

# Geometry of Differentiability

In single variable calculus, derivatives were closely related to the slope of the tangent line to a graph at a point. We used this idea of the slope of the tangent line to define derivatives as a limit of slopes of secant lines,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

## VISUAL

In the other direction, we were able to use differentiation rules to more easily find the equation for the tangent line to a graph at a point.

**Example 1.** *We'll find the equation for the tangent line to the graph of  $f(x) = x^3 + 2x + 1$  at  $x = 2$ .*

*We can find the slope of the tangent line by computing  $f'(2)$ . Using differentiation rules, we have*

$$f'(x) = \boxed{3x^2 + 2}.$$

*Plugging in  $x = 2$ , we have  $f'(2) = \boxed{14}$ .*

*Since the tangent line will have to pass through the point  $(2, f(2))$ , we compute*

$$f(2) = \boxed{13}.$$

*So, the tangent line will pass through the point  $(2, 13)$ , and will have slope 14. Writing the equation of the line in point-slope form, we have*

$$y - 13 = \boxed{14(x - 2)}.$$

When a single variable function is differentiable, we can use the above method to find an equation for the tangent line. In addition, the tangent line provides us with a good linear approximation for the function.

We would like to do something analogous for multi-variable functions, but this raises a few questions. What would be the equivalent of the tangent line? What does it mean for a function to be differentiable?

As we begin to explore these questions, we'll focus on functions  $\mathbb{R}^2 \rightarrow \mathbb{R}$ , so that we can visualize their graphs.

## Geometric Interpretation of Differentiability

In single variable calculus, we could get a good sense of whether a function was differentiable by looking at its graph.

**Problem 1** For each of the graphs, determine whether the given function is differentiable at  $x = a$ .

GRAPH PARABOLA

**Multiple Choice:**

- (a) differentiable ✓
- (b) not differentiable

GRAPH JUMP DISCONTINUITY

**Multiple Choice:**

- (a) differentiable
- (b) not differentiable ✓

GRAPH ABSOLUTE VALUE

**Multiple Choice:**

- (a) differentiable
- (b) not differentiable ✓

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If there is a discontinuity or some sort of corner or cusp in the graph at a point, then the function will not be differentiable at that point. Roughly speaking, if we “zoom in” on the graph of a function near a point, and the graph looks very close to a line, then the function will be differentiable at that point.

PARABOLA ZOOM

We’ll extend this idea to make our first, informal definition of differentiability for a function from  $\mathbb{R}^2$  to  $\mathbb{R}$ .

**Definition 1.** (Informal Definition) Consider a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Suppose for some point  $(a, b)$  in  $\mathbb{R}^2$ , if we zoom in on the graph of  $f$  near the point  $(a, b)$ , the graph of  $f$  looks like a plane. Then  $f$  is differentiable at  $(a, b)$ .

Although this definition provides us with nice geometric intuition for determining if a function is differentiable, it's not at all precise or rigorous. Eventually, we'll need a more formal definition of differentiability, so we'll return to this concept later.

For now, let's use this informal definition to investigate differentiability for a couple of functions.

**Example 2.** Consider the function  $f(x, y) = xy + 2x + y$ , graphed below.

ZOOMABLE GRAPH

Is  $f$  differentiable at  $(0, 0)$ ?

**Multiple Choice:**

- (a) Yes. ✓
- (b) No.

**Example 3.** Consider the function

$$f(x, y) = \begin{cases} 1 - |y| & \text{if } |y| \leq |x| \\ 1 - |x| & \text{if } |y| > |x| \end{cases},$$

graphed below.

ZOOMABLE GRAPH

Is  $f$  differentiable at  $(0, 0)$ ?

**Multiple Choice:**

- (a) Yes.
- (b) No. ✓

## The tangent plane

If a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is differentiable, when we zoom in, the graph will look like a plane. Because of this, there will be a plane that's a good linear approximation for the function near that point. We can use the partial derivatives with respect to  $x$  and  $y$  to find an equation for this plane, which we call the tangent plane.

**Definition 2.** If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is differentiable at  $(a, b)$ , then the tangent plane to the graph of  $f$  at  $(a, b)$  is defined by the equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Let's revisit our previous examples, and see what happens with the tangent plane.

**Example 4.** Consider the function  $f(x, y) = xy + 2x + y$ , which we found is differentiable at the point  $(0, 0)$ . Let's find an equation for the tangent plane to the graph of  $f$  at  $(0, 0)$ .

We begin by finding the partial derivatives with respect to  $x$  and  $y$ .

$$\begin{aligned} f_x(x, y) &= y + 2 \\ f_y(x, y) &= x + 1 \end{aligned}$$

At  $(0, 0)$ , we have

$$\begin{aligned} f_x(0, 0) &= 2, \\ f_y(0, 0) &= 1. \end{aligned}$$

Finding the value of  $f$  at  $(0, 0)$ , we have

$$f(0, 0) = 0.$$

Putting all of this together, we obtain an equation for the tangent plane to the graph of  $f$  at  $(0, 0)$ .

$$\begin{aligned} z &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &= 2x + y \end{aligned}$$

**Example 5.** Consider the function

$$f(x, y) = \begin{cases} 1 - |y| & \text{if } |y| \leq |x| \\ 1 - |x| & \text{if } |y| > |x| \end{cases},$$

which we found is not differentiable  $(0, 0)$ . Even though this function is not differentiable, let's see what happens when we try to find an equation for the tangent plane.

To compute the partial derivatives with respect to  $x$  and  $y$ , we'll need to use the limit definition.

$$\begin{aligned}
 f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1 - |0|) - (1 - |0|)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{0}{h} \\
 &= 0. f_y(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0, 0 + h) - f(0, 0)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1 - |0|) - (1 - |0|)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{0}{h} \\
 &= 0.
 \end{aligned}$$

We can also find  $f(0, 0)$ ,

$$f(0, 0) = 1.$$

So, we have all of the necessary pieces to find an equation for the (nonexistent) “tangent plane”:

$$\begin{aligned}
 z &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\
 &= 1.
 \end{aligned}$$

However, we decided that the function wasn’t differentiable at  $(0, 0)$ , so the graph does not have a tangent plane at the point.

This example brings up a couple of important points.

- It’s possible for the partial derivatives of a function to all exist, and yet the function is not differentiable.
- It’s possible that we can find the equation  $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ , and yet  $f$  has no tangent plane at the point  $(a, b)$ .

For these reasons, differentiability is a much more subtle concept in multivariable calculus than it was in single variable calculus, and our next task will be to find a formal definition for differentiability.