Homework 8: Differentiability

Graded Problems

Problem 1 *Consider the linear transformation* $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^2$ *given by*

$$T(x, y, z) = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(a) *Compute* $D\mathbf{T}(0, 0, 0)$.

(b) *Explain why your answer to (a) makes sense in the context of linear approximations.*

Problem 2 *Let* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *be the function given by*

$$\mathbf{f}(x, y) = (x^2 + y, 3y).$$

In this problem you will prove according to the limit definition that $\mathbf{f}$ *is differentiable at the point* $\mathbf{a} = (1, 2)$. In each part, do all calculations by hand and show appropriate amounts of work. Do not use technology.

(a) *Find the derivative matrix* $D\mathbf{f}$ *of* $\mathbf{f}$.

(b) *To prove* $\mathbf{f}$ *is differentiable, you must show it has a good linear approximation at the point* $\mathbf{a} = (1, 2)$. *Using your answer from part (a), determine the formula for this linear approximation. (No simplifications necessary.)*

(c) *Use Definition 3.8 to prove that* $\mathbf{f}$ *is differentiable at* $\mathbf{a} = (1, 2)$. *In this step it would be very useful to simplify the numerator of the "difference quotient" as much as possible, and then make some substitutions on the top and bottom to help you evaluate the limit.*

Learning outcomes:
Author(s):

Professional Problem

Problem 3 *The first draft of your project is due this week.*

Completion Packet

Problem 4 *Find the gradient of each function at the given point.*

(a) $f(x, y) = xe^{xy} + x^2 y$, *at the point* $(1, 1)$.

(b) $g(x, y) = \sin(x^2 + y) + y\cos(x)$, *at the point* $(0, \pi)$.

Problem 5 *Find the matrix of partial derivatives of the following functions.*

(a) $f(x, y, z) = \ln(x^2 y) + xyz$

(b) $\mathbf{f}(x, y, z) = \left(\dfrac{\sqrt{x^2 + y^2}}{z}, z^2 y^3\right)$

(c) $\mathbf{f}(x) = (\ln(x), xe^x, \sin(x))$

Problem 6 *Consider the function* $\mathbf{f}(x, y, z) = \left(\dfrac{1}{x^2 + y^2}, \dfrac{xy}{z}\right)$.

(a) *Compute the partial derivatives of* $\mathbf{f}$.

(b) *Show that* $\mathbf{f}$ *is differentiable on its domain.*

Problem 7 *Find the point where the plane tangent to* $z = x^2 + y^2$ *at* $(1, 1, 2)$ *intersects the* z*-axis.*

Problem 8 *Because the partial derivatives of a function are necessary to construct the limit in the definition of differentiability, a sort of converse to Theorem 3.10 is: If* $\mathbf{f} : D \subset \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable at* $\mathbf{a} \in \mathbb{R}^n$*, then the partial derivatives of* $\mathbf{f}(\mathbf{x})$ *must all be defined at* $\mathbf{a}$*. Use this result to show the following functions are not differentiable at the indicated point.*

(a) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, *at the point* $(0, 0, 0)$.

(b) $g(x, y) = |x - y|$, *at the point* $(3, 3)$.