Multivariable Calculus

July 5, 2018

Contents

Part I

Week 9: Curl

Curl of a Vector Field

Imagine the vector field below represents fluid flow:

Desmos link: `https://www.desmos.com/calculator/vhuoyka1ys`

If we fix the center point of each + above, which way will they rotate? (clockwise/counter-clockwise)

We can describe this concept as microscopic rotation or local rotation, and it turns out that the *curl* of a vector field measures this local rotation.

In this activity, we define curl and focus on computation. In the next activity, we discuss the geometric significance of curl and how it represents local rotation.

Definition of Curl

A curl is an example of an *operator*, which is a mathematical object you've seen before. Roughly speaking, it's a "function" on functions. That is, it takes a function as an input, and produces a function as an output. Here, we're using "function" very broadly - a function could be scalar-valued, a path, or even a vector field!

To prove that you've seen operators before, let's look at a specific example:

Problem 1 *What does* $\dfrac{d}{dt}g(t)$ *mean?*

Multiple Choice:

(a) *Multiply* $g(t)$ *by the fraction* $\dfrac{d}{dt}$.

(b) *Take the derivative of* g *with respect to* t. ✓

Problem 2 *What does* $\dfrac{d}{dt}$ *mean?*

Multiple Choice:

Learning outcomes:
Author(s):

(a) *The same thing as* $\frac{1}{t}$.

(b) *Take the derivative with respect to* t. ✓

Problem 3 *It turns out that* $\frac{d}{dt}$ *is an example of an* operator.

To introduce the curl, we need to talk about another operator, ∇ *which we call the* del operator.

What does $\nabla(g(x, y, z))$ *mean?*

Multiple Choice:

(a) *The change in* g.

(b) $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$ ✓

Problem 4 *From this, we can deduce that* ∇ *should mean* $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. *Note that this is an operator.*

Definition 1. *The* del operator *in* $\mathbb{R}^n$ *is* $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, ..., \frac{\partial}{\partial x_n}\right)$.

There's one more ingredient that we need to review in order to define the curl of a vector field, the cross product.

Problem 5 *If* $\mathbf{v} = (1, 2, 3)$ *and* $\mathbf{w} = (4, 5, 6)$, *what is* $\mathbf{v} \times \mathbf{w}$? $\boxed{(-3, 6, -3)}$.

Problem 6 *Note that this is computed as the determinant*

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

Problem 7 *Given a vector field* $\mathbf{F} = (M(x,y,z), N(x,y,z), P(x,y,z))$*, how might we interpret* $\nabla \times \mathbf{F}$*?*

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \boxed{\dfrac{\partial}{\partial x}} & \boxed{\dfrac{\partial}{\partial y}} & \boxed{\dfrac{\partial}{\partial z}} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\end{aligned}$$

Based on this, we give our definition for the curl of a three-dimensional vector field:

Definition 2. *The* curl *of a three-dimensional vector field* $\mathbf{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$ *is*

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \boxed{M} & \boxed{N} & \boxed{P} \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\end{aligned}$$

Note that this input is a vector field $\mathbf{F}$ in $\mathbb{R}^3$, and the output is another vector field in $\mathbf{R}^3$.

Problem 8 *Let* $\mathbf{F} = (e^y, xz, 3z)$*. Compute the curl* $\nabla \times \mathbf{F}$*.*

$$\nabla \times \mathbf{F} = \boxed{(-x, 0, z - e^y)}$$

Note that we have only defined the curl for three-dimensional vector fields. However, by being a bit clever, we can extend this definition to two-dimensional vector fields.

Definition 3. *If the three-dimensional vector field* $\mathbf{F}$ *has the form* $\mathbf{F}(x,y,z) = (M(x,y), N(x,y), 0)$*, then* $\nabla \times \mathbf{F}$ *is often called the* two-dimensional curl *of* $\mathbf{F}$*.*

Moreover, if $\mathbf{G}(x,y) = (M(x,y), N(x,y))$ *is a vector field in* $\mathbb{R}^2$*, then we define the* curl *of* $\mathbf{G}$ *as the curl of the three-dimensional vector field* $\widetilde{\mathbf{G}}(x,y,z) = (M(x,y), N(x,y), 0)$*.*

It turns out, the curl of a two-dimensional vector field can be written in a simpler form.

Proposition 1. *The two-dimensional curl of* $\mathbf{F}(x, y) = (M(x, y), N(x, y), 0)$ *is*

$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Proof From the definition of the curl, we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Since the third component, P, of our vector field is identically 0, we have

$$\nabla \times \mathbf{F} = \left(\boxed{0} - \frac{\partial N}{\partial z}, -\left(\boxed{0} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Both $M(x, y)$ and $N(x, y)$ are constant with respect to z, so we then have

$$\nabla \times \mathbf{F} = \left(\boxed{0}, \boxed{0}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right),$$

as desired. ■

Sometimes we refer to $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ as the curl $\nabla \times \mathbf{F}$ if $\mathbf{F}$ is two-dimensional, instead of writing out the entire vector.

Note that we've only defined the curl of a vector field for two- and three-dimensional vector fields. Why doesn't it make sense to define the curl of a four-dimensional (or higher!) vector field?

Multiple Choice:

(a) We only exist in three dimensions.

(b) The cross product is only defined in $\mathbf{R}^3$. ✓

Problem 9 *Given* $\mathbf{F}(x, y) = (y, 0)$*, compute the curl* $\nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, -1)}$$

Problem 10 *Given* $\mathbf{F}(x, y) = (-y, 0)$*, compute the curl* $\nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 1)}$$

Let's look at this example, $\mathbf{F}(x,y) = (-y, 0)$. It turns out that this is the vector field from the beginning of this activity:

We imagined that the center of the plus signs were fixed, and determined that the vector field would rotate the plus signs counterclockwise. We claimed that this local rotation had something to do with the curl of the vector field, which we computed to be $\nabla \times \mathbf{F} = (0, 0, 1)$.

In the next activity, we'll study the geometric significance of the curl, and why the curl measures this "microscopic" rotation.

Summary

In this section, we defined the curl of a two- or three-dimensional vector field, which can be computed as follows:

- For a three-dimensional vector field, $\mathbf{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$, we have $\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$
- The two-dimensional curl of $\mathbf{F}(x,y) = (M(x,y), N(x,y), 0)$ can be computed as
$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

In the next activity, we will discuss the geometric significance of the curl, and how it relates to the local rotation of the vector field.

Geometric Significance of Curl

Consider the vector field $\mathbf{F}(x, y) = (-y, 0)$.

We can compute the curl of this vector field,

$$\nabla \times \mathbf{F} = (0, 0, 1)$$

Imagine that we fix a point (representing a particle) in this vector field, but allow it to rotate. If imagine the vector field acting as a force on this particle, which way will it cause the particle to rotate?

(VECTOR FIELD)

Here, we see that the vector field is applying a greater force to the "top" of the particle than to the "bottom." this will cause the particle to rotate counterclockwise. We describe this type of rotation as *local rotation* or *microscopic rotation*, since it's the rotation when we "zoom in" on the particle.

It turns out that the curl of a vector field provides a measure of this local rotation - but how are these connected? We will answer this question in this section, discussing the geometric significance of the curl.

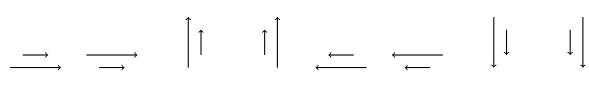
Geometric Significance of Two-dimensional Curl

Recall that, for a two-dimensional vector field $\mathbf{F}(x, y) = (M, N)$, we can compute the curl as

$$\nabla \times \mathbf{F} = (0, 0, N_x - M_y)$$

where N_x is the partial derivative of N with respect to x, and M_y is the partial derivative of M with respect to y. We'll start by considering how M_y and N_x contribute to local rotation.

First let's consider the case where $M_y < 0$. In this case, the x-component of the vector field $\mathbf{F}$ is *decreasing* as we move in the positive y direction. Select all pictures which match this situation.



Select All Correct Answers:

Learning outcomes:
Author(s):

(a) (a) ✓

(b) (b)

(c) (c)

(d) (d)

(e) (e)

(f) (f) ✓

(g) (g)

(h) (h)

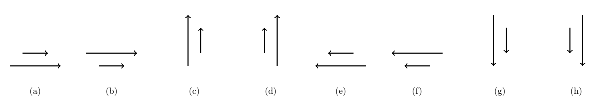
If $M_y < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

Now, let's consider the case where $N_x > 0$. This means that the y-component of the vector field $\mathbf{F}$ is increasing as we move in the positive x direction. Select all pictures which match this situation.



Select All Correct Answers:

(a) (a)

(b) (b)

(c) (c)

(d) (d) ✓

(e) (e)

(f) (f)

(g) (g) ✓

(h) (h)

If $N_x < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

We've seen that the signs of N_x and M_y correspond to the direction of local rotation, with $N_x > 0$ and $M_y < 0$ contributing to counterclockwise rotation.

In general, we have that the sign of $N_x - M_y$ corresponds to the direction of local rotation in the plane. In particular, we have the following correspondences:

$$\begin{aligned}
N_x - M_y > 0 &\longleftrightarrow \text{counterclockwise local rotation} \\
N_x - M_y < 0 &\longleftrightarrow \text{clockwise local rotation} \\
N_x - M_y = 0 &\longleftrightarrow \text{no local rotation}
\end{aligned}$$

Remembering that $N_x - M_y$ is the *curl* of the two-dimensional vector field $\mathbf{F}$, we now have that the sign of the curl tells us the direction of local rotation for two-dimensional vector fields.

We have a special term for a vector field that never has any local rotation: we call such a vector field *irrotational*.

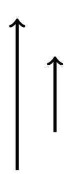
Furthermore, the length of the curl,

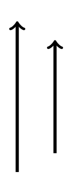
$$\|(0, 0, N_x - M_y)\| = |N_x - M_y|,$$

corresponds to the speed of rotation.

For example, in which case will the particle spin faster?

Multiple Choice:

(a)  ✓

(b) 

Note that this corresponds to a larger value of N_x (the change in the y-component of $\mathbf{F}$ as we move in the positive x direction).

We now apply our knowledge of the geometric significance of the curl in a couple of examples.

Example 1. *Consider the vector field* $\mathbf{F}(x, y) = (-y, x^2)$*. Compute the curl of* $\mathbf{F}$*, and describe the local rotation of the vector field at the points* $(1, 0)$ *and* $(-4, 1)$*.*

Explanation. *We begin by computing the curl of* $\mathbf{F}$*,*

$$\nabla \times \mathbf{F} = (0, 0, \boxed{2x + 1}).$$

At the point $(1, 0)$*, we have* $(\nabla \times \mathbf{F})(1, 0) = (0, 0, \boxed{3})$*. Looking at the third component, we see that the sign of* $N_x - M_y$ *at* $(1, 0)$ *is* (positive/ negative/ zero)*. Thus, the local rotation of the vector field at the point* $(1, 0)$ *is* (clockwise/ counterclockwise/ no rotation)*.*

At the point $(-4, 1)$*, we have* $(\nabla \times \mathbf{F})(-4, 1) = (0, 0, \boxed{-7})$*. Looking at the third component, we see that the sign of* $N_x - M_y$ *at* $(-4, 1)$ *is* (positive/ negative/ zero)*. Thus, the local rotation of the vector field at the point* $(-4, 1)$ *is* (clockwise/ counterclockwise/ no rotation)*.*

Looking at a graph of the vector field, we can see that this local rotation is reflected in the graph.

(ADD GRAPH, WITH ROTATION?)

Example 2. *Let* $\mathbf{F}(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$*. Compute the curl* $\nabla \times \mathbf{F}$*, and interpret it geometrically.*

Explanation. *Computing our partial derivatives, we have*

$$N_x = \boxed{\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2}}$$

and

$$M_y = \boxed{\frac{-x^2 - y^2 + 2x^2}{(x^2 + y^2)^2}}.$$

Then, the curl of $\mathbf{F}$ *is*

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0}).$$

Thus, we see that there is no local rotation at any point in the vector field. This is particularly interesting once we look at a graph of the vector field.

(GRAPH)

From the graph of the vector field, there certainly seems to be some larger scale, global rotation of the vector field. However, our computation showed that there *is no local rotation. This example illustrates an important distinction: curl measures* local *rotation of a vector field, which is a different concept from* global *rotation.*

In this section, we saw how the curl of a vector field corresponded to local rotation for a two-dimensional vector field. In the next section, we describe how the curl of a vector field corresponds to local rotation for *three*-dimensional vector fields.

Geometric Significance of Three-dimensional Curl

For a three dimensional vector field $\mathbf{F}(x, y, z) = (M, N, P)$, we can compute the curl of $\mathbf{F}$ as

$$\nabla \times \mathbf{F} = (P_y - N_z, \boxed{-P_x + M_z}, \boxed{N_x - M_y}).$$

Here, the situation is more complicated than in two dimensions. In the plane, there are only two possible ways to rotate: clockwise and counterclockwise. In $\mathbb{R}^3$, there are infinitely many different ways to rotate, since we have infinitely many choices of axes. Yikes!

Fortunately, three-dimensional curl still tells about local rotation. In this case, we imagine local rotation as rotation of an infinitesimal (tiny) sphere. This sphere can rotate in infinitely many different ways, depending on which axis we rotate around.

When we look at the components of the curl, this tells us about rotation perpendicular to each of the axes, ignoring rotation in any other direction. Specifically,

$$\begin{aligned}
N_x - M_y \text{ (the } z\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } z\text{-axis)}\\
P_y - N_z \text{ (the } x\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } x\text{-axis)}\\
-P_x + M_z \text{ (the } y\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } y\text{-axis)}
\end{aligned}$$

Once again, the sign tells us the direction of rotation, with positive sign corresponding to counterclockwise rotation (viewed from the positive axes).

Furthermore, the length of the curl, $\|\nabla \times \mathbf{F}\|$, tells us the speed of rotation, and the direction of $\nabla \times \mathbf{F}$ tells us the axis of rotation.

In $\mathbb{R}^3$, we would like to be able to describe the direction of rotation around a given axis. However, this can be tricky, since it's a matter of perspective. Imagine rotation in the xy-plane. If the rotation is clockwise viewed from above, then it will be counterclockwise from below! Fortunately, curl follows the *right hand rule*:

> If you point your right thumb in the direct of $\nabla \times \mathbf{F}$, then your fingers will curl in the direction of local rotation.

We now put this to use in an example.

Problem 1 *Consider the vector field* $\mathbf{F}(x,y,z) = \left(0, \dfrac{-z}{(y^2+z^2)^{3/2}}, \dfrac{y}{(y^2+z^2)^{3/2}}\right)$. *Compute the curl of* $\mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{\left(\frac{-1}{(y^2+z^2)^{3/2}}, 0, 0\right)}$$

Problem 2 *What is the axis of local rotation (at any point)?*

Multiple Choice:

(a) *The x-axis.* ✓

(b) *The y-axis.*

(c) *The z-axis.*

(d) *Some other line.*

Problem 3 *Viewed from the positive x-axis, what is the direction of local rotation (at any point)?*

Multiple Choice:

(a) *Clockwise.* ✓

(b) *Counterclockwise.*

Problem 4 *How does the speed of local rotation change as we move closer to the origin?*

Multiple Choice:

(a) *Stays the same.*

(b) *Gets slower.*

(c) *Gets faster.* ✓

We've now seen how the curl describes the local rotation of a three-dimensional vector field. In the next section, we'll cover some connections of the curl to previous topics.

Summary

In this section, we studied the geometric significance of the curl. We found that the curl gives a measure of the local rotation of a vector field.

For a two-dimensional vector field, the sign of the curl told us the direction of rotation. Specifically, we have the following correspondence.

$$\begin{aligned} N_x - M_y > 0 &\longleftrightarrow \text{counterclockwise local rotation} \\ N_x - M_y < 0 &\longleftrightarrow \text{clockwise local rotation} \\ N_x - M_y = 0 &\longleftrightarrow \text{no local rotation} \end{aligned}$$

The magnitude $|N_x - M_y|$ corresponds to speed of rotation.

For a three-dimensional vector field, the components of the curl tell us about local rotation perpendicular to the axes. We also have:

- The length of the curl, $\|\nabla \times \mathbf{F}\|$, corresponds to the speed of rotation.
- The direction of the curl vector $\nabla \times \mathbf{F}$ gives the axis of rotation.
- Curl follows the right hand rule: if you point your thumb in the direction of $\nabla \times \mathbf{F}$, your fingers curl in the direction of local rotation.

In the next section, we'll consider the curl of a conservative vector field, and how the curl connects to Green's Theorem.

Connections of Curl with Older Material

We've defined the curl of a two or three dimensional vector field, and we found that this gives a measure of the local rotation of a vector field.

In this section, we discuss connections of the curl to previous topics from the course. In particular, we find the curl of a conservative vector field, and we restate Green's Theorem in terms of curl.

Curl of a Conservative Vector Field

In this section, we prove that the curl of a conservative vector field will always be zero. Thus, conservative vector fields are irrotational.

Theorem 1. *Suppose* $\mathbf{F}$ *is a* C^1 *conservative vector field in* $\mathbb{R}^3$*, so there is a function* $f : \mathbb{R}^3 \to \mathbb{R}$ *such that* $\mathbf{F} = \nabla f$*. Then* $\nabla \times \mathbf{F} = \mathbf{0}$*.*

Proof Suppose $\mathbf{F}(x, y, z) = (M, N, P)$ is a C^1 conservative vector field, with $\mathbf{F} = \nabla f$. Then we must have

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \boxed{(M, N, P)}.$$

Computing the curl of $\mathbf{F}$, we have

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, -\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right),\end{aligned}$$

substituting in for the components M, N, and P.

Now, we will use Clairaut's Theorem to simplify this vector. Since $\mathbf{F} = (M, N, P)$ is a C^1 vector field, the partial derivatives of its components ($\frac{\partial M}{\partial y}$, $\frac{\partial M}{\partial z}$, etc.) exist and are continuous. This means that all second-order partial derivatives of f exist and are continuous. Then, by Clairaut's Theorem, the order of differentiation for the second-order mixed partials doesn't matter. In particular, we

Learning outcomes:
Author(s):

have

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} &= \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial z \partial y} &= \frac{\partial^2 f}{\partial y \partial z}.\end{aligned}$$

Using this fact in our computation of the curl, we now have

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, -\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) \\ &= \boxed{(0,0,0)}\end{aligned}$$

Thus, we have shown that the curl of a conservative vector field is zero. ■

So, conservative vector fields are irrotational. A reasonable follow-up question would be: if the curl of a vector field is zero, is the vector field necessarily conservative? We'll leave this as an open question for the reader, with the suggestion that you think about how you can use past results, and what hypotheses are necessary for this converse to be true.

Curl and Green's Theorem

In this section, we see that we've actually already seen the curl of a vector field. It turns out that the curl showed up in Green's Theorem, we just didn't know that it was the curl yet.

Recall the statement of Green's Theorem:

Theorem 2. *Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise C^1 curves. Orient the boundary ∂D so that D is on the left as one travels along ∂D.*

Let $\mathbf{F}(x,y) = (M,N)$ be a C^1 vector field defined on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

The integrand of the double integral, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, should now look familiar. This mysterious quantity is actually the two-dimensional curl of the vector field $\mathbf{F}$!

Using this realization, we can now restate Green's Theorem in terms of the curl of $\mathbf{F}$.

Theorem 3. *Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise C^1 curves. Orient the boundary ∂D so that D is on the left as one travels along ∂D.*

Let $\mathbf{F}(x, y) = (M, N)$ be a C^1 vector field defined on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F}\, dxdy.$$

Now, let's think a bit more about what Green's Theorem is saying here.

The vector line integral, $\oint_C \mathbf{F} \cdot d\mathbf{s}$, computes the global circulation of the vector field around the boundary of the region.

The double integral $\iint_D \nabla \times \mathbf{F}\, dxdy$ is computed by integrating curl over the region D. We can think of this as "adding up" the local rotation of the vector field.

Thus, we can think of Green's Theorem as saying that the global circulation of the vector field around the boundary is equal to the total local rotation across the region. If you think about it, this does make some sense!

Summary

In this activity, we connected the curl of a vector field to concepts we've covered previously. In particular, we showed that the curl of a C^1 conservative vector field is zero, and we restated Green's Theorem in terms of curl.

Online Homework

Online Problems

Problem 1 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (-2y\cos(3x), 3x\sin(-2y), 0)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 2\cos(3x) - 3\sin(2y)}$$

Find the curl of $\mathbf{F}$ *at the point* $(x, y, z) = (\pi, \pi, \pi)$.

$$(\nabla \times \mathbf{F})(\pi, \pi, \pi) = \boxed{(0, 0, -2}$$

Is $\mathbf{F}$ *a conservative vector field?*

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

(c) *Not enough information.*

Justify your answer.

Free Response:

Problem 2 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (yz, 2xz, 3xy)$.

$$\nabla \times \mathbf{F} = \boxed{(x, -2y, z)}$$

Find the curl of $\mathbf{F}$ *at the point* $(x, y, z) = (0, 0, 0)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}$$

Is $\mathbf{F}$ *irrotational?*

Multiple Choice:

Learning outcomes:
Author(s):

(a) *Yes.*

(b) *No.* ✓

(c) *Not enough information.*

Problem 3 *Compute the curl of the vector field* $\mathbf{F}(x,y,z) = (x^2, y^3, z^4)$.

$$\nabla \times \mathbf{F} = \boxed{(0,0,0)}.$$

Find the curl of $\mathbf{F}$ *at the point* $(x,y,z) = (1,2,3)$.

$$(\nabla \times \mathbf{F})(1,2,3) = \boxed{(0,0,0)}.$$

Is $\mathbf{F}$ *irrotational?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

(c) *Not enough information.*

Problem 4 *Compute the two-dimensional curl of the vector field* $\mathbf{F}(x,y) = (-xy, xy)$.

$$\nabla \times \mathbf{F} = \left(0, 0, \boxed{x+y}\right)$$

Describe the local rotation of $\mathbf{F}$ *at the point* $(1,1)$.

Multiple Choice:

(a) *Counterclockwise.* ✓

(b) *Clockwise.*

(c) *No rotation.*

Describe the local rotation of $\mathbf{F}$ *at the point* $(-1,1)$.

Multiple Choice:

(a) *Counterclockwise.*

(b) *Clockwise.*

(c) *No rotation.* ✓

Describe the local rotation of **F** *at the point* $(-1, -1)$.

Multiple Choice:

(a) *Counterclockwise.*

(b) *Clockwise.* ✓

(c) *No rotation.*

Problem 5 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (2x - y, -x + 4y)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is **F** *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 6 *Find a potential function* f *for* **F**, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y) = \boxed{x^2 - xy + 2y^2}$$

Problem 7 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (2y, 3x)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{1})$$

Is **F** *conservative?*

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

Problem 8 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (2x, 3y)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 9 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y) = \boxed{x^2 + \frac{3}{2}y^2}$$

Problem 10 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (-4x + y\cos(x), \sin(x))$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 11 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y) = \boxed{-2x^2 + y\sin(x)}$$

Problem 12 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (\sin(x), y^2, e^z)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 13 *Find a potential function f for $\mathbf{F}$, so that $\nabla f = \mathbf{F}$.*

$$f(x, y, z) = \boxed{-\cos(x) + \frac{1}{2}y^2 + e^z}$$

Written Homework

Written Problems

Problem 1 *Prove that, for C^1 vector fields $\mathbf{F}$ and $\mathbf{G}$ in $\mathbf{R}^3$,*

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

(This shows that the curl is an additive operator, *so the curl of a sum is the sum of the curls.)*

Problem 2 (a) *Compute the curl of the vector field $\mathbf{F} = (x, y, z)$. Explain why your answer makes sense geometrically.*

(b) *Suppose we have a C^1 vector field $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$. Compute the curl of $\mathbf{F}$, and explain why your answer makes sense geometrically.*

Problem 3 (a) *Consider the function $f(x, y, z) = e^x \sin(y) + z^2 \cos(y)$. Compute ∇f, and verify that $\nabla \times (\nabla f) = \mathbf{0}$.*

(b) *Prove that for any C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$, the curl of the gradient of f is zero. That is, prove that*

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Professional Problem

Problem 4 *Consider a wheel W centered at a point on the z-axis, rotating about the z-axis. In this problem, we will investigate how the rotation of this wheel relates to the curl of a vector field describing its motion.*

Learning outcomes:
Author(s):

Let P be a point on the wheel W, of distance d from the center. The rotation of the wheel can be describe by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of the wheel W. You may assume that ω is constant. Let $\mathbf{x}(x, y, z)$ give the position of P at time t.

(a) *Carefully explain why $\mathbf{x}'(t)$ is orthogonal to both $\mathbf{w}$ and $\mathbf{x}(t)$. Then use the angle θ in the figure to show that $\mathbf{x}'(t) = \mathbf{w} \times \mathbf{x}(t)$. We can therefore define a "velocity field $\mathbf{F}(x, y, z) = \mathbf{w} \times (x, y, z)$, which describes the motion of the wheel W. In other words, every point on W has velocity given by $\mathbf{F}(x, y, z)$.*

(b) *Show that $\mathbf{x}'(t) = (-\omega y, \omega x)$.*

(c) *Show that $\nabla \times \mathbf{F} = 2\mathbf{w}$. Hence if the motion of an object is described by the velocity field $\mathbf{F}$, the curl vector points in the direction of the axis of (positive) rotation, and its length is proportional to the angular speed of the rotation.*

ADD IMAGE

Hint: *What is the linear speed of P in terms of $\mathbf{x}$?*

Recall that angular speed equals linear speed divided by radius. How you can write this in terms of ω, d, and $\mathbf{x}$?

Be sure your solution addresses why $\mathbf{x}'(t)$ is $\mathbf{w} \times \mathbf{x}(t)$, and not $\mathbf{x}(t) \times \mathbf{w}$.

Extra Problems

Online Problems

Problem 1 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (x, y, z)$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

Problem 2 *Compute the 2-dimensional curl of the vector field* $\mathbf{F}(x, y) = (xy^2, y\sin(x^2))$. *Your answer should be given as a vector.*

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 2xy(\cos(x^2) - 1))}$$

Problem 3 *Compute the 2-dimensional curl of the vector field* $\mathbf{F}(x, y) = (e^{xy}, \cos(y))$. *Your answer should be given as a vector.*

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, -xe^{xy})}$$

Problem 4 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (y, z, x)$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(-1, -1, -1)}$$

Problem 5 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (yz, xz, xy)$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

Learning outcomes:
Author(s):

Problem 6 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (\sin(x), \cos(y), xy)$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(x, -y, 0)}$$

Problem 7 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (0, x^3, \sin(x + y)$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(\cos(x + y), -\cos(x + y), 3x^2)}$$

Problem 8 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (e^{xy}, \cos(z), e^y)$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(e^y + \sin(z), 0, -xe^{xy})}$$

Problem 9 *Compute the 2-dimensional curl of the vector field* $\mathbf{F}(x, y) = (y, x)$. *Your answer should be given as a vector.*

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

Problem 10 *Compute the 2-dimensional curl of the vector field* $\mathbf{F}(x, y) = (y, -x)$. *Your answer should be given as a vector.*

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, -2)}$$

Problem 11 *Compute the 2-dimensional curl of the vector field* $\mathbf{F}(x, y) = (x^2y + 3xy, xy^4 - 2x^2y)$. *Your answer should be given as a vector.*

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, -x^2 - 3x - 4xy + y^4)}$$

Problem 12 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (yz, 2xz, 3xy)$.

$$\nabla \times \mathbf{F} = \boxed{(x, -2y, z)}$$

Find the curl of $\mathbf{F}$ *at the point* $(x, y, z) = (0, 0, 0)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}$$

Is $\mathbf{F}$ *irrotational?*

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

(c) *Not enough information.*

Problem 13 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (x^2, y^3, z^4)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}.$$

Find the curl of $\mathbf{F}$ *at the point* $(x, y, z) = (1, 2, 3)$.

$$(\nabla \times \mathbf{F})(1, 2, 3) = \boxed{(0, 0, 0)}.$$

Is $\mathbf{F}$ *irrotational?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

(c) *Not enough information.*

Problem 14 *Compute the two-dimensional curl of the vector field* $\mathbf{F}(x, y) = (-xy, xy)$.

$$\nabla \times \mathbf{F} = \left(0, 0, \boxed{x + y}\right)$$

Describe the local rotation of $\mathbf{F}$ *at the point* $(1, 1)$.

Multiple Choice:

(a) *Counterclockwise.* ✓

(b) *Clockwise.*

(c) *No rotation.*

Describe the local rotation of $\mathbf{F}$ *at the point* $(-1, 1)$.

Multiple Choice:

(a) *Counterclockwise.*

(b) *Clockwise.*

(c) *No rotation.* ✓

Describe the local rotation of $\mathbf{F}$ *at the point* $(-1, -1)$.

Multiple Choice:

(a) *Counterclockwise.*

(b) *Clockwise.* ✓

(c) *No rotation.*

Problem 15 *Find the 2-dimensional curl of the vector field* $\mathbf{F}(x, y) = (-y, x)$. *Your answer should be given as a vector.*

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 2)}$$

Is this vector field irrotational?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Describe the direction of the local rotation of this vector field.

Multiple Choice:

(a) *Clockwise*

(b) *Counterclockwise* ✓

(c) *No rotation*

Plot this vector field. INCLUDE GRAPHING STUFF

Find the 2-dimensional curl of the vector field $\mathbf{G}(x, y) = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{-y}{x^2 + y^2} \right)$. *Your answer should be given as a vector.*

$$\nabla \times \mathbf{G} = \boxed{(0, 0, 0)}$$

Is this vector field irrotational?

Multiple Choice:

(a) *Yes* ✓

(b) *No*

Describe the direction of the local rotation of this vector field.

Multiple Choice:

(a) *Clockwise*

(b) *Counterclockwise*

(c) *No rotation* ✓

Plot this vector field. INCLUDE GRAPHING STUFF

Although the vector field $\mathbf{F}$ *and* $\mathbf{G}$ *have the same flow lines, we see that one is irrotational and the other is not. Why does this happen?*

Multiple Choice:

(a) *Math is broken.*

(b) *Curl describes local rotation, not global rotation.* ✓

Problem 16 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (-2y\cos(3x), 3x\sin(-2y), 0)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 2\cos(3x) - 3\sin(2y))}$$

Find the curl of $\mathbf{F}$ *at the point* $(x, y, z) = (\pi, \pi, \pi)$.

$$(\nabla \times \mathbf{F})(\pi, \pi, \pi) = \boxed{(0, 0, -2}$$

Is $\mathbf{F}$ *a conservative vector field?*

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

(c) *Not enough information.*

Justify your answer.

Free Response:

Problem 17 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (ze^{xz} + z\sin(x), xe^{xy}, -\cos(x))$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, x^2 e^{xz}, xye^{xy} + e^{xy})}$$

Is this vector field conservative?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Problem 18 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (2x - y, -x + 4y)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 19 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y) = \boxed{x^2 - xy + 2y^2}$$

Problem 20 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (2y, 3x)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{1})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.*

(b) *No.* ✓

Problem 21 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (2x, 3y)$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 22 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y) = \boxed{x^2 + \frac{3}{2}y^2}$$

Problem 23 *Compute the curl of the vector field* $\mathbf{F}(x, y) = (-4x + y\cos(x), \sin(x))$.

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0})$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 24 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y) = \boxed{-2x^2 + y\sin(x)}$$

Problem 25 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (\sin(x), y^2, e^z)$.

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 0)}$$

Is $\mathbf{F}$ *conservative?*

Multiple Choice:

(a) *Yes.* ✓

(b) *No.*

Problem 26 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y, z) = \boxed{-\cos(x) + \frac{1}{2}y^2 + e^z}$$

Problem 27 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (x, y, z)$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

Is this vector field conservative?

Multiple Choice:

(a) *Yes* ✓

(b) *No*

Problem 28 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y, z) = \boxed{\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2}$$

Problem 29 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (z, 4yz, x^2 + 3y)$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(3 - 4y, 1 - 2x, 0)}$$

Is this vector field conservative?

Multiple Choice:

(a) *Yes*

(b) *No* ✓

Problem 30 *Compute the curl of the vector field* $\mathbf{F}(x, y, z) = (ye^{xy} + z\cos(x), xe^{xy}, \sin(x))$.

$$\nabla \times \mathbf{F}(x, y, z) = \boxed{(0, 0, 0)}$$

Is this vector field conservative?

Multiple Choice:

(a) *Yes* ✓

(b) *No*

Problem 31 *Find a potential function* f *for* $\mathbf{F}$, *so that* $\nabla f = \mathbf{F}$.

$$f(x, y, z) = \boxed{e^{xy} + \sin(x)z}$$

Written Problems

Problem 32 *Prove that, for* C^1 *vector fields* $\mathbf{F}$ *and* $\mathbf{G}$ *in* $\mathbf{R}^3$,

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

(This shows that the curl is an additive operator, *so the curl of a sum is the sum of the curls.)*

Problem 33 *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be vector fields in* $\mathbb{R}^3$, *and let* a *and* b *be real numbers. Prove that*

$$\nabla \times (a\mathbf{F} + b\mathbf{G}) = a(\nabla \times \mathbf{F}) + b(\nabla \times \mathbf{G}).$$

Problem 34 *Let* $\mathbf{F}$ *be a vector field in* $\mathbb{R}^3$, *and let* $f : \mathbb{R}^3 \to \mathbb{R}$ *be a scalar valued function. Prove that*

$$\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}.$$

Problem 35 *Let* $\mathbf{F}(x, y, z) = \dfrac{c\mathbf{r}}{\|\mathbf{r}\|^3}$, *where* c *is constant and* $\mathbf{r} = (x, y, z)$. *Prove that the curl of* $\mathbf{F}$, $\nabla \times \mathbf{F}$, *is zero.*

Problem 36 *Let* $\mathbf{F}(x, y, z) = (x, y, z)$, *and let* $\mathbf{a} \in \mathbb{R}^3$ *be a vector with constant entries. Prove that*

$$\nabla \times (\mathbf{a} \times \mathbf{F}) = 2\mathbf{a}.$$

Problem 37 (a) *Compute the curl of the vector field* $\mathbf{F} = (x, y, z)$. *Explain why your answer makes sense geometrically.*

(b) *Suppose we have a* C^1 *vector field* $\mathbf{F}(x, y, z) = (f(x), g(y), h(z))$. *Compute the curl of* $\mathbf{F}$, *and explain why your answer makes sense geometrically.*

Problem 38 (a) *Consider the function* $f(x, y, z) = e^x \sin(y) + z^2 \cos(y)$. *Compute* ∇f, *and verify that* $\nabla \times (\nabla f) = \mathbf{0}$.

(b) *Prove that for any* C^2 *function* $f : \mathbb{R}^3 \to \mathbb{R}$, *the curl of the gradient of* f *is zero. That is, prove that*

$$\nabla \times (\nabla f) = \mathbf{0}.$$
