

# The Dot Product

In this section we review the dot product on vectors. This also includes the angle between vectors and the projection of one vector onto another.

## The Dot Product

We begin with the definition of the dot product.

**Definition 1.** The dot product of two vectors  $\vec{v} = (v_1, v_2, \dots, v_n)$  and  $\vec{w} = (w_1, w_2, \dots, w_n)$  in  $\mathbb{R}^n$  is

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n.$$

Notice that the dot product takes two vectors and outputs a scalar.

**Example 1.**  $(1, 6) \cdot (-3, -6) = -3 - 36 = -39$

$$(1, 2, 3) \cdot (7, -2, 4) = 7 - 4 + 12 = 15$$

$$(1, 7, -3) \cdot (3, 0, 1) = 3 + 0 - 3 = 0$$

We can also compute the dot product using the magnitude (or length) of the vectors and the angle in between them.

**Proposition 1.** If  $\vec{v}$  and  $\vec{w}$  are vectors in  $\mathbb{R}^n$ , then

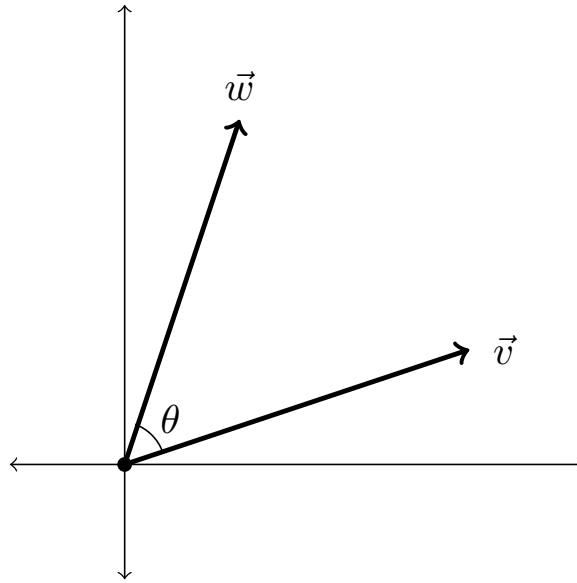
$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta,$$

where  $\|\vec{v}\|$  and  $\|\vec{w}\|$  are the lengths of the vectors  $\vec{v}$  and  $\vec{w}$ , respectively, and  $\theta$  is the angle between  $\vec{v}$  and  $\vec{w}$ .

This is illustrated in the picture below.

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Learning outcomes:  
Author(s):



This provides us with a geometric interpretation of the dot product: it gives us a measure of “how much” in the same direction two vectors are (taking their lengths into account). This also gives us a useful way to compute the angle between two vectors.

**Example 2.** Consider the vectors  $(1, 4)$  and  $(-2, 2)$ . We have

$$(1, 4) \cdot (-2, 2) = -2 + 8 = 6,$$

$$\|(1, 4)\| = \sqrt{1^2 + 4^2} = \sqrt{17},$$

$$\|(-2, 2)\| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}.$$

From  $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$ , we then have

$$6 = \sqrt{17}\sqrt{8} \cos \theta.$$

Solving for  $\theta$ , we obtain the angle between the vectors as

$$\theta = \arccos\left(\frac{6}{\sqrt{17}\sqrt{8}}\right) \approx 59.04^\circ$$

Furthermore, note that for nonzero vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$ , their dot product is 0 if and only if  $\cos(\theta) = 0$ . This means that  $\theta$  would have to be  $90^\circ$  or  $270^\circ$ , meaning that  $\vec{v}$  and  $\vec{w}$  are perpendicular.

**Proposition 2.** Two nonzero vectors  $\vec{v}$  and  $\vec{w}$  in  $\mathbb{R}^n$  are perpendicular if and only if  $\vec{v} \cdot \vec{w} = 0$ .

This provides us with a very useful algebraic method for determining if two vectors are perpendicular.

**Example 3.** The vectors  $(1, 7, -3)$  and  $(3, 0, 1)$  in  $\mathbb{R}^3$  are perpendicular, since

$$(1, 7, -3) \cdot (3, 0, 1) = 3 + 0 - 3 = 0.$$

By taking the dot product of a vector with itself, we get an important relationship between the dot product and the length of a vector.

**Proposition 3.** Let  $\vec{v}$  be a vector in  $\mathbb{R}^n$ . Then

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2.$$

This can be shown directly, or using the fact that the angle between  $\vec{v}$  and itself is 0.

## Projection of one vector onto another

We can also use the dot product to define the projection of one vector onto another.

**Definition 2.** For vectors  $\vec{a}$  and  $\vec{b}$  in  $\mathbb{R}^n$ , we define the vector projection of  $\vec{a}$  onto  $\vec{b}$  as

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b}.$$

**Example 4.** We can use this to find the projection of  $(2, 4, 3)$  onto  $(1, -1, 1)$ .

$$\begin{aligned} \text{proj}_{(1, -1, 1)}(2, 4, 3) &= \frac{(2, 4, 3) \cdot (1, -1, 1)}{(1, -1, 1) \cdot (1, -1, 1)}(1, -1, 1) \\ &= \frac{2 - 4 + 3}{1 + 1 + 1}(1, -1, 1) \\ &= \frac{1}{3}(1, -1, 1) \\ &= \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) \end{aligned}$$

## Summary

In this section we reviewed the dot product on vectors, the angle between vectors, and the projection of one vector onto another.