

Directional Derivatives

In order to find how a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ changes with each of the input variables, we defined the partial derivatives of f . For example, when $n = 2$, we defined the partial derivative of f with respect to x to be

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

Here, we thought of y as a constant, which made f only a function of x , and reduced us to a single variable derivative. This told us how a small change in x would affect the value of f , if we kept y constant. In other words, the partial derivatives described how the function f was changing in the positive x -direction and in the positive y -direction.

PICTURE

But what if we want to find how f changes if we change both x and y ? One possible way to do this would be to change x and y by the same amount, which would be equivalent to finding how f changes as we move along the line $y = x$.

PICTURE

Alternatively, we could change y by twice as much as x . This would be equivalent to finding how f changes as we move along the line $y = 2x$.

PICTURE

As you can see, there are many different ways that we can change x and y , corresponding to different directions in the xy -plane. In order to determine how f changes as we move in all of these different directions, we will now define directional derivatives.

Directional derivatives

We would like to compute the instantaneous rate of change of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ as we move in some given direction in \mathbb{R}^n . We will model our definition after partial derivatives and single variable derivatives, and use a unit vector \vec{v} to describe the direction.

Definition 1. Consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a point $\vec{a} \in \mathbb{R}^n$, and a direction given by a unit vector $\vec{v} \in \mathbb{R}^n$. Then we define the directional derivative of f at

Learning outcomes:
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\vec{a} in the direction of \vec{v} to be

$$D_{\vec{v}}f(\vec{a}) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h},$$

provided this limit exists.

Noticing that by looking at $f(\vec{a} + h\vec{v})$, we are finding the value of f when we move a small distance, h , in the direction of \vec{v} from the point \vec{a} .

VISUAL

When computing directional derivatives, it's important to remember that the direction must be given by a *unit* vector. Otherwise, the length of the vector will change the value of the limit above. If you'd like to find a directional derivative in a direction given by a non-unit vector \vec{w} , you should normalize \vec{w} to unit length.

Example 1. We'll compute the directional derivative of $f(x, y) = x^2y + y^2$ at $\vec{a} = (2, 0)$, in the direction of $(3, 4)$.

Since $(3, 4)$ isn't a unit vector, we need to normalize it. Since $\|(3, 4)\| = \sqrt{3^2 + 4^2} = 5$, we'll use the vector $\vec{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$ to compute our desired directional derivative.

$$\begin{aligned} D_{\vec{v}}f(\vec{a}) &= \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left((2, 0) + h\left(\frac{3}{5}, \frac{4}{5}\right)\right) - f(2, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left(2 + \frac{3}{5}h, \frac{4}{5}h\right) - f(2, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(2 + \frac{3}{5}h\right)^2 \cdot \frac{4}{5}h + \left(\frac{4}{5}h\right)^2 - 0}{h} \\ &= \lim_{h \rightarrow 0} \left(2 + \frac{3}{5}h\right)^2 \cdot \frac{4}{5} + \left(\frac{4}{5}\right)^2 h \\ &= 4 \cdot \frac{4}{5} \\ &= \frac{16}{5}. \end{aligned}$$

Fortunately, we won't always need to resort to evaluating directional derivatives using the limit definition. We'll soon see how we can use the gradient to compute directional derivatives.