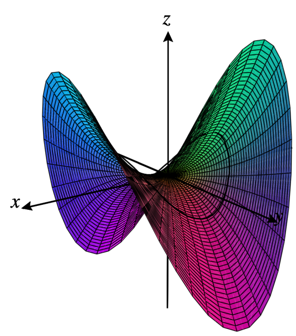
Lagrange Multipliers

We've seen how we can optimize a function subject to a constraint using substitution, and we've seen that it can be very difficult to correctly handle the constraints! Fortunately, there is a tool that we can use to simplify this process. This tool is called "Lagrange multipliers."

Gradients

Suppose we wish to optimize a function $f : \mathbb{R}^n \to \mathbb{R}$ subject to some constraint $g(\vec{x}) = C$, where C is a constant, and $g : \mathbb{R}^n \to \mathbb{R}$ is a function. For now, we'll focus on the case $n = 2$, so we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a constraint $g(x, y) = C$. The graph of f will be a surface in $\mathbb{R}^3$,



and the graph of $g(x, y) = C$ is a curve in $\mathbb{R}^2$.

Learning outcomes: Understand the justification for Lagrange multipliers, and use them to solve constrained optimization problems.

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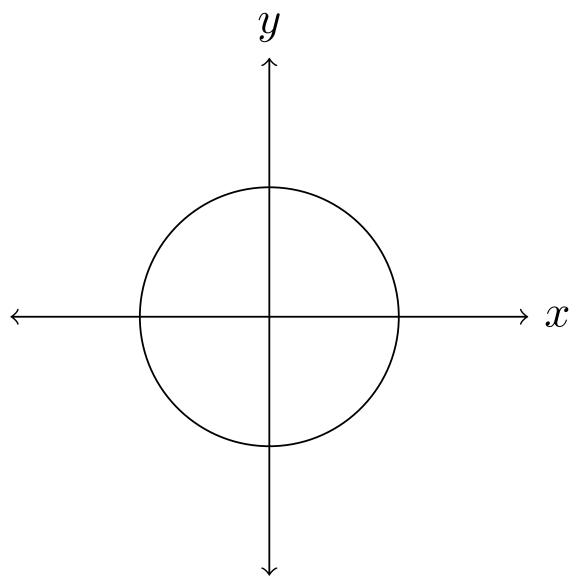
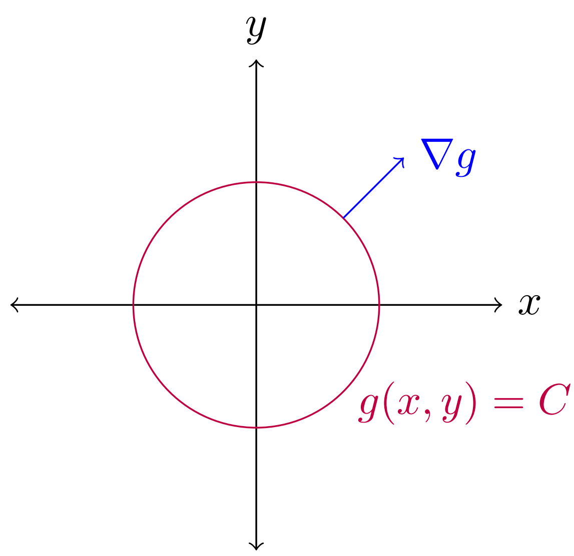
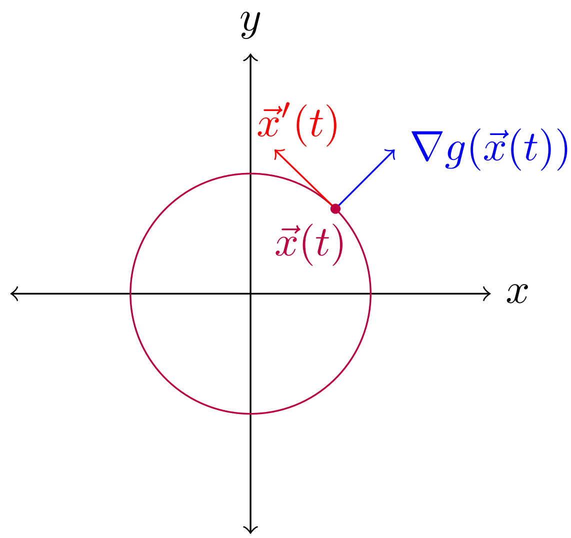


Now, suppose that the maximum value of $f(x, y)$ subject to the constraint $g(x, y) = C$ occurs at some point (a, b). Suppose we parametrize the curve $g(x, y) = C$ as $\vec{x}(t)$, with $(a, b) = \vec{x}(t_0)$. We can view $g(x, y) = C$ as a level curve of the function $g(x, y)$, and then the gradient of $g(x, y)$ will always be perpendicular to the curve $g(x, y) = C$.



More precisely, for any point $(x, y) = \vec{x}(t)$ on the curve $g(x, y) = C$, we'll have $\nabla g(x, y)$ is perpendicular to $\vec{x}'(t)$. That is, $\nabla g(\vec{x}(t)) \perp \vec{x}'(t)$ for all t.



Next, let's turn our attention back to f. If f has an absolute maximum subject to $g(x, y) = C$, then $f(\vec{x}(t))$ has an absolute maximum at $t = t_0$. This means that $f(\vec{x}(t))$ has a critical point at $t = t_0$, so $\frac{d}{dt} f(\vec{x}(t))|_{t=t_0} = 0$. Using the chain rule, we can rewrite this as

$$\nabla f(\vec{x}(t_0)) \cdot \vec{x}'(t) = 0.$$

So, both $\nabla f(a, b)$ and $\nabla g(a, b)$ are perpendicular to $\vec{x}'(t)$. Since we are considering vectors in $\mathbb{R}^2$, this means that $\nabla f(a, b)$ and $\nabla g(a, b)$ are parallel, so we can write

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

for some constant λ.

So, we can find candidate points for the absolute maximum (and similarly, the absolute minimum) of $f(x, y)$ subject to the constraint $g(x, y) = C$ by finding points where

$$\nabla f(a, b) = \lambda \nabla g(a, b).$$

This observation generalizes to $\mathbb{R}^n$.

Proposition 1. *Consider $\mathcal{C}^1$ functions $f, g : \mathbb{R}^n \to \mathbb{R}$, and let C be a constant. If f has an absolute maximum or absolute minimum at $\vec{a}$ subject to the constraint $g(\vec{x}) = C$, then there exists some scalar λ such that*

$$\nabla f(\vec{a}) = \lambda \nabla g(\vec{a}).$$

The constant λ is called a Lagrange Multiplier.

We can leverage this theorem into a method for finding absolute extrema of a function subject to a constraint, which we call the *method of Lagrange multipliers.*

To find the absolute extrema of a function $f(\vec{x})$ subject to a constraint $g(\vec{x}) = C$:

(a) Compute the gradients $\nabla f(\vec{x})$ and $\nabla g(\vec{x})$.

(b) Solve the system of equations

$$\begin{cases} \nabla f(\vec{x}) = \lambda \nabla g(\vec{x}) \\ g(\vec{x}) = C \end{cases}$$

for $\vec{x}$ and λ.

(c) The solutions to the system of equations in (2) are the critical points of $f(\vec{x})$ subject to $g(\vec{x}) = C$. Classify these critical points in order to determine the absolute extrema.

If $g(\vec{x}) = C$ is compact, we can determine the absolute extrema by comparing the values of f at the critical points. In this case, absolute extrema are guaranteed to exist by the Extreme Value Theorem.

We'll see how this process works in a couple of examples. First, we repeating an optimization problem which was previously done with substitution.

Example 1. *We'll find the absolute maximum and absolute minimum of $f(x, y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 1$. Notice that $x^2 + y^2 = 1$ is a compact region, so absolute extrema will exist.*

If we let $g(x, y) = x^2 + y^2$, our constraint is $g(x, y) = 1$.

Computing the gradients of f and g, we have

$$\begin{aligned} \nabla f(x, y) &= \boxed{(2x, -2y)}, \\ \nabla g(x, y) &= \boxed{(2x, 2y)}. \end{aligned}$$

Setting up the system of equations

$$\begin{cases} \nabla f(\vec{x}) = \lambda \nabla g(\vec{x}) \\ g(\vec{x}) = 1 \end{cases},$$

we have

$$\begin{aligned} \nabla(2x, -2y) &= \lambda(2x, 2y) \\ x^2 + y^2 &= 1 \end{aligned}$$

We can rewrite this as a system of three equations,

$$\begin{cases} 2x = 2\lambda x \\ 2y = -2\lambda y \\ x^2 + y^2 = 1 \end{cases}.$$

From the first equation, we have either $x = 0$ *or* $\lambda = 1$.

If $x = 0$, *the third equation gives us* $y = \pm 1$. *Thus, we obtain the critical points* $(0, \pm 1)$.

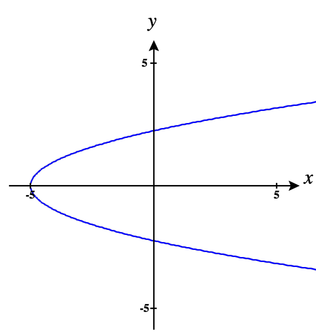
If $\lambda = 1$, *the second equation gives us* $y = 0$. *Then, the third equation gives us* $x = \pm 1$. *This gives us the critical points* $(\pm 1, 0)$.

Since we know that f *will have an absolute maximum and minimum subject to our constraint, we will compare the values at the critical points to determine the absolute maximum and minimum.*

$$\begin{aligned} f(0,-1) &= -1 \\ f(0,1) &= -1 \\ f(-1,0) &= 1 \\ f(1,0) &= 1 \end{aligned}$$

We see that the absolute maximum of f *subject to our constraint is* 1, *and this occurs at the points* $(\pm 1, 0)$. *The absolute minimum of* f *subject to our constraint is* -1, *and this occurs at the points* $(0, \pm 1)$.

Example 2. *Next, we'll attempt to optimize* $f(x, y) = 12 - x^2 - y^2$ *subject to the constraint* $y^2 = x + 5$. *Here, we'll need to be a bit more careful, since the curve defined by* $y^2 = x + 5$ *is not compact, as it is unbounded.*



However, Lagrange multipliers will still be helpful for finding critical points. We can rewrite our constaint as $y^2 - x = 5$, *and take* $g(x, y) = y^2 - x$. *So, we are optimizing* $f(x, y)$ *subject to the constraint* $g(x, y) = 5$.

We begin by finding the gradients of f and g.

$$\nabla f(x,y) = \boxed{(-2x,-2y)}$$
$$\nabla g(x,y) = \boxed{(-1,2y)}$$

Next, we solve the system

$$\begin{cases} \nabla f(\vec{x}) = \lambda \nabla g(\vec{x}) \\ g(\vec{x}) = 5 \end{cases},$$

which is

$$\begin{cases} (-2x,-2y) = \lambda(-1,2y) \\ y^2 - x = 5 \end{cases}.$$

We can rewrite this system as the three equations

$$\begin{cases} -2x = -\lambda \\ -2y = 2\lambda y \\ y^2 - x = 5 \end{cases}.$$

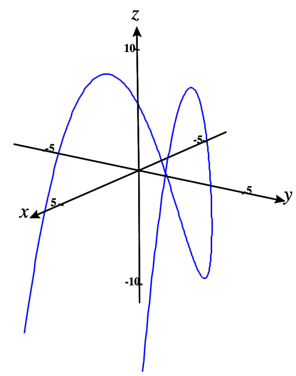
From the second equation, we have either $y = 0$, or $\lambda = -1$.

If $y = 0$, the third equation gives us $x = -5$. So, we have a critical point $(-5,0)$.

If $\lambda = -1$, the first equation gives us $x = -\frac{1}{2}$. Then the third equation gives us $y = \pm\frac{3}{\sqrt{2}}$. So, we have the critical points $\left(-\frac{1}{2}, \pm\frac{3}{\sqrt{2}}\right)$.

In this case, we can't determine the absolute maximum and absolute minimum by plugging in these points, since we aren't optimizing over a compact region.

However, looking at the graph of f over the curve $g(x,y) = 5$, we can see that the absolute maximum occurs at the points $\left(-\frac{1}{2}, \pm\frac{3}{\sqrt{2}}\right)$. Although there is a local minimum at $(-5,0)$, this is not an absolute minimum, as there is no absolute minimum.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