Homework 9: Properties of Derivatives

Graded Problems

Problem 1 (a) *Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function. Prove that $f_{xyzx} = f_{zxxy}$, where these are fourth-order partial derivatives. You are free to use Theorem 4.3 for this problem, but not Theorem 4.5.*

(b) *Verify the result of (a) for the function $f(x, y, z) = x^3y^2z$.*

Problem 2 *Suppose we have differentiable functions $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\begin{aligned} \mathbf{g}(1,2) &= (3,0), \\ D\mathbf{f}(x,y) &= \begin{pmatrix} xe^y & x^2y \\ 0 & yx^2+1 \end{pmatrix}, \\ D(\mathbf{f}\circ\mathbf{g})(1,2) &= \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}. \end{aligned}$$

Find $D\mathbf{g}(1,2)$.

Professional Problem

Problem 3 (a) *Consider differentiable functions $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = 0$ when $y = g(x)$. Prove that if $\partial f/\partial y \neq 0$, then*

$$\frac{dg}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

(b) *Consider the equation $\sin(x) + \cos(y) = 0$. Make a suitable choice of f, and use the result of (a) to compute $\frac{dy}{dx}$ in terms of x and y.*

(c) *Use implicit differentiation to verify your answer to (b).*

Learning outcomes:
Author(s):

Completion Packet

Problem 4 *Consider the functions* $\mathbf{f}(x, y) = (x^2+y, xy-\sin(xy))$ *and* $\mathbf{g}(x, y) = (e^{xy}, x^2y)$. *Compute* $D(\mathbf{f} + \mathbf{g})$ *in two ways:*

(a) *By computing the derivative of* $\mathbf{f} + \mathbf{g}$

(b) *By computing the derivatives of* $\mathbf{f}$ *and* $\mathbf{g}$, *and using the sum rule.*

Problem 5 *Consider the functions* $f(x, y, z) = xy^2z^3$ *and* $g(x, y, z) = xyz$.

(a) *Verify that*
$$D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}).$$

(b) *Verify that*
$$D(f/g)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}.$$

Problem 6 *Find all second-order partial derivatives of the function* $f(x, y) = \ln(xy)$.

Problem 7 *Find all second-order partial derivatives of the function* $g(u, v) = e^{u^2+v^2}$.

Problem 8 *Find all second-order partial derivatives of the function* $h(x, y, z) = xy^2z^3$.

Problem 9 *Consider the functions* $y(s, t) = e^s + e^{st} + e^t$ *and* $\mathbf{x}(t) = (t, t^2)$. *Compute* $D(\mathbf{x} \circ y)$ *in two ways:*

(a) *Determining a formula for the composition* $\mathbf{x} \circ y$, *then computing the total derivative.*

(b) *Computing total derivatives of* $\mathbf{x}$ *and* y, *and using the chain rule.*

Problem 10 *Consider functions* $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ *and* $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^3$ *such that* $\mathbf{f}(x, y, z) = (x^2y + e^z, e^x + y)$ *and* $D\mathbf{g}(x, y) = \begin{pmatrix} y & x \\ 0 & 1 \\ 0 & xy \end{pmatrix}$. *For each of the given total derivatives, either explain why they do not exist, compute them, or explain what additional information we would need to compute them.*

(a) $D(\mathbf{f} \circ \mathbf{g})$

(b) $D(\mathbf{g} \circ \mathbf{f})$

Problem 11 *Consider functions* $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ *and* $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $\mathbf{f}(x, y, z) = (\sin(xyz), \cos(xyz))$ *and* $D\mathbf{g}(x, y) = \begin{pmatrix} e^{xy} & 0 \\ 0 & e^{xy} \end{pmatrix}$. *For each of the given total derivatives, either explain why they do not exist, compute them, or explain what additional information we would need to compute them.*

(a) $D(\mathbf{f} \circ \mathbf{g})$

(b) $D(\mathbf{g} \circ \mathbf{f})$

Problem 12 *Consider the function* $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$ *given by* $\mathbf{g}(\rho, \theta, \phi) = (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$, *which changes spherical coordinates to Cartesian coordinates. For any differentiable function* $f : \mathbb{R}^3 \to \mathbb{R}$, *compute* $\partial f/\partial\rho$, $\partial f/\partial\theta$, *and* $\partial f/\partial\phi$ *in terms of* $\partial f/\partial x$, $\partial f/\partial y$, *and* $\partial f/\partial z$.