Parametric Curves

We've dealt with several ways to describe curves in $\mathbb{R}^2$:

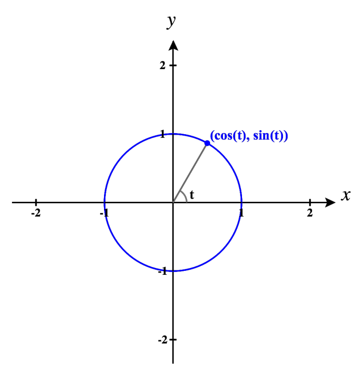
- As the graph of a function. For example, $f(x) = x^2$.
- As the set of points satisfying an equation. For example, the points (x, y) such that $x^2 + y^2 = 1$.
- As the set of points satisfying an equation in another coordinate system. For example, $r = \sin(\theta)$ in polar coordinates.

Another way that we can describe a curve is using *parametric equations*. When describing a curve using parametric equations, we define x and y in terms of a third variable, often t, called the *parameter*. We often think about t as representing time, and imagine the curve being drawn out as t increases. This gives us another way to describe curves in $\mathbb{R}^2$, and potentially describe some new and strange curves.

We can describe the unit circle in $\mathbb{R}^2$ with the parametric equations

$$\begin{aligned} x &= \cos(t), \\ y &= \sin(t), \end{aligned}$$

for $0 \leq t \leq 2\pi$.



Learning outcomes:
Author(s):

We can think of t as giving the angle that a point makes with the positive axis. It can also be helpful to imagine t as representing time, and the parametric equations tracing out the circle as time passes.

YOUTUBE LINK: unit circle 1

Consider the parametric equations for the unit circle in $\mathbb{R}^2$:

$$x = \cos(t),$$
$$y = \sin(t),$$

for $0 \leq t \leq 2\pi$.

We can combine these equations into a single vector,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \leq t \leq 2\pi.$$

We can visualize the vectors $\vec{x}(t)$ tracing out the unit circle as t goes from 0 to 2π.

YOUTUBE VIDEO: unit circle 2

Notice that we are blurring the distinction between vectors and points. Although we can think of $vecx(t)$ as a position vector, we would more commonly think of $\vec{x}(t)$ as a point on a curve. Although this might seem a bit sloppy, it will prove very useful throughout the course. Although intuitively we might prefer to use points, a lot of the computations tools that we'll require are more appropriately used with vectors.

We have defined a function $\vec{x}$ from the interval $[0, 2\pi] \subset \mathbb{R}$ to $\mathbb{R}^2$, and this idea provides the motivation behind our definition for paths.

Parametric Curves in $\mathbb{R}^n$

Definition 1. *A* path *in $\mathbb{R}^n$ is a continuous function*

$$\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n,$$

where $I \subset \mathbb{R}$ is an interval.

This is also called a parametrized curve *or* parametric curve. *When we wish to emphasize that $\vec{x}(t)$ is a vector, we'll call it the* position vector.

We'll focus on the cases $n = 2$ and $n = 3$ in this course.

We defined a path as a continuous function, however, we haven't said what it means for a multivariable function to be continuous. We'll come back to this later, and we'll give a rigorous definition for continuity. For now, this should fit with your intuition: you can draw the path without lifting your pencil from the paper.

Sometimes we care more about the image of a path than how the path is drawn out, and then we refer to a curve.

Definition 2. *A* curve C *in* $\mathbb{R}^n$ *is the image of some path* $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$.

In this case, we say that $\vec{x}$ *is a* parametrization *for the curve* C.

The difference between a curve and a path is largely a matter of perspective: when working with a curve, we pay attention to *what* is drawn; when working with a path, we care about *how* it is drawn.

Example 1. *There are many different parametrizations for a given curve.*

Consider again the unit circle C *in* $\mathbb{R}^2$. *Which of the following are parametrizations for* C*?*

Select All Correct Answers:

(a) $\vec{x}(t) = (\cos(t), \sin(t))$ *for* $0 \le t \le 2\pi$ ✓

(b) $\vec{x}(t) = (\sin(t), \cos(t))$ *for* $0 \le t \le \pi$

(c) $\vec{x}(t) = (t, \pm\sqrt{1-t^2})$ *for* $-1 \le t \le 1$

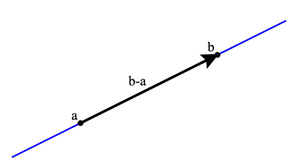
(d) $\vec{x}(t) = (\sin(2\pi t), \cos(2\pi t))$ *for* $0 \le t \le 1$ ✓

(e) $\vec{x}(t) = (\cos(t), \sin(t))$ *for* $-10 \le t \le 10$ ✓

Example 2. *In this example, we review a parametrization for the line through points* $\vec{a}$ *and* $\vec{b}$ *in* $\mathbb{R}^n$.

Given points $\vec{a}$ *and* $\vec{b}$ *in* $\mathbb{R}^n$, *we obtain a vector starting at* $\vec{a}$ *and ending at* $\vec{b}$ *by taking* $\vec{b} - \vec{a}$. *This vector is parallel to the line through* $\vec{a}$ *and* $\vec{b}$. *Then, taking scalar multiples* $t(\vec{b} - \vec{a})$ *for* $t \in \mathbb{R}$, *we have a line parallel to the line through* $\vec{a}$ *and* $\vec{b}$. *Finally, we add one of the points,* $\vec{a}$, *to ensure that our line passes through these two points. Thus, we arrive at our parametrization,*

$$\vec{l}(t) = \vec{a} + t(\vec{b} - \vec{a}) \text{ for } t \in \mathbb{R}.$$

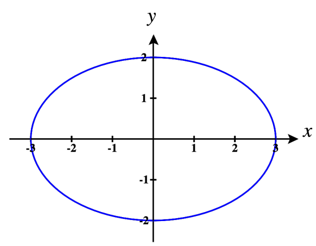


Example 3. *In this example, we see how we can obtain new transformations from old ones, using linear algebra and simple transformations.*

Recall the parametrization for the unit circle in $\mathbb{R}^2$,

$$\vec{x}(t) = (\cos(t), \sin(t)) \text{ for } 0 \le t \le 2\pi.$$

Now, consider the ellipse below.



We can think of this ellipse as the result of stretching the unit circle horizontally by a factor of 3 *and vertically by a factor of* 2*. That is, we are applying the linear transformation*

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

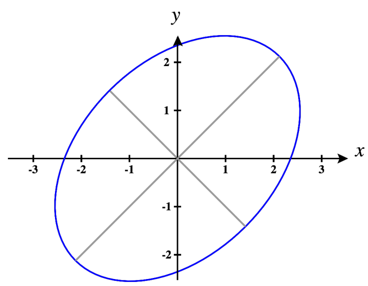
We can apply this to the parametrization for the unit circle, in order to parametrization for the ellipse.

$$\begin{aligned} \vec{(y)}(t) &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \\ &= (3\cos(t), 2\sin(t)). \end{aligned}$$

Thus, we have a parametrization for the ellipse given by

$$\vec{y}(t) = (3\cos(t), 2\sin(t)) \text{ for } 0 \le t \le 2\pi.$$

Next, consider the following ellipse.



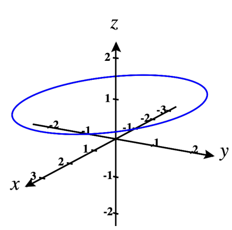
We can obtain this from our previous ellipse by counterclockwise rotation of $\pi/4$. The matrix for this linear transformation is

$$\begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \boxed{1/\sqrt{2}} & \boxed{-1/\sqrt{2}} \\ \boxed{1/\sqrt{2}} & \boxed{1/\sqrt{2}} \end{pmatrix}.$$

Applying this rotation to our parametrization for the previous ellipse, we obtain a parametrization for our new ellipse.

$$\vec{z}(t) = \boxed{(3/\sqrt{2}\cos(t) - 2/\sqrt{2}\sin(t), 3/\sqrt{2}\cos(t) + 2/\sqrt{2}\sin(t))} \textit{ for } 0 \leq t \leq 2\pi$$

Finally, we consider an ellipse in $\mathbb{R}^3$, shown below.



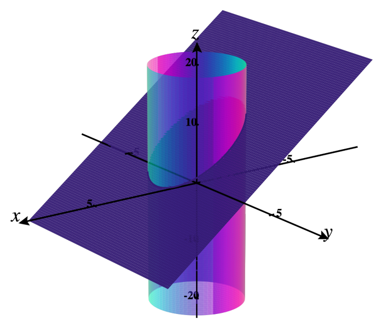
This ellipse is parallel to the xy-plane, and will have constant z-coordinate of 1. *Note the similarity to the first ellipse we considered. A parametrization for* *this ellipse can be obtained by taking the parametrization $\vec{y}$ for our first ellipse in $\mathbb{R}^2$, and appending the constant z-coordinate.*

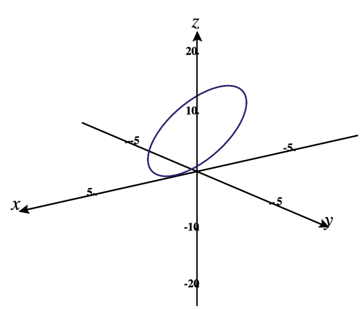
$$\vec{a}(t) = \boxed{(3\cos(t), 2\sin(t), 1)} \text{ for } 0 \leq t \leq 2\pi$$

Examples in $\mathbb{R}^3$

In this section, we give examples of parametrizations of a couple of more complicated curves in $\mathbb{R}^3$, taking advantage of our previous experience with cylindrical coordinates.

Example 4. *We'll parametrize the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = 7 - 3x$ in $\mathbb{R}^3$, pictured below.*





Our x and y coordinates must satisfy $x^2 + y^2 = 4$, which would define a circle, if we were in $\mathbb{R}^2$. Recalling our parametrizations for circles, these coordinates *can be written as*

$$x(t) = 2\cos(t)$$
$$y(t) = 2\sin(t)$$

for $0 \leq t \leq 2\pi$.

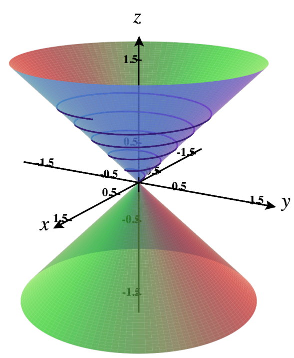
It remains to write the z*-coordinate in terms of the parameter* t*. Turning our attention to the equation for the plane,* $z = 7 - 3x$*, we have* z *expressed in terms of* x*. Since we have expressed* x *in terms of* t*, we can make this substitution to describe* z *in terms of* t*,*

$$z(t) = \boxed{7 - 6\cos(t)}.$$

Putting all of this together, we have a parametrization for this intersection given by

$$\vec{x}(t) = \boxed{(2\cos(t), 2\sin(t), 7 - 6\cos(t))} \text{ for } 0 \leq t \leq 2\pi.$$

Example 5. *Consider the curve below, which lies on the cone* $z^2 = x^2 + y^2$*, and makes five rotations around the* z*-axis as the height ranges from* 0 *to* 1*. We'll refer to this curve as a "tornado."*



We'll parametrize this curve by thinking about it in cylindrical coordinates, using the height as the parameter.

First, let's consider what's happening with the z*-coordinate. Since the height of the tornado rangers from* 0 *to* 1*, so will* z*. We'll set* $z = t$*, with* $0 \leq t \leq 1$*, and express* x *and* y *in terms of* t *as well.*

Now, we turn our attention to the angle θ*. As the height ranges from* 0 *to* 1*, the tornado makes five revolutions, so* θ *should range from* 0 *to* 10π*. Thus, expressing* θ *in terms of* t*, we let* $\theta = 10\pi t$*.*

Next, we consider the radius r*. Since we are on the cone* $z^2 = x^2 + y^2$*, we have* $z^2 = r^2$*. Since* $z \geq 0$*, we have* $z = r$*. Thus, we can write* r *in terms of* t *as* $r = t$*.*

Finally, putting all of this together with $x = r\cos\theta$ *and* $y = r\sin\theta$*, we have a parametrization for the tornado given by*

$$\vec{x}(t) = \boxed{(t\cos(10\pi\theta), t\sin(10\pi\theta), t)} \text{ for } t \in [0, 1].$$