

Written Homework

Written Homework

Problem 1 Prove that, for C^1 vector fields \mathbf{F} and \mathbf{G} in \mathbb{R}^3 ,

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

(This shows that the curl is an additive operator, so the curl of a sum is the sum of the curls.)

Problem 2 (a) Compute the curl of the vector field $\mathbf{F} = (x, y, z)$. Explain why your answer makes sense geometrically.

(b) Suppose we have a C^1 vector field $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$. Compute the curl of \mathbf{F} , and explain why your answer makes sense geometrically.

Problem 3 (a) Consider the function $f(x, y, z) = e^x \sin(y) + z^2 \cos(y)$. Compute ∇f , and verify that $\nabla \times (\nabla f) = \mathbf{0}$.

(b) Prove that for any C^2 function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, the curl of the gradient of f is zero. That is, prove that

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Professional Problem

Problem 4 Consider a wheel W centered at a point on the z -axis, rotating about the z -axis. In this problem, we will investigate how the rotation of this wheel relates to the curl of a vector field describing its motion.

Learning outcomes:
Author(s):

Let P be a point on the wheel W , of distance d from the center. The rotation of the wheel can be describe by the vector $\mathbf{w} = \omega \mathbf{k}$, where ω is the angular speed of the wheel W . You may assume that ω is constant. Let $\mathbf{x}(x, y, z)$ give the position of P at time t .

- (a) Carefully explain why $\mathbf{x}'(t)$ is orthogonal to both \mathbf{w} and $\mathbf{x}(t)$. Then use the angle θ in the figure to show that $\mathbf{x}'(t) = \mathbf{w} \times \mathbf{x}(t)$. We can therefore define a “velocity field $\mathbf{F}(x, y, z) = \mathbf{w} \times (x, y, z)$, which describes the motion of the wheel W . In other words, every point on W has velocity given by $\mathbf{F}(x, y, z)$.
- (b) Show that $\mathbf{x}'(t) = (-\omega y, \omega x)$.
- (c) Show that $\nabla \times \mathbf{F} = 2\mathbf{w}$. Hence if the motion of an object is described by the velocity field \mathbf{F} , the curl vector points in the direction of the axis of (positive) rotation, and its length is proportional to the angular speed of the rotation.

ADD IMAGE

Hint: What is the linear speed of P in terms of \mathbf{x} ?

Recall that angular speed equals linear speed divided by radius. How you can write this in terms of ω , d , and \mathbf{x} ?

Be sure your solution addresses why $\mathbf{x}'(t)$ is $\mathbf{w} \times \mathbf{x}(t)$, and not $\mathbf{x}(t) \times \mathbf{w}$.