

Derivative Rules

$f(x)$	$f'(x)$
$u(x)v(x)$	$u'v + uv'$
$\frac{u(x)}{v(x)}$	$\frac{u'v - uv'}{v^2}$
$u(v(x))$	$u'(v(x))v'(x)$
$x^n$	$nx^{n-1}$
$\ln(u(x))$	$\frac{u'(x)}{u(x)}$
$\sin(ax)$	$a \cos(ax)$
$\cos(ax)$	$-a \sin(ax)$
$\tan(ax)$	$a \sec^2(ax)$
$\cot(ax)$	$-a \csc^2(ax)$
$\sec(ax)$	$a \sec(ax) \tan(ax)$
$\csc(ax)$	$-a \csc(ax) \cot(ax)$

Trigonometric Identities

$1 = \sin^2(x) + \cos^2(x)$   
 $\sin(2x) = 2 \sin(x) \cos(x)$   
 $\cos(2x) = \cos^2(x) - \sin^2(x)$   
 $\sin^2(x) = \frac{1 - \cos(2x)}{2}$   
 $\cos^2(x) = \frac{1 + \cos(2x)}{2}$

Partial Fraction Decomposition

Given the LHS in the denominator, substitute the RHS.

$(ax + b)^k \rightarrow \frac{A_1}{ax + b} + \dots + \frac{A_k}{(ax + b)^k}$   
 $(ax^2 + bx + c)^k \rightarrow \frac{A_1x + B_1}{ax^2 + bx + c} + \dots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$

Integration Techniques

$\int u \, dv = uv - \int v \, du$   
 $\int f(g(x)) \frac{dg(x)}{dx} \, dx = \int f(u) \, du$   
where  $u = g(x)$ .

Trigonometric Substitutions

Form	Substitution
$a^2 - b^2x^2$	$x = \frac{a}{b} \sin(\theta)$
$a^2 + b^2x^2$	$x = \frac{a}{b} \tan(\theta)$
$b^2x^2 - a^2$	$x = \frac{a}{b} \sec(\theta)$

L'Hôpital's Rule

If  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$  or  $\pm\infty$ , then

$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ .

Continuity

$f(x)$  continuous at  $c$  iff  $\lim_{x \rightarrow c} f(x) = f(c)$ .  
 $f(x)$  is continuous on  $I : (a, b)$  if it is continuous for all  $x \in I$ .  
 $f(x)$  is continuous on  $I : [a, b]$  if it is

continuous for all  $x \in I$ , but only right continuous at  $a$  and left continuous at  $b$ .

Intermediate Value Theorem

If  $f(x)$  is continuous on  $I : [a, b]$  and  $f(a) \leq c \leq f(b)$ , then  $\exists x \in I : f(x) = c$ .

Differentiability

$f(x)$  is differentiable at  $x = x_0$  iff  $f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$  exists. This defines the derivative  $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ .

Differentiability implies continuity.

Mean Value Theorem

If  $f(x)$  is continuous and differentiable on  $I : [a, b]$ , then

$\exists c \in I : f'(c) = \frac{f(b) - f(a)}{b - a}$ .

Definite Integrals

$A = \int_a^b f(x) \, dx$

Fundamental Theorem of Calculus

$\int_a^b \frac{d}{dx} F(x) \, dx = F(b) - F(a)$   
 $\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$

Taylor Polynomials

$f(x) \approx p_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$

Taylor Series

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$

Maclaurin Series:  $x_0 = 0$ .

Common Maclaurin Series

Function	Series Term	Conv.
$e^x$	$\frac{x^n}{n!}$	all $x$
$\sin(x)$	$(-1)^n \frac{x^{2n+1}}{(2n+1)!}$	all $x$
$\cos(x)$	$(-1)^n \frac{x^{2n}}{(2n)!}$	all $x$
$\frac{1}{1-x}$	$x^n$	$(-1, 1)$
$\frac{1}{1+x^2}$	$(-1)^n x^{2n}$	$(-1, 1)$
$\ln(1+x)$	$(-1)^{n+1} \frac{x^n}{n}$	$(-1, 1]$

Power Series:  $\sum_{n=0}^{\infty} c_n (x - x_0)^n$

Series Tests

For a series of the form  $\sum_{i=i_0}^{\infty} a_i$ :

Alternating Series Test

Given  $a_i = (-1)^i b_i$  and  $b_i > 0$ .  
If  $b_{i+1} \leq b_i$  &  $\lim_{i \rightarrow \infty} b_i = 0$ , then convergent, else inconclusive.

Ratio Test

Given  $\rho = \lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right|$ :  
 $\rho < 1$  : convergent  
 $\rho > 1$  : divergent  
 $\rho = 1$  : inconclusive

Multivariable Functions

$f : \mathbb{R}^n \rightarrow \mathbb{R}$

Level Curves

$L_c(f) = \{(x, y) : f(x, y) = c\}$

If  $f(x, y) \rightarrow L$  as  $(x, y) \rightarrow (x_0, y_0)$ , then  $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L$  along any smooth curve.

The limit does not exist if  $L$  changes along different smooth curves.

**Partial Derivatives:** w.r.t one variable, others held constant.

**Gradient:**  $\nabla = \langle \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n} \rangle$

Multivariable Chain Rule

For  $f = f(x(t_1, \dots, t_n))$  with  $\mathbf{x} = [x_1 \dots x_m]$   
 $\frac{\partial f}{\partial t_i} = \nabla f \cdot \partial_{t_i} \mathbf{x}$ .

Directional Derivatives

$\nabla_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$

where  $\mathbf{u}$  is a unit vector and the slope is given by  $\|\nabla_{\mathbf{u}} f\|$ . If  $\nabla_{\mathbf{u}} f = 0$ ,  $\mathbf{u}$  is tangent to the level curve at  $\mathbf{x}_0$ .

$\max_{\|\mathbf{u}\|=1} \nabla_{\mathbf{u}} f = \nabla f$

If  $\nabla f \neq 0$ ,  $\nabla f$  is a normal vector to the level curve at  $\mathbf{x}_0$ .

Critical Points

$(x_0, y_0)$  is a critical point if  $\nabla f(x_0, y_0) = \mathbf{0}$  or if  $\nabla f(x_0, y_0)$  is undefined.

Classification of Critical Points

$D = f_{xx}f_{yy} - (f_{xy})^2$

$D > 0$  and  $f_{xx} < 0$ : local maxima

$D > 0$  and  $f_{xx} > 0$ : local minima

$D < 0$ : saddle point

$D = 0$ : inconclusive

Double Integrals

The volume of the solid enclosed between the surface  $z = f(x, y)$  and the region  $\Omega$  is defined by

$V = \iint_{\Omega} f(x, y) \, dA$ .

If  $\Omega$  is a region bounded by  $a \leq x \leq b$  and  $c \leq y \leq d$ , then

$\iint_{\Omega} f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy$   
 $= \int_a^b \int_c^d f(x, y) \, dy \, dx$

Type I Regions

∬Ω f(x, y) dA = ∫a b ∫g1(x) g2(x) f(x, y) dy dx

Bounded left & right by:
x = a and x = b

Bounded below & above by:
y = g1(x) and y = g2(x)
where g1(x) ≤ g2(x) for a ≤ x ≤ b:

Type II Regions

∬Ω f(x, y) dA = ∫c d ∫h1(x) h2(x) f(x, y) dx dy

Bounded left & right by:
x = h1(y) and x = h2(y)

Bounded below & above by:
y = c and y = d
where h1(y) ≤ h2(y) for c ≤ y ≤ d.
To integrate, solve the inner integrals first.

Vector Valued Functions

r : R → R^n

The domain of r(t) is the intersection of the domains of its components.
The orientation of r(t) is the direction of motion along the curve as the value of the parameter increases.
Limits, derivatives and integrals are all component-wise. Each component has its own constant of integration.

Parametric Lines

l(t) = P0 + tv

where l(t) passes through P0, and is parallel to v.

Tangent Lines

If r(t) is differentiable at t0 and r'(t0) ≠ 0
l(t) = r(t0) + tr'(t0).

Curves of Intersection

Choose one of the variables as the parameter, and express the remaining variables in terms of that parameter.

Arc Length

S = ∫a b ||r'(t)|| dt

Ordinary Differential Equations

Order: highest derivative in DE.
Autonomous DE: does not depend explicitly on the independent variable.

Qualitative Analysis

dy/dt = f(y)

A fixed point is the value of y for which f(y) = 0.

Stability of Fixed Points

Given a positive/negative perturbation from a fixed point, that point is
Stable: if both tend toward FP
Unstable: if both tend away from FP
Semi-Stable: if one tends toward FP, and another tends away from FP

Directly Integrable ODEs

For dy/dx = f(x):
y(x) = ∫ f(x) dx.

Separable ODEs

For dy/dx = p(x)q(y):
∫ 1/q(y) dy = ∫ p(x) dx.

Linear ODEs

For dy/dx + p(x)y = q(x), use the integrating factor: I(x) = e^∫ p(x)dx, so that
y(x) = 1/I(x) ∫ I(x)q(x) dx.

Exact ODEs

P(x, y) + Q(x, y) dy/dx = 0 has the solution Ψ(x, y) = c iff it is exact, namely, when Py = Qx, where P = Ψx and Q = Ψy. Then

Ψ(x, y) = ∫ P(x, y) dx + f(y)
Ψ(x, y) = ∫ Q(x, y) dy + g(x)

and f(y) and g(x) can be determined by solving these equations simultaneously.

Second-Order ODEs

a2(x)y'' + a1(x)y' + a0(x)y = F(x)

Initial Values

y(x0) = y0 y'(x0) = y1

Boundary Values

y(x0) = y0 y(x1) = y1

Reduction of Order

y2(x) = v(x) y1(x)
v(x) can be determined by substituting y2 into the ODE, using w(x) = v'(x).

General Solution

y(x) = yH(x) + yP(x)

Homogeneous Solution

yH(x) = e^λx

Real Distinct Roots

yH(x) = c1e^λ1x + c2e^λ2x

Real Repeated Roots

yH(x) = c1e^λx + c2te^λx

Complex Conjugate Roots

Given λ = α ± βi:
yH(x) = e^αx (c1 cos(βx) + c2 sin(βx))

Particular Solution

See table below. Substitute yP into the nonhomogeneous ODE, and solve the undetermined coefficients.

Spring and Mass Systems

my'' + γy' + ky = f(t)

Newton's Law: F = my''

Spring force: Fs = -ky

Damping force: Fd = -γy'

m : mass k : spring constant
γ : damping f(t) : external force

Electrical Circuits

The sum of voltages around a loop equals 0.

v(t) - iR - L di/dt - q/C = 0
L d^2q/dt^2 + R dq/dt + 1/C q = v(t)
where i = dq/dt.

Voltage drop across various elements:

vR = iR
vC = q/C
vL = L di/dt

R : resistance C : capacitance
L : inductance v(t) : voltage supply

Table with 2 columns: F(x) and yP(x). Rows include: a constant, a polynomial of degree n, e^kx, cos(ωx) or sin(ωx), a combination of the above, linearly dependent to yH(x), and A, sum of Ai xi, Ae^kx, A0 cos(ωx) + A1 sin(ωx), a combination of the above, multiply yP(x) by x until linearly independent.