Week 4: Homework

Concept Check

Problem 1 *In your own words, what is an open set?*

Free Response:

Problem 2 *True or False: If a set is not open, then it's closed.*

Multiple Choice:

(a) *True*

(b) *False* ✓

Problem 3 *Provide a specific counterexample.*

Free Response:

Problem 4 *Complete the definition.*

A continuous vector field is called $\boxed{pathindependent}$ *if* $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s} = \int_D \mathbf{F}\cdot d\mathbf{s}$ *for any two simple, piecewise* C^1*, oriented curves* C *and* D *with the same start and end points.*

Problem 5 *Complete the statement of the Fundamental Theorem of Line Integrals.*

Learning outcomes:
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Let $f : X \to \mathbb{R}$ be C^1, where $X \subset \mathbb{R}^n$ is $\boxed{\textit{open}}$ and $\boxed{\textit{connected}}$. *Then if C is any piecewise C^1 curve from $\mathbf{A}$ to $\mathbf{B}$, then*

$$\int_C \nabla f \cdot d\sim = \boxed{f(B) - f(A)}$$

Problem 6 *Complete the theorem statement.*

Let $F : X \to \mathbb{R}^n$ be a C^1 vector field, where $X \subset \mathbb{R}^n$ is $\boxed{\textit{open}}$ and $\boxed{\textit{connected}}$. *If $\mathbf{F}$ is conservative, then $D\mathbf{F}$ is symmetric.*

Online Homework

Problem 7 *(Stolen from webwork) Determine whether the vector field $\mathbf{F}(x, y) = (2x \sin y, x^2 \cos y + 2y)$ is conservative.*

We first compute

$$f(x, y) = \int 2x \sin y \, dx = \boxed{x^2 \sin(y)} + g(y).$$

Problem 8 *The partial derivative of the right-hand side with respect to y is*

$$\frac{\partial f}{\partial y} = \boxed{x^2 \cos(y)} + g'(y).$$

Problem 9 *We conclude that*

$$g(y) = \boxed{y^2}.$$

Problem 10 *The vector field is conservative, and the potential function is*

$$f(x, y) = \boxed{x^2 \sin(y) + y^2}$$

Problem 11 *(Stolen from webwork) For each of the following vector fields* $\mathbf{F}$*, decide whether it is conservative or not. If it is conservative, give a potential function* f*. If it is not conservative, write "DNE".*

$$\mathbf{F}(x, y) = (12x - 4y)\mathbf{i} + (-4x + 8y)\mathbf{j}$$

$$f(x, y) = \boxed{6x^2 - 4xy + 4y^2}$$

$$\mathbf{F}(x, y) = 6y\mathbf{i} + 7x\mathbf{j}$$

$$f(x, y) = \boxed{DNE}$$

$$\mathbf{F}(x, y, z) = 6x\mathbf{i} + 7y\mathbf{j} + \mathbf{k}$$

$$f(x, y, z) = \boxed{3x^2 + \frac{7y^2}{z}}$$

$$\mathbf{F}(x, y) = (6 \sin y)\mathbf{i} + (-8y + 6x \cos y)\mathbf{j}$$

$$f(x, y) = \boxed{6x \sin(y) - 4y^2}$$

$$\mathbf{F}(x, y, z) = 6x^2\mathbf{i} - 4y^2\mathbf{j} + 4z^2\mathbf{k}$$

$$f(x, y, z) = \boxed{\frac{1}{3}(6x^3 - 4y^3 + 4z^3)}$$

Problem 12 *(Stolen from webwork) For each set, indicate if it's connected and/or simply connected (select all which apply).*

(a) $\mathbb{R} \setminus \{0\}$

Select All Correct Answers:

(i) *Connected*

(ii) *Simply connected*

(iii) *Neither* ✓

(b) $\mathbb{R}^2 \setminus \{(0, 0)\}$

Select All Correct Answers:

(i) *Connected* ✓

(ii) *Simply connected*

(iii) *Neither*

(c) $\mathbb{R}^3 \setminus \{(0,0,0)\}$

Select All Correct Answers:

(i) *Connected* ✓
(ii) *Simply connected* ✓
(iii) *Neither*

(d) $\mathbb{R} \setminus \{x : |x| \leq 1\}$

Select All Correct Answers:

(i) *Connected*
(ii) *Simply connected*
(iii) *Neither* ✓

(e) $\mathbb{R}^2 \setminus \{(x,y) : |(x,y)| \leq 1\}$

Select All Correct Answers:

(i) *Connected* ✓
(ii) *Simply connected*
(iii) *Neither*

(f) $\mathbb{R}^3 \setminus \{(x,y,z) : |(x,y,z)| \leq 1\}$

Select All Correct Answers:

(i) *Connected* ✓
(ii) *Simply connected* ✓
(iii) *Neither*

(g) $\mathbb{R}^2 \setminus \{(x,y) : x = y\}$

Select All Correct Answers:

(i) *Connected*
(ii) *Simply connected*
(iii) *Neither* ✓

(h) $\mathbb{R}^3 \setminus \{(x,y,z) : x = y = z\}$

Select All Correct Answers:

(i) *Connected* ✓

(ii) *Simply connected*
(iii) *Neither*

Problem 13 *(Stolen from webwork) Determine whether the given set is open, connected, and/or simply connected. Select all that apply.*

(a) $\{(x, y) : x > 1, y < 2\}$

Select All Correct Answers:

(i) *Open* ✓
(ii) *Connected* ✓
(iii) *Simply connected* ✓

(b) $\{(x, y) : 2x^2 + y^2 < 1\}$

Select All Correct Answers:

(i) *Open* ✓
(ii) *Connected* ✓
(iii) *Simply connected* ✓

(c) $\{(x, y) : x^2 - y^2 < 1\}$

Select All Correct Answers:

(i) *Open* ✓
(ii) *Connected* ✓
(iii) *Simply connected* ✓

(d) $\{(x, y) : x^2 - y^2 > 1\}$

Select All Correct Answers:

(i) *Open* ✓
(ii) *Connected*
(iii) *Simply connected*

(e) $\{(x, y) : 1 < x^2 + y^2 < 4\}$

Select All Correct Answers:

(i) *Open* ✓

(ii) *Connected* ✓

(iii) *Simply connected*

Problem 14 *(Stolen from webwork) For each problem, determine if the set is open, closed, connected, and/or simply connected. Select all that apply.*

(a) $\{(x,y) : x^2+y^2<1\} \cup \{(x,y) : (x-3)^2+y^2<2\} \subset \mathbb{R}^2$

Select All Correct Answers:

(i) *Open* ✓

(ii) *Closed*

(iii) *Connected*

(iv) *Simply connected*

(b) $\{(x,y) : x^2+y^2<1\} \cap \{(x,y) : (x-3)^2+y^2<1\} \subset \mathbb{R}^2$

Select All Correct Answers:

(i) *Open* ✓

(ii) *Closed* ✓

(iii) *Connected* ✓

(iv) *Simply connected* ✓

(c) $\{(x,y,z) : x,y,z \geq 1\} \subset \mathbb{R}^3$

Select All Correct Answers:

(i) *Open*

(ii) *Closed* ✓

(iii) *Connected* ✓

(iv) *Simply connected* ✓

(d) $\{(x,y,z) : x^2+y^2+z^2<1\} \cup \{(x,y,z) : (x-1)^2+y^2+z^2<1\} \subset \mathbb{R}^3$

Select All Correct Answers:

(i) *Open* ✓

(ii) *Closed*

(iii) *Connected* ✓

(iv) *Simply connected* ✓

(e) $\{(x, y) : x^2 + y^2 + z^2 = 1\} \cap \{(x, y, z) : z < 0\} \subset \mathbb{R}^3$

Select All Correct Answers:

(i) *Open*
(ii) *Closed*
(iii) *Connected* ✓
(iv) *Simply connected* ✓

Written Homework

Problem 15 *(Stolen from Colley) Of the two vector fields*

$$\mathbf{F} = xy^2z^3\mathbf{i} + 2x^2y\mathbf{j} + 3x^2y^2z^2\mathbf{k}$$

and

$$\mathbf{G} = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$$

one is conservative and one is not. Determine which is which, and, for the conservative field, find a scalar potential function.

Problem 16 *(Stolen from Colley) Find all functions $M(x, y)$ such that the vector field*

$$\mathbf{F} = M(x, y)\mathbf{i} + (x \sin y - y \cos x)\mathbf{j}$$

is conservative.

Problem 17 *Sketch both of the following sets (in two different pictures) and determine whether they are open, closed, neither, or both and whether they are disconnected, connected, and/or simply connected. State all which apply and justify your answer using appropriate definitions and theorems.*

(a) $A = \{x^2 + y^2 < 1\} \cup \{x = 0\} \subset \mathbb{R}^2$
(b) $B = \{x^2 + y^2 = 1\} \cup \{x = 0\} \subset \mathbb{R}^2$

Professional Problem

Problem 18 *Write a careful proof of this statement:*

Theorem 1. *Let $D_1, D_2, ..., D_n$ be closed sets in $\mathbb{R}^n$. Then their intersection*

$$D = D_1 \cap D_2 \cap \cdots \cap D_n$$

is a closed set as well.

For this professional problem, be very careful to use notation and definitions correctly. Because you have already seen most of the necessary elements, your proof should be elegant, well-organized, and concise.

Hint: *Take a similar theorem we proved in workshop about open sets, and use De Morgan's Laws for sets. Incidentally, this theorem would still be true even with an intersection of infinitely many closed sets!*