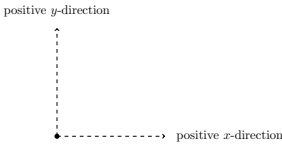


# Directional Derivatives

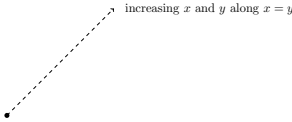
In order to find how a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  changes with each of the input variables, we defined the partial derivatives of  $f$ . For example, when  $n = 2$ , we defined the partial derivative of  $f$  with respect to  $x$  to be

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

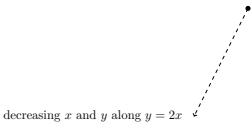
Here, we thought of  $y$  as a constant, which made  $f$  only a function of  $x$ , and reduced us to a single variable derivative. This told us how a small change in  $x$  would affect the value of  $f$ , if we kept  $y$  constant. In other words, the partial derivatives described how the function  $f$  was changing in the positive  $x$ -direction and in the positive  $y$ -direction.



But what if we want to find how  $f$  changes if we change both  $x$  and  $y$ ? One possible way to do this would be to increase  $x$  and  $y$  by the same amount, which would be equivalent to finding how  $f$  changes as we increase  $x$  and  $y$  along the line  $y = x$ .



Alternatively, we could decrease  $y$  by twice as much as  $x$ . This would be equivalent to finding how  $f$  changes as decreasing  $x$  and  $y$  along the line  $y = 2x$ .



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Learning outcomes:  
Author(s):

As you can see, there are many different ways that we can change  $x$  and  $y$ , corresponding to different directions in the  $xy$ -plane. In order to determine how  $f$  changes as we move in all of these different directions, we will now define directional derivatives.

## Directional derivatives

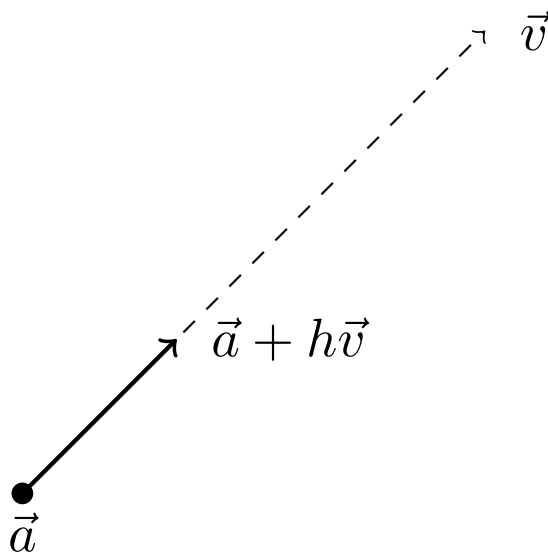
We would like to compute the instantaneous rate of change of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  at a point as we move in some given direction in  $\mathbb{R}^n$ . We will model our definition after partial derivatives and single variable derivatives, and use a unit vector  $\vec{v}$  to describe the direction.

**Definition 1.** Consider a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , a point  $\vec{a} \in \mathbb{R}^n$ , and a direction given by a unit vector  $\vec{v} \in \mathbb{R}^n$ . Then we define the directional derivative of  $f$  at  $\vec{a}$  in the direction of  $\vec{v}$  to be

$$D_{\vec{v}}f(\vec{a}) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h},$$

provided this limit exists.

Noticing that by looking at  $f(\vec{a} + h\vec{v})$ , we are finding the value of  $f$  when we move a small distance,  $h$ , in the direction of  $\vec{v}$  from the point  $\vec{a}$ .



When computing directional derivatives, it's important to remember that the direction must be given by a *unit* vector. Otherwise, the length of the vector will

change the value of the limit above. If you'd like to find a directional derivative in a direction given by a non-unit vector  $\vec{w}$ , you should normalize  $\vec{w}$  to unit length.

**Example 1.** We'll compute the directional derivative of  $f(x, y) = x^2y + y^2$  at  $\vec{a} = (2, 0)$ , in the direction of  $(3, 4)$ .

Since  $(3, 4)$  isn't a unit vector, we need to normalize it. Since  $\|(3, 4)\| = \sqrt{3^2 + 4^2} = 5$ , we'll use the vector  $\vec{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$  to compute our desired directional derivative.

$$\begin{aligned} D_{\vec{v}}f(\vec{a}) &= \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left((2, 0) + h\left(\frac{3}{5}, \frac{4}{5}\right)\right) - f(2, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left(2 + \frac{3}{5}h, \frac{4}{5}h\right) - f(2, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(2 + \frac{3}{5}h\right)^2 \cdot \frac{4}{5}h + \left(\frac{4}{5}h\right)^2 - 0}{h} \\ &= \lim_{h \rightarrow 0} \left(2 + \frac{3}{5}h\right)^2 \cdot \frac{4}{5} + \left(\frac{4}{5}\right)^2 h \\ &= 4 \cdot \frac{4}{5} \\ &= \frac{16}{5}. \end{aligned}$$

Fortunately, we won't always need to resort to evaluating directional derivatives using the limit definition. We'll soon see how we can use the gradient to compute directional derivatives.