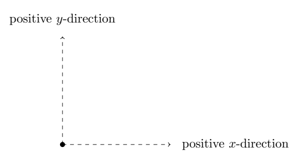
Directional Derivatives

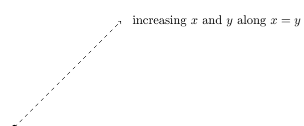
In order to find how a function $f : \mathbb{R}^n \to \mathbb{R}$ changes with each of the input variables, we defined the partial derivatives of f. For example, when $n = 2$, we defined the partial derivative of f with respect to x to be

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

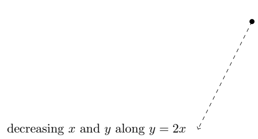
Here, we thought of y as a constant, which made f only a function of x, and reduced us to a single variable derivative. This told us how a small change in x would after the value of f, if we kept y constant. In other words, the partial derivatives described how the function f was changing in the positive x-direction and in the positive y-direction.



But what if we want to find how f changes if we change both x and y? One possible way to do this would be to increase x and y by the same amount, which would be equivalent to finding how f changes as we increase x and y along the line $y = x$.



Alternatively, we could decrease y by twice as much as x. This would be equivalent to finding how f changes as decreasing x and y along the line $y = 2x$.



Learning outcomes: Understand the idea behind directional derivatives, and use the limit definition to compute them.

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As you can see, there are many different ways that we can change x and y, corresponding to different directions in the xy-plane. In order to determine how f changes as we move in all of these different directions, we will now define directional derivatives.

Directional derivatives

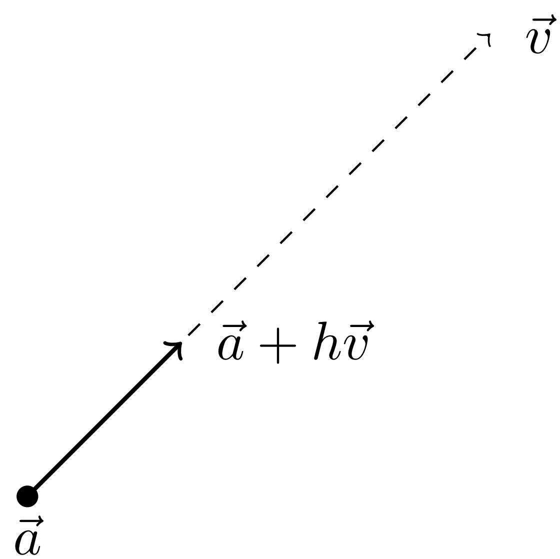
We would like to compute the instantaneous rate of change of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point as we move in some given direction in $\mathbb{R}^n$. We will model our definition after partial derivatives and single variable derivatives, and use a unit vector $\vec{v}$ to describe the direction.

Definition 1. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, a point $\vec{a} \in \mathbb{R}^n$, and a direction given by a unit vector $\vec{v} \in \mathbb{R}^n$. Then we define the* directional derivative *of f at $\vec{a}$ in the direction of $\vec{v}$ to be*

$$D_{\vec{v}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h},$$

provided this limit exists.

Noticing that by looking at $f(\vec{a} + h\vec{v})$, we are finding the value of f when we move a small distance, h, in the direction of $\vec{v}$ from the point $\vec{a}$.



When computing directional derivatives, it's important to remember that the direction must be given by a *unit* vector. Otherwise, the length of the vector will change the value of the limit above. If you'd like to find a directional derivative in a direction given by a non-unit vector $\vec{w}$, you should normalize $\vec{w}$ to unit length.

Example 1. *We'll compute the directional derivative of $f(x, y) = x^2y + y^2$ at $\vec{a} = (2, 0)$, in the direction of $(3, 4)$.*

Since $(3, 4)$ isn't a unit vector, we need to normalize it. Since $\|(3, 4)\| = \sqrt{3^2 + 4^2} = 5$, we'll use the vector $\vec{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$ to compute our desired directional derivative.

$$
\begin{aligned}
D_{\vec{v}} f(\vec{a}) &= \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h} \\
&= \lim_{h \to 0} \frac{f\left((2, 0) + h\left(\frac{3}{5}, \frac{4}{5}\right)\right) - f(2, 3)}{h} \\
&= \lim_{h \to 0} \frac{f\left(2 + \frac{3}{5}h, \frac{4}{5}h\right) - f(2, 0)}{h} \\
&= \lim_{h \to 0} \frac{\left(2 + \frac{3}{5}h\right)^2 \cdot \frac{4}{5}h + \left(\frac{4}{5}h\right)^2 - 0}{h} \\
&= \lim_{h \to 0} \left(2 + \frac{3}{5}h\right)^2 \cdot \frac{4}{5} + \left(\frac{4}{5}\right)^2 h \\
&= 4 \cdot \frac{4}{5} \\
&= \frac{16}{5}.
\end{aligned}
$$

Fortunately, we won't always need to resort to evaluating directional derivatives using the limit definition. We'll soon see how we can use the gradient to compute directional derivatives.