The Dot Product

In this section we review the dot product on vectors. This also includes the angle between vectors and the projection of one vector onto another.

The Dot Product

We begin with the definition of the dot product.

Definition 1. *The* dot product *of two vectors* $\vec{v} = (v_1, v_2, ..., v_n)$ *and* $\vec{w} = (w_1, w_2, ..., w_n)$ *in* $\mathbb{R}^n$ *is*

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Notice that the dot product takes two vectors and outputs a scalar.

Example 1. $(1, 6) \cdot (-3, -6) = -3 - 36 = -39$

$(1, 2, 3) \cdot (7, -2, 4) = 7 - 4 + 12 = 15$

$(1, 7, -3) \cdot (3, 0, 1) = 3 + 0 - 3 = 0$

We can also compute the dot product using the magnitude (or length) of the vectors and the angle in between them.

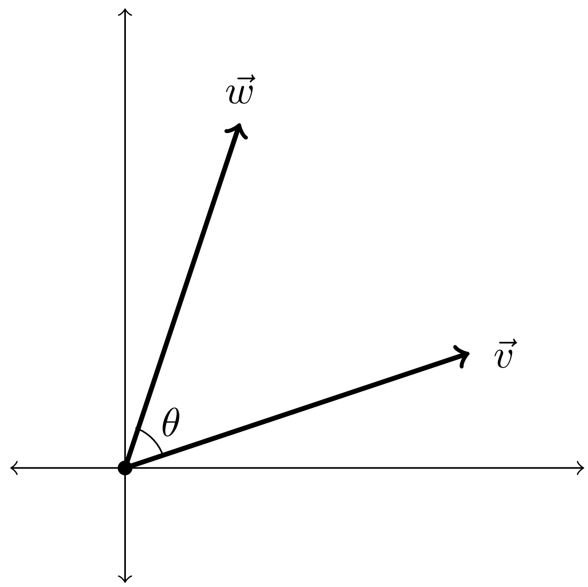
Proposition 1. *If* $\vec{v}$ *and* $\vec{w}$ *are vectors in* $\mathbb{R}^n$*, then*

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta,$$

where $\|\vec{v}\|$ *and* $\|\vec{w}\|$ *are the lengths of the vectors* $\vec{v}$ *and* $\vec{w}$*, respectively, and* θ *is the angle between* $\vec{v}$ *and* $\vec{w}$*.*

This is illustrated in the picture below.

Learning outcomes:
Author(s):



This provides us with a geometric interpretation of the dot product: it gives us a measure of "how much" in the same direction two vectors are (taking their lengths into account). This also gives us a useful way to compute the angle between two vectors.

Example 2. *Consider the vectors* $(1,4)$ *and* $(-2,2)$*. We have*

$$\begin{aligned}(1,4)\cdot(-2,2) &= -2+8 = 6,\\ \|(1,4)\| &= \sqrt{1^2+4^2} = \sqrt{17},\\ \|(-2,2)\| &= \sqrt{(-2)^2+2^2} = \sqrt{8}.\end{aligned}$$

From $\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$*, we then have*

$$6 = \sqrt{17}\sqrt{8}\cos\theta.$$

Solving for θ*, we obtain the angle between the vectors as*

$$\theta = \arccos\left(\frac{6}{\sqrt{17}\sqrt{8}}\right) \approx 59.04^\circ$$

Furthermore, note that for nonzero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, their dot product is 0 if and only if $\cos(\theta) = 0$. This means that θ would have to be 90° or 270°, meaning that $\vec{v}$ and $\vec{w}$ are perpendicular.

Proposition 2. *Two nonzero vectors* $\vec{v}$ *in* $\vec{w}$ *in* $\mathbb{R}^n$ *are perpendicular if and only if* $\vec{v}\cdot\vec{w} = 0$*.*

This provides us with a very useful algebraic method for determining if two vectors are perpendicular.

Example 3. *The vectors* $(1, 7, -3)$ *and* $(3, 0, 1)$ *in* $\mathbb{R}^3$ *are perpendicular, since*

$$(1, 7, -3) \cdot (3, 0, 1) = 3 + 0 - 3 = 0.$$

By taking the dot product of a vector with itself, we get an important relationship between the dot product and the length of a vector.

Proposition 3. *Let* $\vec{v}$ *be a vector in* $\mathbb{R}^n$*. Then*

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2.$$

This can be shown directly, or using the fact that the angle between $\vec{v}$ and itself is 0.

Projection of one vector onto another

We can also use the dot product to define the projection of one vector onto another.

Definition 2. *For vectors* $\vec{a}$ *and* $\vec{b}$ *in* $\mathbb{R}^n$*, we define the* vector projection *of* $\vec{a}$ *onto* $\vec{b}$ *as*

$$proj_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\vec{b}.$$

Example 4. *We can use this to find the projection of* $(2, 4, 3)$ *onto* $(1, -1, 1)$*.*

$$\begin{aligned} proj_{(1,-1,1)}(2, 4, 3) &= \frac{(2, 4, 3) \cdot (1, -1, 1)}{(1, -1, 1) \cdot (1, -1, 1)}(1, -1, 1) \\ &= \frac{2 - 4 + 3}{1 + 1 + 1}(1, -1, 1) \\ &= \frac{1}{3}(1, -1, 1) \\ &= \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) \end{aligned}$$

Summary

In this section we reviewed the dot product on vectors, the angle between vectors, and the projection of one vector onto another.