

Geometric Interpretation of Partial Derivatives

We've defined the partial derivatives of a function as follows.

Definition 1. Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. For $1 \leq i \leq n$, we define the partial derivative of f with respect to x_i to be

$$f_{x_i}(x_1, \dots, x_n) = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h},$$

provided this limit exists.

In other words, if we treat all variables except for x_i as constants, and differentiate with respect to x_i , we get the partial derivative with respect to x_i .

When computing a partial derivative with respect to x_i , we're looking at the instantaneous rate of change of f with respect to x_i , if we keep the rest of the variables constant. Roughly speaking, we're asking: how does increasing x_i a tiny bit affect the value of f ?

We can see the partial derivatives reflected in the shape of the graph of f . So that we can visualize the graph of f , we'll focus on a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, so we're considering the partial derivative of f with respect to x , and with respect to y .

Suppose at the point $(1, 2)$, we have that $f_x(1, 2) > 0$ and $f_y(1, 2) > 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will increase. If we move a tiny amount in the positive y direction, the value of f will increase as well.

YouTube link: <https://www.youtube.com/watch?v=bRB4HpPdNc>

Similarly, suppose at the point $(1, 2)$, we have that $f_x(1, 2) < 0$ and $f_y(1, 2) < 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will decrease. If we move a tiny amount in the positive y direction, the value of f will decrease as well.

YouTube link: <https://www.youtube.com/watch?v=17cKrUCjG9U>

Now, let's consider the case where $f_x(1, 2) > 0$ and $f_y(1, 2) < 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will increase. But, if we move a tiny amount in the positive y direction, the value of f will decrease.

Learning outcomes:
Author(s):

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YouTube link: <https://www.youtube.com/watch?v=IEWHvz46pXw>

Next, let's suppose that $f_x(1, 2) > 0$ and $f_y(1, 2) = 0$. As expected, f increases as we move a tiny amount in the positive x direction. On the other hand, the graph of f has flattened out as we move in the y direction. However, this doesn't mean that it's constant! It's just the instantaneous rate of change that's 0 at that one point.

YouTube link: <https://www.youtube.com/watch?v=ukDxOLVvk24Y>

Now, let's look at a case where $f_x(1, 2) = 0$ and $f_y(1, 2) = 0$. As before, this does not mean that f is constant. This just means that the rates of change are both instantaneously 0. Points with this property will be important later in the course, when we study optimization.

YouTube link: <https://www.youtube.com/watch?v=i6dsWWlQjNQ>