

## Homework 11: Taylor's Theorem

### Graded Problems

**Problem 1** Give a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  with  $a > 0$  and  $\det(A) > 0$  for which the quadratic form  $p(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$  is **NOT** positive definite. Does this mean the theorem in problem 6 is incorrect?

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**Problem 2** Prove the following theorem: Let  $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$  be a symmetric  $2 \times 2$  matrix. If  $\det(A) < 0$ , then the quadratic form  $p(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$  is indefinite, regardless of the value of  $a$ .

(Hint: think about the cases  $a > 0$ ,  $a = 0$  and  $a < 0$ . In two cases you can apply Sylvester's Theorem. In the third case, you'll have to do some work by hand to show  $p(x, y)$  can have both positive and negative values.)

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### Professional Problem

**Problem 3** Complete the online peer review form posted on moodle.

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### Completion Packet

**Problem 4** Find the symmetric matrix that represents each quadratic form.

(a)  $r(x_1, x_2, x_3, x_4) = x_3^2 - x_2x_3 + x_1x_4$

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Learning outcomes:  
Author(s):

$$(b) \quad t(\mathbf{x}) = \mathbf{x}^T \begin{bmatrix} 6 & 1 & 8 & -2 \\ 0 & 5 & 1 & 9 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

**Problem 5** Prove the following theorem without using Sylvester's theorem:  
 Let  $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$  be a symmetric  $2 \times 2$  matrix. If  $a > 0$  and  $\det(A) > 0$ , then the quadratic form  $p(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$  is positive definite.

(Hint: Write out  $p$  in terms of the variables  $x$  and  $y$ , then complete the square with respect to  $x$  and collect the remaining terms.)

- Problem 6**
- (a) Write the Taylor series for  $f(x) = \sin(x)$  centered at  $x = 0$ .
  - (b) Find the second-order Taylor approximation for  $f(x, y) = \sin(xy)$  centered at  $(0, 0)$ , using your answer to part (a).
  - (c) Verify your answer to part (b), by computing the second-order Taylor approximation for  $f(x, y) = \sin(xy)$  directly.

- Problem 7**
- (a) Write the Taylor series for  $f(x) = e^x$  centered at  $x = 0$ .
  - (b) Find the second-order Taylor approximation for  $f(x, y) = e^{x^2+y^2}$  centered at  $(0, 0)$ , using your answer to part (a).
  - (c) Verify your answer to part (b), by computing the second-order Taylor approximation for  $f(x, y) = e^{x^2+y^2}$  directly.

**Problem 8** Compute the Hessian matrix for each function at the given point.

- (a)  $f(x, y) = \sqrt{xy}$  at  $(1, 1)$
- (b)  $f(x, y) = \cos(x) + x^2 \sin(y)$  at  $(0, \pi)$
- (c)  $f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 + 1}$  at  $(0, 0, 0)$

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**Problem 9** Consider the function  $f(x, y) = \frac{x}{x+y}$  and the point  $\mathbf{a} = (1, 2)$ .

- (a) Find the first-order Taylor polynomial of  $f$  at  $\mathbf{a}$ .
  - (b) Find the second-order Taylor polynomial of  $f$  at  $\mathbf{a}$ .
  - (c) Express the second-order Taylor polynomial using the derivative matrix and the Hessian matrix, as in formula (10) of section 4.1 of the textbook.
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**Problem 10** Consider the function  $f(x, y) = x^2 e^y$  and the point  $\mathbf{a} = (1, 0)$ .

- (a) Find the first-order Taylor polynomial of  $f$  at  $\mathbf{a}$ .
  - (b) Find the second-order Taylor polynomial of  $f$  at  $\mathbf{a}$ .
  - (c) Express the second-order Taylor polynomial using the derivative matrix and the Hessian matrix, as in formula (10) of section 4.1 of the textbook.
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