Multivariable Calculus

March 20, 2019

Contents

Part I

Review

Vectors

In this section, we review some basics about vectors. This includes the definition of a vector, basic vector operations, standard basis vectors, and notation.

Vectors

In linear algebra, we often worked with vectors. We begin by recalling the (algebraic) definition of a vector in $\mathbb{R}^n$.

Definition 1. *A* vector *in $\mathbb{R}^n$ is an ordered n-tuple of real numbers. That is, a vector $\vec{v}$ may be written as*

$$\vec{v} = (a_1, a_2, ..., a_n)$$

where $a_1, a_2, ..., a_n$ are real numbers.

We call the numbers a_i the components *or* entries *of the vector. We call n the* dimension *of the vector $\vec{v}$, and say that $\vec{v}$ is n-*dimensional*.*

We write the vector with an arrow above it, as $\vec{v}$, in order to make the distinction between vectors and *scalars*, which are just real numbers. Some other common notations for vectors are $\mathbf{v}$ and $\hat{v}$. It's important to make this distinction between vectors and scalars, so you should make use of one of these notations for vectors.

Example 1. $\vec{v} = (1, 3)$ *is a vector in* $\mathbb{R}^2$.

$\vec{w} = (-1, 5, 0)$ *is a vector in* $\mathbb{R}^3$.

$\vec{x} = (1, -2, 3)$ *is a vector in* $\mathbb{R}^3$.

$\vec{y} = (-6, \pi, 1/24, -0.5, 3)$ *is a vector in* $\mathbb{R}^5$.

It's sometimes convenient to write a vector as a column vector instead (particularly when working with linear transformations, which we'll review in a later

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Learning outcomes:
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section). We could write

$$\vec{v} = (a_1, a_2, ..., a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

or

$$\vec{v} = (a_1, a_2, ..., a_n) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

The choice between square brackets and parentheses is just a difference in notation, they mean the same thing, and you should feel free to use either.

Example 2. *We write the following vectors as column vectors.*

$\vec{v} = (1, 3) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

$\vec{w} = (-1, 5, 0) = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}.$

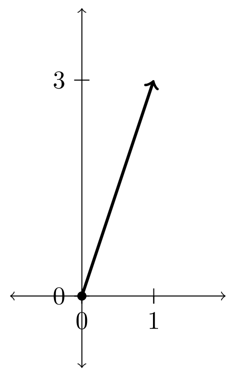
$\vec{x} = (1, -2, 3) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$

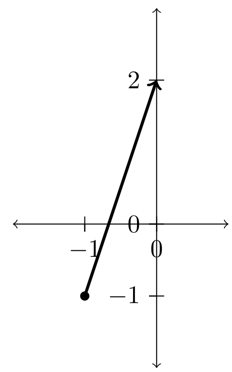
$\vec{y} = (-6, \pi, 1/24, -0.5, 3) = \begin{bmatrix} -6 \\ \pi \\ 1/24 \\ -0.5 \\ 3 \end{bmatrix}.$

The Geometric Perspective

We also can think of a vector geometrically, as giving a direction and magnitude, but without a fixed position.

In two or three dimensions, it is useful to visualize a vector as an arrow in $\mathbb{R}^n$. We might visualize a vector $\vec{v} = (1, 3)$ in $\mathbb{R}^2$ as the arrow starting at the origin and ending at the point $(1, 3)$, thus giving a direction and a magnitude. However, we typically don't think of a vector as having a set location. We could also visualize the vector $\vec{v}$ as starting at the point $(-1, -1)$ and ending at the point $(0, 2)$. Note that this arrow would have the same direction and magnitude as the one starting at the origin, thus they represent the same vector.





In four or higher dimensions, visualizing anything becomes very difficult. It can still be useful to think of a vector $(1,2,3,4,5)$ in $\mathbb{R}^5$ as starting at the origin and ending at the point $(1,2,3,4,5)$, but you probably won't be able to have a very clear picture of this in your head.

This concept will probably seem more useful once you think about a displacement vector.

Definition 2. *Given points $P_1 = (x_1, ..., x_n)$ and $P_2 = (y_1, ..., y_n)$ in $\mathbb{R}^n$, the* displacement vector *from P_1 to P_2 is*

$$\vec{P_1P_2} = (y_1 - x_1, ..., y_n - x_n).$$

This is the vector that starts at P_1 and ends at P_2.

Notice that the notation $(a_1, ..., a_n)$ that we use for a vector in $\mathbb{R}^n$ is identical to the notation that we'd use for a point in $\mathbb{R}^n$. Since both vectors and points in $\mathbb{R}^n$ are defined as n-tuples of points, they are, in some sense, the same thing. The difference between the two comes when we consider the context and geometric significance of the vector or point that we're working with. As we move into multivariable calculus, we'll often blur the distinction between a vector and a point, and sometimes think of a vector as a point and vice versa. This will be greatly simplify notation, and we promise that it won't be as confusing as it sounds!

Vector Operations

Before defining some basic vector operations, we define what it means for two vectors to be equal. This is done by comparing the components of the vectors.

Definition 3. *Two vectors $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ in $\mathbb{R}^n$ are* equal *if their corresponding components are equal, so $a_1 = b_1$, $a_2 = b_2$, ... , $a_n = b_n$.*

Notice that, in order to be equal, two vectors must have the same dimension and the same entries in the same order. Thus, the vectors $(1,3)$ and $(1,3,0)$ are not equal.

We now define addition of two vectors of the same dimension, which is done componentwise.

Definition 4. *Let $\vec{a} = (a_1, a_2, ..., a_n)$ and $\vec{b} = (b_1, b_2, ..., b_n)$ be vectors in $\mathbb{R}^n$. We define $\vec{a} + \vec{b}$ to be the vector in $\mathbb{R}^n$ given by*

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n).$$

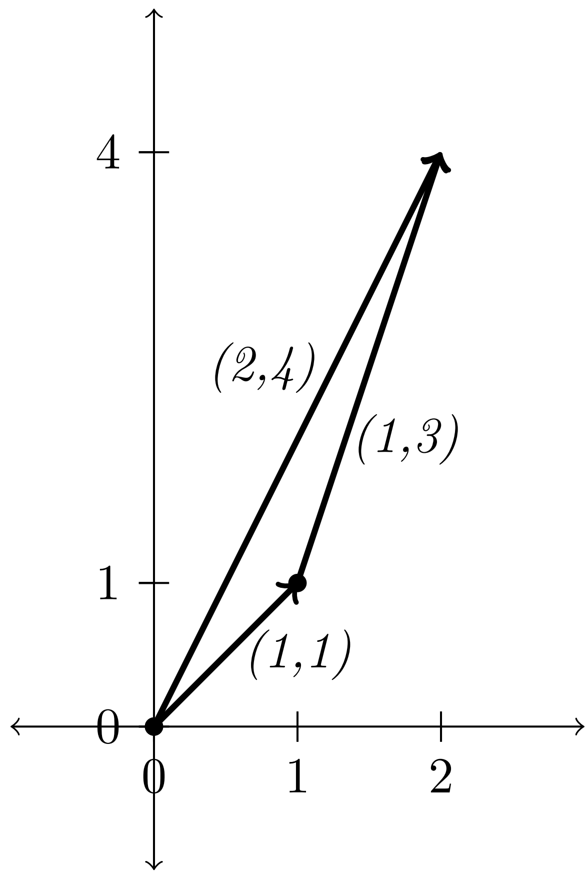
Note that we can only add two vectors if they have the same dimension.

Example 3. *Adding the vectors $\vec{a} = (1, -8, 2)$ and $\vec{b} = (3, -1, -2)$, we obtain*

$$\begin{aligned} \vec{a} + \vec{b} &= (1, -8, 2) + (3, -1, -2) \\ &= (1 + 3, -8 - 1, 2 - 2) \\ &= (4, -9, 0). \end{aligned}$$

Geometrically, we can add vectors by placing the start point of the second vector at the end point of the first vector, and drawing an arrow from the start point of the first vector to the end point of the second vector.

Example 4. *In this example, we add the vectors $(1,1)$ and $(1,3)$. Adding these vectors algebraically, we obtain $(2,4)$. We can also see this geometrically by placing the start point of the vector $(1,3)$ at the end of the vector $(1,1)$ (so at the point $(1,1)$), and drawing the vector from the origin to the end point of the vector $(1,3)$, which is now at $(2,4)$.*



Another vector operation is scalar multiplication. Here, we multiply a vector by a real number, possibly changing the length of the vector.

Definition 5. *Let $\vec{a} = (a_1, ..., a_n)$ be a vector in $\mathbb{R}^n$, and let r be a real number (also called a scalar). We define the* scalar product *$r\vec{a}$ to be*

$$r\vec{a} = (ra_1, ..., ra_n).$$

Thus, we see that scalar multiplication is defined by multiplying each component of the vector by the scalar r.

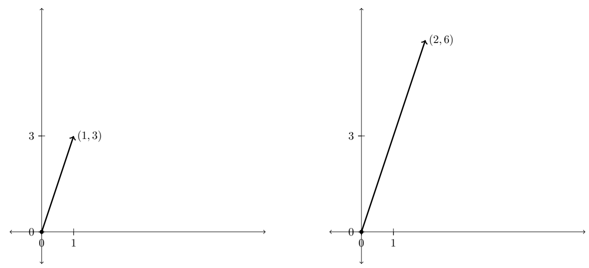
Example 5. $3(1, 5, -2) = (3, 15, -6)$

$-1(1, 1, 1) = (-1, -1, -1)$

$0(6, 2, 4) = (0, 0, 0)$

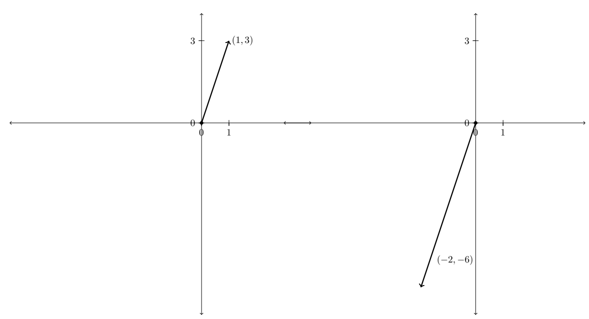
Now, let's look at what scalar multiplication does geometrically. Consider the vector $(1, 3)$.

When we multiply $(1, 3)$ by 2, we obtain $(2, 6)$, which is twice as long as $(1, 3)$ and goes in the same direction.



When we multiply $(1, 3)$ by $\frac{1}{2}$, we obtain $\left(\frac{1}{2}, \frac{3}{2}\right)$, which is half as long as $(1, 3)$ and goes in the same direction.

If we multiply $(1, 3)$ by -2, we obtain $(-2, -6)$, which is twice as long as $(1, 3)$ and goes in the exact opposite direction.



Thus, we have seen that multiplying a vector by a scalar changes the length of a vector, but not the direction (except for reversing it, if the scalar is negative).

Properties

Now, let's recall some useful properties of vector addition and scalar multiplication.

Proposition 1. *Suppose $\vec{a}, \vec{b}, \vec{c}$ are vectors in $\mathbb{R}^n$ and k, l are real numbers. Then*

(a) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ *(vector addition is* commutative*);*

(b) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ *(vector addition is* associative*);*

(c) $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$*, where* $\vec{0} = (0, ..., 0)$ *is the* zero vector *in* $\mathbb{R}^n$*;*

(d) $(k + l)\vec{a} = k\vec{a} + l\vec{a}$*;*

(e) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$ *(with the previous property, scalar multiplication is* distributive*);*

(f) $k(l\vec{a}) = (kl)\vec{a} = l(k\vec{a})$*;*

(g) $1\vec{a} = \vec{a}$*.*

These properties tell us different kinds of algebraic manipulations that we can do with vectors.

Standard Basis Vectors

It's often useful to write things in terms of the standard basis vectors for $\mathbb{R}^n$.

Definition 6. *The vectors* $\vec{e}_1 = (1, 0, ...0)$*,* $\vec{e}_2 = (0, 1, 0, ..., 0)$*, ...,* $\vec{e}_n = (0, ..., 0, 1)$ *in* $\mathbb{R}^n$ *are called the* standard basis vectors *for* $\mathbb{R}^n$*.*

Note that any vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of the standard unit vectors. For example, in $\mathbb{R}^4$,

$$\begin{aligned}(1, 5, -3, 6) &= 1(1, 0, 0, 0) + 5(0, 1, 0, 0) - 3(0, 0, 1, 0) + 6(0, 0, 0, 1) \\ &= 1\vec{e}_1 + 5\vec{e}_2 - 3\vec{e}_3 + 6\vec{e}_4.\end{aligned}$$

In $\mathbb{R}^2$, we sometimes write the standard basis vectors as $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. This gives us a new notation for vectors, for example we could write

$$(3, 4) = 3\mathbf{i} + 4\mathbf{j}.$$

Similarly, in $\mathbb{R}^3$, we sometimes write the standard basis vectors as $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$. We can then write

$$(2, 3, 4) = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

Summary

In this section, we reviewed some basics about vectors, including the definition of a vector, basic vector operations, standard basis vectors, notation, and the geometric perspective.

The Dot Product

In this section we review the dot product on vectors. This also includes the angle between vectors and the projection of one vector onto another.

The Dot Product

We begin with the definition of the dot product.

Definition 7. *The* dot product *of two vectors* $\vec{v} = (v_1, v_2, ..., v_n)$ *and* $\vec{w} = (w_1, w_2, ..., w_n)$ *in* $\mathbb{R}^n$ *is*

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Notice that the dot product takes two vectors and outputs a scalar.

Example 6. $(1, 6) \cdot (-3, -6) = -3 - 36 = -39$

$(1, 2, 3) \cdot (7, -2, 4) = 7 - 4 + 12 = 15$

$(1, 7, -3) \cdot (3, 0, 1) = 3 + 0 - 3 = 0$

We can also compute the dot product using the magnitude (or length) of the vectors and the angle in between them.

Proposition 2. *If* $\vec{v}$ *and* $\vec{w}$ *are vectors in* $\mathbb{R}^n$*, then*

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta,$$

where $\|\vec{v}\|$ *and* $\|\vec{w}\|$ *are the lengths of the vectors* $\vec{v}$ *and* $\vec{w}$*, respectively, and* θ *is the angle between* $\vec{v}$ *and* $\vec{w}$*.*

This is illustrated in the picture below.

Learning outcomes:
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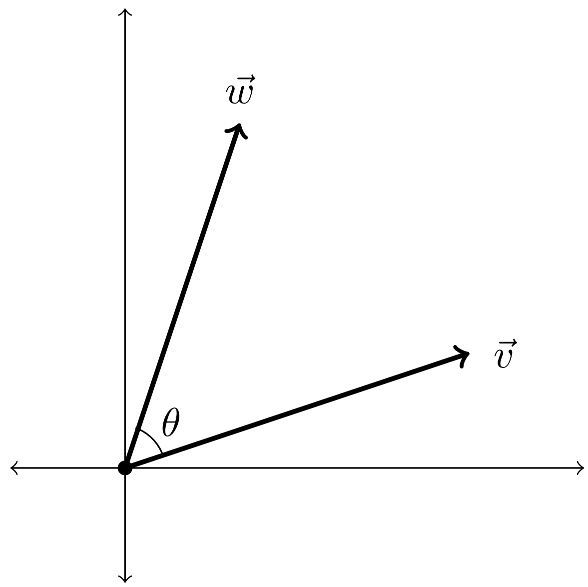

This provides us with a geometric interpretation of the dot product: it gives us a measure of "how much" in the same direction two vectors are (taking their lengths into account). This also gives us a useful way to compute the angle between two vectors.

Example 7. *Consider the vectors* $(1,4)$ *and* $(-2,2)$*. We have*

$$\begin{aligned}(1,4)\cdot(-2,2) &= -2+8=6,\\ \|(1,4)\| &= \sqrt{1^2+4^2}=\sqrt{17},\\ \|(-2,2)\| &= \sqrt{(-2)^2+2^2}=\sqrt{8}.\end{aligned}$$

From $\vec{v}\cdot\vec{w}=\|\vec{v}\|\,\|\vec{w}\|\cos\theta$*, we then have*

$$6=\sqrt{17}\sqrt{8}\cos\theta.$$

Solving for θ*, we obtain the angle between the vectors as*

$$\theta=\arccos\left(\frac{6}{\sqrt{17}\sqrt{8}}\right)\approx 59.04^\circ$$

Furthermore, note that for nonzero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, their dot product is 0 if and only if $\cos(\theta)=0$. This means that θ would have to be 90° or 270°, meaning that $\vec{v}$ and $\vec{w}$ are perpendicular.

Proposition 3. *Two nonzero vectors* $\vec{v}$ *in* $\vec{w}$ *in* $\mathbb{R}^n$ *are perpendicular if and only if* $\vec{v}\cdot\vec{w}=0$*.*

This provides us with a very useful algebraic method for determining if two vectors are perpendicular.

Example 8. *The vectors* $(1, 7, -3)$ *and* $(3, 0, 1)$ *in* $\mathbb{R}^3$ *are perpendicular, since*

$$(1, 7, -3) \cdot (3, 0, 1) = 3 + 0 - 3 = 0.$$

By taking the dot product of a vector with itself, we get an important relationship between the dot product and the length of a vector.

Proposition 4. *Let* $\vec{v}$ *be a vector in* $\mathbb{R}^n$*. Then*

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2.$$

This can be shown directly, or using the fact that the angle between $\vec{v}$ and itself is 0.

Projection of one vector onto another

We can also use the dot product to define the projection of one vector onto another.

Definition 8. *For vectors* $\vec{a}$ *and* $\vec{b}$ *in* $\mathbb{R}^n$*, we define the* vector projection *of* $\vec{a}$ *onto* $\vec{b}$ *as*

$$proj_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\vec{b}.$$

Example 9. *We can use this to find the projection of* $(2, 4, 3)$ *onto* $(1, -1, 1)$*.*

$$\begin{aligned} proj_{(1,-1,1)}(2, 4, 3) &= \frac{(2, 4, 3) \cdot (1, -1, 1)}{(1, -1, 1) \cdot (1, -1, 1)}(1, -1, 1) \\ &= \frac{2 - 4 + 3}{1 + 1 + 1}(1, -1, 1) \\ &= \frac{1}{3}(1, -1, 1) \\ &= \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) \end{aligned}$$

Summary

In this section we reviewed the dot product on vectors, the angle between vectors, and the projection of one vector onto another.

The Cross Product

In this section, we review the vector cross product, including the geometric perspective of the cross product, the area of a parallelogram, and the volume of parallelepiped.

The Cross Product

The cross product is fundamentally different from the dot product in a couple of ways. The cross product is defined only on vectors in $\mathbb{R}^3$, while the dot product is defined in $\mathbb{R}^n$ for any positive integer n. Furthermore, the cross product takes two vectors and produces another vector, while the dot product takes two vectors and produces a scalar.

We now give the algebraic definition of the cross product.

Definition 9. *Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$. The* cross product *of $\vec{a}$ and $\vec{b}$, denoted $\vec{a} \times \vec{b}$, is defined to be*

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Equivalently, we can compute the cross product as

$$\vec{a} \times \vec{b} = det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

where

$$\mathbf{i} = (1, 0, 0),$$
$$\mathbf{j} = (0, 1, 0),$$
$$\mathbf{k} = (0, 0, 1).$$

Learning outcomes:
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Example 10.

$$
\begin{aligned}
(3,2,-1)\times(9,0,2) &= det\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & -1 \\ 9 & 0 & 2 \end{pmatrix} \\
&= (2\cdot 2)\mathbf{i} - (0\cdot -1)\mathbf{i} + (-1\cdot 9)\mathbf{j} - (2\cdot 3)\mathbf{j} + (3\cdot 0)\mathbf{k} - (9\cdot -1)\mathbf{k} \\
&= 4\mathbf{i} - 15\mathbf{j} + 9\mathbf{k} \\
&= (4,-15,9)
\end{aligned}
$$

The cross product has some nice algebraic properties, which can be very useful.

Proposition 5. *Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in $\mathbb{R}^3$, and let $k \in \mathbb{R}$ be a scalar. The cross product has the following properties:*

(a) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ *(the cross product is* anticommutative*);*

(b) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$*;*

(c) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ *(with the previous property, the cross product is* distributive *over vector addition);*

(d) $k(\vec{a} \times \vec{b}) = (k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b})$.

In particular, it's important to remember that the cross product is *not* commutative, so the order of the vectors matters!

Geometry of the Cross Product

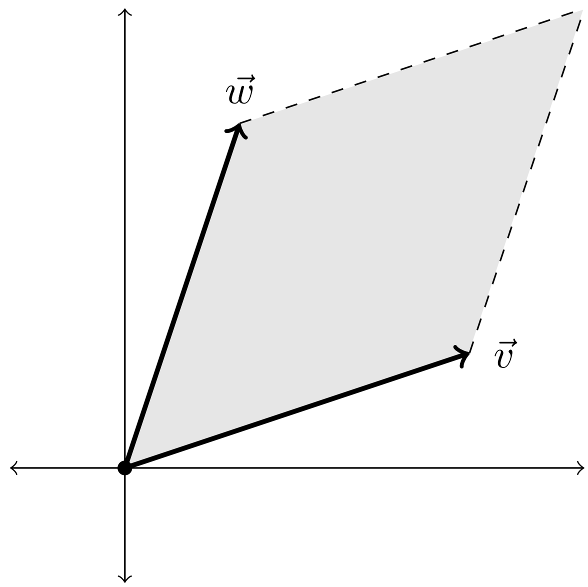
It's often easiest to compute cross products algebraically, but it's easier to understand their significance from a geometric perspective. We now discuss some of the geometric properties of the cross product.

Proposition 6. *Let $\vec{a}$ and $\vec{b}$ be vectors in $\mathbb{R}^3$, and consider their cross product $\vec{a} \times \vec{b}$.*

- *The* magnitude *of the vector $\vec{a} \times \vec{b}$ can be computed as*

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin(\theta),$$

  where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$. Furthermore, this magnitude is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.



- *The vector* $\vec{a} \times \vec{b}$ *is always perpendicular to both* $\vec{a}$ *and* $\vec{b}$*, and follows the* right-hand rule*. That is, if you take your right hand and orient it so you can curl your fingers from the vector* $\vec{a}$ *to the* $\vec{b}$*, your thumb will be pointing in the same direction as the cross product* $\vec{a} \times \vec{b}$*.*

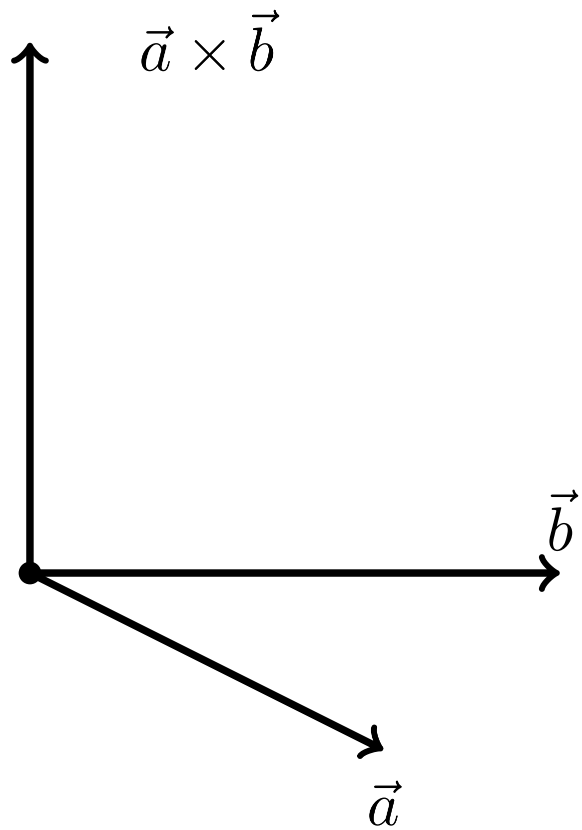
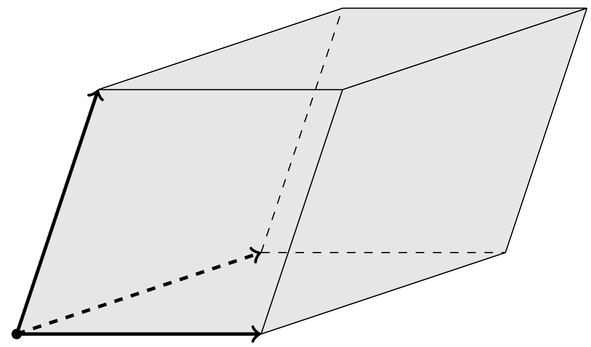


Image this image in $\mathbb{R}^3$, so that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Volume of a Parallelepiped

We can use the cross product and dot product together to compute the volume of a parallelepiped.



The volume of the parellelepiped can be computed as the area of the base times the height. We've seen that the area of the base can be computed as the magnitude of a cross product, $\|\vec{a} \times \vec{b}\|$. The height of the parallelepiped can be computed as $\|\vec{c}\| \, |\cos(\theta)|$, where θ is the angle between the vector $\vec{c}$ and a line perpendicular to the base. We then have that the volume is $\|\vec{a} \times \vec{b}\| \, \|\vec{c}\| \, |\cos(\theta)|$, which we can recognize as the absolute value of the dot product of the vectors $\vec{a} \times \vec{b}$ and $\vec{c}$. Thus we have the following proposition.

Proposition 7. *The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be computed as $|(\vec{a} \times \vec{b}) \cdot \vec{c}|$.*

Summary

We've reviewed the cross product, including its properties and geometric perspective, including its use in finding the area of parallelograms and volume of parallelepipeds.

Matrices

In this section, we review matrices, including the determinant and the linear transformation represented by a matrix.

Matrices

We begin with the definition of a matrix.

Definition 10. *An $m \times n$* matrix *A is a rectangular array of numbers a_{ij}, with m rows and n columns:*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where the a_{ij} are real numbers for i and j integers with $1 \leq i \leq m$ and $1 \leq j \leq n$. The numbers a_{ij} are called the entries *of the matrix A.*

Note that for an entry a_{ij}, the subscript ij describes the location of a_{ij} in the matrix A: i gives the row, and j gives the column.

We can also think of a matrix as a "vector of vectors" in two different ways. If we imagine that the columns of A are vectors in $\mathbb{R}^n$, then the matrix of A can be viewed as a vector of column vectors. If we imagine that the rows of A are vectors in $\mathbb{R}^n$, then the matrix A can be viewed as a vector of row vectors.

Matrix Operations

Here, we'll define matrix addition and matrix multiplication.

In order to be able to add two matrices, they need to have the exact same dimensions. That is, they both need to be $m \times n$ matrices for some fixed values of m and n. When we have two matrices with the same dimensions, we define their sum component-wise or entry-wise.

Learning outcomes:
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Definition 11. *Let A and B be two $m \times n$ matrices, with*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}.$$

Then we define the matrix sum $A + B$ *to be*

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn}. \end{pmatrix},$$

That is, $A+B$ is the $m \times n$ matrix obtained by adding the corresponding entries of A and B.

Example 11. *We can add the matrices $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$ as follows:*

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} \\ &= \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix} \end{aligned}$$

Example 12. *We cannot add the matrices $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 \\ 10 & 11 \end{pmatrix}$, because their dimensions don't match.*

As you might expect, matrix addition has some nice properties which are inherited from addition of real numbers. We list some of them here.

Proposition 8. *Let A, B, and C be $m \times n$ matrices. Then we have*

(a) $A + B = B + A$ *(matrix addition is* commutative*);*

(b) $A + (B + C) = (A + B) + C$ *(matrix addition is* associative*).*

Furthermore, there is an $m \times n$ matrix O, called the zero matrix*, such that $A + O = A$ for any $m \times n$ matrix A. All of the entries of the zero matrix are the real number 0.*

We've seen that matrix addition works in a very natural way, and multiplying a matrix by a scalar (or real number) is similarly nice. We now define scalar multiplication for matrices.

Definition 12. *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

be an $m \times n$ matrix, and let $k \in \mathbb{R}$ be a scalar. Then the scalar product *of k and A, denoted kA, is*

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}.$$

That is, we obtain the scalar product by multiplying each entry in A by the scalar k.

Example 13. *We can compute the scalar product of 2 and the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ as follows:*

$$\begin{aligned} 2A &= 2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 2\cdot 1 & 2\cdot 2 & 2\cdot 3 \\ 2\cdot 4 & 2\cdot 5 & 2\cdot 6 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}. \end{aligned}$$

We now list some nice properties of scalar multiplication.

Proposition 9. *Let A and B be $m \times n$ matrices, and let k and l be scalars in $\mathbb{R}$. Then*

(a) $(k+l)A = kA + lA$ *(scalar multiplication is* distributive *over scalar addition);*

(b) $k(A+B) = kA + kB$ *(scalar multiplication is* distributive *over matrix addition);*

(c) $k(lA) = (kl)A = l(kA)$.

We'll now define matrix multiplication, which can be a bit trickier to work with than matrix addition or scalar multiplication. Here are some important things to remember about matrix multiplication:

- Not all matrices can be multiplied. In order to compute the product AB of two matrices A and B, the number of columns in A needs to be the same as the number of rows in B.
- Matrix multiplication is *not* commutative. In fact, its possible that the matrix product AB exists but the product BA does not.

Definition 13. *Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix, with*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}.$$

Note that we are assuming the number of columns in A is the same as the number of rows in B.

We define the matrix product *of A and B, denoted AB, to be the $m \times p$ matrix*

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + a_{12}b_{2p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + a_{22}b_{2p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + a_{m2}b_{2p} + \cdots + a_{mn}b_{np} \end{pmatrix}$$

Equivalently, we could define the ijth entry of AB to be the dot product of the ith row of A with the jth column of B. This makes sense, since the number of columns in A is the same as the number of rows in B (both n), which ensures that the ith row of A and the jth column of B are both vectors in $\mathbb{R}^n$.

This definition can seem a bit convoluted, and it's easier to understand how matrix multiplication works by going through an example.

Example 14. *We can compute the product AB of the matrices $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$*

and $B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$ *as follows:*

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}, \\ &= \begin{pmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{pmatrix}, \\ &= \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}. \end{aligned}$$

Note that we multiplied a 2×3 *matrix by a* 3×2 *matrix, and we obtained a* 2×2 *matrix.*

We can also compute the product BA *for the same matrices as above.*

$$\begin{aligned} BA &= \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \\ &= \begin{pmatrix} 7 \cdot 1 + 8 \cdot 4 & 7 \cdot 2 + 8 \cdot 5 & 7 \cdot 3 + 8 \cdot 6 \\ 9 \cdot 1 + 10 \cdot 4 & 9 \cdot 2 + 10 \cdot 5 & 9 \cdot 3 + 10 \cdot 6 \\ 11 \cdot 1 + 12 \cdot 4 & 11 \cdot 2 + 12 \cdot 5 & 11 \cdot 3 + 12 \cdot 6 \end{pmatrix}, \\ &= \begin{pmatrix} 39 & 54 & 69 \\ 49 & 68 & 81 \\ 59 & 82 & 105 \end{pmatrix}. \end{aligned}$$

Note that we multiplied a 3×2 *matrix by a* 2×3 *matrix, and we obtained a* 3×3 *matrix.*

Note that in this case $AB \neq BA$*; matrix multiplication is not commutative, so the order of the matrices matters!*

Although matrix multiplication is not commutative, it still has some nice algebraic properties. We list some of them here.

Proposition 10. *Let* A*,* B*, and* C *be matrices of dimensions such that the following operations are defined, and let* k *be a scalar. Then*

(a) $A(BC) = (AB)C$ *(matrix multiplication is* associative*);*

(b) $k(AB) = (kA)B = A(kB)$*;*

(c) $A(B + C) = AB + AC$*;*

(d) $(A + B)C = AC + BC$ *(with the previous property, matrix multiplication is* distributive *over matrix addition).*

Determinants

When we have a square matrix (meaning an $n \times n$ matrix, where the number of rows and number of columns are the same), we can compute an important number, called the determinant of the matrix. It turns out that this single number can tell us some important things about the matrix!

We begin by defining the determinant of a 2×2 matrix.

Definition 14. *Consider the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define the* determinant *of the matrix A to be*

$$det(A) = ad - bc.$$

We also sometimes use the notation $|A|$ for the determinant of the matrix A.

Note that the determinate of a 2×2 matrix is just a number, not a matrix. We compute the determinant in a couple of examples.

Example 15. *We'll compute the determinant of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.*

$$\begin{aligned} det(A) &= 1 \cdot 4 - 2 \cdot 3 \\ &= -2. \end{aligned}$$

We've defined the determinant of 2×2 matrices, but we haven't defined the determinant of a larger square matrix yet. It turns out that the determinant is defined *inductively*. This means that the determinant of a 3×3 matrix is defined using determinants of 2×2 matrices, the determinant of a 4×4 matrix is defined using determinants of 3×3 matrices, the determinant of a 5×5 matrix is defined using determinants of 4×4 matrices, and so on. This means in order to compute the determinant of a large square matrix, we often need to compute the determinants of many smaller matrices.

We now give the definition of the determinant of an $n \times n$ matrix.

Definition 15. *Let A be an $n \times n$ matrix, with entries a_{ij}. We defined the* determinant *of A to be the number computed by*

$$det(A) = (-1)^{1+1} a_{11} det(A_{11}) + (-1)^{1+2} a_{12} det(A_{12}) + \cdots + (-1)^{1+n} a_{1n} det(A_{1n}),$$

where A_{ij} is the $(n-1) \times (n-1)$ submatrix of A which we obtain by deleting the ith row and $j$the column from A.

This definition is pretty confusing if you read through it without seeing an example, but this actually follows a nice pattern. This pattern is easier to see with an example.

Example 16. *We compute the determinant of the 4×4 matrix,*

$$A = \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that we begin by writing this in terms of determinants of 3×3 matrices. But in order to compute the determinant of each 3×3 matrix, we write it in terms of 2×2 matrices! This winds up being a lot of determinants to compute.

$$\begin{aligned} det(A) = (-1)^{1+1} 1 det \begin{pmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{1+2} 4 det \begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ + (-1)^{1+3} 2 det \begin{pmatrix} 0 & 0 & 1 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{1+4} (-1) det \begin{pmatrix} 0 & 0 & -2 \\ -3 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

We now compute the determinant of each of the 3×3 submatrices, which we *compute using determinants of* 2×2 *matrices.*

$$\begin{aligned}
det\begin{pmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (-1)^{1+1}0det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2}(-2)det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + (-1)^{1+3}1det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= 1 \cdot 0 \cdot (1 \cdot 1 - 0 \cdot 0) + -1 \cdot (-2) \cdot (0 \cdot 1 - 0 \cdot 0) + 1 \cdot 1 \cdot (0 \cdot 0 - 1 \cdot 0) \\
&= 0 \\
det\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (-1)^{1+1}0det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2}(-2)det\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + (-1)^{1+3}1det\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \\
&= 1 \cdot 0 \cdot (1 \cdot 1 - 0 \cdot 0) + -1 \cdot (-2) \cdot (-3 \cdot 1 - 0 \cdot 0) + 1 \cdot 1 \cdot (-3 \cdot 0 - 1 \cdot 0) \\
&= 6 \\
det\begin{pmatrix} 0 & 0 & 1 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (-1)^{1+1}0det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2}0det\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + (-1)^{1+3}1det\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \\
&= 1 \cdot 0 \cdot (0 \cdot 1 - 0 \cdot 0) + -1 \cdot 0 \cdot (-3 \cdot 1 - 0 \cdot 0) + 1 \cdot 1 \cdot (-3 \cdot 0 - 0 \cdot 0) \\
&= 0 \\
det\begin{pmatrix} 0 & 0 & -2 \\ -3 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} &= (-1)^{1+1}0det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-1)^{1+2}0det\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \\
&\quad + (-1)^{1+3}(-2)det\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \\
&= 1 \cdot 0 \cdot (0 \cdot 0 - 1 \cdot 0) + -1 \cdot 0 \cdot (-3 \cdot 0 - 1 \cdot 0) + 1 \cdot (-2) \cdot (-3 \cdot 0 - 0 \cdot 0) \\
&= 0
\end{aligned}$$

Substituting these in to our computation of the determinant of A*, we then obtain*

$$\begin{aligned}
det(A) &= 1 \cdot 1 \cdot 0 + (-1) \cdot 4 \cdot (6) + 1 \cdot 2 \cdot 0 + (-1) \cdot (-1) \cdot 0 \\
&= -24.
\end{aligned}$$

We sometimes call this method of computing a determinant as “expanding along the first row.” This is because we can also compute the determinant of a matrix by similarly expanding along a different row, or even a column.

Proposition 11. *We can similarly compute the determinant of an* $n \times n$ *matrix*

A by expanding along any row or column. Expanding along the ith row, we have

$$det(A) = (-1)^{i+1} a_{i1}\, det(A_{i1}) + (-1)^{i+2} a_{i2}\, det(A_{i2}) + \cdots + (-1)^{i+n} a_{in}\, det(A_{in}).$$

Expanding along the jth column, we have

$$det(A) = (-1)^{1+j} a_{1j}\, det(A_{1j}) + (-1)^{2+j} a_{2j}\, det(A_{2j}) + \cdots + (-1)^{n+j} a_{nj}\, det(A_{nj}).$$

Once again, A_{ij} is the $(n-1)\times(n-1)$ submatrix obtained from A by deleting the ith row and jth column.

It can be useful to think about which row or column will be easiest to expand along. In particular, choosing a row or column with a lot of zeros greatly simplifies computation.

Example 17. *We'll once again compute the determinant of the 4×4 matrix*

$$A = \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

this time by expanding along the second column. Note that this column is a good choice, since there's only one nonzero element. We have

$$det(A) = (-1)^{1+2}(4)\, det\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{2+2}(0)\, det\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{3+2}(0)\, det\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We'll only compute the determinant of the submatrix $\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; we won't bother computing the others since their determinants will be multiplied by 0.

$$\begin{aligned} det\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= (-1)^{3+1}(0)\, det\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} + (-1)^{3+2}(0)\, det\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} + (-1)^{3+3}(1)\, det\begin{pmatrix} 0 & -2 \\ -3 & 1 \end{pmatrix} \\ &= 0 + 0 + (1)(1)(0\cdot 1 - (-2)\cdot(-3)), \\ &= -6. \end{aligned}$$

Once again, we don't bother computing the determinants which will be multiplied by zero. Note that we chose to expand across the last row, since it had two zeroes. Expanding along the first column would also have been a reasonable choice.

Returning to our computation of the determinant of A, we have

$$\begin{aligned} det(A) &= (-1)^{1+2}(4)\, det\begin{pmatrix} 0 & -2 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{2+2}(0)\, det\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (-1)^{3+2}(0)\, det\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (-1)(4)(-6) + 0 + 0 + 0, \\ &= 24. \end{aligned}$$

Notice that this matching our previous computation, expanding along the first row.

One of the most powerful uses of the determinant is to tell us whether or not a matrix is invertible. Recall that an $n \times n$ matrix A is *invertible* if there is a matrix B such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.

Proposition 12. *An $n \times n$ matrix A is invertible if and only if its determinant is nonzero.*

This gives us a convenient way to test if a matrix is invertible, without needing to produce an explicit inverse.

Example 18. *We found that the determinant of the matrix*

$$A = \begin{pmatrix} 1 & 4 & 2 & -1 \\ 0 & 0 & -2 & 1 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is 24*. Since this is nonzero, the matrix A is invertible.*

On the other hand, you can verify that the determinant of the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & -2 \\ -3 & 1 & -1 & 0 \\ -1 & 3 & 1 & 2 \end{pmatrix}$$

is 0*. Thus, the matrix B is not invertible.*

Linear Transformations

One of the most important uses of matrices is to represent linear transformations. Recall the definition of a linear transformation.

Definition 16. *A function T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a linear transformation if for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ and all scalars $k \in \mathbb{R}$, we have*

(a) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$;

(b) $T(k\vec{v}) = kT(\vec{v})$.

We can view an $m \times n$ matrix A as representing a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ as follows. We write vectors as column vectors, or, equivalently, $n \times 1$ or $m \times 1$ matrices. For an input column vector $\vec{v}$ in $\mathbb{R}^n$, we multiply $\vec{v}$ by A

on the left, using matrix multiplication. This produces an $m \times 1$ matrix, or, equivalently, a column vector in $\mathbb{R}^m$. Thus, we can define a function

$$T_A(\vec{v}) = A\vec{v}.$$

Using properties of matrix multiplication, we have that this is a linear transformation. Thus, we have the linear transformation associated to a matrix.

Example 19. *Consider the linear transformation T_A from $\mathbb{R}^3$ to $\mathbb{R}^2$ corresponding to the 2×3 matrix*

$$A = \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix}.$$

Let investigate the images of several vectors in $\mathbb{R}^3$ under the linear transformation T_A.

$$\begin{aligned} T_A((1,2,3)) &= \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + -5 \cdot 2 + 3 \cdot 3 \\ 2 \cdot 1 + 0 \cdot 2 + -1 \cdot 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T_A((1,-1,2)) &= \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + -5 \cdot -1 + 3 \cdot 1 \\ 2 \cdot 1 + 0 \cdot -1 + -1 \cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T_A((1,0,0)) &= \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + -5 \cdot 0 + 3 \cdot 0 \\ 2 \cdot 1 + 0 \cdot 0 + -1 \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

$$T_A((0,1,0)) = \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \cdot 0 + -5 \cdot 1 + 3 \cdot 0 \\ 2 \cdot 0 + 0 \cdot 1 + -1 \cdot 0 \end{pmatrix}$$
$$= \begin{pmatrix} -5 \\ 0 \end{pmatrix}$$

$$T_A((0,0,1)) = \begin{pmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \cdot 0 + -5 \cdot 0 + 3 \cdot 1 \\ 2 \cdot 0 + 0 \cdot 0 + -1 \cdot 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Notice that when we apply the linear transformation to the standard unit vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$, we obtain the columns of A as the output vector. This observation can be used to reconstruct a matrix from a given linear transformation.

Proposition 13. *Given any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$, there is an $m \times n$ matrix such that $T = T_A$.*

Furthermore, the columns of A can be obtained by applying T to the standard unit vectors. More specifically, the jth column of A is given by $T(\vec{e}_j)$.

We can see how this is useful through an example.

Example 20. *Consider the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates a vector by 30° counterclockwise. We can see geometrically that, for the standard unit vectors $\vec{e}_1$ and $\vec{e}_2$ in $\mathbb{R}^2$, we have*

$$T((1,0)) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right),$$
$$T((0,1)) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

These tell us the columns of the matrix corresponding to the linear transformation, so we then know that the rotation can be represented by the matrix

$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Summary

In this section, we reviewed matrix operations and properties, determinants, and linear transformations.

Although we've reviewed some of the most important concepts from linear algebra, there is still a lot of material that we weren't able to include here. Make sure you refer back to your linear algebra textbook if there's anything else you need to review!

Representations of Lines and Planes

In this section, we review the different ways we can represent lines and planes, including parametric representations.

Representations of Lines

When you think of describing a line algebraically, you might think of the standard form

$$y = mx + b,$$

where m is the slope and b is the y-intercept. This is often called *slope-intercept* form.

In addition to slope-intercept form, there are several other ways to represent lines. For example, you may remember using *point-slope* form in single variable calculus. We can describe a line of slope m going through a point (x_0, y_0) with the equation

$$y - y_0 = m(x - x_0).$$

It's important to note that there are many different possible choices for the point (x_0, y_0). Because of this, unlike slope-intercept form, point-slope form does not give a unique representation of a line.

In linear algebra, we saw that we could parametrize a line using a vector $\vec{v} = (v_1, v_2)$ giving the direction of the line, and a point (x_0, y_0) that the line passes through. We parametrize the line as

$$\begin{aligned} \vec{x}(t) &= (v_1, v_2)t + (x_0, y_0), \\ &= (v_1 t + x_0, v_2 t + y_0). \end{aligned}$$

Note that this representation works a bit differently from the previous two representations. In slope-intercept form and point-slope form, the line was the set of points (x, y) satisfying the given equation. However, in the parametrization, we plug in values for the parameter t in order to get points on the line.

Unlike slope-intercept form and point-slope form, the parametrization of a line can easily be generalized to three or more dimensions. That is, a line in $\mathbb{R}^n$ through the point $\vec{a}$ and in the direction of the vector $\vec{v}$ can be parametrized as

$$\vec{x}(t) = \vec{v}t + \vec{a},$$

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for $t \in \mathbb{R}$.

If we would like to describe a line in higher dimensions using equations (rather than a parametrization), we would need more than one equation. For example, in $\mathbb{R}^3$, we would require two equations to determine a line.

Representations of Planes

We also have multiple ways to represent planes. Here, we'll focus on planes in $\mathbb{R}^3$.

Recall that a plane can be determined by two vectors (giving the "direction" of the plane) and a point that the plane passes through. We can use this to give a parametrization for the plane through the point $\vec{a}$ and parallel to the vectors $\vec{v}$ and $\vec{w}$:

$$\vec{x}(s,t) = \vec{v}s + \vec{w}t + \vec{a},$$

for s and t in $\mathbb{R}$. Note that we require two parameters for the parametrization of the plane.

We can also describe a plane using a single linear equation in x, y, and z. For example,

$$2x + 4y - z = 9$$

defines a plane. A standard way to do this is using a point on the plane and a normal vector to the plane. Recall that a normal vector is perpendicular to every vector in the plane. If $\vec{n} = (n_1, n_2, n_3)$ is a normal vector to a plane passing through the point $\vec{a} = (a_1, a_2, a_3)$, the plane is defined by the equation

$$\vec{n} \cdot (\vec{x} - \vec{a}) = 0.$$

This can be rewritten as

$$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0.$$

Summary

We reviewed various representations of lines and planes, including parametrizations.

Part II

Coordinate Systems and Functions

Review of Coordinate Systems

In this activity, we review coordinate systems that you've seen before, in preparation for introducing new coordinate systems in subsequent sections.

Cartesian Plane

The coordinates that you're probably most comfortable with are standard two-dimensional coordinates, also called Cartesian coordinate system on the plane.

In Cartesian coordinates, we describe a point using an x-coordinate and a y-coordinate. We write a point as (x, y), where the x-coordinate describes the horizontal displacement of the point, and the y-coordinate describes the vertical displacement of the point.

Learning outcomes:
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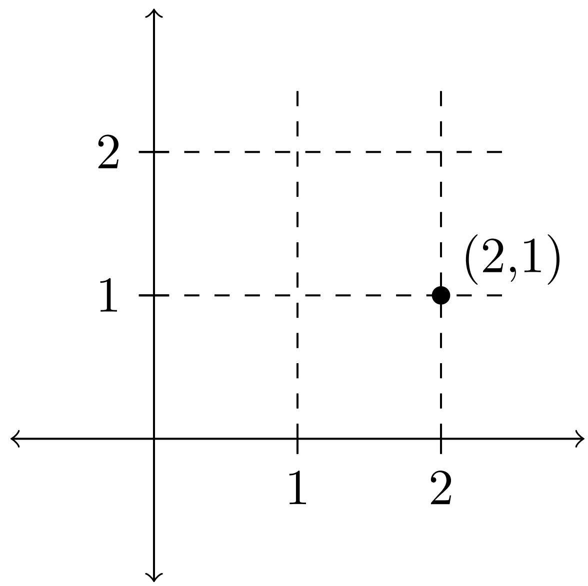


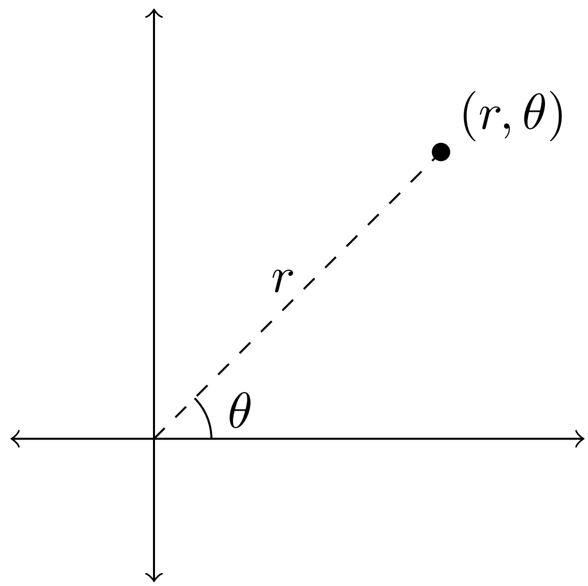
Polar Coordinates

You've also seen polar coordinates.

In polar coordinates, we describe a point with an r-coordinate and a θ-coordinate. The r coordinate gives the distance between the point and the origin, and the θ-coordinate gives the angle (in radians) between the positive x-axis and the segment connecting the origin and the point.

We can switch between cartesian and polar coordinates using the equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$



Problem 1 *Write the point $(r, \theta) = (5, \pi/3)$ in cartesian coordinates.*

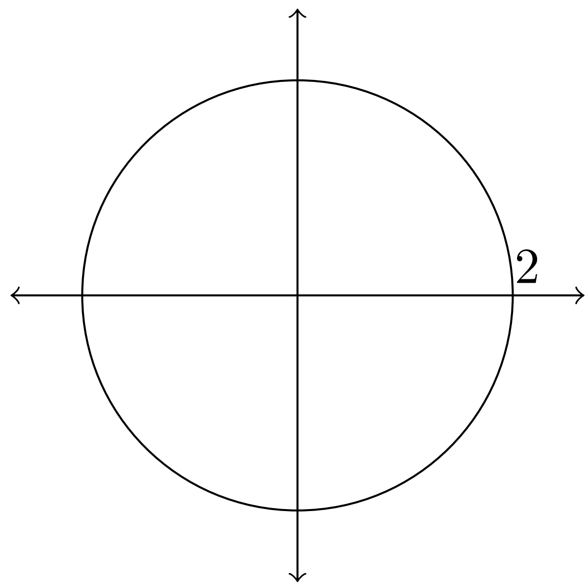
$$(x, y) = \boxed{(5/2, 5\sqrt{3}/2)}$$

Write the point $(x, y) = (-2, 2)$ in polar coordinates.

$$(r, \theta) = \boxed{(\sqrt{8}, 3\pi/4)}$$

Example 21. *Recall that we can describe a circle of radius 2 using Cartesian points as the set of points (x, y) satisfying*

$$x^2 + y^2 = 4.$$



We would like to write describe this circle using polar coordinates.

By definition, the circle of radius 2 *centered at the origin consists of the points which are distance* 2 *from the origin. Because of this, for any point on the circle, we have*

$$r = \boxed{2}.$$

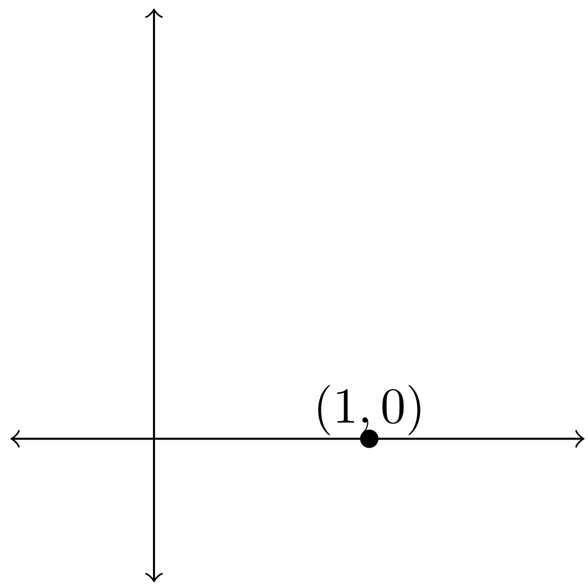
There are points on the circle making every possible angle with the positive x-*axis, so we don't need any restrictions on* θ. *If, however, we only wanted part of the circle, we would accomplish this by restricting* θ. *For example, to get the top half of the circle, we would make the restriction* $0 \leq \theta \leq \pi$.

Thus, in polar coordinates, the circle of radius 2 *centered at the origin can be described as the set of points* (r, θ) *such that*

$$r = 2.$$

There's an important difference between Cartesian coordinates and polar coordinates: Cartesian coordinates are *unique*, while polar coordinates are not. This means that, given a point P in the plane, there's only one way to describe this point as (x, y) using Cartesian coordinates. However, there are many ways to write the point as (r, θ), using polar coordinates.

Take, for example, the point $(1, 0)$, written in Cartesian coordinates.



This point is on the x-axis and is distance 1 from the origin. Thus, perhaps the most obvious way to represent this point in polar coordinates is as $(r, \theta) = (1, 0)$ (coincidentally, the same as in Cartesian coordinates). But we could also describe the angle as 2π, 4π, -2π, etc. So, we could also write the point in polar coordinates as $(r, \theta) = (1, 2\pi)$, and so on.

Perhaps more surprisingly, we can describe this point as $(-1, \pi)$. Imagine making an angle of π with the positive x-axis (so we're on the negative x-axis), then going backwards past the origin. This also gets you to our point. Using equivalent angles, we can also represent the point as $(-1, 3\pi)$, $(-1, -\pi)$, and so on.

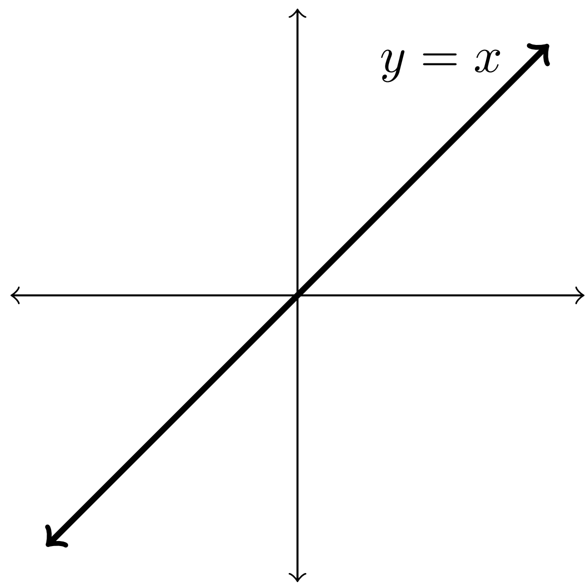
There are some times in working with polar coordinates when we would like to be able to represent points uniquely, and in these situations, we often make restrictions

$$0 \leq r,$$
$$0 \leq \theta < 2\pi.$$

However, even with these restrictions, there still is a point that has multiple representations! Namely, the origin can be written as $(r, \theta) = (0, \theta)$ for any angle θ.

Depending on the situation and context, different people may use different restrictions or conventions for their ranges for r and θ. For this reason, it's good to specify what values you're allowing, to avoid being misunderstood!

Example 22. *Let's consider the line described in Cartesian coordinates as the set of points (x, y) such that $y = x$. We'll figure out how to describe this line in polar coordinates.*



Let's restrict our polar coordinates to $0 \leq r$ *and* $0 \leq \theta < 2\pi$. *Perhaps your first guess is to describe the line as the points* (r, θ) *such that*

$$\theta = \pi/4.$$

Which shape does this describe?

Multiple Choice:

(a) *A point.*

(b) *Half of the line.* ✓

(c) *The whole line.*

(d) *A different line.*

(e) *A circle.*

Describing the line as $\theta = \pi/4$ *is a reasonable first guess, as we can see that many of the points make an angle* $\pi/4$ *with the positive* x*-axis. However, with the restriction that* $r \geq 0$*, this leaves out half of the line! In order to describe the entire line, we have a couple of options. One option would be to relax our restriction on* r*, and allow negative values as well. This would certainly give us the whole line. If, however, we would like to keep this restriction that* $r \geq 0$*, we could also include points with* $\theta = 5\pi/4$*, which will give us the other half of the line.*

Which of the following describe the line $y = x$ *in polar coordinates? Select all that work.*

Select All Correct Answers:

(a) *The points* (r, θ) *such that* $\theta = \pi/4$*, where* $r \geq 0$*.*

(b) *The points* (r, θ) *such that* $\theta = \pi/4$*, where* r *can be any real number.* ✓

(c) *The points* (r, θ) *such that* $\theta = \pi/4$ *or* $\theta = -\pi/4$*, where* $r \geq 0$*.*

(d) *The points* (r, θ) *such that* $\theta = \pi/4$ *or* $\theta = -\pi/4$*, where* r *can be any real number.*

(e) *The points* (r, θ) *such that* $\theta = \pi/4$ *or* $\theta = 5\pi/4$*, where* $r \geq 0$*.* ✓

(f) *The points* (r, θ) *such that* $\theta = \pi/4$ *or* $\theta = 5\pi/4$*, where* r *can be any real number.* ✓

Recall that the relationship between Cartesian and polar coordinates:

$$\begin{aligned} x &= \boxed{r\cos\theta}, \\ y &= \boxed{r\sin\theta}. \end{aligned}$$

Recall the following equations describing the relationship between Cartesian and polar coordinates, which can be useful for converting between these two coordinate systems.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

Example 23. *Consider the set of points* (r, θ) *such that* $r = 2\cos\theta$*. What does this set of points look like?*

It's not very clear from $r = 2\cos\theta$ *what shape this is describing, so let's try converting this to Cartesian coordinates, and see if we get something we recognize.*

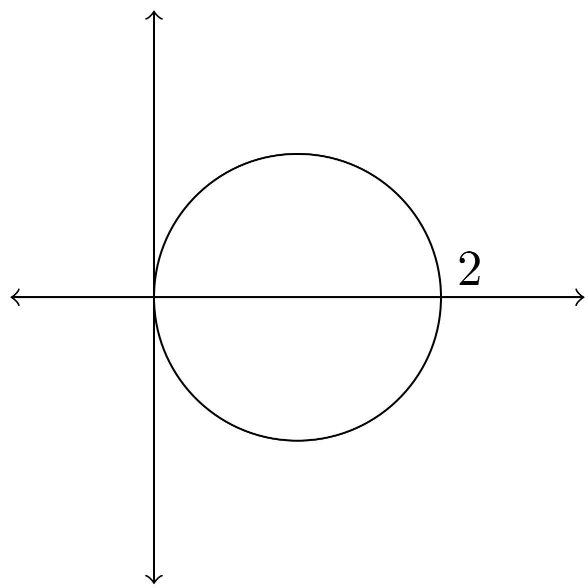
Using the conversion equations above, we have that $r^2 = \boxed{x^2 + y^2}$*, and* $\cos\theta = \dfrac{x}{r}$*. Making substitutions using these facts, we have:*

$$\begin{aligned} r &= 2\cos\theta \\ r &= 2\frac{x}{r} \\ r^2 &= 2x \\ x^2 + y^2 &= 2x \end{aligned}$$

We now have an equation solely in terms of x and y, but maybe it isn't quite recognizable yet. But if we do a bit more algebra...

$$\begin{aligned} x^2 + y^2 &= 2x \\ (x^2 - 2x + 1) + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

Now, we can see that this is a circle of radius $\boxed{1}$ *centered at* $\boxed{(1,0)}$.



Linear Change of Coordinates

In Linear Algebra, we saw how different coordinate systems arose through linear change of coordinates. You may remember this referred to as "slanty space."

When we write a point in Cartesian coordinates as (x, y), we can think of this as a linear combination of the standard basis vectors:

$$(x, y) = x(1, 0) + y(0, 1).$$

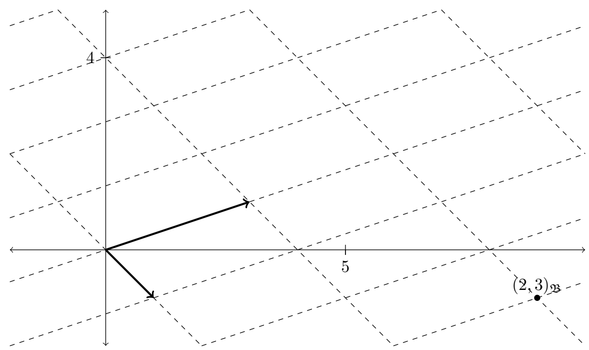
Of course, we can just as well write a point as a linear combination of vectors from a different basis, say $(3, 1)$ and $(1, -1)$. Let's call this basis $\mathfrak{B}$. For example, we can write the vector $(9, -1)$ as

$$(9, -1) = 2(3, 1) + 3(1, -1).$$

Taking the coefficients, in $\mathfrak{B}$-coordinates, we would write this point as

$$(2,3)_{\mathfrak{B}}.$$

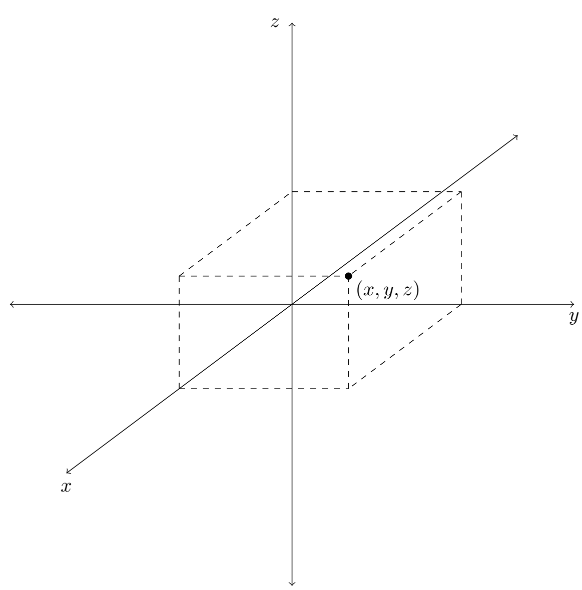
Note that we write $\mathfrak{B}$ in the subscript, in order to remind us that these are $\mathfrak{B}$-coordinates, rather than standard Cartesian coordinates.



With linear changes of coordinates, it's easy to make a mistake and forget which coordinates you're using. Make sure to keep careful track!

Three-Dimensional Coordinates

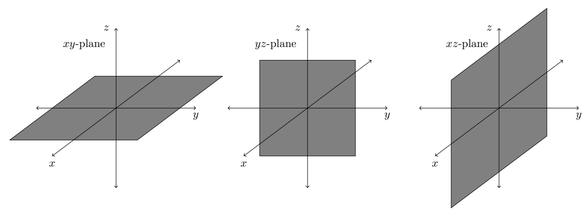
In Linear Algebra, we also worked in three-dimensional Cartesian coordinates, (x, y, z) in $\mathbb{R}^3$.



It's important to remember that the x, y, and z axes follow the right hand rule. That is, if you take your right hand, and point your pointer finger in the direction of the positive x-axis, point your middle finger in the direction of the positive y-axis, then your thumb points in the direction of the positive z-axis.

Another way to say this is that if you point the fingers of your right hand in the direction of the positive x-axis and curl them to point in the direction of the positive y-axis, your thumb points in the direction of the positive z-axis.

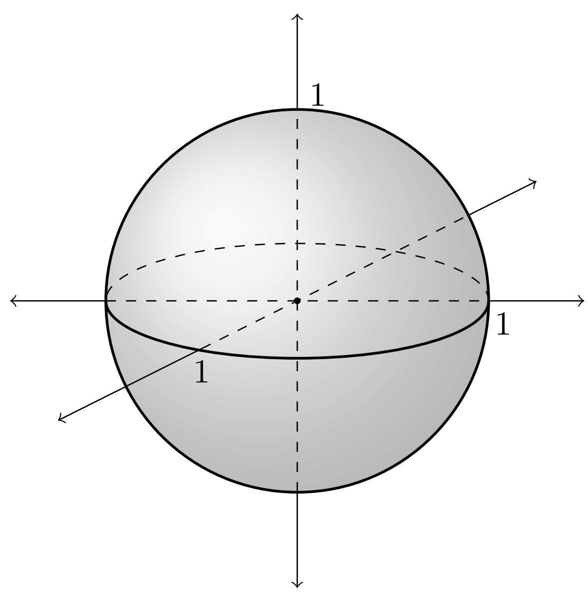
We'll often refer to the *coordinate planes* in $\mathbb{R}^3$. These are the three planes we obtain by setting each of the coordinates to be zero.



More precisely, the xy-plane is the set of points (x, y, z) such that $z = 0$, the yz-plane is the set of points such that $x = 0$, and the xz-plane is the set of points such that $y = 0$.

Similarly to in the plane, we can describe sets of points in $\mathbb{R}^3$ using equations.

Example 24. *The set of points (x, y, z) such that $x^2 + y^2 + z^2 = 1$ is the sphere of radius 1 centered at the origin in $\mathbb{R}^3$.*



Conclusion

In this activity, we reviewed coordinate systems that you've seen before: standard two-dimensional coordinates, polar coordinates, coordinates with respect to a given set of basis vectors, and three-dimensional coordinates.

Cylindrical Coordinates

In this activity, we introduce cylindrical coordinates, a new coordinate system on $\mathbb{R}^3$. We also discuss how to convert between cylindrical and Cartesian coordinates.

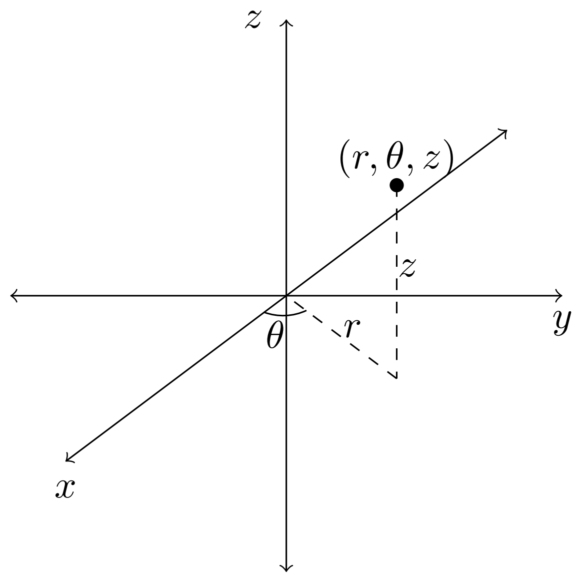
Cylindrical Coordinates

We've seen how points in $\mathbb{R}^2$ can be written using polar coordinates. Polar coordinates can be useful for describing shapes that are difficult to describe in Cartesian coordinates.

We'd now like to extend this idea to $\mathbb{R}^3$, using a coordinate system called *cylindrical coordinates*. Like polar coordinates, cylindrical coordinates will be useful for describing shapes in $\mathbb{R}^3$ that are difficult to describe using Cartesian coordinates. Later in the course, we will also see how cylindrical coordinates can be useful in multivariable Calculus, when differentiating or integrating in Cartesian coordinates is difficult or impossible.

Cylindrical coordinates are really just a simple extension of polar coordinates. For points in the xy-plane, we describe them using r and θ, where r is the distance from the origin and θ is the angle with the positive x-axis. We then tack on a z-coordinate, the exact same as the z-coordinate in Cartesian coordinates, which tells us the vertical displacement of the point.

Learning outcomes:
Author(s):



Example 25. *We'll convert the point $(x, y, z) = (1, 1, 1)$ to cylindrical coordinates.*

We can figure out r and θ by just considering the x- and y-coordinates of the point, $(1, 1)$. Then this becomes equivalent to representing the point in polar coordinates, so we have

$$(r, \theta) = \boxed{(\sqrt{2}, \pi/4)}.$$

For last coordinate, z, notice that this is telling us the height of the point, which is the exact same as the z-coordinate of the point written in Cartesian coordinates! So, our z coordinate is $\boxed{1}$, and the point $(x, y, z) = (1, 1, 1)$ can be written in cylinderical coordinates as

$$(r, \theta, z) = \boxed{(\sqrt{2}, \pi/4, 1)}.$$

You may use the applet below to experiment with how changing the different coordinates changes the point given in cylindrical coordinates.

MATH INSIGHT APPLET

Uniqueness

When we studied polar coordinates, we saw that there were many different ways to represent a point. For example, the point $(x, y) = (0, 1)$ could be written as $(r, \theta) = (1, \pi/2)$, $(1, 5\pi/2)$, or even $(-1, 3\pi/2)$. And the origin was especially devious, it could be written as $(0, \theta)$ for any angle θ.

Because of this and the relationship between polar and cylindrical coordinates, it's not surprisingly that cylindrical coordinates have similar issues with uniqueness. For example, the point $(0, 1, 1)$ can be written as $(r, \theta, z) = (1, \pi/2, 1)$, $(1, 5\pi/2, 1)$, $(-1, 3\pi/w, 1)$, and so on. Any point on the z-axis can be written as $(0, \theta, z)$, where z is its z-coordinate, and θ is any angle.

Problem 2 *Which of the following, written in cylindrical coordinates, is equivalent to the point $(x, y, z) = (1, 1, 1)$? Select all that apply.*

Select All Correct Answers:

(a) $(1, 1, 1)$

(b) $(1, \pi/4, 1)$

(c) $\sqrt{2}, \pi/4, 1)$ ✓

(d) $(-1, 3\pi/4, 1)$

(e) $-\sqrt{2}, \pi/4, 1)$

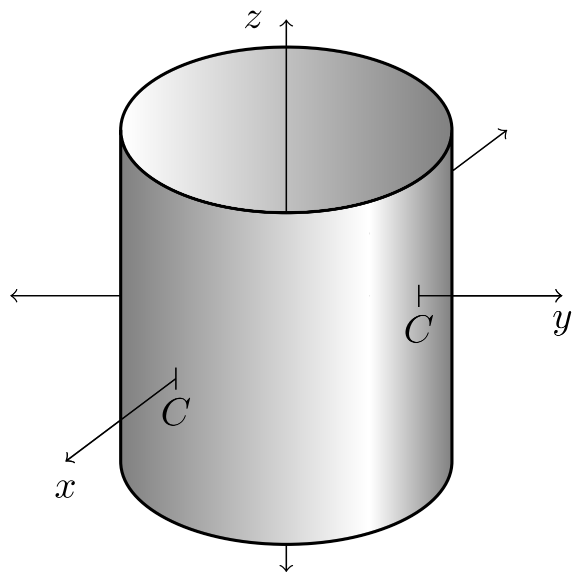
(f) $-\sqrt{2}, -3\pi/4, 1)$ ✓

As with polar coordinates, in situations where uniqueness is important, we will often make the restrictions $r \geq 0$ and $0 \leq \theta < 2\pi$.

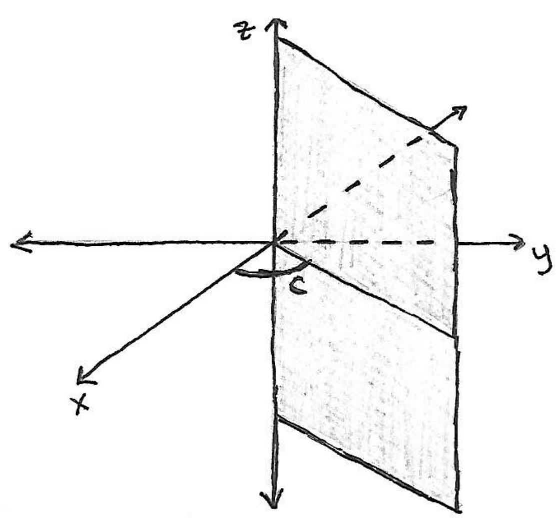
Constant-Coordinate Surfaces

Let's look at what happens in cylindrical coordinates when we set each of the coordinates r, θ, z to be constant, with the standard restrictions that $0 \leq r$ and $0 \leq \theta \leq \pi/2$. This can give us insight to how cylindrical coordinates behave.

We'll begin by examining the set of points (r, θ, z), where $r = C$ is a constant. We have that $r = C$ is constant, which means that the distance between any such point and the z axis is constant, C. Also, θ and z can be anything. This will give us the cylinder of radius C, centered at the z-axis.

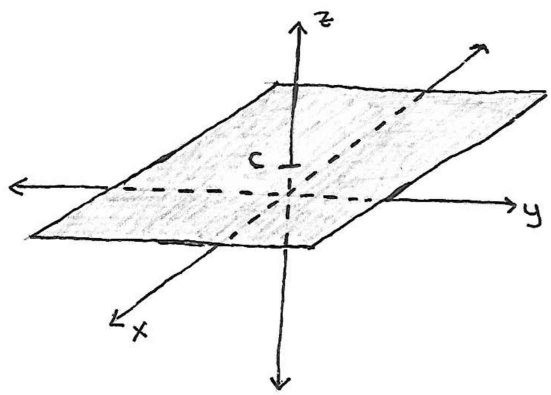


Next, we'll investigate the set of points (r, θ, z), where $\theta = C$ is constant. Let's consider the projection of this point onto the xy-plane. The projection will make an angle C with the positive x-axis, and have distance $r \geq 0$ from the origin. The height of the point can be any real number. From these observations, we conclude that the set of such points is the following half plane in $\mathbb{R}^3$.



Note that if we didn't have the restriction $r \geq 0$, we would get an entire plane rather than a half plane.

Finally, we'll consider the set of points (r, θ, z), where $z = C$ is constant. Since $z = C$, we will only have points at height C. Varying r and θ will then give us all points in the plane at height C parallel to the xy-plane, as below.



Converting between Cartesian and cylindrical coordinates

Perhaps not surprisingly, converting between Cartesian coordinates and cylindrical coordinates is very similar to how we converted between Cartesian coordinates and polar coordinates. That is, we can use the equations:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z, \\ r^2 &= x^2 + y^2, \\ \tan\theta &= \frac{y}{x}. \end{aligned}$$

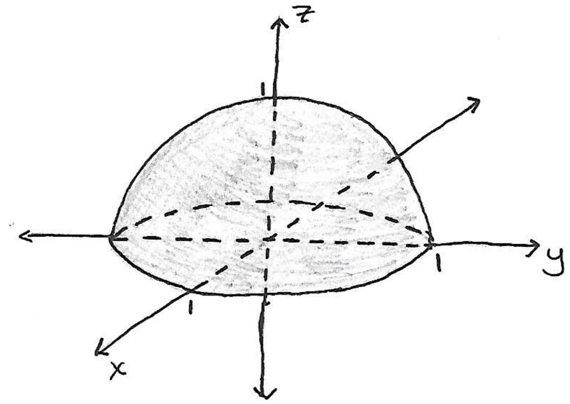
Example 26. *We'll convert $z = \sqrt{1 - r^2}$ to Cartesian coordinates.*
Using $r^2 = x^2 + y^2$, we have

$$z = \sqrt{1 - x^2 - y^2}.$$

You may recognize this as the top half of the sphere of radius 1 *centered at the origin. You could also rewrite this as*

$$x^2 + y^2 + z^2 = 1,$$

keeping in mind that $z \geq 0$.



Example 27. *We'll convert* $(x-2)^2 + y^2 = 1$ *(where* z *can be anything) to cylindrical coordinates. Note that this is the cylinder of radius 1, centered at the vertical line through* $(2, 0, 0)$.

Expanding the expression, we have

$$x^2 - 4x + 1 + y^2 = 1.$$

Substituting $r^2 = x^2 + y^2$ *and subtracting* 1 *from each side, we obtain*

$$r^2 - 4x = 0.$$

We then substitute $x = r\cos\theta$.

$$r^2 - 4r\cos\theta = 0.$$

Dividing both sides by r, *we have*

$$r - 4\cos\theta = 0,$$

or

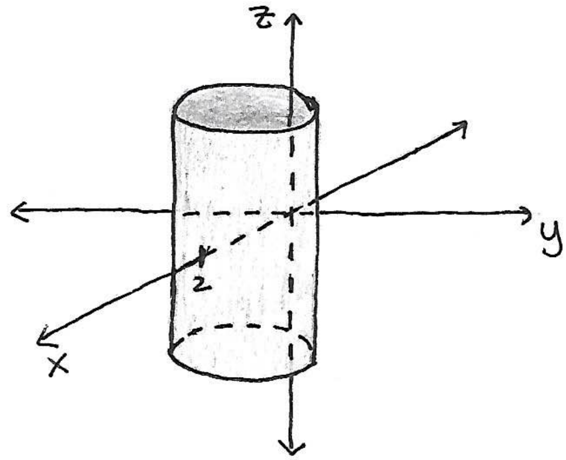
$$r = 4\cos\theta,$$

and z *can be anything.*

When we divided by r, *we implicitly assumed that* r *was not* 0. *This means that we might accidentally be omitting the origin, but if we take* $\theta = \pi/2$, *we have*

$$r = 4\cos(0) = 0,$$

so the origin is already included in the surface $r = 4\cos\theta$.



Conclusion

We introduced cylindrical coordinates and how to convert between cylindrical coordinates and Cartesian coordinates, and we discussed the uniqueness of cylindrical coordinates.

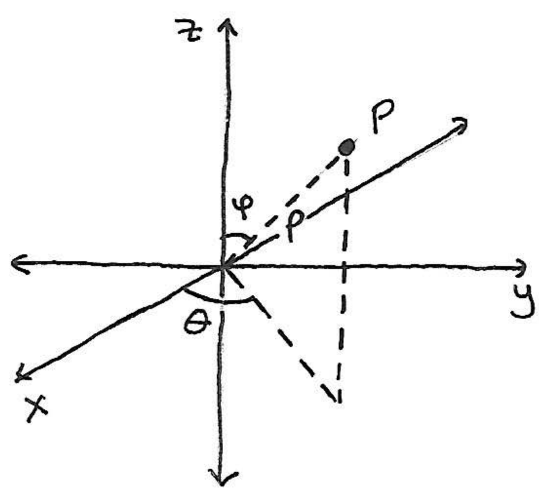
Spherical Coordinates

In this activity, we introduce spherical coordinates, a new coordinate system on $\mathbb{R}^3$. We also discuss how to convert between spherical and Cartesian coordinates.

Spherical Coordinates

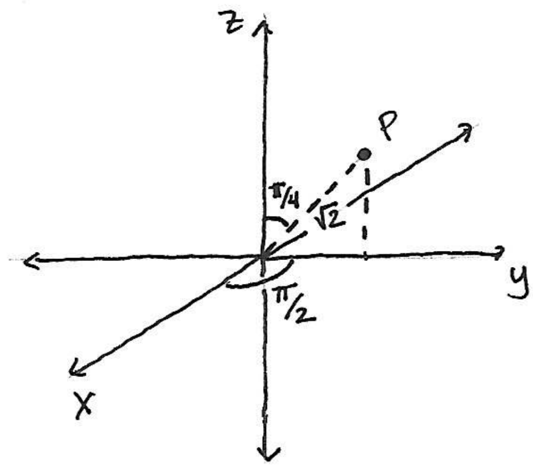
We've seen how to express points in $\mathbb{R}^3$ using Cartesian coordinates and using cylindrical coordinates. We'll now introduce a new coordinate system, called *spherical coordinates*.

Given a point P in $\mathbb{R}^3$, imagine drawing a line segment from the origin to P. In spherical coordinates, we write P as (ρ, θ, ϕ). Here, ρ is the length of the segment (also the distance between P and the origin). The second coordinate, θ, is angle between the positive x-axis and the projection of the segment onto the xy-plane. The third coordinate, ϕ, is the angle between the segment and the positive z-axis.



Example 28. *We'll write the point P in spherical coordinates, where P is given by $(x, y, z) = (0, 1, 1)$ in Cartesian coordinates.*

Learning outcomes:
Author(s):



The distance between P and the origin is

$$\sqrt{0^2 + 1^2 + 1^2} = \boxed{\sqrt{2}},$$

so $\rho = \boxed{\sqrt{2}}$.

The angle between the positive x-axis and the projection of P onto the xy-plane is $\boxed{\pi/2}$ *(in radians), so* $\theta = \boxed{\pi/2}$.

The angle between P and the positive z-axis is $\boxed{\pi/4}$ *(in radians), so* $\phi = \boxed{\pi/4}$.

Thus we can write P in spherical coordinates as $\boxed{(\sqrt{2}, \pi/2, \pi/4)}$.

Although we will be consistent with our definitions of θ and ϕ as above, it's important to know that some people reverse the roles of θ and ϕ. This is particularly common among physicists.

Uniqueness

As with polar and cylindrical coordinates, there are issues of uniqueness with spherical coordinates that we do not encounter in Cartesian coordinates.

Let's take for the example the point $(x, y, z) = (0, 1, 1)$, written in Cartesian coordinates. We've seen the canonical way to write this point in spherical coordinates, as $(\sqrt{2}, \pi/2, \pi/4)$. However, we could also write this as $(\sqrt{2}, 5\pi/2, \pi/4)$, $(\sqrt{2}, -3\pi/2, \pi/4)$, or even $(-\sqrt{2}, 3\pi/2, 5\pi/4)$.

Because of this issue, we'll commonly use the restrictions

$$\begin{aligned}
0 \leq &\rho \\
0 \leq &\theta < 2\pi \\
0 \leq &\phi \leq \pi
\end{aligned}$$

when working with spherical coordinates in order to improve the uniqueness situation. Unfortunately, there are still multiple ways to represent the origin in spherical coordinates.

Problem 3 *Which of the following represent the origin in spherical coordinates? Select all that apply.*

Select All Correct Answers:

(a) $(0, 0, 0)$ ✓

(b) $(0, \pi/2, 0)$ ✓

(c) $(0, 0, \pi/4)$ ✓

(d) $(0, \pi/2, \pi/4)$ ✓

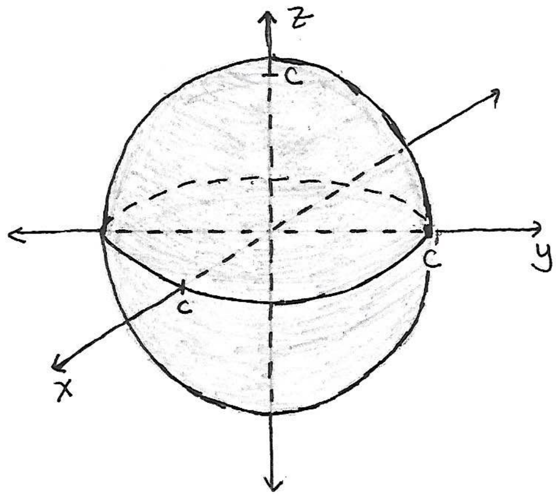
You may use the following applet to experiment with how the different coordinates change a point written in spherical coordinates.

MATH INSIGHT APPLET

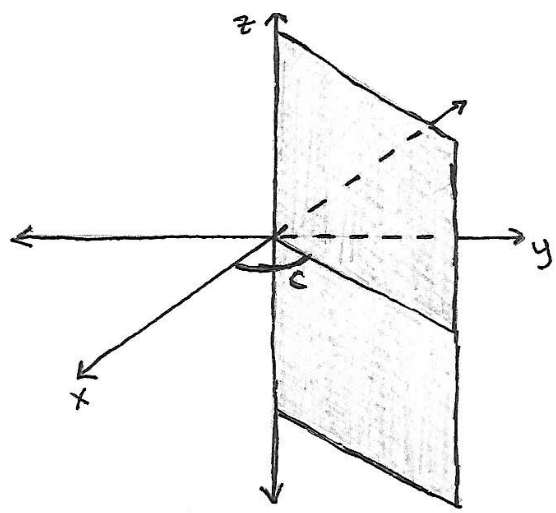
Constant-Coordinate Surfaces

As we did with cylindrical coordinates, we'll see what happens when we set each of the coordinates to be constant.

Consider the set of points (ρ, θ, ϕ), where $\rho = C$ is constant. This means that the distance between the origin and any such point is C. Varying the angles θ and ϕ gives us all such points, which make a sphere of radius C.



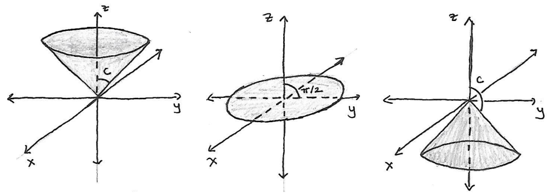
Now, consider the set of points (ρ, θ, ϕ), where $\theta = C$ is constant. This means that the projection of any such point onto the xy-plane will make an angle C with the positive x-axis. Varying ρ gives us points at various distances from the origin, and varying ϕ gives us points making various angles with the positive z-axis. With the restrictions $\rho \geq 0$ and $0 \leq \phi \leq \pi$, we obtain a half plane, as below.



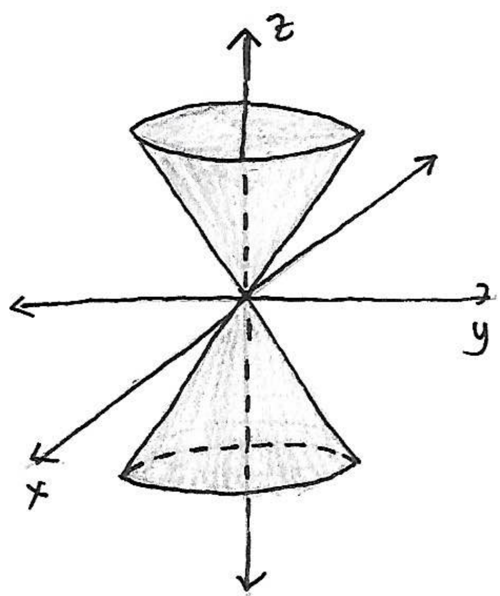
Notice that if we relaxed the restrictions on ρ and ϕ, we could obtain the entire plane.

Finally, we consider the set of points (ρ, θ, ϕ), where $\phi = C$ is constant. This means that every such point has an angle C with the positive z-axis. Varying

ρ and θ, with the restriction $\rho \geq 0$, we get the surfaces below, depending on if $C < \pi/2$, $C = pi/2$, or $C > pi/2$.



Looking at the surfaces when $C > \pi/2$ or $C < \pi/2$, we would commonly call these surfaces "cones." However, in most mathematics, "cone" is more commonly used to describe the surface below, which you might call a double cone.



Note that if you relax the restriction $\rho \geq 0$, you'll get cone (or double cone) above when $C \neq 0$.

It may seem strange that mathematicians prefer this double cone to the seemingly simpler cones that you're used to. However, it turns out that the double cone is easier to describe algebraically.

You can use the following applet to see what happens when you vary the value of the constant C for each of the constant-coordinate surfaces above:

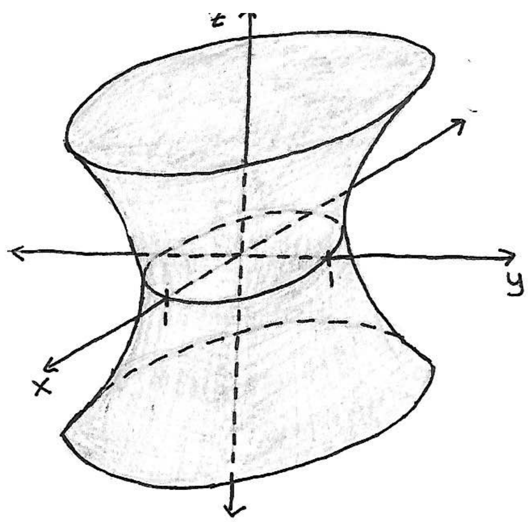
MATH INSIGHT APPLET

Converting Between Spherical and Cartesian Coordinates

When converting between spherical coordinates and Cartesian coordinates, it can be useful to use the following equations, which describe the relationship between the two coordinate systems.

$$\begin{aligned} x &= \rho\cos\theta\sin\phi \\ y &= \rho\sin\theta\sin\phi \\ z &= \rho\cos\phi \\ \rho^2 &= x^2+y^2+z^2 \end{aligned}$$

Example 29. *We'll convert $x^2+y^2-z^2=1$ from Cartesian coordinates to spherical coordinates. This surface is called an* elliptic hyperboloid*, and its graph is shown below. We'll learn how to identify this and other surfaces later in the course.*



Making the substitutions $x=\rho\cos\theta\sin\phi$, $y=\rho\sin\theta\sin\phi$, and $z=\rho\cos\phi$, we have

$$\rho^2\cos^2\theta\sin^2\phi+\rho^2\sin^2\theta\sin^2\phi-\rho^2\cos^2\phi=1.$$

We can factor $\rho^2\sin^2\phi$ out of the first two terms and obtain

$$\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)-\rho^2\cos^2\phi=1.$$

Recalling that $\cos^2\theta+\sin^2\theta=1$, we can simplify this to

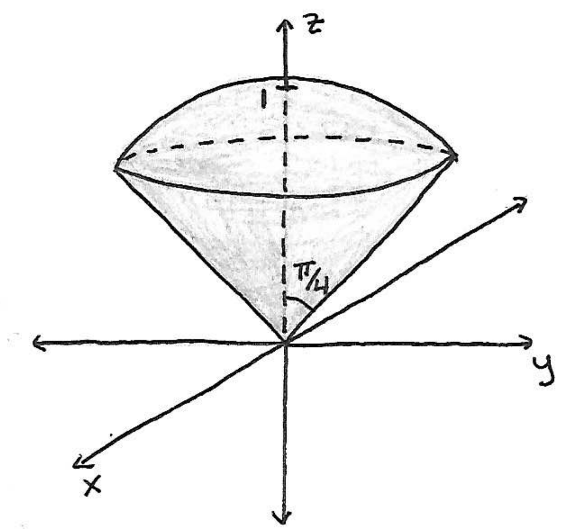
$$\rho^2\sin^2\phi-\rho^2\cos^2\phi=1.$$

Recalling the double angle formula $\cos(2\phi) = \cos^2(\phi) - \sin^2(\phi)$*, we can simplify this to*

$$\rho^2 \cos(2\phi) = 1.$$

Example 30. *Sketch the set of points* (ρ, θ, ϕ) *(in spherical coordinates) such that* $0 \leq \rho \leq 1$ *and* $0 \leq \phi \leq \pi/4$.

The condition $0 \leq \rho \leq 1$ *means that we'll have only points within distance* 1 *of the origin. The condition* $0 \leq \phi \leq \pi/4$ *means that we'll have only points within angle* $\pi/4$ *from the* z*-axis. Putting these conditions together, we have the solid "ice-cream cone" region sketched below.*



Spherical coordinates in $\mathbb{R}^n$ [OPTIONAL]

Since we've seen polar coordinates in $\mathbb{R}^2$, and cylindrical and spherical coordinates in $\mathbb{R}^3$, you might be wondering if there are similar coordinate systems in $\mathbb{R}^4$, $\mathbb{R}^5$, and so on.

It is possible to define spherical coordinates in $\mathbb{R}^n$ for any n, and you can find a description here.

Conclusion

We introduced spherical coordinates and how to convert between spherical coordinates and Cartesian coordinates, and we discussed the uniqueness of spherical coordinates.

Functions

In this activity, we cover the definition of a function. We also cover several important definitions and properties of functions, including domain, codomain, range, surjective and injective functions, and component functions.

Definition of a Function

You've certainly seen many functions before. For example, you've worked with linear functions, such as

$$f(x) = 3x + 2,$$

quadratic functions, such as

$$h(t) = -4.9t^2 + 20t + 5,$$

and more complicated functions such as

$$g(x) = e^{5\sin(x^2)} + \ln\cot x.$$

You've seen functions of more than one variable in the form of linear transformations, such as

$$\begin{aligned} &T : \mathbb{R}^2 \to \mathbb{R}^3, \\ &T(x, y) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Not surprisingly, in multivariable calculus, we'll be studying functions of more than one variable. Before starting to work with these functions, we now cover some of the fundamental definitions and properties related to functions in general, beginning with the definition of a function.

Definition 17. *For sets X and Y, a* function $f : X \to Y$ *from X to Y assigns an element of Y to each element of X.*

We call X the domain *of f, and Y the* codomain *of f.*

VIDEO

We commonly think of X as giving the set of inputs to a function, and Y as containing the outputs. Each input coming from the set X has to have some

Learning outcomes:
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corresponding output, but some elements of Y might not actually occur as outputs of the function.

Problem 4 *Which of the following are functions? Select All that apply.*

Select All Correct Answers:

(a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$ ✓

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(a, b) = a - b$ ✓

(c) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = \pm x$

(d) $f : \mathbb{R} \to \mathbb{R}^2$ *defined by* $f(x) = (x, x)$ ✓

If we would like to refer to the elements in the codomain which actually do occur as outputs, we call this the range of f.

Definition 18. *The* range *of a function $f : X \to Y$ is the set of elements $y \in Y$ such that there is some $x \in X$ with $f(x) = y$. That is, there is some input x that has y as an output. In set notation, we write*

$$\textit{Range } f = \{y \in Y \,:\, y = f(x) \textit{ for some } x \in X\}.$$

VIDEO

Problem 5 *What is the range of the function $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, x)$?*

Multiple Choice:

(a) $\mathbb{R}^2$

(b) $\mathbb{R}$

(c) $\{(a, b) \in \mathbb{R}^2 \,:\, a = b\}$ ✓

(d) $\{(a, b) \in \mathbb{R}^2 \,:\, a = b\}$

Sometimes we work with functions that aren't defined on all of $\mathbb{R}^n$. When the domain of f is a subset D of $\mathbb{R}^n$, we write

$$f : D \subset \mathbb{R}^n \to \mathbb{R}^m.$$

When we're working with functions on subsets of $\mathbb{R}^n$, we'll frequently want to work with the largest possible set that the function is defined on. We call this the *natural domain* of the function.

Problem 6 *What is the natural domain of the function* $f(x, y) = \dfrac{x}{x-y}$*?*

Multiple Choice:

(a) $\mathbb{R}^2$

(b) $\mathbb{R}^2 \setminus \{(0, 0)\}$

(c) $\mathbb{R}^2 \setminus \{(a, b) \,:\, a = 0 \text{ or } b = 0\}$

(d) $\mathbb{R}^2 \setminus \{(x, y) \,:\, a = b\}$ ✓

Types of Functions

In some special situations, every element of Y really does appear as an output of the function f. In this case, we say that f is onto, or surjective.

Definition 19. *A function $f : X \to Y$ is* onto*, or* surjective*, if for every element $y \in Y$, there is some $x \in X$ such that $f(x) = y$. We can also write this condition as*

$$Y = \textit{Range } f.$$

VIDEO

Problem 7 *Which of the following functions are onto? Select all that apply.*

Select All Correct Answers:

(a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(a, b) = a - b$ ✓

(c) $f : \mathbb{R} \to \mathbb{R}^2$ *defined by* $f(x) = (x, x)$

(d) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = x^2y^2$

Another important type of function is a one-to-one, or injective, function. For a one-to-one function, different inputs always go to different outputs.

Definition 20. *A function $f : X \to Y$ is* one-to-one*, or* injective*, if whenever $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$, then we must have $x_1 = x_2$.*

Another way to say this is that whenever $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

VIDEO

Problem 8 *Which of the following functions are injective? Select all that apply.*

Select All Correct Answers:

(a) $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(a, b) = a - b$

(c) $f : \mathbb{R} \to \mathbb{R}^2$ *defined by* $f(x) = (x, x)$ ✓

(d) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = x^2y^2$

Component Functions

When we're trying to understand the behavior of a function $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$, it can sometimes be helpful to split $\mathbb{R}^m$ into its components. From this, we get the component functions of f.

Definition 21. *Let* $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ *be a function. The* component functions *of* f *are scalar-valued functions* $f_i : X \subset \mathbb{R}^n \to \mathbb{R}$ *for* $1 \leq i \leq m$ *such that*

$$f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), ..., f_m(\vec{x})).$$

Conclusion

We covered the definition of a function. We also covered several important definitions and properties of functions, including domain, codomain, range, surjective and injective functions, and component functions.

Graphing Functions

In this activity, we give the formal definition of the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. We discuss strategies for figuring out the shapes of the graph of such functions, using contour curves, level curves, and sections.

Definition of the Graph of a Function

You might already have an intuitive idea of what the graph of a function $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ should be, but perhaps don't know the formal definition, or how to figure out what the graph of an arbitrary function looks like. We'll begin with the definition of the graph, before discussing how to actually produce graphs.

Definition 22. *Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function. The* graph *of f is the set of points*

$$\textit{Graph } f = \{(\vec{x}, f(\vec{x}) : \vec{x} \in X\}$$

in $\mathbb{R}^3$.

We typically visualize a point in the graph as lying over the point $\vec{x}$ in the plane at a height $f(\vec{x})$.

Note that this is similar to the graph of a function from a subset of $\mathbb{R}$ to $\mathbb{R}$. The graph of a function $X \subset \mathbb{R}^n \to \mathbb{R}$ can be defined similarly, but this tends to be less useful once $n \geq 3$, since it's hard to visualize four or more dimensions!

Strategies for Graphing

It can be much trickier to sketch the graph of a function $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ than it was to sketch the graphs of functions $\mathbb{R} \to \mathbb{R}$. One common strategy that people will initially try is plotting individual points to try to get a sense of the graph. However, for graphs in $\mathbb{R}^3$, you would need a lot of points to get a representative sample of the plane. For this reason, *plotting points alone is not an effective strategy.* However, plotting a single point here or there can be helpful.

We've now told you what doesn't work for graphing functions in $\mathbb{R}^3$, so now we should probably tell you what does work. The essential idea of all of these

Learning outcomes:
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strategies is that we know you're pretty comfortable graphing in $\mathbb{R}^2$, so we're going to take advantage of that experience.

We'll begin with contour curves, which are obtained by setting the z-coordinate to be constant. Think of this as taking horizontal slices of the graph.

Definition 23. *Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function. The* contour curve *of the function f at height C is the set of points in $\mathbb{R}^3$ obtained by taking the intersection of the graph of f with the plane $z = C$.*

PICTURE/VIDEO EXAMPLE

We can also consider the level curves of a function, which are closely related to contour curves.

Definition 24. *Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function. The* level curve *of the function $f$at height C is the set of points in $\mathbb{R}^2$ satisfying $C = f(x, y)$.*

After reading this definition, you're probably thinking "hey, aren't contour curves and level curves the exact same thing?" They're certainly closely related. The key difference is that level curves exist in the plane, $\mathbb{R}^2$, while contour curves exist in three-space, $\mathbb{R}^3$. Since they're in the plane, level curves are usually easier to draw. However, contour curves are more useful for figuring out the shape of a graph. For these reasons, it can be useful to go back and forth between level curves and contour curves.

PICTURE/VIDEO EXAMPLE

We can think of contour curves as taking slices of the graph where z is constant. It can also be useful to take slices of the graph where x or y is constant. We call these slices sections of the graph.

Definition 25. *Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function, and let C be a constant.*

The section *of the graph of f by $x = C$ is the set of points*

$$\{(C, y, z) \in \mathbb{R}^3 \, : \, z = f(C, y)\}$$

The section *of the graph of f by $y = C$ is the set of points*

$$\{(x, C, z) \in \mathbb{R}^3 \, : \, z = f(x, C)\}$$

Note that, like contour curves, sections exist in $\mathbb{R}^3$.

PICTURE/VIDEO EXAMPLE

EXTRA EXAMPLE

Level Surfaces

So far, we have focused on graphing functions from subsets of $\mathbb{R}^2$ to $\mathbb{R}$, so the graphs are in $\mathbb{R}^3$.

We now turn our attention to the graphs of functions from subsets of $\mathbb{R}^3$ to $\mathbb{R}$. Note that the graph of such a function will exist in $\mathbb{R}^4$. Since the world we live in only has three physical dimensions, it can be very difficult to visualize a four dimensional object! Fortunately, there are various tricks that can be used to get some sense of what a four dimensional object looks like. We cover one of them here.

When we had a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, we could get a sense of the graph by looking at its level curves, which were curves in the same plane.

For a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$, we can adopt a similar approach. We can once again consider the level sets, which are obtained by taking the output to be some constant:

$$f(x, y, z) = C.$$

In this case, the level sets will be level surfaces, which live in $\mathbb{R}^3$. By graphing several level surfaces, we can see what a slice of the graph of f looks like at various heights, giving us some sense of how the overall graph behaves. Of course, because this graph exists in four dimensions, we still probably won't be able to visualize this perfectly.

To see how this can help us visualize the four-dimensional graph of a function $f : \mathbb{R}^3 \to \mathbb{R}$, we give an example.

Example 31. *Consider the function*

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Find the level surfaces at heights -1, 0, 1, 2, *and* 3. *Use these level surfaces to describe the graph of* f.

We'll begin with the level surface at height -1. *This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$-1 = \sqrt{x^2 + y^2 + z^2}.$$

There are no points that satisfy this equation, so the level surface is empty.

Now we'll consider the level surface at height 0. *This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$0 = \sqrt{x^2 + y^2 + z^2}.$$

The only point which satisfies this equation is the origin, so the level "surface" is the single point $(0, 0, 0)$.

Let's look at the level surface at height 1. *This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$1 = \sqrt{x^2 + y^2 + z^2}.$$

Squaring both sides, we can rewriting this as

$$1 = x^2 + y^2 + z^2.$$

The graph of this equation is the sphere of radius 1 *centered at the origin, which is our level surface.*

Let's look at the level surface at height 2*. This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$2 = \sqrt{x^2 + y^2 + z^2}.$$

Squaring both sides, we can rewriting this as

$$4 = x^2 + y^2 + z^2.$$

The graph of this equation is the sphere of radius 2 *centered at the origin, which is our level surface.*

Let's look at the level surface at height 3*. This is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *such that*

$$3 = \sqrt{x^2 + y^2 + z^2}.$$

Squaring both sides, we can rewriting this as

$$9 = x^2 + y^2 + z^2.$$

The graph of this equation is the sphere of radius 3 *centered at the origin, which is our level surface.*

We graph our level surfaces below.

PICTURE

We can see that the level surfaces are spheres whose radii increase linearly with the height. So, we can describe the graph of f *as some sort of four-dimensional cone.*

Conclusion

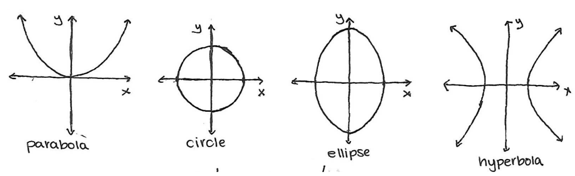
In this activity, we gave the formal definition of the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. We discussed strategies for figuring out the shapes of the graph of such functions, using contour curves, level curves, and sections.

Quadric Surfaces

In this activity, we introduce and classify quadric surfaces, which form an important family of surfaces.

Definition of a Quadric Surface

You might remember studying conic sections, such as parabolas, circles, ellipses, and hyperbolas. These are curves in the plane that arise through polynomial equations of degree two in two variables.



Quadric Surfaces are the three dimensional analogue of conic sections. That is, a quadric surface is the set of points in $\mathbb{R}^3$ satisfying some polynomial of degree two in three variables.

Definition 26. *A* quadric surface *is the set of points* (x, y, z) *in* $\mathbb{R}^3$ *satisfying the equation*

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + J = 0,$$

where $A, B, C, D, E, F, G, H, I, J \in \mathbb{R}$ *are constants.*

Simple Forms

Dealing with quadric surfaces in general can be computationally cumbersome, so we'll focus on quadric surfaces in some simple forms.

Example 32. *The set of points satisfying*

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

for some constants $a, b, c \in \mathbb{R}$*, is called an* ellipsoid.

Learning outcomes:
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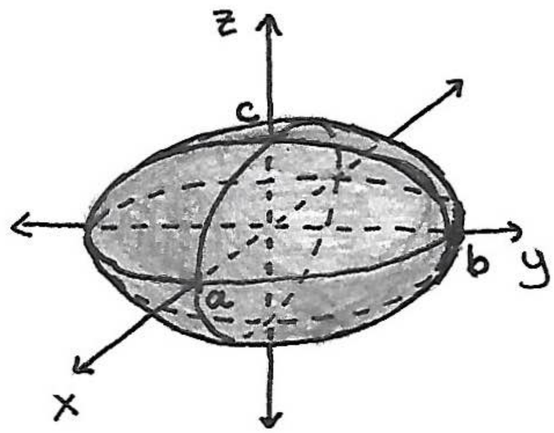


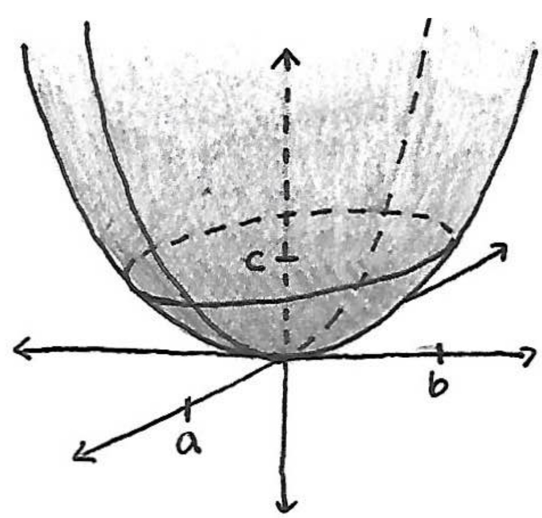
An ellipsoid is kind of like a three dimensional ellipse. In fact, the sections and contour curves of such an an ellipsoid are ellipses.

In the special case that $a = b = c$, this ellipsoid is a sphere of radius a.

Example 33. *The set of points satisfying*

$$z/c = x^2/a^2 + y^2/b^2,$$

for some constants $a, b, c \in \mathbb{R}$, is called an elliptic paraboloid.

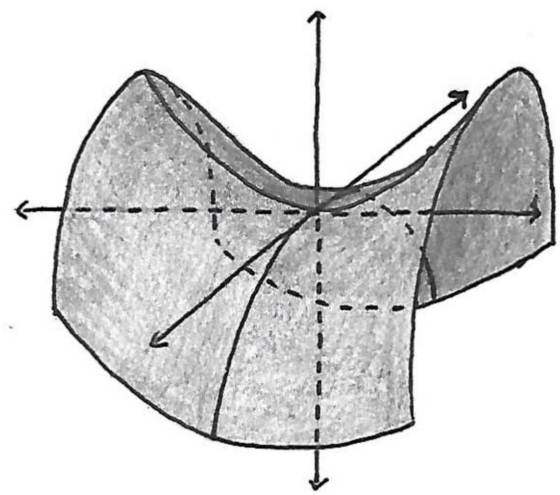


The contour curves of such an elliptic paraboloid are ellipses, however the sections are parabolas which all open in the same direction.

Example 34. *The set of points satisfying*

$$z/c = y^2/b^2 - x^2/a^2,$$

for some constants $a, b, c \in \mathbb{R}$*, is called a* hyperbolic paraboloid*.*

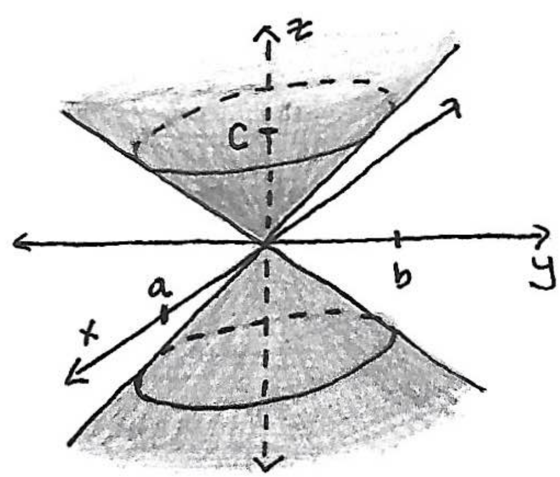


The contour curves of such a hyperbolic paraboloid are hyperbolas, and the sections are parabolas opening in opposite directions for x *and* y *sections. This surface is often described as a "saddle".*

Example 35. *The set of points satisfying*

$$z^2/c^2 = x^2/a^2 + y^2/b^2,$$

for some constants $a, b, c \in \mathbb{R}$*, is called an* elliptic cone*.*

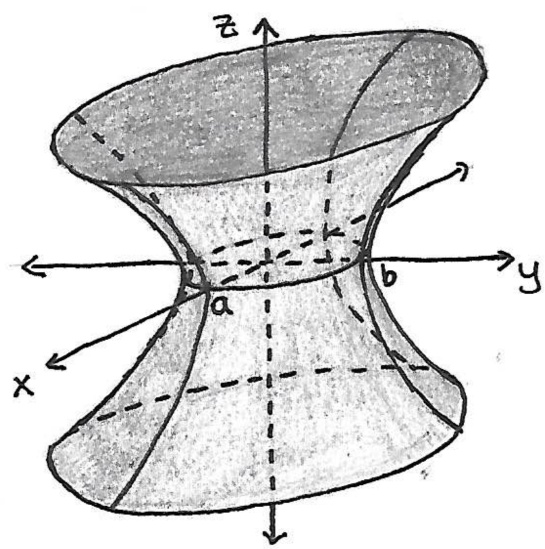


The contour curves of such an elliptic cone are ellipses, and the sections by $x = 0$ *and* $y = 0$ *are pairs of intersecting lines.*

Example 36. *The set of points satisfying*

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1.$$

for some constants $a, b, c \in \mathbb{R}$*, is called a* hyperboloid of one sheet*.*

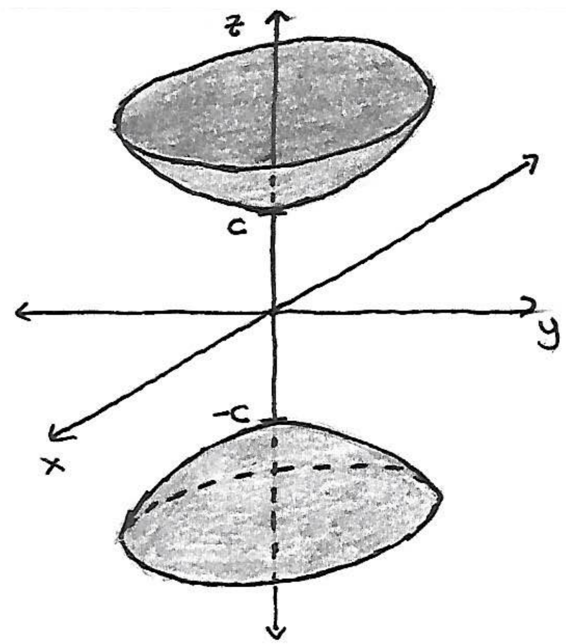


The contour curves of such a hyperboloid are ellipses, and the sections are hyperbolas.

Example 37. *The set of points satisfying*

$$z^2/c^2 - x^2/a^2 - y^2/b^2 = 1.$$

for some constants $a, b, c \in \mathbb{R}$*, is called a* hyperboloid of two sheets*.*



The contour curves of such a hyperboloid are ellipses, and the sections are hyperbolas. We describe this as the hyperboloid "of two sheets" since it has two disconnected pieces, as opposed to the hyperboloid of one sheet, which has only one.

APPLET/INTERACTIVE VARYING PARAMETERS

Some Other Forms

Although we won't really work with quadric surfaces in their most general form, we will consider quadric surfaces that are translations of the forms given above.

For example, the graph of the equation

$$(x-3)^2 + \frac{(y+2)^2}{3} + \frac{(z-1)^2}{2} = 1$$

is an ellipsoid centered at $(3, -2, 1)$.

PICTURE

However, equations describing quadric surfaces might not always be given to you in easily identifiable forms. In these cases, you might have to do some algebra in order to get the equation into a form where it can be identified as a particular quadric surface. These manipulations will frequently involve completing the square.

We now work through an example of identifying a quadric surface given in a non-standard form.

Example 38. *Identify the type of quadric surface determined by the equation*

$$-4x^2 + 2y^2 + z^2 + 8x + 4y + 4z = 2,$$

and sketch a graph of this surface.

Our strategy for writing this equation in a recognizable form will be to group terms involving x, group terms involving y, and group terms involving z. We'll then complete the square for each variable.

Grouping terms by variable, we have

$$(-4x^2 + 8x) + (2y^2 + 4y) + (z^2 + 4z) = 2.$$

For each of these grouping, we factor out the leading coefficient, obtaining

$$\boxed{-4}(x^2 - 2x) + \boxed{2}(y^2 + 2y) + (z^2 + 4z) = 2.$$

We now add or subtract as needed to make the quadratics into squares, getting

$$-4(x^2 - 2x + 1) + 2(y^2 + 2y + 1) + (z^2 + 4z + 2) = \boxed{4}.$$

We factor the quadratics to get

$$-4(\boxed{x-1})^2 + 2(\boxed{y+1})^2 + (\boxed{z+2})^2 = 4.$$

Finally, we divide by the constant on the right, to get the final form

$$-(\boxed{x-1})^2 + \frac{(\boxed{y+1})^2}{2} + \frac{(\boxed{z+2})^2}{4} = 1.$$

We can see that this quadric surface is centered at $\boxed{(1,-1,-2)}$, *but maybe it still isn't apparent which quadric surface this determines.*

Notice that this form is similar to our standard form for a hyperboloid of one sheet, except here it's the x-term that's subtracted instead of the z-term. This is because this is, in fact, a hyperboloid of one sheet, it just happens to be "around" a line parallel to the x-axis, rather than a vertical line.

Let's look at a section, in order to help with our sketch. Taking the section $x = 1$, we have an ellipse parallel to the yz-plane, centered at $(1,-1,-2)$, with radii $\sqrt{2}$ and 2.

Combining our observations, we can sketch the graph of this hyperboloid as below.

PICTURE

Conclusion

In this activity, we introduced and classified quadric surfaces, which form an important family of surfaces.

Part III

Curves in $\mathbb{R}^n$

Parametric Curves

In this activity, we parametrize curves in $\mathbb{R}^n$, focusing on the cases $n = 2$ and $n = 3$.

Review of Parametrizations in $\mathbb{R}^2$

We've dealt with several ways to describe curves in $\mathbb{R}^2$:

- As the graph of a function. For example, $f(x) = x^2$.
- As the set of points satisfying an equation. For example, the points (x, y) such that $x^2 + y^2 = 1$.
- As the set of points satisfying an equation in another coordinate system. For example, $r = \sin(\theta)$ in polar coordinates.

Another way that we can describe a curve is using *parametric equations*. In parametric equations, we define x and y in terms of a third variable, usually t, called the *parameter*. This gives us another way to describe curves in $\mathbb{R}^2$, and potentially describe some new and strange curves.

Example 39. *We can describe the unit circle in $\mathbb{R}^2$ with the parametric equations*

$$\begin{aligned} x &= \cos(t), \\ y &= \sin(t), \end{aligned}$$

for $0 \leq t \leq 2\pi$.

PICTURE

We can think of t as giving the angle that a point makes with the positive axis. It can also be helpful to imagine t as representing time, and the parametric equations tracing out the circle as time passes.

Learning outcomes:
Author(s):

Parametrizing Curves in $\mathbb{R}^n$

Consider the parametric equations for the unit circle in $\mathbb{R}^2$:

$$x = \cos(t),$$
$$y = \sin(t),$$

for $0 \leq t \leq 2\pi$.

We can combine these equations into a single vector,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

This defines a function $\vec{x}$ from the interval $[0, 2\pi] \subset \mathbb{R}$ to $\mathbb{R}^2$, and is the motivation behind our definition for paths.

Definition 27. *A* path *in $\mathbb{R}^n$ is a continuous function*

$$\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n,$$

where $I \subset \mathbb{R}$ is an interval.

This is also called a parametrized curve *or* parametric curve*.*

We'll focus on the cases $n = 2$ and $n = 3$ in this course.

We defined a path as a continuous function, however, we haven't said what it means for a multivariable function to be continuous. We'll come back to this later, and we'll give a rigorous definition for continuity. For now, this should fit with your intuition: you can draw the path without lifting your pencil from the paper.

Sometimes we can more about the image of a path than how the path is drawn out, and then we refer to a curve.

Definition 28. *A* curve *in $\mathbb{R}^n$ is the image of a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$.*

We say that $\vec{x}$ is a parametrization *for the curve.*

The difference between a curve and a path is largely a matter of perspective: when working with a curve, we pay attention to *what* is drawn; when working with a path, we care about *how* it is drawn.

Example 40. *There are many different parametrizations for a given curve.*

Consider again the unit circle C in $\mathbb{R}^2$. Which of the following are parametrizations for C?

Select All Correct Answers:

(a) $\vec{x}(t) = (\cos(t), \sin(t))$ *for* $0 \leq t \leq 2\pi$ ✓

(b) $\vec{x}(t) = (\sin(t), \cos(t))$ *for* $0 \leq t \leq \pi$

(c) $\vec{x}(t) = (t, \pm\sqrt{1-t^2})$ *for* $-1 \leq t \leq 1$

(d) $\vec{x}(t) = (\sin(2\pi t), \cos(2\pi t))$ *for* $0 \leq t \leq 1$ ✓

(e) $\vec{x}(t) = (\cos(t), \sin(t))$ *for* $-10 \leq t \leq 10$ ✓

Example 41. *In this example, we review how to parametrize the line through points* $\vec{a}$ *and* $\vec{b}$ *in* $\mathbb{R}^n$*.*

Given points $\vec{a}$ *and* $\vec{b}$ *in* $\mathbb{R}^n$*, we obtain a vector starting at* $\vec{a}$ *and ending at* $\vec{b}$ *by taking* $\vec{b} - \vec{a}$*. This vector is parallel to the line through* $\vec{a}$ *and* $\vec{b}$*. Then, taking scalar multiples* $t(\vec{b} - \vec{a})$ *for* $t \in \mathbb{R}$*, we have a line parallel to the line through* $\vec{a}$ *and* $\vec{b}$*. Finally, we add one of the points,* $\vec{a}$*, to ensure that our line passes through these two points. Thus, we arrive at our parametrization,*

$$\vec{l}(t) = \vec{a} + t(\vec{b} - \vec{a}) \text{ for } t \in \mathbb{R}.$$

PICTURE

Example 42. *In this example, we see how we can obtain new transformations from old ones, using linear algebra and simple transformations.*

Recall the parametrization for the unit circle in $\mathbb{R}^2$*,*

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Now, consider the ellipse below.

PICTURE

We can think of this ellipse as the result of stretching the unit circle horizontally by a factor of 3 *and vertically by a factor of* 2*. That is, we are applying the linear transformation*

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

We can apply this to the parametrization for the unit circle, in order to parametrization for the ellipse.

$$\begin{aligned} \vec{(y)}(t) &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \\ &= (3\cos(t), 2\sin(t)). \end{aligned}$$

Thus, we have a parametrization for the ellipse given by

$$\vec{y}(t) = (3\cos(t), 2\sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

Next, consider the following ellipse.

PICTURE

We can obtain this from our previous ellipse by counterclockwise rotation of $\pi/4$. The matrix for this linear transformation is

$$\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \boxed{1/\sqrt{2}} & \boxed{-1/\sqrt{2}} \\ \boxed{1/\sqrt{2}} & \boxed{1/\sqrt{2}} \end{pmatrix}.$$

Applying this rotation to our parametrization for the previous ellipse, we obtain a parametrization for our new ellipse.

$$\vec{z}(t) = \boxed{(3/\sqrt{2}\cos(t) - 2/\sqrt{2}\sin(t), 3/\sqrt{2}\cos(t) + 2/\sqrt{2}\sin(t))} \text{ for } 0 \le t \le 2\pi$$

Finally, we consider an ellipse in $\mathbb{R}^3$, shown below.

PICTURE

This ellipse is parallel to the xy-plane, and will have constant z-coordinate. Note the similarity to the first ellipse we considered. A parametrization for this ellipse can be obtained by taking the parametrization $\vec{y}$ for our first ellipse in $\mathbb{R}^2$, and appending the constant z-coordinate.

$$\vec{a}(t) = \boxed{(3\cos(t), 2\sin(t))} \text{ for } 0 \le t \le 2\pi$$

Examples in $\mathbb{R}^3$

In this section, we give examples of parametrizations of a couple of more complicated curves in $\mathbb{R}^3$, taking advantage of our previous experience with cylindrical coordinates.

Example 43. *We'll parametrize the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = 7 - 3x$ in $\mathbb{R}^3$, pictured below.*

PICTURE

Our x and y coordinates must satisfy $x^2 + y^2 = 4$, which would define a circle, if we were in $\mathbb{R}^2$. Recalling our parametrizations for circles, these coordinates can be written as

$$x(t) = 2\cos(t)$$
$$y(t) = 2\sin(t)$$

for $0 \le t \le 2\pi$.

It remains to write the z-coordinate in terms of the parameter t. Turning our attention to the equation for the plane, $z = 7 - 3x$, we have z expressed in terms *of x. Since we have expressed x in terms of t, we can make this substitution to describe z in terms of t,*

$$z(t) = \boxed{7 - 6\cos(t)}.$$

Putting all of this together, we have a parametrization for this intersection given by

$$\vec{x}(t) = \boxed{(2\cos(t), 2\sin(t), 7 - 6\cos(t))} \text{ for } 0 \le t \le 2\pi.$$

Geogebra link: https://tube.geogebra.org/m/dxywtu7x

Example 44. *Consider the curve below, which lies on the cone $z^2 = x^2 + y^2$, and makes five rotations around the z-axis as the height ranges from 0 to 1. We'll refer to this curve as a "tornado."*

PICTURE

We'll parametrize this curve by thinking about it in cylindrical coordinates, using the height as the parameter.

First, let's consider what's happening with the z-coordinate. Since the height of the tornado rangers from 0 to 1, so will z. We'll set $z = t$, with $0 \le t \le 1$, and express x and y in terms of t as well.

Now, we turn our attention to the angle θ. As the height ranges from 0 to 1, the tornado makes five revolutions, so θ should range from 0 to 10π. Thus, expressing θ in terms of t, we let $\theta = 10\pi t$.

Next, we consider the radius r. Since we are on the cone $z^2 = x^2 + y^2$, we have $z^2 = r^2$. Since $z \ge 0$, we have $z = r$. Thus, we can write r in terms of t as $r = t$.

Finally, putting all of this together with $x = r\cos\theta$ and $y = r\sin\theta$, we have a parametrization for the tornado given by

$$\vec{x}(t) = \boxed{(t\cos(10\pi\theta), t\sin(10\pi\theta), t)} \text{ for } t \in [0, 1].$$

Geogebra link: https://tube.geogebra.org/m/tdmnpcg5

Conclusion

In this activity, we parametrized curves in $\mathbb{R}^n$, focusing on the cases $n = 2$ and $n = 3$.

Velocity and Speed

In this activity, we learn how to find the velocity and speed of a parametrized curve in $\mathbb{R}^n$.

Derivatives

Consider a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$ which parametrizes a curve in $\mathbb{R}^n$. We often think about this as a particle tracing out the curve as time, given by t, passes. We would like to be able to understand and describe the motion of the particle on the curve, and find its velocity and speed, in particular. In order to do this, we need to figure out how to differentiate a path.

Before we define the derivative of a path, we quickly review the single variable definition of a derivative, given in Calculus I.

Given a single variable function $f(x)$, we found the instantaneous rate of change at x of this function by taking the derivative of f at x. The derivative also told us the slope of the tangent line at x. In order to compute this, we imagined finding the slope of secant lines getting closer and closer to the point. Taking a limit, we obtained the slope of the tangent line.

PICTURE

The slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$ is given by $\dfrac{f(x+h)-f(x)}{h}$, so we defined the derivative of f at x to be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

We use the same idea for a path $\vec{x}$ in $\mathbb{R}^n$. We consider secant vectors from $\vec{x}(t)$ to $\vec{x}(t+h)$ as $h \to 0$.

PICTURE

Scaling these vectors to account for the change in the parameter and taking a limit, we arrive at the definition of the derivative.

Definition 29. *Let $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$ be a path in $\mathbb{R}^n$. We define the derivative of $\vec{x}$ at t to be*

$$\vec{x}'(t) = \lim_{h\to 0} \frac{\vec{x}(t+h)-\vec{x}(t)}{h},$$

Learning outcomes:
Author(s):

if the limit exists.

We also call $\vec{x}'(t)$ the velocity vector *of $\vec{x}$, and write it as $\vec{v}(t)$.*

When we first defined derivatives in Calculus I, we spent weeks figuring out how to compute them. We started computing using only the limit definition, then we introduced the power rule, the product rule, the chain rule, and so on. Fortunately, we don't need to repeat this process in Multivariable Calculus: we can take advantage of our previous experience computing derivatives. In order to see why this is the case, let's take another look at our definition for the derivative of a path.

We have $\vec{x}'(t) = \lim_{h\to 0} \dfrac{\vec{x}(t+h) - \vec{x}(t)}{h}$ for a path $\vec{x}$. We can write out the path $\vec{x}$ in terms of its components, so

$$\vec{x}(t) = (x_1(t), ..., x_n(t)).$$

Substituting this into the limit, we have

$$\begin{aligned} \vec{x}'(t) &= \lim_{h\to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}, \\ &= \lim_{h\to 0} \frac{(x_1(t+h), ..., x_n(t+h)) - (x_1(t), ..., x_n(t))}{h}, \\ &= \lim_{h\to 0} \frac{(x_1(t+h) - x_1(t), ..., x_n(t+h) - x_1(t))}{h}. \end{aligned}$$

Dividing through by the scalar h and bringing the limit inside of the vector, we have

$$\begin{aligned} \vec{x}'(t) &= \lim_{h\to 0} \left(\frac{x_1(t+h) - x_1(t)}{h}, ..., \frac{x_n(t+h) - x_1(t)}{h} \right), \\ &= \left(\lim_{h\to 0} \frac{x_1(t+h) - x_1(t)}{h}, ..., \lim_{h\to 0} \frac{x_n(t+h) - x_1(t)}{h} \right). \end{aligned}$$

At this point you should be somewhat skeptical. We haven't defined limits of vectors, much less described how to manipulate them. We'll come back to this in a few weeks in much more detail. For now, hopefully it makes sense that looking at what a vector approaches depends on what its components approach, and you'll allow us this sleight of hand.

Looking at the limits inside of the components, they should look familiar. They're derivatives of single variable functions! That is, we now have

$$\vec{x}'(t) = (x_1'(t), ..., x_n'(t)).$$

This means that we can differentiate a path by differentiating its components, thus taking advantage of our knowledge of single variable derivatives.

Proposition 14. *We can differentiate a path by differentiating its components. That is,*

$$\vec{x}'(t) = (x_1'(t), ..., x_n'(t)).$$

Example 45. *Consider the path $\vec{x}(t) = (\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$, which parametrizes the unit circle in $\mathbb{R}^2$. We compute the derivative of this path,*

$$\vec{x}'(t) = \left(\frac{d}{dt}\cos(t), \frac{d}{dt}\sin(t)\right), \qquad = (-\sin(t), \cos(t)).$$

Consider the path $\vec{y}(t) = (t^2, t^3)$ for $0 \leq t \leq 1$.

$$y'(t) = \boxed{(2t, 3t^2)}$$

Consider the path $\vec{z}(t) = \left(t, e^{t^2}\right)$ for $-\infty < t < \infty$.

$$z'(t) = \boxed{(1, 2te^{t^2})}$$

Velocity and Speed

We defined the derivative $\vec{x}'$ of a path $\vec{x}$, thinking of a limit of scaled secant vectors. Taking the limit of these vectors, our derivative gives us a vector which is tangent to the path.

PICTURE

The direction of $\vec{x}'$ gives us the direction of instantaneous of a particle moving along the path, and the length of $\vec{x}'$ tells us the speed of the particle. Recall that we sometimes refer to $\vec{x}'$ as the velocity vector, and write it as $\vec{v}$.

Definition 30. *Consider a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$.*

The velocity vector *of $\vec{x}$ at t is $\vec{v}(t) = \vec{x}'(t)$. The velocity vector is tangent to $\vec{x}$ at $\vec{x}(t)$.*

The speed *of $\vec{x}$ at t is $\|\vec{x}'(t)\| = \|\vec{v}(t)\|$.*

Example 46. *Consider the path $\vec{x}(t) = (\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$, which parametrizes the unit circle in $\mathbb{R}^2$. We previously computed the velocity of this path as*

$$\vec{v}(t) = \vec{x}'(t) = (-\sin(t), \cos(t)).$$

We can then compute the speed of $\vec{x}$ as

$$\begin{aligned} \|\vec{x}'(t)\| &= \|(-\sin(t), \cos(t))\|, \\ &= \sqrt{(-\sin(t))^2 + (\cos(t))^2}, \\ &= \sqrt{1}, \\ &= 1. \end{aligned}$$

Consider the path $\vec{y}(t) = (\cos(t^2), \sin(t^2))$ for $0 \le t \le \sqrt{2\pi}$. This also parametrizes the unit circle in $\mathbb{R}^2$. The velocity vector of this path is

$$\vec{y}'(t) = \boxed{(-2t\sin(t^2), 2t\cos(t^2))}.$$

The speed of this path is

$$\|\vec{y}'(t)\| = \boxed{2t}.$$

Although both of these paths parametrize the unit circle counterclockwise and starting and ending at $(1,0)$, they do so in different ways. The first path, $\vec{x}$, traverses the unit circle at constant speed. The second path, $\vec{y}$, travels very slowly at first, then the speed increases as it travels around the circle.

Consider a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$. The velocity of this path gives us a vector $\vec{x}'(t)$ tangent to the curve at $\vec{x}(t)$. The tangent line to $\vec{x}$ at $\vec{x}(t)$ passes through the point $\vec{x}(t)$ and is parallel to the vector $\vec{x}'(t)$. This allows us to parametrize the tangent line, however we need to be very careful to distinguish between the parameter for the *line* and the parameter for the *path*. We do this by taking the parameter for our curve to be t_0 at our chosen point, so we are working with the point $\vec{x}(t_0)$ and the tangent vector $\vec{x}'(t_0)$.

Proposition 15. *Consider a path $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$. We can parametrize the line tangent to $\vec{x}$ at $\vec{x}(t_0)$ as*

$$l(t) = \vec{x}(t_0) + t\vec{x}'(t_0) \text{ for } -\infty < t < \infty.$$

Note that it's particularly important to allow the parameter t to be any real number, otherwise we will be missing part of the line.

Conclusion

In this activity, we learned how to find the velocity and speed of a parametrized curve in $\mathbb{R}^n$.

Properties of Velocity and Speed

In this activity, we explore some important properties of the velocity and speed of a parametrized curve.

Differentiation Laws

In single variable calculus, we used the product rule to differentiate products of functions. Although we can't take the product of two vectors in general, we do have the dot product and cross product, and we would like to understand how differentiation interacts with these products. Fortunately, they turn out to be very similar to the product rule from single variable calculus.

Proposition 16. *Consider paths $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$. For t such that $\vec{x}'(t)$ and $\vec{y}'(t)$ both exist, we have*

$$(\vec{x}\cdot\vec{y})'(t) = \vec{x}'(t)\cdot\vec{y}(t) + \vec{x}(t)\cdot\vec{y}'(t).$$

If $n = 3$, we also have

$$(\vec{x}\times\vec{y})'(t) = \vec{x}'(t)\times\vec{y}(t) + \vec{x}(t)\times\vec{y}'(t).$$

Proof We prove this result for the dot product, and leave the proof for the cross product as an exercise.

Suppose t is such that both $\vec{x}'(t)$ and $\vec{y}'(t)$ exist, and write $\vec{x}(t) = (x_1(t), ..., x_n(t))$ and $\vec{y}(t) = (y_1(t), ..., y_n(t))$. Then we have

$$(\vec{x}\cdot\vec{y})(t) = x_1(t)y_1(t) + \cdots + x_n(t)y_n(t).$$

Using the single variable product rule and regrouping, we have

$$\begin{aligned}(\vec{x}\cdot\vec{y})'(t) &= \frac{d}{dt}\left(x_1(t)y_1(t) + \cdots + x_n(t)y_n(t)\right),\\ &= x_1'(t)y_1(t) + x_1(t)y_1'(t) + \cdots + x_n'(t)y_n(t) + x_n(t)y_n'(t),\\ &= \left(x_1'(t)y_1(t) + \cdots + x_n'(t)y_n(t)\right) + \left(x_1(t)y_1'(t) + \cdots + x_n(t)y_n'(t)\right).\end{aligned}$$

Notice that the left summand is $\vec{x}'(t)\cdot\vec{y}(t)$ and the right summand is $\vec{x}(t)\cdot\vec{y}'(t)$. Thus, we arrive at our result,

$$(\vec{x}\cdot\vec{y})'(t) = \vec{x}'(t)\cdot\vec{y}(t) + \vec{x}(t)\cdot\vec{y}'(t).$$

■

Learning outcomes:
Author(s):

Constant Speed Path

To finish up our unit on parametrized paths, we consider the special case where a path is constant distance from the origin. In this case, the path $\vec{x}$ is always perpendicular to its derivative. This makes sense intuitively, if you imagine a particle on the path moving in the direction of its velocity vector. If the velocity vector $\vec{v}$ were not perpendicular to $\vec{x}$, a particle moving a tiny distance along the path would have to move either closer to the origin or farther from the origin.

PICTURE

Proposition 17. *If $\vec{x}(t)$ has constant length, then $\vec{x}(t)$ is perpendicular to $\vec{x}'(t)$, for all t such that $\vec{x}'(t)$ is defined.*

We leave the proof of this proposition as an exercise. It's helpful to think about how the dot product $\vec{x}(t) \cdot \vec{x}(t)$ relates to the length of $\vec{x}(t)$.

Conclusion

In this activity, we explore some important properties of the velocity and speed of a parametrized curve.

The Length of a Curve

In this activity, we learn how to compute the length of a curve given a smooth parametrization.

The Length of a Path

Suppose we have a path $\vec{x} : I \to \mathbb{R}^n$, where I is the interval $[a, b]$, and suppose we want to find the length of the corresponding curve C.

PICTURE

We can approximate the curve with a lot of short line segments, and then compute the total length of the line segments to estimate the length of the curve.

PICTURE

One way that we can do this is by subdividing the interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal length, and then take the line segments connecting $\vec{x}(t_{i-1})$ and $\vec{x}(t_i)$.

PICTURE

As n increases, our line segments get shorter and shorter, giving us a more accurate approximation of the length of the curve. If $\vec{x}$ is a smooth parametrization of C, when we take the limit as $n \to \infty$, we will find the exact length of the curve.

PICTURE

Let's use this idea to find a formula for the length of a curve parametrized by a smooth path $\vec{x}(t)$. The length of the segment connecting $\vec{x}(t_{i-1})$ and $\vec{x}(t_i)$ can be computed as $\|\vec{x}(t_i) - \vec{x}(t_{i-1})\|$, so we have that the length of the curve is

$$\begin{aligned} L(\vec{x}) &\approx \sum_{i=1}^{n} \|\vec{x}(t_i) - \vec{x}(t_{i-1})\| \\ &= \sum_{i=1}^{n} \|\frac{\vec{x}(t_i) - \vec{x}(t_{i-1})}{\delta t}\| \delta t, \end{aligned}$$

where we both multiply and divide by δt, the length of each subinterval.

Learning outcomes:
Author(s):

As $n \to \infty$, the length of the subintervals, δt, goes to 0, and $\dfrac{\vec{x}(t_i) - \vec{x}(t_{i-1})}{\delta t}$ goes to $\vec{x}'(t_i)$. This gives us

$$L(\vec{x}) = \lim_{n \to \infty} \sum_{i=1}^{n} \|\vec{x}'(t_i)\| \delta.$$

Recognizing this as an integral, we arrive at

$$L(\vec{x}) = \int_a^b \|\vec{x}'(t)\| dt.$$

Although we started with the goal of finding the length of a *curve*, however the result that we came up with could depend on the choice of parametrization $\vec{x}(t)$ of the curve. So, for now, we will use this idea to define the length of a path.

Definition 31. *Let $\vec{x} : I \to \mathbb{R}^n$ be a C^1 path defined on the interval $I = [a, b]$. The* length *of the path $\vec{x}$ is*

$$L(\vec{x}) = \int_a^b \|\vec{x}'(t)\| dt.$$

In the next section, we will explore how the choice of the parametrization affects the computation of the length of the corresponding curve. For now, we'll compute the length of paths in a couple of examples.

Example 47. *Consider the path $\vec{x} : [0, 2\pi] \to \mathbb{R}^2$ defined by $\vec{x}(t) = (\cos(t), \sin(t))$.*

PICTURE

We compute the length of this path.

$$\begin{aligned} L(\vec{x}) &= \int_a^b \|\vec{x}'(t)\| dt \\ &= \int_0^{2\pi} \|(-\sin(t), \cos(t))\| dt \\ &= \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt \\ &= \int_0^{2\pi} \boxed{1} dt \\ &= \boxed{2pi} \end{aligned}$$

Remembering that this is a parametrization for the unit circle, this matches with what we know to be the circumference of the circle.

Example 48. *Consider the path $\vec{x} : [0, 3\pi] \to \mathbb{R}^3$ defined by $\vec{x}(t) = (\cos(t), \sin(t), t)$.*

PICTURE

We compute the length of this path.

$$\begin{aligned} L(\vec{x}) &= \int_a^b \|\vec{x}'(t)\| dt \\ &= \int_0^{3\pi} \|\boxed{(-\sin(t), \cos(t), 1)}\| dt \\ &= \int_0^{3\pi} \boxed{\sqrt{2}} dt \\ &= \boxed{3\sqrt{2}\pi} \end{aligned}$$

Because of the square roots that appear when we take the magnitude of $\vec{x}'(t)$, arclength integrals for arbitrary curves are often messy to compute. However, in these cases, at least we can write down an integral representing the length of the curve, and perhaps use technology to either evaluate or approximate the integral.

The Length of a Curve

In one of the previous examples, we found the length of the path $\vec{x}(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$. we found that $L(\vec{x}) = 2\pi$, matching what we know to be the circumference of the circle of radius 1.

PICTURE

Let's see what happens when we take a different parametrization for the same curve, this time computing the length of $\vec{y}(t) = (\cos(t), \sin(t))$ for $t \in [0, 4\pi]$. In this case, we get

$$\begin{aligned} L(\vec{x}) &= \int_a^b \|\vec{x}'(t)\| dt \\ &= \int_0^{4\pi} \|(-\sin(t), \cos(t))\| dt \\ &= \int_0^{4\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt \\ &= \int_0^{4\pi} 1\, dt \\ &= \boxed{4pi}. \end{aligned}$$

Although $\vec{y}$ is another C^1 parametrization of the unit circle, we get a different result for the length of the path!

The issue here is that while $\vec{x}$ traces around the unit circle once, the path $\vec{y}$ traces around the unit circle twice. When we are computing $L(\vec{y})$, we are really computing the distance traced by the path $\vec{y}$, which is why we get twice the length of the actual curve.

Because of this, if we want to compute the length of a curve, we need to be careful with our choice of parametrization, to make sure that we are only tracing over the curve once.

The proof of the following theorem is left as an exercise. A curve is *simple* if it does not intersect itself.

Theorem 1. *Let $\vec{x}(t)$, $a \leq t \leq b$, and $\vec{y}(s)$, $c \leq s \leq d$ be smooth and simple parametrizations of the same curve C. Then $L(\vec{x}) = L(\vec{y})$.*

In this case, we define the length of C to be

$$L(C) = L(\vec{x}) = L(\vec{y}).$$

We also might encounter a situation where we want to compute the length of a curve which is not C^1, but is piecewise C^1. In this case, we can compute the length of the curve by computing the lengths of the pieces, and adding them together.

PICTURE

Arclength Function

In this activity, we introduce the arclength function, and parametrize a curve with respect to arclength.

The Arclength Function

Suppose we're considering the arclength of some curve $\vec{x}(t)$. Let's fix some starting point at time $t = a$, but instead of fixing the endpoint as well, we'll let that vary, so we'll think of it as a variable. For each choice of endpoint, we can compute the arclength along the curve from our fixed start point to the endpoint. This gives us a special function, which we call the *arclength function*:

$$s(t) = \int_a^t \|\vec{x}'(\tau)\| d\tau.$$

Note that we're using τ for the variable of integration, to avoid confusion with the t used for the endpoint.

COOL INTERACTIVE

Let's see what happens when we differentiate the arclength function. First, recall part of the Fundamental Theorem of Calculus: if f is a continuous function, and we define $F(x) = \displaystyle\int_a^x f(u)\, du$, then

$$F'(x) = \boxed{f(x)}.$$

Applying this to the arclength function, $s(t) = \displaystyle\int_a^t \|\vec{x}'(\tau)\|\, d\tau$, we have

$$s'(t) = \|\vec{x}'(t)\|.$$

This means that the derivative of the arclength function is the *speed* of the parametrization, or the speed of a particle moving along the path. If we remember that the arclength function is computing the distance traveled along the path, it makes sense that its derivative should be speed.

Learning outcomes:
Author(s):

Parametrization with Respect to Arclength

One of the most important uses of the arclength function is to reparametrize a curve according to *arclength*. This means that we find a special parametrization $\vec{x}(s)$ for $0 \leq s \leq L$, where L is the length of the curve. This parametrization has the property that $\vec{x}(s)$ is always the point distance s along the curve.

We'll see that the arclength parametrization is very useful, but unfortunately it is often computationally difficult to find. The process for reparametrizating with respect to arclength is:

(a) Find a parametrization $\vec{x}(t)$ for the curve.

(b) Find the arclength function $s(t)$.

(c) Invert the arclength function. That is, write the parameter t in terms of the arclength s.

(d) Substitute the expression for t into the original parametrization $\vec{x}(t)$, which gives you a parametrization for the curve with respect to arclength, s.

These steps sound simple enough, but are complicated by the fact that arclength integrals rarely simplify nicely, so it can be difficult to write down an inverse. Below, we work through an example where arclength can be simplified nicely.

Example 49. *Consider the helix $\vec{x}(t) = (3\cos(t), 3\sin(t), 4t)$ for $0 \leq t \leq 8\pi$. We'll reparametrize this helix with respect to arclength.*

GRAPH

We're already given a parametrization for the curve, so we'll begin by finding the arclength function $s(t) = \int_0^t \|\vec{x}'(\tau)\|\, d\tau$. First, we'll find the velocity.

$$\vec{x}'(t) = \boxed{(-3\sin(t), 3\cos(t), 4)}$$

Then, we can find the speed.

$$\|\vec{x}'(t)\| = \boxed{5}$$

Now, we find the arclength function by integrating speed.

$$\begin{aligned} s(t) &= \int_0^t \|\vec{x}'(\tau)\|\, d\tau \\ &= \int_0^t 5 d\tau \\ &= \boxed{5t} \end{aligned}$$

We now find the inverse of the arclength function, to find the parameter t in terms of arclength s. Working for $s = 5t$, we have

$$t = \boxed{s/5}.$$

We substitute this into our original parametrization, and we have

$$\vec{x}(s) = \boxed{(3\cos(s/5), 3\sin(s/5), \frac{4}{5}s)}.$$

This gives us the parametrization with respect to arclength.

We will soon see that the arclength parametrization can be used to examine the inherent geometry of the curve, independent of how quickly or slowly a given path traverses the curve. This is because the arclength parametrization always has unit speed. Let's investigate why this is true.

Let's suppose we obtain the arclength parametrization $\vec{x}(s)$ from some parametrization $\vec{x}(t)$. Since s is a function of t, we can use the chain rule to find the velocity of $\vec{x}(t)$. In particular,

$$\vec{x}'(t) = \vec{x}'(s) \cdot s'(t).$$

Earlier, we found that $s'(t) = \|\vec{x}'(t)\|$. Substituting, we have

$$\vec{x}'(t) = \vec{x}'(s)\|\vec{x}'(t)\|.$$

Taking the magnitude of both sides, we have

$$\|\vec{x}'(t)\| = \|\vec{x}'(s)\| \cdot \|\vec{x}'(t)\|.$$

From this, we see that

$$\|\vec{x}'(s)\| = 1.$$

So, the speed of the arclength parametrization is always 1.

Curvature

In this activity, we define the curvature of a path and discuss its geometric significance.

Definition of Curvature

We have an intuitive idea for what it means for a curve to be "curvy."

EXAMPLES OF VERY CURVY AND NOT VERY CURVY CURVES

However, we don't yet have a way to represent "curviness" mathematically. In this section, we'll define the *curvature* of a curve, which will allow us to quantify "curviness."

In order to ensure that our definition is independent of the parametrization, we'll need to work with the arclength parametrization $\vec{x}(s)$. Recall that this parametrization traverses the curve at unit speed.

Let's look at the behavior of the unit tangent vector as we traverse various curves. Recall that the unit tangent vector is the velocity divided by the speed, so

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}.$$

Since the arclength parametrization has unit speed, we have

$$\vec{T}(s) = \vec{x}'(s).$$

SUPER AWESOME INTERACTIVE

We see that the unit tangent vector changes very quickly when we're curving sharply, and the unit tangent vector doesn't change when we're going straight. So, we'll use the change in the unit tangent vector to measure "curviness."

Definition 32. *Suppose the arclength parametrization of a curve C is $\vec{x}(s)$. Then we define the* curvature *of C at time s to be*

$$\kappa(s) = \|\vec{T}'(s)\|.$$

Unfortunately, as we've seen, it's not always easy to find an arclength parametrization for a curve. Fortunately, we can still compute the curvature without finding an arclength parametrization.

Learning outcomes:
Author(s):

Suppose we have a parametrization $\vec{x}(t)$ of a curve. Then, thinking of s as a function of t and using the chain rule,

$$\vec{T}'(t) = \vec{T}'(s)s'(t).$$

Then $\vec{T}'(s) = \dfrac{\vec{T}'(t)}{s'(t)}$. Recalling that $s'(t)$ is the speed of the parametrization $\vec{x}(t)$, we can compute the curvature as follows.

Proposition 18. *Let $\vec{x}(t)$ be a parametrization of a curve C. Then*

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}.$$

We'll typically use this equation to compute curvature.

Computing Curvature

Example 50. *We'll compute the curvature of a circle of radius $a > 0$, parametrized by $\vec{x}(t) = (a\cos(t), a\sin(t))$.*

In order to find the unit tangent vector, we'll need to compute the velocity and speed.

$$\vec{x}'(t) = \boxed{(-a\sin(t), a\cos(t))}$$

Then the speed of $x(t)$ is

$$\|\vec{x}'(t)\| = \boxed{a}.$$

Dividing the velocity by the speed, we obtain the unit tangent vector,

$$\vec{T}(t) = \boxed{(-\sin(t), \cos(t))}.$$

Now, we find $\vec{T}'(t)$.

$$\vec{T}'(t) = (-cos(t), -sin(t))$$

The magnitude of this vector is $\boxed{1}$.

Finally, we compute the curvature at time t.

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|} = \boxed{\frac{1}{a}}$$

Note that the curvature is independent of the time t, so independent of our position on the circle. This matches with the symmetry of the circle.

PICTURE

Example 51. *We'll compute the curvature of the path* $\vec{x}(t) = (t\cos(t), t\sin(t), t^2)$*, for* $t \geq 0$*.*

In order to find the unit tangent vector, we'll need to compute the velocity and speed.

$$\vec{x}'(t) = \boxed{(-t\sin(t), t\cos(t), 2t)}$$

Then the speed of $x(t)$ *is*

$$\|\vec{x}'(t)\| = \boxed{\sqrt{5}t}.$$

Dividing the velocity by the speed, we obtain the unit tangent vector,

$$\vec{T}(t) = \boxed{(\frac{-1}{\sqrt{5}}\sin(t), \frac{1}{\sqrt{5}}\cos(t), \frac{2}{\sqrt{5}})}.$$

Now, we find $\vec{T}'(t)$*.*

$$\vec{T}'(t) = (\frac{-1}{\sqrt{5}}\cos(t), \frac{-1}{\sqrt{5}}\sin(t), 0)$$

The magnitude of this vector is $\boxed{\sqrt{5}}$*.*

Finally, we compute the curvature at time t*.*

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|} = \boxed{\frac{1}{t}}$$

Osculating Circle

In a previous example, we found that the curvature of the circle of radius a is constant, and $\kappa = \frac{1}{a}$. Another way to say this is that the radius of the circle is "one over the curvature," or $a = \frac{1}{\kappa}$. We can extend this idea to other curves as well: given $\vec{x}(t)$, $r = \frac{1}{\kappa(t)}$ is the radius of the circle which "best fits" the graph at time t. We call this circle the *osculating circle* at that point.

PICTURE/INTERACTIVE

Defining the Moving Frame

In this section, we introduce the moving frame of a path, which is also called the TNB frame. This is a set of three mutually perpendicular unit vectors (an orthonormal set) which provide a consistent reference frame for a particle moving along a path.

Imagine yourself walking around, and think about the following three directions:

- the direction that you're looking ("ahead')
- the direction that your head is pointing ("up")
- the direction to your right ("right")

Together, the directions ahead, up, and right define your own, personal reference frame. You can describe locations relative to your reference frame:

> "the classroom is up a floor, three doors ahead, and on the right."

However, relative to the rest of the universe, these directions change as you walk around. If you turn around to face in the opposite direction, ahead and right would be pointing in the opposite directions that they were before. If you fly across the world to Australia, the up direction is now in a different direction. So, this frame of reference is unique to your position, and how you are moving around.

Our goal is to define a similar reference frame for a particle moving along a path in $\mathbb{R}^3$, which will consist of three orthonormal vectors.

Defining the mooving frame

Suppose we have a path $\vec{x}(t)$ in $\mathbb{R}^3$, and we want to define a reasonable reference frame for a particle moving along this path, which will consist of three mutually orthogonal unit vectors, hence an orthonormal basis for $\mathbb{R}^3$. We'll construct this reference frame one vector at a time, thinking about how we can encode the motion of a particle along the path.

The first vector of our moving frame will match the "ahead" direction of our analogy: we want a vector that tells us the direction in which the particle is

Learning outcomes:
Author(s):

moving. Since the velocity vector $\vec{x}'(t)$ points in the direction of instantaneous motion along the path, this gives us the correct direction for our first vector. However, the velocity vector isn't necessarily a unit vector. In order to obtain a unit vector in the same direction as the velocity vector, we divide $\vec{x}'(t)$ by its length, obtaining the *unit tangent vector*:

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}.$$

VIDEO/INTERACTIVE

For the second vector of our moving frame, we'd like to give the direction in which a particle moving along the path is turning. This doesn't quite match up with the direction "right" from our analogy, since you might be turning to either the left or the right. Thinking back to our definition of curvature, we were able to see how a particle was turning by looking at the change in the unit tangent vector. That is, we considered $\vec{T}'(t)$. As it turns out, $\vec{T}'(t)$ will always be perpendicular to $\vec{T}(t)$.

Proposition 19. $\vec{T}'(t) \perp \vec{T}(t)$

Proof Since $\vec{T}(t)$ is a unit vector, we have

$$\vec{T}(t) \cdot \vec{T}(t) = \boxed{1}$$

for all t. Differentiating both sides of this equation, we have

$$\frac{d}{dt}(\vec{T}(t) \cdot \vec{T}(t)) = \boxed{0}.$$

Using properties of derivatives, we then have

$$\vec{T}'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}'(t) = 0,$$

so

$$\vec{T}'(t) \cdot \vec{T}(t) = 0.$$

This means that $T'(t)$ and $T(t)$ are perpendicular for all t. ■

Since $\vec{T}'(t)$ will always be perpendicular to $\vec{T}(t)$, it's a great candidate for the second direction in our moving frame. However, $\vec{T}'(t)$ won't always be a unit vector, so we'll need to divide by its length to normalize. This gives us the *unit normal vector*,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}.$$

VIDEO/INTERACTIVE

So far we have the first two vectors of our moving frame, so we just need to find the third. Let's think about how many potential candidates there are for the last vector in our moving frame.

If we have two orthonormal vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$, how many unit vectors in $\mathbb{R}^3$ are perpendicular to both of these vectors?

Multiple Choice:

(a) 0

(b) 1

(c) 2 ✓

(d) 3

(e) infinitely many

We'll choose the third and final vector $\vec{B}$ in our moving frame so that respects the right hand rule with the first two vectors.

PICTURE/GIF

In order to do this, we define the *unit binormal vector* to be

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

What is the relationship between $\vec{T}(t) \times \vec{N}(t)$ and $\vec{N}(t) \times \vec{T}(t)$? Select all that apply.

Select All Correct Answers:

(a) They're the same vector.

(b) They have the same length. ✓

(c) They point in the same direction.

(d) They point in opposite directions. ✓

(e) One is -1 times the other. ✓

So, we need to be careful about the order of this cross product, in order to choose our unit binormal vector consistently.

Together, $\vec{T}$, $\vec{N}$ and $\vec{B}$ are three mutually perpendicular unit vectors. They form an orthonormal basis for $\mathbb{R}^3$, which changes as we move along the path.

VIDEO/INTERACTIVE

We now summarize the above derivations in our definition of the moving frame.

Definition 33. *Given a path $\vec{x}(t)$, we define the* moving frame *of the path to be the triple $(\vec{T}, \vec{N}, \vec{B})$.*

$\vec{T}$ *is the* unit tangent vector*,*

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}.$$

$\vec{N}$ *is the* unit normal vector*,*

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}.$$

$\vec{B}$ *is the* unit binormal vector*,*

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

The moving frame is also called the TNB frame*.*

Moving Frame Computations

Recall our definition of the moving frame:

Definition 34. *Given a path $\vec{x}(t)$, we define the* moving frame *of the path to be the triple $(\vec{T}, \vec{N}, \vec{B})$.*

$\vec{T}$ is the unit tangent vector*,*

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}.$$

$\vec{N}$ is the unit normal vector*,*

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}.$$

$\vec{B}$ is the unit binormal vector*,*

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

The moving frame is also called the TNB frame*.*

We'll now work through some moving frame computations. These computations can sometimes get quite nasty, so it's important to simplify as you go, and plug in points when possible.

Examples

Example 52. *We'll compute the moving frame for the path $\vec{x}(t) = (\cos(t), \sin(t), t)$ in general, and when $t = 0$.*

In order to find the unit tangent vector, we first need to find the velocity vector.

$$\vec{x}'(t) = \boxed{(-\sin(t), \cos(t), 1)}$$

Computing the length, we have

$$\|\vec{x}'(t)\| = \boxed{\sqrt{2}}.$$

Learning outcomes:
Author(s):

From these, we compute

$$\begin{aligned}\vec{T}(t) &= \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} \\ &= \boxed{(-\frac{1}{\sqrt{2}}\sin(t), \frac{1}{\sqrt{2}}\cos(t), \frac{1}{\sqrt{2}})}.\end{aligned}$$

Plugging in $t = 0$, we have

$$\vec{T}(0) = \boxed{(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}.$$

Next, we need to find the unit binormal vector. For this, we first need to find $\vec{T}'(t)$.

$$\vec{T}'(t) = \boxed{(-\frac{1}{\sqrt{2}}\cos(t), -\frac{1}{\sqrt{2}}\sin(t), 0)}$$

Computing the length, we have

$$\|\vec{T}'(t)\| = \boxed{\frac{1}{\sqrt{2}}}.$$

From these, we compute

$$\begin{aligned}\vec{N}(t) &= \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \\ &= \boxed{(-\cos(t), -\sin(t), 0)}.\end{aligned}$$

Plugging in $t = 0$, we have

$$\vec{N}(t) = \boxed{(-1, 0, 0)}.$$

Finally, we need to find the unit binormal vector.

$$\begin{aligned}\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\ &= \boxed{(\frac{1}{\sqrt{2}}\sin(t), -\frac{1}{\sqrt{2}}\cos(t), \frac{1}{\sqrt{2}})}\end{aligned}$$

Plugging in $t = 0$, we have

$$\vec{B}(0) = \boxed{(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}.$$

PICTURE/VIDEO/INTERACTIVE

Example 53. *We'll compute the moving frame for the path $\vec{x}(t) = (1, t, t^2)$ when $t = 1$.*

Since we only need the moving frame for one specific value of t, we'll plug in this value as soon as we can, to help simplify computation. However, we need to make sure that we've taken all necessary derivatives before plugging in $t = 1$.

In order to find the unit tangent vector, we first need to find the velocity vector.

$$\vec{x}'(t) = \boxed{(0, 1, 2t)}$$

If we only needed to find $\vec{T}(1)$, we could plug in $t = 1$ at this point. However, we will eventually need to differentiate $\vec{T}(t)$ to find $\vec{N}(t)$, so we'll hold off on plugging in $t = 1$ for now.

Next, we find the length of $\vec{x}'(t)$.

$$\|\vec{x}'(t)\| = \boxed{\sqrt{1 + 4t^2}}$$

Then, we have that the unit tangent vector is

$$\begin{aligned}\vec{T}(t) &= \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} \\ &= \boxed{(0, \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}})}.\end{aligned}$$

Plugging in $t = 1$, we have

$$\vec{T}(1) = \boxed{(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})}.$$

Next, we differentiate $\vec{T}(t)$. Take a moment to revel in gratitude that we're doing this computation for you.

$$\vec{T}'(t) = \left(0, \frac{-4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}\right)$$

At this point, we don't have anymore derivatives to take, so we'll plug in $t = 1$ before continuing our computation.

$$\vec{T}'(1) = \boxed{(0, \frac{-4}{5^{3/2}}, \frac{2}{5^{3/2}})}$$

The length of this vector is

$$\|\vec{T}'(1)\| = \boxed{2/5}.$$

From these, we compute the unit normal vector when $t = 1$*,*

$$\begin{aligned}\vec{N}(1) &= \frac{\vec{T}'(1)}{\|\vec{T}'(1)\|} \\ &= \boxed{(0, \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}})}.\end{aligned}$$

Finally, we compute the unit binormal vector when $t = 1$*.*

$$\begin{aligned}\vec{B}(1) &= \vec{T}(1) \times \vec{N}(1) \\ &= \boxed{(1, 0, 0)}\end{aligned}$$

Notice how helpful it was to plug in $t = 1$ *as early as we could!*

PICTURE/VIDEO/INTERACTIVE

Decomposition of Acceleration

Recall our definition of the moving frame:

Definition 35. *Given a path* $\vec{x}(t)$*, we define the* moving frame *of the path to be the triple* $(\vec{T}, \vec{N}, \vec{B})$.

$\vec{T}$ *is the* unit tangent vector*,*

$$\vec{T}(t) = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|}.$$

$\vec{N}$ *is the* unit normal vector*,*

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}.$$

$\vec{B}$ *is the* unit binormal vector*,*

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

The moving frame is also called the TNB frame*.*

Throughout our study of paths, we've found a lot of ways that we can describe the behavior of the path, in addition to the moving frame:

- The velocity vector, $\vec{v}(t) = \vec{x}'(t)$.
- The speed, $s'(t) = \|\vec{x}'(t)\|$.
- The acceleration, $\vec{a}(t) = \vec{x}''(t)$.
- The arclength function, $s(t) = \displaystyle\int_a^t \|\vec{x}'(\tau)\| d\tau$.
- The parametrization with respect to arclength, $\vec{x}(s)$.
- The curvature, $\kappa(t) = \|\vec{T}'(s)\| = \dfrac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}$.
- The osculating circle and osculating plane (DID I EVER DEFINE THIS?).

We'll now explore the connections between these concepts. In particular, we'll derive a useful decomposition of the acceleration vector as a linear combination of the unit tangent and unit normal vectors.

Learning outcomes:
Author(s):

Curvature and torsion

We begin with the following result, which connects the curvature and unit normal vector with the derivative of the unit tangent vector with respect to arclength.

Proposition 20.

$$\frac{d\vec{T}}{ds} = \kappa \vec{N}$$

Proof First, thinking of arclength s as a function of t and using the chain rule, we have

$$\frac{d}{dt}\vec{T}(s(t)) = \vec{T}'(s(t))s'(t).$$

Recognizing $s'(t)$ as the speed, we can rewrite this as

$$\frac{d}{dt}\vec{T}(t) = \vec{T}'(s)\|\vec{x}'(t)\|.$$

Solving for $\vec{T}'(s)$, we have

$$\frac{d\vec{T}}{ds} = \frac{\vec{T}'(t)}{\|\vec{x}'(t)\|}.$$

Turning to the other side of the equality, recall that we defined the curvature to be $\kappa(t) = \|\vec{T}'(s)\|$, and we found that we could also compute this as $\dfrac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}$.

We defined the unit normal vector to be $\dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$.

Putting all of this together, we have

$$\begin{aligned} \kappa(t)\vec{N}(t) &= \frac{\|\vec{T}'(t)\|}{\|\vec{x}'(t)\|}\frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \\ &= \frac{\vec{T}'(t)}{\|\vec{x}'(t)\|} \\ &= \frac{d\vec{T}}{ds}, \end{aligned}$$

proving our result. ■

It turns out that there is a similar result relating the normal vector with the derivative of the unit binormal vector with respect to arclength:

$$\frac{d\vec{B}}{ds} = \tau \vec{N}.$$

We haven't talked about the coefficient τ yet, but this is another important property of curves, called *torsion.*

Together, the curvature and torsion carry a lot of important information about the curve. In fact, the curvature and torsion completely determine the curve!

Decomposition of acceleration

Recall that the unit tangent vector $\vec{T}$ points in the direction of instantaneous motion, and the unit normal vector $\vec{N}$ points in the direction that a path is turning. So, it shouldn't be too surprising that the acceleration vector is always a linear combination of $\vec{T}$ and $\vec{N}$. However, it's very surprising that we can recognize the coefficients in terms of things we've seen before!

Proposition 21. *Let $\vec{x}(t)$ be a C^2 path in $\mathbb{R}^3$. Then the acceleration vector can be written as*

$$\vec{a} = s''\vec{T} + (s')^2\kappa\vec{N},$$

where s is the arclength function (so s' is the speed), $\vec{T}$ is the unit tangent vector, κ is the curvature, and $\vec{N}$ is the unit normal vector.

Proof We begin with some observations.

Recall that we defined the unit tangent vector as $\vec{T}$ as $\vec{T} = \dfrac{\vec{v}}{\|\vec{v}\|}$, where $\vec{v}(t) = \vec{x}'(t)$ is the velocity vector. Replacing the speed $\|\vec{v}\|$ with s', this gives us

$$\vec{T} = s'\vec{v}.$$

Similarly, we defined the unit normal vector as $\vec{N} = \dfrac{\vec{T}'}{\|\vec{T}'\|}$, so we can write

$$\vec{T}' = \|\vec{T}'\|\vec{N}.$$

We have that the curvature is $\kappa = \dfrac{\|\vec{T}'\|}{s'}$, so we can rewrite this as

$$\begin{aligned} \vec{T}' &= \|\vec{T}'\|\vec{N} \\ &= (s'\kappa)\vec{N}. \end{aligned}$$

Finally, we turn to our acceleration vector. We recall that acceleration is the derivative of velocity, and use the product rule with the above observations to obtain

$$\begin{aligned}
\vec{a} &= \vec{v}' \\
&= \frac{d}{dt}(s'\vec{T}) \\
&= s''\vec{T} + s'\vec{T}' \\
&= s''\vec{T} + s'(s'\kappa)\vec{N} \\
&= s''\vec{T} + (s')^2\kappa\vec{N}.
\end{aligned}$$

■

Immediately from this result, we can make a lot of important observations.

Proposition 22. *Let $\vec{x}(t)$ be a C^2 path in $\mathbb{R}^3$. Then:*

- *$\vec{a}$ is a linear combination of $\vec{T}$ and $\vec{N}$.*
- *$\vec{a}$ is always in the osculating plane.*
- *Since $(s')^2 \geq 0$ and $\kappa \geq 0$, the acceleration $\vec{a}$ points in the direction that we're turning.*
- *If $\kappa = 0$, then $\vec{a}$ is parallel to $\vec{T}$.*
- *If speed is constant, then $s'' = 0$, so $\vec{a}$ is parallel to $\vec{N}$.*

We leave the proofs of these facts as an exercise.

Summary of notation for parametric curves

There are a lot of symbols to keep track of when studying the geometry of parametric curves. To make matters worse, most of them have multiple names. For example, the derivative of $\vec{x}(t)$ can be denoted by either $\vec{x}'(t)$ or $\dot{\mathbf{x}}(x)$, but we often call it $\vec{v}(t)$ because it represents velocity. Given a parametrization $\vec{x}(t)$, $t \in [a, b]$, which represents motion of a particle along a curve C, we list most of the related functions and their interpretations.

Position vector

- Notation: $\vec{x}(t)$, $\mathbf{x}(t)$
- Represents the position of a particle at time t

Tangent vector

- Notation: $\vec{x}'(t)$, $\dot{\mathbf{x}}(t)$, $\vec{v}(t)$, $\mathbf{x}'(t)$, $\mathbf{v}(t)$
- Derivative of $\vec{x}(t)$; also called the velocity vector.
- Its direction shows the direction of instantaneous motion, and its length ($||\vec{v}(t)|| = ||\vec{x}'(t)||$ etc.) is the instantaneous speed.

Unit tangent vector

- Notation: $\vec{T}(t)$
- Computed as $\vec{x}'(t)/||\vec{x}'(t)||$, $\vec{v}(t)/||\vec{v}(t)||$, etc.

Derivative of the unit tangent vector

- Notation: $\vec{T}'(t)$, $d\vec{T}/dt$
- Derivative of unit tangent vector with respect to time.
- Not necessarily a unit vector.
- Must be perpendicular to $\vec{T}(t)$: because $\vec{T}$ is a unit vector, $\vec{T} \cdot \vec{T} = 1$; differentiating both sides with respect to t gives $2\vec{T} \cdot \vec{T}' = 0$.

Unit normal vector

- Notation: $\vec{N}(t)$
- Computed as $\vec{T}'(t)/||\vec{T}'(t)||$.
- Perpendicular to $\vec{T}$, since it's a scaled version of $\vec{T}'$.

Binormal vector

- Notation: $\vec{B}(t)$
- Computed as $\vec{B} = \vec{T} \times \vec{N}$.
- Is a unit vector, since the angle between $\vec{T}$ and $\vec{N}$ is $\theta = \pi/2$ and therefore $||\vec{T} \times \vec{N}|| = ||\vec{T}||\,||\vec{N}||\sin\theta = 1$.

Distance Traveled

- Notation: s, $s(t)$
- Written as s when it's treated as a variable.
- We often view s as a function of time, and compute the arclength function $s(t) = \displaystyle\int_a^t ||\vec{x}'(u)||\,du$.

Speed

- Notation: ds/dt, $s'(t)$
- Computed as $ds/dt = s'(t) = ||\vec{x}'(t)|| = ||\vec{v}(t)||$ as proven by applying the Fundamental Theorem of Calculus to the definition of $s(t)$.

Curvature

- Notation: κ
- Measures how quickly a curve "turns" at a given point: $\kappa = \left| | \frac{d\vec{T}}{ds} \right| |$.
- Using the chain rule we can write κ as a function of t:

$$\kappa(t) = \left|\left|\frac{d\vec{T}}{ds}\right|\right| = \left|\left|\frac{d\vec{T}}{dt}\frac{dt}{ds}\right|\right| = \left|\left|\frac{d\vec{T}/dt}{ds/dt}\right|\right| = \frac{||\vec{T}'(t)||}{s'(t)} = \frac{||\vec{T}'(t)||}{||\vec{x}'(t)||}$$

- Many other formulas exist, e.g. for curves which are the graphs of functions $y = f(x)$.

Acceleration vector

- Notation: $\vec{x}''(t)$, $\ddot{\mathbf{x}}(t)$, $\vec{x}''(t)$, $\vec{a}(t)$, $\vec{a}(t)$
- Second derivative of $\vec{x}(t)$ and first derivative of velocity.
- If we rewrite $\vec{T}(t) = \vec{v}(t)/||\vec{v}(t)|| = \vec{v}(t)/s'(t)$ as $\vec{v} = s'\vec{T}$, we can apply the product rule to calculate:

$$\vec{a}(t) = s''(t)\vec{T}(t) + s'(t)\vec{T}'(t) = s''\vec{T} + s'||\vec{T}'||\vec{N} = s''\vec{T} + (s')^2\kappa\vec{N}$$

  The second equation comes from rewriting $\vec{N}(t) = \vec{T}'(t)/||\vec{T}'(t)||$ as $\vec{T}' = ||\vec{T}'||\vec{N}$ and substituting. The third equation used the fact that $\kappa = ||\vec{T}'||/s'$, so that $||\vec{T}'|| = \kappa s'$. We've made the scalar functions red and the vectors black to emphasize that acceleration is a linear combination of $\vec{T}$ and $\vec{N}$ (or $\vec{T}'$).

I COPIED THIS FROM THE HANDOUT (except for formatting, some notation, and a little rewriting)... HOPEFULLY THAT'S FINE?

SOME SORT OF RANDOMIZED NOTATION QUIZ?

Part IV

Limits and Derivatives

Introduction to Limits

We would like to eventually define derivatives and integrals for functions on $\mathbb{R}^n$, but before we do this, we'll need to study limits.

In single variable calculus, we gave a formal, epsilon-delta definition of limits.

Definition 36. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We write*

$$\lim_{x \to a} f(x) = L$$

if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all x with $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon$.

The idea here is that if x gets close enough to a, then $f(x)$ is guaranteed to get close to L. This leads us to our second, informal definition of a limit.

Definition 37. *(Informal definition) We say that*

$$\lim_{x \to a} f(x) = L$$

if $f(x)$ approaches L as x approaches a.

When we think of x approaching a along the number line, $\mathbb{R}$, we can approach a from two directions: left and right.

VISUAL

This lead us to the idea of left and right limits.

When we have a function whose domain is a subset of $\mathbb{R}^n$, there are infinitely many possible ways to approach a point. We can approach a point along infinitely many different lines, and we can also "zig-zag" or "spiral" into a point.

VISUAL

This makes limits in $\mathbb{R}^n$ particularly challenging!

Learning outcomes:
Author(s):

Showing that limits do not exist

For now, we'll work with an informal, intuitive definition of limits of functions $f : \mathbb{R}^n \to \mathbb{R}$. We'll revisit this definition, and provide a formal, epsilon-delta definition, in a later section. We model our informal definition after our definition from single variable calculus.

Definition 38. *Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that*

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$$

if $f(\vec{x})$ gets close to L as $\vec{x}$ gets close to $\vec{a}$.

Note that since f is a function from $\mathbb{R}^n$ to $\mathbb{R}$, the inputs of f are points/vectors in $\mathbb{R}^n$, and the outputs of f are numbers in $\mathbb{R}$.

An important consequence of this definition is that if we approach the point $\vec{a}$ along any path, the value of the function f should always approach the limit L (if the limit exists). This provides us with an important tool for showing that some limits do not exist.

Proposition 23. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, and a point $\vec{a} \in \mathbb{R}^n$. Suppose there are continuous paths $vecx(t)$ and $\vec{y}(t)$ such that $\vec{x}(t_1) = \vec{y}(t_2) = \vec{a}$, and suppose that*

$$\lim_{t \to t_1} f(\vec{x}(t)) \neq \lim_{t \to t_2} f(\vec{y}(t))$$

(or one of these limits does not exist). Then $\lim_{\vec{x} \to \vec{a}} f(x)$ does not exist.

Note that the contrapositive of this statement is false: if we find two paths along which a function has the same limit, this does not guarantee that the overall limit exists.

We will prove this proposition once we give the epsilon-delta definition of a limit, but for now, we'll use it to show that some limits do not exist.

Example 54. *Consider the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. We will show that $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.*

First, let's see what happens when we approach the origin along the x-axis. In this case, the y-coordinate will always be 0. More specifically, we approach along the path $\vec{x}(t) = (t, 0)$, and let $t \to 0$. We find the limit of f along this path:

$$\begin{aligned}
\lim_{t \to 0} f(\vec{x}(t)) &= \lim_{t \to 0} f(t, 0) \\
&= \lim_{t \to 0} \frac{t^2 - 0^2}{t^2 + 0^2} \\
&= \lim_{t \to 0} 1 \\
&= 1.
\end{aligned}$$

Next, let's see what happens when we approach the origin along the y-axis. That is, we'll consider the path $\vec{y}(t) = (0, t)$, and take $t \to 0$. We find the limit of f along this path:

$$\begin{aligned}\lim_{t\to 0} f(\vec{y}(t)) &= \lim_{t\to 0} f(0,t) \\ &= \lim_{t\to 0} \frac{0^2 - t^2}{0^2 + t^2} \\ &= \boxed{-1}.\end{aligned}$$

Thus, we have found two paths along which f approaches different values. This means that $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ does not exist.

We can see this behavior reflected in the graph of f.

GRAPH

Example 55. *Consider the function $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$. We'll investigate whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.*

First, let's see what happens when we approach the origin along the x- and y-axes.

Along the x-axis, we use the path $\vec{x}(t) = (t, 0)$, and we have

$$\begin{aligned}\lim_{t\to 0} f(\vec{x}(t)) &= \lim_{t\to 0} f(t,0) \\ &= \lim_{t\to 0} \frac{t^2 0}{t^4 + 0^2} \\ &= \boxed{0}.\end{aligned}$$

Along the y-axis, we use the path $\vec{y}(t) = (0, t)$, and we have

$$\begin{aligned}\lim_{t\to 0} f(\vec{y}(t)) &= \lim_{t\to 0} f(0,t) \\ &= \lim_{t\to 0} \frac{0^2 t}{0^4 + t^2} \\ &= \boxed{0}.\end{aligned}$$

Based on these limits, what can we conclude about $\lim_{(x,y)\to(0,0)} f(x,y)$?

Multiple Choice:

(a) *It exists, and equals 0.*

(b) *It doesn't exist.*

(c) *We still don't know if it exists or not.* ✓

Next, let's see what happens when we approach the origin along any line $y = mx$. We can parametrize this line as $\vec{z}(t) = (t, mt)$. Along this line, we have

$$\begin{aligned}
\lim_{t\to 0} f(\vec{z}(t)) &= \lim_{t\to 0} f(t, mt) \\
&= \lim_{t\to 0} \frac{t^2 mt}{t^4 + (mt)^2} \\
&= \lim_{t\to 0} \frac{mt^3}{t^4 + (mt)^2} \\
&= \lim_{t\to 0} \frac{mt}{t^2 + m^2} \\
&= \boxed{0}.
\end{aligned}$$

Based on the above limits, what can we conclude about $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$*?*

Multiple Choice:

(a) *It exists, and equals* 0.

(b) *It doesn't exist.*

(c) *We still don't know if it exists or not.* ✓

Finally, let's see what happens when we approach the origin along the parabola $y = x^2$, which can be parametrized as $\vec{w}(t) = (t, t^2)$. Along this path, we have

$$\begin{aligned}
\lim_{t\to 0} f(\vec{q}(t)) &= \lim_{t\to 0} f(t, t^2) \\
&= \lim_{t\to 0} \frac{t^2 t^2}{t^4 + (t^2)^2} \\
&= \lim_{t\to 0} \frac{t^4}{t^4 + mt^4} \\
&= \boxed{1/2}.
\end{aligned}$$

Based on the above limits, what can we conclude about $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$*?*

Multiple Choice:

(a) *It exists, and equals* 0.

(b) *It doesn't exist.* ✓

(c) *We still don't know if it exists or not.*

In the previous example, we saw that we might get the same limit approaching along any line through the origin, but it's still possible that the overall limit might not exist. Thus, we won't be able to show that limits exist by examining specific paths, and we'll need to find other methods to evaluate limits.

Evaluating Limits

We've seen how we can approach along paths to show that some limits do not exist, but we still don't have any methods for showing that multivariable limits do exist. Our first tool for doing this will be the epsilon-delta definition of a limit, which will allow us to formally prove that a limit exists.

Unfortunately, the epsilon-delta approach has some draw backs. Epsilon-delta proofs can be difficult, and they often require you to either guess or compute the value of a limit prior to starting the proof! So, we will want some easier methods for evaluating limits. One such method will be changing coordinates in a way that reduces our limit to a single variable limit.

Epsilon-delta definition

Let's begin by recalling the epsilon-delta definition of a limit from single variable calculus.

Definition 39. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We write*

$$\lim_{x \to a} f(x) = L$$

if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all x with $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon$.

VISUAL

We use this definition to guide our formal definition of a limit for multivariable functions.

Definition 40. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We write*

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$$

if for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $\vec{x}$ with $0 < \|\vec{x} - \vec{a}\| < \delta$, we have $|f(x) - L| < \epsilon$.

Because the inputs here are points in $\mathbb{R}^n$, when we take points "close to" $\vec{a}$, we do this in terms of distance. Recall that we can compute the distance between points $\vec{x} = (x_1, ..., x_n)$ and $\vec{y} = (y_1, ..., y_n)$ as

$$\|\vec{x} - \vec{y}\| = \sqrt{(x_1 - y_1)^2 + (x_n - y_n)^2}.$$

Learning outcomes:
Author(s):

So when we're considering all $\vec{x}$ such that $\|\vec{x}-\vec{a}\| < \delta$, we're taking all points within a distance δ of $\vec{a}$. This is called the *open ball of radius δ centered at $\vec{a}$.*

PICTURES

When we add in the condition that $0 < \|\vec{x}-\vec{a}\|$, we are excluding the point a itself.

Example 56. *We'll prove that $\lim_{\vec{x}\to(1,2,3)}(2x-y+5z) = 15$, using the epsilon-delta definition of a limit.*

Given $\epsilon > 0$, we'd like to find δ such that $0 < \|(x,y,z)-(1,2,3)\| < \delta$ guarantees $|(2x-y+5z)-15| < \epsilon$. Let's see what we can get from $0 < \|(x,y,z)-(1,2,3)\| < \delta$. We have

$$0 < \sqrt{(x-1)^2+(y-2)^2+(z-3)^2} < \delta.$$

Using the triangle inequality (DOES THIS SHOW UP IN THE REVIEW?), we have

$$0 < \sqrt{(x-1)^2} + \sqrt{(y-2)^2} + \sqrt{(z-3)^2} < \delta,$$

which we can rewrite as

$$0 < |x-1| + |y-2| + |z-3| < \delta.$$

From this, we have $|x-1| < \delta$, $|y-2| < \delta$, and $|z-3| < \delta$.

Now, let's look at $|(2x-y+5z)-15|$, and try to rewrite this to make use of what we've found so far. We have

$$\begin{aligned} |(2x-y+5z)-15| &= |(2x-2)+(-y+2)+(5z-15)| \\ &= |2(x-1)-(y-2)+5(z-3)| \\ &\geq |2(x-1)|+|y-2|+|5(z-3)|, \end{aligned}$$

using the triangle inequality. By properties of absolute values, and using our above observations, we have

$$\begin{aligned} |2(x-1)|+|y-2|+|5(z-3)| &= 2|x-1|+|y-2|+5|z-3| \\ &= 2\delta+\delta+5\delta \\ &= 8\delta. \end{aligned}$$

So if we choose $\delta = \epsilon/8$, we'll have $|(2x-y+5z)-15| < \epsilon$ for all (x,y,z) with $0 < \|(x,y,z)-(1,2,3)\| < \delta$.

SHOULD I DO THE WHOLE PROOF?

Now that we have a formal definition of limits, let's revisit our result about approaching a point along various paths.

Proposition 24. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, and a point $\vec{a} \in \mathbb{R}^n$. Suppose there are continuous paths $vecx(t)$ and $\vec{y}(t)$ such that $\vec{x}(t_1) = \vec{y}(t_2) = \vec{a}$, and suppose that*

$$\lim_{t\to t_1} f(\vec{x}(t)) \neq \lim_{t\to t_2} f(\vec{y}(t))$$

(or one of these limits does not exist). Then $\lim_{\vec{x}\to\vec{a}} f(x)$ *does not exist.*

Proof (I'M NOT SURE THIS IS ACTUALLY WORTHWHILE...SO I STOPPED WRITING IT)

In this proof, we'll handle the case where both limits exist. Suppose $\lim_{t\to t_1} f(\vec{x}(t)) = L$ and $\lim_{t\to t_2} f(\vec{y}(t)) = M$. Let $\epsilon = \frac{|L-M|}{4}$.

We will do a proof by contradiction. Suppose $\lim_{\vec{x}\to\vec{a}} f(x) = N$. Then there exists δ such that

Since $\lim_{t\to t_1} f(\vec{x}(t)) = L$, there exists some $\delta_1 > 0$ such that for all t with $0 < |t-t_1| < \delta_1$, we have $|f(\vec{x}(t)) - L| < \epsilon$. Since $\vec{x}$ is continuous, we can also find some δ_1' such that $\|\vec{x}(t) - \vec{a}\| < \delta$ for $|t - t_1| < \delta_1'$.

Since $\lim_{t\to t_2} f(\vec{y}(t)) = M$, there exists some $\delta_2 > 0$ such that for all t with $0 < |t-t_2| < \delta_2$, we have $|f(\vec{x}(t)) - L| < \epsilon$.

■

Changing coordinates

Although doing a delta-epsilon proof can be effective for proving that a limit exists and what it's equal to, we still need to predict the value of a limit before starting such a proof. So, we'd like some other techniques for showing that multivariable limits exist, and for evaluating them.

One strategy for evaluating limits is to change coordinates in a way that reduces our multivariable limit to a single variable limit.

Suppose we're taking the limit of a function $f(x, y)$ as $(x, y) \to (0, 0)$, so we're approaching the origin. In polar coordinates (θ, r), approaching the origin is equivalent to taking $r \to 0$. It doesn't matter what θ does; as long as r goes to 0, we will be approaching the origin.

VISUAL

This makes polar coordinates a common and convenient choice for a change of variables to evaluate limits.

Example 57. *We'll use polar coordinates to evaluate the limit* $\lim_{(x,y)\to(0,0)} xy$. *Changing to polar coordinates, we have*

$$\begin{aligned}\lim_{(x,y)\to(0,0)} xy &= \lim_{r\to 0} r\cos\theta \cdot r\sin\theta \\ &= \lim_{r\to 0} r^2 \cos\theta \sin\theta.\end{aligned}$$

We will use the squeeze theorem to evaluate this limit. Since $-1 \le \cos\theta\sin\theta \le 1$ *for all* θ*, we have*

$$-r^2 \le r^2\cos\theta\sin\theta \le r^2.$$

Since $\lim_{r\to 0} -r^2 = \lim_{r\to 0} r^2 = 0$*, by the squeeze theorem, we have*

$$\lim_{r\to 0} r^2\cos\theta\sin\theta = 0.$$

Thus, $\lim_{(x,y)\to(0,0)} xy = 0$*.*

Similarly, when the domain of a function is $\mathbb{R}^3$, we can use spherical coordinates to evaluate a limit approaching the origin.

Example 58. *We will use spherical coordinates to evaluate the limit* $\lim_{(x,y,z)\to(0,0,0)} \frac{x^3}{x^2+y^2+z^2}$*. Changing to spherical coordinates, we have*

$$\begin{aligned}
\lim_{(x,y,z)\to(0,0,0)} \frac{x^3}{x^2+y^2+z^2} &= \lim_{\rho\to 0} \frac{(\rho\cos\theta\sin\phi)^3}{(\rho\cos\theta\sin\phi)^2+(\rho\sin\theta\sin\phi)^2+(\rho\cos\phi)^2} \\
&= \lim_{\rho\to 0} \frac{\rho^3\cos^3\theta\sin^3\phi}{\rho^2\cos^2\theta\sin^2\phi+\rho^2\sin^2\theta\sin^2\phi+\rho^2\cos^2\phi} \\
&= \lim_{\rho\to 0} \frac{\rho^3\cos^3\theta\sin^3\phi}{(\rho^2\cos^2\theta+\rho^2\sin^2\theta)\sin^2\phi+\rho^2\cos^2\phi} \\
&= \lim_{\rho\to 0} \frac{\rho^3\cos^3\theta\sin^3\phi}{\rho^2\sin^2\phi+\rho^2\cos^2\phi} \\
&= \lim_{\rho\to 0} \frac{\rho^3\cos^3\theta\sin^3\phi}{\rho^2} \\
&= \lim_{\rho\to 0} \rho\cos^3\theta\sin^3\phi.
\end{aligned}$$

Since $-1 \le \cos^3\theta\sin^3\phi \le 1$*, we have*

$$-|\rho| \le \rho\cos^3\theta\sin^3\phi \le |\rho|.$$

Since $\lim_{\rho\to 0} -|\rho| = \lim_{\rho\to 0} |\rho| = 0$*, by the squeeze theorem, we have*

$$\lim_{\rho\to 0} \rho\cos^3\theta\sin^3\phi = 0,$$

So $\lim_{(x,y,z)\to(0,0,0)} \frac{x^3}{x^2+y^2+z^2}$*.*

In other situations, a different change of coordinates might be more useful. For example, linear changes of coordinates might be used.

Example 59. *We'll use a change of coordinates to evaluate* $\lim_{(x,y)\to(0,0)}(x-y)e^{-(x-y)^2}$*. Here, it's convenient to use the linear change of coordinates* $y = x - y$ *and* $v = x + y$*. Notice that* $(x,y) \to (0,0)$ *is equivalent to* $(u,v) \to (0,0)$*. From this, we have*

$$\lim_{(x,y)\to(0,0)}(x-y)e^{-(x-y)^2} = \lim_{(u,v)\to(0,0)} ue^{-u^2}.$$

Notice that the expression on the right depends only on u*, and not on* v*. Because of this, we can evaluate this limit as*

$$\begin{aligned}\lim_{(u,v)\to(0,0)} ue^{-u^2} &= \lim_{u\to 0} ue^{-u^2} \\ &= \lim_{u\to 0}\frac{u}{e^{u^2}} &&= \boxed{0}.\end{aligned}$$

Thus, we have $\lim_{(x,y)\to(0,0)}(x-y)e^{-(x-y)^2} = 0$.

We can also change coordinates to help us show that certain limits do not exist.

Example 60. *We will use polar coordinates to show that* $\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$ *doesn't exist. Changing to polar coordinates, we have*

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2} &= \lim_{r\to 0}\frac{r\cos\theta\cdot r\sin\theta}{r^2} \\ &= \lim_{r\to 0}\cos\theta\sin\theta.\end{aligned}$$

If we take $\theta = 0$*, this is equivalent to approaching the origin along the* x*-axis, and we have* $\cos\theta\sin\theta = 0$*.*

If we take $\theta = \pi/4$*, this is equivalent to approaching the origin along the line* $y = x$*, and we have* $\cos\theta\sin\theta = \frac{1}{2}$*.*

Since we get different values approaching the origin along different paths, we see that $\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$ *does not exist.*

Note that the change of coordinates here wasn't absolutely necessary. We could have directly evaluated the limits approaching along the x*-axis and along the line* $y = x$*, and seen that the limit does not exist. However, sometimes it's easier to see that a limit doesn't exist by attempting a change to polar coordinates, and finding that we end up with a limit that depends on the value of* θ*.*

In all of the above cases, we considered limits approaching the origin. We can use similar techniques to evaluate limits approaching other points, often by translating coordinate systems.

Example 61. *Consider the limit* $\lim_{(x,y)\to(2,3)}(x-2)(y-3)$. *Here, it's convenient to use polar coordinates translated so they're centered at the point* $(2,3)$. *That is, we let* $x = 2 + r\cos\theta$ *and* $y = 3 + r\sin\theta$, *so that* $r = 0$ *when* $(x,y) = (2,3)$. *Then we can evaluate our limit as*

$$\lim_{(x,y)\to(2,3)}(x-2)(y-3) = \lim_{r\to 0}(2 + r\cos\theta - 2)(3 + r\sin\theta - 3)$$

$$= \lim_{r\to 0} r\cos\theta \cdot r\sin\theta \qquad = \lim_{r\to 0} r^2\cos\theta\sin\theta$$

Notice that $-r^2 \leq r^2\cos\theta\sin\theta \leq r^2$, *and* $\lim_{r\to 0} -r^2 = \lim_{r\to 0} r^2 = 0$. *Then, by the squeeze theorem,* $\lim_{r\to 0} r^2\cos\theta\sin\theta = 0$. *Thus, we have*

$$\lim_{(x,y)\to(2,3)}(x-2)(y-3) = 0.$$

SHOULD THIS INCLUDE SOMETHING ABOUT THE REQUIREMENTS FOR A CHANGE OF COORDINATES?

Continuity and Limits in General

So far, we've seen how we can show that limits exist using a delta-epsilon proof, or by changing coordinates. In single variable calculus, we were often able to evaluate limits by direct substitution. For example, we could evaluate

$$\begin{aligned}\lim_{x\to 2} x^2 + x\sin(\pi x) &= 2^2 + 2\sin(\pi \cdot 2)\\ &= \boxed{4}.\end{aligned}$$

We are able to do this because the function $f(x) = x^2 + x\sin(\pi x)$ is continuous. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

For continuous functions, we can evaluate limits by simply plugging in the value.

Once we define continuity for multivariable functions, and determine which functions are continuous, we can use similar methods to evaluate multivariable limits.

Continuity

We define continuity similarly to how we did in single variable calculus.

Definition 41. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is* continuous *at $\vec{x} = \vec{a}$ in $\mathbb{R}^n$ if*

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a}).$$

Also similar to single variable calculus, virtually all of the common functions that we work with are continuous on their domains. That is, anywhere that they're defined, they are continuous.

Theorem 2. *The following functions are continuous on their domains:*

- *polynomials*
- *root functions*
- *rational functions*

Learning outcomes:
Author(s):

- *trigonometric functions and inverse trigonometric functions*
- *exponential functions and logarithmic functions*

Furthermore, all of the ways that we'd like to combine continuous functions will result in another continuous function.

Theorem 3. *If f is continuous at $\vec{x} = \vec{a}$, and k is a real number, then kf is also continuous at $\vec{a}$.*

If f and g are continuous at $\vec{x} = \vec{a} \in \mathbb{R}^n$, then $f + g$ is also continuous at $\vec{a}$.

If f and g are continuous at $\vec{x} = \vec{a} \in \mathbb{R}^n$, then fg is also continuous at $\vec{a}$.

If g is continuous at $\vec{x} = \vec{a}$, and f is continuous at $g(\vec{a})$, then $f \circ g$ is continuous at $\vec{a}$.

Example 62. *Which of the following functions is continuous at $(0, 0)$? Select all that apply.*

Select All Correct Answers:

(a) $f(x, y) = 3x^3 + 2xy^2 + x + 1$ ✓

(b) $g(x, y) = \sin(x) \cos(x)$ ✓

(c) $h(x, y) = \dfrac{x^2 + y^2 + 1}{x + y + 1}$ ✓

(d) $i(x, y) = \dfrac{x^2 y}{x^2 + y^2}$

(e) $j(x, y) = \tan(xy)$ ✓

(f) $k(x, y) = e^{\sin(x+y)}$ ✓

(g) $l(x, y) = \ln(x^2 + y^2)$

(h) $m(x, y) = \dfrac{1}{\ln(x^2 + y^2 + 2)}$ ✓

Example 63. *Evaluate the following limits, or enter "DNE" if they do not* *exist.*

$$\lim_{(x,y)\to(0,0)} 3x^3 + 2xy^2 + x + 1 = \boxed{1}$$

$$\lim_{(x,y)\to(0,0)} \sin(x)\cos(x) = \boxed{0}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+1}{x+y+1} = \boxed{1}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \boxed{0}$$

$$\lim_{(x,y)\to(0,0)} \tan(xy) = \boxed{0}$$

$$\lim_{(x,y)\to(0,0)} e^{\sin(x+y)} = \boxed{1}$$

$$\lim_{(x,y)\to(0,0)} \ln(x^2+y^2) = \boxed{DNE}$$

$$\lim_{(x,y)\to(0,0)} \frac{1}{\ln(x^2+y^2+2)} = \boxed{1/\ln(2)}$$

Note that even if a function is discontinuous at a point, it's still possible that the limit exists. In this case, you'll need to use a change of coordinates or other method to evaluate the limit.

Limits in General

So far, we've defined limits of scalar-valued functions, $\mathbb{R}^n \to \mathbb{R}$. We've seen how we can evaluate these limits, or show that they do not exist. However, we've yet to deal with more general multivariable functions, $\mathbb{R}^n \to \mathbb{R}^m$.

Fortunately, limits in the more general setting turn out to be an easy extension of the limits that we've already defined. That is, if we have a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$, we can write $\vec{f}$ in terms of its coordinate functions,

$$\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), ..., f_m(\vec{x})).$$

Then, we can use the limits of the coordinate functions to define a limit of f.

Definition 42. *Suppose we have a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ with $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), ..., f_m(\vec{x}))$. Then we define*

$$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \left(\lim_{\vec{x}\to\vec{a}} f_1(\vec{x}), \lim_{\vec{x}\to\vec{a}} f_2(\vec{x}), ..., \lim_{\vec{x}\to\vec{a}} f_m(\vec{x}) \right),$$

provided each of these limits exist.

So, we can evaluate the limit of a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ by taking the limit of the component functions. Because of this, the results and methods that we've used for limits of scalar functions carry naturally over to this more general setting.

Example 64. *Evaluate the following limits, or enter "DNE" if they do not exist.*

$$\lim_{(x,y,z)\to(1,2,3)} (x^2 + y, z, xz) = \boxed{(3,3,3)}$$

$$\lim_{(x,y)\to(0,1)} \left(\sin(x+y), \frac{1}{\ln(y)}\right) = \boxed{DNE}$$

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3 y}{x^2+y^2}, \frac{xy^3}{x^2+y^2}\right) = \boxed{(0,0)}$$

Partial Derivatives

Now that we've defined limits of multivariable function, we're ready to begin to explore how multivariable functions change, using derivatives. Let's recall how we found derivatives in single variable calculus, where they gave us a way to compute the instantaneous rate of change of a function.

We defined the *derivative* of a function f at a to be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This definition arose out of geometric considerations. The instantaneous rate of change of a function at a point can be found as the slope of the tangent line to the graph of the function at that point. Our key insight was that, as we get very close to a, the slope of secant lines will approach the slope of the tangent line. This lead us to our limit definition for the derivative. The quotient $\dfrac{f(a+h) - f(a)}{h}$ gives us the slope of the secant line between the points $(a, f(a))$ and $(a+h, f(a+h))$. As we take $h \to 0$, these points get closer together, and the slope of the secant line approaches the slope of the tangent line.

VISUAL

We're beginning to study how multivariable functions change, and we'd like to do this using derivatives. With multivariable functions, it's not clear what this should mean. For single variable functions the question was simple: if we change x, what happens to y? But with multivariable functions, we have multiple inputs, and we could change them in a variety of ways.

For example, consider the function $f(x, y) = x^2 + y$ at $(x, y) = (1, 1)$. We could look at how this function changes if we increase x by a little bit, and leave $y = 1$. We could also look at how this function changes if we increase y by a little bit, and leave $x = 1$. We could also look at how this function changes if we increase x and y by the same amount, or increase y by twice as much as x, or infinitely many other ways.

Because of the breadth of possibilities, it's hard to decide what a multivariable derivative should be. We'll revisit this question later, but for now, we'll see how a function $f : \mathbb{R}^n \to \mathbb{R}$ changes with respect to one input variable at a time.

Learning outcomes:
Author(s):

Definition of Partial Derivatives

In order to study how a function $f : \mathbb{R}^n \to \mathbb{R}$ changes with respect to one input variable, we keep the other variables constant, and change only that variable. This leads us to our definition of partial derivatives. For clarity, we'll begin with the $n = 2$ case, before introducing more general partial derivatives.

Definition 43. *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$.

We define the partial derivative of f with respect to x *to be*

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

provided this limit exists.

We define the partial derivative of f with respect to y *to be*

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

provided this limit exists.

Note that these are single variable limits, so we haven't taken advantage of our new multivariable limits yet.

Let's look at how we can compute partial derivatives, similarly to how we computed single variable derivatives using limits.

Example 65. *We'll compute the partial derivatives of the function* $f(x, y) = x^2 + xy + y$ *at the point* $(1, 2)$.

$$\begin{aligned} f_x(1,2) &= \lim_{h\to 0} \frac{f(1+h, 2) - f(1, 2)}{h} \\ &= \lim_{h\to 0} \frac{((1+h)^2 + (1+h)\cdot 2 + 2) - (1^2 + 1\cdot 2 + 2)}{h} \\ &= \lim_{h\to 0} \frac{((1+2h+h^2) + (2+2h) + 2) - 5}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 4h}{h} \\ &= \lim_{h\to 0} (h+4) \\ &= 4 \end{aligned}$$

$$\begin{aligned}
f_y(1,2) &= \lim_{h\to 0} \frac{f(1,2+h) - f(1,2)}{h} \\
&= \lim_{h\to 0} \frac{(1^2 + 1\cdot(2+h) + (2+h)) - (1^2 + 1\cdot 2 + 2)}{h} \\
&= \lim_{h\to 0} \frac{2h}{h} \\
&= \lim_{h\to 0} 2 \\
&= 2
\end{aligned}$$

More generally, for functions $f : \mathbb{R}^n \to \mathbb{R}$, partial derivatives are defined similarly.

Definition 44. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. For $1 \le i \le n$, we define the* partial derivative of f with respect to x_i *to be*

$$f_{x_i}(x_1, ..., x_n) = \lim_{h\to 0} \frac{f(x_1, ..., x_{i-1}, x_i + h, x_{i+1}, ..., x_n) - f(x_1, ..., x_n)}{h},$$

provided this limit exists.

ADD PIECEWISE EXAMPLE

Computing Partial Derivatives

When we compute partial derivatives, we're really just taking all but one of the input variables to be constant, and computing a single variable derivative with respect to the remaining variable. Because of this, all of the differentiation rules that we learned in single variable calculus will also apply to partial derivatives. This greatly simplifies computation of partial derivatives.

Example 66. *We'll compute the partial derivatives of $f(x, y) = x^2 + xy + y$.*

Thinking of y as a constant and differentiating with respect to x, we obtain the partial derivative with respect to x:

$$f_x(x, y) = 2x + y.$$

Thinking of x as a constant and differentiating with respect to y, we obtain the partial derivative with respect to y:

$$f_y(x, y) = x + 1.$$

Example 67. *For each of the following functions, compute the partial derivatives.*

$f(x, y) = x^2y^2 + x^2 + y^2$

$$f_x(x, y) = \boxed{2xy^2 + 2x}$$
$$f_y(x, y) = \boxed{2x^2y + 2y}$$

$g(x, y) = \sin(xy)$

$$g_x(x, y) = \boxed{y\cos(xy)}$$
$$g_y(x, y) = \boxed{x\cos(xy)}$$

$h(x, y, z) = xyz + xe^{xy}$

$$h_x(x, y, z) = \boxed{yz} + e^{xy} + xye^{xy}$$
$$h_y(x, y, z) = \boxed{xz} + x^2e^{xy}$$
$$h_z(x, y, z) = \boxed{xy}$$

Geometric Interpretation of Partial Derivatives

We've defined the partial derivatives of a function as follows.

Definition 45. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. For $1 \leq i \leq n$, we define the* partial derivative of f with respect to x_i *to be*

$$f_{x_i}(x_1, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, ..., x_{i-1}, x_i + h, x_{i+1}, ..., x_n) - f(x_1, ..., x_n)}{h},$$

provided this limit exists.

In other words, if we treat all variables except for x_i as constants, and differentiate with respect to x_i, we get the partial derivative with respect to x_i.

When computing a partial derivative with respect to x_i, we're looking at the instantaneous rate of change of f with respect to x_i, if we keep the rest of the variables constant. Roughly speaking, we're asking: how does increasing x_i a tiny bit affect the value of f?

We can see the partial derivatives reflected in the shape of the graph of f. So that we can visualize the graph of f, we'll focus on a function $f : \mathbb{R}^2 \to \mathbb{R}$, so we're considering the partial derivative of f with respect to x, and with respect to y.

Suppose at the point $(1, 2)$, we have that $f_x(1, 2) > 0$ and $f_y(1, 2) > 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will increase. If we move a tiny amount in the positive y direction, the value of f will increase as well.

INTERACTIVE

Similarly, suppose at the point $(1, 2)$, we have that $f_x(1, 2) < 0$ and $f_y(1, 2) < 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will decrease. If we move a tiny amount in the positive y direction, the value of f will decrease as well.

INTERACTIVE

Now, let's consider the case where $f_x(1, 2) > 0$ and $f_y(1, 2) < 0$. Then, around the point $(1, 2)$, if we move a tiny amount in the positive x direction, the value of f will increase. But, if we move a tiny amount in the positive y direction, the value of f will decrease.

Learning outcomes:
Author(s):

INTERACTIVE

Next, let's suppose that $f_x(1, 2) > 0$ and $f_y(1, 2) = 0$. As expected, f increases as we move a tiny amount in the positive x direction. On the other hand, the graph of f has flattened out as we move in the y direction. However, this doesn't mean that it's constant! It's just the instantaneous rate of change that's 0 at that one point.

INTERACTIVE

Now, let's look at a case where $f_x(1, 2) = 0$ and $f_y(1, 2) = 0$. As before, this does not mean that f is constant. This just means that the rates of change are both instantaneously 0. Points with this property will be important later in the course, when we study optimization.

INTERACTIVE

Geometry of Differentiability

In single variable calculus, derivatives were closely related to the slope of the tangent line to a graph at a point. We used this idea of the slope of the tangent line to define derivatives as a limit of slopes of secant lines,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

VISUAL

In the other direction, we were able to use differentiation rules to more easily find the equation for the tangent line to a graph at a point.

Example 68. *We'll find the equation for the tangent line to the graph of $f(x) = x^3 + 2x + 1$ at $x = 2$.*

We can find the slope of the tangent line by computing $f'(2)$. Using differentiation rules, we have

$$f'(x) = \boxed{3x^2 + 2}.$$

Plugging in $x = 2$, we have $f'(2) = \boxed{14}$.

Since the tangent line will have to pass through the point $(2, f(2))$, we compute

$$f(2) = \boxed{13}.$$

So, the tangent line will pass through the point $(2, 13)$, and will have slope 14. Writing the equation of the line in point-slope form, we have

$$y - 13 = \boxed{14(x - 2)}.$$

When a single variable function is differentiable, we can use the above method to find an equation for the tangent line. In addition, the tangent line provides us with a good linear approximation for the function.

We would like to do something analogous for multi-variable functions, but this raises a few questions. What would be the equivalent of the tangent line? What does it mean for a function to be differentiable?

As we begin to explore these questions, we'll focus on functions $\mathbb{R}^2 \to \mathbb{R}$, so that we can visualize their graphs.

Learning outcomes:
Author(s):

Geometric Interpretation of Differentiability

In single variable calculus, we could get a good sense of whether a function was differentiable by looking at its graph.

Problem 9 *For each of the graphs, determine whether the given function is differentiable at $x = a$.*

GRAPH PARABOLA

Multiple Choice:

(a) *differentiable* ✓

(b) *not differentiable*

GRAPH JUMP DISCONTINUITY

Multiple Choice:

(a) *differentiable*

(b) *not differentiable* ✓

GRAPH ABSOLUTE VALUE

Multiple Choice:

(a) *differentiable*

(b) *not differentiable* ✓

If there is a discontinuity or some sort of corner or cusp in the graph at a point, then the function will not be differentiable at that point. Roughly speaking, if we "zoom in" on the graph of a function near a point, and the graph looks very close to a line, then the function will be differentiable at that point.

PARABOLA ZOOM

We'll extend this idea to make our first, informal definition of differentiability for a function from $\mathbb{R}^2$ to $\mathbb{R}$.

Definition 46. *(Informal Definition) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose for some point (a, b) in $\mathbb{R}^2$, if we zoom in on the graph of f near the point (a, b), the graph of f looks like a plane. Then f is differentiable at (a, b).*

Although this definition provides us with nice geometric intuition for determining if a function is differentiable, it's not at all precise or rigorous. Eventually, we'll need a more formal definition of differentiability, so we'll return to this concept later.

For now, let's use this informal definition to investigate differentiability for a couple of functions.

Example 69. *Consider the function $f(x, y) = xy + 2x + y$, graphed below.*

ZOOMABLE GRAPH

Is f differentiable at $(0, 0)$?

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Example 70. *Consider the function*

$$f(x, y) = \begin{cases} 1 - |y| & \text{if } |y| \leq |x| \\ 1 - |x| & \text{if } |y| > |x| \end{cases},$$

graphed below.

ZOOMABLE GRAPH

Is f differentiable at $(0, 0)$?

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

The tangent plane

If a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, when we zoom in, the graph will look like a plane. Because of this, there will be a plane that's a good linear approximation for the function near that point. We can use the partial derivatives with respect to x and y to find an equation for this plane, which we call the tangent plane.

Definition 47. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then the* tangent plane *to the graph of f at (a, b) is defined by the equation*

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Let's revisit our previous examples, and see what happens with the tangent plane.

Example 71. *Consider the function $f(x,y) = xy + 2x + y$, which we found is differentiable at the point $(0,0)$. Let's find an equation for the tangent plane to the graph of f at $(0,0)$.*

We begin by finding the partial derivatives with respect to x and y.

$$\begin{aligned} f_x(x,y) &= \boxed{y+2} \\ f_y(x,y) &= \boxed{x+1} \end{aligned}$$

At $(0,0)$, we have

$$\begin{aligned} f_x(0,0) &= \boxed{2}, \\ f_y(0,0) &= \boxed{1}. \end{aligned}$$

Finding the value of f at $(0,0)$, we have

$$f(0,0) = \boxed{0}.$$

Putting all of this together, we obtain an equation for the tangent plane to the graph of f at $(0,0)$.

$$\begin{aligned} z &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= \boxed{2x+y} \end{aligned}$$

Example 72. *Consider the function*

$$f(x,y) = \begin{cases} 1-|y| & \textit{if } |y| \leq |x| \\ 1-|x| & \textit{if } |y| > |x| \end{cases},$$

which we found is not differentiable $(0,0)$. Even though this function is not differentiable, let's see what happens when we try to find an equation for the tangent plane.

To compute the partial derivatives with respect to x and y, we'll need to use the limit definition.

$$\begin{aligned}
f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{(1-|0|)-(1-|0|)}{h} \\
&= \lim_{h\to 0} \frac{0}{h} \\
&= 0. f_y(0,0) \qquad\qquad = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{(1-|0|)-(1-|0|)}{h} \\
&= \lim_{h\to 0} \frac{0}{h} \\
&= 0.
\end{aligned}$$

We can also find $f(0,0)$*,*

$$f(0,0) = 1.$$

So, we have all of the necessary pieces to find an equation for the (nonexistent) "tangent plane":

$$\begin{aligned}
z &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\
&= 1.
\end{aligned}$$

However, we decided that the function wasn't differentiable at $(0,0)$*, so the graph does not have a tangent plane at the point.*

This example brings up a couple of important points.

- It's possible for the partial derivatives of a function to all exist, and yet the function is not differentiable.
- It's possible that we can find the equation $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, and yet f has no tangent plane at the point (a,b).

For these reasons, differentiability is a much more subtle concept in multivariable calculus than it was in single variable calculus, and our next task will be to find a formal definition for differentiability.

Differentiability of Functions of Two Variables

So far, we have an informal definition of differentiability for functions $f\mathbb{R}^2 \to \mathbb{R}$: if the graph of f "looks like" a plane near a point, then f is differentiable at that point.

Definition 48. *(Informal Definition) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose for some point (a, b) in $\mathbb{R}^2$, if we zoom in on the graph of f near the point (a, b), the graph of f looks like a plane. Then f is differentiable at (a, b).*

In the case where a function is differentiable at a point, we defined the tangent plane at that point.

Definition 49. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then the* tangent plane *to the graph of f at (a, b) is defined by the equation*

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We would like a formal, precise definition of differentiability. The key idea behind this definition is that a function should be differentiable if the plane above is a "good" linear approximation. To see what this means, let's revisit the single variable case.

In single variable calculus, a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = a$ if the following limit exists:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

This limit exists if and only if

$$\lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right) = 0.$$

In turn, this is true if and only if

$$\lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = 0.$$

If we let $L(x) = f(a) + f'(a)(x - a)$, this is equivalent to

$$\lim_{x\to a} \frac{f(x) - L(x)}{x - a} = 0.$$

Learning outcomes:
Author(s):

Recall that $L(x)$, as defined above, is the linear approximation to f at $x = a$. This is also a function whose graph is the tangent line to f at $x = a$. So, roughly speaking, we have shown that a single variable function is differentiable if and only the difference between $f(x)$ and its linear approximation goes to 0 quickly as x approaches a.

This idea will inform our definition for differentiability of multivariable functions: a function will be differentiable at a point if it has a good linear approximation, which will mean that the difference between the function and the linear approximation gets small quickly as we approach the point.

Formal definition of differentiability

We are now in position to give our formal definition of differentiability for a function $f : \mathbb{R}^2 \to \mathbb{R}$. We'll make our definition so that a function is differentiable at a point if the difference between the function and the linear approximation

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

gets small "quickly". Here, "quickly" is relative to how $\vec{x}$ is approaching $\vec{a}$, so relative to the distance $\|\vec{x} - \vec{a}\|$ between these points.

Notice that the function $h(x, y)$ matches the equation for the tangent plane, when the function f is differentiable.

Definition 50. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, and suppose that the partial derivatives f_x and f_y are defined at the point $(x, y) = (a, b)$. Define the linear function*

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We say that f is differentiable *at $(x, y) = (a, b)$ if*

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0.$$

If either of the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ do not exist, or the above limit does not exist or is not 0, then f is not differentiable at (a, b).

We had previously used our informal definition of differentiability to determine that the function $f(x, y) = xy + 2x + y$ is differentiable at $(0, 0)$. Let's verify this using our new, formal definition of differentiability.

Example 73. *We'll show that the function $f(x, y) = xy+2x+y$ is differentiable at $(0, 0)$. In order to do this, we first need to find the function $h(x, y)$. This repeats earlier work, where we found the tangent plane to $f(x, y) = xy + 2x + y$ at $(0, 0)$.*

We begin by finding the partial derivatives with respect to x and y.

$$f_x(x,y) = \boxed{y+2}$$
$$f_y(x,y) = \boxed{x+1}$$

At $(0,0)$, we have

$$f_x(0,0) = \boxed{2},$$
$$f_y(0,0) = \boxed{1}.$$

Finding the value of f at $(0,0)$, we have

$$f(0,0) = \boxed{0}.$$

Putting all of this together, we obtain an equation for the function $h(x,y)$.

$$\begin{aligned} h(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= \boxed{2x+y} \end{aligned}$$

Now, we show that f is differentiable at $(a,b) = (0,0)$, by evaluating the limit

$$\begin{aligned} \lim_{(x,y)\to(a,b)} \frac{f(x,y)-h(x,y)}{\|(x,y)-(a,b)\|} &= \lim_{(x,y)\to(0,0)} \frac{(xy+2x+y)-(2x+y)}{\sqrt{(x-0)^2+(y-0)^2}} \\ &= \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}. \end{aligned}$$

Switching to polar coordinates, we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{r\cos\theta \cdot r\sin\theta}{\sqrt{r^2}} \\ &= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{|r|}. \end{aligned}$$

Since $-1 \leq \cos\theta\sin\theta \leq 1$, we have

$$-|r| \leq \frac{r^2\cos\theta\sin\theta}{|r|} \leq |r|.$$

Since $\lim_{r} \to 0 - |r| = \lim_{r} \to 0|r| = 0$, by the squeeze theorem, we have

$$\lim_{r\to} \frac{r^2\cos\theta\sin\theta}{|r|} = 0.$$

Thus, we have shown that $\lim_{(x,y)\to(0,0)} \dfrac{f(x,y)-h(x,y)}{\|(x,y)-(0,0)\|} = 0$, showing that f is differentiable at $(0,0)$.

The Gradient

We've given a formal definition for differentiability of a function $f : \mathbb{R}^2 \to \mathbb{R}$,

Definition 51. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, and suppose that the partial derivatives f_x and f_y are defined at the point $(x, y) = (a, b)$. Define the linear function*

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We say that f is differentiable *at $(x, y) = (a, b)$ if*

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0.$$

If either of the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ do not exist, or the above limit does not exist or is not 0, then f is not differentiable at (a, b).

The idea behind this definition is that $h(x, y)$ will be a "good" linear approximation to $f(x, y)$ near (a, b) if f is differentiable at (a, b).

We would now like to define differentiability for scalar-valued functions of more than two variables, so functions $\mathbb{R}^n \to \mathbb{R}$. This definition will closely resemble our definition above, which handles the case $n = 2$. For example, in the case $n = 3$, we will use the linear function

$$h(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).$$

For larger n, we'll define a similar function h, but this notation will quickly become unwieldy! In order to simplify notation, we'll now introduce a new object to organize our partial derivatives: the gradient of a scalar-valued function.

The gradient

In order to organize our information about partial derivatives, and streamline our definition of differentiability for functions $\mathbb{R}^n \to \mathbb{R}$, we now define the gradient of a scalar-valued function.

Definition 52. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. The* gradient *of f is the function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$, defined by*

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right).$$

Learning outcomes:
Author(s):

The gradient vector will be a useful computation tool in general, not only for defining differentiability.

Example 74. *For $f(x,y,z) = x^2 + ye^z$, we can compute the partial derivatives*

$$f_x(x,y,z) = 2x,$$
$$f_y(x,y,z) = e^z,$$
$$f_z(x,y,z) = ye^z.$$

Then the gradient of f is

$$\nabla f = (2x, e^z ye^z).$$

Problem 10 *Find the gradient of each function.*

$f(x,y,z) = \sin(xyz)$

$$\nabla f(x,y,z) = \boxed{(yz\cos(xyz), xz\cos(xyz), xy\cos(xyz))}$$

$g(x,y) = x^2e^y + y$

$$\nabla g(x,y) = \boxed{(2xe^y, x^2e^y + 1)}$$

$h(x_1,x_2,x_3,x_4) = x_1^2x_2 + x_1x_3 + x_2x_4$

$$\nabla h(x_1,x_2,x_3,x_4) = \boxed{(2x_1x_2 + x_3, x_1^2 + x_4, x_1, x_2)}$$

Differentiability

Now that we've defined the gradient, let's revisit our definition of differentiability for a function from $\mathbb{R}^2$ to $\mathbb{R}$. We used the function

$$h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Looking at the terms $f_x(a,b)(x-a) + f_y(a,b)(y-b)$, we can rewrite this as a dot product of two vectors:

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) = (f_x(a,b), f_y(a,b)) \cdot (x-a, y-b).$$

The first vector is the gradient of f evaluated at (a,b), so we can rewrite this as

$$(f_x(a,b), f_y(a,b)) \cdot (x-a, y-b) = \nabla f(a,b) \cdot (x-a, y-b).$$

If we take $\vec{x} = (x,y)$ and $\vec{a} = (a,b)$, we can write this as

$$\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}).$$

With these notational changes in mind, we now define differentiability for a function $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 53. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. For a point $\vec{a} \in \mathbb{R}^n$, define*

$$h(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}).$$

We say that f is differentiable *at $\vec{a}$ if*

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - h(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0.$$

If f is differentiable, we say that $h(\vec{x})$ is the tangent hyperplane *to f at $\vec{a}$.*

If any of the partial derivatives of f do not exist, or the above limit does not exist or is not 0, then f is not differentiable at $\vec{a}$.

Example 75. *We'll use this definition of differentiability to prove that the function $f(x, y, z) = xy + z$ is differentiable at $(1, 1, 1)$.*

First, we find the gradient of f.

$$\nabla f(x, y, z) = \boxed{(y, x, 1)}$$

At the point $(1, 1, 1)$, we have

$$\nabla f(1, 1, 1) = \boxed{(1, 1, 1)}.$$

From this, we find the formula for $h(x, y, z)$.

$$\begin{aligned} h(x, y, z) &= f(1, 1, 1) + \nabla f(1, 1, 1) \cdot ((x, y, z) - (1, 1, 1)) \\ &= 2 + (1, 1, 1) \cdot (x - 1, y - 1, z - 1) \\ &= \boxed{x + y + z - 1} \end{aligned}$$

Next, we evaluate the limit

$$\begin{aligned} \lim_{(x,y,z) \to (1,1,1)} \frac{f(x, y, z) - h(x, y, z)}{\|(x, y, z) - (1, 1, 1)\|} &= \lim_{(x,y,z) \to (1,1,1)} \frac{(xy + z) - (x + y + z - 1)}{\sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2}} \\ &= \lim_{(x,y,z) \to (1,1,1)} \frac{xy - x - y + 1}{\sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2}}. \end{aligned}$$

To evaluate this limit, we switch to translated spherical coordinates

$$\begin{aligned} x &= 1 + \rho \cos\theta \sin\phi, \\ y &= 1 + \rho \sin\theta \sin\phi, \\ z &= 1 + \rho \cos\phi. \end{aligned}$$

Making this change, we obtain

$$\begin{aligned} \lim_{(x,y,z) \to (1,1,1)} \frac{xy - x - y + 1}{\sqrt{(x - 1)^2 + (y - 1)^2 + (z - 1)^2}} &= \lim_{\rho \to 0} \frac{(1 + \rho\cos\theta\sin\phi)(1 + \rho\sin\theta\sin\phi) - (1 + \rho\cos\theta\sin\phi) - (1}{|\rho|} \\ &= \lim_{\rho \to 0} \frac{\rho^2 \cos\theta \sin\theta \sin^2\phi}{|\rho|}. \end{aligned}$$

Since $-|\rho| \leq \dfrac{\rho^2 \cos\theta \sin\theta \sin^2\phi}{|\rho|} \leq |\rho|$, *we use the squeeze theorem to obtain*

$$\lim_{\rho \to 0} \frac{\rho^2 \cos\theta \sin\theta \sin^2\phi}{|\rho|} = 0.$$

Thus, we have shown that $f(x, y, z) = xy + z$ *is differentiable at* $(1, 1, 1)$.

Differentiability in General

We've defined differentiability for scalar-valued functions, $\mathbb{R}^n \to \mathbb{R}$.

Definition 54. *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. For a point $\vec{a} \in \mathbb{R}^n$, define*

$$h(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}).$$

We say that f is differentiable *at $\vec{a}$ if*

$$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x}) - h(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0.$$

If f is differentiable, we say that $h(\vec{x})$ is the tangent hyperplane *to f at $\vec{a}$.*

If any of the partial derivatives of f do not exist, or the above limit does not exist or is not 0, then f is not differentiable at $\vec{a}$.

We'd now like to define differentiability for vector-valued functions, $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$.

In order to define differentiability for scalar-valued functions $f : \mathbb{R}^n \to \mathbb{R}$, we organized our partial derivatives into a vector, the gradient of f.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right)$$

We would like to do something similar for a vector-valued function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$, and organize all of the partial derivatives into a single object. However, for a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$, we not only have partial derivatives with respect to all of the different variables, we have partial derivatives of all of the component functions with respect to all of the different variables! This leads us to the derivative matrix.

The derivative matrix

Definition 55. *Let $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$, and suppose all of the partial derivatives of $\vec{f}$ exist. Write $\vec{f}$ in terms of its component functions,*

$$\vec{f}(x_1, ..., x_n) = (f_1(x_1, .., x_n), ..., f_m(x_1, ..., x_n)).$$

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Author(s):

We define the derivative matrix *of $\vec{f}$ to be the $m \times n$ matrix with $\dfrac{\partial f_i}{\partial x_j}$ as the ijth entry. That is,*

$$D\vec{f}(x_1, ..., x_n) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

It can be hard to remember whether the variable or the component changes across the rows or columns. Here are a couple of ways to remember which way it goes:

- The derivative matrix represents a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.
- The rows are gradients of the component functions.

Example 76. *We'll find the derivative matrix of the function $\vec{f}(x, y, z) = (x^2 + yz, xyz)$. Since $\vec{f} : \mathbb{R}^3 \to \mathbb{R}^2$, the derivative matrix will be a 2×3 matrix.*

$$\begin{aligned} D\vec{f}(x, y, z) &= \begin{pmatrix} \dfrac{\partial}{\partial x}(x^2 + yz) & \dfrac{\partial}{\partial y}(x^2 + yz) & \dfrac{\partial}{\partial z}(x^2 + yz) \\ \dfrac{\partial}{\partial x}(xyz) & \dfrac{\partial}{\partial y}(xyz) & \dfrac{\partial}{\partial z}(xyz) \end{pmatrix} \\ &= \begin{pmatrix} 2x & z & y \\ yz & xz & xy \end{pmatrix} \end{aligned}$$

Problem 11 *Find the derivative matrix of each of the following functions.*

$\vec{f}(x, y) = (x^2 + y^2, \sin(xy), e^{x+y})$

$$D\vec{f}(x, y) = \begin{pmatrix} \boxed{2x} & \boxed{2y} \\ \boxed{y\cos(xy)} & \boxed{x\cos(xy)} \\ \boxed{e^{x+y}} & \boxed{e^{x+y}} \end{pmatrix}$$

$\vec{g}(x, y, z) = (x^2 z + yz^2, x + y + z, x + y^2 + z)$

$$D\vec{g}(x, y, z) = \begin{pmatrix} \boxed{2xz} & \boxed{z^2} & \boxed{x^2 + 2yz} \\ \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{2y} & \boxed{1} \end{pmatrix}$$

Differentiability

We can now generalize our definition of differentiability for scalar-valued functions, by replacing the gradient with the derivative matrix.

Definition 56. *Consider a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$. For a point $\vec{a} \in \mathbb{R}^n$, define*

$$\vec{h}(\vec{x}) = \vec{f}(\vec{a}) + D\vec{f}(\vec{a})(\vec{x} - \vec{a}).$$

We say that $\vec{f}$ is differentiable *at $\vec{a}$ if*

$$\lim_{\vec{x} \to \vec{a}} \frac{\vec{f}(\vec{x}) - \vec{h}(\vec{x})}{\|\vec{x} - \vec{a}\|} = \vec{0}.$$

If any of the partial derivatives of $\vec{f}$ do not exist, or the above limit does not exist or is not 0*, then $\vec{f}$ is not differentiable at $\vec{a}$.*

Note one of the quirks of multivariable differentiation: if the derivative matrix exist, it's still possible for the function to not be differentiable.

For vector-valued functions, we can also reduce differentiability to differentiability of its component functions.

Theorem 4. *A function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if and only if its component functions are all differentiable.*

This theorem can quickly be proved from the definitions of differentiability in these two cases.

A criterion for differentiability

Checking differentiability using the limit definitions that we've found can be a huge pain! It would be much nicer if we could tell if a function is differentiable just by looking at the partial derivatives. Fortunately, this is possible in some cases.

Theorem 5. *Consider a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that all of the partial derivatives, $\partial f_i / \partial x_j$, exist and are continuous in a neighborhood of a point $\vec{a}$. Then $\vec{f}$ is differentiable at $\vec{a}$.*

An analogous result holds for scalar-valued functions. This theorem requires a very important note: its converse is *false*. That is, if one or more of the partial derivatives of a function is discontinuous, it's still possible that the function is differentiable. In this case, you would probably need to resort to the limit definition to determine differentiability.

Example 77. *We'll use the above theorem to show that $\vec{g}(x, y, z) = (x^2z + yz^2, x + y + z, x + y^2 + z)$ is differentiable at all points in $\mathbb{R}^3$.*

First, we find all of the partial derivatives of g.

$$\frac{\partial g_1}{\partial x} = \boxed{2xz}$$
$$\frac{\partial g_1}{\partial y} = \boxed{z^2}$$
$$\frac{\partial g_1}{\partial z} = \boxed{x^2 + 2yz}$$
$$\frac{\partial g_2}{\partial x} = \boxed{1}$$
$$\frac{\partial g_2}{\partial y} = \boxed{1}$$
$$\frac{\partial g_2}{\partial z} = \boxed{1}$$
$$\frac{\partial g_3}{\partial x} = \boxed{1}$$
$$\frac{\partial g_3}{\partial y} = \boxed{2y}$$
$$\frac{\partial g_2}{\partial z} = \boxed{1}$$

All of these functions are polynomials, hence continuous at all points $\mathbb{R}^3$. Since the partial derivatives of $\vec{g}$ all exist and are continuous on $\mathbb{R}^3$, by the theorem above, $\vec{g}$ is differentiable at all points in $\mathbb{R}^3$.

Higher Order Partial Derivatives

In this activity, we introduce higher order partial derivatives, and discuss their geometric meaning.

Higher Order Partials

Back in single variable Calculus, we were able to use the second derivative to get information about a function. For instance, the second derivative gave us valuable information about the shape of the graph. More specifically, we could use the second derivative to determine the concavity.

- If $f''(x) > 0$ on an interval, then the graph of f is concave up on that interval.
- If $f''(x) < 0$ on an interval, then the graph of f is concave down on that interval.
- If f'' changes signs at a point, then the graph of f has an inflection point at that point.

Furthermore, we were able to use second derivative in conjunction with roots of the first derivative to find local maxima and minima. We could also think about the second derivative as the rate-of-change of a rate-of-change, or describing some sort of acceleration or deceleration.

Since the second derivative contains so much useful information, we would like to come up with a way to define second derivatives for multivariable functions! At first, this might seem simple: just take the derivative of the derivative. But when we took the total derivative of a multivariable function $f : \mathbb{R}^m \to \mathbb{R}^n$, we got a matrix of partial derivatives, and it's not at all clear how we would differentiate a matrix.

For now, we'll settle for defining second order partial derivatives, and we'll have to wait until later in the course to define more general second order derivatives. Fortunately, second order partial derivatives work exactly like you'd expect: you simply take the partial derivative of a partial derivative.

Example 78. *Consider the function $f(x, y) = 2x^2 + 4xy - 7y^2$. We'll start by*

Learning outcomes:
Author(s):

computing the first order partial derivatives of f, with respect to x and y.

$$f_x(x, y) = \boxed{6x + 4y}$$
$$f_y(x, y) = \boxed{4x - 14y}$$

We can then compute the second order partial derivatives f_{xx} and f_{yy} by differentiating with respect to x again, and with respect to y again.

$$f_{xx}(x, y) = \boxed{6}$$
$$f_{yy}(x, y) = \boxed{-14}$$

However, this isn't the only way that we could take second order partial derivatives! We could differentiate with respect to x first, and with respect to y second, to get f_{xy}. We could also differentiate with respect to y first, and with respect to x second, to get f_{yx}.

$$f_{xy}(x, y) = \boxed{4}$$
$$f_{yx}(x, y) = \boxed{4}$$

Notice that we got the same result for f_{xy} and f_{yx}, so it didn't end up mattering what order we took these derivatives in. In turns out that this is not a coincidence, and it's a consequence of Clairaut's Theorem, which we'll talk about in the next section.

There are a few different ways that we can denote second order partials. We can denote the second order partial of f that we get by differentiating with respect to x twice as any of the following.

$$f_{xx}(x, y) \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x^2}$$

For the second order partial of f that we get by differentiating with respect to x first, then differentiating with respect to y, we denote this as below.

$$f_{xy}(x, y) \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y \partial x}$$

To remember what order to take these derivatives for all of these notations, start with the variable closest to f, and work your way out.

We can similarly define and compute third order partials, fourth order partials, and so on.

Example 79. *COMPUTE SOME PARTIAL DERIVATIVES*

Clairaut's Theorem

In previous examples, we've seen that it doesn't matter what order you use to take higher order partial derivatives, you seem to wind up with the same answer no matter what. This isn't an amazing coincidence where we randomly chose functions that happened to have this property; this turns out to be true for many functions. Clairaut's Theorem gives us this result.

Theorem 6. *(Clairaut's Theorem) Suppose we have a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$. Suppose further that* all *of the second order mixed partial derivatives of $f(x_1, ..., x_n)$ exist and are continuous on an open disc around $\mathbf{a} \in D$. Then, for any x_i and x_j,*

$$f_{x_i x_j}(\mathbf{a}) = f_{x_j x_i}(\mathbf{a}).$$

Proof prove things ■

We can prove a similar result for even higher order partial derivatives. Before we do that, we'll introduce a new definition to make it easier to describe how "nice" functions are.

Definition 57. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class $\mathcal{C}^k$ if all of the partial derivatives of f up to and including the kth order exist and are continuous.*

If f is of class $\mathcal{C}^k$ for all k, then we say f is of class $\mathcal{C}^\infty$.

If f is continuous, we say f is of class $\mathcal{C}^0$.

We'll often write this as $f \in \mathcal{C}^3$ and say "f is $\mathcal{C}$ three," for example, if f is of class $\mathcal{C}^3$.

Problem 12 *If $f(x, y) = x^2+y^2+2xy$, which of the following are true? Select all that apply.*

Select All Correct Answers:

(a) f is $\mathcal{C}^0$. ✓

(b) f is $\mathcal{C}^1$. ✓

(c) f is $\mathcal{C}^2$. ✓

(d) f is $\mathcal{C}^3$. ✓

(e) f is $\mathcal{C}^\infty$. ✓

We can generalize Clairaut's Theorem to kth order derivatives for $\mathcal{C}^k$ functions.

Theorem 7. *If f is $\mathcal{C}^k$, then the kth order mixed partials can be computed in any order.*

For example, if f is $\mathcal{C}^{12}$, we have

$$\begin{aligned} f_{xyxyzyzzxyxz} &= f_{xxxxyyyyzzzz} \\ &= f_{xyzxyzxyzxyz}. \end{aligned}$$

Problem 13 *If f is $\mathcal{C}^5$, which of the following are guaranteed to be equal to f_{zzyzx}? Select all that apply,*

Select All Correct Answers:

(a) f_{xxzxy}

(b) f_{xyzzz} ✓

(c) f_{xzzzy} ✓

(d) f_{zzxzy} ✓

(e) f_{yyzyx}

Geometric Significance

Remembering back to single variable calculus, we could use the first and second derivatives of a function to figure out the shape of the graph.

More precisely, we used the first derivative to determine where a function was increasing and where it was decreasing. If $f'(x) > 0$ on some interval, then $f(x)$ is increasing on that interval. If $f'(x) < 0$ on some interval, then $f(x)$ is decreasing on that interval.

PICTURE

We used the second derivative to determine the concavity of a function. If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval. If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval.

PICTURE

Partial derivatives can give us similar information about the graph of a multivariable function, although the situation is a bit more nuanced. Let's consider a function $f : \mathbb{R}^2 \to \mathbb{R}$, so we can visualize its graph. We've already seen that the partial derivative with respect to x tells us how f changes as x changes.

More specifically, suppose $f_x(x, y_0) > 0$ for x in some open interval containing x_0. Then $f(x, y)$ is increasing as we move *in the positive x-direction* from the point (x_0, y_0).

PICTURE

Similarly, suppose $f_x(x, y_0) < 0$ for x in some open interval constaining x_0. Then $f(x, y)$ is increasing as we move in the positive x-direction from the point (x_0, y_0).

PICTURE

Suppose $f_x(x, y_0) = 0$ for x in some open interval constaining x_0. Then $f(x, y)$ is constant as we move in the positive x-direction from the point (x_0, y_0).

PICTURE

The partial derivative with respect to y, f_y, can tell us where f is increasing, decreasing, or constant as we move in the positive y direction.

PICTURE

Now, let's look at what the second-order partial derivatives tell us about the graph of the function $f(x, y)$. It shouldn't be too surprising that the sign of f_{xx} tells us about the concavity of f as we move in the positive x direction.

PICTURE

Similarly, the sign of f_{yy} tells us about the concavity of f as we move in the positive y direction.

PICTURE

But what do the mixed partials, f_{xy} and f_{yx}, tell us about the graph of f? Let's consider f_{xy}, which is the partial derivative of f_x with respect to y. The partial derivative f_x tells us the rate of change of f as we move in the positive x direction. Then, the partial derivative of f_x with respect to y tells us how f_x changes as we move in the positive y direction. That is, we look at how the rate of change in the x direction changes as we move in the y direction. This is a lot to unravel!

PICTURE/VIDEO/INTERACTIVE

We can similarly think of f_{yx} as the change in f_y as we move in the positive x direction.

PICTURE/VIDEO/INTERACTIVE

Conclusion

In this activity, we considered higher order partial derivatives, and found that the order of differentiation doesn't matter for "nice" functions using Clairaut's theorem. We also considered the geometric information carried by second order partial derivatives.

Differentiation Properties

In this activity, we explore some of the properties of differentiation. This include how derivatives interact with addition and scalar multiplication, as well as product and quotient rules for scalar valued functions.

Linearity of the derivative

As you may recall from single variable calculus, "the derivative of the sum is the sum of the derivatives." That is, if we have differentiable functions $f(x)$ and $g(x)$, we compute the derivative of the sum $f(x) + g(x)$ by taking the sum of the derivatives $f'(x)$ and $g'(x)$. For example,

$$\begin{aligned} \frac{d}{dx}\left(\sin(x) + x^2\right) &= \frac{d}{dx}\left(\sin(x)\right) + \frac{d}{dx}\left(x^2\right) \\ &= \cos(x) + 2x. \end{aligned}$$

An analogous result holds in multivariable calculus, for the derivative matrix.

Proposition 25. *Suppose* $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ *and* $\mathbf{g} : Y \subset \mathbb{R}^n \to \mathbb{R}^m$ *are differentiable functions on subsets* X *and* Y *of* $\mathbb{R}^n$*, respectively. Then* $\mathbf{f} + \mathbf{g}$ *is differentiable on* $X \cap Y$*, and* $D(\mathbf{f} + \mathbf{g}) = D\mathbf{f} + D\mathbf{g}$ *on* $X \cap Y$*.*

Proof We will begin by showing that $D(\mathbf{f} + \mathbf{g}) = D\mathbf{f} + D\mathbf{g}$ on $X \cap Y$. Since $X \cap Y$ is contained in the domains of both $\mathbf{f}$ and $\mathbf{g}$, $D\mathbf{f}$ and $D\mathbf{g}$ both exist on $X \cap Y$.

Write $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_m(\mathbf{x}))$ and $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), ..., g_m(\mathbf{x}))$ in terms of their component functions. Then, we have

$$\begin{aligned} (\mathbf{f} + \mathbf{g})(\mathbf{x}) &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \\ &= (f_1(\mathbf{x}), ..., f_m(\mathbf{x})) + (g_1(\mathbf{x}), ..., g_m(\mathbf{x})) \\ &= (f_1(\mathbf{x}) + g_1(\mathbf{x}), ..., f_m(\mathbf{x}) + g_m(\mathbf{x})) \\ &= ((f_1 + g_1)(\mathbf{x}), ..., (f_m + g_m)(\mathbf{x})), \end{aligned}$$

giving us the component functions of $\mathbf{f} + \mathbf{g}$.

Using the component functions found above, we take the derivative matrix of

Learning outcomes:
Author(s):

$\mathbf{f}+\mathbf{g}$, using the linearity of partial derivatives.

$$\begin{aligned}
D(\mathbf{f}+\mathbf{g}) &= \begin{pmatrix} \dfrac{\partial(f_1+g_1)}{\partial x_1} & \dfrac{\partial(f_1+g_1)}{\partial x_2} & \cdots & \dfrac{\partial(f_1+g_1)}{\partial x_n} \\ \dfrac{\partial(f_2+g_2)}{\partial x_1} & \dfrac{\partial(f_2+g_2)}{\partial x_2} & \cdots & \dfrac{\partial(f_2+g_2)}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial(f_m+g_m)}{\partial x_1} & \dfrac{\partial(f_m+g_m)}{\partial x_2} & \cdots & \dfrac{\partial(f_m+g_m)}{\partial x_n} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}+\dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2}+\dfrac{\partial g_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n}+\dfrac{\partial g_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1}+\dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2}+\dfrac{\partial g_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n}+\dfrac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1}+\dfrac{\partial g_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2}+\dfrac{\partial g_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n}+\dfrac{\partial g_m}{\partial x_n} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix} + \begin{pmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} & \cdots & \dfrac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial g_m}{\partial x_1} & \dfrac{\partial g_m}{\partial x_2} & \cdots & \dfrac{\partial g_m}{\partial x_n} \end{pmatrix} \\
&= D\mathbf{f}+D\mathbf{g}.
\end{aligned}$$

Thus, we have $D(\mathbf{f}+\mathbf{g}) = D\mathbf{f}+D\mathbf{g}$ on $X \cap Y$.

Next, we need to show that $\mathbf{f}+\mathbf{g}$ is differentiable on $X \cap Y$. In order to show this, we will show that the following limit evaluates to 0, for $\mathbf{a} \in X \cap Y$. This is done by separating the $\mathbf{f}$ and $\mathbf{g}$ terms, using the triangle inequality, and using the fact that $\mathbf{f}$ and $\mathbf{g}$ are both differentiable on $X \cap Y$.

$$
\begin{aligned}
&\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|(\mathbf{f}+\mathbf{g})(\mathbf{x}) - ((\mathbf{f}+\mathbf{g})(\mathbf{a}) + D(\mathbf{f}+\mathbf{g})(\mathbf{a})(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} \\
&\quad = \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - \mathbf{g}(\mathbf{a}) - (D\mathbf{f}(\mathbf{a}) + D\mathbf{g}(\mathbf{a}))(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} \\
&\quad = \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - \mathbf{g}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a}) - D\mathbf{g}(\mathbf{a})(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} \\
&\quad = \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{a}) - D\mathbf{g}(\mathbf{a})(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} \\
&\quad \leq \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})\| + \|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{a}) - D\mathbf{g}(\mathbf{a})(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} \\
&\quad \leq \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} + \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{a}) - D\mathbf{g}(\mathbf{a})(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} \\
&\quad = \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - (\mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a}))\|}{\|\mathbf{x}-\mathbf{a}\|} + \lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{g}(\mathbf{x}) - (\mathbf{g}(\mathbf{a}) + D\mathbf{g}(\mathbf{a})(\mathbf{x}-\mathbf{a}))\|}{\|\mathbf{x}-\mathbf{a}\|} \\
&\quad = 0 + 0 \\
&\quad = 0
\end{aligned}
$$

■

You may also recall the constant multiple rule from single variable calculus. For example,

$$
\begin{aligned}
\frac{d}{dx}(4x^2) &= 4\left(\frac{d}{dx}(x^2)\right) \\
&= 4(2x) \\
&= 8x.
\end{aligned}
$$

We have an analogous result in multivariable calculus.

Proposition 26. *Suppose $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable function on a subset X of $\mathbb{R}^n$, and let k be any constant. Then $k\mathbf{f}$ is also differentiable on X, and $D(k\mathbf{f}) = kD\mathbf{f}$ on X.*

The proof of this proposition is somewhat similar to the previous theorem, and it is left as an exercise.

Although these are important results, they actually aren't particularly useful for computing derivative matrices. In practice, you'd compute the derivative matrix of a sum of functions by first adding the components, then differentiating. We include an example to demonstrate this.

Example 80. *Consider the functions $\mathbf{f}, \mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{f}(x, y) = (x^2, xy)$ and $\mathbf{g}(x, y) = (xy^2, y)$.*

We will compute the derivative matrix $D(\mathbf{f}+\mathbf{g})$ in two ways: first directly, and then by using the sum rule proved above.

The function $\mathbf{f}+\mathbf{g}$ is given by

$$(\mathbf{f}+\mathbf{g})(x,y) = \boxed{(x^2+xy^2, xy+y)}.$$

We can then compute the derivative matrix $D(\mathbf{f}+mathbf{g})$ directly as

$$D(\mathbf{f}+\mathbf{g}) = \begin{pmatrix} \boxed{2x+y^2} & \boxed{2xy} \\ \boxed{y} & \boxed{x+1} \end{pmatrix}.$$

We will now use our multivariable sum rule. We have

$$D\mathbf{f}(x,y) = \begin{pmatrix} \boxed{2x} & \boxed{0} \\ \boxed{y} & \boxed{x} \end{pmatrix},$$

and

$$D\mathbf{g}(x,y) = \begin{pmatrix} \boxed{y^2} & \boxed{2xy} \\ \boxed{0} & \boxed{1} \end{pmatrix}.$$

Adding these matrices, we then have that

$$\begin{aligned} D(\mathbf{f}+\mathbf{g})(x,y) &= D\mathbf{f}(x,y) + D\mathbf{g}(x,y) \\ &= \begin{pmatrix} \boxed{2x+y^2} & \boxed{2xy} \\ \boxed{y} & \boxed{x+1} \end{pmatrix}. \end{aligned}$$

We get the correct answer using either method, but using the sum rule doesn't seem to provide much (if any) of a computational advantage over computing the derivative matrix directly. Nonetheless, it is mathematically important that the derivative seems to "distribute" over addition.

Product and Quotient laws

We will now try to find multi-variable analogs to the product and quotient rules from single variable calculus. Let's start by considering when these rules might make sense.

Suppose we have functions $\mathbf{f}(x,y) = (x^2, xy)$ and $\mathbf{g}(x,y) = (xy^2, y)$. If we wanted to define the product of these functions, what would that mean? The outputs of $\mathbf{f}$ and $\mathbf{g}$ are vectors, so we'd be trying to multiply two vectors - but there isn't really a clear multiplication on vectors,! We could try multiplying component-wise, taking the dot product, or taking the cross product, and in different settings, these all might be reasonable things to do. We could work on finding product rules for all of the different ways we could "multiply" two vectors (these can be found in the exercises), but we'll save some time, and focus on a case where we do have one clear choice for multiplication: *scalar-valued functions*.

Proposition 27. *Suppose* $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}$ *and* $\mathbf{g} : Y \subset \mathbb{R}^n \to \mathbb{R}$ *are differentiable scalar-valued functions on subsets* X *and* Y *of* $\mathbb{R}^n$*, respectively. Then* fg *is differentiable on* $X \cap Y$*, and*

$$D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a})$$

for $\mathbf{a}$ *in* $X \cap Y$*.*

Similarly, we have a multi-variable quotient rule for scalar valued functions.

Proposition 28. *Suppose* $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}$ *and* $\mathbf{g} : Y \subset \mathbb{R}^n \to \mathbb{R}$ *are differentiable scalar-valued functions on subsets* X *and* Y *of* $\mathbb{R}^n$*, respectively. Then* $\dfrac{f}{g}$ *is differentiable on* $X \cap Y$*, and*

$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{(g(\mathbf{a}))^2}$$

for $\mathbf{a}$ *in* $X \cap Y$*.*

We'll leave the proofs of these results as exercises, as they are similar to the single variable proofs.

Chain Rule

In this activity, we introduce the multi-variable chain rule, and we use it to compute derivatives of compositions of functions.

The Chain Rule

Let's begin by recalling the chain rule from single variable calculus. If we have differentiable functions f and g, then we can compute the derivative of the composition as $(f \circ g)'(x) = f'(g(x))g'(x)$.

Example 81. *Let $f(x) = \sin(x)$ and $g(x) = x^2$. Then $f'(x) = \cos(x)$, $g'(x) = 2x$, and we can differentiate the composition $(f \circ g)(x) = \sin(x^2)$ using the chain rule:*

$$\begin{aligned}(f \circ g)'(x) &= f'(g(x))g'(x) \\ &= \cos(g(x)) \cdot 2x \\ &= \cos(x^2) \cdot 2x.\end{aligned}$$

We can also use the chain rule to differentiate the composition $(g \circ f)(x) = \sin^2(x)$.

$$\begin{aligned}(g \circ f)'(x) &= g'(f(x))f'(x) \\ &= 2(f(x))\cos(x) \\ &= 2\sin(x)\cos(x)\end{aligned}$$

The multi-variable chain rule is similar, with the derivative matrix taking the place of the single variable derivative, so that the chain rule will involve matrix multiplication. We also need to pay extra attention to whether the composition of functions is even defined.

Theorem 8. *Suppose $\mathbf{f} : Y \subset \mathbb{R}^p \to \mathbb{R}^n$ and $\mathbf{g} : X \subset \mathbb{R}^m \to \mathbb{R}^p$ are defined on open sets $Y \subset \mathbb{R}^p$ and $X \subset \mathbb{R}^m$, respectively. Suppose that $\mathbf{g}(X) \subset Y$, so the image of $\mathbf{g}$ is contained in the domain of $\mathbf{f}$. Suppose further that $\mathbf{g}$ is differentiable at some point $\mathbf{x}_0 \in X$, and that $\mathbf{f}$ is differentiable at $\mathbf{y}_0 = \mathbf{g}(\mathbf{x}_0) \in Y$.*

Then the composition $\mathbf{f} \circ \mathbf{g}$ is differentiable at $\mathbf{x}_0$, and

$$\begin{aligned}D(\mathbf{f} \circ \mathbf{g})(\mathbf{x}_0) &= D\mathbf{f}(\mathbf{y}_0)D\mathbf{g}(\mathbf{x}_0) \\ &= D\mathbf{f}(\mathbf{g}(\mathbf{x}_0))D\mathbf{g}(\mathbf{x}_0).\end{aligned}$$

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Although the conditions sound complicated, essentially they're just requiring that all of the derivatives mentioned actually exist. Note the similarities to the single variable chain rule.

Proof super great proof of chain rule ■

A Special Case

We'll now consider a special case of the chain rule, when we have a composition $f \circ g$ of functions $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$. Note that f is a scalar function, and we can think of $\mathbf{g}$ as a curve in $\mathbb{R}^n$.

Let's look at what the chain rule tells us in this case. For any $x_0 \in \mathbb{R}$, we have

$$D(f \circ \mathbf{g})(x_0) = Df(\mathbf{g}(x_0))D\mathbf{g}(x_0).$$

Writing $\mathbf{g}(x) = (g_1(x), ..., g_n(x))$ in terms of its components, we have

$$D\mathbf{g} = \begin{pmatrix} \partial g_1/\partial x \\ \vdots \\ \partial g_n/\partial x \end{pmatrix}.$$

Since $\mathbf{g}$ only has one input variable, we can rewrite this as

$$D\mathbf{g}(x) = \begin{pmatrix} g_1'(x) \\ \vdots \\ g_n'(x) \end{pmatrix}.$$

Now that we've sorted out $D\mathbf{g}$, let's consider Df. Since f is a scalar-valued function, Df will consist of only one row,

$$Df = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \end{pmatrix}.$$

For $Df(\mathbf{g}(x_0))$, we would evaluate these partial derivatives at $\mathbf{g}(x_0)$:

$$Df(\mathbf{g}(x_0)) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{g}(x_0)) & \cdots & \dfrac{\partial f}{\partial x_n}(\mathbf{g}(x_0)) \end{pmatrix}.$$

Now let's turn our attention back to the composition $f \circ \mathbf{g}$. Putting together our results from above, we have

$$\begin{aligned} D(f \circ \mathbf{g})(x_0) &= Df(\mathbf{g}(x_0))D\mathbf{g}(x_0) \\ &= \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{g}(x_0)) & \cdots & \dfrac{\partial f}{\partial x_n}(\mathbf{g}(x_0)) \end{pmatrix} \begin{pmatrix} g_1'(x_0) \\ \vdots \\ g_n'(x_0) \end{pmatrix} \\ &= \left(\left(\dfrac{\partial f}{\partial x_1}(\mathbf{g}(x_0)) \right) g_1'(x_0) + \cdots + \left(\dfrac{\partial f}{\partial x_n}(\mathbf{g}(x_0)) \right) g_n'(x_0) \right) \end{aligned}$$

Since $f \circ \mathbf{g} : \mathbb{R} \to \mathbb{R}$ is a single variable function, its derivative matrix at x_0 only has one entry, which is $\dfrac{df \circ \mathbf{g}}{dx}(x_0)$. So, we can rewrite the above as

$$\frac{df \circ \mathbf{g}}{dx}(x_0) = \left(\frac{\partial f}{\partial x_1}(\mathbf{g}(x_0))\right) g_1'(x_0) + \cdots + \left(\frac{\partial f}{\partial x_n}(\mathbf{g}(x_0))\right) g_n'(x_0).$$

This gives us a special case of the Chain Rule, that can be useful when we have a composition of functions $\mathbb{R} \to \mathbb{R}^n \to \mathbb{R}$.

Examples

Example 82. *Let's consider the functions* $\mathbf{f}(x, y) = (x^2 + y^2, xy)$ *and* $\mathbf{g}(x, y) = (3xy, x - y, 7y^2)$*. First, let's decide how we can compose these functions.*

Which composition(s) exist?

Multiple Choice:

(a) *Neither* $\mathbf{f} \circ \mathbf{g}$ *nor* $\mathbf{g} \circ \mathbf{f}$ *exists.*

(b) $\mathbf{f} \circ \mathbf{g}$ *exists, but* $\mathbf{g} \circ \mathbf{f}$ *does not.*

(c) $\mathbf{g} \circ \mathbf{f}$ *exists, but* $\mathbf{f} \circ \mathbf{g}$ *does not.* ✓

(d) $\mathbf{f} \circ \mathbf{g}$ *and* $\mathbf{g} \circ \mathbf{f}$ *both exist.*

We'll compute the derivative matrix $D(\mathbf{g} \circ \mathbf{f}$ *in two ways: using the chain rule, and directly.*

Let's begin by using the chain rule. We'll have

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = D\mathbf{g}(\mathbf{f}(\mathbf{x}))D\mathbf{f}(\mathbf{x}),$$

so we'll start by computing the derivative matrices $D\mathbf{f}$ *and* $D\mathbf{g}$*.*

$$D\mathbf{f}(x, y) = \begin{pmatrix} \boxed{2x} & \boxed{2y} \\ \boxed{y} & \boxed{x} \end{pmatrix}$$

$$D\mathbf{g}(x, y) = \begin{pmatrix} \boxed{3y} & \boxed{3x} \\ \boxed{1} & \boxed{-1} \\ \boxed{0} & \boxed{14y} \end{pmatrix}$$

Now, for $D\mathbf{g}(\mathbf{f}(\mathbf{x}))$*, we need to input* $\mathbf{f}(x, y)$ *into* $D\mathbf{g}$*.*

$$\begin{aligned} D\mathbf{g}(\mathbf{f}(x, y)) &= D\mathbf{g}(x^2 + y^2, xy) \\ &= \begin{pmatrix} \boxed{3(xy)} & \boxed{3(x^2 + y^2)} \\ \boxed{1} & \boxed{-1} \\ \boxed{0} & \boxed{14(xy)} \end{pmatrix} \end{aligned}$$

To compute $D(\mathbf{g} \circ \mathbf{f})(x, y)$, we multiply matrices, and obtain

$$\begin{aligned} D(\mathbf{g} \circ \mathbf{f})(x, y) &= \begin{pmatrix} \boxed{3(xy)} & \boxed{3(x^2+y^2)} \\ \boxed{1} & \boxed{-1} \\ \boxed{0} & \boxed{14(xy)} \end{pmatrix} \begin{pmatrix} \boxed{2x} & \boxed{2y} \\ \boxed{y} & \boxed{x} \end{pmatrix} \\ &= \begin{pmatrix} \boxed{9x^2y+3y^3} & \boxed{3x^3+9xy^2} \\ \boxed{2x-y} & \boxed{2y-x} \\ \boxed{14xy^2} & \boxed{14x^2y} \end{pmatrix}. \end{aligned}$$

Let's verify our answer, by computing the $D(\mathbf{g} \circ \mathbf{f})(x, y)$ directly, without using the chain rule. We'll begin by finding $\mathbf{g} \circ \mathbf{f}$.

$$\begin{aligned} (\mathbf{g} \circ \mathbf{f})(x, y) &= \mathbf{g}(f(x, y)) \\ &= \mathbf{g}(x^2 + y^2, xy) \\ &= \boxed{(3(x^2+y^2)(xy), (x^2+y^2) - (xy), 7(xy)^2)}. \end{aligned}$$

This simplifies to $(\mathbf{g} \circ \mathbf{f})(x, y) = (3x^3y + 3xy^3, x^2 + y^2 - xy, 7x^2y^2)$. We can then compute the derivative matrix.

$$(\mathbf{g} \circ \mathbf{f})(x, y) = \begin{pmatrix} \boxed{9x^2y+3y^3} & \boxed{3x^3+9xy^2} \\ \boxed{2x-y} & \boxed{2y-x} \\ \boxed{14xy^2} & \boxed{14x^2y} \end{pmatrix}$$

We see that this gives the same result as using the chain rule.

Example 83. *special case*

Example 84. *Consider the function $g : \mathbb{R}^2 \to \mathbb{R}^2$, given by $\mathbf{g}(r, \theta) = (r \cos\theta, r \sin\theta)$. We can think of this function as converting from polar coordinates to Cartesian coordinates. Suppose we have a function $f : \mathbb{R}^2 \to \mathbb{R}$, and we want to know how f changes with respect to polar coordinates. We could find this by computing*

$$D(f \circ \mathbf{g}) = \begin{pmatrix} \dfrac{\partial(f \circ \mathbf{g})}{\partial r} & \dfrac{\partial(f \circ \mathbf{g})}{\partial \theta} \end{pmatrix}.$$

For specific functions f, we could probably compute this derivative matrix directly. However, if we use the chain rule, we can find a general formula for the derivative matrix. The chain rule tells us

$$D(f \circ \mathbf{g})(r, \theta) = Df(\mathbf{g}(r, \theta)) D\mathbf{g}(r, \theta).$$

We don't know precisely what Df will be, but we can write it in terms of its partial derivatives:

$$Df = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{pmatrix}.$$

Given $\mathbf{g}(r,\theta) = (r\cos\theta, r\sin\theta)$*, we can compute the derivative matrix of* $\mathbf{g}$*.*

$$D\mathbf{g}(r,\theta) = \begin{pmatrix} \partial g_1/\partial r & \partial g_1/\partial\theta \\ \partial g_2 \partial r & \partial g_2/\partial\theta \end{pmatrix}$$
$$= \begin{pmatrix} \boxed{\cos\theta} & \boxed{-r\sin\theta} \\ \boxed{\sin\theta} & \boxed{r\cos\theta} \end{pmatrix}$$

Then, from the chain rule, we have

$$\begin{aligned} D(f\circ\mathbf{g})(r,\theta) &= Df(\mathbf{g}(r,\theta))D\mathbf{g}(r,\theta) \\ &= \begin{pmatrix} \frac{\partial f}{\partial x}(\mathbf{g}(r,\theta)) & \frac{\partial f}{\partial y}(\mathbf{g}(r,\theta)) \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\frac{\partial f}{\partial x}(\mathbf{g}(r,\theta)) + \sin\theta\frac{\partial f}{\partial y}(\mathbf{g}(r,\theta)) & -r\sin\theta\frac{\partial f}{\partial x}(\mathbf{g}(r,\theta)) + r\cos\theta\frac{\partial f}{\partial y}(\mathbf{g}(r,\theta)) \end{pmatrix} \end{aligned}$$

Since

$$D(f\circ\mathbf{g}) = \begin{pmatrix} \frac{\partial(f\circ\mathbf{g})}{\partial r} & \frac{\partial(f\circ\mathbf{g})}{\partial\theta} \end{pmatrix},$$

we then have that

$$\begin{aligned} \frac{\partial(f\circ\mathbf{g})}{\partial r} &= \cos\theta\frac{\partial f}{\partial x}(\mathbf{g}(r,\theta)) + \sin\theta\frac{\partial f}{\partial y}(\mathbf{g}(r,\theta)) \\ &= \cos\theta\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) + \sin\theta\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial(f\circ\mathbf{g})}{\partial\theta} &= -r\sin\theta\frac{\partial f}{\partial x}(\mathbf{g}(r,\theta)) + r\cos\theta\frac{\partial f}{\partial y}(\mathbf{g}(r,\theta)) \\ &= -r\sin\theta\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) + r\cos\theta\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta). \end{aligned}$$

Conclusion

Part V

Optimization

Part VI

Vector Fields and Line Integrals

Vector Fields

In this activity, we introduced vector fields. We give examples and learn how to graph them.

Motivation

Suppose we want to look at temperature patterns across a region at some specific time. We might represent this by taking the temperature at various locations, and then plotting each temperature at its location on a map.

PICTURE

Now, suppose we want to represent wind patterns across the same region at some specific time. We can represent this by plotting the wind speeds at various locations, but this doesn't tell the whole story - we'd also like to represent the direction in which the wind is blowing.

In order to do this, we can instead plot a *vector* at each location. The direction of the vector will tell us the direction of the wind, and the length of the vector will tell us the wind speed.

PICTURE

Although we only plot the vectors at a few points, we know that there are windspeed vectors that could be drawn at every point in the region. However, we can use this plot to infer the behavior of windspeed in general, without checking the windspeed everywhere in the region.

This is an example of a vector field. There are many other important examples of vector fields, and they can be used to represent fluid flow, gravitational fields,

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Author(s):

and magnetic fields.

PICTURES/EXAMPLES

Definition

We've seen that a vector field consists of vectors placed at each point in some region, and we can think of this as assigning a vector to each point in the region. This sounds like a function, and provides the intuition behind our definition of a vector field.

Definition 58. *A* vector field *is a function* $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}n$.

If $\vec{F}$ *is a continuous function, then we say that* $\vec{F}$ *is a* continuous vector field*.*

In this definition, we're thinking about the inputs as points and the outputs as vectors, even though both are in $\mathbb{R}^n$.

ADD EXAMPLES

Graphing and Scaling

If the vectors in our vector field are particularly long, graphing our vector field can quickly turn into a cluttered mess. In these situations, it can be useful to scale the vectors, so that we can more clearly see the behavior of our vector field.

EXAMPLE

In fact, most graphing software will automatically scale vector fields, whether you want them to or not.

Topology

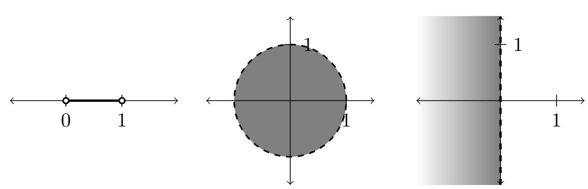
In this activity, we introduce some basic ideas from the area of mathematics called "topology." We introduce these ideas in order to be able to correctly state the theorems in the next activities, on conservative vector fields.

Topology is focused on the shape and spaces, and how to distinguish between different shapes and spaces. However, it takes a much more flexible perspective that you might have seen in geometry classes. In topology, deformations like stretching, shrinking, or bending a space aren't viewed as changing a space. In particular, distance between points isn't important. Instead, topology employs a much looser idea of closeness - provided by open sets, which are defined in the first section.

In general, there are a lot of interesting and weird topological spaces which are not subspaces of any $\mathbb{R}^n$! However, for our purposes, focusing on subspaces of $\mathbb{R}^n$ will be sufficient, so our definitions might look different from some of the definitions you would see in a topology textbook.

Open Sets

You've probably seen many open sets before, without even realizing it! For example, the open interval $(0, 1)$ is an open set, as is the disc $\{(x, y) \mid x^2+y^2 < 1\}$ and the open half-plane $\{(x, y) \mid x < 0\}$.



Some open sets.

We begin by defining the most basic type of open set, called an open ball.

Definition 59. *In $\mathbb{R}^n$, we call $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}$ the* open ball *of radius $r > 0$ centered at $\mathbf{x}$.*

In words, this is the set of points within a fixed distance r of a center point $\mathbf{x}$.

Learning outcomes:
Author(s): Melissa Lynn

Example 85. *In* $\mathbb{R}^1$*, an open ball is simply an open interval. For example,* $(1,3)$ *is the ball* $B_1(2)$ *of radius* 1 *centered at* 2*.*

YouTube link: `https://www.youtube.com/watch?v=QXy9aLiBfSs`

In $\mathbb{R}^2$*, an open ball is the inside of a circle (not including the boundary). For example,* $B_2((0,3))$ *is the inside of the circle of radius* 2 *centered at* $(0,3)$*.*

YouTube link: `https://www.youtube.com/watch?v=r9OMlNgPqZA`

In $\mathbb{R}^3$*, and open ball is the inside of a sphere of radius (not including the boundary). For example,* $B_1((0,0,0))$ *is the inside of the sphere of radius* 1 *centered at the origin.*

YouTube link: `https://www.youtube.com/watch?v=xSHf7KQN6Lc`

In each problem, describe the open ball.

Problem 14 *In* $\mathbb{R}^1$, $B_3(1)$ *is the*

Multiple Choice:

(a) *open interval* ✓

(b) *inside of a circle*

(c) *inside of a sphere*

Problem 14.1 $(\boxed{-2}, \boxed{4})$.

Problem 15 *In* $\mathbb{R}^2$, $B_2((1,1))$ *is the*

Multiple Choice:

(a) *open interval*

(b) *inside of a circle* ✓

(c) *inside of a sphere*

Problem 15.1 *of radius* $\boxed{2}$ *centered at* $(\boxed{1}, \boxed{1})$.

Problem 16 *In $\mathbb{R}^3$, $B_4((1,2,3))$ is the*

Multiple Choice:

(a) *open interval*

(b) *inside of a circle*

(c) *inside of a sphere* ✓

Problem 16.1 *of radius* $\boxed{4}$ *centered at* $(\boxed{1}, \boxed{2}, \boxed{3})$.

Now we can define open sets in general, using our new idea of open balls.

Definition 60. *A set $U \subset \mathbb{R}^n$ is* open *if for every $\mathbf{a} \in U$, there is a radius $r > 0$ such that $B_r(\mathbf{a}) \subset U$.*

In words, for any point $\mathbf{a}$ in U, we can find a radius r small enough that the entire ball of radius r centered at $\mathbf{a}$ is contained in U.

YouTube link: `https://www.youtube.com/watch?v=S-cLB6dLCQo`

Even if we don't have an open set, it's possible that there might be some points in U with this special property. We call these interior points.

Definition 61. *Let $U \subset \mathbb{R}^n$. A point $\mathbf{a} \in U$ is an* interior point *if there is a radius $r > 0$ such that $B_r(\mathbf{a}) \subset U$.*

So, we can restate the definition of an open set as "every point is an interior point."

Problem 17 *For each of the following, determine whether or not the set is open.*

(a) $\{(x, y) : x^2 + y^2 < 1\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open* ✓

(ii) *not open*

(b) $\{(x, y) : x^2 + y^2 \leq 1\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open*

(ii) *not open* ✓

(c) $\mathbb{R}^2$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open* ✓

(ii) *not open*

(d) $\emptyset$ *in* $\mathbb{R}^n$

Multiple Choice:

(i) *open* ✓

(ii) *not open*

(e) $\{(x, y) : x \geq 0 \text{ and } y \geq 0\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open*

(ii) *not open* ✓

(f) $\{(x, y) : x < y\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open* ✓

(ii) *not open*

We finish this section by proving an important result about open sets: that the union of a collection of open sets is itself an open set.

Theorem 9. *Suppose $\{U_i\}$ is a collection of open sets, where each U_i is open. Let $U = \bigcup U_i$, the union of all the U_i. Then U is an open set.*

Proof Let $\mathbf{a} \in U$. We will show that $\mathbf{a}$ is an interior point.

Since $\mathbf{a} \in U = \bigcup U_i$, there is some i such that $\mathbf{a} \in U_i$. Since U_i is open, there is a radius $r > 0$ such that $B_r(\mathbf{a})$ is contained entirely in U_i. Then, since $U_i \subset U$, we have that $B_r(\mathbf{a}) \subset U$. This shows that $\mathbf{a}$ is an interior point, and that U is an open set.

■

Closed Sets

We now introduce closed sets, which are defined via their relationship to open sets.

Definition 62. *A set $X \subset \mathbb{R}^n$ is* closed *if its complement is open.*

Recall that the *complement* of a subset X of $\mathbb{R}^n$ consists of the elements of $\mathbb{R}^n$ which are not in X. That is,

$$\mathbb{R}^n \setminus X = X^C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in X\}.$$

$\mathbb{R}^n \setminus X$ and X^C are both common notations for complements. X^C has the advantage of being succinct, while $\mathbb{R}^n \setminus X$ has the advantage of referring to the larger set $\mathbb{R}^n$, which can help avoid confusion.

We now give a couple examples of closed sets.

Example 86. *The closed interval $[1, 3]$ is a closed set in $\mathbb{R}$, since its complement, $(-\infty, 1) \cup (3, \infty)$, is an open set.*

The set $\{(x, y) : x^2+y^2 \leq 1\} \subset \mathbb{R}^2$, since its complement, $\{(x, y) : x^2+y^2 > 1\}$, is an open set.

It's very important to remember that, even though closed sets are defined in relation to open sets, "closed" is not the same as "not open". It is possible for a set to be neither closed nor open, and it's possible for a set to be both open and closed.

Example 87. *The interval $[0, 1)$ in $\mathbb{R}$ is neither closed nor open. It is not open, since $0 \in [0, 1)$ is not an interior point. It is also not closed, because its complement, $(-\infty, 0) \cup [1, \infty)$, is not open ($1$ is not an interior point).*

$\mathbb{R}^2$ (in $\mathbb{R}^2$) is both closed and open. We've seen that its open, and we've seen that its complement, $\emptyset$, is also open. Hence $\mathbb{R}^2$ is both open and closed.

We've used the word "boundary" to refer to the "edge" of a set. This intuitive idea is useful in working with example, however we do need a rigorous definition of the boundary of a set, which we now give.

Definition 63. *A point $\mathbf{x} \in \mathbb{R}^n$ is a* boundary point *of a set $X \in \mathbb{R}^n$ if every $B_r(\mathbf{x})$ (for $r > 0$) contains both points in X and points not in X.*

The set of all boundary points of X is called the boundary *of X.*

Example 88. *The boundary of $\{(x, y) : x^2+y^2 < 1\} \subset \mathbb{R}^2$ is $\{(x, y) : x^2+y^2 = 1\}$.*

The boundary of the interval $(0, 1] \subset \mathbb{R}$ is $\{0, 1\}$.

From working with examples of open and closed sets, you may have guessed that closed sets tend to contain their boundary points, while open sets do not. This is supported by the following theorem.

Theorem 10. *A set $X \subset \mathbb{R}^n$ is closed if and only if it contains all of its boundary points.*

Proof If X is closed, then its complement X^C is open. Consider a boundary point x of X. For any radius $r > 0$, $B_r(x)$ contains points in X and points in X^C. Since X^C is open, this means that $x \notin X^C$. Thus $x \in X$, as desired.

Working in the other direction, suppose that X contains all of its boundary points. We will show that the complement of X, X^C, is open. Let $x \in X^C$. Since x is not a boundary point of X, and any open ball centered at x contains at least one point not in X (namely, x), there must be an open ball $B_r(x)$ with $r > 0$ containing no points in X. Then $B_r(x) \subset X^C$. This shows that X^C is open, and hence X is closed. ■

Problem 18 *For each of the following, determine if the set is open, closed, both, or neither.*

(a) $\{(x, y) : x^2 + y^2 > 1\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open* ✓
(ii) *closed*
(iii) *both open and closed*
(iv) *neither open nor closed*

(b) $\{(x, y) : x^2 + y^2 \leq 1\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open*
(ii) *closed* ✓
(iii) *both open and closed*
(iv) *neither open nor closed*

(c) $\mathbb{R}^2$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open*
(ii) *closed*

(iii) *both open and closed* ✓
(iv) *neither open nor closed*

(d) $\emptyset$ *in* $\mathbb{R}^n$

Multiple Choice:

(i) *open*
(ii) *closed*
(iii) *both open and closed* ✓
(iv) *neither open nor closed*

(e) $\{(x, y) \,:\, x > 0 \text{ and } y \geq 0\}$ *in* $\mathbb{R}^2$

Multiple Choice:

(i) *open*
(ii) *closed*
(iii) *both open and closed*
(iv) *neither open nor closed* ✓

(f) $\{(x, y) \,:\, x \leq y\}$ *in* $\mathbb{R}^2$

Multiple Choice:

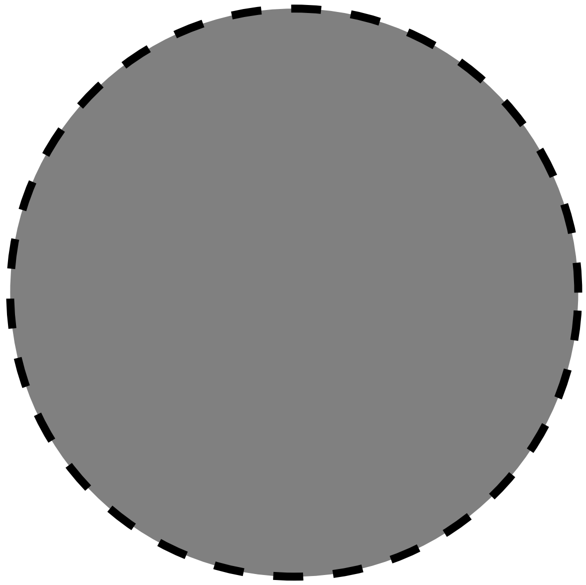
(i) *open*
(ii) *closed* ✓
(iii) *both open and closed*
(iv) *neither open nor closed*

Connected Sets

We now define what it means for a set to be path-connected. This is generally consistent with your intuitive idea of what "connected" should mean, but we are now able to make this mathematically rigorous.

Definition 64. *A set* $X \subset \mathbb{R}^n$ *is path-connected if any two points can be connected by a path which lies entirely in* X*.*

Problem 19 *For each of the following sets, determine whether or not they are path-connected.*

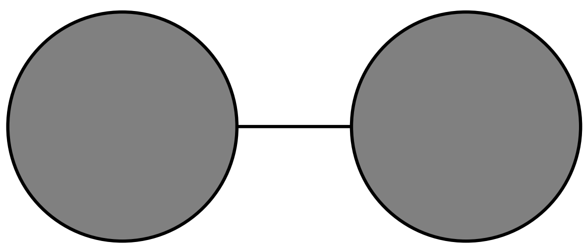


(a)

Multiple Choice:

(i) *path-connected* ✓

(ii) *not path-connected*

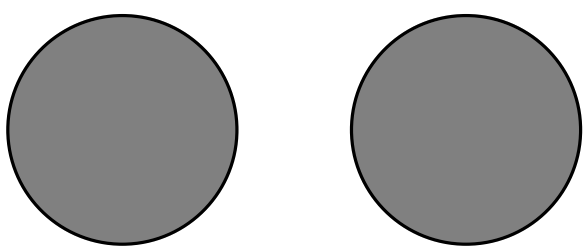


(b)

Multiple Choice:

(i) *path-connected* ✓

(ii) *not path-connected*



(c)

Multiple Choice:

(i) *path-connected*

(ii) *not path-connected* ✓

Simply Connected Sets

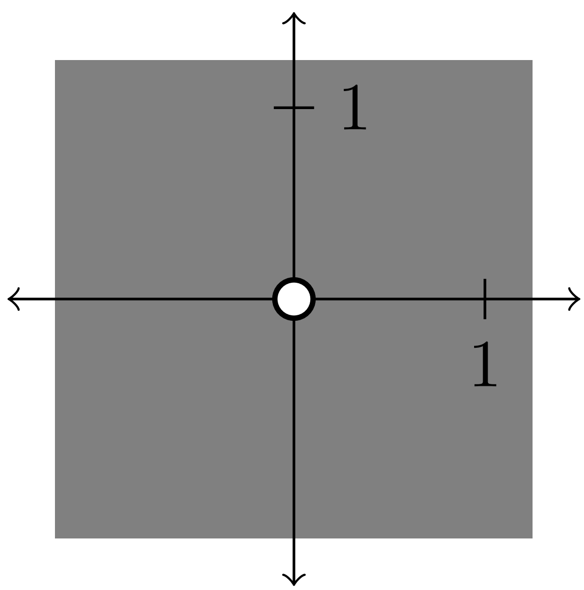
The last topological concept we will cover is when a path-connected set is "simply connected." Intuitively, this depends on whether or not the set has holes. Our definition for simply connected is a bit hand wavy, but this will serve our purpose just fine. It can be made rigorous using continuous maps.

Definition 65. *A path-connected set $X \subset \mathbb{R}^n$ is* simply connected *if any closed path (i.e., loop) can be shrunk to a point, where the shrinking occurs entirely in X.*

Note that a set needs to be path-connected in order to be considered simply connected.

Problem 20 *For each of the following sets, determine whether or not they are simply connected.*

(a) $\mathbb{R}^2 \setminus \{(0,0)\}$

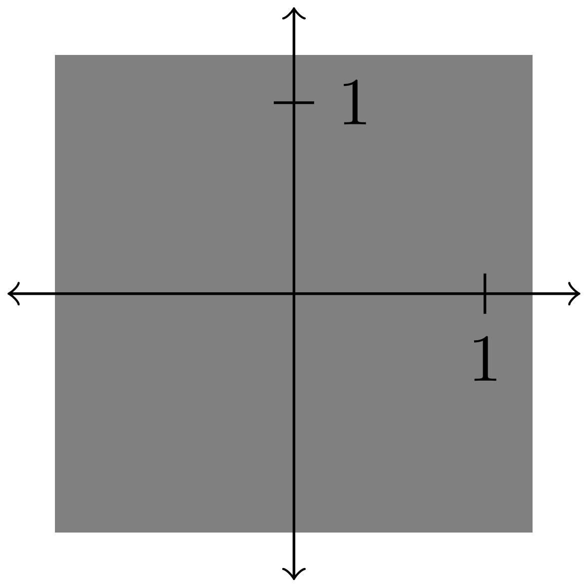


Multiple Choice:

(i) *simply connected*

(ii) *not simply connected* ✓

(b) $\mathbb{R}^2$

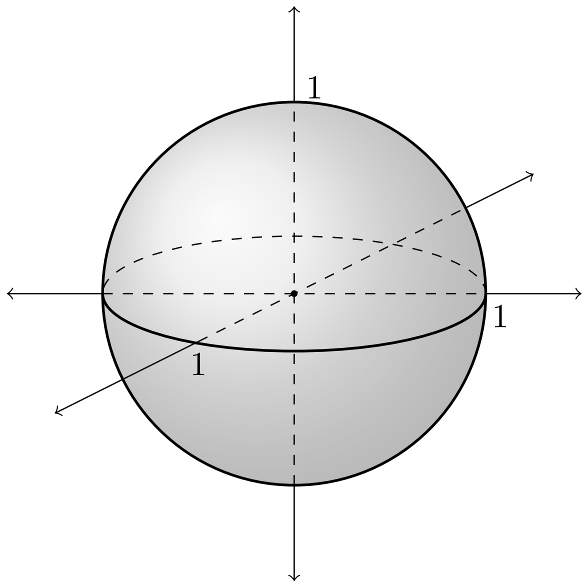


Multiple Choice:

(i) *simply connected* ✓

(ii) *not simply connected*

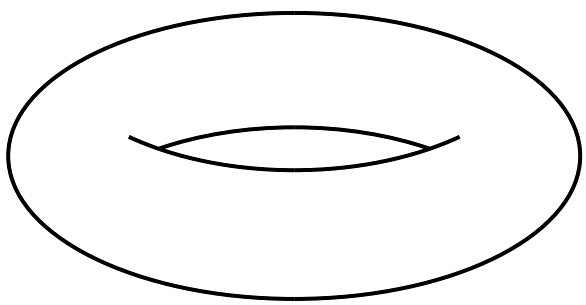
(c) $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$



Multiple Choice:

(i) *simply connected* ✓
(ii) *not simply connected*

(d) *A torus (the surface of a donut)*



Multiple Choice:

(i) *simply connected*
(ii) *not simply connected* ✓

Summary

In this activity, we introduced the following topological terms.

- open ball
- open set
- interior point
- closed set
- boundary point
- boundary of a set
- path-connected
- simply connected

We also established the following results.

- The union of a collection of open sets is itself open.
- A set is closed if and only if it contains all of its boundary points.

These terms will allow us to correctly state some theorems about vector fields, which have special requirements on the domain. These theorems will be covered in the next two activities.

Path Independence and FTLI

In the previous activity, we introduced some basic definitions from topology. These definitions are necessary in order to correctly state theorems in this section and the next, so pay attention to the hypotheses of these theorems!

We begin this activity by defining what it means for a vector field to be path independent. We then state and prove the Fundamental Theorem of Line Integrals. This theorem gives us an easy way to compute vector line integrals of a gradient field, provided the domain is open and connected. We then use the Fundamental Theorem of Line Integrals to show that conservative vector fields are path independent (under the appropriate conditions), and finish by discussing the various methods we now have for computing vector line integrals.

Path Independence

In this section, we introduce the idea of "path independence" for a vector field. We see an example of a vector field that is not path independent, and vector field that is path independent.

Example 89. *Consider the vector field* $\boldsymbol{F}(x,y) = (y,0)$*. Let* $\boldsymbol{x}(t)$ *be the path from* $(1,0)$ *to* $(0,1)$ *along a straight line. Let* $\boldsymbol{y}(t)$ *be the path from* $(1,0)$ *to* $(0,1)$ *counterclockwise around the unit circle. Compute* $\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s}$ *and* $\int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s}$*. Are they equal?*

Explanation. *We'll begin by parametrizing our paths.*

$$\boldsymbol{x}(t) = (\boxed{1-t}, \boxed{t}) \qquad \text{for } t \in [0,1]$$

$$\boldsymbol{y}(t) = (\boxed{\cos(t)}, \boxed{\sin(t)}) \qquad \text{for } t \in [0, \frac{\pi}{2}]$$

Now, we compute these line integrals using the definition

$$\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt.$$

Learning outcomes:
Author(s): Melissa Lynn

Integrating along $\boldsymbol{x}$, we have

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\ &= \int_{\boxed{0}}^{\boxed{1}} \text{given}(t, 0) \cdot (-1, 1)\, dt \\ &= \int_{\boxed{0}}^{\boxed{1}} -t\, dt \\ &= \boxed{-\frac{1}{2}}\end{aligned}$$

Integrating along $\boldsymbol{y}$, we have

$$\begin{aligned}\int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\ &= \int_0^{\pi/2} (\sin(t), 0) \cdot (\boxed{-\sin(t)}, \boxed{\cos(t)})\, dt \\ &= \int_0^{\pi/2} \boxed{-\sin^2(t)}\, dt \\ &= \boxed{-\frac{\pi}{4}}\end{aligned}$$

Comparing $\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s}$ and $\int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s}$, we see that they are

Multiple Choice:

(a) *Equal.*

(b) *Not equal.* ✓

Now, let's investigate integrating a different vector field along those same paths.

Example 90. *Consider the vector field $\boldsymbol{F}(x, y) = (y, x)$. Let $\boldsymbol{x}(t)$ be the path from $(1, 0)$ to $(0, 1)$ along a straight line. Let $\boldsymbol{y}(t)$ be the path from $(1, 0)$ to $(0, 1)$ counterclockwise around the unit circle. Compute $\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s}$ and $\int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s}$. Are they equal?*

Explanation. *Once again, we begin by parametrizing our paths.*

$$\begin{aligned}\boldsymbol{x}(t) &= (1 - t, t) && \text{for } t \in [0, 1] \\ \boldsymbol{y}(t) &= (\cos(t), \sin(t)) && \text{for } t \in [0, \frac{\pi}{2}]\end{aligned}$$

Next, we compute these line integrals using the definition

$$\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt.$$

Integrating along $\boldsymbol{x}$*, we have*

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\ &= \int_0^1 (\boxed{t}, \boxed{1-t}) \cdot (-1, 1)\, dt \\ &= \int_0^1 \boxed{1-2t}\, dt \\ &= \boxed{0}\end{aligned}$$

Integrating along $\boldsymbol{y}$*, we have*

$$\begin{aligned}\int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\ &= \int_0^{\pi/2} (\boxed{\sin(t)}, \boxed{\cos(t)}) \cdot (-\sin(t), \cos(t))\, dt \\ &= \int_0^{\pi/2} \boxed{-\sin^2(t) + \cos^2(t)}\, dt \\ &= \boxed{0}\end{aligned}$$

Comparing $\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s}$ *and* $\int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s}$*, we see that they are*

Multiple Choice:

(a) *Equal.* ✓

(b) *Not equal.*

In fact, it turns out that if we integrate $\mathbf{F}(x, y) = (y, x)$ along *any* path starting at $(1, 0)$ and ending at $(0, 1)$, the integral will be 0. So, the integral does not depend on the path we take to get between these two points. A vector field with this property is called *path independent*.

Definition 66. *A continuous vector field is called* path independent *if* $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_D \boldsymbol{F} \cdot d\boldsymbol{s}$ *for any two simple, piecewise* C^1*, oriented curves* C *and* D *with the same start and end points.*

Let's review the meaning of the requirements on the curves C and D.

A curve is *simple* if it (isn't too "bumpy." / doesn't intersect itself, except the start and end point can be the same. ✓/ is smooth.)

A curve is C^1 if it (is continuous. / is differentiable. / has continuous partial derivatives. ✓)

A curve is *oriented* if it (has a specified direction. ✓/ knows which way is North.)

Our next question is: how do we know if a vector field is path independent? Of course, it's impossible to check every line integral over every possible curve between any two points - we would never finish these computations! Instead, we need some theorems that give us conditions under which a vector field is path independent. Our first step towards these theorems is the Fundamental Theorem of Line Integrals.

Fundamental Theorem of Line Integrals

We now introduce the Fundamental Theorem of Line Integrals, which gives us a slick way to compute the integral of a gradient vector field over a piecewise C^1 curve. In particular, note the conditions on the domain X: it must be open and connected.

Theorem 11. ***Fundamental Theorem of Line Integrals***

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1, where X is open and connected. Then if C is any piecewise C^1 curve from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$\int_C \nabla f \cdot d\boldsymbol{s} = f(\boldsymbol{B}) - f(\boldsymbol{A})$$

This should look vaguely familiar - it resembles the Fundamental Theorem of Calculus from single variable calculus, also called the evaluation theorem.

We now prove the Fundamental Theorem of Line Integrals in the special case where we have a *simple* parametrization of the curve C.

Proof Let $\mathbf{x}(t)$ be a simple parametrization of C, where $t \in [a, b]$, $\mathbf{x}(a) = \mathbf{A}$, and $\mathbf{x}(b) = \mathbf{B}$ (so the starting point is $\mathbf{A}$, and the ending point is $\mathbf{B}$).

Then we compute the line integral as:

$$\int_C \nabla f \cdot d\mathbf{s} = \int_{\mathbf{x}} \nabla f \cdot d\mathbf{s}.$$

By the definition, we can compute this line integral as

$$\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

The integrand here should look familiar from one of the multivariable versions of the chain rule - it's the derivative of $f(\mathbf{x}(t))$. Making this replacement, we have

$$\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \frac{d}{dt}(f(\mathbf{x}(t)))\,dt.$$

Now, we can apply the Fundamental Theorem of Calculus (from single variable) to evaluate this integral, since an antiderivative for the integrand will be given by $f(\mathbf{x}(t))$. From this we obtain:

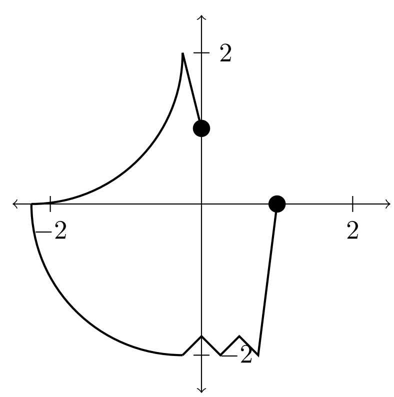
$$\begin{aligned}\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} &= (f(\mathbf{x}(t)))|_a^b \\ &= f(\mathbf{x}(b)) - f(\mathbf{x}(a)) \\ &= f(\mathbf{B}) - f(\mathbf{A})\end{aligned}$$

In the last step, we use that $\mathbf{A}$ and $\mathbf{B}$ are the start an end points of $\mathbf{x}(t)$, respectively.

Thus, we have proven the Fundamental Theorem of Line Integrals (when we have a simple parametrization of the curve C). ■

Before discussing how the Fundamental Theorem of Line Integrals relates to path independence, let's look at how this helps us compute integrals of gradient vector fields.

Example 91. *Let $\boldsymbol{F}(x,y) = (y,x)$. Observe that $\boldsymbol{F} = \nabla f$, where $f(x,y) = xy$. Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ for the curve C below, starting at $(0,1)$ and ending at $(1,0)$.*



Explanation. *We certainly would like to avoid parametrizing this curve! So, we will use the Fundamental Theorem of Line Integrals to compute this integral.*

First, let's verify that $\boldsymbol{F} = \nabla f$ for $f(x, y) = xy$.

$$\frac{\partial}{\partial x} xy = \boxed{y}$$
$$\frac{\partial}{\partial y} xy = \boxed{x}$$

Thus, we have $\nabla f(x, y) = (y, x) = \boldsymbol{F}(x, y)$.

We can then use the Fundamental Theorem of Line Integrals to compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$.

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_C \nabla f \cdot d\boldsymbol{s} \\
&= f(0, 1) - f(1, 0) \\
&= \boxed{0}
\end{aligned}$$

Now, it turns out that we can use the Fundamental Theorem of Line Integrals to prove the following corollary about the relationship between conservative vector fields and path independence. Note once again that we require the domain to be open and connected.

Corollary 1. *If $\boldsymbol{F}$ is a conservative vector field (so $\boldsymbol{F} = \nabla f$ for some f) defined on an open and connected domain X, then $\boldsymbol{F}$ is path independent.*

Proof Let C and D be two curves with starting point $\mathbf{A}$ and ending point $\mathbf{B}$. We will show that $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_D \mathbf{F} \cdot d\mathbf{s}$.

Recall that "$\mathbf{F}$ is conservative" means that $\mathbf{F} = \nabla f$ for some function f, which will enable us to use the Fundamental Theorem of Line Integrals (FTLI). Then we have:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_C \nabla f \cdot d\mathbf{s} \\
&= f(\mathbf{B}) - f(\mathbf{A}) && \text{(by FTLI)} \\
&= \int_D \nabla f \cdot d\mathbf{s} && \text{(also by FTLI)} \\
&= \int_D \mathbf{F} \cdot d\mathbf{s}.
\end{aligned}$$

Thus, we have shown that $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_D \mathbf{F} \cdot d\mathbf{s}$, and so have shown that $\mathbf{F}$ is path independent. ■

This corollary, with the Fundamental Theorem of Line Integrals, gives us a new tool for computing line integrals.

Strategies for Computing Line Integrals

We now have a few options for computing line integrals:

(a) Using the original definition:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt$$

(b) If $\mathbf{F}$ is conservative (so $\mathbf{F} = \nabla f$ for some f):

(i) We can use the Fundamental Theorem of Line Integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{B}) - f(\mathbf{A})$$

(ii) Since the vector field is path independent, we can find an *easier* path with the same start and end points, and integrate over that path.

Let's look at how these different methods can be used in an example.

Example 92. *Let* $\boldsymbol{F}(x, y) = (2xy^2, 2x^2y)$, *and consider* $\boldsymbol{x}(t) = (2\cos(\pi t), \sin(\pi t^2))$ *for* $t \in [0, 1]$. *Compute* $\displaystyle\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s}$.

Explanation. *We'll evaluate this integral in three different ways.*

First, let's evaluate using the definition of vector line integrals.

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \boldsymbol{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\ &= \int_0^1 (4\cos(\pi t)\sin^2(\pi t^2), 8\cos^2(\pi t)\sin(\pi t^2)) \cdot (\boxed{-2\pi\sin(\pi t)}, \boxed{2\pi t\cos(\pi t^2)})\, dt \\ &= \int_0^1 \boxed{-8\pi\cos(\pi t)\sin(\pi t)\sin^2(\pi t^2) + 16\pi t\cos(\pi t^2)\cos^2(\pi t)\sin(\pi t^2)}\, dt\end{aligned}$$

This is an integral that it might be possible to figure out how to compute, but we certainly do not want to! We can use a computer algebra system to see that the result is 0*, but to compute the integral by hand, we will turn to our other methods.*

For our alternate methods, we need to find a potential function $f(x, y)$ *such that* $\boldsymbol{F} = \nabla f$. *It turns out that* $f(x, y) = x^2y^2$ *works. Let's verify this.*

$$\frac{\partial}{\partial x}(x^2y^2) = \boxed{2xy^2}$$
$$\frac{\partial}{\partial y}(x^2y^2) = \boxed{2x^2y}$$

Now that we have our function $f(x, y) = x^2y^2$ *such that* $\boldsymbol{F} = \nabla f$*, we will use the Fundamental Theorem of Line Integrals to evaluate. Note the start and end points of our curve*

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{x}(0) = (\boxed{2}, \boxed{0}) \\ \boldsymbol{B} &= \boldsymbol{x}(1) = (\boxed{-2}, \boxed{0})\end{aligned}$$

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_{\boldsymbol{x}} \nabla f \cdot d\boldsymbol{s} \\ &= f(\boldsymbol{B}) - f(\boldsymbol{A}) \\ &= \boxed{0}\end{aligned}$$

Note that this is a much easier computation than the integral we had from the first method.

Finally, we compute the line integral using the third method. We have already shown that $\mathbf{F}$ *is a conservative vector field (by finding* f *such that* $\mathbf{F} = \nabla f$*), and hence we know that* $\mathbf{F}$ *is path independent. So we can compute this integral by instead integrating over an easier path with the same start and end points,* $(2, 0)$ *and* $(-2, 0)$*, respectively. Let's choose the straight line from* $(2, 0)$ *to* $(-2, 0)$*, and parametrize this curve.*

$$\boldsymbol{y}(t) = (\boxed{t}, 0) \qquad \textit{for } t \in [-2, 2]$$

Now we can integrate over y *instead, which will be a much easier computation.*

$$\begin{aligned}\int_{\boldsymbol{x}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_{\boldsymbol{y}} \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= \int_{-2}^{2} \boldsymbol{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\ &= \int_{-2}^{2} (\boxed{0}, \boxed{0}) \cdot (1, 0)\, dt \\ &= \int_{-2}^{2} \boxed{0}\, dt \\ &= \boxed{0}\end{aligned}$$

So we've seen that we can compute this line integral in a few different ways, using the fact that the vector field is conservative.

Depending on the particular problem or example, any one of these methods might be easier than the others. You should practice trying these different methods, and see which you prefer! However, remember that for the second and third options, we need to first verify that the vector field $\mathbf{F}$ is conservative. This usually means finding a potential function f such that $\mathbf{F} = \nabla f$.

Summary

In this activity, we introduced the following terms.

- Path independent

We also established the following results.

- The Fundamental Theorem of Line Integrals.
- If $\mathbf{F}$ is a conservative vector field defined on an open and connected domain X, then $\mathbf{F}$ is path independent.

These results provide us with new methods to compute some vector line integrals.

In the next activity, we will discuss how we can determine whether or not a vector field is conservative.

Conservative Vector Fields

In the previous activity, we proved the Fundamental Theorem of Line Integrals:

Theorem 12. ***Fundamental Theorem of Line Integrals***

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1, where X is open and connected. Then if C is any piecewise C^1 curve from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$\int_C \nabla f \cdot d\boldsymbol{s} = \boxed{f(B) - f(A)}$$

We were able to use the Fundamental Theorem of Line Integrals to easily compute line integrals of conservative vector fields, and we were also able to prove the following corollary.

Corollary 2. *If $\boldsymbol{F}$ is a conservative vector field (so $\boldsymbol{F} = \nabla f$ for some f) defined on an open and connected domain X, then $\boldsymbol{F}$ is* $\boxed{\textit{pathindependent}}$.

Both of these important results require that we have a conservative vector field. In this activity, we will discuss how we can determine whether or not a vector field is conservative, so that we can check if those results can be applied.

Finding a Potential Function

Recall the definition of a conservative vector field.

Definition 67. *A vector field* $\mathbf{F}$ *is* conservative *if there is a C^1 scalar-valued function f such that $\mathbf{F} = \nabla f$. Then f is called a* potential function *for* $\mathbf{F}$.

So, one way to show that a vector field $\mathbf{F}$ is conservative is by finding such a potential function f.

For simple examples, you might be able to do this by guessing. However, more complicated examples require a more systematic approach. The approach is easiest to understand through examples, so we'll work through a couple before describing the steps for the general case.

Example 93. *Find a potential function for the vector field $\mathbf{F}(x, y) = (2xy^3 + 1, 3x^2y^2 - y^{-2})$.*

Learning outcomes:
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Explanation. *First, note that if there there is a function f such that $\nabla f = \mathbf{F}$, then*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2xy^3 + 1, 3x^2y^2 - y^{-2}\right)$$

Let's start by looking at the x-term. We must have $\dfrac{\partial f}{\partial x} = 2xy^3 + 1$. Integrating with respect to x, we have

$$\begin{aligned} f(x,y) &= \int 2xy^3 + 1\,dx \\ &= x^2y^3 + x + g(y) \end{aligned}$$

The first part of this expression, $x^2y^3 + x$, is an antiderivative for $2xy^3 + 1$ with respect to x. The second part of the expression, $g(y)$, is the "constant" for the integral. It's possible that there are some terms which depend only on y, hence are constant with respect to x, and writing $g(y)$ takes these terms into account.

At this point, we know that f has the form $f(x,y) = x^2y^3 + x + g(y)$, but we still need to figure out what $g(y)$ is. For this, we use the y-term of the vector field $\mathbf{F}$. From this, we have $\dfrac{\partial f}{\partial y} = 3x^2y^2 - y^{-2}$. Since we know $f(x,y) = x^2y^3 + x + g(y)$, we must have

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^2y^3 + x + g(y)\right) \\ &= 3x^2y^2 + g'(y). \end{aligned}$$

Comparing this with $\dfrac{\partial f}{\partial y} = 3x^2y^2 - y^{-2}$, we must have $g'(y) = -y^{-2}$. Then we find that

$$\begin{aligned} g(y) &= \int -y^{-2}\,dy \\ &= y^{-1} + C \end{aligned}$$

Hence, any potential function would have the form $f(x,y) = x^2y^3 + x + y^{-1} + C$. Choosing $C = 0$, we obtain a specific potential function $f(x,y) = x^2y^3 + x + y^{-1}$.

We now work through finding a potential function for a three dimensional vector field.

Example 94. *Find a potential function for the vector field $\mathbf{F}(x,y,z) = (2xy, x^2 + z + 2y, y + \cos(z))$.*

Explanation. *First, note that a potential function $f(x,y,z)$ would have to satisfy*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\boxed{2xy}, \boxed{x^2 + z + 2y}, \boxed{y + \cos(z)}\right)$$

We begin by considering the x-component, noticing that $\dfrac{\partial f}{\partial x} = 2xy$. *We integrate with respect to x.*

$$\begin{aligned} f(x,y,z) &= \int 2xy\,dx \\ &= \boxed{x^2y} + g(y,z) \end{aligned}$$

Here, $g(y,z)$ is a function of only y and z, hence constant with respect to x. We now differentiate with respect to y, in order to compare to the y-component of the vector field.

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}\left(x^2y + g(y,z)\right) \\ &= \boxed{x^2} + \frac{\partial g}{\partial y} \end{aligned}$$

Comparing this with $\dfrac{\partial f}{\partial y} = x^2 + z + 2y$, *we have* $\dfrac{\partial g}{\partial y} = z + 2y$. *We integrate this with respect to y.*

$$\begin{aligned} g(y,z) &= \int z + 2y\,dy \\ &= \boxed{yz + y^2} + h(z) \end{aligned}$$

Here, h is a function of only z, hence is constant with respect to y. We now know that f has the form $f(x,y,z) = x^2y + yz + y^2 + h(z)$. So, our final task is to find $h(z)$. We differentiate f with respect to z in order to compare with the z-component of the vector field.

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2y + yz + y^2 + h(z)\right) \qquad = \boxed{y} + h'(z)$$

Comparing this with $\dfrac{\partial f}{\partial z} = y + \cos(z)$, *we have $h'(z) = \cos(z)$. Integrating with respect to z, we obtain*

$$\begin{aligned} h(z) &= \int \cos(z)\,dz \\ &= \boxed{\sin(z)} + C \end{aligned}$$

Where C is a constant. Thus, any potential function would have the form $f(x,y,z) = \boxed{x^2y + yz + y^2 + \sin(z)} + C$. Choosing $C = 0$, we have a specific potential function $f(x,y,z) = \boxed{x^2y + yz + y^2 + \sin(z)}$.

Summarizing the steps we take in each of the above examples, we have the following process for finding a potential function for a conservative vector field $\mathbf{F}(x_1, x_2, ..., x_n)$.

(a) Integrate the first component of $\mathbf{F}$ with respect to x_1, in order to find the terms of $f(x_1, x_2, ..., x_n)$ which depend on x_1. From this, we can write $f(x_1, x_2, ..., x_n) = (x_1\text{-terms}) + f_1(x_2, ..., x_n)$.

(b) Differentiate $f(x_1, x_2, ..., x_n) = (x\text{-terms}) + f_1(x_2, ..., x_n)$ with respect to x_2. Compare this to the second component of $\mathbf{F}$ in order to determine an expression for $\dfrac{\partial f_1}{\partial x_2}$. Integrate this expression with respect to x_2, so we can write $f_1(x_2, ..., x_n) = (x_2\text{-terms}) + f_2(x_3, ..., x_n)$. Hence we have $f(x_1, x_2, ..., x_n) = (x_1\text{- and } x_2\text{-terms}) + f_2(x_3, ..., x_n)$.

(c) Repeat this process until all components are used.

So far, we've only seen cases where a potential function exists. However, we would also like to be able to show that a vector field is *not* conservative. Let's look at what happens in our process when we have a vector field which is not conservative.

Example 95. *Try (and fail) to find a potential function for the vector field* $\mathbf{F}(x, y) = (-y, x)$.

Explanation. *If a potential function existed, it would have to satisfy*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (\boxed{-y}, \boxed{x})$$

We begin with $\dfrac{\partial f}{\partial x} = -y$. *Integrating with respect to* x, *we have*

$$\begin{aligned} f(x, y) &= \int -y\, dx \\ &= \boxed{-yx} + g(y) \end{aligned}$$

Differentating with respect to y, *we then obtain*

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-yx + g(y)\right) \\ &= \boxed{-x} + g(y) \end{aligned}$$

When we compare this to the y*-component of the vector field* $\mathbf{F}$ *in order to determine* $g(y)$, *we would have to have* $x = -x + g(y)$. *But this is impossible! Thus we see that our method has broken down, and we are not able to find a potential function.*

Here, we see that the system breaks down, and we aren't able to produce a potential function. This is good, since it turns out the vector field isn't conservative. However, we would an easy way to prove that it isn't conservative. The following theorem gives us a quick way to prove that a vector field is not conservative.

Theorem 13. *Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and connected. If $\mathbf{F}$ is conservative, then $D\mathbf{F}$ is symmetric.*

The contrapositive of this theorem states:

Theorem 14. *Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and connected. If $D\mathbf{F}$ is not $\boxed{\textit{symmetric}}$ then $\mathbf{F}$ is not $\boxed{\textit{conservative}}$.*

Thus, provided we have a C^1 vector field and the domain is open and connected, we can show a vector field is not conservative by showing that its derivative matrix is not symmetric.

Example 96. *Show that the vector field $\mathbf{F}(x, y) = (-y, x)$ is not conservative.*

Explanation. *First, note that $\mathbf{F}$ is a C^1 vector field with domain $\mathbb{R}^2$. Since $\mathbb{R}^2$ is open and connected, our theorem applies. We compute the derivative matrix $D\mathbf{F}$.*

$$\begin{aligned} D\mathbf{F} &= \begin{pmatrix} \frac{\partial}{\partial x}(-y) & \frac{\partial}{\partial y}(-y) \\ \frac{\partial}{\partial x}x & \frac{\partial}{\partial y}x \end{pmatrix} \\ &= \begin{pmatrix} \boxed{0} & \boxed{-1} \\ \boxed{1} & \boxed{0} \end{pmatrix} \end{aligned}$$

Since this matrix is not symmetric, $\mathbf{F}$ is not a conservative vector field.

Note how much simpler this is than trying to find a potential function. We now prove our theorem, showing that a conservative C^1 vector field on an open and connected domain has symmetric derivative.

Proof Let $\mathbf{F}$ be a C^1 vector field defined on an open connected domain $X \subset \mathbb{R}^n$. If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for some scalar-valued function f on X. This means

$$\mathbf{F}(x_1, ..., x_n) = \nabla f(x_1, ..., x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right).$$

Then the derivative matrix of $\mathbf{F}$ is

$$D\mathbf{F} = D(\nabla f) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

By Clairaut's Theorem, the mixed partials are equal, so this matrix is symmetric. ■

For each of the given vector fields $\mathbf{F}$, determine whether or not it's conservative. If it is conservative, find a potential function. If $\mathbf{F}$ is not conservative, compute the derivative matrix of $\mathbf{F}$ in order to prove that it is not conservative.

Problem 21 $\mathbf{F}(x, y) = (2xy + y^2 + e^y, x^2 + 2xy + xe^y)$

Multiple Choice:

(a) *conservative* ✓

(b) *not conservative*

Problem 21.1 $f(x, y) = \boxed{x^2y + y^2x + e^yx}$

Problem 22 $\mathbf{F}(x, y) = (x^2y + e^{x^2}, \sin(x) + y^3)$

Multiple Choice:

(a) *conservative*

(b) *not conservative* ✓

Problem 22.1 $D\mathbf{F}(x, y) = \begin{pmatrix} 2xy + 2xe^{x^2} & \boxed{x^2} \\ \boxed{\cos(x)} & 3y^2 \end{pmatrix}$

We've seen that if a vector field is conservative, then its derivative matrix is symmetric. But is the converse true? That is, if the derivative matrix is symmetric, does that mean that the vector field is conservative? We'll answer this question in the next chapter.

Summary

In this activity, we reviewed the following terms.

- Conservative

- Potential function

We also established the following results.

- A systematic method for finding a potential function for a given vector field (thus showing that the vector field is conservative).
- Let $\mathbf{F}$ be a C^1 vector field, and let X be open and connected. If $D\mathbf{F}$ is not symmetric then $\mathbf{F}$ is not conservative.

These provide us with methods for showing that vector field isn't conservative, and for providing a potential function when it is conservative.

Part VII

Double Integrals

Part VIII

Curl and Divergence

Curl of a Vector Field

Imagine the vector field below represents fluid flow:

Desmos link: `https://www.desmos.com/calculator/vhuoyka1ys`

If we fix the center point of each + above, which way will they rotate? (clockwise / counter-clockwise ✓)

We can describe this concept as microscopic rotation or local rotation, and it turns out that the *curl* of a vector field measures this local rotation.

In this activity, we define curl and focus on computation. In the next activity, we discuss the geometric significance of curl and how it represents local rotation.

Definition of Curl

A curl is an example of an *operator*, which is a mathematical object you've seen before. Roughly speaking, it's a "function" on functions. That is, it takes a function as an input, and produces a function as an output. Here, we're using "function" very broadly - a function could be scalar-valued, a path, or even a vector field!

To prove that you've seen operators before, let's look at a specific example:

Problem 23 *What does $\frac{d}{dt}g(t)$ mean?*

Multiple Choice:

(a) *Multiply $g(t)$ by the fraction $\frac{d}{dt}$.*

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(b) *Take the derivative of g with respect to t.* ✓

Problem 23.1 *What does* $\frac{d}{dt}$ *mean?*

Multiple Choice:

(a) *The same thing as* $\frac{1}{t}$.

(b) *Take the derivative with respect to t.* ✓

Problem 23.1.1 *It turns out that* $\frac{d}{dt}$ *is an example of an* operator.

To introduce the curl, we need to talk about another operator, ∇ *which we call the* del operator.

What does $\nabla(g(x, y, z))$ *mean?*

Multiple Choice:

(a) *The change in g.*

(b) $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$ ✓

Problem 23.1.1.1 *From this, we can deduce that* ∇ *should mean* $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. *Note that this is an operator.*

Definition 68. *The* del operator *in* $\mathbb{R}^n$ *is* $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, ..., \frac{\partial}{\partial x_n}\right)$.

There's one more ingredient that we need to review in order to define the curl of a vector field, the cross product.

Problem 24 *If* $\mathbf{v} = (1, 2, 3)$ *and* $\mathbf{w} = (4, 5, 6)$, *what is* $\mathbf{v} \times \mathbf{w}$? $\boxed{(-3, 6, -3)}$.

Problem 24.1 *Note that this is computed as the determinant*

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

Problem 24.1.1 *Given a vector field* $\mathbf{F} = (M(x,y,z), N(x,y,z), P(x,y,z))$, *how might we interpret* $\nabla \times \mathbf{F}$*?*

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \boxed{\dfrac{\partial}{\partial x}} & \boxed{\dfrac{\partial}{\partial y}} & \boxed{\dfrac{\partial}{\partial z}} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \end{aligned}$$

Based on this, we give our definition for the curl of a three-dimensional vector field:

Definition 69. *The* curl *of a three-dimensional vector field* $\mathbf{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$ *is*

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \boxed{M} & \boxed{N} & \boxed{P} \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \end{aligned}$$

Note that this input is a vector field $\mathbf{F}$ in $\mathbb{R}^3$, and the output is another vector field in $\mathbf{R}^3$.

Problem 25 *Let* $\mathbf{F} = (e^y, xz, 3z)$*. Compute the curl* $\nabla \times \mathbf{F}$*.*

$$\nabla \times \mathbf{F} = \boxed{(-x, 0, z - e^y)}$$

Note that we have only defined the curl for three-dimensional vector fields. However, by being a bit clever, we can extend this definition to two-dimensional vector fields.

Definition 70. *If the three-dimensional vector field* $\mathbf{F}$ *has the form* $\mathbf{F}(x, y, z) = (M(x, y), N(x, y), 0)$*, then* $\nabla \times \mathbf{F}$ *is often called the* two-dimensional curl *of* $\mathbf{F}$*.*

Moreover, if $\mathbf{G}(x, y) = (M(x, y), N(x, y))$ *is a vector field in* $\mathbb{R}^2$*, then we define the* curl *of* $\mathbf{G}$ *as the curl of the three-dimensional vector field* $\widetilde{\mathbf{G}}(x, y, z) = (M(x, y), N(x, y), 0)$*.*

It turns out, the curl of a two-dimensional vector field can be written in a simpler form.

Proposition 29. *The two-dimensional curl of* $\mathbf{F}(x, y) = (M(x, y), N(x, y), 0)$ *is*

$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Proof From the definition of the curl, we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Since the third component, P, of our vector field is identically 0, we have

$$\nabla \times \mathbf{F} = \left(\boxed{0} - \frac{\partial N}{\partial z}, -\left(\boxed{0} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Both $M(x, y)$ and $N(x, y)$ are constant with respect to z, so we then have

$$\nabla \times \mathbf{F} = \left(\boxed{0}, \boxed{0}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right),$$

as desired. ■

Sometimes we refer to $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ as the curl $\nabla \times \mathbf{F}$ if $\mathbf{F}$ is two-dimensional, instead of writing out the entire vector.

Note that we've only defined the curl of a vector field for two- and three-dimensional vector fields. Why doesn't it make sense to define the curl of a four-dimensional (or higher!) vector field?

Multiple Choice:

(a) We only exist in three dimensions.

(b) The cross product is only defined in $\mathbf{R}^3$. ✓

Problem 26 *Given* $\mathbf{F}(x, y) = (y, 0)$, *compute the curl* $\nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, -1)}$$

Problem 27 *Given* $\mathbf{F}(x, y) = (-y, 0)$, *compute the curl* $\nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 1)}$$

Let's look at this example, $\mathbf{F}(x, y) = (-y, 0)$. It turns out that this is the vector field from the beginning of this activity:

We imagined that the center of the plus signs were fixed, and determined that the vector field would rotate the plus signs counterclockwise. We claimed that this local rotation had something to do with the curl of the vector field, which we computed to be $\nabla \times \mathbf{F} = (0, 0, 1)$.

In the next activity, we'll study the geometric significance of the curl, and why the curl measures this "microscopic" rotation.

Summary

In this section, we defined the curl of a two- or three-dimensional vector field, which can be computed as follows:

- For a three-dimensional vector field, $\mathbf{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$, we have $\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$
- The two-dimensional curl of $\mathbf{F}(x, y) = (M(x, y), N(x, y), 0)$ can be computed as
$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

In the next activity, we will discuss the geometric significance of the curl, and how it relates to the local rotation of the vector field.

Geometric Significance of Curl

Consider the vector field $\mathbf{F}(x, y) = (-y, 0)$.

We can compute the curl of this vector field,

$$\nabla \times \mathbf{F} = (0, 0, 1)$$

Imagine that we fix a point (representing a particle) in this vector field, but allow it to rotate. If imagine the vector field acting as a force on this particle, which way will it cause the particle to rotate?

(VECTOR FIELD)

Here, we see that the vector field is applying a greater force to the "top" of the particle than to the "bottom." this will cause the particle to rotate counterclockwise. We describe this type of rotation as *local rotation* or *microscopic rotation*, since it's the rotation when we "zoom in" on the particle.

It turns out that the curl of a vector field provides a measure of this local rotation - but how are these connected? We will answer this question in this section, discussing the geometric significance of the curl.

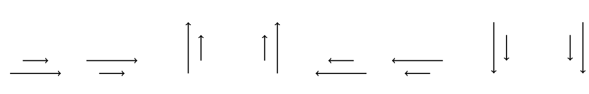
Geometric Significance of Two-dimensional Curl

Recall that, for a two-dimensional vector field $\mathbf{F}(x, y) = (M, N)$, we can compute the curl as

$$\nabla \times \mathbf{F} = (0, 0, N_x - M_y)$$

where N_x is the partial derivative of N with respect to x, and M_y is the partial derivative of M with respect to y. We'll start by considering how M_y and N_x contribute to local rotation.

First let's consider the case where $M_y < 0$. In this case, the x-component of the vector field $\mathbf{F}$ is *decreasing* as we move in the positive y direction. Select all pictures which match this situation.



Select All Correct Answers:

Learning outcomes:
Author(s): Melissa Lynn

(a) (a) ✓

(b) (b)

(c) (c)

(d) (d)

(e) (e)

(f) (f) ✓

(g) (g)

(h) (h)

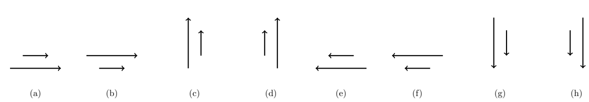
If $M_y < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

Now, let's consider the case where $N_x > 0$. This means that the y-component of the vector field $\mathbf{F}$ is increasing as we move in the positive x direction. Select all pictures which match this situation.



Select All Correct Answers:

(a) (a)

(b) (b)

(c) (c)

(d) (d) ✓

(e) (e)

(f) (f)

(g) (g) ✓

(h) (h)

If $N_x < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

We've seen that the signs of N_x and M_y correspond to the direction of local rotation, with $N_x > 0$ and $M_y < 0$ contributing to counterclockwise rotation.

In general, we have that the sign of $N_x - M_y$ corresponds to the direction of local rotation in the plane. In particular, we have the following correspondences:

$$\begin{aligned}
N_x - M_y > 0 &\longleftrightarrow \text{counterclockwise local rotation} \\
N_x - M_y < 0 &\longleftrightarrow \text{clockwise local rotation} \\
N_x - M_y = 0 &\longleftrightarrow \text{no local rotation}
\end{aligned}$$

Remembering that $N_x - M_y$ is the *curl* of the two-dimensional vector field $\mathbf{F}$, we now have that the sign of the curl tells us the direction of local rotation for two-dimensional vector fields.

We have a special term for a vector field that never has any local rotation: we call such a vector field *irrotational*.

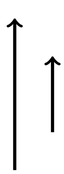
Furthermore, the length of the curl,

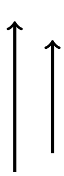
$$\|(0, 0, N_x - M_y)\| = |N_x - M_y|,$$

corresponds to the speed of rotation.

For example, in which case will the particle spin faster?

Multiple Choice:

(a)  ✓

(b) 

Note that this corresponds to a larger value of N_x (the change in the y-component of $\mathbf{F}$ as we move in the positive x direction).

We now apply our knowledge of the geometric significance of the curl in a couple of examples.

Example 97. *Consider the vector field* $\mathbf{F}(x, y) = (-y, x^2)$*. Compute the curl of* $\mathbf{F}$*, and describe the local rotation of the vector field at the points* $(1, 0)$ *and* $(-4, 1)$*.*

Explanation. *We begin by computing the curl of* $\mathbf{F}$*,*

$$\nabla \times \mathbf{F} = (0, 0, \boxed{2x + 1}).$$

At the point $(1, 0)$*, we have* $(\nabla \times \mathbf{F})(1, 0) = (0, 0, \boxed{3})$*. Looking at the third component, we see that the sign of* $N_x - M_y$ *at* $(1, 0)$ *is (positive ✓/ negative/ zero). Thus, the local rotation of the vector field at the point* $(1, 0)$ *is (clockwise / counterclockwise ✓/ no rotation).*

At the point $(-4, 1)$*, we have* $(\nabla \times \mathbf{F})(-4, 1) = (0, 0, \boxed{-7})$*. Looking at the third component, we see that the sign of* $N_x - M_y$ *at* $(-4, 1)$ *is (positive/ negative ✓/ zero). Thus, the local rotation of the vector field at the point* $(-4, 1)$ *is (clockwise ✓/ counterclockwise/ no rotation).*

Looking at a graph of the vector field, we can see that this local rotation is reflected in the graph.

(ADD GRAPH, WITH ROTATION?)

Example 98. *Let* $\mathbf{F}(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$*. Compute the curl* $\nabla \times \mathbf{F}$*, and interpret it geometrically.*

Explanation. *Computing our partial derivatives, we have*

$$N_x = \boxed{\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2}}$$

and

$$M_y = \boxed{\frac{-x^2 - y^2 + 2x^2}{(x^2 + y^2)^2}}.$$

Then, the curl of $\mathbf{F}$ *is*

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0}).$$

Thus, we see that there is no local rotation at any point in the vector field. This is particularly interesting once we look at a graph of the vector field.

(GRAPH)

From the graph of the vector field, there certainly seems to be some larger scale, global rotation of the vector field. However, our computation showed that there *is no local rotation. This example illustrates an important distinction: curl measures* local *rotation of a vector field, which is a different concept from* global *rotation.*

In this section, we saw how the curl of a vector field corresponded to local rotation for a two-dimensional vector field. In the next section, we describe how the curl of a vector field corresponds to local rotation for *three*-dimensional vector fields.

Geometric Significance of Three-dimensional Curl

For a three dimensional vector field $\mathbf{F}(x, y, z) = (M, N, P)$, we can compute the curl of $\mathbf{F}$ as

$$\nabla \times \mathbf{F} = (P_y - N_z, \boxed{-P_x + M_z}, \boxed{N_x - M_y}).$$

Here, the situation is more complicated than in two dimensions. In the plane, there are only two possible ways to rotate: clockwise and counterclockwise. In $\mathbb{R}^3$, there are infinitely many different ways to rotate, since we have infinitely many choices of axes. Yikes!

Fortunately, three-dimensional curl still tells about local rotation. In this case, we imagine local rotation as rotation of an infinitesimal (tiny) sphere. This sphere can rotate in infinitely many different ways, depending on which axis we rotate around.

When we look at the components of the curl, this tells us about rotation perpendicular to each of the axes, ignoring rotation in any other direction. Specifically,

$$\begin{aligned} N_x - M_y \text{ (the } z\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } z\text{-axis)} \\ P_y - N_z \text{ (the } x\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } x\text{-axis)} \\ -P_x + M_z \text{ (the } y\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } y\text{-axis)} \end{aligned}$$

Once again, the sign tells us the direction of rotation, with positive sign corresponding to counterclockwise rotation (viewed from the positive axes).

Furthermore, the length of the curl, $\|\nabla \times \mathbf{F}\|$, tells us the speed of rotation, and the direction of $\nabla \times \mathbf{F}$ tells us the axis of rotation.

In $\mathbb{R}^3$, we would like to be able to describe the direction of rotation around a given axis. However, this can be tricky, since it's a matter of perspective. Imagine rotation in the xy-plane. If the rotation is clockwise viewed from above, then it will be counterclockwise from below! Fortunately, curl follows the *right hand rule*:

> If you point your right thumb in the direct of $\nabla \times \mathbf{F}$, then your fingers will curl in the direction of local rotation.

We now put this to use in an example.

Problem 28 *Consider the vector field* $\mathbf{F}(x, y, z) = \left(0, \frac{-z}{(y^2+z^2)^{3/2}}, \frac{y}{(y^2+z^2)^{3/2}}\right)$. *Compute the curl of* $\mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{(\frac{-1}{(y^2+z^2)^{3/2}}, 0, 0)}$$

Problem 28.1 *What is the axis of local rotation (at any point)?*

Multiple Choice:

(a) *The x-axis.* ✓

(b) *The y-axis.*

(c) *The z-axis.*

(d) *Some other line.*

Problem 28.1.1 *Viewed from the positive x-axis, what is the direction of local rotation (at any point)?*

Multiple Choice:

(a) *Clockwise.* ✓

(b) *Counterclockwise.*

Problem 28.1.1.1 *How does the speed of local rotation change as we move closer to the origin?*

Multiple Choice:

(a) *Stays the same.*

(b) *Gets slower.*

(c) *Gets faster.* ✓

We've now seen how the curl describes the local rotation of a three-dimensional vector field. In the next section, we'll cover some connections of the curl to previous topics.

Summary

In this section, we studied the geometric significance of the curl. We found that the curl gives a measure of the local rotation of a vector field.

For a two-dimensional vector field, the sign of the curl told us the direction of rotation. Specifically, we have the following correspondence.

$$\begin{aligned} N_x - M_y > 0 &\longleftrightarrow \text{counterclockwise local rotation} \\ N_x - M_y < 0 &\longleftrightarrow \text{clockwise local rotation} \\ N_x - M_y = 0 &\longleftrightarrow \text{no local rotation} \end{aligned}$$

The magnitude $|N_x - M_y|$ corresponds to speed of rotation.

For a three-dimensional vector field, the components of the curl tell us about local rotation perpendicular to the axes. We also have:

- The length of the curl, $\|\nabla \times \mathbf{F}\|$, corresponds to the speed of rotation.
- The direction of the curl vector $\nabla \times \mathbf{F}$ gives the axis of rotation.
- Curl follows the right hand rule: if you point your thumb in the direction of $\nabla \times \mathbf{F}$, your fingers curl in the direction of local rotation.

In the next section, we'll consider the curl of a conservative vector field, and how the curl connects to Green's Theorem.

Connections of Curl with Older Material

We've defined the curl of a two or three dimensional vector field, and we found that this gives a measure of the local rotation of a vector field.

In this section, we discuss connections of the curl to previous topics from the course. In particular, we find the curl of a conservative vector field, and we restate Green's Theorem in terms of curl.

Curl of a Conservative Vector Field

In this section, we prove that the curl of a conservative vector field will always be zero. Thus, conservative vector fields are irrotational.

Theorem 15. *Suppose* $\mathbf{F}$ *is a* C^1 *conservative vector field in* $\mathbb{R}^3$*, so there is a function* $f : \mathbb{R}^3 \to \mathbb{R}$ *such that* $\mathbf{F} = \nabla f$*. Then* $\nabla \times \mathbf{F} = \mathbf{0}$.

Proof Suppose $\mathbf{F}(x, y, z) = (M, N, P)$ is a C^1 conservative vector field, with $\mathbf{F} = \nabla f$. Then we must have

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \boxed{(M, N, P)}.$$

Computing the curl of $\mathbf{F}$, we have

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, -\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right),\end{aligned}$$

substituting in for the components M, N, and P.

Now, we will use Clairaut's Theorem to simplify this vector. Since $\mathbf{F} = (M, N, P)$ is a C^1 vector field, the partial derivatives of its components ($\frac{\partial M}{\partial y}$, $\frac{\partial M}{\partial z}$, etc.) exist and are continuous. This means that all second-order partial derivatives of f exist and are continuous. Then, by Clairaut's Theorem, the order of differentiation for the second-order mixed partials doesn't matter. In particular, we

Learning outcomes:
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have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z}$$
$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}.$$

Using this fact in our computation of the curl, we now have

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \\ &= \boxed{(0,0,0)} \end{aligned}$$

Thus, we have shown that the curl of a conservative vector field is zero. ■

So, conservative vector fields are irrotational. A reasonable follow-up question would be: if the curl of a vector field is zero, is the vector field necessarily conservative? We'll leave this as an open question for the reader, with the suggestion that you think about how you can use past results, and what hypotheses are necessary for this converse to be true.

Curl and Green's Theorem

In this section, we see that we've actually already seen the curl of a vector field. It turns out that the curl showed up in Green's Theorem, we just didn't know that it was the curl yet.

Recall the statement of Green's Theorem:

Theorem 16. *Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise C^1 curves. Orient the boundary ∂D so that D is on the left as one travels along ∂D.*

Let $\mathbf{F}(x, y) = (M, N)$ be a C^1 vector field defined on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

The integrand of the double integral, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$, should now look familiar. This mysterious quantity is actually the two-dimensional curl of the vector field $\mathbf{F}$!

Using this realization, we can now restate Green's Theorem in terms of the curl of $\mathbf{F}$.

Theorem 17. *Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise C^1 curves. Orient the boundary ∂D so that D is on the left as one travels along ∂D.*

Let $\mathbf{F}(x, y) = (M, N)$ be a C^1 vector field defined on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F} \, dxdy.$$

Now, let's think a bit more about what Green's Theorem is saying here.

The vector line integral, $\oint_C \mathbf{F} \cdot d\mathbf{s}$, computes the global circulation of the vector field around the boundary of the region.

The double integral $\iint_D \nabla \times \mathbf{F} \, dxdy$ is computed by integrating curl over the region D. We can think of this as "adding up" the local rotation of the vector field.

Thus, we can think of Green's Theorem as saying that the global circulation of the vector field around the boundary is equal to the total local rotation across the region. If you think about it, this does make some sense!

Summary

In this activity, we connected the curl of a vector field to concepts we've covered previously. In particular, we showed that the curl of a C^1 conservative vector field is zero, and we restated Green's Theorem in terms of curl.

Part IX
Surface Integrals

Part X
Triple Integrals