The Gradient and Level Sets

We've defined the directional derivatives of a function, which allow us to determine how a function is changing in various directions.

Definition 1. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, a point $\vec{a} \in \mathbb{R}^n$, and a direction given by a unit vector $\vec{v} \in \mathbb{R}^n$. Then we define the* directional derivative *of f at $\vec{a}$ in the direction of $\vec{v}$ to be*

$$D_{\vec{v}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h},$$

provided this limit exists.

However, we would like an easier way to evaluate directional derivatives, that doesn't require the limit definition.

Such a method will require use of the gradient of the function. Recall our definition of the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 2. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. The* gradient *of f is the function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$, defined by*

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right).$$

We'll see that this vector turns out to be closely related to directional derivatives.

The gradient and directional derivatives

Let's suppose we have a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, and consider our definition of the directional derivative a function f at $\vec{a}$ in the direction of a unit vector $\vec{v}$, which was

$$D_{\vec{v}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}.$$

We'll rewrite this definition by considering another function, $F(h) = f(\vec{a} + h\vec{v})$. Notice that F is a single variable function, and when we rewrite the directional

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derivative, we have

$$\begin{aligned} D_{\vec{v}} f(\vec{a}) &= \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h} \\ &= \lim_{h \to 0} \frac{F(h) - F(0)}{h - 0} \\ &= F'(0). \end{aligned}$$

So, the directional derivative is just the derivative of this single variable function $F(h)$ evaluated at 0. Revisiting our definition of $F(h)$, we can use the chain rule to find the derivative of F.

$$\begin{aligned} \frac{d}{dh} F(h) &= \nabla f(\vec{a} + h\vec{v}) \cdot \frac{d}{dh} (\vec{a} + h\vec{v}) \\ &= \nabla f(\vec{a} + h\vec{v}) \cdot \vec{v} \end{aligned}$$

Evaluating at $h = 0$, we have

$$\begin{aligned} D_{\vec{v}} f(\vec{a}) &= F'(0) \\ &= \nabla f(\vec{a}) \cdot \vec{v}. \end{aligned}$$

Thus, we have arrived at the following result.

Theorem 1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a} \in \mathbb{R}^n$. Then $D_{\vec{v}} f(\vec{a})$ exists for all unit vectors $\vec{v} \in \mathbb{R}^n$, and*

$$D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}.$$

Let's use this result to compute some directional derivatives.

Example 1. *We'll compute the directional derivative of $f(x, y) = x^2 y + y^2$ at $\vec{a} = (2, 0)$, in the direction of $(3, 4)$. (We previously computed this directional derivative using the limit definition.)*

Since $(3, 4)$ isn't a unit vector, we need to normalize it. Since $\|(3, 4)\| = \sqrt{3^2 + 4^2} = 5$, we'll use the vector $\vec{v} = \left(\frac{3}{5}, \frac{4}{5} \right)$ to compute our desired directional derivative.

Next, we'll need the gradient of f.

$$\nabla f(x, y) = (2xy, x^2 + 2y)$$

Since the partial derivatives of f are polynomials, they are continuous, so f is differentiable. Thus, we can use the above theorem to compute the directional derivative.

Then, we can compute the directional derivative as

$$\begin{aligned} D_{\vec{v}} f(\vec{a}) &= \nabla f(\vec{a}) \cdot \vec{v} \\ &= \nabla f(2,0) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) \\ &= (0,4) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) \\ &= \frac{16}{5}. \end{aligned}$$

This matches what we had previously computed using the definition of directional derivatives.

Problem 1 *Compute the directional derivative of* $f(x,y,z) = 3xy + xz^2$ *at* $\vec{a} = (2,0,1)$, *in the direction of* $(2,2,1)$.

$$D_{\vec{v}} f(\vec{a}) = \boxed{6}$$

Compute the directional derivative of $f(x,y,z) = 3xy + xz^2$ *at* $\vec{a} = (2,0,1)$, *in the direction of* $(-2,1,-1)$.

$$D_{\vec{v}} f(\vec{a}) = \boxed{0}$$

The gradient and level sets

We've shown that for a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we can compute directional derivatives as

$$D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}.$$

What does this mean for the possible values for a directional derivative? Recall that the dot product can be computed as

$$\nabla f(\vec{a}) \cdot \vec{v} = \|\nabla f(\vec{a})\| \, \|\vec{v}\| \cos\theta,$$

Where θ is the angle between the two vectors. Since $\vec{v}$ is a unit vector, we have

$$\|\nabla f(\vec{a})\| \, \|\vec{v}\| \cos\theta = \|\nabla f(\vec{a})\| \cos\theta.$$

Since $-1 \leq \cos\theta \leq 1$, we have that

$$-\|\nabla f(\vec{a})\| \leq D_{\vec{v}} f(\vec{a}) \leq \|\nabla f(\vec{a})\|.$$

In particular, the largest that $D_{\vec{v}}f(\vec{a})$ can be is $\|\nabla f(\vec{a})\|$, and this occurs when $\vec{v}$ points in the same direction as $\nabla f(\vec{a})$ (so $\theta = 0$). Thus, the gradient points in the direction of greatest increase.

VISUAL

On the other hand, the minimum value that $D_{\vec{v}}f(\vec{a})$ can have is $-\|\nabla f(\vec{a})\|$, and this occurs when $\vec{v}$ points in the opposite direction from $\nabla f(\vec{a})$, in the direction of $-\nabla f(\vec{a})$. Thus, $-\nabla f(\vec{a})$ points in the direction of greatest decrease.

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Additionally, from $D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$, we can see that $\vec{v}$ is perpendicular to $\nabla f(\vec{a})$ if and only if $D_{\vec{v}}f(\vec{a}) = 0$. But what does it mean for $D_{\vec{v}}f(\vec{a})$? This means that there is no instantaneous change in f in the direction of $\vec{v}$, which means that $\vec{v}$ will be a tangent vector to a level curve.

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We'll state this observation more formally, and prove that the gradient is perpendicular to the level curves.

Theorem 2. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, and suppose f is of class $\mathcal{C}^1$. For some constant c, consider the level set*

$$S = \{\vec{x} \in \mathbb{R}^n \ : \ f(\vec{x}) = c\}.$$

Then, for any point $\vec{x}_0$ in S, the gradient $\nabla f(\vec{x}_0)$ is perpendicular to S.

Proof Let $\vec{x}(t)$ be a parametrization of S, and let $\vec{x}_0 = \vec{x}(t_0)$. In order to show that $\nabla f(\vec{x}_0)$ is perpendicular to S, we will show that the gradient $\nabla f(\vec{x}_0)$ is perpendicular to the velocity vector $\vec{x}'(t_0)$. By the definition of S,

$$f(\vec{x}(t)) = c,$$

for all t. Differentiating both sides of this identity, and using the chain rule on the left side, we obtain

$$\nabla f(\vec{x}(t)) \cdot \vec{x}'(t) = 0.$$

Plugging in $t = t_0$, this gives us

$$\nabla f(\vec{x}(t_0)) \cdot \vec{x}'(t_0) = 0,$$

which we can rewrite as

$$\nabla f(\vec{x}_0) \cdot \vec{x}'(t_0) = 0.$$

Thus, we have shown that $\nabla f(\vec{x}_0)$ is perpendicular to the level set S.

THIS PROOF ONLY WORKS WHEN IT'S A LEVEL CURVE, NOT A LEVEL SURFACE ETC. ∎