

# Lagrange Multipliers

We've seen how we can optimize a function subject to a constraint using substitution, and we've seen that it can be very difficult to correctly handle the constraints! Fortunately, there is a tool that we can use to simplify this process. This tool is called "Lagrange multipliers."

## Gradients

Suppose we wish to optimize a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  subject to some constraint  $g(\vec{x}) = C$ , where  $C$  is a constant, and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function. For now, we'll focus on the case  $n = 2$ , so we have a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  and a constraint  $g(x, y) = C$ . The graph of  $f$  will be a surface in  $\mathbb{R}^3$ , and the graph of  $g(x, y) = C$  is a curve in  $\mathbb{R}^2$ .

SOME PICTURES SOMEWHERE IN HERE, REDO THIS IN TERMS OF CRITICAL POINTS?

Now, suppose that the maximum value of  $f(x, y)$  subject to the constraint  $g(x, y) = C$  occurs at some point  $(a, b)$ . Suppose we parametrize the curve  $g(x, y) = C$  as  $\vec{x}(t)$ , with  $(a, b) = \vec{x}(t_0)$ . We can view  $g(x, y) = C$  as a level curve of the function  $g(x, y)$ , and then the gradient of  $g(x, y)$  will always be perpendicular to the curve  $g(x, y) = C$ . More precisely, for any point  $(x, y) = \vec{x}(t)$  on the curve  $g(x, y) = C$ , we'll have  $\nabla g(x, y)$  is perpendicular to  $\vec{x}'(t)$ . That is,  $\nabla g(\vec{x}(t)) \perp \vec{x}'(t)$  for all  $t$ .

Next, let's turn our attention back to  $f$ . If  $f$  has an absolute maximum subject to  $g(x, y) = C$ , then  $f(\vec{x}(t))$  has an absolute maximum at  $t = t_0$ . This means that  $f(\vec{x}(t))$  has a critical point at  $t = t_0$ , so  $\frac{d}{dt}f(\vec{x}(t))|_{t=t_0} = 0$ . Using the chain rule, we can rewrite this as

$$\nabla f(\vec{x}(t_0)) \cdot \vec{x}'(t) = 0.$$

So, both  $\nabla f(a, b)$  and  $\nabla g(a, b)$  are perpendicular to  $\vec{x}'(t)$ . Since we are considering vectors in  $\mathbb{R}^2$ , this means that  $\nabla f(a, b)$  and  $\nabla g(a, b)$  are parallel, so we can write

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

for some constant  $\lambda$ .

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Learning outcomes:  
Author(s):

So, we can find candidate points for the absolute maximum (and similarly, the absolute minimum) of  $f(x, y)$  subject to the constraint  $g(x, y) = C$  by finding points where

$$\nabla f(a, b) = \lambda \nabla g(a, b).$$

This observation generalizes to  $\mathbb{R}^n$ .

**Proposition 1.** Consider  $\mathcal{C}^1$  functions  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ , and let  $C$  be a constant. If  $f$  has an absolute maximum or absolute minimum at  $\vec{a}$  subject to the constraint  $g(\vec{x}) = C$ , then there exists some scalar  $\lambda$  such that

$$\nabla f(\vec{a}) = \lambda \nabla g(\vec{a}).$$

The constant  $\lambda$  is called a Lagrange Multiplier.

We can leverage this theorem into a method for finding absolute extrema of a function subject to a constraint, which we call the *method of Lagrange multipliers*.

To find the absolute extrema of a function  $f(\vec{x})$  subject to a constraint  $g(\vec{x}) = C$ :

- (a) Compute the gradients  $\nabla f(\vec{x})$  and  $\nabla g(\vec{x})$ .
- (b) Solve the system of equations

$$\begin{cases} \nabla f(\vec{x}) = \lambda \nabla g(\vec{x}) \\ g(\vec{x}) = C \end{cases}$$

for  $\vec{x}$  and  $\lambda$ .

- (c) The solutions to the system of equations in (2) are the critical points of  $f(\vec{x})$  subject to  $g(\vec{x}) = C$ . Classify these critical points in order to determine the absolute extrema.

If  $g(\vec{x}) = C$  is compact, we can determine the absolute extrema by comparing the values of  $f$  at the critical points. In this case, absolute extrema are guaranteed to exist by the Extreme Value Theorem.

We'll see how this process works in a couple of examples. First, we repeating an optimization problem which was previously done with substitution.

**Example 1.** We'll find the absolute maximum and absolute minimum of  $f(x, y) = x^2 - y^2$  subject to the constraint  $x^2 + y^2 = 1$ . Notice that  $x^2 + y^2 = 1$  is a compact region, so absolute extrema will exist.

If we let  $g(x, y) = x^2 + y^2$ , our constraint is  $g(x, y) = 1$ .

Computing the gradients of  $f$  and  $g$ , we have

$$\begin{aligned} \nabla f(x, y) &= \boxed{(2x, -2y)}, \\ \nabla g(x, y) &= \boxed{(2x, 2y)}. \end{aligned}$$

Setting up the system of equations

$$\begin{cases} \nabla f(\vec{x}) = \lambda \nabla g(\vec{x}) \\ g(\vec{x}) = 1 \end{cases},$$

we have

$$\begin{aligned} \nabla(2x, -2y) &= \lambda(2x, 2y) \\ x^2 + y^2 &= 1 \end{aligned}$$

We can rewrite this as a system of three equations,

$$\begin{cases} 2x = 2\lambda x \\ 2y = -2\lambda y \\ x^2 + y^2 = 1 \end{cases}.$$

From the first equation, we have either  $x = 0$  or  $\lambda = 1$ .

If  $x = 0$ , the third equation gives us  $y = \pm 1$ . Thus, we obtain the critical points  $(0, \pm 1)$ .

If  $\lambda = 1$ , the second equation gives us  $y = 0$ . Then, the third equation gives us  $x = \pm 1$ . This gives us the critical points  $(\pm 1, 0)$ .

Since we know that  $f$  will have an absolute maximum and minimum subject to our constraint, we will compare the values at the critical points to determine the absolute maximum and minimum.

$$\begin{aligned} f(0, -1) &= -1 \\ f(0, 1) &= -1 \\ f(-1, 0) &= 1 \\ f(1, 0) &= 1 \end{aligned}$$

We see that the absolute maximum of  $f$  subject to our constraint is 1, and this occurs at the points  $(\pm 1, 0)$ . The absolute minimum of  $f$  subject to our constraint is  $-1$ , and this occurs at the points  $(0, \pm 1)$ .

**Example 2.** Next, we'll attempt to optimize  $f(x, y) = 12 - x^2 - y^2$  subject to the constraint  $y^2 = x + 5$ . Here, we'll need to be a bit more careful, since the curve defined by  $y^2 = x + 5$  is not compact, as it is unbounded.

PICTURE

However, Lagrange multipliers will still be helpful for finding critical points. We can rewrite our constraint as  $y^2 - x = 5$ , and take  $g(x, y) = y^2 - x$ . So, we are optimizing  $f(x, y)$  subject to the constraint  $g(x, y) = 5$ .

## Lagrange Multipliers

We begin by finding the gradients of  $f$  and  $g$ .

$$\nabla f(x, y) = \boxed{(-2x, -2y)}$$

$$\nabla g(x, y) = \boxed{(-1, 2y)}$$

Next, we solve the system

$$\begin{cases} \nabla f(\vec{x}) = \lambda \nabla g(\vec{x}) \\ g(\vec{x}) = 5 \end{cases},$$

which is

$$\begin{cases} (-2x, -2y) = \lambda(-1, 2y) \\ y^2 - x = 5 \end{cases}.$$

We can rewrite this system as the three equations

$$\begin{cases} -2x = -\lambda \\ -2y = 2\lambda y \\ y^2 - x = 5 \end{cases}.$$

From the second equation, we have either  $y = 0$ , or  $\lambda = -1$ .

If  $y = 0$ , the third equation gives us  $x = -5$ . So, we have a critical point  $(-5, 0)$ .

If  $\lambda = -1$ , the first equation gives us  $x = \frac{1}{2}$ . Then the third equation gives us  $y = \pm \frac{3}{\sqrt{2}}$ . So, we have the critical points  $\left(-\frac{1}{2}, \pm \frac{3}{\sqrt{2}}\right)$ .

In this case, we can't determine the absolute maximum and absolute minimum by plugging in these points, since we aren't optimizing over a compact region.

However, looking at the graph of  $f$  over the curve  $g(x, y) = 5$ , we can see that the absolute maximum occurs at the points  $\left(-\frac{1}{2}, \pm \frac{3}{\sqrt{2}}\right)$ . Although there is a local minimum at  $(-5, 0)$ , this is not an absolute minimum, as there is no absolute minimum.

GRAPH