatrix $\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; we won't bother computing the others since their determinants will be multiplied by 0.

$$\begin{aligned} det\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (-1)^{3+1}(0)\, det\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} + (-1)^{3+2}(0)\, det\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} + (-1)^{3+3}(1)\, det\begin{pmatrix} 0 & -2 \\ -3 & 1 \end{pmatrix} \\ &= 0 + 0 + (1)(1)(0\cdot 1 - (-2)\cdot(-3)), \\ &= -6. \end{aligned}$$

Once again, we don't bother computing the determinants which will be multiplied by zero. Note that we chose to expand across the last row, since it had two zeroes. Expanding along the first column would also have been a reasonable choice.

Returning to our computation of the determinant of A, we have

$$\begin{aligned} det(A) &= (-1)^{1+2}(4)\, det\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{2+2}(0)\, det\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{3+2}(0)\, det\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (-1)(4)(-6) + 0 + 0 + 0, \\ &= 24. \end{aligned}$$

Notice that this matching our previous computation, expanding along the first row.

One of the most powerful uses of the determinant is to tell us whether or not a matrix is invertible. Recall that an $n \times n$ matrix A is *invertible* if there is a matrix B such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.

Proposition 12. *An $n \times n$ matrix A is invertible if and only if its determinant is nonzero.*

This gives us a convenient way to test if a matrix is invertible, without needing to produce an explicit inverse.

Example 18. *We found that the determinant of the matrix*

$$A = \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is 24. *Since this is nonzero, the matrix A is invertible.*

On the other hand, you can verify that the determinant of the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & -2 \\ -3 & 1 & -1 & 0 \\ -1 & 3 & 1 & 2 \end{pmatrix}$$

is 0. *Thus, the matrix B is not invertible.*

Linear Transformations

One of the most important uses of matrices is to represent linear transformations. Recall the definition of a linear transformation.

Definition 16. *A function T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a linear transformation if for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ and all scalars $k \in \mathbb{R}$, we have*

(a) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$;

(b) $T(k\vec{v}) = kT(\vec{v})$.

We can view an $m \times n$ matrix A as representing a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ as follows. We write vectors as column vectors, or, equivalently, $n \times 1$ or $m \times 1$ matrices. For an input column vector $\vec{v}$ in $\mathbb{R}^n$, we multiply $\vec{v}$ by A

on the left, using matrix multiplication. This produces an $m \times 1$ matrix, or, equivalently, a column vector in $\mathbb{R}^m$. Thus, we can define a function

$$T_A(\vec{v}) = A\vec{v}.$$

Using properties of matrix multiplication, we have that this is a linear transformation. Thus, we have the linear transformation associated to a matrix.

Example 19. *Consider the linear transformation T_A from $\mathbb{R}^3$ to $\mathbb{R}^2$ corresponding to the 2×3 matrix*

$$A = \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix}.$$

Let investigate the images of several vectors in $\mathbb{R}^3$ under the linear transformation T_A.

$$\begin{aligned} T_A((1,2,3)) &= \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + -5 \cdot 2 + 3 \cdot 3 \\ 2 \cdot 1 + 0 \cdot 2 + -1 \cdot 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T_A((1,-1,2)) &= \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + -5 \cdot -1 + 3 \cdot 1 \\ 2 \cdot 1 + 0 \cdot -1 + -1 \cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T_A((1,0,0)) &= \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + -5 \cdot 0 + 3 \cdot 0 \\ 2 \cdot 1 + 0 \cdot 0 + -1 \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
T_A((0,1,0)) &= \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot 0 + -5 \cdot 1 + 3 \cdot 0 \\ 2 \cdot 0 + 0 \cdot 1 + -1 \cdot 0 \end{pmatrix} \\
&= \begin{pmatrix} -5 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
T_A((0,0,1)) &= \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot 0 + -5 \cdot 0 + 3 \cdot 1 \\ 2 \cdot 0 + 0 \cdot 0 + -1 \cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ -1 \end{pmatrix}
\end{aligned}$$

Notice that when we apply the linear transformation to the standard unit vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$, we obtain the columns of A as the output vector. This observation can be used to reconstruct a matrix from a given linear transformation.

Proposition 13. *Given any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$, there is an $m \times n$ matrix such that $T = T_A$.*

Furthermore, the columns of A can be obtained by applying T to the standard unit vectors. More specifically, the jth column of A is given by $T(\vec{e}_j)$.

We can see how this is useful through an example.

Example 20. *Consider the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates a vector by 30° counterclockwise. We can see geometrically that, for the standard unit vectors $\vec{e}_1$ and $\vec{e}_2$ in $\mathbb{R}^2$, we have*

$$\begin{aligned}
T((1,0)) &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \\
T((0,1)) &= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).
\end{aligned}$$

These tell us the columns of the matrix corresponding to the linear transformation, so we then know that the rotation can be represented by the matrix

$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Summary

In this section, we reviewed matrix operations and properties, determinants, and linear transformations.

Although we've reviewed some of the most important concepts from linear algebra, there is still a lot of material that we weren't able to include here. Make sure you refer back to your linear algebra textbook if there's anything else you need to review!

Representations of Lines and Planes

In this section, we review the different ways we can represent lines and planes, including parametric representations.

Representations of Lines

When you think of describing a line algebraically, you might think of the standard form

$$y = mx + b,$$

where m is the slope and b is the y-intercept. This is often called *slope-intercept* form.

In addition to slope-intercept form, there are several other ways to represent lines. For example, you may remember using *point-slope* form in single variable calculus. We can describe a line of slope m going through a point (x_0, y_0) with the equation

$$y - y_0 = m(x - x_0).$$

It's important to note that there are many different possible choices for the point (x_0, y_0). Because of this, unlike slope-intercept form, point-slope form does not give a unique representation of a line.

In linear algebra, we saw that we could parametrize a line using a vector $\vec{v} = (v_1, v_2)$ giving the direction of the line, and a point (x_0, y_0) that the line passes through. We parametrize the line as

$$\begin{aligned} \vec{x}(t) &= (v_1, v_2)t + (x_0, y_0), \\ &= (v_1 t + x_0, v_2 t + y_0). \end{aligned}$$

Note that this representation works a bit differently from the previous two representations. In slope-intercept form and point-slope form, the line was the set of points (x, y) satisfying the given equation. However, in the parametrization, we plug in values for the parameter t in order to get points on the line.

Unlike slope-intercept form and point-slope form, the parametrization of a line can easily be generalized to three or more dimensions. That is, a line in $\mathbb{R}^n$ through the point $\vec{a}$ and in the direction of the vector $\vec{v}$ can be parametrized as

$$\vec{x}(t) = \vec{v}t + \vec{a},$$

Learning outcomes:
Author(s):

for $t \in \mathbb{R}$.

If we would like to describe a line in higher dimensions using equations (rather than a parametrization), we would need more than one equation. For example, in $\mathbb{R}^3$, we would require two equations to determine a line.

Representations of Planes

We also have multiple ways to represent planes. Here, we'll focus on planes in $\mathbb{R}^3$.

Recall that a plane can be determined by two vectors (giving the "direction" of the plane) and a point that the plane passes through. We can use this to give a parametrization for the plane through the point $\vec{a}$ and parallel to the vectors $\vec{v}$ and $\vec{w}$:

$$\vec{x}(s,t) = \vec{v}s + \vec{w}t + \vec{a},$$

for s and t in $\mathbb{R}$. Note that we require two parameters for the parametrization of the plane.

We can also describe a plane using a single linear equation in x, y, and z. For example,

$$2x + 4y - z = 9$$

defines a plane. A standard way to do this is using a point on the plane and a normal vector to the plane. Recall that a normal vector is perpendicular to every vector in the plane. If $\vec{n} = (n_1, n_2, n_3)$ is a normal vector to a plane passing through the point $\vec{a} = (a_1, a_2, a_3)$, the plane is defined by the equation

$$\vec{n} \cdot (\vec{x} - \vec{a}) = 0.$$

This can be rewritten as

$$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0.$$

Summary

We reviewed various representations of lines and planes, including parametrizations.

Part II

Week 1: Coordinate Systems

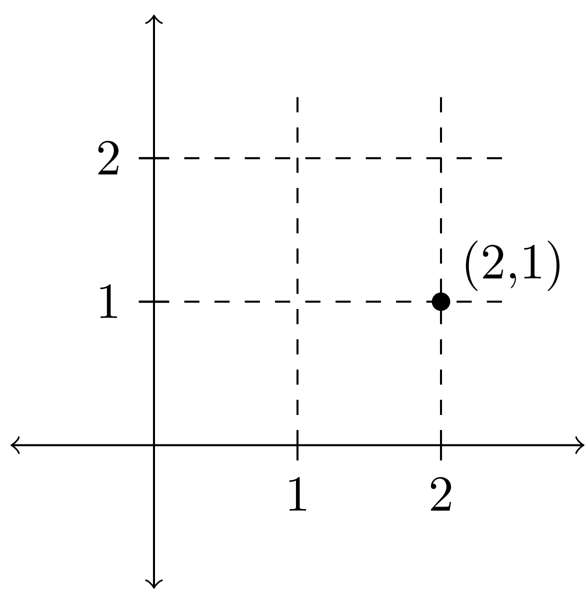
Review of Coordinate Systems

In this activity, we review coordinate systems that you've seen before, in preparation for introducing new coordinate systems in subsequent sections.

Cartesian Plane

The coordinates that you're probably most comfortable with are standard two-dimensional coordinates, also called Cartesian coordinate system on the plane.

In Cartesian coordinates, we describe a point using an x-coordinate and a y-coordinate. We write a point as (x, y), where the x-coordinate describes the horizontal displacement of the point, and the y-coordinate describes the vertical displacement of the point.



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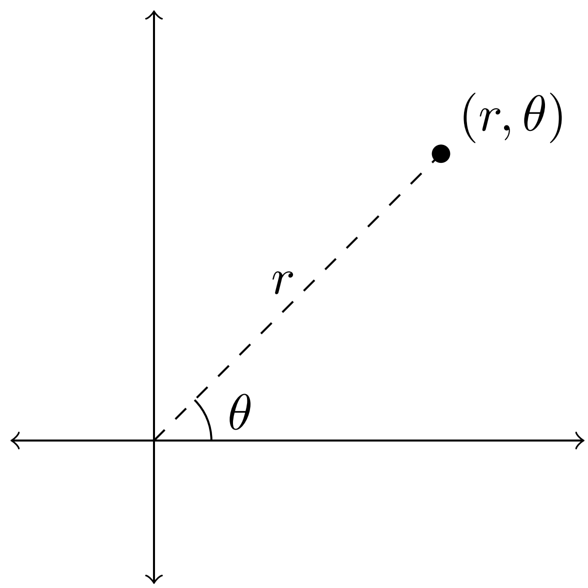
Polar Coordinates

You've also seen polar coordinates.

In polar coordinates, we describe a point with an r-coordinate and a θ-coordinate. The r coordinate gives the distance between the point and the origin, and the θ-coordinate gives the angle (in radians) between the positive x-axis and the segment connecting the origin and the point.

We can switch between cartesian and polar coordinates using the equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$



Problem 1 *Write the point $(r,\theta) = (5, \pi/3)$ in cartesian coordinates.*

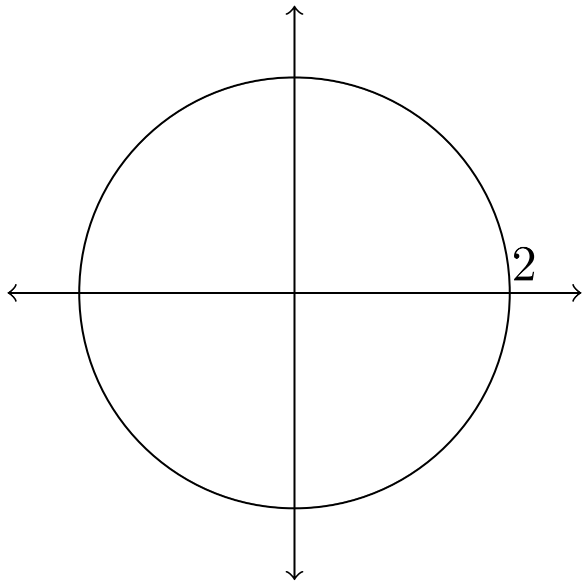
$$(x, y) = \boxed{(5/2, 5\sqrt{3}/2)}$$

Write the point $(x, y) = (-2, 2)$ in polar coordinates.

$$(r, \theta) = \boxed{(\sqrt{8}, 3\pi/4)}$$

Example 21. *Recall that we can describe a circle of radius 2 using Cartesian points as the set of points (x, y) satisfying*

$$x^2 + y^2 = 4.$$



We would like to write describe this circle using polar coordinates.

By definition, the circle of radius 2 *centered at the origin consists of the points which are distance* 2 *from the origin. Because of this, for any point on the circle, we have*

$$r = \boxed{2}.$$

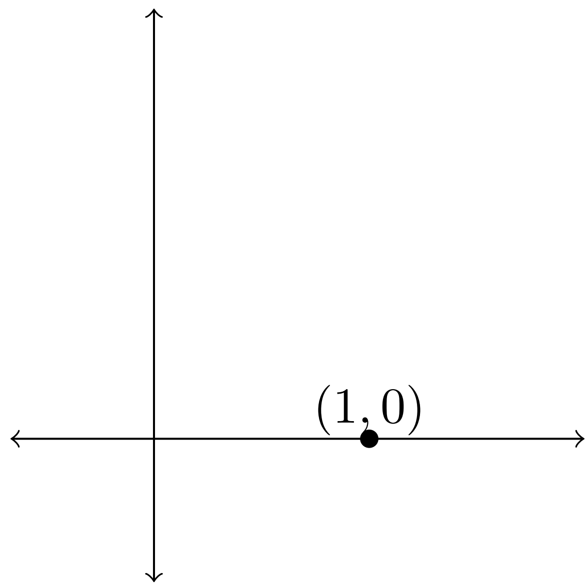
There are points on the circle making every possible angle with the positive x*-axis, so we don't need any restrictions on* θ*. If, however, we only wanted part of the circle, we would accomplish this by restricting* θ*. For example, to get the top half of the circle, we would make the restriction* $0 \leq \theta \leq \pi$*.*

Thus, in polar coordinates, the circle of radius 2 *centered at the origin can be described as the set of points* (r, θ) *such that*

$$r = 2.$$

There's an important difference between Cartesian coordinates and polar coordinates: Cartesian coordinates are *unique*, while polar coordinates are not. This means that, given a point P in the plane, there's only one way to describe this point as (x, y) using Cartesian coordinates. However, there are many ways to write the point as (r, θ), using polar coordinates.

Take, for example, the point $(1, 0)$, written in Cartesian coordinates.



This point is on the x-axis and is distance 1 from the origin. Thus, perhaps the most obvious way to represent this point in polar coordinates is as $(r, \theta) = (1, 0)$ (coincidentally, the same as in Cartesian coordinates). But we could also describe the angle as 2π, 4π, -2π, etc. So, we could also write the point in polar coordinates as $(r, \theta) = (1, 2\pi)$, and so on.

Perhaps more surprisingly, we can describe this point as $(-1, \pi)$. Imagine making an angle of π with the positive x-axis (so we're on the negative x-axis), then going backwards past the origin. This also gets you to our point. Using equivalent angles, we can also represent the point as $(-1, 3\pi)$, $(-1, -\pi)$, and so on.

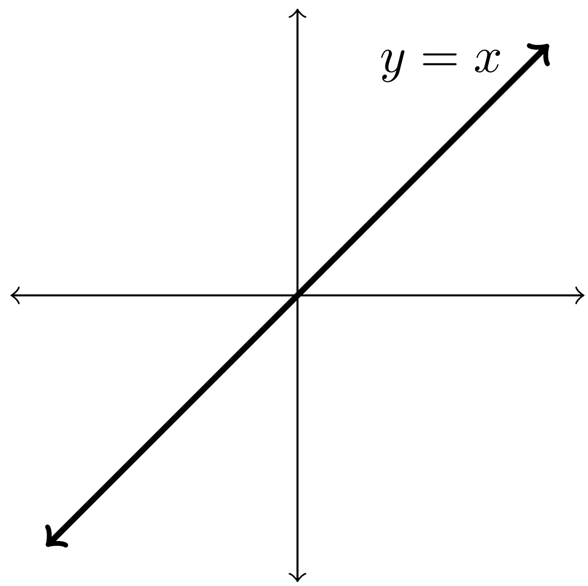
There are some times in working with polar coordinates when we would like to be able to represent points uniquely, and in these situations, we often make restrictions

$$\begin{gathered} 0 \leq r, \\ 0 \leq \theta < 2\pi. \end{gathered}$$

However, even with these restrictions, there still is a point that has multiple representations! Namely, the origin can be written as $(r, \theta) = (0, \theta)$ for any angle θ.

Depending on the situation and context, different people may use different restrictions or conventions for their ranges for r and θ. For this reason, it's good to specify what values you're allowing, to avoid being misunderstood!

Example 22. *Let's consider the line described in Cartesian coordinates as the set of points (x, y) such that $y = x$. We'll figure out how to describe this line in polar coordinates.*



Let's restrict our polar coordinates to $0 \leq r$ and $0 \leq \theta < 2\pi$. Perhaps your first guess is to describe the line as the points (r, θ) such that

$$\theta = \pi/4.$$

Which shape does this describe?

Multiple Choice:

(a) *A point.*

(b) *Half of the line.* ✓

(c) *The whole line.*

(d) *A different line.*

(e) *A circle.*

Describing the line as $\theta = \pi/4$ is a reasonable first guess, as we can see that many of the points make an angle $\pi/4$ with the positive x-axis. However, with the restriction that $r \geq 0$, this leaves out half of the line! In order to describe the entire line, we have a couple of options. One option would be to relax our restriction on r, and allow negative values as well. This would certainly give us the whole line. If, however, we would like to keep this restriction that $r \geq 0$, we could also include points with $\theta = 5\pi/4$, which will give us the other half of the line.

Which of the following describe the line $y = x$ in polar coordinates? Select all that work.

Select All Correct Answers:

(a) *The points* (r, θ) *such that* $\theta = \pi/4$*, where* $r \geq 0$.

(b) *The points* (r, θ) *such that* $\theta = \pi/4$*, where* r *can be any real number.* ✓

(c) *The points* (r, θ) *such that* $\theta = \pi/4$ *or* $\theta = -\pi/4$*, where* $r \geq 0$.

(d) *The points* (r, θ) *such that* $\theta = \pi/4$ *or* $\theta = -\pi/4$*, where* r *can be any real number.*

(e) *The points* (r, θ) *such that* $\theta = \pi/4$ *or* $\theta = 5\pi/4$*, where* $r \geq 0$. ✓

(f) *The points* (r, θ) *such that* $\theta = \pi/4$ *or* $\theta = 5\pi/4$*, where* r *can be any real number.* ✓

Recall that the relationship between Cartesian and polar coordinates:

$$x = \boxed{r\cos\theta},$$
$$y = \boxed{r\sin\theta}.$$

Recall the following equations describing the relationship between Cartesian and polar coordinates, which can be useful for converting between these two coordinate systems.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

Example 23. *Consider the set of points* (r, θ) *such that* $r = 2\cos\theta$*. What does this set of points look like?*

It's not very clear from $r = 2\cos\theta$ *what shape this is describing, so let's try converting this to Cartesian coordinates, and see if we get something we recognize.*

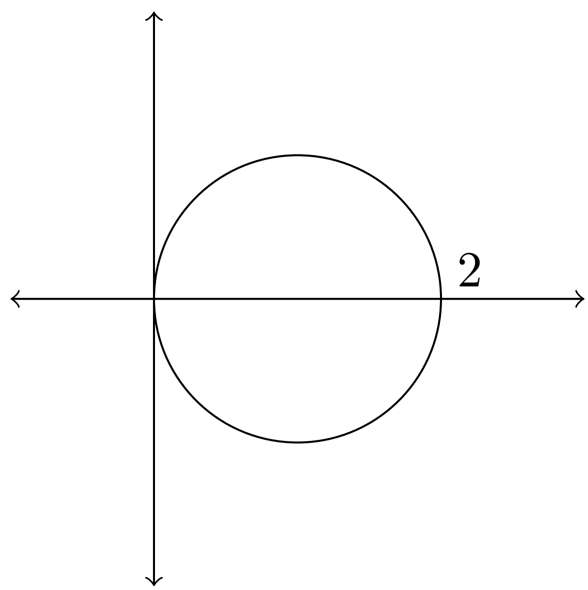
Using the conversion equations above, we have that $r^2 = \boxed{x^2 + y^2}$*, and* $\cos\theta = \dfrac{x}{r}$*. Making substitutions using these facts, we have:*

$$\begin{aligned} r &= 2\cos\theta \\ r &= 2\frac{x}{r} \\ r^2 &= 2x \\ x^2 + y^2 &= 2x \end{aligned}$$

We now have an equation solely in terms of x and y, but maybe it isn't quite recognizable yet. But if we do a bit more algebra...

$$\begin{aligned} x^2 + y^2 &= 2x \\ (x^2 - 2x + 1) + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

Now, we can see that this is a circle of radius $\boxed{1}$ *centered at* $\boxed{(1,0)}$.



Linear Change of Coordinates

In Linear Algebra, we saw how different coordinate systems arose through linear change of coordinates. You may remember this referred to as "slanty space."

When we write a point in Cartesian coordinates as (x, y), we can think of this as a linear combination of the standard basis vectors:

$$(x, y) = x(1, 0) + y(0, 1).$$

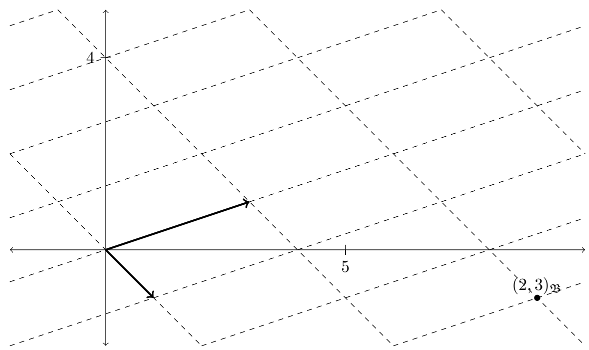
Of course, we can just as well write a point as a linear combination of vectors from a different basis, say $(3, 1)$ and $(1, -1)$. Let's call this basis $\mathfrak{B}$. For example, we can write the vector $(9, -1)$ as

$$(9, -1) = 2(3, 1) + 3(1, -1).$$

Taking the coefficients, in $\mathfrak{B}$-coordinates, we would write this point as

$$(2,3)_{\mathfrak{B}}.$$

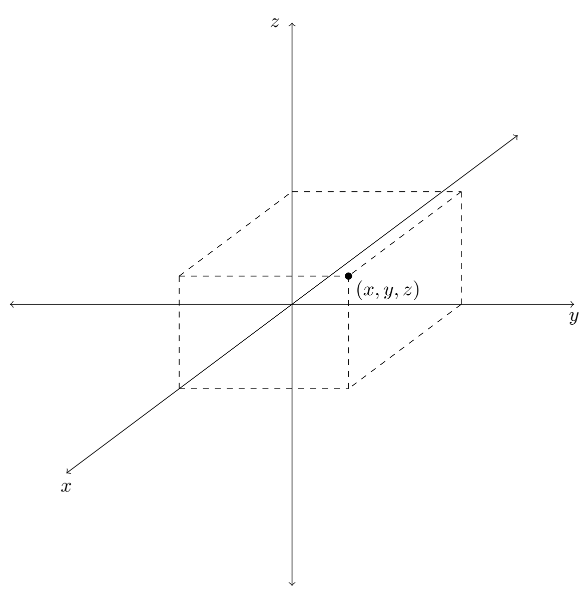
Note that we write $\mathfrak{B}$ in the subscript, in order to remind us that these are $\mathfrak{B}$-coordinates, rather than standard Cartesian coordinates.



With linear changes of coordinates, it's easy to make a mistake and forget which coordinates you're using. Make sure to keep careful track!

Three-Dimensional Coordinates

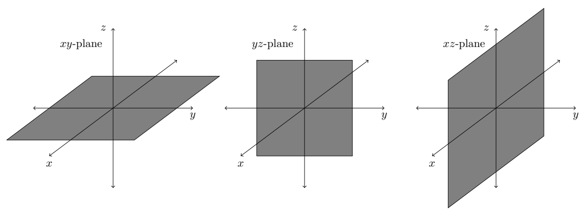
In Linear Algebra, we also worked in three-dimensional Cartesian coordinates, (x, y, z) in $\mathbb{R}^3$.



It's important to remember that the x, y, and z axes follow the right hand rule. That is, if you take your right hand, and point your pointer finger in the direction of the positive x-axis, point your middle finger in the direction of the positive y-axis, then your thumb points in the direction of the positive z-axis.

Another way to say this is that if you point the fingers of your right hand in the direction of the positive x-axis and curl them to point in the direction of the positive y-axis, your thumb points in the direction of the positive z-axis.

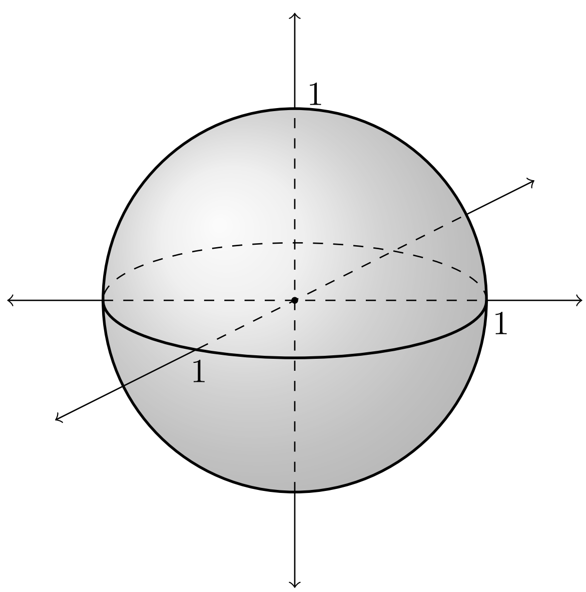
We'll often refer to the *coordinate planes* in $\mathbb{R}^3$. These are the three planes we obtain by setting each of the coordinates to be zero.



More precisely, the xy-plane is the set of points (x, y, z) such that $z = 0$, the yz-plane is the set of points such that $x = 0$, and the xz-plane is the set of points such that $y = 0$.

Similarly to in the plane, we can describe sets of points in $\mathbb{R}^3$ using equations.

Example 24. *The set of points* (x, y, z) *such that* $x^2 + y^2 + z^2 = 1$ *is the sphere of radius 1 centered at the origin in* $\mathbb{R}^3$.



Conclusion

In this activity, we reviewed coordinate systems that you've seen before: standard two-dimensional coordinates, polar coordinates, coordinates with respect to a given set of basis vectors, and three-dimensional coordinates.

Cylindrical Coordinates

In this activity, we introduce cylindrical coordinates, a new coordinate system on $\mathbb{R}^3$. We also discuss how to convert between cylindrical and Cartesian coordinates.

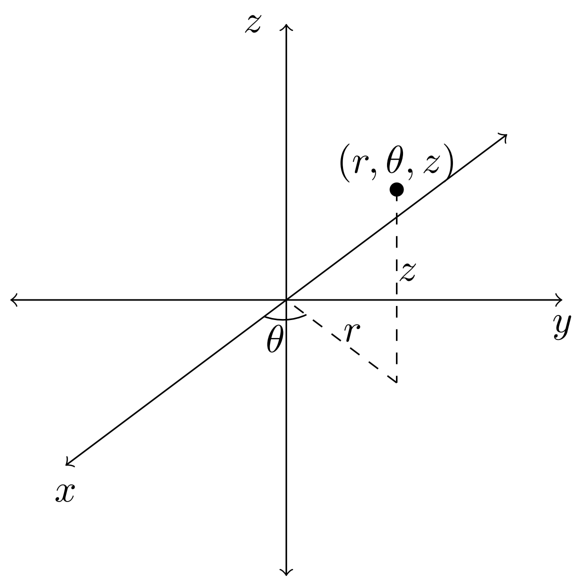
Cylindrical Coordinates

We've seen how points in $\mathbb{R}^2$ can be written using polar coordinates. Polar coordinates can be useful for describing shapes that are difficult to describe in Cartesian coordinates.

We'd now like to extend this idea to $\mathbb{R}^3$, using a coordinate system called *cylindrical coordinates*. Like polar coordinates, cylindrical coordinates will be useful for describing shapes in $\mathbb{R}^3$ that are difficult to describe using Cartesian coordinates. Later in the course, we will also see how cylindrical coordinates can be useful in multivariable Calculus, when differentiating or integrating in Cartesian coordinates is difficult or impossible.

Cylindrical coordinates are really just a simple extension of polar coordinates. For points in the xy-plane, we describe them using r and θ, where r is the distance from the origin and θ is the angle with the positive x-axis. We then tack on a z-coordinate, the exact same as the z-coordinate in Cartesian coordinates, which tells us the vertical displacement of the point.

Learning outcomes:
Author(s):



Example 25. *We'll convert the point $(x, y, z) = (1, 1, 1)$ to cylindrical coordinates.*

We can figure out r and θ by just considering the x- and y-coordinates of the point, $(1, 1)$. Then this becomes equivalent to representing the point in polar coordinates, so we have

$$(r, \theta) = \boxed{(\sqrt{2}, \pi/4)}.$$

For last coordinate, z, notice that this is telling us the height of the point, which is the exact same as the z-coordinate of the point written in Cartesian coordinates! So, our z coordinate is $\boxed{1}$, and the point $(x, y, z) = (1, 1, 1)$ can be written in cylinderical coordinates as

$$(r, \theta, z) = \boxed{(\sqrt{2}, \pi/4, 1)}.$$

You may use the applet below to experiment with how changing the different coordinates changes the point given in cylindrical coordinates.

MATH INSIGHT APPLET

Uniqueness

When we studied polar coordinates, we saw that there were many different ways to represent a point. For example, the point $(x, y) = (0, 1)$ could be written as $(r, \theta) = (1, \pi/2)$, $(1, 5\pi/2)$, or even $(-1, 3\pi/2)$. And the origin was especially devious, it could be written as $(0, \theta)$ for any angle θ.

Because of this and the relationship between polar and cylindrical coordinates, it's not surprisingly that cylindrical coordinates have similar issues with uniqueness. For example, the point $(0, 1, 1)$ can be written as $(r, \theta, z) = (1, \pi/2, 1)$, $(1, 5\pi/2, 1)$, $(-1, 3\pi/w, 1)$, and so on. Any point on the z-axis can be written as $(0, \theta, z)$, where z is its z-coordinate, and θ is any angle.

Problem 2 *Which of the following, written in cylindrical coordinates, is equivalent to the point $(x, y, z) = (1, 1, 1)$? Select all that apply.*

Select All Correct Answers:

(a) $(1, 1, 1)$

(b) $(1, \pi/4, 1)$

(c) $\sqrt{2}, \pi/4, 1)$ ✓

(d) $(-1, 3\pi/4, 1)$

(e) $-\sqrt{2}, \pi/4, 1)$

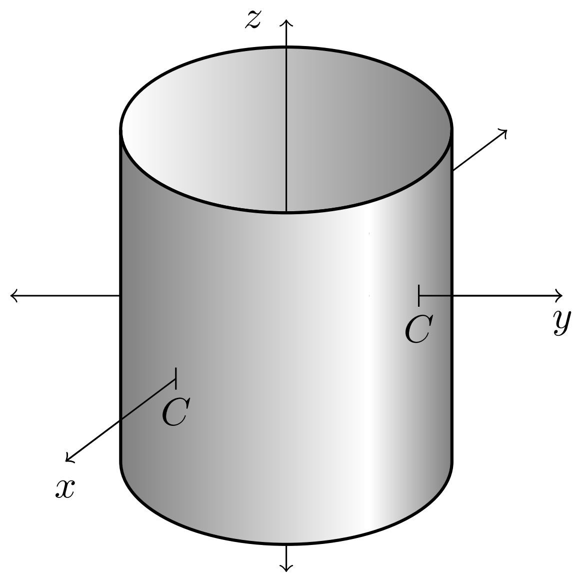
(f) $-\sqrt{2}, -3\pi/4, 1)$ ✓

As with polar coordinates, in situations where uniqueness is important, we will often make the restrictions $r \geq 0$ and $0 \leq \theta < 2\pi$.

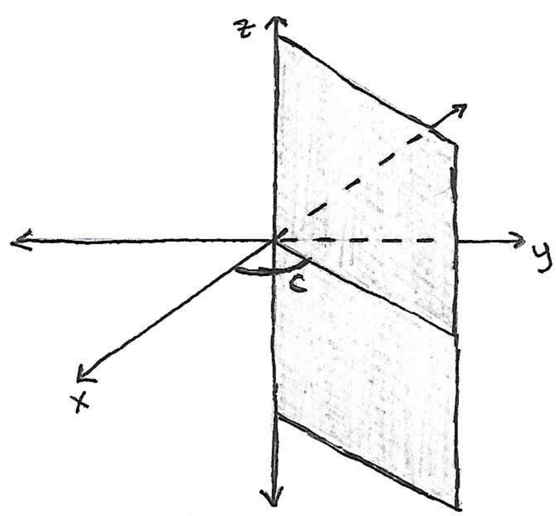
Constant-Coordinate Surfaces

Let's look at what happens in cylindrical coordinates when we set each of the coordinates r, θ, z to be constant, with the standard restrictions that $0 \leq r$ and $0 \leq \theta \leq \pi/2$. This can give us insight to how cylindrical coordinates behave.

We'll begin by examining the set of points (r, θ, z), where $r = C$ is a constant. We have that $r = C$ is constant, which means that the distance between any such point and the z axis is constant, C. Also, θ and z can be anything. This will give us the cylinder of radius C, centered at the z-axis.

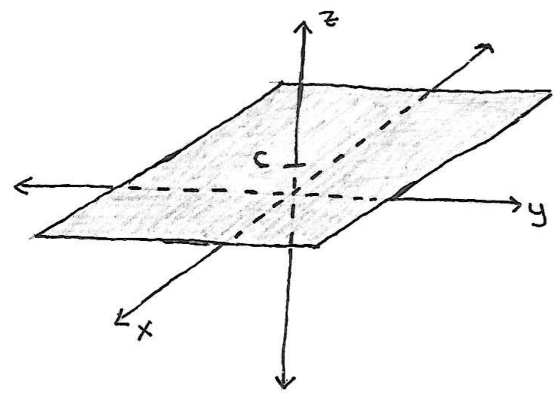


Next, we'll investigate the set of points (r, θ, z), where $\theta = C$ is constant. Let's consider the projection of this point onto the xy-plane. The projection will make an angle C with the positive x-axis, and have distance $r \geq 0$ from the origin. The height of the point can be any real number. From these observations, we conclude that the set of such points is the following half plane in $\mathbb{R}^3$.



Note that if we didn't have the restriction $r \geq 0$, we would get an entire plane rather than a half plane.

Finally, we'll consider the set of points (r, θ, z), where $z = C$ is constant. Since $z = C$, we will only have points at height C. Varying r and θ will then give us all points in the plane at height C parallel to the xy-plane, as below.



Converting between Cartesian and cylindrical coordinates

Perhaps not surprisingly, converting between Cartesian coordinates and cylindrical coordinates is very similar to how we converted between Cartesian coordinates and polar coordinates. That is, we can use the equations:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z, \\ r^2 &= x^2 + y^2, \\ \tan\theta &= \frac{y}{x}. \end{aligned}$$

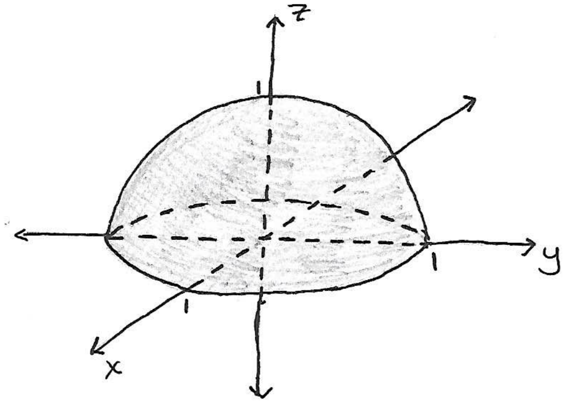
Example 26. *We'll convert $z = \sqrt{1 - r^2}$ to Cartesian coordinates.*
Using $r^2 = x^2 + y^2$, we have

$$z = \sqrt{1 - x^2 - y^2}.$$

You may recognize this as the top half of the sphere of radius 1 centered at the origin. You could also rewrite this as

$$x^2 + y^2 + z^2 = 1,$$

keeping in mind that $z \geq 0$.



Example 27. *We'll convert* $(x-2)^2 + y^2 = 1$ *(where* z *can be anything) to cylindrical coordinates. Note that this is the cylinder of radius 1, centered at the vertical line through* $(2, 0, 0)$*.*

Expanding the expression, we have

$$x^2 - 4x + 1 + y^2 = 1.$$

Substituting $r^2 = x^2 + y^2$ *and subtracting* 1 *from each side, we obtain*

$$r^2 - 4x = 0.$$

We then substitute $x = r\cos\theta$*.*

$$r^2 - 4r\cos\theta = 0.$$

Dividing both sides by r*, we have*

$$r - 4\cos\theta = 0,$$

or

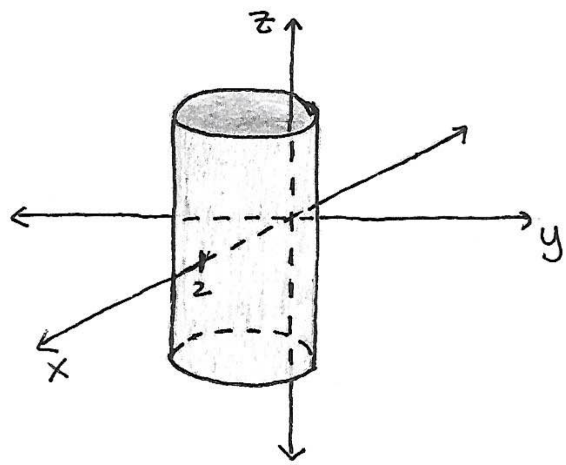
$$r = 4\cos\theta,$$

and z *can be anything.*

When we divided by r*, we implicitly assumed that* r *was not* 0*. This means that we might accidentally be omitting the origin, but if we take* $\theta = \pi/2$*, we have*

$$r = 4\cos(0) = 0,$$

so the origin is already included in the surface $r = 4\cos\theta$*.*



Conclusion

We introduced cylindrical coordinates and how to convert between cylindrical coordinates and Cartesian coordinates, and we discussed the uniqueness of cylindrical coordinates.

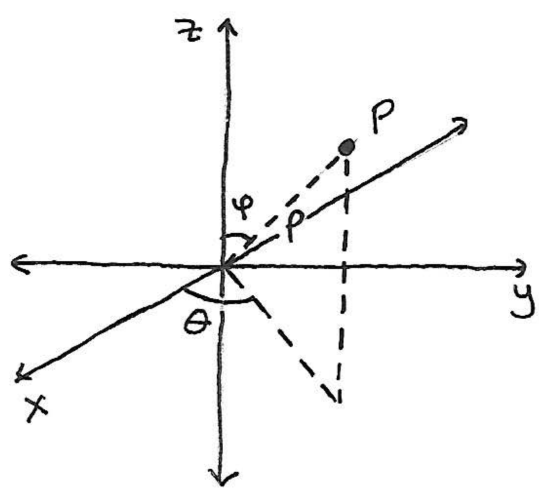
Spherical Coordinates

In this activity, we introduce spherical coordinates, a new coordinate system on $\mathbb{R}^3$. We also discuss how to convert between spherical and Cartesian coordinates.

Spherical Coordinates

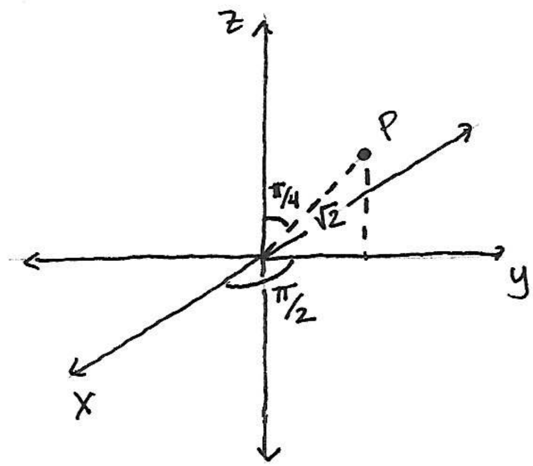
We've seen how to express points in $\mathbb{R}^3$ using Cartesian coordinates and using cylindrical coordinates. We'll now introduce a new coordinate system, called *spherical coordinates*.

Given a point P in $\mathbb{R}^3$, imagine drawing a line segment from the origin to P. In spherical coordinates, we write P as (ρ, θ, ϕ). Here, ρ is the length of the segment (also the distance between P and the origin). The second coordinate, θ, is angle between the positive x-axis and the projection of the segment onto the xy-plane. The third coordinate, ϕ, is the angle between the segment and the positive z-axis.



Example 28. *We'll write the point P in spherical coordinates, where P is given by $(x, y, z) = (0, 1, 1)$ in Cartesian coordinates.*

Learning outcomes:
Author(s):



The distance between P and the origin is

$$\sqrt{0^2+1^2+1^2} = \boxed{\sqrt{2}},$$

so $\rho = \boxed{\sqrt{2}}$.

The angle between the positive x-axis and the projection of P onto the xy-plane is $\boxed{\pi/2}$ *(in radians), so* $\theta = \boxed{\pi/2}$.

The angle between P and the positive z-axis is $\boxed{\pi/4}$ *(in radians), so* $\phi = \boxed{\pi/4}$.

Thus we can write P in spherical coordinates as $\boxed{(\sqrt{2}, \pi/2, \pi/4)}$.

Although we will be consistent with our definitions of θ and ϕ as above, it's important to know that some people reverse the roles of θ and ϕ. This is particularly common among physicists.

Uniqueness

As with polar and cylindrical coordinates, there are issues of uniqueness with spherical coordinates that we do not encounter in Cartesian coordinates.

Let's take for the example the point $(x, y, z) = (0, 1, 1)$, written in Cartesian coordinates. We've seen the canonical way to write this point in spherical coordinates, as $(\sqrt{2}, \pi/2, \pi/4)$. However, we could also write this as $(\sqrt{2}, 5\pi/2, \pi/4)$, $(\sqrt{2}, -3\pi/2, \pi/4)$, or even $(-\sqrt{2}, 3\pi/2, 5\pi/4)$.

Because of this issue, we'll commonly use the restrictions

$$\begin{aligned} 0 \leq& \rho \\ 0 \leq& \theta < 2\pi \\ 0 \leq& \phi \leq \pi \end{aligned}$$

when working with spherical coordinates in order to improve the uniqueness situation. Unfortunately, there are still multiple ways to represent the origin in spherical coordinates.

Problem 3 *Which of the following represent the origin in spherical coordinates? Select all that apply.*

Select All Correct Answers:

(a) $(0, 0, 0)$ ✓

(b) $(0, \pi/2, 0)$ ✓

(c) $(0, 0, \pi/4)$ ✓

(d) $(0, \pi/2, \pi/4)$ ✓

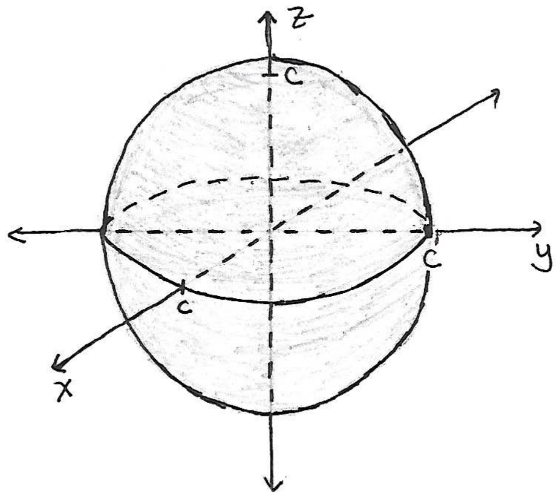
You may use the following applet to experiment with how the different coordinates change a point written in spherical coordinates.

MATH INSIGHT APPLET

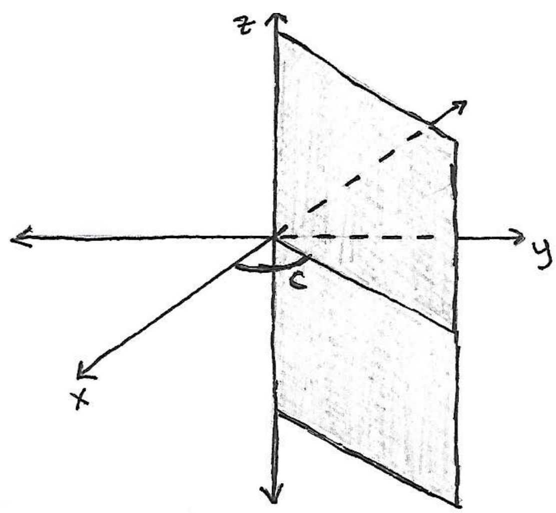
Constant-Coordinate Surfaces

As we did with cylindrical coordinates, we'll see what happens when we set each of the coordinates to be constant.

Consider the set of points (ρ, θ, ϕ), where $\rho = C$ is constant. This means that the distance between the origin and any such point is C. Varying the angles θ and ϕ gives us all such points, which make a sphere of radius C.



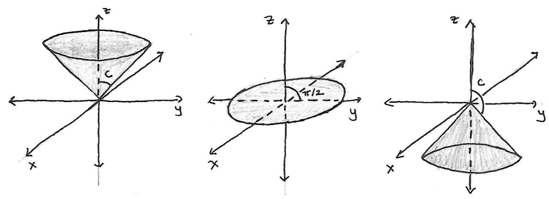
Now, consider the set of points (ρ, θ, ϕ), where $\theta = C$ is constant. This means that the projection of any such point onto the xy-plane will make an angle C with the positive x-axis. Varying ρ gives us points at various distances from the origin, and varying ϕ gives us points making various angles with the positive z-axis. With the restrictions $\rho \geq 0$ and $0 \leq \phi \leq \pi$, we obtain a half plane, as below.



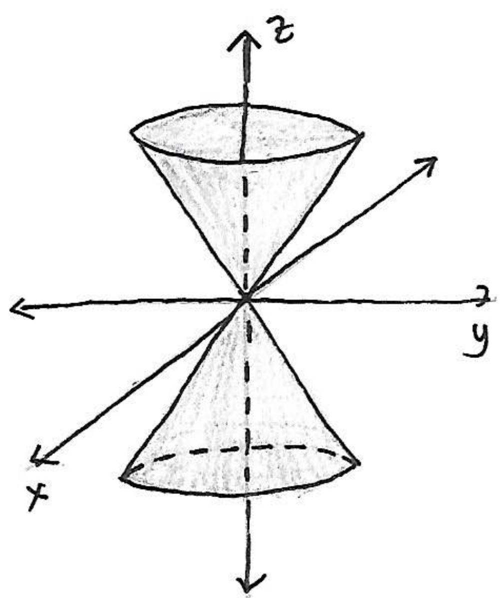
Notice that if we relaxed the restrictions on ρ and ϕ, we could obtain the entire plane.

Finally, we consider the set of points (ρ, θ, ϕ), where $\phi = C$ is constant. This means that every such point has an angle C with the positive z-axis. Varying

ρ and θ, with the restriction $\rho \geq 0$, we get the surfaces below, depending on if $C < \pi/2$, $C = pi/2$, or $C > pi/2$.



Looking at the surfaces when $C > \pi/2$ or $C < \pi/2$, we would commonly call these surfaces "cones." However, in most mathematics, "cone" is more commonly used to describe the surface below, which you might call a double cone.



Note that if you relax the restriction $\rho \geq 0$, you'll get cone (or double cone) above when $C \neq 0$.

It may seem strange that mathematicians prefer this double cone to the seemingly simpler cones that you're used to. However, it turns out that the double cone is easier to describe algebraically.

You can use the following applet to see what happens when you vary the value of the constant C for each of the constant-coordinate surfaces above:

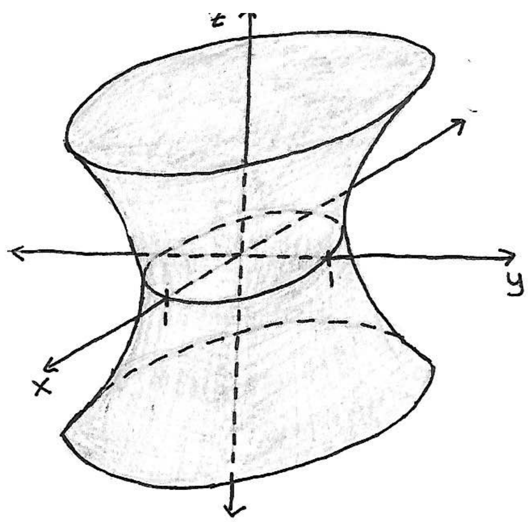
MATH INSIGHT APPLET

Converting Between Spherical and Cartesian Coordinates

When converting between spherical coordinates and Cartesian coordinates, it can be useful to use the following equations, which describe the relationship between the two coordinate systems.

$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$
$$\rho^2 = x^2 + y^2 + z^2$$

Example 29. *We'll convert $x^2 + y^2 - z^2 = 1$ from Cartesian coordinates to spherical coordinates. This surface is called an* elliptic hyperboloid*, and its graph is shown below. We'll learn how to identify this and other surfaces later in the course.*



Making the substitutions $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$, we have

$$\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi - \rho^2\cos^2\phi = 1.$$

We can factor $\rho^2\sin^2\phi$ out of the first two terms and obtain

$$\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) - \rho^2\cos^2\phi = 1.$$

Recalling that $\cos^2\theta + \sin^2\theta = 1$, we can simplify this to

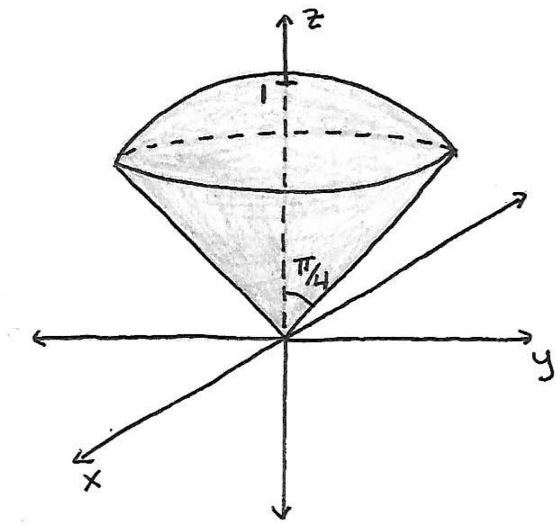
$$\rho^2\sin^2\phi - \rho^2\cos^2\phi = 1.$$

Recalling the double angle formula $\cos(2\phi) = \cos^2(\phi) - \sin^2(\phi)$*, we can simplify this to*

$$\rho^2 \cos(2\phi) = 1.$$

Example 30. *Sketch the set of points* (ρ, θ, ϕ) *(in spherical coordinates) such that* $0 \leq \rho \leq 1$ *and* $0 \leq \phi \leq \pi/4$*.*

The condition $0 \leq \rho \leq 1$ *means that we'll have only points within distance* 1 *of the origin. The condition* $0 \leq \phi \leq \pi/4$ *means that we'll have only points within angle* $\pi/4$ *from the* z*-axis. Putting these conditions together, we have the solid "ice-cream cone" region sketched below.*



Spherical coordinates in $\mathbb{R}^n$ [OPTIONAL]

Since we've seen polar coordinates in $\mathbb{R}^2$, and cylindrical and spherical coordinates in $\mathbb{R}^3$, you might be wondering if there are similar coordinate systems in $\mathbb{R}^4$, $\mathbb{R}^5$, and so on.

It is possible to define spherical coordinates in $\mathbb{R}^n$ for any n, and you can find a description here.

Conclusion

We introduced spherical coordinates and how to convert between spherical coordinates and Cartesian coordinates, and we discussed the uniqueness of spherical coordinates.

Online Homework

Problem 4 *Find the Cartesian coordinates of the point $(\pi/2, \pi, 2)$, given in cylindrical coordinates.*

$$(x, y, z) = \boxed{(0, -\pi/2, 2)}$$

Problem 5 *Find the Cartesian coordinates of the point $(2, \pi, \pi/2)$, given in spherical coordinates.*

$$(x, y, z) = \boxed{(-2, 0, 0)}$$

Problem 6 *Find cylindrical coordinates for the point $(0, -1, 3)$, written in Cartesian coordinates. Your answer should satisfy $0 \leq r$ and $0 \leq \theta < 2\pi$.*

$$(r, \theta, z) = \boxed{(1, \pi, 3)}$$

Problem 7 *Find spherical coordinates for the point $\left(-\sqrt{2}, \sqrt{2}, 2\sqrt{3}\right)$, written in Cartesian coordinates. Your answer should satisfy $0 \leq \rho$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \phi$.*

$$(\rho, \theta, \phi) = \boxed{(4, 3\pi/4, \pi/6)}$$

Problem 8 *Consider the surface described in Cartesian coordinates by*

$$2z^2 = x^2 + y^2.$$

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Author(s):

Describe this surface with an equation in cylindrical coordinates, of the form $0 = f(r, \theta, z)$.

$$0 = \boxed{r^2 - 2z^2}$$

FIGURE OUT HOW TO HANDLE THIS!!! What type of shape is this?

Multiple Choice:

(a) *Plane*

(b) *Cylinder*

(c) *Sphere*

(d) *Cone* ✓

(e) *Other*

Problem 9 *Consider the surface described in Cartesian coordinates by*

$$2z^2 = x^2 + y^2.$$

Describe this surface with an equation in spherical coordinates, of the form $0 = f(\rho, \theta, \phi)$.

$$0 = \boxed{\rho^2 \sin^2 \phi - 2\cos^2 \phi}$$

FIGURE OUT HOW TO HANDLE THIS!!! What type of shape is this?

Multiple Choice:

(a) *Plane*

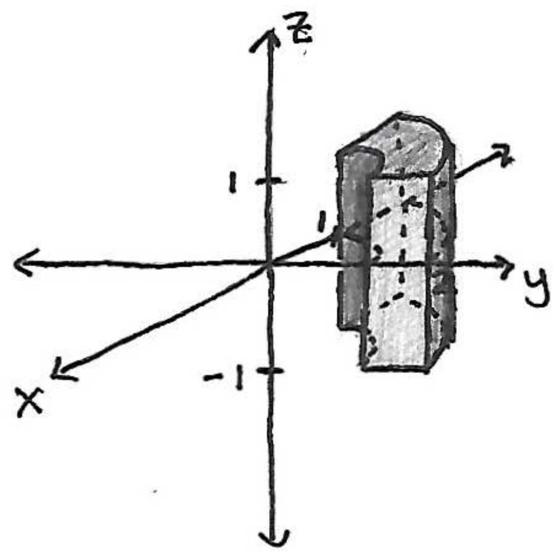
(b) *Cylinder*

(c) *Sphere*

(d) *Cone* ✓

(e) *Other*

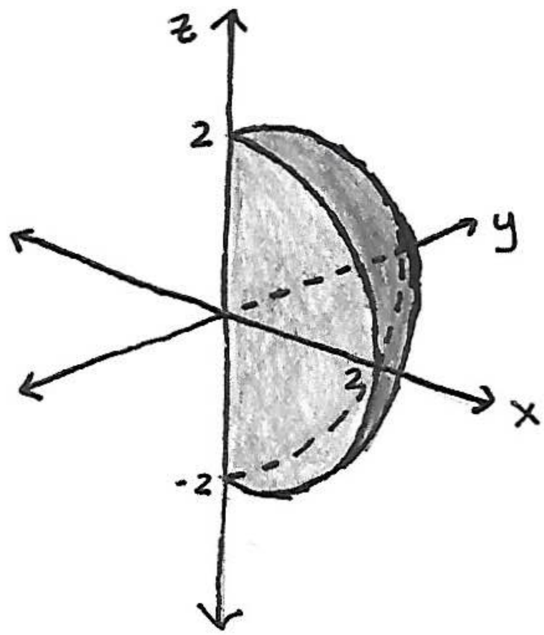
Problem 10 *Consider the following region in* $\mathbb{R}^3$.



This region is the set of points (r, θ, z), in cylindrical coordinates, satisfying the inequalities

$$\boxed{1} \leq r \leq \boxed{2}$$
$$\boxed{\pi/2} \leq \theta \leq \boxed{\pi}$$
$$\boxed{-1} \leq z \leq \boxed{1}$$

Problem 11 *Consider the following region in* $\mathbb{R}^3$.



This region is the set of points (ρ, θ, ϕ), in spherical coordinates, satisfying the inequalities

$$\boxed{0} \leq \rho \leq \boxed{2}$$
$$\boxed{0} \leq \theta \leq \boxed{pi/2}$$
$$\boxed{0} \leq \phi \leq \boxed{pi}$$

Problem 12 *For each of the following equations in cylindrical coordinates, select the type of shape they define.*

FIGURE OUT CORRECT ANSWERS

$r = \cos\theta$

Multiple Choice:

(a) *plane*

(b) *cylinder*

(c) *sphere*

(d) *other*

$z = r\cos\theta$

Multiple Choice:

(a) *plane*

(b) *cylinder*

(c) *sphere*

(d) *other*

$z = -r$

Multiple Choice:

(a) *plane* ✓

(b) *cylinder*

(c) *sphere*

(d) *other*

Problem 13 *For each of the following equations in spherical coordinates, select the type of shape they define.*

FIGURE OUT CORRECT ANSWERS

$\rho = \cos\phi$

Multiple Choice:

(a) *plane*

(b) *cylinder*

(c) *sphere*

(d) *other*

$\rho = \sin\theta$

Multiple Choice:

(a) *plane*

(b) *cylinder*

(c) *sphere*

(d) *other*

$\rho \cos\theta \sin\phi = 1$

Multiple Choice:

(a) *plane* ✓

(b) *cylinder*

(c) *sphere*

(d) *other*

Written Homework

Written Homework

Problem 14 *Consider the surface described by $(r-3)^2+z^2=1$ in cylindrical coordinates, with the restriction $r \geq 0$.*

(a) *Sketch the intersection of the surface with the half-plane $\theta = 0$.*

(b) *Sketch the intersection of the surface with the half-plane $\theta = \frac{\pi}{2}$.*

(c) *Sketch the intersection of the surface with the plane $z = 0$.*

(d) *Sketch the surface.*

Problem 15 *Sketch the region in $\mathbb{R}^3$ with cylindrical coordinates satisfying the inequality*

$$r \leq z \leq 4 - 2r$$

Problem 16 (a) *Given a function f, consider the graphs of the equations $r = f(\theta)$ and $r = 2f(\theta)$, in polar coordinates. How are these graphs related?*

(b) *Given a function f, consider the graphs of the equations $\rho = f(\theta, \phi)$ and $\rho = 2f(\theta, \phi)$, in spherical coordinates. How are these graphs related?*

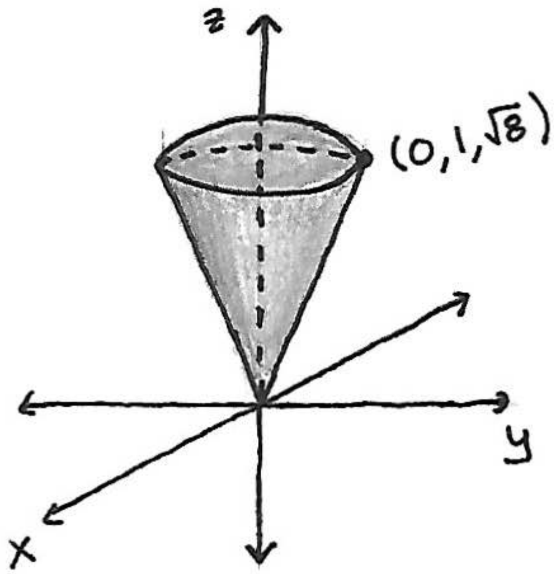
(c) *Given a function f, consider the graphs of the equations $r = f(\theta)$ and $r = -f(\theta)$, in polar coordinates. How are these graphs related?*

(d) *Given a function f, consider the graphs of the equations $\rho = f(\theta, \phi)$ and $\rho = -f(\theta, \phi)$, in spherical coordinates. How are these graphs related?*

Learning outcomes:
Author(s):

Professional Problem

Problem 17 *Consider the solid in $\mathbb{R}^3$ shown below.*



(a) *Describe the solid using spherical coordinates.*

(b) *Describe the solid using cylindrical coordinates.*

Your solution should be professionally written, following the instructions given in class and in the Professional Problem information sheet. In particular, be sure to focus on the following (which is not *an exhaustive list!):*

- ***Structure:*** *State your solution in brief, and then concisely explain why it is correct.*
- ***Formatting:*** *Be sure to follow the formatting requirements on Moodle. Staple the checklist to your problem, it will be used as part of your assessment.*
- ***Figures:*** *Include a clearly labeled figure for each part of the problem. The labels should help to demonstrate why your solution is correct. Label everything which is necessary, and nothing which is not. Refer to the figure in your solution as necessary.*
- ***LaTeX:*** *(optional) If you wish you type your solution in LaTeX, use the tex file on Moodle.*

Part III

Week 2: Multivariable Functions and Graphing

Functions

In this activity, we cover the definition of a function. We also cover several important definitions and properties of functions, including domain, codomain, range, surjective and injective functions, and component functions.

Definition of a Function

You've certainly seen many functions before. For example, you've worked with linear functions, such as

$$f(x) = 3x + 2,$$

quadratic functions, such as

$$h(t) = -4.9t^2 + 20t + 5,$$

and more complicated functions such as

$$g(x) = e^{5\sin(x^2)} + \ln\cot x.$$

You've seen functions of more than one variable in the form of linear transformations, such as

$$T : \mathbb{R}^2 \to \mathbb{R}^3,$$
$$T(x, y) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Not surprisingly, in multivariable calculus, we'll be studying functions of more than one variable. Before starting to work with these functions, we now cover some of the fundamental definitions and properties related to functions in general, beginning with the definition of a function.

Learning outcomes:
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Definition 17. *For sets X and Y, a* function $f : X \to Y$ *from X to Y assigns an element of Y to each element of X.*

We call X the domain *of f, and Y the* codomain *of f.*

VIDEO

We commonly think of X as giving the set of inputs to a function, and Y as containing the outputs. Each input coming from the set X has to have some corresponding output, but some elements of Y might not actually occur as outputs of the function.

Problem 18 *Which of the following are functions? Select All that apply.*

Select All Correct Answers:

(a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$ ✓

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(a, b) = a - b$ ✓

(c) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = \pm x$

(d) $f : \mathbb{R} \to \mathbb{R}^2$ *defined by* $f(x) = (x, x)$ ✓

If we would like to refer to the elements in the codomain which actually do occur as outputs, we call this the range of f.

Definition 18. *The* range *of a function $f : X \to Y$ is the set of elements $y \in Y$ such that there is some $x \in X$ with $f(x) = y$. That is, there is some input x that has y as an output. In set notation, we write*

$$\textit{Range } f = \{y \in Y \,:\, y = f(x) \textit{ for some } x \in X\}.$$

VIDEO

Problem 19 *What is the range of the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, x)$?*

Multiple Choice:

(a) $\mathbb{R}^2$

(b) $\mathbb{R}$

(c) $\{(a, b) \in \mathbb{R}^2 \,:\, a = b\}$ ✓

(d) $\{(a, b) \in \mathbb{R}^2 \,:\, a = b\}$

Sometimes we work with functions that aren't defined on all of $\mathbb{R}^n$. When the domain of f is a subset D of $\mathbb{R}^n$, we write

$$f : D \subset \mathbb{R}^n \to \mathbb{R}^m.$$

When we're working with functions on subsets of $\mathbb{R}^n$, we'll frequently want to work with the largest possible set that the function is defined on. We call this the *natural domain* of the function.

Problem 20 *What is the natural domain of the function* $f(x, y) = \dfrac{x}{x-y}$*?*

Multiple Choice:

(a) $\mathbb{R}^2$

(b) $\mathbb{R}^2 \setminus \{(0, 0)\}$

(c) $\mathbb{R}^2 \setminus \{(a, b) \, : \, a = 0 \text{ or } b = 0\}$

(d) $\mathbb{R}^2 \setminus \{(x, y) \, : \, a = b\}$ ✓

Types of Functions

In some special situations, every element of Y really does appear as an output of the function f. In this case, we say that f is onto, or surjective.

Definition 19. *A function $f : X \to Y$ is* onto*, or* surjective*, if for every element $y \in Y$, there is some $x \in X$ such that $f(x) = y$. We can also write this condition as*

$$Y = \text{Range } f.$$

VIDEO

Problem 21 *Which of the following functions are onto? Select all that apply.*

Select All Correct Answers:

(a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(a, b) = a - b$ ✓

(c) $f : \mathbb{R} \to \mathbb{R}^2$ *defined by* $f(x) = (x, x)$

(d) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = x^2y^2$

Another important type of function is a one-to-one, or injective, function. For a one-to-one function, different inputs always go to different outputs.

Definition 20. *A function* $f : X \to Y$ *is* one-to-one*, or* injective*, if whenever* $f(x_1) = f(x_2)$ *for* $x_1, x_2 \in X$*, then we must have* $x_1 = x_2$*.*

Another way to say this is that whenever $x_1 \neq x_2$*, we have* $f(x_1) \neq f(x_2)$*.*

VIDEO

Problem 22 *Which of the following functions are injective? Select all that apply.*

Select All Correct Answers:

(a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(a, b) = a - b$

(c) $f : \mathbb{R} \to \mathbb{R}^2$ *defined by* $f(x) = (x, x)$ ✓

(d) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = x^2y^2$

Component Functions

When we're trying to understand the behavior of a function $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$, it can sometimes be helpful to split $\mathbb{R}^m$ into its components. From this, we get the component functions of f.

Definition 21. *Let* $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ *be a function. The* component functions *of* f *are scalar-valued functions* $f_i : X \subset \mathbb{R}^n \to \mathbb{R}$ *for* $1 \leq i \leq m$ *such that*

$$f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), ..., f_m(\vec{x})).$$

Conclusion

We covered the definition of a function. We also covered several important definitions and properties of functions, including domain, codomain, range, surjective and injective functions, and component functions.

Graphing Functions

In this activity, we give the formal definition of the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. We discuss strategies for figuring out the shapes of the graph of such functions, using contour curves, level curves, and sections.

Definition of the Graph of a Function

You might already have an intuitive idea of what the graph of a function $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ should be, but perhaps don't know the formal definition, or how to figure out what the graph of an arbitrary function looks like. We'll begin with the definition of the graph, before discussing how to actually produce graphs.

Definition 22. *Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function. The* graph *of f is the set of points*

$$\textit{Graph } f = \{(\vec{x}, f(\vec{x}) : \vec{x} \in X\}$$

in $\mathbb{R}^3$.

We typically visualize a point in the graph as lying over the point $\vec{x}$ in the plane at a height $f(\vec{x})$.

Note that this is similar to the graph of a function from a subset of $\mathbb{R}$ to $\mathbb{R}$. The graph of a function $X \subset \mathbb{R}^n \to \mathbb{R}$ can be defined similarly, but this tends to be less useful once $n \geq 3$, since it's hard to visualize four or more dimensions!

Strategies for Graphing

It can be much trickier to sketch the graph of a function $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ than it was to sketch the graphs of functions $\mathbb{R} \to \mathbb{R}$. One common strategy that people will initially try is plotting individual points to try to get a sense of the graph. However, for graphs in $\mathbb{R}^3$, you would need a lot of points to get a representative sample of the plane. For this reason, *plotting points alone is not an effective strategy.* However, plotting a single point here or there can be helpful.

We've now told you what doesn't work for graphing functions in $\mathbb{R}^3$, so now we should probably tell you what does work. The essential idea of all of these

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strategies is that we know you're pretty comfortable graphing in $\mathbb{R}^2$, so we're going to take advantage of that experience.

We'll begin with contour curves, which are obtained by setting the z-coordinate to be constant. Think of this as taking horizontal slices of the graph.

Definition 23. *Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function. The* contour curve *of the function f at height C is the set of points in $\mathbb{R}^3$ obtained by taking the intersection of the graph of f with the plane $z = C$.*

PICTURE/VIDEO EXAMPLE

We can also consider the level curves of a function, which are closely related to contour curves.

Definition 24. *Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function. The* level curve *of the function $f$at height C is the set of points in $\mathbb{R}^2$ satisfying $C = f(x, y)$.*

After reading this definition, you're probably thinking "hey, aren't contour curves and level curves the exact same thing?" They're certainly closely related. The key difference is that level curves exist in the plane, $\mathbb{R}^2$, while contour curves exist in three-space, $\mathbb{R}^3$. Since they're in the plane, level curves are usually easier to draw. However, contour curves are more useful for figuring out the shape of a graph. For these reasons, it can be useful to go back and forth between level curves and contour curves.

PICTURE/VIDEO EXAMPLE

We can think of contour curves as taking slices of the graph where z is constant. It can also be useful to take slices of the graph where x or y is constant. We call these slices sections of the graph.

Definition 25. *Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function, and let C be a constant.*

The section *of the graph of f by $x = C$ is the set of points*

$$\{(C, y, z) \in \mathbb{R}^3 \, : \, z = f(C, y)\}$$

The section *of the graph of f by $y = C$ is the set of points*

$$\{(x, C, z) \in \mathbb{R}^3 \, : \, z = f(x, C)\}$$

Note that, like contour curves, sections exist in $\mathbb{R}^3$.

PICTURE/VIDEO EXAMPLE

EXTRA EXAMPLE

Level Surfaces

So far, we have focused on graphing functions from subsets of $\mathbb{R}^2$ to $\mathbb{R}$, so the graphs are in $\mathbb{R}^3$.

We now turn our attention to the graphs of functions from subsets of $\mathbb{R}^3$ to $\mathbb{R}$. Note that the graph of such a function will exist in $\mathbb{R}^4$. Since the world we live in only has three physical dimensions, it can be very difficult to visualize a four dimensional object! Fortunately, there are various tricks that can be used to get some sense of what a four dimensional object looks like. We cover one of them here.

When we had a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, we could get a sense of the graph by looking at its level curves, which were curves in the same plane.

For a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$, we can adopt a similar approach. We can once again consider the level sets, which are obtained by taking the output to be some constant:

$$f(x, y, z) = C.$$

In this case, the level sets will be level surfaces, which live in $\mathbb{R}^3$. By graphing several level surfaces, we can see what a slice of the graph of f looks like at various heights, giving us some sense of how the overall graph behaves. Of course, because this graph exists in four dimensions, we still probably won't be able to visualize this perfectly.

To see how this can help us visualize the four-dimensional graph of a function $f : \mathbb{R}^3 \to \mathbb{R}$, we give an example.

Example 31. *Consider the function*

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Find the level surfaces at heights -1*,* 0*,* 1*,* 2*, and* 3*. Use these level surfaces to describe the graph of* f*.*

We'll begin with the level surface at height -1*. This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$-1 = \sqrt{x^2 + y^2 + z^2}.$$

There are no points that satisfy this equation, so the level surface is empty.

Now we'll consider the level surface at height 0*. This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$0 = \sqrt{x^2 + y^2 + z^2}.$$

The only point which satisfies this equation is the origin, so the level "surface" is the single point $(0, 0, 0)$*.*

Let's look at the level surface at height 1*. This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$1 = \sqrt{x^2 + y^2 + z^2}.$$

Squaring both sides, we can rewriting this as

$$1 = x^2 + y^2 + z^2.$$

The graph of this equation is the sphere of radius 1 *centered at the origin, which is our level surface.*

Let's look at the level surface at height 2*. This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$2 = \sqrt{x^2 + y^2 + z^2}.$$

Squaring both sides, we can rewriting this as

$$4 = x^2 + y^2 + z^2.$$

The graph of this equation is the sphere of radius 2 *centered at the origin, which is our level surface.*

Let's look at the level surface at height 3*. This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$3 = \sqrt{x^2 + y^2 + z^2}.$$

Squaring both sides, we can rewriting this as

$$9 = x^2 + y^2 + z^2.$$

The graph of this equation is the sphere of radius 3 *centered at the origin, which is our level surface.*

We graph our level surfaces below.

PICTURE

We can see that the level surfaces are spheres whose radii increase linearly with the height. So, we can describe the graph of f *as some sort of four-dimensional cone.*

Conclusion

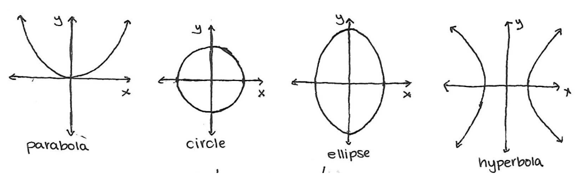
In this activity, we gave the formal definition of the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. We discussed strategies for figuring out the shapes of the graph of such functions, using contour curves, level curves, and sections.

Quadric Surfaces

In this activity, we introduce and classify quadric surfaces, which form an important family of surfaces.

Definition of a Quadric Surface

You might remember studying conic sections, such as parabolas, circles, ellipses, and hyperbolas. These are curves in the plane that arise through polynomial equations of degree two in two variables.



Quadric Surfaces are the three dimensional analogue of conic sections. That is, a quadric surface is the set of points in $\mathbb{R}^3$ satisfying some polynomial of degree two in three variables.

Definition 26. *A* quadric surface *is the set of points (x, y, z) in $\mathbb{R}^3$ satisfying the equation*

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + J = 0,$$

where $A, B, C, D, E, F, G, H, I, J \in \mathbb{R}$ are constants.

Simple Forms

Dealing with quadric surfaces in general can be computationally cumbersome, so we'll focus on quadric surfaces in some simple forms.

Example 32. *The set of points satisfying*

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

for some constants $a, b, c \in \mathbb{R}$, is called an ellipsoid.

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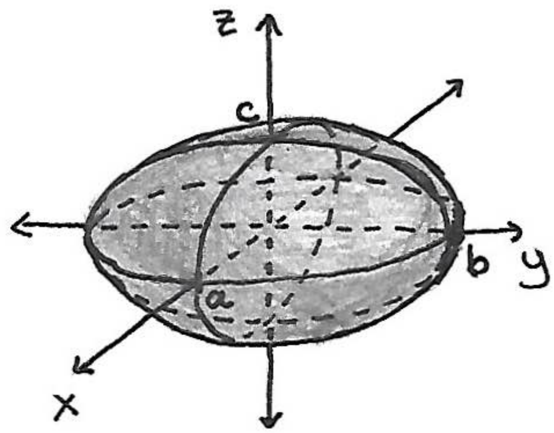


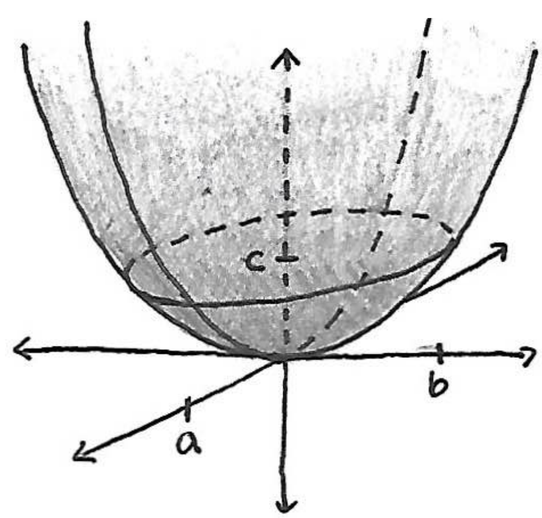
An ellipsoid is kind of like a three dimensional ellipse. In fact, the sections and contour curves of such an an ellipsoid are ellipses.

In the special case that $a = b = c$, this ellipsoid is a sphere of radius a.

Example 33. *The set of points satisfying*

$$z/c = x^2/a^2 + y^2/b^2,$$

for some constants $a, b, c \in \mathbb{R}$, is called an elliptic paraboloid.

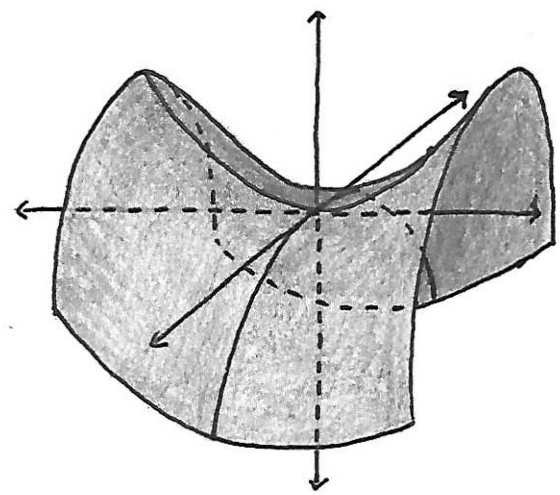


The contour curves of such an elliptic paraboloid are ellipses, however the sections are parabolas which all open in the same direction.

Example 34. *The set of points satisfying*

$$z/c = y^2/b^2 - x^2/a^2,$$

for some constants $a, b, c \in \mathbb{R}$*, is called a* hyperbolic paraboloid*.*

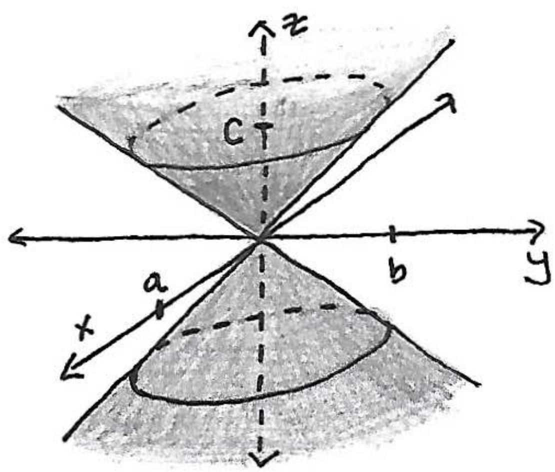


The contour curves of such a hyperbolic paraboloid are hyperbolas, and the sections are parabolas opening in opposite directions for x *and* y *sections. This surface is often described as a "saddle".*

Example 35. *The set of points satisfying*

$$z^2/c^2 = x^2/a^2 + y^2/b^2,$$

for some constants $a, b, c \in \mathbb{R}$*, is called an* elliptic cone*.*

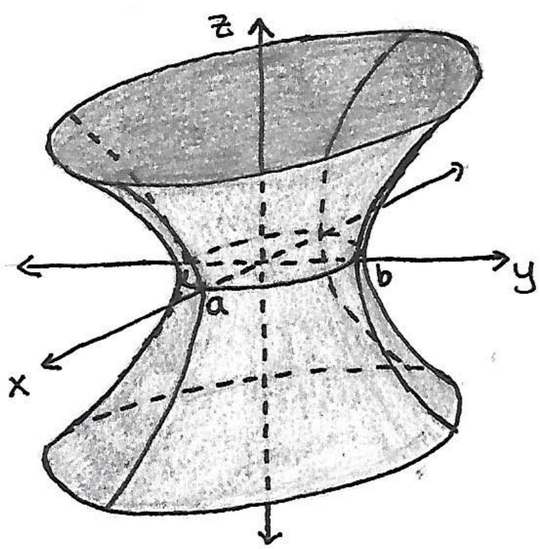


The contour curves of such an elliptic cone are ellipses, and the sections by $x = 0$ and $y = 0$ are pairs of intersecting lines.

Example 36. *The set of points satisfying*

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1.$$

for some constants $a, b, c \in \mathbb{R}$, is called a hyperboloid of one sheet*.*

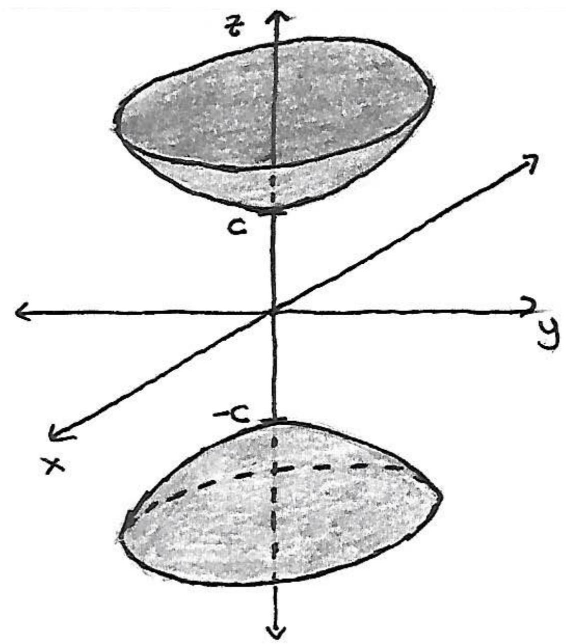


The contour curves of such a hyperboloid are ellipses, and the sections are hyperbolas.

Example 37. *The set of points satisfying*

$$z^2/c^2 - x^2/a^2 - y^2/b^2 = 1.$$

for some constants $a, b, c \in \mathbb{R}$, is called a hyperboloid of two sheets*.*



The contour curves of such a hyperboloid are ellipses, and the sections are hyperbolas. We describe this as the hyperboloid "of two sheets" since it has two disconnected pieces, as opposed to the hyperboloid of one sheet, which has only one.

APPLET/INTERACTIVE VARYING PARAMETERS

Some Other Forms

Although we won't really work with quadric surfaces in their most general form, we will consider quadric surfaces that are translations of the forms given above.

For example, the graph of the equation

$$(x-3)^2 + \frac{(y+2)^2}{3} + \frac{(z-1)^2}{2} = 1$$

is an ellipsoid centered at $(3, -2, 1)$.

PICTURE

However, equations describing quadric surfaces might not always be given to you in easily identifiable forms. In these cases, you might have to do some algebra in order to get the equation into a form where it can be identified as a particular quadric surface. These manipulations will frequently involve completing the square.

We now work through an example of identifying a quadric surface given in a non-standard form.

Example 38. *Identify the type of quadric surface determined by the equation*

$$-4x^2 + 2y^2 + z^2 + 8x + 4y + 4z = 2,$$

and sketch a graph of this surface.

Our strategy for writing this equation in a recognizable form will be to group terms involving x, group terms involving y, and group terms involving z. We'll then complete the square for each variable.

Grouping terms by variable, we have

$$(-4x^2 + 8x) + (2y^2 + 4y) + (z^2 + 4z) = 2.$$

For each of these grouping, we factor out the leading coefficient, obtaining

$$\boxed{-4}(x^2 - 2x) + \boxed{2}(y^2 + 2y) + (z^2 + 4z) = 2.$$

We now add or subtract as needed to make the quadratics into squares, getting

$$-4(x^2 - 2x + 1) + 2(y^2 + 2y + 1) + (z^2 + 4z + 2) = \boxed{4}.$$

We factor the quadratics to get

$$-4(\boxed{x-1})^2 + 2(\boxed{y+1})^2 + (\boxed{z+2})^2 = 4.$$

Finally, we divide by the constant on the right, to get the final form

$$-(\boxed{x-1})^2 + \frac{(\boxed{y+1})^2}{2} + \frac{(\boxed{z+2})^2}{4} = 1.$$

We can see that this quadric surface is centered at $\boxed{(1,-1,-2)}$*, but maybe it still isn't apparent which quadric surface this determines.*

Notice that this form is similar to our standard form for a hyperboloid of one sheet, except here it's the x-term that's subtracted instead of the z-term. This is because this is, in fact, a hyperboloid of one sheet, it just happens to be "around" a line parallel to the x-axis, rather than a vertical line.

Let's look at a section, in order to help with our sketch. Taking the section $x = 1$, we have an ellipse parallel to the yz-plane, centered at $(1,-1,-2)$, with radii $\sqrt{2}$ and 2.

Combining our observations, we can sketch the graph of this hyperboloid as below.

PICTURE

Conclusion

In this activity, we introduced and classified quadric surfaces, which form an important family of surfaces.

Online Homework

Problem 23 *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x, y) = x^2 + 4y^2 - 2.$$

What is the domain of f?

Multiple Choice:

(a) $\mathbb{R}$

(b) $\mathbb{R} \setminus \{0\}$

(c) $[0, \infty)$

(d) $(0, \infty)$

(e) $\mathbb{R}^2$ ✓

(f) $\mathbb{R}^2 \setminus \{(0, 0)\}$

What is the range of f?

$$\text{Range } f = \boxed{[-2, \infty)}$$

Is f onto?

Multiple Choice:

(a) *yes*

(b) *no* ✓

Problem 23.1 *We would like to restrict the codomain of the function f so that it becomes onto. We'll describe our new codomain as the set of numbers a in $\mathbb{R}$ such that some condition holds. Which condition gives us the largest possible codomain such that f is onto?*

Multiple Choice:

(a) $a \in \mathbb{R}$

Learning outcomes:
Author(s):

(b) $a \geq 0$

(c) $a > 0$

(d) $a \neq 0$

(e) $a = 0$

(f) $a \geq 2$

(g) $a > 2$

(h) $a \neq 2$

(i) $a = 2$

(j) $a \geq -2$ ✓

(k) $a > -2$

(l) $a \neq -2$

(m) $a = -2$

Is f one-to-one?

Multiple Choice:

(a) *yes*

(b) *no* ✓

Problem 23.2 *We would like to restrict the domain of the function f, so that it becomes one-to-one. We'll describe our new domain as the set of points (x, y) in $\mathbb{R}^2$ such that some condition(s) hold. Which condition(s) give us the largest possible domain such that f is one-to-one?*

Select All Correct Answers:

(a) $x \neq 0$

(b) $x \geq 0$ ✓

(c) $x > 0$

(d) $y \neq 0$

(e) $y \geq 0$ ✓

(f) $y > 0$

Problem 24 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by*

$$f(\vec{x}) = 3\vec{x} + \boldsymbol{i} - 2\boldsymbol{j}.$$

Find the component functions of f in terms of x, y, and z.

$$f_1(x, y, z) = \boxed{3x + 1}$$
$$f_2(x, y, z) = \boxed{3y - 2}$$
$$f_3(x, y, z) = \boxed{3z}$$

Problem 25 *Consider the linear function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(\vec{x}) = A\vec{x}$, where*

$$A = \begin{pmatrix} 1 & 5 & 2 \\ -2 & 0 & 1 \end{pmatrix},$$

and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

(a) *Determine the component functions of f in terms of x_1, x_2, and x_3.*

$$f_1(x_1, x_2, x_3) = \boxed{x_1 + 5x_2 + 2x_3}$$
$$f_2(x_1, x_2, x_3) = \boxed{-2x_1 + x_3}$$

(b) *Is f one-to-one?*

Multiple Choice:

(i) *Yes*
(ii) *No* ✓

(c) *Is f onto?*

Multiple Choice:

(i) *Yes* ✓
(ii) *No*

Problem 26 *Consider the function*

$$f(x, y) = xy.$$

What is the shape of the level curve at height 0 of f?

Multiple Choice:

(a) *Empty*
(b) *A single line*
(c) *Two intersecting lines* ✓
(d) *Two parallel lines*
(e) *Circle*
(f) *Ellipse*
(g) *Parabola*
(h) *Hyperbola*

What is the shape of the level curve at height 1 of f?

Multiple Choice:

(a) *Empty*
(b) *A single line*
(c) *Two intersecting lines*
(d) *Two parallel lines*
(e) *Circle*
(f) *Ellipse*
(g) *Parabola*
(h) *Hyperbola* ✓

What is the shape of the level curve at height -1 of f?

Multiple Choice:

(a) *Empty*

(b) *A single line*

(c) *Two intersecting lines*

(d) *Two parallel lines*

(e) *Circle*

(f) *Ellipse*

(g) *Parabola*

(h) *Hyperbola* ✓

What is the shape of the level curve at height 2 of f?

Multiple Choice:

(a) *Empty*

(b) *A single line*

(c) *Two intersecting lines*

(d) *Two parallel lines*

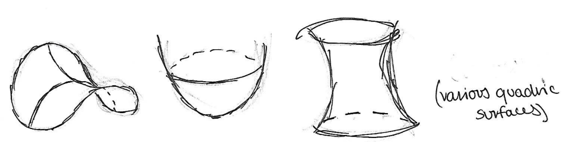
(e) *Circle*

(f) *Ellipse*

(g) *Parabola*

(h) *Hyperbola* ✓

Which of the following is the graph of f?



Problem 27 *Consider the function*

$$f(x, y) = |x|.$$

What is the shape of the level curve at height 0 of f?

Multiple Choice:

(a) *Empty*

(b) *A single line* ✓

(c) *Two intersecting lines*

(d) *Two parallel lines*

(e) *Circle*

(f) *Ellipse*

(g) *Parabola*

(h) *Hyperbola*

What is the shape of the level curve at height 1 of f?

Multiple Choice:

(a) *Empty*

(b) *A single line*

(c) *Two intersecting lines*

(d) *Two parallel lines* ✓

(e) *Circle*

(f) *Ellipse*

(g) *Parabola*

(h) *Hyperbola*

What is the shape of the level curve at height -1 of f?

Multiple Choice:

(a) *Empty* ✓

(b) *A single line*

(c) *Two intersecting lines*

(d) *Two parallel lines*

(e) *Circle*

(f) *Ellipse*

(g) *Parabola*

(h) *Hyperbola*

What is the shape of the level curve at height 2 of f?

Multiple Choice:

(a) *Empty*

(b) *A single line*

(c) *Two intersecting lines*

(d) *Two parallel lines* ✓

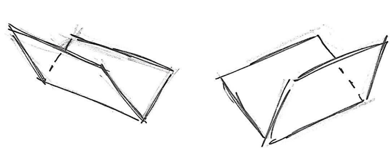
(e) *Circle*

(f) *Ellipse*

(g) *Parabola*

(h) *Hyperbola*

Which of the following is the graph of f?



+ graph
|x+y|
|x−y|

Problem 28 *Which of the following is the graph of the ellipsoid*

$$\frac{x^2}{9} + y^2 + \frac{z^2}{4} = 1?$$

PICTURES

Is there a function $f(x, y)$ such that the graph of f is the ellipsoid above?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Problem 28.1 *Why is this impossible?*

Multiple Choice:

(a) *It wouldn't be one-to-one.*

(b) *It wouldn't be onto.*

(c) *There would be multiple inputs with the same output.*

(d) *A single input would need to have two outputs.* ✓

Problem 29 *Classify the quadric surface defined by the equation*

$$x^2 + 4y^2 + z^2 + 8y = 0.$$

Multiple Choice:

(a) *Ellipsoid* ✓

(b) *Elliptic Paraboloid*

(c) *Hyperbolic Paraboloid*

(d) *Elliptic Cone*

(e) *Hyperboloid of One Sheet*

(f) *Hyperboloid of Two Sheets*

It is centered at $\boxed{(0, -1, 0)}$.

Problem 29.1 *Which of the following is the graph of the quadric surface given above?*

GRAPHS

Problem 30 *Classify the quadric surface defined by the equation*

$$2x^2 + 2y^2 - 8y - z + 4 = 0$$

Multiple Choice:

(a) *Ellipsoid*

(b) *Elliptic Paraboloid* ✓

(c) *Hyperbolic Paraboloid*

(d) *Elliptic Cone*

(e) *Hyperboloid of One Sheet*

(f) *Hyperboloid of Two Sheets*

It is centered at $\boxed{(0, 2, -4)}$.

Problem 30.1 *Which of the following is the graph of the quadric surface given above?*

GRAPHS

Written Homework

Written Homework

Problem 31 *Consider the function*

$$f(x,y,z) = \frac{4}{\sqrt{9-x^2-y^2-z^2}}.$$

(a) *What is the domain of f? Describe this domain as a region in $\mathbb{R}^3$.*

(b) *What is the range of f?*

Problem 32 *Consider the function*

$$f(x) = x^2 + y^2 - 4.$$

(a) *Draw at least five level curves of f.*

(b) *Use these level curves to sketch the graph of f.*

Problem 33 *Draw the graph of the surface in $\mathbb{R}^3$ determined by the equation*

$$x = y^2/4 - z^2/9.$$

Use level curves and/or sections to justify why your drawing is correct.

Professional Problem

Problem 34 (a) *Consider the function $g : \mathbb{R}^3 \to \mathbb{R}$ given by*

$$g(x,y,z) = x^2 + y^2.$$

Draw at least three level surfaces of g, which will be surface in $\mathbb{R}^3$. What do you notice about these level surfaces?

Learning outcomes:
Author(s):

(b) *Suppose you have a function* $g : \mathbb{R}^3 \to \mathbb{R}$*, such that* g *depends on* x *and* y*, but does not depend on* z*. What can you say about the level surfaces of* g*?*

(c) *Suppose you have a function* $g : \mathbb{R}^3 \to \mathbb{R}$*, such that* g *depends on* y *and* z*, but does not depend on* x*. What can you say about the level surfaces of* g*?*

(d) *Suppose you have a function* $g : \mathbb{R}^3 \to \mathbb{R}$*, such that* g *depends on* x*, but does not depend on* y *or* z*. What can you say about the level surfaces of* g*?*

Part IV

Week 3: Parametric Curves

Parametric Curves

In this activity, we parametrize curves in $\mathbb{R}^n$, focusing on the cases $n = 2$ and $n = 3$.

Review of Parametrizations in $\mathbb{R}^2$

We've dealt with several ways to describe curves in $\mathbb{R}^2$:

- As the graph of a function. For example, $f(x) = x^2$.
- As the set of points satisfying an equation. For example, the points (x, y) such that $x^2 + y^2 = 1$.
- As the set of points satisfying an equation in another coordinate system. For example, $r = \sin(\theta)$ in polar coordinates.

Another way that we can describe a curve is using *parametric equations*. In parametric equations, we define x and y in terms of a third variable, usually t, called the *parameter*. This gives us another way to describe curves in $\mathbb{R}^2$, and potentially describe some new and strange curves.

Example 39. *We can describe the unit circle in $\mathbb{R}^2$ with the parametric equations*

$$\begin{aligned} x &= \cos(t), \\ y &= \sin(t), \end{aligned}$$

for $0 \leq t \leq 2\pi$.

PICTURE

We can think of t as giving the angle that a point makes with the positive axis. It can also be helpful to imagine t as representing time, and the parametric equations tracing out the circle as time passes.

Learning outcomes:
Author(s):

Parametrizing Curves in $\mathbb{R}^n$

Consider the parametric equations for the unit circle in $\mathbb{R}^2$:

$$x = \cos(t),$$
$$y = \sin(t),$$

for $0 \leq t \leq 2\pi$.

We can combine these equations into a single vector,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

This defines a function $\vec{x}$ from the interval $[0, 2\pi] \subset \mathbb{R}$ to $\mathbb{R}^2$, and is the motivation behind our definition for paths.

Definition 27. *A* path *in $\mathbb{R}^n$ is a continuous function*

$$\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n,$$

where $I \subset \mathbb{R}$ is an interval.

This is also called a parametrized curve *or* parametric curve.

We'll focus on the cases $n = 2$ and $n = 3$ in this course.

We defined a path as a continuous function, however, we haven't said what it means for a multivariable function to be continuous. We'll come back to this later, and we'll give a rigorous definition for continuity. For now, this should fit with your intuition: you can draw the path without lifting your pencil from the paper.

Sometimes we can more about the image of a path than how the path is drawn out, and then we refer to a curve.

Definition 28. *A* curve *in $\mathbb{R}^n$ is the image of a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$.*

We say that $\vec{x}$ is a parametrization *for the curve.*

The difference between a curve and a path is largely a matter of perspective: when working with a curve, we pay attention to *what* is drawn; when working with a path, we care about *how* it is drawn.

Example 40. *There are many different parametrizations for a given curve.*

Consider again the unit circle C in $\mathbb{R}^2$. Which of the following are parametrizations for C?

Select All Correct Answers:

(a) $\vec{x}(t) = (\cos(t), \sin(t))$ *for* $0 \leq t \leq 2\pi$ ✓

(b) $\vec{x}(t) = (\sin(t), \cos(t))$ *for* $0 \leq t \leq \pi$

(c) $\vec{x}(t) = (t, \pm\sqrt{1-t^2})$ *for* $-1 \leq t \leq 1$

(d) $\vec{x}(t) = (\sin(2\pi t), \cos(2\pi t))$ *for* $0 \leq t \leq 1$ ✓

(e) $\vec{x}(t) = (\cos(t), \sin(t))$ *for* $-10 \leq t \leq 10$ ✓

Example 41. *In this example, we review how to parametrize the line through points $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$.*

Given points $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^n$, we obtain a vector starting at $\vec{a}$ and ending at $\vec{b}$ by taking $\vec{b} - \vec{a}$. This vector is parallel to the line through $\vec{a}$ and $\vec{b}$. Then, taking scalar multiples $t(\vec{b} - \vec{a})$ for $t \in \mathbb{R}$, we have a line parallel to the line through $\vec{a}$ and $\vec{b}$. Finally, we add one of the points, $\vec{a}$, to ensure that our line passes through these two points. Thus, we arrive at our parametrization,

$$\vec{l}(t) = \vec{a} + t(\vec{b} - \vec{a}) \text{ for } t \in \mathbb{R}.$$

PICTURE

Example 42. *In this example, we see how we can obtain new transformations from old ones, using linear algebra and simple transformations.*

Recall the parametrization for the unit circle in $\mathbb{R}^2$,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Now, consider the ellipse below.

PICTURE

We can think of this ellipse as the result of stretching the unit circle horizontally by a factor of 3 and vertically by a factor of 2. That is, we are applying the linear transformation

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

We can apply this to the parametrization for the unit circle, in order to parametrization for the ellipse.

$$\begin{aligned} \vec{(}y)(t) &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \\ &= (3\cos(t), 2\sin(t)). \end{aligned}$$

Thus, we have a parametrization for the ellipse given by

$$\vec{y}(t) = (3\cos(t), 2\sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Next, consider the following ellipse.

PICTURE

We can obtain this from our previous ellipse by counterclockwise rotation of $\pi/4$. The matrix for this linear transformation is

$$\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \boxed{1/\sqrt{2}} & \boxed{-1/\sqrt{2}} \\ \boxed{1/\sqrt{2}} & \boxed{1/\sqrt{2}} \end{pmatrix}.$$

Applying this rotation to our parametrization for the previous ellipse, we obtain a parametrization for our new ellipse.

$$\vec{z}(t) = \boxed{(3/\sqrt{2}\cos(t) - 2/\sqrt{2}\sin(t), 3/\sqrt{2}\cos(t) + 2/\sqrt{2}\sin(t))} \textit{ for } 0 \le t \le 2\pi$$

Finally, we consider an ellipse in $\mathbb{R}^3$, shown below.

PICTURE

This ellipse is parallel to the xy-plane, and will have constant z-coordinate. Note the similarity to the first ellipse we considered. A parametrization for this ellipse can be obtained by taking the parametrization $\vec{y}$ for our first ellipse in $\mathbb{R}^2$, and appending the constant z-coordinate.

$$\vec{a}(t) = \boxed{(3\cos(t), 2\sin(t))} \textit{ for } 0 \le t \le 2\pi$$

Examples in $\mathbb{R}^3$

In this section, we give examples of parametrizations of a couple of more complicated curves in $\mathbb{R}^3$, taking advantage of our previous experience with cylindrical coordinates.

Example 43. *We'll parametrize the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = 7 - 3x$ in $\mathbb{R}^3$, pictured below.*

PICTURE

Our x and y coordinates must satisfy $x^2 + y^2 = 4$, which would define a circle, if we were in $\mathbb{R}^2$. Recalling our parametrizations for circles, these coordinates can be written as

$$x(t) = 2\cos(t)$$
$$y(t) = 2\sin(t)$$

for $0 \le t \le 2\pi$.

It remains to write the z-coordinate in terms of the parameter t. Turning our attention to the equation for the plane, $z = 7 - 3x$, we have z expressed in terms *of x. Since we have expressed x in terms of t, we can make this substitution to describe z in terms of t,*

$$z(t) = \boxed{7 - 6\cos(t)}.$$

Putting all of this together, we have a parametrization for this intersection given by

$$\vec{x}(t) = \boxed{(2\cos(t), 2\sin(t), 7 - 6\cos(t))} \text{ for } 0 \leq t \leq 2\pi.$$

Example 44. *Consider the curve below, which lies on the cone $z^2 = x^2 + y^2$, and makes five rotations around the z-axis as the height ranges from 0 to 1. We'll refer to this curve as a "tornado."*

PICTURE

We'll parametrize this curve by thinking about it in cylindrical coordinates, using the height as the parameter.

First, let's consider what's happening with the z-coordinate. Since the height of the tornado rangers from 0 to 1, so will z. We'll set $z = t$, with $0 \leq t \leq 1$, and express x and y in terms of t as well.

Now, we turn our attention to the angle θ. As the height ranges from 0 to 1, the tornado makes five revolutions, so θ should range from 0 to 10π. Thus, expressing θ in terms of t, we let $\theta = 10\pi t$.

Next, we consider the radius r. Since we are on the cone $z^2 = x^2 + y^2$, we have $z^2 = r^2$. Since $z \geq 0$, we have $z = r$. Thus, we can write r in terms of t as $r = t$.

Finally, putting all of this together with $x = r\cos\theta$ and $y = r\sin\theta$, we have a parametrization for the tornado given by

$$\vec{x}(t) = \boxed{(t\cos(10\pi\theta), t\sin(10\pi\theta), t)} \text{ for } t \in [0, 1].$$

Conclusion

In this activity, we parametrized curves in $\mathbb{R}^n$, focusing on the cases $n = 2$ and $n = 3$.

Velocity and Speed

In this activity, we learn how to find the velocity and speed of a parametrized curve in $\mathbb{R}^n$.

Derivatives

Consider a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$ which parametrizes a curve in $\mathbb{R}^n$. We often think about this as a particle tracing out the curve as time, given by t, passes. We would like to be able to understand and describe the motion of the particle on the curve, and find its velocity and speed, in particular. In order to do this, we need to figure out how to differentiate a path.

Before we define the derivative of a path, we quickly review the single variable definition of a derivative, given in Calculus I.

Given a single variable function $f(x)$, we found the instantaneous rate of change at x of this function by taking the derivative of f at x. The derivative also told us the slope of the tangent line at x. In order to compute this, we imagined finding the slope of secant lines getting closer and closer to the point. Taking a limit, we obtained the slope of the tangent line.

PICTURE

The slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$ is given by $\dfrac{f(x+h)-f(x)}{h}$, so we defined the derivative of f at x to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

We use the same idea for a path $\vec{x}$ in $\mathbb{R}^n$. We consider secant vectors from $\vec{x}(t)$ to $\vec{x}(t+h)$ as $h \to 0$.

PICTURE

Scaling these vectors to account for the change in the parameter and taking a limit, we arrive at the definition of the derivative.

Definition 29. *Let $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$ be a path in $\mathbb{R}^n$. We define the derivative of $\vec{x}$ at t to be*

$$\vec{x}'(t) = \lim_{h \to 0} \frac{\vec{x}(t+h)-\vec{x}(t)}{h},$$

Learning outcomes:
Author(s):

if the limit exists.

We also call $\vec{x}'(t)$ the velocity vector *of $\vec{x}$, and write it as $\vec{v}(t)$.*

When we first defined derivatives in Calculus I, we spent weeks figuring out how to compute them. We started computing using only the limit definition, then we introduced the power rule, the product rule, the chain rule, and so on. Fortunately, we don't need to repeat this process in Multivariable Calculus: we can take advantage of our previous experience computing derivatives. In order to see why this is the case, let's take another look at our definition for the derivative of a path.

We have $\vec{x}'(t) = \lim_{h\to 0} \dfrac{\vec{x}(t+h) - \vec{x}(t)}{h}$ for a path $\vec{x}$. We can write out the path $\vec{x}$ in terms of its components, so

$$\vec{x}(t) = (x_1(t), ..., x_n(t)).$$

Substituting this into the limit, we have

$$\begin{aligned}
\vec{x}'(t) &= \lim_{h\to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}, \\
&= \lim_{h\to 0} \frac{(x_1(t+h), ..., x_n(t+h)) - (x_1(t), ..., x_n(t))}{h}, \\
&= \lim_{h\to 0} \frac{(x_1(t+h) - x_1(t), ..., x_n(t+h) - x_1(t))}{h}.
\end{aligned}$$

Dividing through by the scalar h and bringing the limit inside of the vector, we have

$$\begin{aligned}
\vec{x}'(t) &= \lim_{h\to 0} \left(\frac{x_1(t+h) - x_1(t)}{h}, ..., \frac{x_n(t+h) - x_1(t)}{h} \right), \\
&= \left(\lim_{h\to 0} \frac{x_1(t+h) - x_1(t)}{h}, ..., \lim_{h\to 0} \frac{x_n(t+h) - x_1(t)}{h} \right).
\end{aligned}$$

At this point you should be somewhat skeptical. We haven't defined limits of vectors, much less described how to manipulate them. We'll come back to this in a few weeks in much more detail. For now, hopefully it makes sense that looking at what a vector approaches depends on what its components approach, and you'll allow us this sleight of hand.

Looking at the limits inside of the components, they should look familiar. They're derivatives of single variable functions! That is, we now have

$$\vec{x}'(t) = (x_1'(t), ..., x_n'(t)).$$

This means that we can differentiate a path by differentiating its components, thus taking advantage of our knowledge of single variable derivatives.

Proposition 14. *We can differentiate a path by differentiating its components. That is,*

$$\vec{x}'(t) = (x_1'(t), ..., x_n'(t)).$$

Example 45. *Consider the path $\vec{x}(t) = (\cos(t), \sin(t))$ for $0 \le t \le 2\pi$, which parametrizes the unit circle in $\mathbb{R}^2$. We compute the derivative of this path,*

$$\vec{x}'(t) = \left(\frac{d}{dt}\cos(t), \frac{d}{dt}\sin(t)\right), \qquad = (-\sin(t), \cos(t)).$$

Consider the path $\vec{y}(t) = (t^2, t^3)$ for $0 \le t \le 1$.

$$y'(t) = \boxed{(2t, 3t^2)}$$

Consider the path $\vec{z}(t) = \left(t, e^{t^2}\right)$ for $-\infty < t < \infty$.

$$z'(t) = \boxed{(1, 2te^{t^2})}$$

Velocity and Speed

We defined the derivative $\vec{x}'$ of a path $\vec{x}$, thinking of a limit of scaled secant vectors. Taking the limit of these vectors, our derivative gives us a vector which is tangent to the path.

PICTURE

The direction of $\vec{x}'$ gives us the direction of instantaneous of a particle moving along the path, and the length of $\vec{x}'$ tells us the speed of the particle. Recall that we sometimes refer to $\vec{x}'$ as the velocity vector, and write it as $\vec{v}$.

Definition 30. *Consider a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$.*

The velocity vector *of $\vec{x}$ at t is $\vec{v}(t) = \vec{x}'(t)$. The velocity vector is tangent to $\vec{x}$ at $\vec{x}(t)$.*

The speed *of $\vec{x}$ at t is $\|\vec{x}'(t)\| = \|\vec{v}(t)\|$.*

Example 46. *Consider the path $\vec{x}(t) = (\cos(t), \sin(t))$ for $0 \le t \le 2\pi$, which parametrizes the unit circle in $\mathbb{R}^2$. We previously computed the velocity of this path as*

$$\vec{v}(t) = \vec{x}'(t) = (-\sin(t), \cos(t)).$$

We can then compute the speed of $\vec{x}$ as

$$\begin{aligned} \|\vec{x}'(t)\| &= \|(-\sin(t), \cos(t))\|, \\ &= \sqrt{(-\sin(t))^2 + (\cos(t))^2}, \\ &= \sqrt{1}, \\ &= 1. \end{aligned}$$

Consider the path $\vec{y}(t) = (\cos(t^2), \sin(t^2))$ *for* $0 \leq t \leq \sqrt{2\pi}$. *This also parametrizes the unit circle in* $\mathbb{R}^2$. *The velocity vector of this path is*

$$\vec{y}'(t) = \boxed{(-2t\sin(t^2), 2t\cos(t^2))}.$$

The speed of this path is

$$\|\vec{y}'(t)\| = \boxed{2t}.$$

Although both of these paths parametrize the unit circle counterclockwise and starting and ending at $(1, 0)$, *they do so in different ways. The first path,* $\vec{x}$, *traverses the unit circle at constant speed. The second path,* $\vec{y}$, *travels very slowly at first, then the speed increases as it travels around the circle.*

Consider a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$. The velocity of this path gives us a vector $\vec{x}'(t)$ tangent to the curve at $\vec{x}(t)$. The tangent line to $\vec{x}$ at $\vec{x}(t)$ passes through the point $\vec{x}(t)$ and is parallel to the vector $\vec{x}'(t)$. This allows us to parametrize the tangent line, however we need to be very careful to distinguish between the parameter for the *line* and the parameter for the *path*. We do this by taking the parameter for our curve to be t_0 at our chosen point, so we are working with the point $\vec{x}(t_0)$ and the tangent vector $\vec{x}'(t_0)$.

Proposition 15. *Consider a path* $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$. *We can parametrize the line tangent to* $\vec{x}$ *at* $\vec{x}(t_0)$ *as*

$$l(t) = \vec{x}(t_0) + t\vec{x}'(t_0) \text{ for } -\infty < t < \infty.$$

Note that it's particularly important to allow the parameter t to be any real number, otherwise we will be missing part of the line.

Conclusion

In this activity, we learned how to find the velocity and speed of a parametrized curve in $\mathbb{R}^n$.

Properties of Velocity and Speed

In this activity, we explore some important properties of the velocity and speed of a parametrized curve.

Differentiation Laws

In single variable calculus, we used the product rule to differentiate products of functions. Although we can't take the product of two vectors in general, we do have the dot product and cross product, and we would like to understand how differentiation interacts with these products. Fortunately, they turn out to be very similar to the product rule from single variable calculus.

Proposition 16. *Consider paths $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$. For t such that $\vec{x}'(t)$ and $\vec{y}'(t)$ both exist, we have*

$$(\vec{x} \cdot \vec{y})'(t) = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}'(t).$$

If $n = 3$, we also have

$$(\vec{x} \times \vec{y})'(t) = \vec{x}'(t) \times \vec{y}(t) + \vec{x}(t) \times \vec{y}'(t).$$

Proof We prove this result for the dot product, and leave the proof for the cross product as an exercise.

Suppose t is such that both $\vec{x}'(t)$ and $\vec{y}'(t)$ exist, and write $\vec{x}(t) = (x_1(t), ..., x_n(t))$ and $\vec{y}(t) = (y_1(t), ..., y_n(t))$. Then we have

$$(\vec{x} \cdot \vec{y})(t) = x_1(t)y_1(t) + \cdots + x_n(t)y_n(t).$$

Using the single variable product rule and regrouping, we have

$$\begin{aligned}(\vec{x} \cdot \vec{y})'(t) &= \frac{d}{dt}\left(x_1(t)y_1(t) + \cdots + x_n(t)y_n(t)\right), \\ &= x_1'(t)y_1(t) + x_1(t)y_1'(t) + \cdots + x_n'(t)y_n(t) + x_n(t)y_n'(t), \\ &= \left(x_1'(t)y_1(t) + \cdots + x_n'(t)y_n(t)\right) + \left(x_1(t)y_1'(t) + \cdots + x_n(t)y_n'(t)\right).\end{aligned}$$

Notice that the left summand is $\vec{x}'(t) \cdot \vec{y}(t)$ and the right summand is $\vec{x}(t) \cdot \vec{y}'(t)$. Thus, we arrive at our result,

$$(\vec{x} \cdot \vec{y})'(t) = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}'(t).$$

■

Learning outcomes:
Author(s):

Constant Speed Path

To finish up our unit on parametrized paths, we consider the special case where a path is constant distance from the origin. In this case, the path $\vec{x}$ is always perpendicular to its derivative. This makes sense intuitively, if you imagine a particle on the path moving in the direction of its velocity vector. If the velocity vector $\vec{v}$ were not perpendicular to $\vec{x}$, a particle moving a tiny distance along the path would have to move either closer to the origin or farther from the origin.

PICTURE

Proposition 17. *If $\vec{x}(t)$ has constant length, then $\vec{x}(t)$ is perpendicular to $\vec{x}'(t)$, for all t such that $\vec{x}'(t)$ is defined.*

We leave the proof of this proposition as an exercise. It's helpful to think about how the dot product $\vec{x}(t) \cdot \vec{x}(t)$ relates to the length of $\vec{x}(t)$.

Conclusion

In this activity, we explore some important properties of the velocity and speed of a parametrized curve.

Part V

Week 27: Conservative Vector Fields and Topology

Topology

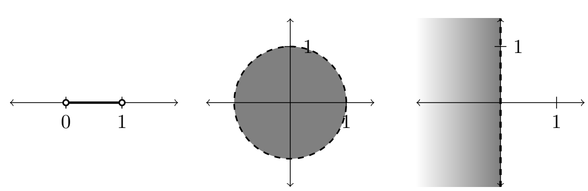
In this activity, we introduce some basic ideas from the area of mathematics called "topology." We introduce these ideas in order to be able to correctly state the theorems in the next activities, on conservative vector fields.

Topology is focused on the shape and spaces, and how to distinguish between different shapes and spaces. However, it takes a much more flexible perspective that you might have seen in geometry classes. In topology, deformations like stretching, shrinking, or bending a space aren't viewed as changing a space. In particular, distance between points isn't important. Instead, topology employs a much looser idea of closeness - provided by open sets, which are defined in the first section.

In general, there are a lot of interesting and weird topological spaces which are not subspaces of any $\mathbb{R}^n$! However, for our purposes, focusing on subspaces of $\mathbb{R}^n$ will be sufficient, so our definitions might look different from some of the definitions you would see in a topology textbook.

Open Sets

You've probably seen many open sets before, without even realizing it! For example, the open interval $(0, 1)$ is an open set, as is the disc $\{(x, y) \mid x^2+y^2 < 1\}$ and the open half-plane $\{(x, y) \mid x < 0\}$.



Some open sets.

Learning outcomes:
Author(s): Melissa Lynn

We begin by defining the most basic type of open set, called an open ball.

Definition 31. *In $\mathbb{R}^n$, we call $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}$ the* open ball *of radius $r > 0$ centered at $\mathbf{x}$.*

In words, this is the set of points within a fixed distance r of a center point $\mathbf{x}$.

Example 47. *In $\mathbb{R}^1$, an open ball is simply an open interval. For example, $(1, 3)$ is the ball $B_1(2)$ of radius 1 centered at 2.*

YouTube link: `https://www.youtube.com/watch?v=QXy9aLiBfSs`

In $\mathbb{R}^2$, an open ball is the inside of a circle (not including the boundary). For example, $B_2((0, 3))$ is the inside of the circle of radius 2 centered at $(0, 3)$.

YouTube link: `https://www.youtube.com/watch?v=r9OMlNgPqZA`

In $\mathbb{R}^3$, and open ball is the inside of a sphere of radius (not including the boundary). For example, $B_1((0, 0, 0))$ is the inside of the sphere of radius 1 centered at the origin.

YouTube link: `https://www.youtube.com/watch?v=xSHf7KQN6Lc`

In each problem, describe the open ball.

Problem 35 *In $\mathbb{R}^1$, $B_3(1)$ is the*

Multiple Choice:

(a) *open interval* ✓

(b) *inside of a circle*

(c) *inside of a sphere*

Problem 35.1 $(\boxed{-2}, \boxed{4})$.

Problem 36 *In $\mathbb{R}^2$, $B_2((1, 1))$ is the*

Multiple Choice:

(a) *open interval*

(b) *inside of a circle* ✓

(c) *inside of a sphere*

Problem 36.1 *of radius* $\boxed{2}$ *centered at* ($\boxed{1}$,$\boxed{1}$).

Problem 37 *In* $\mathbb{R}^3$, $B_4((1,2,3))$ *is the*

Multiple Choice:

(a) *open interval*

(b) *inside of a circle*

(c) *inside of a sphere* ✓

Problem 37.1 *of radius* $\boxed{4}$ *centered at* ($\boxed{1}$,$\boxed{2}$,$\boxed{3}$).

Now we can define open sets in general, using our new idea of open balls.

Definition 32. *A set* $U \subset \mathbb{R}^n$ *is* open *if for every* $\mathbf{a} \in U$*, there is a radius* $r > 0$ *such that* $B_r(\mathbf{a}) \subset U$*.*

In words, for any point $\mathbf{a}$ in U, we can find a radius r small enough that the entire ball of radius r centered at $\mathbf{a}$ is contained in U.

YouTube link: `https://www.youtube.com/watch?v=S-cLB6dLCQo`

Even if we don't have an open set, it's possible that there might be some points in U with this special property. We call these interior points.

Definition 33. *Let* $U \subset \mathbb{R}^n$*. A point* $\mathbf{a} \in U$ *is an* interior point *if there is a radius* $r > 0$ *such that* $B_r(\mathbf{a}) \subset U$*.*

So, we can restate the definition of an open set as "every point is an interior point."

Problem 38 *For each of the following, determine whether or not the set is open.*

(a) $\{(x, y) : x^2 + y^2 < 1\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open* ✓
(ii) *not open*

(b) $\{(x, y) : x^2 + y^2 \leq 1\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open*
(ii) *not open* ✓

(c) $\mathbb{R}^2$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open* ✓
(ii) *not open*

(d) $\emptyset$ *in* $\mathbb{R}^n$

Multiple Choice:

(i) *open* ✓
(ii) *not open*

(e) $\{(x, y) : x \geq 0 \text{ and } y \geq 0\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open*
(ii) *not open* ✓

(f) $\{(x, y) : x < y\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open* ✓
(ii) *not open*

We finish this section by proving an important result about open sets: that the union of a collection of open sets is itself an open set.

Theorem 1. *Suppose* $\{U_i\}$ *is a collection of open sets, where each* U_i *is open. Let* $U = \bigcup U_i$*, the union of all the* U_i*. Then* U *is an open set.*

Proof Let $\mathbf{a} \in U$. We will show that $\mathbf{a}$ is an interior point.

Since $\mathbf{a} \in U = \bigcup U_i$, there is some i such that $\mathbf{a} \in U_i$. Since U_i is open, there is a radius $r > 0$ such that $B_r(\mathbf{a})$ is contained entirely in U_i. Then, since $U_i \subset U$, we have that $B_r(\mathbf{a}) \subset U$. This shows that $\mathbf{a}$ is an interior point, and that U is an open set.

■

Closed Sets

We now introduce closed sets, which are defined via their relationship to open sets.

Definition 34. *A set $X \subset \mathbb{R}^n$ is* closed *if its complement is open.*

Recall that the *complement* of a subset X of $\mathbb{R}^n$ consists of the elements of $\mathbb{R}^n$ which are not in X. That is,

$$\mathbb{R}^n \setminus X = X^C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in X\}.$$

$\mathbb{R}^n \setminus X$ and X^C are both common notations for complements. X^C has the advantage of being succinct, while $\mathbb{R}^n \setminus X$ has the advantage of referring to the larger set $\mathbb{R}^n$, which can help avoid confusion.

We now give a couple examples of closed sets.

Example 48. *The closed interval $[1, 3]$ is a closed set in $\mathbb{R}$, since its complement, $(-\infty, 1) \cup (3, \infty)$, is an open set.*

The set $\{(x, y) : x^2+y^2 \leq 1\} \subset \mathbb{R}^2$, since its complement, $\{(x, y) : x^2+y^2 > 1\}$, is an open set.

It's very important to remember that, even though closed sets are defined in relation to open sets, "closed" is not the same as "not open". It is possible for a set to be neither closed nor open, and it's possible for a set to be both open and closed.

Example 49. *The interval $[0, 1)$ in $\mathbb{R}$ is neither closed nor open. It is not open, since $0 \in [0, 1)$ is not an interior point. It is also not closed, because its complement, $(-\infty, 0) \cup [1, \infty)$, is not open ($1$ is not an interior point).*

$\mathbb{R}^2$ (in $\mathbb{R}^2$) is both closed and open. We've seen that its open, and we've seen that its complement, $\emptyset$, is also open. Hence $\mathbb{R}^2$ is both open and closed.

We've used the word "boundary" to refer to the "edge" of a set. This intuitive idea is useful in working with example, however we do need a rigorous definition of the boundary of a set, which we now give.

Definition 35. *A point* $\mathbf{x} \in \mathbb{R}^n$ *is a* boundary point *of a set* $X \in \mathbb{R}^n$ *if every* $B_r(\mathbf{x})$ *(for* $r > 0$*) contains both points in* X *and points not in* X*.*

The set of all boundary points of X *is called the* boundary *of* X*.*

Example 50. *The boundary of* $\{(x, y) : x^2+y^2 < 1\} \subset \mathbb{R}^2$ *is* $\{(x, y) : x^2+y^2 = 1\}$*.*

The boundary of the interval $(0, 1] \subset \mathbb{R}$ *is* $\{0, 1\}$*.*

From working with examples of open and closed sets, you may have guessed that closed sets tend to contain their boundary points, while open sets do not. This is supported by the following theorem.

Theorem 2. *A set* $X \subset \mathbb{R}^n$ *is closed if and only if it contains all of its boundary points.*

Proof If X is closed, then its complement X^C is open. Consider a boundary point x of X. For any radius $r > 0$, $B_r(x)$ contains points in X and points in X^C. Since X^C is open, this means that $x \notin X^C$. Thus $x \in X$, as desired.

Working in the other direction, suppose that X contains all of its boundary points. We will show that the complement of X, X^C, is open. Let $x \in X^C$. Since x is not a boundary point of X, and any open ball centered at x contains at least one point not in X (namely, x), there must be an open ball $B_r(x)$ with $r > 0$ containing no points in X. Then $B_r(x) \subset X^C$. This shows that X^C is open, and hence X is closed. ■

Problem 39 *For each of the following, determine if the set is open, closed, both, or neither.*

(a) $\{(x, y) : x^2 + y^2 > 1\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open* ✓
(ii) *closed*
(iii) *both open and closed*
(iv) *neither open nor closed*

(b) $\{(x, y) : x^2 + y^2 \leq 1\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open*
(ii) *closed* ✓
(iii) *both open and closed*

   (iv) *neither open nor closed*

(c) $\mathbb{R}^2$ *in* $\mathbb{R}^2$

Multiple Choice:

   (i) *open*
   (ii) *closed*
   (iii) *both open and closed* ✓
   (iv) *neither open nor closed*

(d) $\emptyset$ *in* $\mathbb{R}^n$

Multiple Choice:

   (i) *open*
   (ii) *closed*
   (iii) *both open and closed* ✓
   (iv) *neither open nor closed*

(e) $\{(x, y) : x > 0 \text{ and } y \geq 0\}$ *in* $\mathbb{R}^2$

Multiple Choice:

   (i) *open*
   (ii) *closed*
   (iii) *both open and closed*
   (iv) *neither open nor closed* ✓

(f) $\{(x, y) : x \leq y\}$ *in* $\mathbb{R}^2$

Multiple Choice:

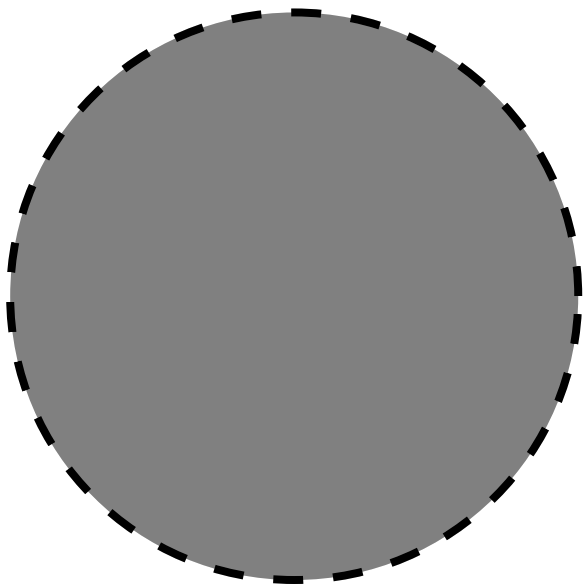
   (i) *open*
   (ii) *closed* ✓
   (iii) *both open and closed*
   (iv) *neither open nor closed*

Connected Sets

We now define what it means for a set to be path-connected. This is generally consistent with your intuitive idea of what "connected" should mean, but we are now able to make this mathematically rigorous.

Definition 36. *A set $X \subset \mathbb{R}^n$ is path-connected if any two points can be connected by a path which lies entirely in X.*

Problem 40 *For each of the following sets, determine whether or not they are path-connected.*

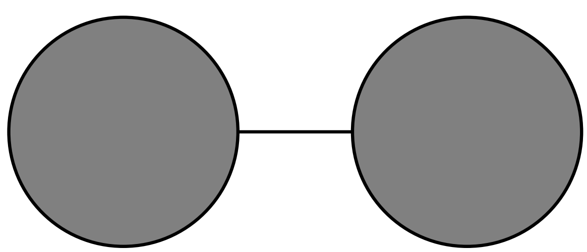


(a)

Multiple Choice:

(i) *path-connected* ✓

(ii) *not path-connected*

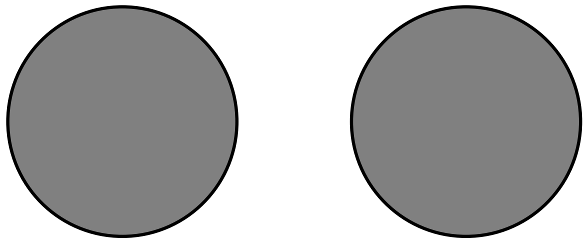


(b)

Multiple Choice:

(i) *path-connected* ✓

(ii) *not path-connected*

(c) 

Multiple Choice:

(i) *path-connected*

(ii) *not path-connected* ✓

Simply Connected Sets

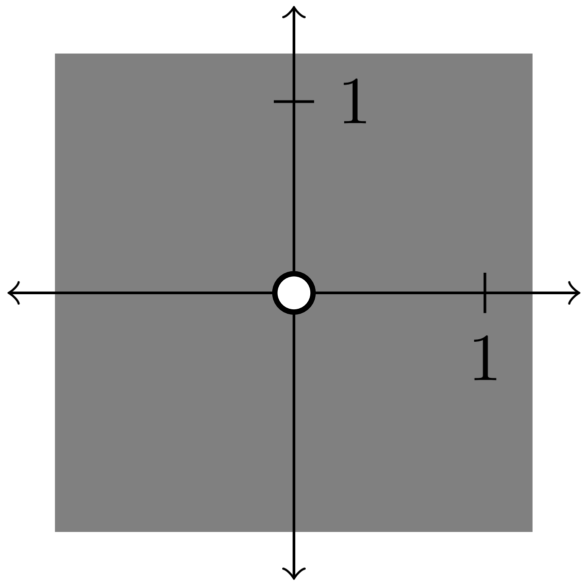
The last topological concept we will cover is when a path-connected set is "simply connected." Intuitively, this depends on whether or not the set has holes. Our definition for simply connected is a bit hand wavy, but this will serve our purpose just fine. It can be made rigorous using continuous maps.

Definition 37. *A path-connected set $X \subset \mathbb{R}^n$ is* simply connected *if any closed path (i.e., loop) can be shrunk to a point, where the shrinking occurs entirely in X.*

Note that a set needs to be path-connected in order to be considered simply connected.

Problem 41 *For each of the following sets, determine whether or not they are simply connected.*

(a) $\mathbb{R}^2 \setminus \{(0,0)\}$

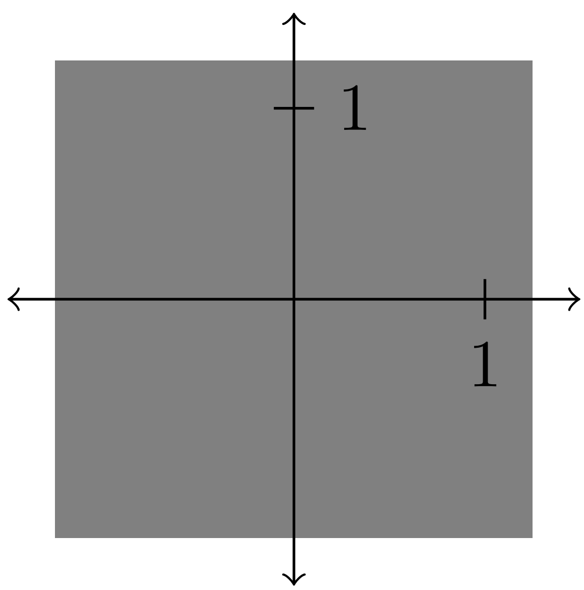


Multiple Choice:

(i) *simply connected*

(ii) *not simply connected* ✓

(b) $\mathbb{R}^2$

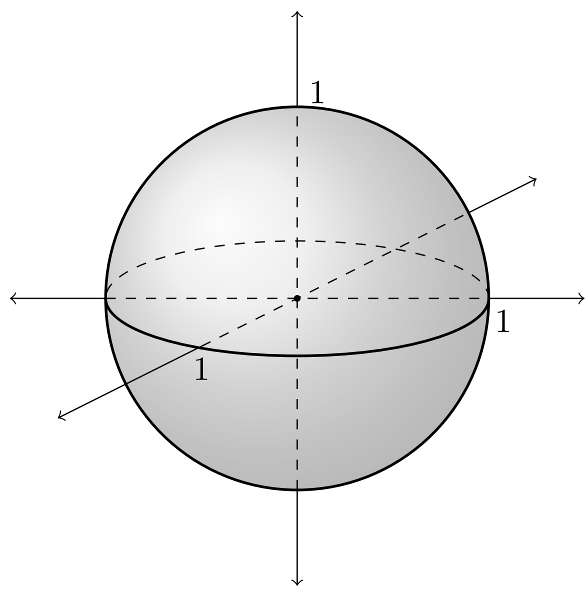


Multiple Choice:

(i) *simply connected* ✓

(ii) *not simply connected*

(c) $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

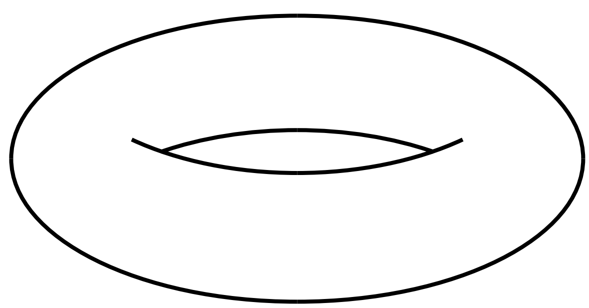


Multiple Choice:

(i) *simply connected* ✓

(ii) *not simply connected*

(d) *A torus (the surface of a donut)*



Multiple Choice:

(i) *simply connected*

(ii) *not simply connected* ✓

Summary

In this activity, we introduced the following topological terms.

- open ball
- open set
- interior point
- closed set
- boundary point
- boundary of a set
- path-connected
- simply connected

We also established the following results.

- The union of a collection of open sets is itself open.
- A set is closed if and only if it contains all of its boundary points.

These terms will allow us to correctly state some theorems about vector fields, which have special requirements on the domain. These theorems will be covered in the next two activities.

Path Independence and FTLI

In the previous activity, we introduced some basic definitions from topology. These definitions are necessary in order to correctly state theorems in this section and the next, so pay attention to the hypotheses of these theorems!

We begin this activity by defining what it means for a vector field to be path independent. We then state and prove the Fundamental Theorem of Line Integrals. This theorem gives us an easy way to compute vector line integrals of a gradient field, provided the domain is open and connected. We then use the Fundamental Theorem of Line Integrals to show that conservative vector fields are path independent (under the appropriate conditions), and finish by discussing the various methods we now have for computing vector line integrals.

Path Independence

In this section, we introduce the idea of "path independence" for a vector field. We see an example of a vector field that is not path independent, and vector field that is path independent.

Example 51. *Consider the vector field* $\boldsymbol{F}(x,y) = (y,0)$. *Let* $\boldsymbol{x}(t)$ *be the path from* $(1,0)$ *to* $(0,1)$ *along a straight line. Let* $\boldsymbol{y}(t)$ *be the path from* $(1,0)$ *to* $(0,1)$ *counterclockwise around the unit circle. Compute* $\displaystyle\int_{\boldsymbol{x}} \boldsymbol{F}\cdot d\boldsymbol{s}$ *and* $\displaystyle\int_{\boldsymbol{y}} \boldsymbol{F}\cdot d\boldsymbol{s}$. *Are they equal?*

Explanation. *We'll begin by parametrizing our paths.*

$$\boldsymbol{x}(t) = (\boxed{1-t}, \boxed{t}) \qquad \textit{for } t \in [0,1]$$

$$\boldsymbol{y}(t) = (\boxed{\cos(t)}, \boxed{\sin(t)}) \qquad \textit{for } t \in [0,\frac{\pi}{2}]$$

Now, we compute these line integrals using the definition

$$\int_{\boldsymbol{x}} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}(\boldsymbol{x}(t))\cdot \boldsymbol{x}'(t)\,dt.$$

Learning outcomes:
Author(s): Melissa Lynn

Integrating along $\boldsymbol{x}$, we have

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F}\cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{x}(t))\cdot \boldsymbol{x}'(t)\,dt \\ &= \int_{\boxed{0}}^{\boxed{1}} \text{given}(t,0)\cdot(-1,1)\,dt \\ &= \int_{\boxed{0}}^{\boxed{1}} -t\,dt \\ &= \boxed{-\frac{1}{2}}\end{aligned}$$

Integrating along $\boldsymbol{y}$, we have

$$\begin{aligned}\int_{\boldsymbol{y}} \boldsymbol{F}\cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{y}(t))\cdot \boldsymbol{y}'(t)\,dt \\ &= \int_0^{\pi/2} (\sin(t),0)\cdot(\boxed{-\sin(t)},\boxed{\cos(t)})\,dt \\ &= \int_0^{\pi/2} \boxed{-\sin^2(t)}\,dt \\ &= \boxed{-\frac{\pi}{4}}\end{aligned}$$

Comparing $\int_{\boldsymbol{x}} \boldsymbol{F}\cdot d\boldsymbol{s}$ and $\int_{\boldsymbol{y}} \boldsymbol{F}\cdot d\boldsymbol{s}$, we see that they are

Multiple Choice:

(a) *Equal.*

(b) *Not equal.* ✓

Now, let's investigate integrating a different vector field along those same paths.

Example 52. *Consider the vector field $\boldsymbol{F}(x,y)=(y,x)$. Let $\boldsymbol{x}(t)$ be the path from $(1,0)$ to $(0,1)$ along a straight line. Let $\boldsymbol{y}(t)$ be the path from $(1,0)$ to $(0,1)$ counterclockwise around the unit circle. Compute $\int_{\boldsymbol{x}} \boldsymbol{F}\cdot d\boldsymbol{s}$ and $\int_{\boldsymbol{y}} \boldsymbol{F}\cdot d\boldsymbol{s}$. Are they equal?*

Explanation. *Once again, we begin by parametrizing our paths.*

$$\boldsymbol{x}(t) = (1-t,t) \qquad \textit{for } t\in[0,1]$$

$$\boldsymbol{y}(t) = (\cos(t),\sin(t)) \qquad \textit{for } t\in[0,\frac{\pi}{2}]$$

Next, we compute these line integrals using the definition

$$\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt.$$

Integrating along $\boldsymbol{x}$, we have

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\ &= \int_0^1 (\boxed{t}, \boxed{1-t}) \cdot (-1, 1)\, dt \\ &= \int_0^1 \boxed{1-2t}\, dt \\ &= \boxed{0}\end{aligned}$$

Integrating along $\boldsymbol{y}$, we have

$$\begin{aligned}\int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\ &= \int_0^{\pi/2} (\boxed{\sin(t)}, \boxed{\cos(t)}) \cdot (-\sin(t), \cos(t))\, dt \\ &= \int_0^{\pi/2} \boxed{-\sin^2(t) + \cos^2(t)}\, dt \\ &= \boxed{0}\end{aligned}$$

Comparing $\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s}$ and $\int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s}$, we see that they are

Multiple Choice:

(a) *Equal.* ✓

(b) *Not equal.*

In fact, it turns out that if we integrate $\mathbf{F}(x, y) = (y, x)$ along *any* path starting at $(1, 0)$ and ending at $(0, 1)$, the integral will be 0. So, the integral does not depend on the path we take to get between these two points. A vector field with this property is called *path independent*.

Definition 38. *A continuous vector field is called* path independent *if* $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_D \boldsymbol{F} \cdot d\boldsymbol{s}$ *for any two simple, piecewise C^1, oriented curves C and D with the same start and end points.*

Let's review the meaning of the requirements on the curves C and D.

A curve is *simple* if it (isn't too "bumpy." / doesn't intersect itself, except the start and end point can be the same. ✓/ is smooth.)

A curve is C^1 if it (is continuous. / is differentiable. / has continuous partial derivatives. ✓)

A curve is *oriented* if it (has a specified direction. ✓/ knows which way is North.)

Our next question is: how do we know if a vector field is path independent? Of course, it's impossible to check every line integral over every possible curve between any two points - we would never finish these computations! Instead, we need some theorems that give us conditions under which a vector field is path independent. Our first step towards these theorems is the Fundamental Theorem of Line Integrals.

Fundamental Theorem of Line Integrals

We now introduce the Fundamental Theorem of Line Integrals, which gives us a slick way to compute the integral of a gradient vector field over a piecewise C^1 curve. In particular, note the conditions on the domain X: it must be open and connected.

Theorem 3. ***Fundamental Theorem of Line Integrals***

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1, where X is open and connected. Then if C is any piecewise C^1 curve from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$\int_C \nabla f \cdot d\boldsymbol{s} = f(\boldsymbol{B}) - f(\boldsymbol{A})$$

This should look vaguely familiar - it resembles the Fundamental Theorem of Calculus from single variable calculus, also called the evaluation theorem.

We now prove the Fundamental Theorem of Line Integrals in the special case where we have a *simple* parametrization of the curve C.

Proof Let $\mathbf{x}(t)$ be a simple parametrization of C, where $t \in [a, b]$, $\mathbf{x}(a) = \mathbf{A}$, and $\mathbf{x}(b) = \mathbf{B}$ (so the starting point is $\mathbf{A}$, and the ending point is $\mathbf{B}$).

Then we compute the line integral as:

$$\int_C \nabla f \cdot d\mathbf{s} = \int_{\mathbf{x}} \nabla f \cdot d\mathbf{s}.$$

By the definition, we can compute this line integral as

$$\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

The integrand here should look familiar from one of the multivariable versions of the chain rule - it's the derivative of $f(\mathbf{x}(t))$. Making this replacement, we have

$$\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \frac{d}{dt}(f(\mathbf{x}(t)))\,dt.$$

Now, we can apply the Fundamental Theorem of Calculus (from single variable) to evaluate this integral, since an antiderivative for the integrand will be given by $f(\mathbf{x}(t))$. From this we obtain:

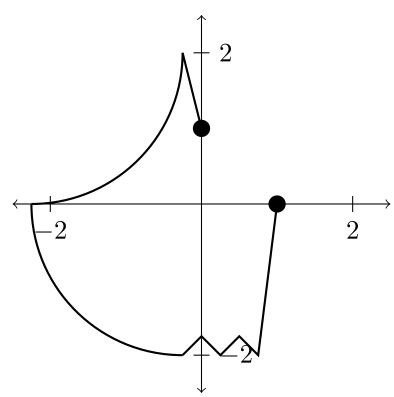
$$\begin{aligned}\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} &= (f(\mathbf{x}(t)))\big|_a^b \\ &= f(\mathbf{x}(b)) - f(\mathbf{x}(a)) \\ &= f(\mathbf{B}) - f(\mathbf{A})\end{aligned}$$

In the last step, we use that $\mathbf{A}$ and $\mathbf{B}$ are the start an end points of $\mathbf{x}(t)$, respectively.

Thus, we have proven the Fundamental Theorem of Line Integrals (when we have a simple parametrization of the curve C). ■

Before discussing how the Fundamental Theorem of Line Integrals relates to path independence, let's look at how this helps us compute integrals of gradient vector fields.

Example 53. *Let $\boldsymbol{F}(x, y) = (y, x)$. Observe that $\boldsymbol{F} = \nabla f$, where $f(x, y) = xy$. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ for the curve C below, starting at $(0, 1)$ and ending at $(1, 0)$.*



Explanation. *We certainly would like to avoid parametrizing this curve! So, we will use the Fundamental Theorem of Line Integrals to compute this integral.*

First, let's verify that $\boldsymbol{F} = \nabla f$ for $f(x,y) = xy$.

$$\frac{\partial}{\partial x}xy = \boxed{y}$$
$$\frac{\partial}{\partial y}xy = \boxed{x}$$

Thus, we have $\nabla f(x,y) = (y,x) = \boldsymbol{F}(x,y)$.

We can then use the Fundamental Theorem of Line Integrals to compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$.

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_C \nabla f \cdot d\boldsymbol{s} \\ &= f(0,1) - f(1,0) \\ &= \boxed{0}\end{aligned}$$

Now, it turns out that we can use the Fundamental Theorem of Line Integrals to prove the following corollary about the relationship between conservative vector fields and path independence. Note once again that we require the domain to be open and connected.

Corollary 1. *If $\boldsymbol{F}$ is a conservative vector field (so $\boldsymbol{F} = \nabla f$ for some f) defined on an open and connected domain X, then $\boldsymbol{F}$ is path independent.*

Proof Let C and D be two curves with starting point $\mathbf{A}$ and ending point $\mathbf{B}$. We will show that $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_D \mathbf{F} \cdot d\mathbf{s}$.

Recall that "$\mathbf{F}$ is conservative" means that $\mathbf{F} = \nabla f$ for some function f, which will enable us to use the Fundamental Theorem of Line Integrals (FTLI). Then we have:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_C \nabla f \cdot d\mathbf{s} \\ &= f(\mathbf{B}) - f(\mathbf{A}) && \text{(by FTLI)} \\ &= \int_D \nabla f \cdot d\mathbf{s} && \text{(also by FTLI)} \\ &= \int_D \mathbf{F} \cdot d\mathbf{s}.\end{aligned}$$

Thus, we have shown that $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_D \mathbf{F} \cdot d\mathbf{s}$, and so have shown that $\mathbf{F}$ is path independent. ■

This corollary, with the Fundamental Theorem of Line Integrals, gives us a new tool for computing line integrals.

Strategies for Computing Line Integrals

We now have a few options for computing line integrals:

(a) Using the original definition:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt$$

(b) If $\mathbf{F}$ is conservative (so $\mathbf{F} = \nabla f$ for some f):

(i) We can use the Fundamental Theorem of Line Integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{B}) - f(\mathbf{A})$$

(ii) Since the vector field is path independent, we can find an *easier* path with the same start and end points, and integrate over that path.

Let's look at how these different methods can be used in an example.

Example 54. *Let* $\boldsymbol{F}(x, y) = (2xy^2, 2x^2y)$*, and consider* $\boldsymbol{x}(t) = (2\cos(\pi t), \sin(\pi t^2))$ *for* $t \in [0, 1]$*. Compute* $\displaystyle\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s}$*.*

Explanation. *We'll evaluate this integral in three different ways.*

First, let's evaluate using the definition of vector line integrals.

$$\begin{aligned}
\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\
&= \int_0^1 (4\cos(\pi t)\sin^2(\pi t^2), 8\cos^2(\pi t)\sin(\pi t^2)) \cdot (\boxed{-2\pi\sin(\pi t)}, \boxed{2\pi t\cos(\pi t^2)})\, dt \\
&= \int_0^1 \boxed{-8\pi\cos(\pi t)\sin(\pi t)\sin^2(\pi t^2) + 16\pi t\cos(\pi t^2)\cos^2(\pi t)\sin(\pi t^2)}\, dt
\end{aligned}$$

This is an integral that it might be possible to figure out how to compute, but we certainly do not want to! We can use a computer algebra system to see that the result is 0*, but to compute the integral by hand, we will turn to our other methods.*

For our alternate methods, we need to find a potential function $f(x, y)$ *such that* $\boldsymbol{F} = \nabla f$*. It turns out that* $f(x, y) = x^2y^2$ *works. Let's verify this.*

$$\frac{\partial}{\partial x}(x^2y^2) = \boxed{2xy^2}$$
$$\frac{\partial}{\partial y}(x^2y^2) = \boxed{2x^2y}$$

Now that we have our function $f(x,y) = x^2y^2$ *such that* $\boldsymbol{F} = \nabla f$, *we will use the Fundamental Theorem of Line Integrals to evaluate. Note the start and end points of our curve*

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{x}(0) = (\boxed{2}, \boxed{0})\\ \boldsymbol{B} &= \boldsymbol{x}(1) = (\boxed{-2}, \boxed{0})\end{aligned}$$

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_{\boldsymbol{x}} \nabla f \cdot d\boldsymbol{s}\\ &= f(\boldsymbol{B}) - f(\boldsymbol{A})\\ &= \boxed{0}\end{aligned}$$

Note that this is a much easier computation than the integral we had from the first method.

Finally, we compute the line integral using the third method. We have already shown that $\mathbf{F}$ *is a conservative vector field (by finding* f *such that* $\mathbf{F} = \nabla f$*), and hence we know that* $\mathbf{F}$ *is path independent. So we can compute this integral by instead integrating over an easier path with the same start and end points,* $(2,0)$ *and* $(-2,0)$*, respectively. Let's choose the straight line from* $(2,0)$ *to* $(-2,0)$*, and parametrize this curve.*

$$\boldsymbol{y}(t) = (\boxed{t}, 0) \qquad \textit{for } t \in [-2,2]$$

Now we can integrate over y *instead, which will be a much easier computation.*

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s}\\ &= \int_{-2}^{2} \boldsymbol{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt\\ &= \int_{-2}^{2} (\boxed{0}, \boxed{0}) \cdot (1,0)\, dt\\ &= \int_{-2}^{2} \boxed{0}\, dt\\ &= \boxed{0}\end{aligned}$$

So we've seen that we can compute this line integral in a few different ways, using the fact that the vector field is conservative.

Depending on the particular problem or example, any one of these methods might be easier than the others. You should practice trying these different methods, and see which you prefer! However, remember that for the second and third options, we need to first verify that the vector field $\mathbf{F}$ is conservative. This usually means finding a potential function f such that $\mathbf{F} = \nabla f$.

Summary

In this activity, we introduced the following terms.

- Path independent

We also established the following results.

- The Fundamental Theorem of Line Integrals.
- If $\mathbf{F}$ is a conservative vector field defined on an open and connected domain X, then $\mathbf{F}$ is path independent.

These results provide us with new methods to compute some vector line integrals.

In the next activity, we will discuss how we can determine whether or not a vector field is conservative.

Conservative Vector Fields

In the previous activity, we proved the Fundamental Theorem of Line Integrals:

Theorem 4. ***Fundamental Theorem of Line Integrals***

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1, where X is open and connected. Then if C is any piecewise C^1 curve from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$\int_C \nabla f \cdot d\boldsymbol{s} = \boxed{f(B) - f(A)}$$

We were able to use the Fundamental Theorem of Line Integrals to easily compute line integrals of conservative vector fields, and we were also able to prove the following corollary.

Corollary 2. *If $\boldsymbol{F}$ is a conservative vector field (so $\boldsymbol{F} = \nabla f$ for some f) defined on an open and connected domain X, then $\boldsymbol{F}$ is $\boxed{\textit{pathindependent}}$.*

Both of these important results require that we have a conservative vector field. In this activity, we will discuss how we can determine whether or not a vector field is conservative, so that we can check if those results can be applied.

Finding a Potential Function

Recall the definition of a conservative vector field.

Definition 39. *A vector field* $\mathbf{F}$ *is* conservative *if there is a C^1 scalar-valued function f such that $\mathbf{F} = \nabla f$. Then f is called a* potential function *for* $\mathbf{F}$.

So, one way to show that a vector field $\mathbf{F}$ is conservative is by finding such a potential function f.

For simple examples, you might be able to do this by guessing. However, more complicated examples require a more systematic approach. The approach is easiest to understand through examples, so we'll work through a couple before describing the steps for the general case.

Example 55. *Find a potential function for the vector field* $\mathbf{F}(x, y) = (2xy^3 + 1, 3x^2y^2 - y^{-2})$.

Learning outcomes:
Author(s): Melissa Lynn

Explanation. *First, note that if there there is a function f such that $\nabla f = \mathbf{F}$, then*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2xy^3+1, 3x^2y^2 - y^{-2}\right)$$

Let's start by looking at the x-term. We must have $\dfrac{\partial f}{\partial x} = 2xy^3+1$. Integrating with respect to x, we have

$$\begin{aligned} f(x,y) &= \int 2xy^3 + 1\, dx \\ &= x^2y^3 + x + g(y) \end{aligned}$$

The first part of this expression, x^2y^3+x, is an antiderivative for $2xy^3+1$ with respect to x. The second part of the expression, $g(y)$, is the "constant" for the integral. It's possible that there are some terms which depend only on y, hence are constant with respect to x, and writing $g(y)$ takes these terms into account.

At this point, we know that f has the form $f(x,y) = x^2y^3+x+g(y)$, but we still need to figure out what $g(y)$ is. For this, we use the y-term of the vector field $\mathbf{F}$. From this, we have $\dfrac{\partial f}{\partial y} = 3x^2y^2 - y^{-2}$. Since we know $f(x,y) = x^2y^3+x+g(y)$, we must have

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^2y^3 + x + g(y)\right) \\ &= 3x^2y^2 + g'(y). \end{aligned}$$

Comparing this with $\dfrac{\partial f}{\partial y} = 3x^2y^2 - y^{-2}$, we must have $g'(y) = -y^{-2}$. Then we find that

$$\begin{aligned} g(y) &= \int -y^{-2}\, dy \\ &= y^{-1} + C \end{aligned}$$

Hence, any potential function would have the form $f(x,y) = x^2y^3+x+y^{-1}+C$. Choosing $C=0$, we obtain a specific potential function $f(x,y) = x^2y^3+x+y^{-1}$.

We now work through finding a potential function for a three dimensional vector field.

Example 56. *Find a potential function for the vector field $\mathbf{F}(x,y,z) = (2xy, x^2 + z + 2y, y + \cos(z))$.*

Explanation. *First, note that a potential function $f(x,y,z)$ would have to satisfy*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\boxed{2xy}, \boxed{x^2+z+2y}, \boxed{y+\cos(z)}\right)$$

We begin by considering the x-component, noticing that $\frac{\partial f}{\partial x} = 2xy$. *We integrate with respect to x.*

$$\begin{aligned} f(x,y,z) &= \int 2xy\,dx \\ &= \boxed{x^2 y} + g(y,z) \end{aligned}$$

Here, $g(y,z)$ is a function of only y and z, hence constant with respect to x. We now differentiate with respect to y, in order to compare to the y-component of the vector field.

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}\left(x^2 y + g(y,z)\right) \\ &= \boxed{x^2} + \frac{\partial g}{\partial y} \end{aligned}$$

Comparing this with $\frac{\partial f}{\partial y} = x^2 + z + 2y$, *we have* $\frac{\partial g}{\partial y} = z + 2y$. *We integrate this with respect to y.*

$$\begin{aligned} g(y,z) &= \int z + 2y\,dy \\ &= \boxed{yz + y^2} + h(z) \end{aligned}$$

Here, h is a function of only z, hence is constant with respect to y. We now know that f has the form $f(x,y,z) = x^2y + yz + y^2 + h(z)$. So, our final task is to find $h(z)$. We differentiate f with respect to z in order to compare with the z-component of the vector field.

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2y + yz + y^2 + h(z)\right) \qquad = \boxed{y} + h'(z)$$

Comparing this with $\frac{\partial f}{\partial z} = y + \cos(z)$, *we have* $h'(z) = \cos(z)$. *Integrating with respect to z, we obtain*

$$\begin{aligned} h(z) &= \int \cos(z)\,dz \\ &= \boxed{\sin(z)} + C \end{aligned}$$

Where C is a constant. Thus, any potential function would have the form $f(x,y,z) = \boxed{x^2y + yz + y^2 + \sin(z)} + C$. *Choosing $C = 0$, we have a specific potential function* $f(x,y,z) = \boxed{x^2y + yz + y^2 + \sin(z)}$.

Summarizing the steps we take in each of the above examples, we have the following process for finding a potential function for a conservative vector field $\mathbf{F}(x_1, x_2, ..., x_n)$.

(a) Integrate the first component of $\mathbf{F}$ with respect to x_1, in order to find the terms of $f(x_1, x_2, ..., x_n)$ which depend on x_1. From this, we can write $f(x_1, x_2, ..., x_n) = (x_1\text{-terms}) + f_1(x_2, ..., x_n)$.

(b) Differentiate $f(x_1, x_2, ..., x_n) = (x\text{-terms}) + f_1(x_2, ..., x_n)$ with respect to x_2. Compare this to the second component of $\mathbf{F}$ in order to determine an expression for $\dfrac{\partial f_1}{\partial x_2}$. Integrate this expression with respect to x_2, so we can write $f_1(x_2, ..., x_n) = (x_2\text{-terms}) + f_2(x_3, ..., x_n)$. Hence we have $f(x_1, x_2, ..., x_n) = (x_1\text{- and } x_2\text{-terms}) + f_2(x_3, ..., x_n)$.

(c) Repeat this process until all components are used.

So far, we've only seen cases where a potential function exists. However, we would also like to be able to show that a vector field is *not* conservative. Let's look at what happens in our process when we have a vector field which is not conservative.

Example 57. *Try (and fail) to find a potential function for the vector field* $\mathbf{F}(x, y) = (-y, x)$.

Explanation. *If a potential function existed, it would have to satisfy*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (\boxed{-y}, \boxed{x})$$

We begin with $\dfrac{\partial f}{\partial x} = -y$. *Integrating with respect to* x, *we have*

$$\begin{aligned} f(x, y) &= \int -y \, dx \\ &= \boxed{-yx} + g(y) \end{aligned}$$

Differentating with respect to y, *we then obtain*

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-yx + g(y)\right) \\ &= \boxed{-x} + g(y) \end{aligned}$$

When we compare this to the y*-component of the vector field* $\mathbf{F}$ *in order to determine* $g(y)$, *we would have to have* $x = -x + g(y)$. *But this is impossible! Thus we see that our method has broken down, and we are not able to find a potential function.*

Here, we see that the system breaks down, and we aren't able to produce a potential function. This is good, since it turns out the vector field isn't conservative. However, we would an easy way to prove that it isn't conservative. The following theorem gives us a quick way to prove that a vector field is not conservative.

Theorem 5. *Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and connected. If $\mathbf{F}$ is conservative, then $D\mathbf{F}$ is symmetric.*

The contrapositive of this theorem states:

Theorem 6. *Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and connected. If $D\mathbf{F}$ is not $\boxed{\textit{symmetric}}$ then $\mathbf{F}$ is not $\boxed{\textit{conservative}}$.*

Thus, provided we have a C^1 vector field and the domain is open and connected, we can show a vector field is not conservative by showing that its derivative matrix is not symmetric.

Example 58. *Show that the vector field $\mathbf{F}(x, y) = (-y, x)$ is not conservative.*

Explanation. *First, note that $\mathbf{F}$ is a C^1 vector field with domain $\mathbb{R}^2$. Since $\mathbb{R}^2$ is open and connected, our theorem applies. We compute the derivative matrix $D\mathbf{F}$.*

$$\begin{aligned} D\mathbf{F} &= \begin{pmatrix} \frac{\partial}{\partial x}(-y) & \frac{\partial}{\partial y}(-y) \\ \frac{\partial}{\partial x}x & \frac{\partial}{\partial y}x \end{pmatrix} \\ &= \begin{pmatrix} \boxed{0} & \boxed{-1} \\ \boxed{1} & \boxed{0} \end{pmatrix} \end{aligned}$$

Since this matrix is not symmetric, $\mathbf{F}$ is not a conservative vector field.

Note how much simpler this is than trying to find a potential function. We now prove our theorem, showing that a conservative C^1 vector field on an open and connected domain has symmetric derivative.

Proof Let $\mathbf{F}$ be a C^1 vector field defined on an open connected domain $X \subset \mathbb{R}^n$. If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for some scalar-valued function f on X. This means

$$\mathbf{F}(x_1, ..., x_n) = \nabla f(x_1, ..., x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right).$$

Then the derivative matrix of $\mathbf{F}$ is

$$D\mathbf{F} = D(\nabla f) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

By Clairaut's Theorem, the mixed partials are equal, so this matrix is symmetric. ■

For each of the given vector fields $\mathbf{F}$, determine whether or not it's conservative. If it is conservative, find a potential function. If $\mathbf{F}$ is not conservative, compute the derivative matrix of $\mathbf{F}$ in order to prove that it is not conservative.

Problem 42 $\mathbf{F}(x, y) = (2xy + y^2 + e^y, x^2 + 2xy + xe^y)$

Multiple Choice:

(a) *conservative* ✓

(b) *not conservative*

Problem 42.1 $f(x, y) = \boxed{x^2y + y^2x + e^yx}$

Problem 43 $\mathbf{F}(x, y) = (x^2y + e^{x^2}, \sin(x) + y^3)$

Multiple Choice:

(a) *conservative*

(b) *not conservative* ✓

Problem 43.1 $D\mathbf{F}(x, y) = \begin{pmatrix} 2xy + 2xe^{x^2} & \boxed{x^2} \\ \boxed{\cos(x)} & 3y^2 \end{pmatrix}$

We've seen that if a vector field is conservative, then its derivative matrix is symmetric. But is the converse true? That is, if the derivative matrix is symmetric, does that mean that the vector field is conservative? We'll answer this question in the next chapter.

Summary

In this activity, we reviewed the following terms.

- Conservative

- Potential function

We also established the following results.

- A systematic method for finding a potential function for a given vector field (thus showing that the vector field is conservative).
- Let $\mathbf{F}$ be a C^1 vector field, and let X be open and connected. If $D\mathbf{F}$ is not symmetric then $\mathbf{F}$ is not conservative.

These provide us with methods for showing that vector field isn't conservative, and for providing a potential function when it is conservative.

Part VI

Week 29: Curl

Curl of a Vector Field

Imagine the vector field below represents fluid flow:

Desmos link: `https://www.desmos.com/calculator/vhuoyka1ys`

If we fix the center point of each + above, which way will they rotate? (clockwise / counter-clockwise ✓)

We can describe this concept as microscopic rotation or local rotation, and it turns out that the *curl* of a vector field measures this local rotation.

In this activity, we define curl and focus on computation. In the next activity, we discuss the geometric significance of curl and how it represents local rotation.

Definition of Curl

A curl is an example of an *operator*, which is a mathematical object you've seen before. Roughly speaking, it's a "function" on functions. That is, it takes a function as an input, and produces a function as an output. Here, we're using "function" very broadly - a function could be scalar-valued, a path, or even a vector field!

To prove that you've seen operators before, let's look at a specific example:

Problem 44 *What does* $\frac{d}{dt}g(t)$ *mean?*

Multiple Choice:

(a) *Multiply* $g(t)$ *by the fraction* $\frac{d}{dt}$.

(b) *Take the derivative of* g *with respect to* t. ✓

Problem 44.1 *What does* $\frac{d}{dt}$ *mean?*

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Multiple Choice:

(a) *The same thing as* $\frac{1}{t}$.

(b) *Take the derivative with respect to* t. ✓

Problem 44.1.1 *It turns out that* $\frac{d}{dt}$ *is an example of an* operator.

To introduce the curl, we need to talk about another operator, ∇ *which we call the* del operator.

What does $\nabla(g(x, y, z))$ *mean?*

Multiple Choice:

(a) *The change in* g.

(b) $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$ ✓

Problem 44.1.1.1 *From this, we can deduce that* ∇ *should mean* $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. *Note that this is an operator.*

Definition 40. *The* del operator *in* $\mathbb{R}^n$ *is* $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, ..., \frac{\partial}{\partial x_n}\right)$.

There's one more ingredient that we need to review in order to define the curl of a vector field, the cross product.

Problem 45 *If* $\mathbf{v} = (1, 2, 3)$ *and* $\mathbf{w} = (4, 5, 6)$, *what is* $\mathbf{v} \times \mathbf{w}$? $\boxed{(-3, 6, -3)}$.

Problem 45.1 *Note that this is computed as the determinant*

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

Problem 45.1.1 *Given a vector field* $\mathbf{F} = (M(x,y,z), N(x,y,z), P(x,y,z))$, *how might we interpret* $\nabla \times \mathbf{F}$*?*

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \boxed{\dfrac{\partial}{\partial x}} & \boxed{\dfrac{\partial}{\partial y}} & \boxed{\dfrac{\partial}{\partial z}} \\ M & N & P \end{vmatrix}$$
$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Based on this, we give our definition for the curl of a three-dimensional vector field:

Definition 41. *The* curl *of a three-dimensional vector field* $\mathbf{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$ *is*

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \boxed{M} & \boxed{N} & \boxed{P} \end{vmatrix}$$
$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Note that this input is a vector field $\mathbf{F}$ in $\mathbb{R}^3$, and the output is another vector field in $\mathbf{R}^3$.

Problem 46 *Let* $\mathbf{F} = (e^y, xz, 3z)$*. Compute the curl* $\nabla \times \mathbf{F}$*.*

$$\nabla \times \mathbf{F} = \boxed{(-x, 0, z - e^y)}$$

Note that we have only defined the curl for three-dimensional vector fields. However, by being a bit clever, we can extend this definition to two-dimensional vector fields.

Definition 42. *If the three-dimensional vector field* $\mathbf{F}$ *has the form* $\mathbf{F}(x, y, z) = (M(x, y), N(x, y), 0)$*, then* $\nabla \times \mathbf{F}$ *is often called the* two-dimensional curl *of* $\mathbf{F}$*.*

Moreover, if $\mathbf{G}(x, y) = (M(x, y), N(x, y))$ *is a vector field in* $\mathbb{R}^2$*, then we define the* curl *of* $\mathbf{G}$ *as the curl of the three-dimensional vector field* $\widetilde{\mathbf{G}}(x, y, z) = (M(x, y), N(x, y), 0)$*.*

It turns out, the curl of a two-dimensional vector field can be written in a simpler form.

Proposition 18. *The two-dimensional curl of* $\mathbf{F}(x, y) = (M(x, y), N(x, y), 0)$ *is*

$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Proof From the definition of the curl, we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Since the third component, P, of our vector field is identically 0, we have

$$\nabla \times \mathbf{F} = \left(\boxed{0} - \frac{\partial N}{\partial z}, -\left(\boxed{0} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Both $M(x, y)$ and $N(x, y)$ are constant with respect to z, so we then have

$$\nabla \times \mathbf{F} = \left(\boxed{0}, \boxed{0}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right),$$

as desired. ■

Sometimes we refer to $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ as the curl $\nabla \times \mathbf{F}$ if $\mathbf{F}$ is two-dimensional, instead of writing out the entire vector.

Note that we've only defined the curl of a vector field for two- and three-dimensional vector fields. Why doesn't it make sense to define the curl of a four-dimensional (or higher!) vector field?

Multiple Choice:

(a) We only exist in three dimensions.

(b) The cross product is only defined in $\mathbf{R}^3$. ✓

Problem 47 *Given* $\mathbf{F}(x, y) = (y, 0)$*, compute the curl* $\nabla \times \mathbf{F}$*.*

$$\nabla \times \mathbf{F} = \boxed{(0, 0, -1)}$$

Problem 48 *Given* $\mathbf{F}(x, y) = (-y, 0)$*, compute the curl* $\nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 1)}$$

Let's look at this example, $\mathbf{F}(x, y) = (-y, 0)$. It turns out that this is the vector field from the beginning of this activity:

We imagined that the center of the plus signs were fixed, and determined that the vector field would rotate the plus signs counterclockwise. We claimed that this local rotation had something to do with the curl of the vector field, which we computed to be $\nabla \times \mathbf{F} = (0, 0, 1)$.

In the next activity, we'll study the geometric significance of the curl, and why the curl measures this "microscopic" rotation.

Summary

In this section, we defined the curl of a two- or three-dimensional vector field, which can be computed as follows:

- For a three-dimensional vector field, $\mathbf{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$, we have $\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$
- The two-dimensional curl of $\mathbf{F}(x, y) = (M(x, y), N(x, y), 0)$ can be computed as
$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

In the next activity, we will discuss the geometric significance of the curl, and how it relates to the local rotation of the vector field.

Geometric Significance of Curl

Consider the vector field $\mathbf{F}(x, y) = (-y, 0)$.

We can compute the curl of this vector field,

$$\nabla \times \mathbf{F} = (0, 0, 1)$$

Imagine that we fix a point (representing a particle) in this vector field, but allow it to rotate. If imagine the vector field acting as a force on this particle, which way will it cause the particle to rotate?

(VECTOR FIELD)

Here, we see that the vector field is applying a greater force to the "top" of the particle than to the "bottom." this will cause the particle to rotate counterclockwise. We describe this type of rotation as *local rotation* or *microscopic rotation*, since it's the rotation when we "zoom in" on the particle.

It turns out that the curl of a vector field provides a measure of this local rotation - but how are these connected? We will answer this question in this section, discussing the geometric significance of the curl.

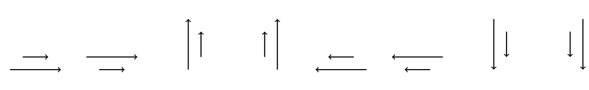
Geometric Significance of Two-dimensional Curl

Recall that, for a two-dimensional vector field $\mathbf{F}(x, y) = (M, N)$, we can compute the curl as

$$\nabla \times \mathbf{F} = (0, 0, N_x - M_y)$$

where N_x is the partial derivative of N with respect to x, and M_y is the partial derivative of M with respect to y. We'll start by considering how M_y and N_x contribute to local rotation.

First let's consider the case where $M_y < 0$. In this case, the x-component of the vector field $\mathbf{F}$ is *decreasing* as we move in the positive y direction. Select all pictures which match this situation.



Select All Correct Answers:

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(a) (a) ✓

(b) (b)

(c) (c)

(d) (d)

(e) (e)

(f) (f) ✓

(g) (g)

(h) (h)

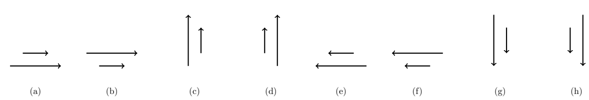
If $M_y < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

Now, let's consider the case where $N_x > 0$. This means that the y-component of the vector field $\mathbf{F}$ is increasing as we move in the positive x direction. Select all pictures which match this situation.



Select All Correct Answers:

(a) (a)

(b) (b)

(c) (c)

(d) (d) ✓

(e) (e)

(f) (f)

(g) (g) ✓

(h) (h)

If $N_x < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

We've seen that the signs of N_x and M_y correspond to the direction of local rotation, with $N_x > 0$ and $M_y < 0$ contributing to counterclockwise rotation.

In general, we have that the sign of $N_x - M_y$ corresponds to the direction of local rotation in the plane. In particular, we have the following correspondences:

$$\begin{aligned} N_x - M_y > 0 &\longleftrightarrow \text{counterclockwise local rotation} \\ N_x - M_y < 0 &\longleftrightarrow \text{clockwise local rotation} \\ N_x - M_y = 0 &\longleftrightarrow \text{no local rotation} \end{aligned}$$

Remembering that $N_x - M_y$ is the *curl* of the two-dimensional vector field $\mathbf{F}$, we now have that the sign of the curl tells us the direction of local rotation for two-dimensional vector fields.

We have a special term for a vector field that never has any local rotation: we call such a vector field *irrotational*.

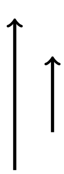
Furthermore, the length of the curl,

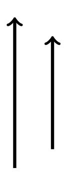
$$\|(0, 0, N_x - M_y)\| = |N_x - M_y|,$$

corresponds to the speed of rotation.

For example, in which case will the particle spin faster?

Multiple Choice:

(a)  ✓

(b) 

Note that this corresponds to a larger value of N_x (the change in the y-component of $\mathbf{F}$ as we move in the positive x direction).

We now apply our knowledge of the geometric significance of the curl in a couple of examples.

Example 59. *Consider the vector field $\mathbf{F}(x, y) = (-y, x^2)$. Compute the curl of $\mathbf{F}$, and describe the local rotation of the vector field at the points $(1, 0)$ and $(-4, 1)$.*

Explanation. *We begin by computing the curl of $\mathbf{F}$,*

$$\nabla \times \mathbf{F} = (0, 0, \boxed{2x + 1}).$$

At the point $(1, 0)$, we have $(\nabla \times \mathbf{F})(1, 0) = (0, 0, \boxed{3})$. Looking at the third component, we see that the sign of $N_x - M_y$ at $(1, 0)$ is (positive ✓/ negative/ zero). Thus, the local rotation of the vector field at the point $(1, 0)$ is (clockwise / counterclockwise ✓/ no rotation).

At the point $(-4, 1)$, we have $(\nabla \times \mathbf{F})(-4, 1) = (0, 0, \boxed{-7})$. Looking at the third component, we see that the sign of $N_x - M_y$ at $(-4, 1)$ is (positive/ negative ✓/ zero). Thus, the local rotation of the vector field at the point $(-4, 1)$ is (clockwise ✓/ counterclockwise/ no rotation).

Looking at a graph of the vector field, we can see that this local rotation is reflected in the graph.

(ADD GRAPH, WITH ROTATION?)

Example 60. *Let $\mathbf{F}(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$. Compute the curl $\nabla \times \mathbf{F}$, and interpret it geometrically.*

Explanation. *Computing our partial derivatives, we have*

$$N_x = \boxed{\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2}}$$

and

$$M_y = \boxed{\frac{-x^2 - y^2 + 2x^2}{(x^2 + y^2)^2}}.$$

Then, the curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0}).$$

Thus, we see that there is no local rotation at any point in the vector field. This is particularly interesting once we look at a graph of the vector field.

(GRAPH)

From the graph of the vector field, there certainly seems to be some larger scale, global rotation of the vector field. However, our computation showed that there *is no local rotation. This example illustrates an important distinction: curl measures* local *rotation of a vector field, which is a different concept from* global *rotation.*

In this section, we saw how the curl of a vector field corresponded to local rotation for a two-dimensional vector field. In the next section, we describe how the curl of a vector field corresponds to local rotation for *three*-dimensional vector fields.

Geometric Significance of Three-dimensional Curl

For a three dimensional vector field $\mathbf{F}(x, y, z) = (M, N, P)$, we can compute the curl of $\mathbf{F}$ as

$$\nabla \times \mathbf{F} = (P_y - N_z, \boxed{-P_x + M_z}, \boxed{N_x - M_y}).$$

Here, the situation is more complicated than in two dimensions. In the plane, there are only two possible ways to rotate: clockwise and counterclockwise. In $\mathbb{R}^3$, there are infinitely many different ways to rotate, since we have infinitely many choices of axes. Yikes!

Fortunately, three-dimensional curl still tells about local rotation. In this case, we imagine local rotation as rotation of an infinitesimal (tiny) sphere. This sphere can rotate in infinitely many different ways, depending on which axis we rotate around.

When we look at the components of the curl, this tells us about rotation perpendicular to each of the axes, ignoring rotation in any other direction. Specifically,

$$\begin{aligned}
N_x - M_y \text{ (the } z\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } z\text{-axis)}\\
P_y - N_z \text{ (the } x\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } x\text{-axis)}\\
-P_x + M_z \text{ (the } y\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } y\text{-axis)}
\end{aligned}$$

Once again, the sign tells us the direction of rotation, with positive sign corresponding to counterclockwise rotation (viewed from the positive axes).

Furthermore, the length of the curl, $\|\nabla \times \mathbf{F}\|$, tells us the speed of rotation, and the direction of $\nabla \times \mathbf{F}$ tells us the axis of rotation.

In $\mathbb{R}^3$, we would like to be able to describe the direction of rotation around a given axis. However, this can be tricky, since it's a matter of perspective. Imagine rotation in the xy-plane. If the rotation is clockwise viewed from above, then it will be counterclockwise from below! Fortunately, curl follows the *right hand rule*:

> If you point your right thumb in the direct of $\nabla \times \mathbf{F}$, then your fingers will curl in the direction of local rotation.

We now put this to use in an example.

Problem 49 *Consider the vector field* $\mathbf{F}(x, y, z) = \left(0, \frac{-z}{(y^2+z^2)^{3/2}}, \frac{y}{(y^2+z^2)^{3/2}}\right)$. *Compute the curl of* $\mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{(\frac{-1}{(y^2+z^2)^{3/2}}, 0, 0)}$$

Problem 49.1 *What is the axis of local rotation (at any point)?*

Multiple Choice:

(a) *The x-axis.* ✓

(b) *The y-axis.*

(c) *The z-axis.*

(d) *Some other line.*

Problem 49.1.1 *Viewed from the positive x-axis, what is the direction of local rotation (at any point)?*

Multiple Choice:

(a) *Clockwise.* ✓

(b) *Counterclockwise.*

Problem 49.1.1.1 *How does the speed of local rotation change as we move closer to the origin?*

Multiple Choice:

(a) *Stays the same.*

(b) *Gets slower.*

(c) *Gets faster.* ✓

We've now seen how the curl describes the local rotation of a three-dimensional vector field. In the next section, we'll cover some connections of the curl to previous topics.

Summary

In this section, we studied the geometric significance of the curl. We found that the curl gives a measure of the local rotation of a vector field.

For a two-dimensional vector field, the sign of the curl told us the direction of rotation. Specifically, we have the following correspondence.

$$\begin{aligned} N_x - M_y > 0 &\longleftrightarrow \text{ counterclockwise local rotation} \\ N_x - M_y < 0 &\longleftrightarrow \text{ clockwise local rotation} \\ N_x - M_y = 0 &\longleftrightarrow \text{ no local rotation} \end{aligned}$$

The magnitude $|N_x - M_y|$ corresponds to speed of rotation.

For a three-dimensional vector field, the components of the curl tell us about local rotation perpendicular to the axes. We also have:

- The length of the curl, $\|\nabla \times \mathbf{F}\|$, corresponds to the speed of rotation.
- The direction of the curl vector $\nabla \times \mathbf{F}$ gives the axis of rotation.
- Curl follows the right hand rule: if you point your thumb in the direction of $\nabla \times \mathbf{F}$, your fingers curl in the direction of local rotation.

In the next section, we'll consider the curl of a conservative vector field, and how the curl connects to Green's Theorem.

Connections of Curl with Older Material

We've defined the curl of a two or three dimensional vector field, and we found that this gives a measure of the local rotation of a vector field.

In this section, we discuss connections of the curl to previous topics from the course. In particular, we find the curl of a conservative vector field, and we restate Green's Theorem in terms of curl.

Curl of a Conservative Vector Field

In this section, we prove that the curl of a conservative vector field will always be zero. Thus, conservative vector fields are irrotational.

Theorem 7. *Suppose* $\mathbf{F}$ *is a* C^1 *conservative vector field in* $\mathbb{R}^3$*, so there is a function* $f : \mathbb{R}^3 \to \mathbb{R}$ *such that* $\mathbf{F} = \nabla f$*. Then* $\nabla \times \mathbf{F} = \mathbf{0}$.

Proof Suppose $\mathbf{F}(x, y, z) = (M, N, P)$ is a C^1 conservative vector field, with $\mathbf{F} = \nabla f$. Then we must have

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \boxed{(M, N, P)}.$$

Computing the curl of $\mathbf{F}$, we have

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, -\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right),\end{aligned}$$

substituting in for the components M, N, and P.

Now, we will use Clairaut's Theorem to simplify this vector. Since $\mathbf{F} = (M, N, P)$ is a C^1 vector field, the partial derivatives of its components ($\frac{\partial M}{\partial y}$, $\frac{\partial M}{\partial z}$, etc.) exist and are continuous. This means that all second-order partial derivatives of f exist and are continuous. Then, by Clairaut's Theorem, the order of differentiation for the second-order mixed partials doesn't matter. In particular, we

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have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z}$$
$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}.$$

Using this fact in our computation of the curl, we now have

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \\ &= \boxed{(0, 0, 0)}\end{aligned}$$

Thus, we have shown that the curl of a conservative vector field is zero. ■

So, conservative vector fields are irrotational. A reasonable follow-up question would be: if the curl of a vector field is zero, is the vector field necessarily conservative? We'll leave this as an open question for the reader, with the suggestion that you think about how you can use past results, and what hypotheses are necessary for this converse to be true.

Curl and Green's Theorem

In this section, we see that we've actually already seen the curl of a vector field. It turns out that the curl showed up in Green's Theorem, we just didn't know that it was the curl yet.

Recall the statement of Green's Theorem:

Theorem 8. *Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise C^1 curves. Orient the boundary ∂D so that D is on the left as one travels along ∂D.*

Let $\mathbf{F}(x, y) = (M, N)$ be a C^1 vector field defined on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

The integrand of the double integral, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, should now look familiar. This mysterious quantity is actually the two-dimensional curl of the vector field $\mathbf{F}$!

Using this realization, we can now restate Green's Theorem in terms of the curl of $\mathbf{F}$.

Theorem 9. *Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise C^1 curves. Orient the boundary ∂D so that D is on the left as one travels along ∂D.*

Let $\mathbf{F}(x, y) = (M, N)$ be a C^1 vector field defined on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F}\, dxdy.$$

Now, let's think a bit more about what Green's Theorem is saying here.

The vector line integral, $\oint_C \mathbf{F} \cdot d\mathbf{s}$, computes the global circulation of the vector field around the boundary of the region.

The double integral $\iint_D \nabla \times \mathbf{F}\, dxdy$ is computed by integrating curl over the region D. We can think of this as "adding up" the local rotation of the vector field.

Thus, we can think of Green's Theorem as saying that the global circulation of the vector field around the boundary is equal to the total local rotation across the region. If you think about it, this does make some sense!

Summary

In this activity, we connected the curl of a vector field to concepts we've covered previously. In particular, we showed that the curl of a C^1 conservative vector field is zero, and we restated Green's Theorem in terms of curl.

Online Homework

Problem 50 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (-2y\cos(3x), 3x\sin(-2y), 0)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 2\cos(3x) - 3\sin(2y)}$$

Find the curl of $\mathbf{F}$ *at the point* $(x, y, z) = (\pi, \pi, \pi)$.

$$(\nabla \times \mathbf{F})(\pi, \pi, \pi) = \boxed{(0, 0, -2}$$

Is $\mathbf{F}$ *a conservative vector field?*

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

(c) *Not enough information.*

Justify your answer.

Free Response:

Problem 51 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (yz, 2xz, 3xy)$.

$$\nabla \times \mathbf{F} = \boxed{(x, -2y, z)}$$

Find the curl of $\mathbf{F}$ *at the point* $(x, y, z) = (0, 0, 0)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}$$

Is $\mathbf{F}$ *irrotational?*

Multiple Choice:

(a) *Yes.*

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Author(s): Melissa Lynn

(b) *No.* ✓

(c) *Not enough information.*

Problem 52 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (x^2, y^3, z^4)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}.$$

Find the curl of $\mathbf{F}$ *at the point* $(x, y, z) = (1, 2, 3)$.

$$(\nabla \times \mathbf{F})(1, 2, 3) = \boxed{(0, 0, 0)}.$$

Is $\mathbf{F}$ *irrotational?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

(c) *Not enough information.*

Problem 53 *Compute the two-dimensional curl of the vector field* $\mathbf{F}(x, y) = (-xy, xy)$.

$$\nabla \times \mathbf{F} = \left(0, 0, \boxed{x + y}\right)$$

Describe the local rotation of $\mathbf{F}$ *at the point* $(1, 1)$.

Multiple Choice:

(a) *Counterclockwise.* ✓

(b) *Clockwise.*

(c) *No rotation.*

Describe the local rotation of $\mathbf{F}$ *at the point* $(-1, 1)$.

Multiple Choice:

(a) *Counterclockwise.*

(b) *Clockwise.*

(c) *No rotation.* ✓

Describe the local rotation of $\mathbf{F}$ *at the point* $(-1, -1)$.

Multiple Choice:

(a) *Counterclockwise.*

(b) *Clockwise.* ✓

(c) *No rotation.*

Problem 54 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (2x - y, -x + 4y)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 54.1 *Find a potential function* f *for* $\mathbf{F}$*, so that* $\nabla f = \mathbf{F}$.

$$f(x, y) = \boxed{x^2 - xy + 2y^2}$$

Problem 55 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (2y, 3x)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{1})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

Problem 56 *Compute the curl of the vector field* $\mathbf{F}(x,y) = (2x, 3y)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 56.1 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x,y) = \boxed{x^2 + \frac{3}{2}y^2}$$

Problem 57 *Compute the curl of the vector field* $\mathbf{F}(x,y) = (-4x + y\cos(x), \sin(x))$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 57.1 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x,y) = \boxed{-2x^2 + y\sin(x)}$$

Problem 58 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (\sin(x), y^2, e^z)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 58.1 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y, z) = \boxed{-\cos(x) + \frac{1}{2}y^2 + e^z}$$

Written Homework

Written Homework

Problem 59 *Prove that, for C^1 vector fields $\mathbf{F}$ and $\mathbf{G}$ in $\mathbf{R}^3$,*

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

(This shows that the curl is an additive operator, *so the curl of a sum is the sum of the curls.)*

Problem 60 (a) *Compute the curl of the vector field $\mathbf{F} = (x, y, z)$. Explain why your answer makes sense geometrically.*

(b) *Suppose we have a C^1 vector field $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$. Compute the curl of $\mathbf{F}$, and explain why your answer makes sense geometrically.*

Problem 61 (a) *Consider the function $f(x, y, z) = e^x \sin(y) + z^2 \cos(y)$. Compute ∇f, and verify that $\nabla \times (\nabla f) = \mathbf{0}$.*

(b) *Prove that for any C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$, the curl of the gradient of f is zero. That is, prove that*

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Professional Problem

Problem 62 *Consider a wheel W centered at a point on the z-axis, rotating about the z-axis. In this problem, we will investigate how the rotation of this wheel relates to the curl of a vector field describing its motion.*

Learning outcomes:
Author(s): Melissa Lynn

Let P be a point on the wheel W, of distance d from the center. The rotation of the wheel can be describe by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of the wheel W. You may assume that ω is constant. Let $\mathbf{x}(x, y, z)$ give the position of P at time t.

(a) *Carefully explain why $\mathbf{x}'(t)$ is orthogonal to both $\mathbf{w}$ and $\mathbf{x}(t)$. Then use the angle θ in the figure to show that $\mathbf{x}'(t) = \mathbf{w} \times \mathbf{x}(t)$. We can therefore define a "velocity field $\mathbf{F}(x, y, z) = \mathbf{w} \times (x, y, z)$, which describes the motion of the wheel W. In other words, every point on W has velocity given by $\mathbf{F}(x, y, z)$.*

(b) *Show that $\mathbf{x}'(t) = (-\omega y, \omega x)$.*

(c) *Show that $\nabla \times \mathbf{F} = 2\mathbf{w}$. Hence if the motion of an object is described by the velocity field $\mathbf{F}$, the curl vector points in the direction of the axis of (positive) rotation, and its length is proportional to the angular speed of the rotation.*

ADD IMAGE

Hint: *What is the linear speed of P in terms of $\mathbf{x}$?*

Recall that angular speed equals linear speed divided by radius. How you can write this in terms of ω, d, and $\mathbf{x}$?

Be sure your solution addresses why $\mathbf{x}'(t)$ is $\mathbf{w} \times \mathbf{x}(t)$, and not $\mathbf{x}(t) \times \mathbf{w}$.