Decomposition of Acceleration

Recall our definition of the moving frame:

Definition 1. *Given a path $\vec{x}(t)$, we define the* moving frame *of the path to be the triple $(\vec{T}, \vec{N}, \vec{B})$.*

$\vec{T}$ *is the* unit tangent vector*,*

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}.$$

$\vec{N}$ *is the* unit normal vector*,*

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}.$$

$\vec{B}$ *is the* unit binormal vector*,*

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

The moving frame is also called the TNB frame*.*

Throughout our study of paths, we've found a lot of ways that we can describe the behavior of the path, in addition to the moving frame:

- The velocity vector, $\vec{v}(t) = \vec{x}'(t)$.
- The speed, $s'(t) = \|\vec{x}'(t)\|$.
- The acceleration, $\vec{a}(t) = \vec{x}''(t)$.
- The arclength function, $s(t) = \displaystyle\int_a^t \|\vec{x}'(\tau)\| d\tau$.
- The parametrization with respect to arclength, $\vec{x}(s)$.
- The curvature, $\kappa(t) = \|\vec{T}'(s)\| = \dfrac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}$.
- The osculating circle and osculating plane (DID I EVER DEFINE THIS?).

We'll now explore the connections between these concepts. In particular, we'll derive a useful decomposition of the acceleration vector as a linear combination of the unit tangent and unit normal vectors.

Learning outcomes:
Author(s):

Curvature and torsion

We begin with the following result, which connects the curvature and unit normal vector with the derivative of the unit tangent vector with respect to arclength.

Proposition 1.
$$\frac{d\vec{T}}{ds} = \kappa\vec{N}$$

Proof First, thinking of arclength s as a function of t and using the chain rule, we have
$$\frac{d}{dt}\vec{T}(s(t)) = \vec{T}'(s(t))s'(t).$$
Recognizing $s'(t)$ as the speed, we can rewrite this as
$$\frac{d}{dt}\vec{T}(t) = \vec{T}'(s)\|\vec{x}'(t)\|.$$
Solving for $\vec{T}'(s)$, we have
$$\frac{d\vec{T}}{ds} = \frac{\vec{T}'(t)}{\|\vec{x}'(t)\|}.$$
Turning to the other side of the equality, recall that we defined the curvature to be $\kappa(t) = \|\vec{T}'(s)\|$, and we found that we could also compute this as $\dfrac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}$.

We defined the unit normal vector to be $\dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$.

Putting all of this together, we have
$$\begin{aligned}\kappa(t)\vec{N}(t) &= \frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}\frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \\ &= \frac{\vec{T}'(t)}{\|\vec{x}'(t)\|} \\ &= \frac{d\vec{T}}{ds},\end{aligned}$$
proving our result. ■

It turns out that there is a similar result relating the normal vector with the derivative of the unit binormal vector with respect to arclength:
$$\frac{d\vec{B}}{ds} = \tau\vec{N}.$$

We haven't talked about the coefficient τ yet, but this is another important property of curves, called *torsion*.

Together, the curvature and torsion carry a lot of important information about the curve. In fact, the curvature and torsion completely determine the curve!

Decomposition of acceleration

Recall that the unit tangent vector $\vec{T}$ points in the direction of instantaneous motion, and the unit normal vector $\vec{N}$ points in the direction that a path is turning. So, it shouldn't be too surprising that the acceleration vector is always a linear combination of $\vec{T}$ and $\vec{N}$. However, it's very surprising that we can recognize the coefficients in terms of things we've seen before!

Proposition 2. *Let $\vec{x}(t)$ be a C^2 path in $\mathbb{R}^3$. Then the acceleration vector can be written as*

$$\vec{a} = s''\vec{T} + (s')^2\kappa\vec{N},$$

where s is the arclength function (so s' is the speed), $\vec{T}$ is the unit tangent vector, κ is the curvature, and $\vec{N}$ is the unit normal vector.

Proof We begin with some observations.

Recall that we defined the unit tangent vector as $\vec{T}$ as $\vec{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$, where $\vec{v}(t) = \vec{x}'(t)$ is the velocity vector. Replacing the speed $\|\vec{v}\|$ with s', this gives us

$$\vec{T} = s'\vec{v}.$$

Similarly, we defined the unit normal vector as $\vec{N} = \dfrac{\vec{T}'}{\|\vec{T}'\|}$, so we can write

$$\vec{T}' = \|\vec{T}'\|\vec{N}.$$

We have that the curvature is $\kappa = \dfrac{\|\vec{T}'\|}{s'}$, so we can rewrite this as

$$\begin{aligned}\vec{T}' &= \|\vec{T}'\|\vec{N} \\ &= (s'\kappa)\vec{N}.\end{aligned}$$

Finally, we turn to our acceleration vector. We recall that acceleration is the derivative of velocity, and use the product rule with the above observations to obtain

$$\begin{aligned}
\vec{a} &= \vec{v}' \\
&= \frac{d}{dt}(s'\vec{T}) \\
&= s''\vec{T} + s'\vec{T}' \\
&= s''\vec{T} + s'(s'\kappa)\vec{N} \\
&= s''\vec{T} + (s')^2\kappa\vec{N}.
\end{aligned}$$

■

Immediately from this result, we can make a lot of important observations.

Proposition 3. *Let $\vec{x}(t)$ be a C^2 path in $\mathbb{R}^3$. Then:*

- *$\vec{a}$ is a linear combination of $\vec{T}$ and $\vec{N}$.*
- *$\vec{a}$ is always in the osculating plane.*
- *Since $(s')^2 \geq 0$ and $\kappa \geq 0$, the acceleration $\vec{a}$ points in the direction that we're turning.*
- *If $\kappa = 0$, then $\vec{a}$ is parallel to $\vec{T}$.*
- *If speed is constant, then $s'' = 0$, so $\vec{a}$ is parallel to $\vec{N}$.*

We leave the proofs of these facts as an exercise.

Summary of notation for parametric curves

There are a lot of symbols to keep track of when studying the geometry of parametric curves. To make matters worse, most of them have multiple names. For example, the derivative of $\vec{x}(t)$ can be denoted by either $\vec{x}'(t)$ or $\dot{\mathbf{x}}(x)$, but we often call it $\vec{v}(t)$ because it represents velocity. Given a parametrization $\vec{x}(t)$, $t \in [a, b]$, which represents motion of a particle along a curve C, we list most of the related functions and their interpretations.

Position vector

- Notation: $\vec{x}(t)$, $\mathbf{x}(t)$
- Represents the position of a particle at time t

Tangent vector

- Notation: $\vec{x}'(t)$, $\dot{\mathbf{x}}(t)$, $\vec{v}(t)$, $\mathbf{x}'(t)$, $\mathbf{v}(t)$
- Derivative of $\vec{x}(t)$; also called the velocity vector.
- Its direction shows the direction of instantaneous motion, and its length ($||\vec{v}(t)|| = ||\vec{x}'(t)||$ etc.) is the instantaneous speed.

Unit tangent vector

- Notation: $\vec{T}(t)$
- Computed as $\vec{x}'(t)/||\vec{x}'(t)||$, $\vec{v}(t)/||\vec{v}(t)||$, etc.

Derivative of the unit tangent vector

- Notation: $\vec{T}'(t)$, $d\vec{T}/dt$
- Derivative of unit tangent vector with respect to time.
- Not necessarily a unit vector.
- Must be perpendicular to $\vec{T}(t)$: because $\vec{T}$ is a unit vector, $\vec{T} \cdot \vec{T} = 1$; differentiating both sides with respect to t gives $2\vec{T} \cdot \vec{T}' = 0$.

Unit normal vector

- Notation: $\vec{N}(t)$
- Computed as $\vec{T}'(t)/||\vec{T}'(t)||$.
- Perpendicular to $\vec{T}$, since it's a scaled version of $\vec{T}'$.

Binormal vector

- Notation: $\vec{B}(t)$
- Computed as $\vec{B} = \vec{T} \times \vec{N}$.
- Is a unit vector, since the angle between $\vec{T}$ and $\vec{N}$ is $\theta = \pi/2$ and therefore $||\vec{T} \times \vec{N}|| = ||\vec{T}||\,||\vec{N}||\sin\theta = 1$.

Distance Traveled

- Notation: s, $s(t)$
- Written as s when it's treated as a variable.
- We often view s as a function of time, and compute the arclength function $s(t) = \displaystyle\int_a^t ||\vec{x}'(u)||\,du$.

Speed

- Notation: ds/dt, $s'(t)$
- Computed as $ds/dt = s'(t) = ||\vec{x}'(t)|| = ||\vec{v}(t)||$ as proven by applying the Fundamental Theorem of Calculus to the definition of $s(t)$.

Curvature

- Notation: κ
- Measures how quickly a curve "turns" at a given point: $\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right|$.
- Using the chain rule we can write κ as a function of t:

$$\kappa(t) = \left|\left|\frac{d\vec{T}}{ds}\right|\right| = \left|\left|\frac{d\vec{T}}{dt}\frac{dt}{ds}\right|\right| = \left|\left|\frac{d\vec{T}/dt}{ds/dt}\right|\right| = \frac{||\vec{T}'(t)||}{s'(t)} = \frac{||\vec{T}'(t)||}{||\vec{x}'(t)||}$$

- Many other formulas exist, e.g. for curves which are the graphs of functions $y = f(x)$.

Acceleration vector

- Notation: $\vec{x}''(t)$, $\ddot{\mathbf{x}}(t)$, $\vec{x}''(t)$, $\vec{a}(t)$, $\vec{a}(t)$
- Second derivative of $\vec{x}(t)$ and first derivative of velocity.
- If we rewrite $\vec{T}(t) = \vec{v}(t)/||\vec{v}(t)|| = \vec{v}(t)/s'(t)$ as $\vec{v} = s'\vec{T}$, we can apply the product rule to calculate:

$$\vec{a}(t) = s''(t)\vec{T}(t) + s'(t)\vec{T}'(t) = s''\vec{T} + s'||\vec{T}'||\vec{N} = s''\vec{T} + (s')^2\kappa\vec{N}$$

  The second equation comes from rewriting $\vec{N}(t) = \vec{T}'(t)/||\vec{T}'(t)||$ as $\vec{T}' = ||\vec{T}'||\vec{N}$ and substituting. The third equation used the fact that $\kappa = ||\vec{T}'||/s'$, so that $||\vec{T}'|| = \kappa s'$. We've made the scalar functions red and the vectors black to emphasize that acceleration is a linear combination of $\vec{T}$ and $\vec{N}$ (or $\vec{T}'$).

I COPIED THIS FROM THE HANDOUT (except for formatting, some notation, and a little rewriting)... HOPEFULLY THAT'S FINE?

SOME SORT OF RANDOMIZED NOTATION QUIZ?