

# Parametric Curves

We've dealt with several ways to describe curves in  $\mathbb{R}^2$ :

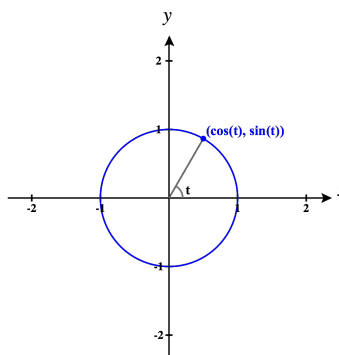
- As the graph of a function. For example,  $f(x) = x^2$ .
- As the set of points satisfying an equation. For example, the points  $(x, y)$  such that  $x^2 + y^2 = 1$ .
- As the set of points satisfying an equation in another coordinate system. For example,  $r = \sin(\theta)$  in polar coordinates.

Another way that we can describe a curve is using *parametric equations*. When describing a curve using parametric equations, we define  $x$  and  $y$  in terms of a third variable, often  $t$ , called the *parameter*. We often think about  $t$  as representing time, and imagine the curve being drawn out as  $t$  increases. This gives us another way to describe curves in  $\mathbb{R}^2$ , and potentially describe some new and strange curves.

We can describe the unit circle in  $\mathbb{R}^2$  with the parametric equations

$$\begin{aligned}x &= \cos(t), \\y &= \sin(t),\end{aligned}$$

for  $0 \leq t \leq 2\pi$ .




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Learning outcomes:

Author(s):

We can think of  $t$  as giving the angle that a point makes with the positive axis. It can also be helpful to imagine  $t$  as representing time, and the parametric equations tracing out the circle as time passes.

YouTube link: <https://www.youtube.com/watch?v=1J6bT8RE3so>

Consider the parametric equations for the unit circle in  $\mathbb{R}^2$ :

$$\begin{aligned}x &= \cos(t), \\y &= \sin(t),\end{aligned}$$

for  $0 \leq t \leq 2\pi$ .

We can combine these equations into a single vector,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

We can visualize the vectors  $\vec{x}(t)$  tracing out the unit circle as  $t$  goes from 0 to  $2\pi$ .

YouTube link: <https://www.youtube.com/watch?v=sYYekBSL4cg>

Notice that we are blurring the distinction between vectors and points. Although we can think of  $\vec{x}(t)$  as a position vector, we would more commonly think of  $\vec{x}(t)$  as a point on a curve. Although this might seem a bit sloppy, it will prove very useful throughout the course. Although intuitively we might prefer to use points, a lot of the computations tools that we'll require are more appropriately used with vectors.

We have defined a function  $\vec{x}$  from the interval  $[0, 2\pi] \subset \mathbb{R}$  to  $\mathbb{R}^2$ , and this idea provides the motivation behind our definition for paths.

## Parametric Curves in $\mathbb{R}^n$

**Definition 1.** A path in  $\mathbb{R}^n$  is a continuous function

$$\vec{x} : I \subset \mathbb{R} \rightarrow \mathbb{R}^n,$$

where  $I \subset \mathbb{R}$  is an interval.

*This is also called a parametrized curve or parametric curve. When we wish to emphasize that  $\vec{x}(t)$  is a vector, we'll call it the position vector.*

We'll focus on the cases  $n = 2$  and  $n = 3$  in this course.

We defined a path as a continuous function, however, we haven't said what it means for a multivariable function to be continuous. We'll come back to this later, and we'll give a rigorous definition for continuity. For now, this should fit with your intuition: you can draw the path without lifting your pencil from the paper.

Sometimes we care more about the image of a path than how the path is drawn out, and then we refer to a curve.

**Definition 2.** A curve  $C$  in  $\mathbb{R}^n$  is the image of some path  $\vec{x} : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$ .

In this case, we say that  $\vec{x}$  is a parametrization for the curve  $C$ .

The difference between a curve and a path is largely a matter of perspective: when working with a curve, we pay attention to *what* is drawn; when working with a path, we care about *how* it is drawn.

**Example 1.** There are many different parametrizations for a given curve.

Consider again the unit circle  $C$  in  $\mathbb{R}^2$ . Which of the following are parametrizations for  $C$ ?

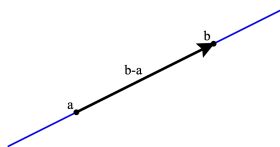
**Select All Correct Answers:**

- (a)  $\vec{x}(t) = (\cos(t), \sin(t))$  for  $0 \leq t \leq 2\pi$  ✓
- (b)  $\vec{x}(t) = (\sin(t), \cos(t))$  for  $0 \leq t \leq \pi$
- (c)  $\vec{x}(t) = (t, \pm\sqrt{1-t^2})$  for  $-1 \leq t \leq 1$
- (d)  $\vec{x}(t) = (\sin(2\pi t), \cos(2\pi t))$  for  $0 \leq t \leq 1$  ✓
- (e)  $\vec{x}(t) = (\cos(t), \sin(t))$  for  $-10 \leq t \leq 10$  ✓

**Example 2.** In this example, we review a parametrization for the line through points  $\vec{a}$  and  $\vec{b}$  in  $\mathbb{R}^n$ .

Given points  $\vec{a}$  and  $\vec{b}$  in  $\mathbb{R}^n$ , we obtain a vector starting at  $\vec{a}$  and ending at  $\vec{b}$  by taking  $\vec{b} - \vec{a}$ . This vector is parallel to the line through  $\vec{a}$  and  $\vec{b}$ . Then, taking scalar multiples  $t(\vec{b} - \vec{a})$  for  $t \in \mathbb{R}$ , we have a line parallel to the line through  $\vec{a}$  and  $\vec{b}$ . Finally, we add one of the points,  $\vec{a}$ , to ensure that our line passes through these two points. Thus, we arrive at our parametrization,

$$\vec{l}(t) = \vec{a} + t(\vec{b} - \vec{a}) \text{ for } t \in \mathbb{R}.$$

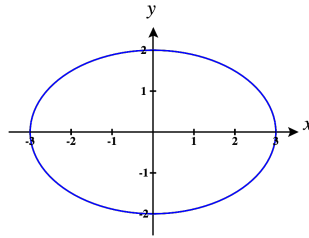


**Example 3.** In this example, we see how we can obtain new transformations from old ones, using linear algebra and simple transformations.

Recall the parametrization for the unit circle in  $\mathbb{R}^2$ ,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Now, consider the ellipse below.



We can think of this ellipse as the result of stretching the unit circle horizontally by a factor of 3 and vertically by a factor of 2. That is, we are applying the linear transformation

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

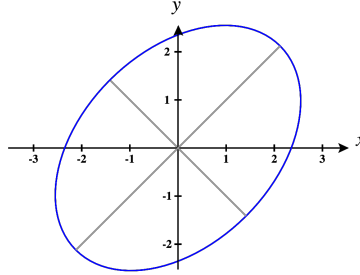
We can apply this to the parametrization for the unit circle, in order to parametrize the ellipse.

$$\begin{aligned} \vec{y}(t) &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \\ &= (3 \cos(t), 2 \sin(t)). \end{aligned}$$

Thus, we have a parametrization for the ellipse given by

$$\vec{y}(t) = (3 \cos(t), 2 \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Next, consider the following ellipse.



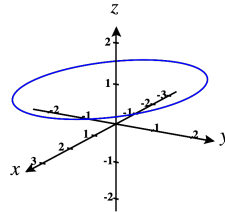
We can obtain this from our previous ellipse by counterclockwise rotation of  $\pi/4$ . The matrix for this linear transformation is

$$\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \boxed{1/\sqrt{2}} & \boxed{-1/\sqrt{2}} \\ \boxed{1/\sqrt{2}} & \boxed{1/\sqrt{2}} \end{pmatrix}.$$

Applying this rotation to our parametrization for the previous ellipse, we obtain a parametrization for our new ellipse.

$$\vec{z}(t) = \boxed{(3/\sqrt{2} \cos(t) - 2/\sqrt{2} \sin(t), 3/\sqrt{2} \cos(t) + 2/\sqrt{2} \sin(t))} \text{ for } 0 \leq t \leq 2\pi$$

Finally, we consider an ellipse in  $\mathbb{R}^3$ , shown below.



This ellipse is parallel to the  $xy$ -plane, and will have constant  $z$ -coordinate of 1. Note the similarity to the first ellipse we considered. A parametrization for

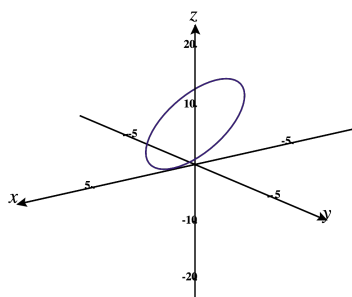
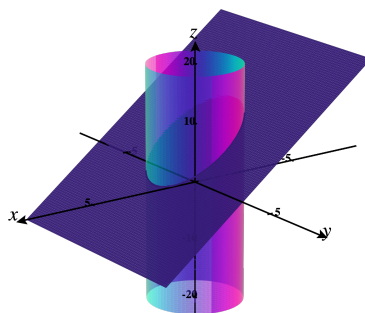
this ellipse can be obtained by taking the parametrization  $\vec{y}$  for our first ellipse in  $\mathbb{R}^2$ , and appending the constant  $z$ -coordinate.

$$\vec{a}(t) = \boxed{(3 \cos(t), 2 \sin(t), 1)} \text{ for } 0 \leq t \leq 2\pi$$

## Examples in $\mathbb{R}^3$

In this section, we give examples of parametrizations of a couple of more complicated curves in  $\mathbb{R}^3$ , taking advantage of our previous experience with cylindrical coordinates.

**Example 4.** We'll parametrize the intersection of the cylinder  $x^2 + y^2 = 4$  and the plane  $z = 7 - 3x$  in  $\mathbb{R}^3$ , pictured below.



Our  $x$  and  $y$  coordinates must satisfy  $x^2 + y^2 = 4$ , which would define a circle, if we were in  $\mathbb{R}^2$ . Recalling our parametrizations for circles, these coordinates

can be written as

$$\begin{aligned}x(t) &= 2 \cos(t) \\ y(t) &= 2 \sin(t)\end{aligned}$$

for  $0 \leq t \leq 2\pi$ .

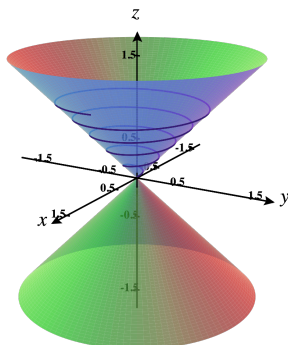
It remains to write the  $z$ -coordinate in terms of the parameter  $t$ . Turning our attention to the equation for the plane,  $z = 7 - 3x$ , we have  $z$  expressed in terms of  $x$ . Since we have expressed  $x$  in terms of  $t$ , we can make this substitution to describe  $z$  in terms of  $t$ ,

$$z(t) = 7 - 6 \cos(t).$$

Putting all of this together, we have a parametrization for this intersection given by

$$\vec{x}(t) = (2 \cos(t), 2 \sin(t), 7 - 6 \cos(t)) \text{ for } 0 \leq t \leq 2\pi.$$

**Example 5.** Consider the curve below, which lies on the cone  $z^2 = x^2 + y^2$ , and makes five rotations around the  $z$ -axis as the height ranges from 0 to 1. We'll refer to this curve as a "tornado."



We'll parametrize this curve by thinking about it in cylindrical coordinates, using the height as the parameter.

First, let's consider what's happening with the  $z$ -coordinate. Since the height of the tornado ranges from 0 to 1, so will  $z$ . We'll set  $z = t$ , with  $0 \leq t \leq 1$ , and express  $x$  and  $y$  in terms of  $t$  as well.

Now, we turn our attention to the angle  $\theta$ . As the height ranges from 0 to 1, the tornado makes five revolutions, so  $\theta$  should range from 0 to  $10\pi$ . Thus, expressing  $\theta$  in terms of  $t$ , we let  $\theta = 10\pi t$ .

Next, we consider the radius  $r$ . Since we are on the cone  $z^2 = x^2 + y^2$ , we have  $z^2 = r^2$ . Since  $z \geq 0$ , we have  $z = r$ . Thus, we can write  $r$  in terms of  $t$  as  $r = t$ .

Finally, putting all of this together with  $x = r \cos \theta$  and  $y = r \sin \theta$ , we have a parametrization for the tornado given by

$$\vec{x}(t) = \boxed{(t \cos(10\pi t), t \sin(10\pi t), t)} \text{ for } t \in [0, 1].$$