# Abstract Mathematical Reasoning

Exam Notes

2021, Semester 1

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## Preface

This document is a succinct reference for the unit MXB102. We hope this document is useful, particularly to those undertaking this unit's final exam. This document is not intended to be a replacement for the lecture notes and unit guide. We would personally like to thank our lecturer, Dr Adrianne Jenner, and all of the tutors for the great amount of support and teachings that we have been able to receive over the course of this unit.

# 1 Logic

| **Name** | **Symbol** | **Truth Table** |
|---|---|---|
| Negation (NOT) | $\neg$ | $\begin{array}{c\|c} P & \neg P \\ \hline \mathsf{T} & \mathsf{F} \\ \mathsf{F} & \mathsf{T} \end{array}$ |
| Conjunction (AND) | $\wedge$ | $\begin{array}{cc\|c} P & Q & P \wedge Q \\ \hline \mathsf{T} & \mathsf{T} & \mathsf{T} \\ \mathsf{T} & \mathsf{F} & \mathsf{F} \\ \mathsf{F} & \mathsf{T} & \mathsf{F} \\ \mathsf{F} & \mathsf{F} & \mathsf{F} \end{array}$ |
| Disjunction (OR) | $\vee$ | $\begin{array}{cc\|c} P & Q & P \vee Q \\ \hline \mathsf{T} & \mathsf{T} & \mathsf{T} \\ \mathsf{T} & \mathsf{F} & \mathsf{T} \\ \mathsf{F} & \mathsf{T} & \mathsf{T} \\ \mathsf{F} & \mathsf{F} & \mathsf{F} \end{array}$ |
| Conditional (IF-THEN) | $\implies$ | $\begin{array}{cc\|c} P & Q & P \implies Q \\ \hline \mathsf{T} & \mathsf{T} & \mathsf{T} \\ \mathsf{T} & \mathsf{F} & \mathsf{F} \\ \mathsf{F} & \mathsf{T} & \mathsf{T} \\ \mathsf{F} & \mathsf{F} & \mathsf{T} \end{array}$ |
| Equivalence (IFF) | $\iff$ or $\equiv$ | $\begin{array}{cc\|c} P & Q & P \iff Q \\ \hline \mathsf{T} & \mathsf{T} & \mathsf{T} \\ \mathsf{T} & \mathsf{F} & \mathsf{F} \\ \mathsf{F} & \mathsf{T} & \mathsf{F} \\ \mathsf{F} & \mathsf{F} & \mathsf{T} \end{array}$ |

Table 1: Boolean operators in order of precedence.

**Theorem 1.0.1** (de Morgan's Law)**.** $\neg(P \wedge Q) \iff \neg P \vee \neg Q.$

**Definition 1.0.1** (Tautology)**.** A tautology is a statement that is always true.

**Definition 1.0.2** (Paradox)**.** A paradox is a statement that is always false.

**Definition 1.0.3** (Converse)**.** The converse of the statement $P \implies Q$ is the statement $Q \implies P$.

**Definition 1.0.4** (Contrapositive)**.** The contrapositive of $P \implies Q$ is the statement $\neg Q \implies \neg P$.

# 2 Sets

| **Name** | **Symbol** | **Definition** |
|---|---|---|
| Empty set | $\varnothing$ | A set with no elements: $\{\}$ |
| Intersection | $\cap$ | $S \cap T = \{x : x \in S \wedge x \in T\}$ |
| Union | $\cup$ | $S \cup T = \{x : x \in S \vee x \in T\}$ |
| Set difference | $\setminus$ or $-$ | $S - T = \{x : x \in S \wedge x \notin T\}$ |
| Subset | $\subseteq$ | If $\#S = \#T, \quad S \subseteq T \equiv \forall x \in S : x \in T$ |
| Strict (or proper) subset | $\subset$ | If $\#S < \#T, \quad S \subset T \equiv \forall x \in S : x \in T$ |
| Complement | $\overline{S}$ | If $S \subseteq T, \quad \overline{S} = T \setminus S$ |

**Definition 2.0.1** (Disjoint)**.** If $S \cap T = \varnothing$, then $S$ and $T$ are disjoint (i.e. two sets are disjoint if their intersection is empty).

## 2.1 Construction of the Natural Numbers

**Definition 2.1.1** ("0")**.** "0" is defined as the empty set, $\varnothing$ or $\{\}$, which is the first element of the natural numbers.

**Definition 2.1.2** (Successor function)**.** The successor function, $S(n)$, is defined as $n \cup \{n\}$.

**Definition 2.1.3** (Addition)**.** Addition (+) is defined recursively by the following axioms:

(A1) "a" + "0" = "a"

(A2) "a" + S("b") = S("a" + "b")

**Definition 2.1.4** (Multiplication)**.** Multiplication ($\cdot$) is defined recursively by the following axioms:

(M1) "a" $\cdot$ "0" = "0"

(M2) "a" $\cdot$ S("b") = "a" + ("a" $\cdot$ "b")

**Definition 2.1.5** (Exponentiation)**.** Exponentiation ($\uparrow$) is defined recursively by the following axioms:

(E1) "a" $\uparrow$ "0" = "1"

(E2) "a" $\uparrow$ S("b") = "a" $\cdot$ ("a" $\uparrow$ "b")

### 2.1.1 Properties of Natural Numbers

**Commutativity of addition** "a" + "b" = "b" + "a"

**Associativity of addition** ("a" + "b") + "c" = "a" + ("b" + "c")

**Commutativity of multiplication** "a" $\cdot$ "b" = "b" $\cdot$ "a"

**Associativity of multiplication** ("a" $\cdot$ "b") $\cdot$ "c" = "a" $\cdot$ ("b" $\cdot$ "c")

**Distributivity of multiplication over addition** "a" $\cdot$ ("b" + "c") = ("a" $\cdot$ "b") + ("a" $\cdot$ "c")

**Cancellation law** If "a" $\cdot$ "b" = "a" $\cdot$ "c" and "a" $\neq$ "0" then "b" = "c"

**Theorem 2.1.1.** $\forall n \in \mathbb{N}\backslash\{0\} : 2^n > n$.

**Theorem 2.1.2.** $\forall n \in \mathbb{N}\backslash\{0\} : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

## 2.2 Cartesian Product

**Definition 2.2.1** (Cartesian product)**.** The Cartesian product of two sets $S$ and $T$ is the set $S \times T$ consisting of all ordered pairs $(x, y)$ where $x \in S$ and $y \in T$.

## 2.3 Quantifiers

| **Name** | **Symbol** | **Usage** |
|---|---|---|
| Universal Quantification (For All) | $\forall$ | $\forall x : P(x)$ |
| Existential Quantification (There Exists) | $\exists$ | $\exists x : P(x)$ |

**Theorem 2.3.1.** $\neg(\forall x : P(x)) \equiv \exists x : \neg P(x)$

**Theorem 2.3.2.** $\neg(\exists x : P(x)) \equiv \forall x : \neg P(x)$

**Definition 2.3.1** (Greater than (>))**.** $x > y \iff \exists a \in \mathbb{N}\backslash\{0\} : x = y + a$

**Definition 2.3.2** (Greater than or equal to ($\geqslant$))**.** $x \geqslant y \iff \exists a \in \mathbb{N} : x = y + a$

# 3 Relations

**Definition 3.0.1** (Binary relation)**.** A binary relation $R$ on $S$ is the subset $S \times S$. If $a,\ b \in S$ and $(a,\ b) \in R$, then $aRb$ ("$a$ is related to $b$ under $R$")

## Relation Properties

**Reflexive** $\forall x \in S : xRx$

**Symmetric** $\forall x,\ y \in S : xRy \implies yRx$

**Antisymmetric** $\forall_{\text{distinct}}\, x,\ y \in S : xRy \implies \neg(yRx)$

**Transitive** $\forall x,\ y,\ z \in S : (xRy \wedge yRz) \implies xRz$

**Definition 3.0.2** (Equivalence relation)**.** A relation is an equivalence relation if it is reflexive, symmetric, and transitive.

**Definition 3.0.3** (Functions as rules)**.** A function $f$ is a rule which maps from some set $S$ (the domain) to some set $T$ (the codomain) (i.e. $f : S \to T$) where each $x \in S$ maps to a unique $f(x) \in T$.

**Definition 3.0.4** (Image)**.** The image of a function $f : S \to T$ is the set $f(S)$, which is equal to $\{f(x) : x \in S\}$.

**Definition 3.0.5** (Functional)**.** A relation is functional if $\forall x \in S \wedge \forall y,\ z \in T : (xRy\ \wedge\ xRz) \implies y = z$.

*Note* 3.0.1. The functional property corresponds to the vertical line test.

**Definition 3.0.6** (Left-total)**.** A relation is left-total if $\forall x \in S : \exists y \in T : xRy$.

**Definition 3.0.7** (Functions as relations)**.** A function $f$ is some relation that is functional and left-total.

## Function Properties

**Injective (one-to-one)** $\forall x,\ z \in S : \forall y \in T : (xRy\ \wedge\ zRy) \implies x = z$

Put in words, every element in the codomain ($T$) is being mapped to by *at most* one element from the domain ($S$).

**Surjective (onto)** $\forall y \in T : \exists x \in S : xRy$

Put in words, every element in the codomain ($T$) is being mapped to by *at least* one element from the domain ($S$).

**Bijective** $R$ is injective and surjective.

**Definition 3.0.8** (Equal functions)**.** Two functions are equal if they have the same domain and codomain for all $x$, so that $f(x) = g(x)$.

**Definition 3.0.9** (Function composition)**.** The composition of two functions $f : X \to Y$ and $g : Y \to Z$, denoted $g \circ f$, maps $X$ to $Z$, and is defined as $(g \circ f)(x) = g(f(x))$.

**Theorem 3.0.1.** *For $f : X \to Y$ and $g : Y \to Z$ and $h : Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$.*

**Definition 3.0.10** (Identity function)**.** The identity function of $X$ is $1_X : X \to X$ and is defined as $1_X(x) = x$ for all $x \in X$.

**Definition 3.0.11** (Inverse function)**.** If $f : X \to Y$ and $g : Y \to X$ are functions where $g \circ f = 1_X$ and $f \circ g = 1_Y$, then $g$ is the inverse of $f$, denoted $f^{-1}$.

**Theorem 3.0.2.** *A function $f$ has an inverse if and only if $f$ is bijective.*

**Definition 3.0.12** (Equivalence class)**.** Let $R$ be an equivalence relation on $S$ and $x \in S$. The equivalence class of $x$ is $[x] = \{y \in S : xRy\}$, i.e. the set of all elements in $S$ that are related to $x$.

$x$ is a representative of the equivalence class $[x]$. There may be different representations of the same equivalence class.

# 4 Number Sets

## 4.1 Rational Numbers

**Definition 4.1.1** (Rational numbers)**.** Define an equivalence relation $\sim$ on $\mathbb{Z} \times (\mathbb{N}\backslash\{0\}$ as $(a,\ b) \sim (c,\ d) \iff ad = bc$. The equivalence classes of $\sim$ are rational numbers. $[(a,\ b)] = \frac{a}{b}$. The set of all rational numbers is $\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}_{>0}\right\}$.

**Definition 4.1.2** (Addition on rational numbers)**.** $(a,\ b) + (c,\ d) = (ad + bc,\ bd)$

**Definition 4.1.3** (Multiplication on rational numbers)**.** $(a,\ b) \cdot (c,\ d) = (ac,\ bd)$

**Theorem 4.1.1.** *There is no smallest positive rational number.*

# 5 Algebraic Structures

### Ring Axioms

**Definition 5.0.1** (Commutativite ring with identity)**.** A commutative ring with identity is a set $S$ with two binary operations $+$ and $\cdot$ such that the following ring axioms are satisfied.

(C1) $+$ is closed: $\forall a,\ b \in S : a + b \in S$.

(A1) $+$ is associative: $\forall a,\ b,\ c \in S : (a + b) + c = a + (b + c)$.

(A2) $+$ is commutative: $\forall a,\ b \in S : a + b = b + a$.

(A3) additive identity: $\forall a \in S : \exists z \in S : a + z = a$.

(A4) additive inverse: $\forall a \in S, \exists b \in S : a + b = z$ (denoted "0").

(C2) $\cdot$ is closed: $\forall a,\ b \in S : a \cdot b \in S$.

(M1) $\cdot$ is associative: $\forall a,\ b,\ c \in S : a \cdot (b \cdot c) = (a \cdot c) \cdot c$.

(M2) $\cdot$ is commutative: $\forall a,\ b \in S : a \cdot b = b \cdot a$.

(M3) multiplicative identity: $\forall a \in S : \exists e \in S : a \cdot e = a$ (denoted "1").

(D) distributivity: $\forall a,\ b,\ c \in S : a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

*Note* 5.0.1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are rings.

**Definition 5.0.2** (Integers)**.** The set of integers ($\mathbb{Z}$) is the smallest ring containing the natural numbers.

*Note* 5.0.2. The integers are the closure of the natural numbers in regards to the ring axioms.

**Definition 5.0.3** (Real numbers)**.** Let $D = \mathbb{Z} \times \{\text{sequence of digits } 0 \dots 9\}$. Define an equivalence relation on $D$ such that

$$x_0.x_1x_2 \dots x_{k-1}x_k\overline{000} = x_0.x_1x_2 \dots x_{k-1}\,(x_k - 1)\,\overline{999}$$

The real numbers ($\mathbb{R}$) is the set of the equivalence classes of D using the equivalence relation above.

**Definition 5.0.4** (Irrational numbers)**.** The irrational numbers ($\mathbb{I}$ or $\overline{\mathbb{Q}}$) are defined as $\mathbb{R}\backslash\mathbb{Q}$.

**Theorem 5.0.1.** *If $a,\ b \in \mathbb{R}$ are such that $ab$ is irrational, then at least one of $a$ or $b$ is irrational.*

*Note* 5.0.3. Let $a,\ b \in \mathbb{R}$. If $ab \in \mathbb{I}$, then either $a \in \mathbb{I}$ or $b \in \mathbb{I}$.

### Field Axioms

**Definition 5.0.5** (Field)**.** A field is a set $S$ which is a commutative ring with identity so that it satisfies all ring axioms and also the following axiom.

(M4) multiplicative inverse: $\forall a \in S\ \{0\} : \exists b \in S : a \cdot b = e$.

*Note* 5.0.4. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields.

# 6 Cardinality

**Definition 6.0.1** (Cardinality)**.** The cardinality of $X$ ($\#X$ or $|X|$), is the number of elements in $X$. If there is a bijection between two sets $X$ and $Y$, then $\#X = \#Y$ or equivalently, $|X| = |Y|$.

**Definition 6.0.2** ($\mathbb{N}_{<n}$)**.** The subset of $\mathbb{N}$ containing all naturals less than $n$.

**Definition 6.0.3** (Finite and infinite sets)**.** If a set $X$ is such that $\#X = \#\mathbb{N}_{<n}$, then the number of elements in $X$ is $n$ and $X$ is a finite set. If there is no $n$ such that there is a bijection between $X$ and $\mathbb{N}_{<n}$, then $X$ is an infinite set.

**Definition 6.0.4** (Countable and uncountable infinities)**.** If there exists a bijection between an infinite set $X$ and $\mathbb{N}$, then $X$ is countably infinite, else uncountably infinite.

**Theorem 6.0.1.** $\#\mathbb{N} = \#\mathbb{Z}$

**Theorem 6.0.2.** $\#(\mathbb{N} \times \mathbb{N}) = \#\mathbb{N}$

**Theorem 6.0.3.** $\#\mathbb{Q} = \#\mathbb{N}$

**Theorem 6.0.4.** $\#\mathbb{N} \neq \#\mathbb{R}$

**Definition 6.0.5** (Power set)**.** The power set of $X$, $\mathcal{P}(X)$, is the set of all subsets of $X$, including $\varnothing$ and $X$.

**Theorem 6.0.5.** *The cardinality of $\mathcal{P}(X)$ is $2^{|X|}$.*

# 7 Proofs

### Proof Structure

1. State the proof method that will be used.
2. If the proof method has conditions show that these conditions have been satisfied before using the proof method.
3. Clearly state any assumptions and definitions as required by the proof.
4. Apply the proof method.
5. Write a sentence summarising the proof method.
6. Conclude the proof with QED, $\square$, or $\blacksquare$.

*Note* 7.0.1 (Disprove). To disprove a statement is to prove its negation.

## 7.1 Direct Proof

1. Use a definition, i.e. even/odd numbers.
2. Show LHS = RHS.
3. For an equivalence ( $\iff$ ) statement, show that both forward ( $\implies$ ) and backward ( $\impliedby$ ) directions are true.

## 7.2 Truth Table, Tautology, and Paradox

1. Draw a truth table for the statement.
2. For a proof by truth table, show that two truth tables are the same.
3. For a proof by tautology, show that the truth table is always true.
4. For a proof by paradox, show that the truth table is always false.

## 7.3 Contradiction

1. Assume that the statement is false.
2. Make a sequence of logical statements that follow from the assumption.
3. Arrive at a contradiction.
4. Since we arrive at a contradiction, the original assumption must have been wrong.
5. Hence the original statement must be true.
6. The ⨳ and $\Longrightarrow\!\Longleftarrow$ symbols are often used to show a contradiction.

## 7.4 Contrapositive

1. Write the original statement as a conditional statement.
2. Rewrite the statement in the contrapositive form.
3. Try to prove the contrapositive statement, usually a direct proof.
4. By proving the contrapositive, you have also proved the original statement.

## 7.5 Contradiction vs. Contrapositive

For the conditional statement: "If $P$, then $Q$".
**By contradiction:** Assume $P$ and $\neg Q$, and find some contradiction.
**By contrapositive:** Assume $\neg Q$, and show $\neg P$.

## 7.6 Induction

1. Find the logical statement $P(n)$ so that the theorem can be written in the form: "Show $P(n)$ is true for all $n \in \mathbb{N}_{\geqslant n_0}$".
2. Prove the base case is true: $P(n)$ is true for $n = n_0$.
3. Write the inductive hypothesis: "Assume $P(n)$ is true for $n = k$".
4. Write the inductive step: "$P(n)$ is true for $n = k + 1$".
5. Prove the inductive step is true, usually a direct proof. Remember to utilise the inductive hypothesis.
6. Conclusion: Since $P(n_0)$ is true, and $P(k+1)$ is true whenever $P(k)$ is true, then the principle of mathematical induction implies $P(n)$ is true for all $n \in \mathbb{N}_{\geqslant n_0}$.

## 7.7 Strong Induction

1. Find the logical statement $P(n)$ so that the theorem can be written in the form: "Show $P(n)$ is true for all $n \in \mathbb{N}_{\geqslant n_0}$".
2. Prove the base case is true: $P(n)$ is true for $n = n_0$.
3. Write the inductive hypothesis: "Assume $P(n)$ is true for $n \leqslant k$".
4. Write the inductive step: "$P(n)$ is true for $n = k + 1$".
5. Prove the inductive step is true, usually a direct proof. Remember to utilise the inductive hypothesis.
6. Conclusion: Since $P(n_0)$ is true, and $P(k+1)$ is true whenever $P(n_0), \dots, P(k)$ is true, then the principle of strong mathematical induction implies $P(n)$ is true for all $n \in \mathbb{N}_{\geqslant n_0}$.

### 7.8 Useful Techniques

1. Use the definition of the numbers, i.e. even and odd numbers.
2. The product of two odd numbers is also odd.
3. For rational numbers assume that $(a, b)$ is fully simplified, so that $a$ and $b$ are co-prime.
4. When using strong induction, remember that $P(k), \ldots, P(n_0)$ are assumed to be true.
5. $a^b = e^{\ln a^b} = e^{b \ln a}$
6. $a^k = a^k + b - b$
7. $a^{k+1} = a \cdot a^k$
8. $\sum_i^{k+1} a_i = a_{k+1} + \sum_i^k a_i$

## 8 Sequences

An infinite sequence on $X$ is a function $a$ with domain $\mathbb{N}_{>0}$ and codomain $X$. That is, $a : \mathbb{N}_{>0} \to X$.

For infinite sequences we write $a_n$ and the sequence can be represented as $\{a_n\}_{n=1}^{\infty}$, where $n$ is the index of the sequence.

### 8.1 Convergence

**Theorem 8.1.1.** *A sequence $\{a_n\}_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$ if*

$$\lim_{n\to\infty} a_n = a \iff \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}_{>0} : \forall n \geqslant n_0 : |a_n - a| < \varepsilon$$

**Theorem 8.1.2.** *A sequence $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$ if*

$$\lim_{n\to\infty} a_n = +\infty \iff \forall N \in \mathbb{R} : \exists n_0 \in \mathbb{N}_{>0} : \forall n \geqslant n_0 : a_n > N$$

**Theorem 8.1.3.** *A sequence $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$ if*

$$\lim_{n\to\infty} a_n = -\infty \iff \forall M \in \mathbb{R} : \exists n_0 \in \mathbb{N}_{>0} : \forall n \geqslant n_0 : a_n < M$$

**Theorem 8.1.4** (Triangle inequality)**.** $|a + b| \leqslant |a| + |b|$.

**Theorem 8.1.5.** *The sequence $\{a_n\}_{n=1}^{\infty}$ converges to $a$ if and only if the sequences $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n-1}\}_{n=1}^{\infty}$ both converge to $a$.*

**Theorem 8.1.6** (Squeeze theorem for sequences)**.** *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be infinite sequences such that $a_n \leqslant b_n \leqslant c_n$ for all $n > n_0$. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = d$, then .*

#### Limit Laws for Sequences

**Theorem 8.1.7.** *Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent infinite sequences on $\mathbb{R}$ that converge to $a$ and $b$ respectively, and let $c \in \mathbb{R}$ be a constant, then*

a) $\lim_{n\to\infty} c = c$.

b) $\lim_{n\to\infty} ca_n = ca$.

c) $\lim_{n\to\infty} (a_n + b_n) = a + b$.

d) $\lim_{n\to\infty} (a_n - b_n) = a - b$.

e) $\lim_{n\to\infty} (a_n b_n) = ab$.

f) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, $(b \neq 0)$.

## 8.2 Cauchy Sequences

**Definition 8.2.1** (Cauchy sequence)**.** A sequence $\{a_n\}_{n=1}^{\infty}$ on $\mathbb{R}$ is a Cauchy sequence if

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}_{>0} : \forall n,\ m \geqslant n_0 : |a_n - a_m| < \varepsilon$$

**Theorem 8.2.1.** *A sequence on $\mathbb{R}$ is convergent if and only if is a Cauchy sequence.*

## 8.3 Monotonicity

**Definition 8.3.1** (Monotone)**.** A sequence is (strictly) monotone if it is (strictly) increasing or (strictly) decreasing.

**Definition 8.3.2** (Eventually)**.** If after removing a finite number of terms from the beginning from a sequence it has a certain property, the original sequence has that property eventually.

**Definition 8.3.3** (Bounded)**.** An infinite sequence is bounded if $L \leqslant a_n \leqslant K$ for all $n$.

**Theorem 8.3.1.** *Every bounded eventually monotone sequence converges.*

## 8.4 Monotonicity Tests

### Difference Test

**Theorem 8.4.1** (Difference test)**.** *A sequence $\{a_n\}_{n=1}^{\infty}$ is*

**Strictly increasing** if $a_{n+1} - a_n > 0$ for all $n$

**Increasing** if $a_{n+1} - a_n \geqslant 0$ for all $n$

**Strictly decreasing** if $a_{n+1} - a_n < 0$ for all $n$

**Decreasing** if $a_{n+1} - a_n \leqslant 0$ for all $n$

### Ratio Test

**Theorem 8.4.2** (Ratio test)**.** *A sequence $\{a_n\}_{n=1}^{\infty}$ where all $a_n$ are positive is*

**Strictly increasing** if $a_{n+1} \div a_n > 1$ for all $n$

**Increasing** if $a_{n+1} \div a_n \geqslant 1$ for all $n$

**Strictly decreasing** if $a_{n+1} \div a_n < 1$ for all $n$

**Decreasing** if $a_{n+1} \div a_n \leqslant 1$ for all $n$

*Note* 8.4.1. If $\lim_{n\to\infty} a_{n+1} \div a_n < 1$, then $a_n \to 0$ as $n \to \infty$.

### Derivative Test

**Theorem 8.4.3** (Derivative test)**.** *If $\{a_n\}_{n=1}^{\infty}$ is an infinite sequence on $\mathbb{R}$ and $a(x)$ is differentiable for $x \in \mathbb{R}_{\geqslant 1}$, then $\{a_n\}_{n=1}^{\infty}$ is*

**Strictly increasing** if $a'(x) > 0$ for all $x \in \mathbb{R}_{\geqslant 1}$

**Increasing** if $a'(x) \geqslant 0$ for all $x \in \mathbb{R}_{\geqslant 1}$

**Strictly decreasing** if $a'(x) < 0$ for all $x \in \mathbb{R}_{\geqslant 1}$

**Decreasing** if $a'(x) \leqslant 0$ for all $x \in \mathbb{R}_{\geqslant 1}$

# 9 Series

## 9.1 Infinite Series

**Definition 9.1.1** (Infinite Series)**.** An infinite series is an expansion that can be written as

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$$

The numbers $a_i$ are the terms of the series.

**Definition 9.1.2** (Partial Sums)**.** The sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ is defined as

$$s_n = \sum_{i=1}^{n} a_i$$

where $s_n$ is the $n$th partial sum of the infinite series.

**Definition 9.1.3** (Convergence of series)**.** The infinite series $\sum_{i=1}^{\infty} a_i$ converges to $A$ if the associated sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ converges to $A$.

**Definition 9.1.4** (Divergence of series)**.** The infinite series $\sum_{i=1}^{\infty} a_i$ diverges to $\pm\infty$, if the associated sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ diverges to $\pm\infty$.

**Theorem 9.1.1.** *If $\sum_{i=1}^{\infty} a_i$ converges, then* $\lim_{i\to\infty} a_i = 0$.

**Theorem 9.1.2.** *Similar to sequences, we can add and subtract convergent infinite series.*

1. Suppose $\sum a_i$ and $\sum b_i$ are both convergent infinite series with the same starting index. Then $\sum a_i \pm b_i$ is also convergent.
2. Suppose $\sum c_i$ is a convergent infinite series and $\sum d_i$ is a divergent infinite series with the same starting index. Then $\sum c_i \pm d_i$ is divergent.
3. Let $\sum e_i$ be an infinite series and let $d \in \mathbb{R}$ be a constant, and $K \in \mathbb{N}$.
   (a) $\sum e_i$ and $\sum de_i$ both converge or both diverge.
   (b) $\sum_{i=i_0}^{\infty} e_i$ and $\sum_{i=K}^{\infty} e_i$ both converge or both diverge.

**Definition 9.1.5** (Absolute Convergence)**.** An infinite series $\sum a_i$ converges (or diverges) absolutely if the series of absolute values $\sum a_i$ also converges (or diverges).

**Theorem 9.1.3.** *If an infinite series is absolutely convergent, then it is also convergent.*

**Definition 9.1.6** (Conditional Convergence)**.** A convergent infinite series that diverges absolutely, converges conditionally.

## 9.2 Convergence Tests

For any infinite series of the form $\sum_{i=i_0}^{\infty} a_i$.

### Divergence Test

$$\text{Does } \lim_{i\to\infty} a_i \neq 0? \begin{cases} \text{YES} & \sum a_i \text{ Diverges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

### $p$-Series

**Conditions** $i_0 = 1$ and $a_i = \frac{1}{i^p}$.

$$\text{Is } p > 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

### Geometric Series

**Conditions** $i_0 = 0$ and $a_i = ar^i$ or $i_0 = 1$ and $a_i = ar^{i-1}$.

$$\text{Is } |r| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges to } \frac{a}{1-r} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

*Note* 9.2.1. The sequence of partial sums is given by $s_n = a\dfrac{1-r^{n+1}}{1-r}$.

### Alternating Series

**Conditions** $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$\text{Is } b_{i+1} \leqslant b_i \ \& \lim_{i\to\infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

### Telescoping Series

**Conditions** Subsequent terms cancel out previous terms in the sum.

$$\text{Does } \lim_{n\to\infty} s_n = s? \begin{cases} \text{YES} & \sum a_i \text{ Converges to } s \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

### Limit Comparison Test

**Conditions** Pick $b_i$ so that $\lim_{i\to\infty} \dfrac{a_i}{b_i} = c > 0$ & $a_i,\, b_n > 0$.

$$\text{Does } \sum_{i=i_0}^{\infty} b_i \text{ converge?} \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

### Comparison Test

**Conditions** Pick $b_i$.

If $\sum_{i=i_0}^{\infty} b_i$ converges:

$$\text{Is } 0 \leqslant a_i \leqslant b_i? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

If $\sum_{i=i_0}^{\infty} b_i$ diverges:

$$\text{Is } 0 \leqslant b_i \leqslant a_i? \begin{cases} \text{YES} & \sum a_i \text{ Diverges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

### Integral Test

**Conditions** Let $a_i = f(i)$, so that $f(x)$ is continuous, positive and decreasing on $[a, \infty)$.

$$\int_a^{\infty} f(x)\,\mathrm{d}x \text{ converges?} \begin{cases} \text{YES} & \sum_{i=a}^{\infty} a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

### Ratio Test

**Conditions** $\forall i : a_i > 0$ and $\lim_{i\to\infty} \dfrac{a_{i+1}}{a_i} \neq 1$.

$$\text{Is } \lim_{i\to\infty} \frac{a_{i+1}}{a_i} < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

$$\text{Is } \lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges Absolutely} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

### Root Test

**Conditions** $\forall i : a_i > 0$ and $\lim_{i\to\infty} \sqrt[i]{a_i} \neq 1$.

$$\text{Is } \lim_{i\to\infty} \sqrt[i]{a_i} < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

$$\text{Is } \lim_{i\to\infty} \sqrt[i]{|a_i|} < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges Absolutely} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

# 10 Limits of Functions

## 10.1 Limits of Functions on $\mathbb{R}$

**Definition 10.1.1** (Finite limits using the $\varepsilon$-$\delta$ definition)**.** Let a function $f(x)$ be defined for all $x$ in an open interval $I$ (which contains $x_0 \in \mathbb{R}$), except for $x_0$ which may or may not be defined.

$$\lim_{x\to x_0} f(x) = L \iff \forall\varepsilon > 0 : \exists\delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

### Limit Laws for Functions

**Theorem 10.1.1.** *The limit of a sum equals the sum of the limits, i.e.*

$$\lim_{x\to x_0} (f(x) + g(x)) = \left(\lim_{x\to x_0} f(x)\right) + \left(\lim_{x\to x_0} g(x)\right)$$

**Theorem 10.1.2.** *The limit of a product equals the product of the limits, i.e.*

$$\lim_{x\to x_0} (f(x) \cdot g(x)) = \left(\lim_{x\to x_0} f(x)\right) \cdot \left(\lim_{x\to x_0} g(x)\right)$$

These are trivially extended to division (if the divisor is non-zero), subtraction, exponentiation, and moving constants in or outside of a limit, i.e.

$$\lim_{x\to x_0} (f(x) - g(x)) = \left(\lim_{x\to x_0} f(x)\right) - \left(\lim_{x\to x_0} g(x)\right) \qquad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$$

$$\lim_{x\to x_0} (f(x))^n = \left(\lim_{x\to x_0} f(x)\right)^n \qquad \lim_{x\to x_0} c \cdot f(x) = c \cdot \lim_{x\to x_0} f(x)$$

**Definition 10.1.2** (Limits towards $\pm\infty$ using the $\varepsilon$-$\delta$ definition)**.** Let a function $f(x)$ be defined for all $x$ in an open interval $I$ (which contains $x_0 \in \mathbb{R}$), except for $x_0$ which may or may not be defined.

$$\lim_{x\to x_0} f(x) = +\infty \iff \forall M \in \mathbb{R} : \exists\delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies f(x) > M$$

$$\lim_{x\to x_0} f(x) = -\infty \iff \forall N \in \mathbb{R} : \exists\delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies f(x) < N$$

**Definition 10.1.3** (Limits towards $\pm\infty$)**.** Let a function $f(x)$ be defined for all $x$ in an infinite open interval $I$ extending in the positive $x$ direction.

$$\lim_{x\to\infty} f(x) = L \iff \forall\varepsilon > 0 : \exists M \in I : x > M \implies |f(x) - L| < \varepsilon$$

Let a function $g(x)$ be defined for all $x$ in an infinite open interval $I$ extending in the negative $x$ direction.

$$\lim_{x\to-\infty} g(x) = L \iff \forall\varepsilon > 0 : \exists N \in I : x < N \implies |g(x) - L| < \varepsilon$$

**Theorem 10.1.3.** $\lim_{x\to x_0} f(x)$ *exists if and only if* $\lim_{x\to {x_0}^+} f(x)$ *and* $\lim_{x\to {x_0}^-} f(x)$ *exist and are equal.*

**Theorem 10.1.4** (Squeeze theorem for functions)**.** *Let an interval $I$ contain $x_0$. If $f,\ g,\ h : I \to \mathbb{R}$ are functions such that $\forall x \in I\backslash\{x_0\},\ f(x) \leqslant g(x) \leqslant h(x)$ and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$, then $\lim_{x\to x_0} g(x) = L$.*

*Note* 10.1.1. The squeeze theorem holds for left- and right-handed and limits to $\pm\infty$.

**Definition 10.1.4** (Function continuity using the $\varepsilon$-$\delta$ definition)**.** A function $f(x) : I \to \mathbb{R}$ is continuous at $c \in I$, if

$$\forall\varepsilon > 0 : \exists\delta > 0 : \forall x \in I : 0 < |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

$f(x)$ is continuous on $I$ if $f(x)$ is continuous for all $x \in I$.

**Definition 10.1.5** (Function continuity as a sequence)**.** A function $f(x) : I \to \mathbb{R}$ is continuous at $c \in I$, if for every sequence $\{a_n\}_{n=1}^{\infty}$ in $I$ that converges to $c$, $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(c)$.

*Note* 10.1.2. "$f(x)$ is continuous at $c$" is equivalent to "$\lim\limits_{x \to c} f(x) = f(c)$".

*Note* 10.1.3. Polynomials are continuous everywhere.

*Note* 10.1.4. sin and cos are continuous everywhere.

**Theorem 10.1.5.** *For a function $f$ continuous at $L$ and another function $g$ where $\lim\limits_{x \to c} g(x) = L$, $\lim\limits_{x \to c}(f \circ g)(x) = f(L)$.*

*Note* 10.1.5. The above theorem still applies when $c$ is $c^{\pm}$ or $\pm\infty$.

**Theorem 10.1.6.** *For two functions $f$ and $g$ that are continuous at $c$,*

a) *$f(x) + g(x)$ is continuous at $c$;*

b) *$f(x) - g(x)$ is continuous at $c$;*

c) *$f(x) \cdot g(x)$ is continuous at $c$; and*

d) *$\dfrac{f(x)}{g(x)}$ is continuous at $c$ when $g(c) \neq 0$.*

**Theorem 10.1.7.** *For a function $g(x)$ which is continuous at $c$ and a function $h(x)$ which is continuous at $g(c)$, $h \circ g$ is continuous at $c$.*

**Theorem 10.1.8** (L'Hôpital rule)**.** *For two differentiable (and therefore continuous) functions $f(x)$ and $g(x)$ except possibly at $x_0$. If $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$, or $\lim\limits_{x \to x_0} f(x) = \pm\infty$ and $\lim\limits_{x \to x_0} g(x) = \pm\infty$, then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ (as long as the limit exists, or diverges to $\pm\infty$).*

*Note* 10.1.6. L'Hôpital's rule also holds for left- and right-handed, and limits that approach $\pm\infty$.

# 11 Addenda

## 11.1 Assumed Knowledge

### 11.1.1 Factorials

$$n! = \prod_{i=1}^{n} i = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$$

### 11.1.2 Quadratic Equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## 11.2 Table of Derivatives

Let $a \in \mathbb{R}$ be a constant, and $n \in \mathbb{Z} \backslash \{0\}$.

| $f(x)$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $a$ | $0$ |
| $x$ | $1$ |
| $x^n$ | $nx^{n-1}$ |
| $e^x$ | $e^x$ |
| $e^{nx}$ | $ne^{nx}$ |
| $e^{g(x)}$ | $g'(x)e^{g(x)}$ |
| $a^x$ | $a^x \ln(a)$ |

| $f(x)$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $\ln(x)$ | $\frac{1}{x}$ |
| $\log_a(x)$ | $\frac{1}{x\ln(a)}$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| $\sqrt[n]{x}$ | $\frac{x^{\frac{1}{n}-1}}{n}$ |

| $f(x)$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\tan(x)$ | $\frac{1}{\cos^2(x)}$ |