Homework 12: Extrema

Graded Problems

Problem 1 *Consider the function $f(x,y) = (y^2 - x)(2y^2 - x)$.*

(a) *Show that $f(x,y)$ has a critical point at the origin.*

(b) *Show that $f(x,y)$ has a local minimum along any line through the origin. That is, show that for constant $(a,b) \neq (0,0)$, that the function $g(t) = f(at, bt)$ has a local minimum at $t = 0$.*

(c) *Show that $f(x,y)$ does not have a local minimum at the origin.*

Problem 2 (a) *Let $x_1, x_2, ..., x_n$ be nonnegative numbers such that their sum is constant. That is, $x_1 + x_2 + \cdots + x_n = C$ for some constant C. Show that the product $x_1 x_2 \cdots x_n$ is a maximum if and only if $x_1 = x_2 = \cdots = x_n = C/n$.*

(b) *Using the result of part (a), show that if $x_1, x_2, ..., x_n$ are nonnegative numbers such that $x_1 + x_2 + \cdots + x_n = 1$, then $x_1 x_2 \cdots x_n \leq 1/n^n$.*

Professional Problem

Problem 3 *Final draft of your project, worth two professional problems.*

Completion Packet

Problem 4 *Find and classify all critical points of each function. When using the Hessian fails to classify points, find another method.*

Learning outcomes:
Author(s):

(a) $f(x, y) = x^3 + y^3 + 12xy$

(b) $f(x, y) = x^2 + y^2 + \dfrac{1}{x^2y^2}$, for $xy \neq 0$

(c) $f(x, y) = (x + y)^2 + x^4$

(d) $f(x, y) = (x + y)e^{-xy}$

(e) $f(x, y, z) = 2x^2 + y^2 + z^2 - xz + xy$

(f) $f(x, y, z) = xy + xz$

Problem 5 *What are the conditions on a, b, c for $f(x, y) = ax^2 + bxy + cy^2$ to have a...*

(a) *...local minimum at the origin?*

(b) *...local maximum at the origin?*

(c) *...saddle point at the origin?*

Problem 6 *For nonzero constants a and b, consider the function $f(x, y) = ax^{-1} + by^{-1} + xy$.*

(a) *Find the (single) critical point of this function.*

(b) *What are the conditions on a and b for the critical point to be a local minimum? A local maximum? A saddle point?*

Problem 7 *Find the shortest distance between a point on the surface $(x - 2)^2 + (y - 3)^2 + z^2 = 1$ and the origin in $\mathbb{R}^3$.*

Problem 8 (a) *Let $x_1, x_2, ..., x_n$ be positive numbers such that their product is constant. That is, $x_1x_2 \cdots x_n = C$ for some constant C. Show that the sum $x_1 + x_2 + \cdots + x_n$ is a* local *minimum if and only if $x_1 = x_2 = \cdots = x_n$.*

(b) *The local minimum which you found in part (a) is, in fact, an absolute minimum (you do not need to show this). Using this fact, show that if $x_1, x_2, ..., x_n$ are positive numbers such that $x_1x_2 \cdots x_n = 1$, then $x_1 + x_2 + \cdots + x_n \geq n$.*
