Virtual whiteboard for Jan'22 math+econ+code

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1 Day 1

1.1 1a. The diet problem

Consider N_{ij}=amount of nutrient i that is brought by one dollar's worth of food j.

Look for q_j = \$ invested in food j
such that we minimize total cost
$\min \sum_j q_j$

subject to the constraint that the minimal intake in each food i is met
$\sum_j N_{ij} q_j \geq d_i$ for each nutrient i
where d_i is the minimum quantity of nutrient i.

To summarize, we need to solve
$\min_{q \geq 0} \sum_j q_j$
s.t. $\sum_j N_{ij} q_j \geq d_i$

In matrix form this is
$\min_{q \geq 0} c^\top q$
s.t. $Nq \geq d$
where $c_j = 1$.

1.1.1 Duality worked out by hand

$\min_{q \geq 0} \sum_j c_j q_j$
s.t. $\sum_j N_{ij} q_j \geq d_i$
transform into an unconstrained minimization problem
$\min_{q \geq 0} \sum_j c_j q_j + \sum_i F\left(d_i - \sum_j N_{ij} q_j\right)$
where $F(z) = 0$ if $z \leq 0$, and $F(z) = +\infty$ if $z > 0$.
How to represent F? How about
$F(z) = \max_{\pi \geq 0} \{\pi z\}$.

$\min_{q \geq 0} \sum_j c_j q_j + \sum_i \max_{\pi_i \geq 0} \pi_i \left(d_i - \sum_j N_{ij} q_j\right)$

$\min_{q\geq 0} \sum_j c_j q_j + \max_{\pi_i \geq 0} \sum_i \pi_i \left(d_i - \sum_j N_{ij} q_j \right)$

$\min_{q_j\geq 0} \max_{\pi_i \geq 0} \sum_j c_j q_j + \sum_i \pi_i \left(d_i - \sum_j N_{ij} q_j \right)$
Assuming $\min\max = \max\min$, we can reformulate this as
$\max_{\pi_i \geq 0} \min_{q_j\geq 0} \sum_j c_j q_j + \sum_i \pi_i \left(d_i - \sum_j N_{ij} q_j \right)$
$\max_{\pi_i \geq 0} \min_{q_j\geq 0} \sum_j c_j q_j + \sum_i \pi_i d_i - \sum_{i,j} \pi_i N_{ij} q_j$
going through the same steps in the opposite direction:
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i + \min_{q_j\geq 0} \sum_j c_j q_j - \sum_{i,j} \pi_i N_{ij} q_j$
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i + \min_{q_j\geq 0} \sum_j q_j \left(c_j - \sum_i \pi_i N_{ij} \right)$
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i + \sum_j \min_{q_j\geq 0} q_j \left(c_j - \sum_i \pi_i N_{ij} \right)$
But $\left(\min_{q_j\geq 0} q_j w_j \right) = -\infty$ if $w_j < 0$ and $= 0$ if $w_j \geq 0$

Therefore
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i$
$s.t.\ c_j \geq \sum_i \pi_i N_{ij}$
that is
$\max_{\pi\geq 0} \pi^\top d$
$s.t.\ N^\top \pi \leq c$

Theorem. Consider the "primal problem"
$\min_{q\geq 0} \sum_j c_j q_j$
s.t. $\sum_j N_{ij} q_j \geq d_i$ $[\pi_i \geq 0]$
and the "dual problem"
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i$
$s.t.\ \sum_i \pi_i N_{ij} \leq c_j$ $[q_j \geq 0]$

Then if either of them is feasible [i.e. there is a variable that meets the constraints] then

(1) the value of the primal problem is equal to the value of the dual problem

(2) complementary slackness:

Assume q is a solution to the primal problem and π is a solution to the dual problem

$\pi_i > 0$ implies $d_i = \sum_j N_{ij} q_j$
$q_j > 0$ implies $c_j = \sum_i \pi_i N_{ij}$

Theorem. If q is feasible for the primal and π is feasible for the dual and if complementary slackness holds, then q is optimal for the primal and π is optimal for the dual.

1.2 1b. The optimal assigment problem

Joint surplus matrix
$\Phi_{xy} = x^\top A y = \sum_{k,l} A_{kl} x_k y_l$

Assume n_x men of type x and m_y women of type y.

Becker-Shapley-Shubik's model of matching.
Assume if man x matches with woman y, then:
x gets surplus α_{xy}
y gets surplus γ_{xy}

Assume utility is transferable, ie if w_{xy} is the transfer from the woman to the man (either positive or negative),
x gets surplus $\alpha_{xy} + w_{xy}$
y gets surplus $\gamma_{xy} - w_{xy}$
This is the Transferable Utility assumption.

w_{xy} is determined at equilibrium.
Regardless of what w_{xy} is, the joint surplus
$\Phi_{xy} = (\alpha_{xy} + w_{xy}) + \left(\gamma_{xy} - w_{xy}\right) = \alpha_{xy} + \gamma_{xy}$ is the same.

Roadmap:
1. Optimality
2. Equilibrium

1.2.1 Optimality

A matching is a $\mu_{xy} \geq 0$ which is the number of men of type x matched with women of type y.
Constraint on μ_{xy}:
$\sum_y \mu_{xy} = n_x$
$\sum_x \mu_{xy} = m_y$

Optimal matching consists of
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy}$
s.t.
$\sum_y \mu_{xy} = n_x \ \ [u_x]$
$\sum_x \mu_{xy} = m_y \ \ [v_y]$

Duality:
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy} + \sum_x \min_{u_x} u_x \left(n_x - \sum_y \mu_{xy}\right) + \sum_y \min_{v_y} v_y \left(m_y - \sum_x \mu_{xy}\right)$
$\min_{u_x, v_y} \sum_x n_x u_x + \sum_y m_y v_y + \max_{\mu \geq 0} \sum_{xy} \mu_{xy} \left(\Phi_{xy} - u_x - v_y\right)$
that is, the dual problem is
$\min_{u_x, v_y} \sum_x n_x u_x + \sum_y m_y v_y$
s.t. $u_x + v_y \geq \Phi_{xy} \ \ \left[\mu_{xy} \geq 0\right]$

By complementary slackness, we have $\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$.

1.2.2 Interpretation as a stable outcome.

We now consider the decentralized version of the problem.

Definition: (μ, u, v) is a stable outcome if

(1) μ is a matching:

$\sum_y \mu_{xy} = n_x \ \ [u_x]$

$\sum_x \mu_{xy} = m_y \ \ [v_y]$

(2) Stability: we have for all x and y that

$u_x + v_y \geq \Phi_{xy}$

[otherwise we would have $u_x + v_y < \Phi_{xy}$ and xy would be a blocking pair, ie there is a way for x to have more than u_x and y to have more than v_y if x and y match together]

(3) Feasibility: if $\mu_{xy} > 0$, then $u_x + v_y = \Phi_{xy}$.

Note that (1) means that μ is feasible for the primal problem

(2) means that (u, v) is feasible for the dual problem

(3) means complementary slackness.

Hence μ is a solution to the primal problem

$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy}$

s.t.

$\sum_y \mu_{xy} = n_x \ \ [u_x]$

$\sum_x \mu_{xy} = m_y \ \ [v_y]$

and (u, v) is a solution to the dual problem

$\min_{u_x, v_y} \sum_x n_x u_x + \sum_y m_y v_y$

s.t. $u_x + v_y \geq \Phi_{xy} \ \ \left[\mu_{xy} \geq 0\right]$

Recovering the transfers. We had that if w_{xy} is the transfer from y to x, then

x gets surplus $\alpha_{xy} + w_{xy}$

y gets surplus $\gamma_{xy} - w_{xy}$

Hence the payoff of x at equilibrium is

$u_x = \max_y \{\alpha_{xy} + w_{xy}\}$

and

$v_y = \max_x \left\{\gamma_{xy} - w_{xy}\right\}$

Hence

$u_x \geq \alpha_{xy} + w_{xy}$ for all x and y

$v_y \geq \gamma_{xy} - w_{xy}$ for all x and y

This yields

$\gamma_{xy} - v_y \leq w_{xy} \leq u_x - \alpha_{xy}$

We have $u_x - \alpha_{xy} \geq \gamma_{xy} - v_y$ – indeed $u_x + v_y \geq \alpha_{xy} + \gamma_{xy}$

When $\mu_{xy} > 0$, we have $w_{xy} = \gamma_{xy} - v_y = u_x - \alpha_{xy}$

When partners can remain unmatched.

Assume that individuals get utility zero if they remain unmatched.

Then a feasible matching imposes

$\sum_y \mu_{xy} \leq n_x$
$\sum_x \mu_{xy} \leq m_y$

An optimal matching solves
$\max_{\mu\geq 0} \sum_{xy} \mu_{xy}\Phi_{xy}$
s.t.
$\sum_y \mu_{xy} \leq n_x \;\; [u_x \geq 0]$
$\sum_x \mu_{xy} \leq m_y \;\; [v_y \geq 0]$

Duality:
$\max_{\mu\geq 0} \sum_{xy} \mu_{xy}\Phi_{xy} + \sum_x \min_{u_x\geq 0} u_x \left(n_x - \sum_y \mu_{xy}\right) + \sum_y \min_{v_y\geq 0} v_y \left(m_y - \sum_x \mu_{xy}\right)$
$\min_{u_x\geq, v_y\geq} \sum_x n_x u_x + \sum_y m_y v_y + \max_{\mu\geq 0} \sum_{xy} \mu_{xy} \left(\Phi_{xy} - u_x - v_y\right)$
that is, the dual problem is
$\min_{u_x\geq, v_y\geq 0} \sum_x n_x u_x + \sum_y m_y v_y$
s.t. $u_x + v_y \geq \Phi_{xy} \;\; \left[\mu_{xy} \geq 0\right]$

By complementary slackness, we have $\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$.
The dual is
$\min_{u_x\geq 0, v_y\geq 0} \sum_x n_x u_x + \sum_y m_y v_y$
s.t. $u_x + v_y \geq \Phi_{xy} \;\; \left[\mu_{xy} \geq 0\right]$

Alternatively, the primal can be expresses as
$\max_{\mu\geq 0} \sum_{xy} \mu_{xy}\Phi_{xy}$
s.t.
$\sum_y \mu_{xy} + \mu_{x0} = n_x \;\; [u_x \geq 0]$
$\sum_x \mu_{xy} + \mu_{0y} = m_y \;\; [v_y \geq 0]$
where μ_{x0} and μ_{0y} act as slackness variable.
Stability interpretation: a stable outcome (μ, u, v) in the problem with singles is such that
(1) μ is a feasible partial matching:
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
(2) Stability holds
$u_x + v_y \geq \Phi_{xy}$
$u_x \geq 0, v_y \geq 0$
(3) Complementary slackness
$\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$
$\mu_{x0} > 0$ i.e $\left(\sum_y \mu_{xy} < n_x\right) \implies u_x = 0$
$\mu_{0y} > 0$ i.e. $\left(\sum_x \mu_{xy} < m_y\right) \implies v_y = 0$

1.2.3 Indivisibilities (finite population)

When $n_x = 1$ for each x and $m_y = 1$ for each y, we should in principle impose an integrality constraint, that is

$$\mu_{xy} \in \{0, 1\}.$$

Then the problem is no longer a linear programming problem, but an integer programming problem.

However, in the bipartite case, one can abstract away from the integrality constraint.

1.2.4 Computation

Consider the problem
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy}$
s.t.
$\sum_y \mu_{xy} \leq n_x \;\; [u_x \geq 0]$
$\sum_x \mu_{xy} \leq m_y \;\; [v_y \geq 0]$
this is of the form
$\max_{\mu \geq 0} \mu^\top \Phi$
$M\mu \leq \binom{n}{m}$
How to convert a matrix into a vector? Take a matrix M, we call $vec\left(M\right)$ its vectorized version,

This can be done by:

* stack the columns: Matlab, Julia, R, Fortran column-major ordering, or Fortran ordering.

* stack the rows: done by NumPy by default, as well as C as well as some other languages: row-major ordering, or C ordering – primary convetion in this course.

Take the first set of constraints $\sum_y \mu_{xy} \leq n_x$. If μ is understood as a matrix, this is

$$\mu 1_Y \leq n$$

by pre-multuplying by the identity this yields

$$I_X \mu 1_Y \leq n$$

And we need to look at $vec\left(I_X \mu 1_Y\right) = matrix.vec\left(\mu\right)$

Fundamental identity is (assuming row-major ordering)

$$vec\left(AXB^\top\right) = \left(A \otimes B\right) vec\left(X\right).$$

Here, our constraints become

$$vec\left(I_X \mu 1_Y\right) = \left(I_X \otimes 1_Y^\top\right) vec\left(\mu\right) \leq n$$

Similarly, we can vectorize the other constraints $\sum_x \mu_{xy} \leq m_y$ into $1_X^\top \mu I_Y \leq m$, that is

$$vec\left(1_X^\top \mu I_Y\right) = \left(1_X^\top \otimes I_Y\right) vec\left(\mu\right) \leq m$$

Hence our optimal assignment problem becomes in a vectorized fashion

$$\begin{aligned} \max_{vec(\mu)\geq 0} \quad & vec(\mu)^\top vec(\Phi) \\ s.t. \quad & \left(I_X \otimes 1_Y^\top\right) vec(\mu) \leq n \\ & \left(1_X^\top \otimes I_Y\right) vec(\mu) \leq m \end{aligned}$$

that is

$$\begin{aligned} \max_{v\geq 0} \quad & v^\top vec(\Phi) \\ s.t. \quad & Mv \leq \begin{pmatrix} n \\ m \end{pmatrix} \end{aligned}$$

where

$$M = \begin{pmatrix} I_X \otimes 1_Y^\top \\ 1_X^\top \otimes I_Y \end{pmatrix}$$

is called the margining matrix.

2 Day 2

About yesterday's exercises

$$\begin{pmatrix} 1/4 & 1/4-1/8 & 1/4+1/8 & \\ & 2/8+1/8 & 1/8-1/8 & \\ & & & \end{pmatrix}$$

networkx
algorithms to detect loops

Birkhoff-von Neumann

$\min \sum_{xy} \mu_{xy} c_{xy}$
$M\mu = \binom{n}{m}$
If $c_{xy} \in \{0, 1\}$, then $\Gamma = \{xy : c_{xy} = 0\}$

$0 = \min \sum_{xy} \mu_{xy} c_{xy}$
$M\mu = \binom{n}{m}$
iff for every xy such that $\mu_{xy} > 0$, then $xy \in \Gamma$
this means that there is a matching between n and m "compatible" with Γ.

The dual to
$\min \sum_{xy} \mu_{xy} c_{xy}$
$M\mu = \binom{n}{m}$
is
$\max_{u,v} \sum_x u_x - \sum_y v_y$
s.t. $u_x - v_y \leq c_{xy}$

$$\Gamma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

2.1 2a. One-dimensional matching

We now assume that types of workers x and firms y belong in $\mathbb{R}$.

Assume that there is the same total mass of workers and firms.

The total mass of workers and firms is normalized to one.

$n(x)$ is the density of probability associated with the distribution of workers

$m(y)$ is the density of probability associated with the distribution of firms

A matching is a distribution of probability $\mu(x, y)$ over pairs x, y. It should satisfy

$\int \mu(x, y)\, dy = n(x)$
$\int \mu(x, y)\, dx = m(y)$

Assume that the economic value created by a CEO of type x with a firm of type y is

$\Phi(x,y)$.

The optimal assignment problem is

$\max_{\mu}\int\Phi(x,y)\,\mu(x,y)\,dxdy$

s.t.

$\int\mu(x,y)\,dy=n(x)\,,x\in R$

$\int\mu(x,y)\,dx=m(y)\,,y\in R$

CEO application (Gabaix and Landier, Tervio):

x is the CEO's talent = extra % return on asset the CEO generates

y is the firm's size (market cap)

In that case, we get that

$\Phi(x,y)=xy$

Agenda:

Goal = predict CEO compensation

Solve for the primal problem

From the solution to the primal problem, we will deduce the solution to the dual problem

The dual problem will give us predictions for CEO compensation

Intuition: If $T(x)$ is the firm that CEO x is matched with, it makes sense to assume that

$T(x)$ is increasing.

If that is the case, what is $T(x)$?

We know that if $X\sim n$ is a random variable distributed according to n, then we $T(X)\sim m$.

We can introduce the cumulative distribution functions (CDF) $F_n(x)=\int_{-\infty}^{x}n(z)\,dz$ and $F_m(y)=\int_{-\infty}^{y}m(z)\,dz$ associated with n and m respectively, and we have

$$\Pr(T(X)\leq y)=F_m(y)$$

take $y=T(x)$ for a fixed value of x, we get

$$\Pr(T(X)\leq T(x))=F_m(T(x))$$

thus

$$\begin{aligned}\Pr(X\leq x)&=F_m(T(x))\\ F_n(x)&=F_m(T(x))\end{aligned}$$

therefore

$$T(x)=F_m^{-1}(F_n(x))$$

In particular, this means that the median CEO – ie the CEO x such $F_n(x) = 1/2$ is matched with the median firm indeed $F_n(x) = 1/2 = F_m(T(x))$ – thus $T(x)$ is the median firm.

Let's see how we can solve for the dual problem. The dual problem is
$\min_{u(x),v(y)} \int u(x)\, n(x)\, dx + \int v(y)\, m(y)\, dy$
s.t. $u(x) + v(y) \geq \Phi(x,y) = xy$

Assume (u,v) is a ** feasible solution ** to the dual problem.
Then for every x and every y,
$u(x) + v(y) \geq xy$
thus
$v(y) \geq \max_x \{xy - u(x)\}$ holds for for every y.
Claim: for any ** optimal solution ** to the dual problem, we have
$v(y) = \max_x \{xy - u(x)\}$ – interpreted as firm y's problem
Obviously, this is symmetric, so we have also
$u(x) = \max_y \{xy - v(y)\}$ – interpreted as worker x's problem.

Let's write down optimality conditions in the firm y's problem
$y = u'(x)$.

But we are in the case where $T(x)$ is known and $T(x) = F_m^{-1}(F_n(x))$.
We have therefore
$u'(x) = F_m^{-1}(F_n(x))$
as a result
$u(x) = \int^x F_m^{-1}(F_n(z))\, dz + cte$.
Deduce v (either by $v(y) = \max_x \{xy - u(x)\}$, or $v(y) = \int^y F_n^{-1}(F_m(z))\, dz + cte$)

Theorem. When Φ is supermodular, i.e. $\partial^2\Phi(x,y)/\partial x\partial y \geq 0$, then the positive assortative matching solution $T(x) = F_m^{-1}(F_n(x))$ is optimal.
Until now we have assumed $\Phi(x,y) = xy$, so $\partial^2\Phi(x,y)/\partial x\partial y = 1 \geq 0$.

We have that if u and v are optimal dual solutions, then
$v(y) = \max_x \{\Phi(x,y) - u(x)\}$ – interpreted as firm y's problem
Obviously, this is symmetric, so we have also
$u(x) = \max_y \{\Phi(x,y) - v(y)\}$ – interpreted as worker x's problem.

By first order condition in the firm's $T(x)$ problem, we have
$u'(x) = \partial_x\Phi(x, T(x))$
Now let's verify that if Φ is supermodular, then T is increasing. Deriving the above wrt x, we have
$u''(x) = \partial^2_{xx}\Phi(x, T(x)) + \partial^2_{xy}\Phi(x, T(x))\, T'(x)$
therefore

$$T'(x) = \frac{u''(x) - \partial^2_{xx}\Phi(x, T(x))}{\partial^2_{xy}\Phi(x, T(x))}$$

The denominator is positive $\partial^2_{xy}\Phi\left(x,T\left(x\right)\right)>0$ by supermodularity

We have $u''\left(x\right)-\partial^2_{xx}\Phi\left(x,T\left(x\right)\right)\geq 0$ by second order conditions. Indeed, $\partial^2_{xx}\Phi\left(x,y\right)-u''\left(x\right)\leq 0$.

Hence

$$T'\left(x\right)\geq 0.$$

Exercise. Model of marriage with taxes.

If single individual get gross income w, then gets net amount $N\left(w\right)$ where N is increasing and concave.

Assume x and y are the gross incomes of two marital partners. Then their combined gross income is $x+y$, and we create two fictious personnas which make $\frac{x+y}{2}$ each. Thus their combined net income is $2N\left(\frac{x+y}{2}\right)$.

Thus we can consider a matching market where the matching surplus is

$\Phi\left(x,y\right)=2N\left(\frac{x+y}{2}\right)$

1) Is there surplus to matching? i.e. do we have $2N\left(\frac{x+y}{2}\right)\geq N\left(x\right)+N\left(y\right)$?

2) Derive the matching equilibrium on this market.

2.2 2b. Semi-discrete optimal transport

Inhabitant's problem

$$u\left(x\right)=\max_j\left\{x^\top y_j-v_j\right\}$$

I would like to get a formula for the market share of fountain j.

Let us compute the aggregate indirect welfare of the consumer. This is

$$\int_{\mathcal{X}}u\left(x\right)n\left(x\right)dx$$

where $n\left(x\right)$ is the density of inhabitants at x. Expressed as a function of the prices, this is

$$F\left(v\right)=\int_{\mathcal{X}}\max_j\left\{x^\top y_j-v_j\right\}n\left(x\right)dx$$

Claim:

$$\frac{\partial F}{\partial v_j}=-D_j\left(v\right)$$

Thus the demand for fountain j is given

$$D_j\left(v\right)=-\frac{\partial F\left(v\right)}{\partial v_j}.$$

Recall that fountain j has fixed capacity q_j. Therefore the market-clearing prices v_j of the fountains are given by

$$D_j\left(v\right)=q_j,$$

that is

$$\frac{\partial F(v)}{\partial v_j} + q_j = 0$$

which we can rewrite as

$$\frac{\partial}{\partial v_j}\left\{F(v) + \sum_k q_k v_k\right\} = 0$$

that is v is obtained by minimizing $S(v) := F(v) + \sum_k q_k v_k$ over v. Thus, v is a solution to

$$\min_v \left\{F(v) + \sum_k q_k v_k\right\}$$

that is

$$\min_v \left\{\int_{\mathcal{X}} \max_j \left\{x^\top y_j - v_j\right\} n(x)\, dx + \sum_k q_k v_k\right\}$$

but we can view this as

$$\begin{aligned} \min_v \quad & \left\{\int_{\mathcal{X}} u(x)\, n(x)\, dx + \sum_k q_k v_k\right\} \\ s.t. \quad & u(x) \geq \max_j \left\{x^\top y_j - v_j\right\} \end{aligned}$$

which reformates as

$$\begin{aligned} \min_v \quad & \left\{\int_{\mathcal{X}} u(x)\, n(x)\, dx + \sum_k q_k v_k\right\} \\ s.t. \quad & u(x) + v_j \geq x^\top y_j \ \forall x, \forall j \end{aligned}$$

Tommaso' suggestion.

$$v_j^{t+1} = v_j^t + \varepsilon\left(D_j(v) - q_j\right).$$

We have $S(v) = F(v) + \sum_k q_k v_k$, and therefore

$$\frac{\partial S(v)}{\partial v_j} = -D_j(v) + q_j$$

thus this algorithm amounts to

$$v_j^{t+1} = v_j^t - \varepsilon \frac{\partial S(v)}{\partial v_j}.$$

This is gradient descent! ie

$$v^{t+1} = v^t - \varepsilon \nabla S(v).$$

Another possibility would be **coordinate descent**.
$q_j = D_j\left(v_j^{t+1}, v_{-j}^t\right)$
Parallel version: Jacobi
Sequential version: Gauss-Seidel

Note that
$q_j = D_j\left(v_j^{t+1}, v_{-j}^t\right)$ is equivalent to
$v_j^{t+1} = \arg\min_{v_j} S\left(v_j, v_{-j}^t\right)$

Final remark. Back to the central planner problem.

The assinment we found consists in mapping inhabitant x with fountain y_j such that $j \in \arg\max_j \left\{x^\top y_j - v_j\right\}$.

Noting that $u(x) = \max_j \left\{x^\top y_j - v_j\right\}$, we have that $\nabla u(x) = y_j$. Thus $T(x) = \nabla u(x)$ is the optimal assignment from inhabitants to fountains, in the sense that it solves the primal problem

$$\begin{aligned}
\max_{\mu(x,y_j)} \quad & -\int |x - y_j|^2 \mu(x, y_j)\, dx \\
s.t. \quad & \int \mu(x, y_j)\, dx = q_j \\
& \sum_j \mu(x, y_j) = n(x).
\end{aligned}$$

Exercise. Implement a coordinate descent (Gauss-Seidel) version of the algorithm.

3 Day 3