

# Decomposition of Acceleration

Recall our definition of the moving frame:

**Definition 1.** Given a path  $\vec{x}(t)$ , we define the moving frame of the path to be the triple  $(\vec{T}, \vec{N}, \vec{B})$ .

$\vec{T}$  is the unit tangent vector,

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}.$$

$\vec{N}$  is the unit normal vector,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}.$$

$\vec{B}$  is the unit binormal vector,

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

The moving frame is also called the TNB frame.

Throughout our study of paths, we've found a lot of ways that we can describe the behavior of the path, in addition to the moving frame.

- The velocity vector,  $\vec{v}(t) = \vec{x}'(t)$ .
- The speed,  $s'(t) = \|\vec{x}'(t)\|$ .
- The acceleration,  $\vec{a}(t) = \vec{x}''(t)$ .
- The arclength function,  $s(t) = \int_a^t \|\vec{x}'(\tau)\| d\tau$ .
- The parametrization with respect to arclength,  $\vec{x}(s)$ .
- The curvature,  $\kappa(t) = \|\vec{T}'(s)\| = \frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}$ .
- The osculating circle and osculating plane.

We'll now explore the connections between these concepts. In particular, we'll derive a useful decomposition of the acceleration vector as a linear combination of the unit tangent and unit normal vectors.

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Learning outcomes: Understand the decomposition of the acceleration vector into tangential and normal components. Be able to relate and manipulate the various properties of parametric curves.

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## Curvature and torsion

We begin with the following result, which connects the curvature and unit normal vector with the derivative of the unit tangent vector with respect to arclength.

**Proposition 1.**

$$\frac{d\vec{T}}{ds} = \kappa \vec{N}$$

**Proof** First, thinking of arclength  $s$  as a function of  $t$  and using the chain rule, we have

$$\frac{d}{dt}\vec{T}(s(t)) = \vec{T}'(s(t))s'(t).$$

Recognizing  $s'(t)$  as the speed, we can rewrite this as

$$\frac{d}{dt}\vec{T}(t) = \vec{T}'(s)\|\vec{x}'(t)\|.$$

Solving for  $\vec{T}'(s)$ , we have

$$\frac{d\vec{T}}{ds} = \frac{\vec{T}'(t)}{\|\vec{x}'(t)\|}.$$

Turning to the other side of the equality, recall that we defined the curvature to be  $\kappa(t) = \|\vec{T}'(s)\|$ , and we found that we could also compute this as  $\frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}$ .

We defined the unit normal vector to be  $\frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$ .

Putting all of this together, we have

$$\begin{aligned}\kappa(t)\vec{N}(t) &= \frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|} \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \\ &= \frac{\vec{T}'(t)}{\|\vec{x}'(t)\|} \\ &= \frac{d\vec{T}}{ds},\end{aligned}$$

proving our result. ■

It turns out that there is a similar result relating the normal vector with the derivative of the unit binormal vector with respect to arclength:

$$\frac{d\vec{B}}{ds} = \tau \vec{N}.$$

We haven't talked about the coefficient  $\tau$  yet, but this is another important property of curves, called *torsion*.

Together, the curvature and torsion carry a lot of important information about the curve. In fact, the curvature and torsion completely determine the curve!

## Decomposition of acceleration

Recall that the unit tangent vector  $\vec{T}$  points in the direction of instantaneous motion, and the unit normal vector  $\vec{N}$  points in the direction that a path is turning. So, it shouldn't be too surprising that the acceleration vector is always a linear combination of  $\vec{T}$  and  $\vec{N}$ . However, it's very surprising that we can recognize the coefficients in terms of things we've seen before!

**Proposition 2.** *Let  $\vec{x}(t)$  be a  $C^2$  path in  $\mathbb{R}^3$ . Then the acceleration vector can be written as*

$$\vec{a} = s''\vec{T} + (s')^2\kappa\vec{N},$$

where  $s$  is the arclength function (so  $s'$  is the speed),  $\vec{T}$  is the unit tangent vector,  $\kappa$  is the curvature, and  $\vec{N}$  is the unit normal vector.

**Proof** We begin with some observations.

Recall that we defined the unit tangent vector as  $\vec{T}$  as  $\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$ , where  $\vec{v}(t) = \vec{x}'(t)$  is the velocity vector. Replacing the speed  $\|\vec{v}\|$  with  $s'$ , this gives us

$$\vec{T} = s'\vec{v}.$$

Similarly, we defined the unit normal vector as  $\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|}$ , so we can write

$$\vec{T}' = \|\vec{T}'\|\vec{N}.$$

We have that the curvature is  $\kappa = \frac{\|\vec{T}'\|}{s'}$ , so we can rewrite this as

$$\begin{aligned}\vec{T}' &= \|\vec{T}'\|\vec{N} \\ &= (s'\kappa)\vec{N}.\end{aligned}$$

Finally, we turn to our acceleration vector. We recall that acceleration is the derivative of velocity, and use the product rule with the above observations to

obtain

$$\begin{aligned}
 \vec{a} &= \vec{v}' \\
 &= \frac{d}{dt}(s'\vec{T}) \\
 &= s''\vec{T} + s'\vec{T}' \\
 &= s''\vec{T} + s'(s'\kappa)\vec{N} \\
 &= s''\vec{T} + (s')^2\kappa\vec{N}.
 \end{aligned}$$

■

Immediately from this result, we can make a lot of important observations.

**Proposition 3.** *Let  $\vec{x}(t)$  be a  $C^2$  path in  $\mathbb{R}^3$ . Then:*

- $\vec{a}$  is a linear combination of  $\vec{T}$  and  $\vec{N}$ .
- $\vec{a}$  is always in the osculating plane.
- Since  $(s')^2 \geq 0$  and  $\kappa \geq 0$ , the acceleration  $\vec{a}$  points in the direction that we're turning.
- If  $\kappa = 0$ , then  $\vec{a}$  is parallel to  $\vec{T}$ .
- If speed is constant, then  $s'' = 0$ , so  $\vec{a}$  is parallel to  $\vec{N}$ .

We leave the proofs of these facts as an exercise.

## Summary of notation for parametric curves

There are a lot of symbols to keep track of when studying the geometry of parametric curves. To make matters worse, most of them have multiple names. For example, the derivative of  $\vec{x}(t)$  can be denoted by either  $\vec{x}'(t)$  or  $\dot{\mathbf{x}}(x)$ , but we often call it  $\vec{v}(t)$  because it represents velocity. Given a parametrization  $\vec{x}(t)$ ,  $t \in [a, b]$ , which represents motion of a particle along a curve  $C$ , we list most of the related functions and their interpretations.

### Position vector

- Notation:  $\vec{x}(t)$ ,  $\mathbf{x}(t)$
- Represents the position of a particle at time  $t$

### Tangent vector

- Notation:  $\vec{x}'(t)$ ,  $\dot{\mathbf{x}}(t)$ ,  $\vec{v}(t)$ ,  $\mathbf{x}'(t)$ ,  $\mathbf{v}(t)$
- Derivative of  $\vec{x}(t)$ ; also called the velocity vector.
- Its direction shows the direction of instantaneous motion, and its length ( $\|\vec{v}(t)\| = \|\vec{x}'(t)\|$  etc.) is the instantaneous speed.

### Unit tangent vector

- Notation:  $\vec{T}(t)$
- Computed as  $\vec{x}'(t)/\|\vec{x}'(t)\|$ ,  $\vec{v}(t)/\|\vec{v}(t)\|$ , etc.

### Derivative of the unit tangent vector

- Notation:  $\vec{T}'(t)$ ,  $d\vec{T}/dt$
- Derivative of unit tangent vector with respect to time.
- Not necessarily a unit vector.
- Must be perpendicular to  $\vec{T}(t)$ : because  $\vec{T}$  is a unit vector,  $\vec{T} \cdot \vec{T} = 1$ ; differentiating both sides with respect to  $t$  gives  $2\vec{T} \cdot \vec{T}' = 0$ .

### Unit normal vector

- Notation:  $\vec{N}(t)$
- Computed as  $\vec{T}'(t)/\|\vec{T}'(t)\|$ .
- Perpendicular to  $\vec{T}$ , since it's a scaled version of  $\vec{T}'$ .

### Binormal vector

- Notation:  $\vec{B}(t)$
- Computed as  $\vec{B} = \vec{T} \times \vec{N}$ .
- Is a unit vector, since the angle between  $\vec{T}$  and  $\vec{N}$  is  $\theta = \pi/2$  and therefore  $\|\vec{T} \times \vec{N}\| = \|\vec{T}\| \|\vec{N}\| \sin \theta = 1$ .

### Distance Traveled

- Notation:  $s$ ,  $s(t)$
- Written as  $s$  when it's treated as a variable.
- We often view  $s$  as a function of time, and compute the arclength function  $s(t) = \int_a^t \|\vec{x}'(u)\| du$ .

**Speed**

- Notation:  $ds/dt$ ,  $s'(t)$
- Computed as  $ds/dt = s'(t) = \|\vec{x}'(t)\| = \|\vec{v}(t)\|$  as proven by applying the Fundamental Theorem of Calculus to the definition of  $s(t)$ .

**Curvature**

- Notation:  $\kappa$
- Measures how quickly a curve “turns” at a given point:  $\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$ .
- Using the chain rule we can write  $\kappa$  as a function of  $t$ :

$$\kappa(t) = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right\| = \left\| \frac{d\vec{T}/dt}{ds/dt} \right\| = \frac{\|\vec{T}'(t)\|}{s'(t)} = \frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}$$

- Many other formulas exist, e.g. for curves which are the graphs of functions  $y = f(x)$ .

**Acceleration vector**

- Notation:  $\vec{x}''(t)$ ,  $\ddot{\mathbf{x}}(t)$ ,  $\vec{x}''(t)$ ,  $\vec{a}(t)$ ,  $\vec{a}(t)$
- Second derivative of  $\vec{x}(t)$  and first derivative of velocity.
- If we rewrite  $\vec{T}(t) = \vec{v}(t)/\|\vec{v}(t)\| = \vec{v}(t)/s'(t)$  as  $\vec{v} = s'\vec{T}$ , we can apply the product rule to calculate:

$$\vec{a}(t) = s''(t)\vec{T}(t) + s'(t)\vec{T}'(t) = s''\vec{T} + s'\|\vec{T}'\|\vec{N} = s''\vec{T} + (s')^2\kappa\vec{N}$$

The second equation comes from rewriting  $\vec{N}(t) = \vec{T}'(t)/\|\vec{T}'(t)\|$  as  $\vec{T}' = \|\vec{T}'\|\vec{N}$  and substituting. The third equation used the fact that  $\kappa = \|\vec{T}'\|/s'$ , so that  $\|\vec{T}'\| = \kappa s'$ . We’ve made the scalar functions red and the vectors black to emphasize that acceleration is a linear combination of  $\vec{T}$  and  $\vec{N}$  (or  $\vec{T}'$ ).

I COPIED THIS FROM THE HANDOUT (except for formatting, some notation, and a little rewriting)... HOPEFULLY THAT’S FINE?

**Problem 1** Which of the following give the speed of a path? Select all that apply.

**Select All Correct Answers:**

*Decomposition of Acceleration*

- (a)  $\|\vec{x}(t)\|$
- (b)  $\|\vec{x}'(t)\|$  ✓
- (c)  $\|\vec{v}(t)\|$  ✓
- (d)  $\|\vec{v}'(t)\|$
- (e)  $s(t)$
- (f)  $s'(t)$  ✓

Which of the following vectors are always containing in the osculating plane?  
Select all that apply.

**Select All Correct Answers:**

- (a)  $\vec{x}(t)$
  - (b)  $\vec{v}(t)$  ✓
  - (c)  $\vec{a}(t)$  ✓
  - (d)  $\vec{T}(t)$  ✓
  - (e)  $\vec{N}(t)$  ✓
  - (f)  $\vec{B}(t)$
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