Calculus and Differential Equations

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1 Integration Techniques

1.1 Table of Derivatives

Let $f(x)$ be a function, and $a \in \mathbb{R}$ be a constant.

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
x^a	ax^{a-1}
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
a^x	$\ln(a)a^x$
e^x	e^x
$\log_a x,\ a \in \mathbb{R}\setminus\{0\}$	$\frac{1}{a\ln x}$
$\ln x$	$\frac{1}{x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
a	0
x	1
$a_1u(x) \pm a_2v(x)$	$a_1\frac{\mathrm{d}u}{\mathrm{d}x} \pm a_2\frac{\mathrm{d}v}{\mathrm{d}x}$
$u(x)v(x)$	$\frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}$
$\frac{u(x)}{v(x)}$	$\frac{\frac{\mathrm{d}u}{\mathrm{d}x}v - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v(x)^2}$
$u(v(x))$	$\frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\csc^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\csc(ax)$	$-a\csc(ax)\cot(ax)$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\arcsin(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\frac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\frac{a}{1+a^2x^2}$
$\operatorname{arccot}(ax)$	$-\frac{a}{1+a^2x^2}$
$\operatorname{arcsec}(ax)$	$\frac{1}{x\sqrt{a^2x^2-1}}$
$\operatorname{arccsc}(ax)$	$-\frac{1}{x\sqrt{a^2x^2-1}}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sinh(ax)$	$a\cosh(ax)$
$\cosh(ax)$	$a\sinh(ax)$
$\tanh(ax)$	$a\,\mathrm{sech}^2(ax)$
$\coth(ax)$	$-a\,\mathrm{csch}^2(ax)$
$\mathrm{sech}(ax)$	$-a\,\mathrm{sech}(ax)\tan(ax)$
$\mathrm{csch}(ax)$	$-a\,\mathrm{csch}(ax)\cot(ax)$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\mathrm{arcsinh}(ax)$	$\frac{a}{\sqrt{1+a^2x^2}}$
$\mathrm{arccosh}(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\mathrm{arctanh}(ax)$	$\frac{a}{1-a^2x^2}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\mathrm{arccoth}(ax)$	$\frac{a}{1-a^2x^2}$
$\mathrm{arcsech}(ax)$	$-\frac{1}{a(1+ax)\sqrt{\frac{1-ax}{1+ax}}}$
$\mathrm{arccsch}(ax)$	$-\frac{1}{ax^2\sqrt{1+\frac{1}{a^2x^2}}}$

Table 1: Derivatives of Elementary Functions

1.2 Trigonometric Identities

1.2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x) + 1 = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$

1.2.2 Double-Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \csc(2x) = \frac{\sec(x)\csc(x)}{2}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad \sec(2x) = \frac{\sec^2(x)\csc^2(x)}{\csc^2(x) - \sec^2(x)}$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} \qquad \cot(2x) = \frac{\cot^2(x) - 1}{2\cot(x)}$$

1.2.3 Power Reducing Identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \csc^2(x) = \frac{2}{1-\cos(2x)}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sec^2(x) = \frac{2}{1+\cos(2x)}$$

$$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)} \qquad \cot^2(x) = \frac{1+\cos(2x)}{1-\cos(2x)}$$

1.3 Partial Fractions

Definition 1.3.1 (Partial Fraction Decomposition)**. Partial fraction decomposition** is a method where a rational function $\frac{P(x)}{Q(x)}$ is rewritten as a sum of fraction.

Factor in denominator	**Term in partial fraction decomposition**
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k,\ k\in\mathbb{N}$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{A}{ax^2+bx+c}$
$(ax^2+bx+c)^k,\ k\in\mathbb{N}$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$

Table 2: Partial Fraction Forms

1.4 Integration by Parts

Theorem 1.4.1.

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

1.5 Integration by Substitution

Theorem 1.5.1.

$$\int f(g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\,,\ \textit{where } u = g(x)$$

1.6 Trigonometric Substitutions

Form	**Substitution**	**Result**	**Domain**
$(a^2-b^2x^2)^n$	$x=\frac{a}{b}\sin(\theta)$	$a^2\cos^2(\theta)$	$\theta\in[-\frac{\pi}{2},\ \frac{\pi}{2}]$
$(a^2+b^2x^2)^n$	$x=\frac{a}{b}\tan(\theta)$	$a^2\sec^2(\theta)$	$\theta\in(-\frac{\pi}{2},\ \frac{\pi}{2})$
$(b^2x^2-a^2)^n$	$x=\frac{a}{b}\sec(\theta)$	$a^2\tan^2(\theta)$	$\theta\in[0,\ \frac{\pi}{2})\cup(\frac{\pi}{2},\ \pi]$

Table 3: Trigonometric substitutions for various forms.

2 Limits, Continuity and Differentiability

2.1 Limits

Theorem 2.1.1 (Limits). $\lim\limits_{x \to x_0} f(x)$ *exists if and only if* $\lim\limits_{x \to {x_0}^+} f(x)$ *and* $\lim\limits_{x \to {x_0}^-} f(x)$ *exist and are equal.*

Definition 2.1.1 (Finite limits using the ε-δ definition).

$$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

Theorem 2.1.2 (L'Hôpital's Rule). *For two differentiable functions* $f(x)$ *and* $g(x)$. *If* $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$, *or* $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = \pm\infty$, *then* $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.

2.2 Continuity

Theorem 2.2.1 (Continuity at a Point). $f(x)$ *is continuous at* c *iff* $\lim\limits_{x \to c} f(x) = f(c)$.

Theorem 2.2.2 (Continuity over an Interval). $f(x)$ *is continuous on* I *if* $f(x)$ *is continuous for all* $x \in I$.

- $f(x)$ *is continuous on* $I : (a,\, b)$ *if it is continuous for all* $x \in I$.
- $f(x)$ *is continuous on* $I : [a,\, b]$ *if it is continuous for all* $x \in I$, *but only right continuous at* a *and left continuous at* b.

If $f(x)$ *is continuous on* $(-\infty,\, \infty)$, $f(x)$ *is continuous everywhere.*

Theorem 2.2.3 (Intermediate Value Theorem). *If* $f(x)$ *is continuous on* $I : [a,\, b]$ *and* c *is any number between* $f(a)$ *and* $f(b)$, *inclusive, then there exists an* $x \in I$ *such that* $f(x) = c$.

2.3 Differentiability

Theorem 2.3.1 (Differentiability). $f(x)$ *is differentiable at* $x = x_0$ *iff*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, it defines the derivative

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Theorem 2.3.2. $f(x)$ *is differentiable on* I *if* $f(x)$ *is differentiable for all* $x_0 \in I$.

Theorem 2.3.3. *Differentiability implies continuity.*

Theorem 2.3.4 (Mean Value Theorem). *If* $f(x)$ *is continuous on* $I : [a,\, b]$ *and differentiable on* I, *then there exists a point* $c \in I$ *such that*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=c} = \frac{f(b) - f(a)}{b - a}$$

3 Definite Integrals

Theorem 3.0.1. *If $f(x)$ is continuous on an interval $I : [a,\, b]$, then the net signed area A between the graph of $f(x)$ and the interval I is*

$$A = \int_a^b f(x)\,\mathrm{d}x$$

Properties of Definite Integrals

Theorem 3.0.2. *Suppose that $f(x)$ and $g(x)$ are continuous on the interval I, with $a,\, b,\, c \in I$ and $k \in \mathbb{R}$ then*

a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0.$

b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$

c) $\displaystyle\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x.$

d) $\displaystyle\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x.$

e) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$

3.1 Riemann Sums

Theorem 3.1.1. *Let A be the area under $f(x)$ on the interval $[a,\, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

where n is the number of rectangles, x_k is the centre of the rectangle k, and Δx_k is the width of the rectangle k. If every rectangle has the same width, then

$$\forall k : \Delta x_k = \frac{b-a}{n}$$

3.2 Fundamental Theorem of Calculus

The fundamental theorem of calculus provides a logical connection between infinite series (definite integrals) and antiderivatives (indefinite integrals).

Theorem 3.2.1 (The Fundamental Theorem of Calculus: Part 1)**.** *If $f(x)$ is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$ then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Equivalently

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x} F(x)\,\mathrm{d}x = F(b) - F(a) \equiv F(x)\big|_a^b$$

Theorem 3.2.2 (The Fundamental Theorem of Calculus: Part 2)**.** *If $f(x)$ is continuous on I then it has an antiderivative on I. In particular, if $a \in I$, then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is an antiderivative of $f(x)$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

Theorem 3.2.3. *Differentiation and integration are inverse operations.*

3.3 Taylor and Maclaurin Polynomials

Theorem 3.3.1 (Taylor Polynomials)**.** *If $f(x)$ is a n differentiable function at x_0, then the nth degree Taylor polynomial for $f(x)$ near x_0, is given by*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Theorem 3.3.2 (Maclaurin Polynomials)**.** *Evaluating a Taylor polynomial near 0, gives the nth degree Maclaurin polynomial for $f(x)$*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Theorem 3.3.3 (Error in Approximation)**.** *Let $R_n(x)$ denote the difference between $f(x)$ and its nth Taylor polynomial, that is*

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = \frac{f^{(n+1)}(s)}{(n+1)!} (x - x_0)^{n+1}$$

where s is between x_0 and x.

4 Taylor and Maclaurin Series

4.1 Infinite Series

Definition 4.1.1 (Taylor Series). If $f(x)$ has derivatives of all orders at x_0, then the Taylor series for $f(x)$ about $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Definition 4.1.2 (Maclaurin Series). If a Taylor series is centred on $x_0 = 0$, it is called a Maclaurin series, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 4.1.3 (Power Series). Both Taylor and Maclaurin series are examples of **power series**, which are defined as follows

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

4.2 Convergence

Theorem 4.2.1 (Convergence of a Taylor Series). *The equality*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

holds at a point x iff

$$\lim_{n\to\infty} \left[f(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \right] = 0$$
$$\lim_{n\to\infty} R_n(x) = 0$$

Definition 4.2.1 (Interval of Convergence). The interval of convergence for a power series is the set of x values for which that series converges.

Definition 4.2.2 (Radius of Convergence). The radius of convergence R is a nonnegative real number or ∞ such that a power series converges if

$$|x - a| < R$$

and diverges if

$$|x - a| > R$$

The behaviour of the power series on the boundary, that is, where $|x - a| = R$, can be determined by substituting $x = R + a$ for the upper boundary, and $x = -R + a$ for the lower boundary.

4.3 Convergence Tests

For any power series of the form $\sum_{i=i_0}^{\infty} a_i$.

Alternating Series

Conditions $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$\text{Is } b_{i+1} \leqslant b_i \;\&\; \lim_{i\to\infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

Ratio Test

$$\text{Is } \lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

The ratio test is inconclusive if $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = 1$.

4.4 Table of Maclaurin Series

Function	**Series**	**Interval of Convergence**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$-\infty < x < \infty$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

Table 4: Maclaurin Series of Common Functions

5 Multivariable Calculus

5.1 Multivariable Functions

Definition 5.1.1. A function is multivariable if its domain consists of several variables. In the reals, these functions are defined

$$f : \mathbb{R}^n \to \mathbb{R}$$

5.2 Level Curves

Definition 5.2.1. Level curves or *contour curves* of a function of two variables is a curve along which the function has constant value.

$$L_c\,(f) = \{(x,\, y) : f\,(x,\, y) = c\}$$

The level curves of a function can be determined by substituting $z = c$, and solving for y.

5.3 Limits and Continuity

Definition 5.3.1 (Finite Limit of Multivariable Functions using the ε-δ Definition)**.**

$$\lim_{(x_1,\, \dots,\, x_n) \to (c_1,\, \dots,\, c_n)} f(x_1,\, \dots,\, x_n) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall (x_1,\, \dots,\, x_n) \in I :$$
$$0 < |x_1 - c_1,\, \dots,\, x_n - c_n| < \delta \implies |f(x_1,\, \dots,\, x_n) - L| < \varepsilon$$

Theorem 5.3.1 (Limits along Smooth Curves)**.** *If $f(x,\, y) \to L$ as $(x,\, y) \to (x_0,\, y_0)$, then $\lim_{(x,\, y) \to (x_0,\, y_0)} = L$ along any smooth curve.*

Theorem 5.3.2 (Existence of a Limit)**.** *If the limit of $f(x,\, y)$ changes along different smooth curves, then $\lim_{(x,\, y) \to (x_0,\, y_0)}$ does not exist.*

Theorem 5.3.3 (Continuity of Multivariable Functions)**.** *A function $f(x_1,\, \dots,\, x_n)$ is continuous at $(c_1,\, \dots,\, c_n)$ iff*

$$\lim_{(x_1,\, \dots,\, x_n) \to (c_1,\, \dots,\, c_n)} f(x_1,\, \dots,\, x_n) = f(c_1,\, \dots,\, c_n)$$

Recognising continuous functions:

- A sum, difference or product of continuous functions is continuous.
- A quotient of continuous functions is continuous expect where the denominator is zero.
- A composition of continuous functions is continuous.

5.4 Partial Derivatives

Definition 5.4.1 (Partial Differentiation)**.** The partial derivative of a multivariable function is its derivative with respect to one of those variables, while the others are held constant.

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1,\, \dots,\, x_{i-1},\, x_i + h,\, x_{i+1},\, \dots,\, x_n) - f(x_1,\, \dots,\, x_n)}{h}$$

5.5 The Gradient Vector

Definition 5.5.1. Let $\boldsymbol{\nabla}$, pronounced "del", denote the vector differential operator defined as follows

$$\boldsymbol{\nabla} = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{bmatrix}$$

5.6 Multivariable Chain Rule

Definition 5.6.1. Let $f = f(\boldsymbol{x}(t_1, \ldots, t_n))$ be the composition of f with $\boldsymbol{x} = [x_1 \quad \cdots \quad x_m]$, then the partial derivative of f with respect to t_i is given by

$$\frac{\partial f}{\partial t_i} = \boldsymbol{\nabla} f \cdot \partial_{t_i} \boldsymbol{x}$$

5.7 Directional Derivatives

Definition 5.7.1. The directional derivative $\boldsymbol{\nabla}_{\boldsymbol{u}} f$ is the rate at which the function f changes in the direction $\boldsymbol{u}$.

$$\begin{aligned} \boldsymbol{\nabla}_{\boldsymbol{u}} f &= \lim_{h \to 0} \frac{f(\boldsymbol{x} + h\boldsymbol{u}) - f(\boldsymbol{x})}{h} \\ &= \boldsymbol{\nabla} f \cdot \boldsymbol{u} \end{aligned}$$

where the slope is given by $\|\boldsymbol{\nabla}_{\boldsymbol{u}} f\|$.

Remark 1. *The directional derivative of f can be denoted in several ways:*

$$\boldsymbol{\nabla}_{\boldsymbol{u}} f = D_{\boldsymbol{u}} f = \partial_{\boldsymbol{u}} f = \frac{\partial f}{\partial \boldsymbol{u}}$$

Theorem 5.7.1 (Direction of Greatest Ascent)**.** *The direction of greatest ascent is given by*

$$\max_{\|\boldsymbol{u}\|=1} \boldsymbol{\nabla}_{\boldsymbol{u}} f = \boldsymbol{\nabla} f$$

where the slope is given by $\|\boldsymbol{\nabla} f\|$.

Theorem 5.7.2 (Direction of Greatest Descent)**.** *The direction of greatest descent is given by*

$$\min_{\|\boldsymbol{u}\|=1} \boldsymbol{\nabla}_{\boldsymbol{u}} f = -\boldsymbol{\nabla} f$$

where the slope is given by $-\|\boldsymbol{\nabla} f\|$.

Proof. Given that $\boldsymbol{u}$ is a unit vector, the dot product definition gives

$$\begin{aligned} \boldsymbol{\nabla}_{\boldsymbol{u}} f &= \boldsymbol{\nabla} f \cdot \boldsymbol{u} \\ &= \|\boldsymbol{\nabla} f\| \|\boldsymbol{u}\| \cos(\theta) \\ &= \|\boldsymbol{\nabla} f\| \cos(\theta) \end{aligned} \tag{1}$$

Equation 1 is maximised when $\cos(\theta)$ is maximised. Thus the maximum slope is given by

$$\max \boldsymbol{\nabla}_{\boldsymbol{u}} f = \|\boldsymbol{\nabla} f\|$$

and the direction of greatest ascent is given by

$$\boldsymbol{u} = \boldsymbol{\nabla} f$$

□

Theorem 5.7.3. *If $\boldsymbol{\nabla} f = 0$, then $\boldsymbol{\nabla} f$ is normal to the level curves of f at any point $(c_1, \ldots, c_n)$.*

5.8 Higher-Order Partial Derivatives

Definition 5.8.1. Higher-order partial derivatives can be denoted using three different notation. The following table shows the mixed partial derivative of $f(x, y)$ w.r.t. x then y.

Leibniz	**Euler**	**Legendre**
$\frac{\partial^2 f}{\partial y \partial x}$	$\partial_{xy} f$	f_{xy}

Table 5: Mixed Partial Derivative Notation

For partial derivatives w.r.t. the same variable, a superscript can be used in Euler notation.

Leibniz	**Euler**	**Legendre**
$\frac{\partial^2 f}{\partial x^2}$	$\partial_x^2 f$	f_{xx}

Table 6: Second-Order Partial Derivative Notation

5.9 Hessian Matrix

Definition 5.9.1. Let the Hessian matrix $\mathbf{H}$ be the matrix of second-order partial derivative operators defined as shown below

$$\mathbf{H} = \begin{bmatrix} \partial_{x_1}^2 & \cdots & \partial_{x_n x_1} \\ \vdots & \ddots & \vdots \\ \partial_{x_1 x_n} & \cdots & \partial_{x_n}^2 \end{bmatrix}$$

5.10 Critical Points

6 Double and Triple Integrals

6.1 Volume under a Two Variable Function

Definition 6.1.1. If f is a function of two variables that is continuous and nonnegative on a region Ω in the xy-plane, then the volume of the solid enclosed between the surface $z = f(x,\ y)$ and the region Ω is defined by

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f({x_k}^*,\ {y_k}^*)\Delta A_k \tag{1}$$

Proof. Using lines parallel to the coordinate axes, the region Ω can be divided into n rectangles, where any rectangles outside Ω are discarded. The area of the kth remaining rectangle at the arbitrary point $({x_k}^*,\ {y_k}^*)$ is given by ΔA_k. Thus the product $f({x_k}^*,\ {y_k}^*)\Delta A_k$ is the volume of the kth rectangular parallelepiped, and the sum of all n volumes over the region Ω approximate the volume V of the entire solid. □

6.2 Double Integral

Definition 6.2.1. By extension of the definite integral of a one variable function expressed in Theorem 3.1.1, the sums in Equation 1 are also called Riemann sums, and the limit is denoted as

$$\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A = \sum_{k=1}^{\infty} f({x_k}^*,\ {y_k}^*)\Delta A_k$$

Properties of Double Integrals

Theorem 6.2.1. *Suppose that $f(x,\ y)$ and $g(x,\ y)$ are continuous on Ω, and Ω can be subdivided into Ω_1 and Ω_2, then*

a) $\displaystyle\iint\limits_{\Omega} kf(x,\ y)\,\mathrm{d}A = k\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A.$

b) $\displaystyle\iint\limits_{\Omega} (f(x,\ y) + g(x,\ y))\,\mathrm{d}A = \iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A + \iint\limits_{\Omega} g(x,\ y)\,\mathrm{d}A.$

c) $\displaystyle\iint\limits_{\Omega} f(x,\ y)\,\mathrm{d}A = \iint\limits_{\Omega_1} f(x,\ y)\,\mathrm{d}A + \iint\limits_{\Omega_2} f(x,\ y)\,\mathrm{d}A.$

6.3 Triple Integrals

Definition 6.3.1. A triple integral is of a function is the net signed volume defined over a finite closed solid region G in an xyz coordinate system.

Theorem 6.3.1. *Divide the bounding box of G into n boxes with sides parallel to the coordinate planes. Discard boxes which contain any points outside of G. Choose an arbitrary point in each remaining box. The volume of the kth remaining box is ΔV_k. The arbitrary point in the kth remaining box is $({x_k}^*,\, {y_k}^*,\, {z_k}^*)$. The Riemann sum is*

$$\iiint\limits_G f(x,\, y,\, z)\,\mathrm{d}V = \sum_{k=1}^{\infty} f({x_k}^*,\, {y_k}^*,\, {z_k}^*)\Delta V_k$$

Properties of Triple Integrals

Theorem 6.3.2. *Suppose that $f(x,\, y,\, z)$ and $g(x,\, y,\, z)$ are continuous on G and G can be subdivided into G_1 and G_2then*

a) $\displaystyle\iiint\limits_G kf(x,\, y,\, z)\,\mathrm{d}V = k\iiint\limits_G f(x,\, y,\, z)\,\mathrm{d}V.$

b) $\displaystyle\iiint\limits_G \left(f(x,\, y,\, z) + g(x,\, y,\, z)\right)\mathrm{d}V = \iiint\limits_G f(x,\, y,\, z)\,\mathrm{d}V + \iiint\limits_G g(x,\, y,\, z)\,\mathrm{d}V.$

c) $\displaystyle\iiint\limits_G f(x,\, y,\, z)\,\mathrm{d}V = \iiint\limits_{G_1} f(x,\, y,\, z)\,\mathrm{d}V + \iiint\limits_{G_2} f(x,\, y,\, z)\,\mathrm{d}V.$

7 Vector-Valued Functions

Definition 7.0.1. A vector-valued function (VVF) is some function with domain $\mathbb{R}$ and codomain $\mathbb{R}^n$. For example,

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3 : \mathbf{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$$

is a vector-valued function where $x,\ y,\ z : \mathbb{R} \to \mathbb{R}$.

Theorem 7.0.1. *The domain of $\mathbf{r}(t)$ is the intersection of the domains of its components.*

Definition 7.0.2 (Orientation)**.** The orientation of $\mathbf{r}(t)$ is the direction of motion along the curve as the value of the parameter increases.

7.1 Limits and Continuity

Theorem 7.1.1 (Limits of VVFs)**.** *The limit of a VFF is the vector of the limits of its components.*

$$\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} x(t),\ \lim_{t\to a} y(t),\ \lim_{t\to a} z(t) \rangle$$

Theorem 7.1.2 (Continuity of VVFs)**.** *The VVF $\mathbf{r}(t)$ is continuous at $t = a$ iff*

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

This follows that a VVF is continuous if each of its components are also continuous.

7.2 Calculus with VVFs

Theorem 7.2.1 (Derivatives of VVFs)**.** *The derivative of a VFF is the vector of the derivatives of its components.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}x(t),\ \frac{\mathrm{d}}{\mathrm{d}t}y(t),\ \frac{\mathrm{d}}{\mathrm{d}t}z(t) \right\rangle$$

Theorem 7.2.2 (Integration of VVFs)**.** *The integral of a VFF is the vector of the integrals of its components.*

$$\int \mathbf{r}(t)\,\mathrm{d}t = \left\langle \int x(t)\,\mathrm{d}t\,,\ \int y(t)\,\mathrm{d}t\,,\ \int z(t)\,\mathrm{d}t \right\rangle$$

Remark 2. *When integrating a VVF, each component has its own constant of integration.*

7.3 Parametrising Lines with VVFs

Definition 7.3.1 (Equation for a Line)**.** A line can be expressed as

$$\mathbf{l}(t) = \boldsymbol{P}_0 + t\boldsymbol{v}$$

where the line $\mathbf{l}(t)$ passes through the point $\boldsymbol{P}_0$, and is parallel to the vector $\boldsymbol{v}$.

Definition 7.3.2 (Tangent Lines)**.** If a VVF $\mathbf{r}(t)$ is differentiable at t_0 and $\mathbf{r}'(t_0) \neq \mathbf{0}$, the tangent line at $t = t_0$ is given by

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0).$$

Remark 3. *Higher-order approximations can be determined using Taylor's formula.*

7.4 Applications of VVFs

Theorem 7.4.1 (Curve of Intersection)**.** *A VVF can be used to determine the curve of intersection between two surfaces. The method is to choose one of the variables (commonly the first) as the parameter, and express the remaining variables in terms of that parameter.*

If the intersection is bounded between two points, the domain can be calculated using the component which was parametrised.

For example, the curve of intersection between

$$y = 2x - 4 \quad \text{and} \quad z = 3x - 1$$

between the points

$$\boldsymbol{P}_1 = (2,\ 0,\ 7) \quad \text{and} \quad \boldsymbol{P}_2 = (3,\ 2,\ 10)$$

is given by

$$\mathbf{r}(t) = \left\langle t,\ 2t - 4,\ 3t - 1 \right\rangle : 2 \leq t \leq 3.$$

Definition 7.4.1 (Arc Length)**.** The arc length S of a smooth continuous VVF $\mathbf{r}(t)$, is the distance along $\mathbf{r}(t)$ between $t = a$ and $t = b$, defined by

$$S = \int_a^b \left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{r}(t) \right\| \mathrm{d}t$$

8 Differential Equations

Definition 8.0.1 (Differential Equations)**.** A differential equation (DE) is an equation which involves the derivatives of one of more unknown functions (called dependent variables), that are with respect to one or more independent variables.

Definition 8.0.2 (Ordinary Differential Equations)**.** An ordinary differential equation (ODE) is a differential equation with derivatives with respect to a single variable.

Definition 8.0.3 (Partial Differential Equations)**.** A partial differential equation (PDE) is a differential equation with derivatives with respect to multiple variables.

Definition 8.0.4 (Order of Differential Equations)**.** The order of a differential equation is the highest derivative in the equation.

Definition 8.0.5 (Autonomous Differential Equations)**.** An autonomous differential equation does not depend explicitly on the independent variable.

Definition 8.0.6 (Linear Differential Equations)**.** A linear differential equation does not have any products of the dependent variable with itself or derivatives. The general form of a linear ODE or order n is

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = f(x).$$

The dependent variable cannot be composed in another function.

8.1 Qualitative Analysis

With qualitative analysis we aim to understand the behaviour of solutions to the ODE. By computing fixed points, we can draw a phase line diagram, and sketch solution curves.

For the autonomous differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

Definition 8.1.1 (Fixed Point)**.** A fixed point is the value of y for which $f(y) = 0$.

Definition 8.1.2 (Stability)**.** By analysing the perturbation of the $f(y)$ near fixed points, we can determine the stability of those fixed points.

Behaviour in Positive/Negative Directions	**Stability**
Both toward fixed point	stable
Both away from fixed point	unstable
One toward and one away from fixed point	semi-stable

Definition 8.1.3 (Phase Plane)**.** Using the information about the behaviour of $f(y)$ around fixed points, $f(y)$ can be plotted against y to construct a phase plane diagram.

Definition 8.1.4 (Phase Line)**.** A phase line is the one-dimensional form of a phase plane, that shows the limiting behaviour of y as $t \to \infty$.

Definition 8.1.5 (Sample Solutions)**.** Using a phase line, we sketch the behaviour of sample solutions of $f(y)$. where the curves asymptote toward stable (also semi-stable) fixed points, and diverge from unstable (also semi-stable) fixed points.

9 First-Order Differential Equations

9.1 Directly Integrable ODEs

For a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$

$$y(x) = \int g(x)\,\mathrm{d}x\,.$$

9.2 Separable ODEs

For a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)q(y)$$

a separation of variables followed by an integration w.r.t. x yields an implicit solution.

$$\int \frac{1}{q(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int p(x)\,\mathrm{d}x\,.$$

9.3 Linear ODEs

For a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

we can use the integrating factor

$$I(x) = e^{\int p(x)\mathrm{d}x}$$

to solve the differential equation

$$y(x) = \frac{1}{I(x)}\int I(x)q(x)\,\mathrm{d}x$$

9.4 Homogeneous ODEs

9.5 Exact ODEs

A differential equation of the form

$$P(x,\,y) + Q(x,\,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact if

$$P_y = Q_x.$$

We find $\psi(x,\, y)$ using

$$\psi(x,\, y) = \int P(x,\, y)\,\mathrm{d}x + f(y)$$
$$\psi(x,\, y) = \int Q(x,\, y)\,\mathrm{d}y + g(x)$$

$f(y)$ and $g(x)$ can be determined by matching these two equations. The general solution to an exact ODE is then

$$\psi(x,\, y) = c$$

10 Second-Order Differential Equations

A linear second-order differential equation is of the form

$$a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = F(x).$$

If $F(x) = 0$, then the equation is homogeneous.

If $F(x) \neq 0$, then the equation is nonhomogeneous.

Definition 10.0.1 (Initial Value Problem)**.** An initial value problem specifies the value for y and its derivative at a single value of the independent variable:

$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

Definition 10.0.2 (Boundary Value Problem)**.** A boundary value problem specifies the value for y at two different values of the independent variable:

$$y(x_0) = y_0, \quad y(x_1) = y_1.$$

Theorem 10.0.1 (Superposition Principle)**.** *Consider a homogeneous ODE*

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0.$$

If $y_1(x),\, y_2(x),\, \ldots,\, y_n(x)$ are solutions of the differential equation, then the linear combination of these solutions

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

is also a solution.

Theorem 10.0.2 (Fundamental Set of Solutions)**.** *The nth order homogeneous ODE with continuous coefficients on an open interval I, has n non-trivial linearly independent solutions that form a fundamental set of solutions on I.*

10.1 Reduction of Order

Reduction of order is a method for finding a second solution to an ODE, given a known solution. The second solution is of the form

$$y_2(x) = v(x)y_1(x).$$

By substituting this form into the differential equation, we will find an ODE for $v(x)$ that we can solve to find the solution $y_2(x)$.

10.2 Homogeneous ODEs

A second-order constant-coefficient homogeneous ODE

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0$$

has solutions of the form

$$y(x) = e^{\lambda x}.$$

10.3 Characteristic Equation

Substituting this form into the ODE gives the characteristic equation

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0.$$

This equation has three distinct cases.

Real Distinct Roots. If $a_1^2 > 4a_0a_2$.

Real Repeated Roots. If $a_1^2 = 4a_0a_2$.

Complex Conjugate Roots. If $a_1^2 < 4a_0a_2$.

10.3.1 Real Distinct Roots

For two real and distinct roots, λ_1 and λ_2, the general solution is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

10.3.2 Real Repeated Roots

For the real repeated root, λ, the general solution is

$$y(x) = c_1 e^{\lambda x} + c_2 t e^{\lambda x}$$

10.3.3 Complex Conjugate Roots

For two complex conjugate roots, $\lambda = \alpha \pm \beta i$, the general solution is

$$y(x) = e^{\alpha x}(c_1 \cos{(\beta x)} + c_2 \sin{(\beta x)})$$

10.4 Nonhomogeneous ODEs