Homework 6: Limits

Completion Packet

Problem 1 *Evaluate the limit, or explain why it doesn't exist.*

$$\lim_{(x,y,z)\to(0,0,0)} 3e^{x^2+y^2+z^2}\cos(xy)$$

Problem 2 *Evaluate the limit, or explain why it doesn't exist.*

$$\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2}$$

Problem 3 *Evaluate the limit, or explain why it doesn't exist.*

$$\lim_{(x,y)\to(1,0)} \frac{|x|}{x^2+y^2}$$

Problem 4 *Evaluate the limit, or explain why it doesn't exist.*

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy}{x^2+y^2+z^2}$$

Problem 5 *Evaluate the limit, or explain why it doesn't exist.*

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2}$$

Learning outcomes:
Author(s):

Problem 6 *Evaluate the limit, or explain why it doesn't exist.*

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$$

Problem 7 *Consider the function* $f(x,y) = \dfrac{xy^2}{(x+y^2)^2}$.

(a) *If you approach* $(0,0)$ *along any line* $y = mx$, *what does* $f(x,y)$ *approach?*

(b) *If you approach* $(0,0)$ *along the parabola* $x = y^2$, *what does* $f(x,y)$ *approach?*

(c) *Does* $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ *exist?*

Problem 8 *Show that the function* $f(x,y) = \cos^2(x+y) + e^{xy}$ *is continuous on its domain.*

Graded Problems

Problem 9 *Consider the function* $f(x,y) = \dfrac{\sin(x+y)}{x+y}$.

(a) *What change of coordinates,* $u(x,y)$ *and* $v(x,y)$, *can you use to evaluate this limit?*

(b) *Evaluate* $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

(c) *Consider the path* $\vec{x}(t) = (t, -t)$. *Explain why* $\displaystyle\lim_{t\to 0} f(\vec{x}(t))$ *does not exist. Explain why this does not contradict your answer to (b).*

Hint: from single variable calculus, what is $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$?

Problem 10 *Evaluate the limit, or explain why it doesn't exist.*

$$\lim_{(x,y)\to(0,0)} \frac{x^2+4xy+y^2}{x^2+y^2}$$

Professional Problem

Problem 11 *In this professional problem, you will prove the following statement.*

> *Let* $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$, *and let* $\varepsilon > 0$ *be given. If* $\delta = \varepsilon/3$, *then* $0 < |(x,y) - (0,0)| < \delta$ *implies that* $|f(x,y) - 0| < \varepsilon$, *and therefore* $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Consider the function

$$f(x,y) = \frac{3x^2y}{x^2+y^2}.$$

By switching to cylindrical coordinates, you could show the limit of f *as* $(x,y) \to (0,0)$ *is 0. Instead, you will show find limit using the* $\varepsilon - \delta$ *definition of a limit. Your goal is this: given any* $\varepsilon > 0$, *you must show there is a* $\delta > 0$ *so that*

$$0 < |(x,y) - (0,0)| < \delta \text{ implies that } |f(x,y) - 0| < \varepsilon.$$

For the given point and function, the choice $\delta = \varepsilon/3$ *will accomplish this.*

Hints: *Think back to the limit proofs you did last year. They were separated into* Think *and* Proof *sections. The* Think *portion has already been done for you, inasmuch as the choice of* δ *has been made for you. However, you still need to understand why this* δ *works in order to provide the proof.*

Below is a list of important facts which are necessary to prove this statement. First, convince yourself that they are true on scratch paper. Then, incorporate these facts, with their justifications, into a complete proof.

- $0 < |(x,y) - (0,0)| < \delta$ *if and only if* $0 < \sqrt{x^2+y^2} < \delta$.
- $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$ *if and only if* $\dfrac{3x^2|y|}{x^2+y^2} < \varepsilon$.
- $\dfrac{3x^2|y|}{x^2+y^2} \leq \dfrac{3(x^2+y^2)|y|}{x^2+y^2}$, *and therefore* $\dfrac{3x^2|y|}{x^2+y^2} \leq 3\sqrt{x^2+y^2}$.

Focuses: *This week, you should focus on these items when writing your solution:*

- ***Organization and Structure:*** *Your proof should be completely self-contained. The facts above are merely suggestions: Their complete and precise statements and justifications should be incorporated into your proof. Write, revise, and rewrite until your entire proof flows together in a logical order. Be concise.*
- ***Explanation:*** *Justify all necessary steps, including those which are not covered by the hints.*

- ***Attention to Mathematical Details:*** *Be very careful with absolute values and strict inequalities. Make sure that your implications follow from each other in the correct order.*
- ***Notation:*** *Use mathematical notation appropriately. Do not use notation in place of english words, or vice versa.* Everything, *including equations, should be part of a complete (but possibly brief) sentence.*