Abstract Mathematical Reasoning

Exam Notes

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Preface

This document is a succinct reference for the unit MXB102. We hope this document is useful, particularly to those undertaking this unit's final exam. This document is not intended to be a replacement for the lecture notes and unit guide. We would personally like to thank our lecturer, Dr Adrianne Jenner, and all of the tutors for the great amount of support and teachings that we have been able to receive over the course of this unit.

1   Logic

Name	Symbol	Truth Table

		P	$\neg P$
Negation (NOT)	$\neg$	T	F
		F	T

		P	Q	$P \wedge Q$
Conjunction (AND)	$\wedge$	T	T	T
		T	F	F
		F	T	F
		F	F	F

		P	Q	$P \vee Q$
Disjunction (OR)	$\vee$	T	T	T
		T	F	T
		F	T	T
		F	F	F

		P	Q	$P \implies Q$
Conditional (IF-THEN)	$\implies$	T	T	T
		T	F	F
		F	T	T
		F	F	T

		P	Q	$P \iff Q$
Equivalence (IFF)	$\iff$ or $\equiv$	T	T	T
		T	F	F
		F	T	F
		F	F	T

Table 1: Boolean operators in order of precedence.

Theorem 1.0.1 (de Morgan's Law). $\neg(P \wedge Q) \iff \neg P \vee \neg Q$.

Definition 1.0.1 (Tautology). A tautology is a statement that is always true.

Definition 1.0.2 (Paradox). A paradox is a statement that is always false.

Definition 1.0.3 (Converse). The converse of the statement $P \implies Q$ is the statement $Q \implies P$.

Definition 1.0.4 (Contrapositive). The contrapositive of $P \implies Q$ is the statement $\neg Q \implies \neg P$.

2   Sets

Name	Symbol	Definition
Empty set	$\varnothing$	A set with no elements: $\{\}$
Intersection	$\cap$	$S \cap T = \{x : x \in S \wedge x \in T\}$
Union	$\cup$	$S \cup T = \{x : x \in S \vee x \in T\}$
Set difference	$\backslash$ or $-$	$S - T = \{x : x \in S \wedge x \notin T\}$
Subset	$\subseteq$	If $\#S = \#T, \quad S \subseteq T \equiv \forall x \in S : x \in T$
Strict (or proper) subset	$\subset$	If $\#S < \#T, \quad S \subset T \equiv \forall x \in S : x \in T$
Complement	$\overline{S}$	If $S \subseteq T, \quad \overline{S} = T \backslash S$

Definition 2.0.1 (Disjoint). If $S \cap T = \varnothing$, then S and T are disjoint (i.e. two sets are disjoint if their intersection is empty).

2.1 Construction of the Natural Numbers

Definition 2.1.1 ("0"). "0" is defined as the empty set, $\varnothing$ or $\{\}$, which is the first element of the natural numbers.

Definition 2.1.2 (Successor function). The successor function, $S(n)$, is defined as $n \cup \{n\}$.

Definition 2.1.3 (Addition). Addition ($+$) is defined recursively by the following axioms:

(A1) "a" + "0" = "a"

(A2) "a" + S("b") = S("a" + "b")

Definition 2.1.4 (Multiplication). Multiplication ($\cdot$) is defined recursively by the following axioms:

(M1) "a" $\cdot$ "0" = "0"

(M2) "a" $\cdot$ S("b") = "a" + ("a" $\cdot$ "b")

Definition 2.1.5 (Exponentiation). Exponentiation ($\uparrow$) is defined recursively by the following axioms:

(E1) "a" $\uparrow$ "0" = "1"

(E2) "a" $\uparrow$ S("b") = "a" $\cdot$ ("a" $\uparrow$ "b")

2.1.1 Properties of Natural Numbers

Commutativity of addition "a" + "b" = "b" + "a"

Associativity of addition ("a" + "b") + "c" = "a" + ("b" + "c")

Commutativity of multiplication "a" $\cdot$ "b" = "b" $\cdot$ "a"

Associativity of multiplication ("a" $\cdot$ "b") $\cdot$ "c" = "a" $\cdot$ ("b" $\cdot$ "c")

Distributivity of multiplication over addition "a" $\cdot$ ("b" + "c") = ("a" $\cdot$ "b") + ("a" $\cdot$ "c")

Cancellation law If "a" $\cdot$ "b" = "a" $\cdot$ "c" and "a" $\neq$ "0" then "b" = "c"

Theorem 2.1.1. $\forall n \in \mathbb{N} \backslash \{0\} : 2^n > n$.

Theorem 2.1.2. $\forall n \in \mathbb{N} \backslash \{0\} : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

2.2 Cartesian Product

Definition 2.2.1 (Cartesian product). The Cartesian product of two sets S and T is the set $S \times T$ consisting of all ordered pairs (x, y) where $x \in S$ and $y \in T$.

2.3 Quantifiers

Name	Symbol	Usage
Universal Quantification (For All)	$\forall$	$\forall x : P(x)$
Existential Quantification (There Exists)	$\exists$	$\exists x : P(x)$

Theorem 2.3.1. $\neg(\forall x : P(x)) \equiv \exists x : \neg P(x)$

Theorem 2.3.2. $\neg(\exists x : P(x)) \equiv \forall x : \neg P(x)$

Definition 2.3.1 (Greater than ($>$)). $x > y \iff \exists a \in \mathbb{N} \backslash \{0\} : x = y + a$

Definition 2.3.2 (Greater than or equal to ($\geqslant$)). $x \geqslant y \iff \exists a \in \mathbb{N} : x = y + a$

3   Relations

Definition 3.0.1 (Binary relation)**.** A binary relation R on S is the subset $S \times S$. If a, $b \in S$ and $(a, b) \in R$, then aRb ("a is related to b under R")

> ## Relation Properties
>
> **Reflexive** $\forall x \in S : xRx$
>
> **Symmetric** $\forall x, y \in S : xRy \implies yRx$
>
> **Antisymmetric** $\forall_{\text{distinct}} x, y \in S : xRy \implies \neg(yRx)$
>
> **Transitive** $\forall x, y, z \in S : (xRy \wedge yRz) \implies xRz$
>
> **Definition 3.0.2** (Equivalence relation)**.** A relation is an equivalence relation if it is reflexive, symmetric, and transitive.

Definition 3.0.3 (Functions as rules)**.** A function f is a rule which maps from some set S (the domain) to some set T (the codomain) (i.e. $f : S \to T$) where each $x \in S$ maps to a unique $f(x) \in T$.

Definition 3.0.4 (Image)**.** The image of a function $f : S \to T$ is the set $f(S)$, which is equal to $\{f(x) : x \in S\}$.

Definition 3.0.5 (Functional)**.** A relation is functional if $\forall x \in S \wedge \forall y, z \in T : (xRy \wedge xRz) \implies y = z$.

Note 3.0.1. The functional property corresponds to the vertical line test.

Definition 3.0.6 (Left-total)**.** A relation is left-total if $\forall x \in S : \exists y \in T : xRy$.

Definition 3.0.7 (Functions as relations)**.** A function f is some relation that is functional and left-total.

> ## Function Properties
>
> **Injective (one-to-one)** $\forall x, z \in S : \forall y \in T : (xRy \wedge zRy) \implies x = z$
>
> > Put in words, every element in the codomain (T) is being mapped to by *at most* one element from the domain (S).
>
> **Surjective (onto)** $\forall y \in T : \exists x \in S : xRy$
>
> > Put in words, every element in the codomain (T) is being mapped to by *at least* one element from the domain (S).
>
> **Bijective** R is injective and surjective.

Definition 3.0.8 (Equal functions)**.** Two functions are equal if they have the same domain and codomain for all x, so that $f(x) = g(x)$.

Definition 3.0.9 (Function composition)**.** The composition of two functions $f : X \to Y$ and $g : Y \to Z$, denoted $g \circ f$, maps X to Z, and is defined as $(g \circ f)(x) = g(f(x))$.

Theorem 3.0.1. *For $f : X \to Y$ and $g : Y \to Z$ and $h : Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$.*

Definition 3.0.10 (Identity function)**.** The identity function of X is $1_X : X \to X$ and is defined as $1_X(x) = x$ for all $x \in X$.

Definition 3.0.11 (Inverse function)**.** If $f : X \to Y$ and $g : Y \to X$ are functions where $g \circ f = 1_X$ and $f \circ g = 1_Y$, then g is the inverse of f, denoted f^{-1}.

Theorem 3.0.2. *A function f has an inverse if and only if f is bijective.*

Definition 3.0.12 (Equivalence class)**.** Let R be an equivalence relation on S and $x \in S$. The equivalence class of x is $[x] = \{y \in S : xRy\}$, i.e. the set of all elements in S that are related to x.

x is a representative of the equivalence class $[x]$. There may be different representations of the same equivalence class.

4 Number Sets

4.1 Rational Numbers

Definition 4.1.1 (Rational numbers)**.** Define an equivalence relation $\sim$ on $\mathbb{Z} \times (\mathbb{N}\backslash\{0\}$ as $(a,\, b) \sim (c,\, d) \iff ad = bc$. The equivalence classes of $\sim$ are rational numbers. $[(a,\, b)] = \frac{a}{b}$. The set of all rational numbers is $\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}_{>0} \right\}$.

Definition 4.1.2 (Addition on rational numbers)**.** $(a,\, b) + (c,\, d) = (ad + bc,\, bd)$

Definition 4.1.3 (Multiplication on rational numbers)**.** $(a,\, b) \cdot (c,\, d) = (ac,\, bd)$

Theorem 4.1.1. *There is no smallest positive rational number.*

5 Algebraic Structures

Ring Axioms

Definition 5.0.1 (Commutativite ring with identity)**.** A commutative ring with identity is a set S with two binary operations $+$ and $\cdot$ such that the following ring axioms are satisfied.

(C1) $+$ is closed: $\forall a,\, b \in S : a + b \in S$.

(A1) $+$ is associative: $\forall a,\, b,\, c \in S : (a + b) + c = a + (b + c)$.

(A2) $+$ is commutative: $\forall a,\, b \in S : a + b = b + a$.

(A3) additive identity: $\forall a \in S : \exists z \in S : a + z = a$.

(A4) additive inverse: $\forall a \in S, \exists b \in S : a + b = z$ (denoted "0").

(C2) $\cdot$ is closed: $\forall a,\, b \in S : a \cdot b \in S$.

(M1) $\cdot$ is associative: $\forall a,\, b,\, c \in S : a \cdot (b \cdot c) = (a \cdot c) \cdot c$.

(M2) $\cdot$ is commutative: $\forall a,\, b \in S : a \cdot b = b \cdot a$.

(M3) multiplicative identity: $\forall a \in S : \exists e \in S : a \cdot e = a$ (denoted "1").

(D) distributivity: $\forall a,\, b,\, c \in S : a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Note 5.0.1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are rings.

Definition 5.0.2 (Integers)**.** The set of integers ($\mathbb{Z}$) is the smallest ring containing the natural numbers.

Note 5.0.2. The integers are the closure of the natural numbers in regards to the ring axioms.

Definition 5.0.3 (Real numbers)**.** Let $D = \mathbb{Z} \times \{$sequence of digits $0 \dots 9\}$. Define an equivalence relation on D such that

$$x_0.x_1 x_2 \dots x_{k-1} x_k \overline{000} = x_0.x_1 x_2 \dots x_{k-1} \left(x_k - 1\right) \overline{999}$$

The real numbers ($\mathbb{R}$) is the set of the equivalence classes of D using the equivalence relation above.

Definition 5.0.4 (Irrational numbers)**.** The irrational numbers ($\mathbb{I}$ or $\overline{\mathbb{Q}}$) are defined as $\mathbb{R}\backslash\mathbb{Q}$.

Theorem 5.0.1. *If $a,\, b \in \mathbb{R}$ are such that ab is irrational, then at least one of a or b is irrational.*

Note 5.0.3. Let $a,\, b \in \mathbb{R}$. If $ab \in \mathbb{I}$, then either $a \in \mathbb{I}$ or $b \in \mathbb{I}$.

Field Axioms

Definition 5.0.5 (Field)**.** A field is a set S which is a commutative ring with identity so that it satisfies all ring axioms and also the following axiom.

(M4) multiplicative inverse: $\forall a \in S \{0\} : \exists b \in S : a \cdot b = e$.

Note 5.0.4. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields.

6 Cardinality

Definition 6.0.1 (Cardinality)**.** The cardinality of X ($\#X$ or $|X|$), is the number of elements in X. If there is a bijection between two sets X and Y, then $\#X = \#Y$ or equivalently, $|X| = |Y|$.

Definition 6.0.2 ($\mathbb{N}_{<n}$)**.** The subset of $\mathbb{N}$ containing all naturals less than n.

Definition 6.0.3 (Finite and infinite sets)**.** If a set X is such that $\#X = \#\mathbb{N}_{<n}$, then the number of elements in X is n and X is a finite set. If there is no n such that there is a bijection between X and $\mathbb{N}_{<n}$, then X is an infinite set.

Definition 6.0.4 (Countable and uncountable infinities)**.** If there exists a bijection between an infinite set X and $\mathbb{N}$, then X is countably infinite, else uncountably infinite.

Theorem 6.0.1. $\#\mathbb{N} = \#\mathbb{Z}$

Theorem 6.0.2. $\#(\mathbb{N} \times \mathbb{N}) = \#\mathbb{N}$

Theorem 6.0.3. $\#\mathbb{Q} = \#\mathbb{N}$

Theorem 6.0.4. $\#\mathbb{N} \neq \#\mathbb{R}$

Definition 6.0.5 (Power set)**.** The power set of X, $\mathcal{P}(X)$, is the set of all subsets of X, including $\varnothing$ and X.

Theorem 6.0.5. *The cardinality of $\mathcal{P}(X)$ is $2^{|X|}$.*

7 Proofs

> **Proof Structure**
>
> 1. State the proof method that will be used.
>
> 2. If the proof method has conditions show that these conditions have been satisfied before using the proof method.
>
> 3. Clearly state any assumptions and definitions as required by the proof.
>
> 4. Apply the proof method.
>
> 5. Write a sentence summarising the proof method.
>
> 6. Conclude the proof with QED, $\square$, or $\blacksquare$.

Note 7.0.1 (Disprove). To disprove a statement is to prove its negation.

7.1 Direct Proof

1. Use a definition, i.e. even/odd numbers.

2. Show LHS = RHS.

3. For an equivalence ($\iff$) statement, show that both forward ($\implies$) and backward ($\impliedby$) directions are true.

7.2 Truth Table, Tautology, and Paradox

1. Draw a truth table for the statement.

2. For a proof by truth table, show that two truth tables are the same.

3. For a proof by tautology, show that the truth table is always true.

4. For a proof by paradox, show that the truth table is always false.

7.3 Contradiction

1. Assume that the statement is false.

2. Make a sequence of logical statements that follow from the assumption.

3. Arrive at a contradiction.

4. Since we arrive at a contradiction, the original assumption must have been wrong.

5. Hence the original statement must be true.

6. The $\lightning$ and $\Longrightarrow\!\!\!\Longleftarrow$ symbols are often used to show a contradiction.

7.4   Contrapositive

1. Write the original statement as a conditional statement.

2. Rewrite the statement in the contrapositive form.

3. Try to prove the contrapositive statement, usually a direct proof.

4. By proving the contrapositive, you have also proved the original statement.

7.5   Contradiction vs. Contrapositive

For the conditional statement: "If P, then Q".
  By contradiction: Assume P and $\neg Q$, and find some contradiction.
  By contrapositive: Assume $\neg Q$, and show $\neg P$.

7.6   Induction

1. Find the logical statement $P(n)$ so that the theorem can be written in the form: "Show $P(n)$ is true for all $n \in \mathbb{N}_{\geqslant n_0}$".

2. Prove the base case is true: $P(n)$ is true for $n = n_0$.

3. Write the inductive hypothesis: "Assume $P(n)$ is true for $n = k$".

4. Write the inductive step: "$P(n)$ is true for $n = k + 1$".

5. Prove the inductive step is true, usually a direct proof. Remember to utilise the inductive hypothesis.

6. Conclusion: Since $P(n_0)$ is true, and $P(k + 1)$ is true whenever $P(k)$ is true, then the principle of mathematical induction implies $P(n)$ is true for all $n \in \mathbb{N}_{\geqslant n_0}$.

7.7   Strong Induction

1. Find the logical statement $P(n)$ so that the theorem can be written in the form: "Show $P(n)$ is true for all $n \in \mathbb{N}_{\geqslant n_0}$".

2. Prove the base case is true: $P(n)$ is true for $n = n_0$.

3. Write the inductive hypothesis: "Assume $P(n)$ is true for $n \leqslant k$".

4. Write the inductive step: "$P(n)$ is true for $n = k + 1$".

5. Prove the inductive step is true, usually a direct proof. Remember to utilise the inductive hypothesis.

6. Conclusion: Since $P(n_0)$ is true, and $P(k + 1)$ is true whenever $P(n_0)$, ..., $P(k)$ is true, then the principle of strong mathematical induction implies $P(n)$ is true for all $n \in \mathbb{N}_{\geqslant n_0}$.

7.8 Useful Techniques

1. Use the definition of the numbers, i.e. even and odd numbers.

2. The product of two odd numbers is also odd.

3. For rational numbers assume that (a, b) is fully simplified, so that a and b are co-prime.

4. When using strong induction, remember that $P(k), \dots, P(n_0)$ are assumed to be true.

5. $a^b = e^{\ln a^b} = e^{b \ln a}$

6. $a^k = a^k + b - b$

7. $a^{k+1} = a \cdot a^k$

8. $\sum_i^{k+1} a_i = a_{k+1} + \sum_i^k a_i$

8 Sequences

An infinite sequence on X is a function a with domain $\mathbb{N}_{>0}$ and codomain X. That is, $a : \mathbb{N}_{>0} \to X$.

For infinite sequences we write a_n and the sequence can be represented as $\{a_n\}_{n=1}^{\infty}$, where n is the index of the sequence.

8.1 Convergence

Theorem 8.1.1. *A sequence $\{a_n\}_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$ if*

$$\lim_{n \to \infty} a_n = a \iff \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}_{>0} : \forall n \geqslant n_0 : |a_n - a| < \varepsilon$$

Theorem 8.1.2. *A sequence $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$ if*

$$\lim_{n \to \infty} a_n = +\infty \iff \forall N \in \mathbb{R} : \exists n_0 \in \mathbb{N}_{>0} : \forall n \geqslant n_0 : a_n > N$$

Theorem 8.1.3. *A sequence $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$ if*

$$\lim_{n \to \infty} a_n = -\infty \iff \forall M \in \mathbb{R} : \exists n_0 \in \mathbb{N}_{>0} : \forall n \geqslant n_0 : a_n < M$$

Theorem 8.1.4 (Triangle inequality). $|a + b| \leqslant |a| + |b|$.

Theorem 8.1.5. *The sequence $\{a_n\}_{n=1}^{\infty}$ converges to a if and only if the sequences $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n-1}\}_{n=1}^{\infty}$ both converge to a.*

Theorem 8.1.6 (Squeeze theorem for sequences). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be infinite sequences such that $a_n \leqslant b_n \leqslant c_n$ for all $n > n_0$. If $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = d$, then .*

Limit Laws for Sequences

Theorem 8.1.7. *Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent infinite sequences on $\mathbb{R}$ that converge to a and b respectively, and let $c \in \mathbb{R}$ be a constant, then*

a) $\lim_{n \to \infty} c = c$.

b) $\lim_{n \to \infty} ca_n = ca$.

c) $\lim_{n \to \infty} (a_n + b_n) = a + b$.

d) $\lim_{n \to \infty} (a_n - b_n) = a - b$.

e) $\lim_{n \to \infty} (a_n b_n) = ab$.

f) $\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = \frac{a}{b}$, ($b \neq 0$).

8.2 Cauchy Sequences

Definition 8.2.1 (Cauchy sequence). *A sequence $\{a_n\}_{n=1}^{\infty}$ on $\mathbb{R}$ is a Cauchy sequence if*

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}_{>0} : \forall n, m \geqslant n_0 : |a_n - a_m| < \varepsilon$$

Theorem 8.2.1. *A sequence on $\mathbb{R}$ is convergent if and only if is a Cauchy sequence.*

8.3 Monotonicity

Definition 8.3.1 (Monotone)**.** A sequence is (strictly) monotone if it is (strictly) increasing or (strictly) decreasing.

Definition 8.3.2 (Eventually)**.** If after removing a finite number of terms from the beginning from a sequence it has a certain property, the original sequence has that property eventually.

Definition 8.3.3 (Bounded)**.** An infinite sequence is bounded if $L \leqslant a_n \leqslant K$ for all n.

Theorem 8.3.1. *Every bounded eventually monotone sequence converges.*

8.4 Monotonicity Tests

Difference Test

Theorem 8.4.1 (Difference test)**.** *A sequence $\{a_n\}_{n=1}^{\infty}$ is*

Strictly increasing if $a_{n+1} - a_n > 0$ for all n

Increasing if $a_{n+1} - a_n \geqslant 0$ for all n

Strictly decreasing if $a_{n+1} - a_n < 0$ for all n

Decreasing if $a_{n+1} - a_n \leqslant 0$ for all n

Ratio Test

Theorem 8.4.2 (Ratio test)**.** *A sequence $\{a_n\}_{n=1}^{\infty}$ where all a_n are positive is*

Strictly increasing if $a_{n+1} \div a_n > 1$ for all n

Increasing if $a_{n+1} \div a_n \geqslant 1$ for all n

Strictly decreasing if $a_{n+1} \div a_n < 1$ for all n

Decreasing if $a_{n+1} \div a_n \leqslant 1$ for all n

Note 8.4.1. If $\lim_{n \to \infty} a_{n+1} \div a_n < 1$, then $a_n \to 0$ as $n \to \infty$.

Derivative Test

Theorem 8.4.3 (Derivative test)**.** *If $\{a_n\}_{n=1}^{\infty}$ is an infinite sequence on $\mathbb{R}$ and $a(x)$ is differentiable for $x \in \mathbb{R}_{\geqslant 1}$, then $\{a_n\}_{n=1}^{\infty}$ is*

Strictly increasing if $a'(x) > 0$ for all $x \in \mathbb{R}_{\geqslant 1}$

Increasing if $a'(x) \geqslant 0$ for all $x \in \mathbb{R}_{\geqslant 1}$

Strictly decreasing if $a'(x) < 0$ for all $x \in \mathbb{R}_{\geqslant 1}$

Decreasing if $a'(x) \leqslant 0$ for all $x \in \mathbb{R}_{\geqslant 1}$

9 Series

9.1 Infinite Series

Definition 9.1.1 (Infinite Series). An infinite series is an expansion that can be written as

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$$

The numbers a_i are the terms of the series.

Definition 9.1.2 (Partial Sums). The sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ is defined as

$$s_n = \sum_{i=1}^{n} a_i$$

where s_n is the nth partial sum of the infinite series.

Definition 9.1.3 (Convergence of series). The infinite series $\sum_{i=1}^{\infty} a_i$ converges to A if the associated sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ converges to A.

Definition 9.1.4 (Divergence of series). The infinite series $\sum_{i=1}^{\infty} a_i$ diverges to $\pm\infty$, if the associated sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ diverges to $\pm\infty$.

Theorem 9.1.1. *If $\sum_{i=1}^{\infty} a_i$ converges, then $\lim_{i \to \infty} a_i = 0$.*

Theorem 9.1.2. *Similar to sequences, we can add and subtract convergent infinite series.*

1. Suppose $\sum a_i$ and $\sum b_i$ are both convergent infinite series with the same starting index. Then $\sum a_i \pm b_i$ is also convergent.

2. Suppose $\sum c_i$ is a convergent infinite series and $\sum d_i$ is a divergent infinite series with the same starting index. Then $\sum c_i \pm d_i$ is divergent.

3. Let $\sum e_i$ be an infinite series and let $d \in \mathbb{R}$ be a constant, and $K \in \mathbb{N}$.

   (a) $\sum e_i$ and $\sum d e_i$ both converge or both diverge.
   (b) $\sum_{i=i_0}^{\infty} e_i$ and $\sum_{i=K}^{\infty} e_i$ both converge or both diverge.

Definition 9.1.5 (Absolute Convergence). An infinite series $\sum a_i$ converges (or diverges) absolutely if the series of absolute values $\sum a_i$ also converges (or diverges).

Theorem 9.1.3. *If an infinite series is absolutely convergent, then it is also convergent.*

Definition 9.1.6 (Conditional Convergence). A convergent infinite series that diverges absolutely, converges conditionally.

9.2 Convergence Tests

For any infinite series of the form $\sum_{i=i_0}^{\infty} a_i$.

Divergence Test

$$\lim_{i\to\infty} a_i \neq 0? \begin{cases} \text{YES} & \sum a_i \text{ Diverges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

p-Series

Conditions $i_0 = 1$ and $a_i = \frac{1}{i^p}$.

$$p > 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

Geometric Series

Conditions $i_0 = 0$ and $a_i = ar^i$ or $i_0 = 1$ and $a_i = ar^{i-1}$.

$$|r| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges to } \frac{a}{1-r} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

Note 9.2.1. The sequence of partial sums is given by $s_n = a\dfrac{1 - r^{n+1}}{1 - r}$.

Alternating Series

Conditions $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$b_{i+1} \leqslant b_i \ \& \ \lim_{i\to\infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

Telescoping Series

Conditions Subsequent terms cancel out previous terms in the sum.

$$\lim_{n\to\infty} s_n = s? \begin{cases} \text{YES} & \sum a_i \text{ Converges to } s \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

Limit Comparison Test

Conditions Pick b_i so that $\lim_{i\to\infty} \frac{a_i}{b_i} = c > 0$ & $a_i, b_n > 0$.

$$\sum_{i=i_0}^{\infty} b_i \text{ convergent?} \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

Comparison Test

Conditions Pick b_i.
If $\sum_{i=i_0}^{\infty} b_i$ converges:

$$0 \leqslant a_i \leqslant b_i? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

If $\sum_{i=i_0}^{\infty} b_i$ diverges:

$$0 \leqslant b_i \leqslant a_i? \begin{cases} \text{YES} & \sum a_i \text{ Diverges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

Integral Test

Conditions Let $a_i = f(i)$, so that $f(x)$ is continuous, positive and decreasing on $[a, \infty)$.
If $\int_a^{\infty} f(x)\,\mathrm{d}x$ converges:

$$\begin{cases} \text{YES} & \sum_{i=a}^{\infty} a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

Ratio Test

Conditions $\forall i : a_i > 0$ and $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} \neq 1$.

$$\lim_{i\to\infty} \frac{a_{i+1}}{a_i} < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

$$\lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges Abs.} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

Root Test

Conditions $\forall i : a_i > 0$ and $\lim_{i\to\infty} \sqrt[i]{a_i} \neq 1$.

$$\lim_{i\to\infty} \sqrt[i]{a_i} < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

$$\lim_{i\to\infty} \sqrt[i]{|a_i|} < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges Abs.} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

10   Limits of Functions

10.1   Limits of Functions on $\mathbb{R}$

Definition 10.1.1 (Finite limits using the ε-δ definition)**.** Let a function $f(x)$ be defined for all x in an open interval I (which contains $x_0 \in \mathbb{R}$), except for x_0 which may or may not be defined.

$$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

Limit Laws for Functions

Theorem 10.1.1. *The limit of a sum equals the sum of the limits, i.e.*

$$\lim_{x \to x_0} (f(x) + g(x)) = \left(\lim_{x \to x_0} f(x) \right) + \left(\lim_{x \to x_0} g(x) \right)$$

Theorem 10.1.2. *The limit of a product equals the product of the limits, i.e.*

$$\lim_{x \to x_0} (f(x) \cdot g(x)) = \left(\lim_{x \to x_0} f(x) \right) \cdot \left(\lim_{x \to x_0} g(x) \right)$$

These are trivially extended to division (if the divisor is non-zero), subtraction, exponentiation, and moving constants in or outside of a limit, i.e.

$$\lim_{x \to x_0} (f(x) - g(x)) = \left(\lim_{x \to x_0} f(x) \right) - \left(\lim_{x \to x_0} g(x) \right) \qquad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$$

$$\lim_{x \to x_0} (f(x))^n = \left(\lim_{x \to x_0} f(x) \right)^n \qquad \lim_{x \to x_0} c \cdot f(x) = c \cdot \lim_{x \to x_0} f(x)$$

Definition 10.1.2 (Limits towards $\pm\infty$ using the ε-δ definition)**.** Let a function $f(x)$ be defined for all x in an open interval I (which contains $x_0 \in \mathbb{R}$), except for x_0 which may or may not be defined.

$$\lim_{x \to x_0} f(x) = +\infty \iff \forall M \in \mathbb{R} : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies f(x) > M$$

$$\lim_{x \to x_0} f(x) = -\infty \iff \forall N \in \mathbb{R} : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies f(x) < N$$

Definition 10.1.3 (Limits towards $\pm\infty$)**.** Let a function $f(x)$ be defined for all x in an infinite open interval I extending in the positive x direction.

$$\lim_{x \to \infty} f(x) = L \iff \forall \varepsilon > 0 : \exists M \in I : x > M \implies |f(x) - L| < \varepsilon$$

Let a function $g(x)$ be defined for all x in an infinite open interval I extending in the negative x direction.

$$\lim_{x \to -\infty} g(x) = L \iff \forall \varepsilon > 0 : \exists N \in I : x < N \implies |g(x) - L| < \varepsilon$$

Theorem 10.1.3. $\lim_{x \to x_0} f(x)$ *exists if and only if* $\lim_{x \to x_0^+} f(x)$ *and* $\lim_{x \to x_0^-} f(x)$ *exist and are equal.*

Theorem 10.1.4 (Squeeze theorem for functions)**.** *Let an interval I contain x_0. If f, g, $h : I \to \mathbb{R}$ are functions such that $\forall x \in I \backslash \{x_0\}$, $f(x) \leqslant g(x) \leqslant h(x)$ and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$, then $\lim_{x \to x_0} g(x) = L$.*

Note 10.1.1. The squeeze theorem holds for left- and right-handed and limits to $\pm\infty$.

Definition 10.1.4 (Function continuity using the ε-δ definition)**.** A function $f(x) : I \to \mathbb{R}$ is continuous at $c \in I$, if

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

$f(x)$ is continuous on I if $f(x)$ is continuous for all $x \in I$.

Definition 10.1.5 (Function continuity as a sequence)**.** A function $f(x) : I \to \mathbb{R}$ is continuous at $c \in I$, if for every sequence $\{a_n\}_{n=1}^{\infty}$ in I that converges to c, $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(c)$.

Note 10.1.2. "$f(x)$ is continuous at c" is equivalent to "$\lim_{x \to c} f(x) = f(c)$".

Note 10.1.3. Polynomials are continuous everywhere.

Note 10.1.4. sin and cos are continuous everywhere.

Theorem 10.1.5. *For a function f continuous at L and another function g where $\lim_{x \to c} g(x) = L$, $\lim_{x \to c}(f \circ g)(x) = f(L)$.*

Note 10.1.5. The above theorem still applies when c is $c^{\pm}$ or $\pm\infty$.

Theorem 10.1.6. *For two functions f and g that are continuous at c,*

a) *$f(x) + g(x)$ is continuous at c;*

b) *$f(x) - g(x)$ is continuous at c;*

c) *$f(x) \cdot g(x)$ is continuous at c; and*

d) *$\dfrac{f(x)}{g(x)}$ is continuous at c when $g(c) \neq 0$.*

Theorem 10.1.7. *For a function $g(x)$ which is continuous at c and a function $h(x)$ which is continuous at $g(c)$, $h \circ g$ is continuous at c.*

Theorem 10.1.8 (L'Hôpital rule)**.** *For two differentiable (and therefore continuous) functions $f(x)$ and $g(x)$ except possibly at x_0. If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$, or $\lim_{x \to x_0} f(x) = \pm\infty$ and $\lim_{x \to x_0} g(x) = \pm\infty$, then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ (as long as the limit exists, or diverges to $\pm\infty$).*

Note 10.1.6. L'Hôpital's rule also holds for left- and right-handed, and limits that approach $\pm\infty$.

11 Addenda

11.1 Assumed Knowledge

11.1.1 Factorials

$$n! = \prod_{i=1}^{n} i = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$$

11.1.2 Quadratic Equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

11.2 Table of Derivatives

Let $a \in \mathbb{R}$ be a constant, and $n \in \mathbb{Z}\backslash\{0\}$.

$f(x)$	$\dfrac{\mathrm{d}f}{\mathrm{d}x}$	$f(x)$	$\dfrac{\mathrm{d}f}{\mathrm{d}x}$	$f(x)$	$\dfrac{\mathrm{d}f}{\mathrm{d}x}$
a	0	$\ln(x)$	$\dfrac{1}{x}$	$\sin(x)$	$\cos(x)$
x	1	$\log_a(x)$	$\dfrac{1}{x\ln(a)}$	$\cos(x)$	$-\sin(x)$
x^n	nx^{n-1}	$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$	$\tan(x)$	$\dfrac{1}{\cos^2(x)}$
e^x	e^x	$\sqrt[n]{x}$	$\dfrac{x^{\frac{1}{n}-1}}{n}$		
e^{nx}	ne^{nx}				
$e^{g(x)}$	$g'(x)e^{g(x)}$				
a^x	$a^x\ln(a)$				