

Derivative Rules

| $f(x)$ | $f'(x)$ |
|---------------------|-------------------------|
| $u(x)v(x)$ | $u'v + uv'$ |
| $\frac{u(x)}{v(x)}$ | $\frac{u'v - uv'}{v^2}$ |
| $u(v(x))$ | $u'(v(x))v'(x)$ |
| x^n | nx^{n-1} |
| $\ln(u(x))$ | $\frac{u'(x)}{u(x)}$ |
| $\sin(ax)$ | $a \cos(ax)$ |
| $\cos(ax)$ | $-a \sin(ax)$ |
| $\tan(ax)$ | $a \sec^2(ax)$ |
| $\cot(ax)$ | $-a \csc^2(ax)$ |
| $\sec(ax)$ | $a \sec(ax) \tan(ax)$ |
| $\csc(ax)$ | $-a \csc(ax) \cot(ax)$ |

Trigonometric Identities

$$1 = \sin^2(x) + \cos^2(x)$$

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Partial Fraction Decomposition

$$\frac{p(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Integration Techniques

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

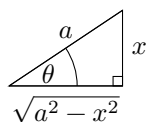
$$\int f(g(x)) \frac{dg(x)}{dx} dx = \int f(u) \frac{du}{dx} dx$$

where $u = g(x)$.

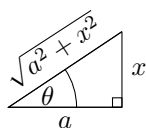
Trigonometric Substitutions

Substitute the appropriate value in and find that the value under the root is now square. Calculate $\frac{dx}{d\theta}$ to change the variable. Use substitution to find the new limits.

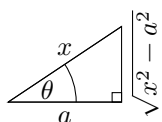
Sine ($a^2 - b^2x^2$, $x = \frac{a}{b} \sin(\theta)$)



Tangent ($a^2 + b^2x^2$, $x = \frac{a}{b} \tan(\theta)$)



Secant ($b^2x^2 - a^2$, $x = \frac{a}{b} \sec(\theta)$)



L'Hôpital's Rule

If $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$ or $\pm\infty$, then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}.$$

Continuity

$f(x)$ continuous at c iff $\lim_{x \rightarrow c} f(x) = f(c)$.

$f(x)$ is continuous on $I : (a, b)$ if it is continuous for all $x \in I$.

$f(x)$ is continuous on $I : [a, b]$ if it is continuous for all $x \in I$, but only right continuous at a and left continuous at b .

Intermediate Value Theorem

If $f(x)$ is continuous on $I : [a, b]$ and $f(a) \leq c \leq f(b)$, then $\exists x \in I : f(x) = c$.

Differentiability

$f(x)$ is differentiable at $x = x_0$ iff

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. This defines the derivative

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Differentiability implies continuity.

Mean Value Theorem

If $f(x)$ is continuous and differentiable on $I : [a, b]$, then

$$\exists c \in I : f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Definite Integrals

$$A = \int_a^b f(x) dx$$

Fundamental Theorem of Calculus

$$\int_a^b \frac{d}{dx} F(x) dx = F(b) - F(a)$$

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Taylor Polynomials / Series

$$f(x) \approx p_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

The Taylor Series is taking the limit as n goes to infinity of p_n .

Maclaurin Series: $x_0 = 0$.

Common Maclaurin Series

| Function | Series Term | Conv. |
|-------------------|---|-----------|
| e^x | $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ | all x |
| $\sin(x)$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ | all x |
| $\cos(x)$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ | all x |
| $\frac{1}{1-x}$ | $\sum_{n=0}^{\infty} x^n$ | $(-1, 1)$ |
| $\frac{1}{1+x^2}$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ | $(-1, 1)$ |
| $\ln(1+x)$ | $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$ | $(-1, 1]$ |

Power Series: $\sum_{n=0}^{\infty} c_n (x - x_0)^n$

Series Tests

For a series of the form $\sum_{i=0}^{\infty} a_i$:

Alternating Series Test

Given $a_i = (-1)^i b_i$ and $b_i > 0$. If $b_{i+1} \leq b_i$ & $\lim_{i \rightarrow \infty} b_i = 0$, then convergent, else inconclusive.

Ratio Test

Given $\rho = \lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right|$:

$\rho < 1$: convergent

$\rho > 1$: divergent

$\rho = 1$: inconclusive

Multivariable Functions

Partial Derivatives are done w.r.t one variable, other variables unchanged.

The **Gradient** is $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots \right\rangle$

Level Curves

$$L_c(f) = \{(x, y) : f(x, y) = c\}$$

If $f(x, y) \rightarrow L$ as $(x, y) \rightarrow (x_0, y_0)$, then

$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L$ along any smooth curve.

The limit does not exist if L changes along different smooth curves.

Multivariable Chain Rule

For $\frac{\partial f}{\partial s}$, create a tree from f to the variable s . On each edge, where the node above is a and below is b , each edge is $\frac{\partial a}{\partial b}$. Traverse each path between f and s and multiply the edges. Add each path together. (i.e. for $f(x, y), x(s, t), y(s, t)$, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$)

Directional Derivatives

$$\mathbf{D}_u f = \nabla_u f = \nabla f \cdot \mathbf{u}$$

where \mathbf{u} is a unit vector and the slope is given by $\|\nabla_u f\|$. If $\nabla_u f = 0$, \mathbf{u} is tangent to the level curve at x_0 .

$$\max_{\|\mathbf{u}\|=1} \nabla_u f = \nabla f$$

If $\nabla f \neq 0$, ∇f is a normal vector to the level curve at x_0 .

Critical Points

(x_0, y_0) is a critical point if $\nabla f(x_0, y_0) = \mathbf{0}$ or if $\nabla f(x_0, y_0)$ is undefined.

Classification of Critical Points

$$D = f_{xx}f_{yy} - (f_{xy})^2$$

$D > 0$ and $f_{xx} < 0$: local maxima

$D > 0$ and $f_{xx} > 0$: local minima

$D < 0$: saddle point

$D = 0$: inconclusive

Double Integrals

The volume of the solid enclosed between the surface $z = f(x, y)$ and the region Ω

is defined by

$$V = \iint_{\Omega} f(x, y) \, dA.$$

If Ω is a region bounded by $a \leq x \leq b$ and $c \leq y \leq d$, then

$$\begin{aligned} \iint_{\Omega} f(x, y) \, dA &= \int_c^d \int_a^b f(x, y) \, dx \, dy \\ &= \int_a^b \int_c^d f(x, y) \, dy \, dx \end{aligned}$$

Type I Regions

$$\iint_{\Omega} f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx$$

Bounded left & right by:

$$x = a \text{ and } x = b$$

Bounded below & above by:

$$y = g_1(x) \text{ and } y = g_2(x)$$

where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$:

Type II Regions

$$\iint_{\Omega} f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy$$

Bounded left & right by:

$$x = h_1(y) \text{ and } x = h_2(y)$$

Bounded below & above by:

$$y = c \text{ and } y = d$$

where $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$.

To integrate, solve the inner integrals first.

Vector Valued Functions

$$\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^n$$

The **domain** of $\mathbf{r}(t)$ is the intersection of the domains of its components.

The **orientation** of $\mathbf{r}(t)$ is the direction of motion along the curve as the value of the parameter increases.

Limits, derivatives and integrals are all *component-wise*. Each component has its own constant of integration.

Parametric Lines

$$\mathbf{l}(t) = \mathbf{P}_0 + t\mathbf{v}$$

where $\mathbf{l}(t)$ passes through \mathbf{P}_0 , and is parallel to \mathbf{v} .

Tangent Lines

If $\mathbf{r}(t)$ is differentiable at t_0 and $\mathbf{r}'(t_0) \neq \mathbf{0}$

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0).$$

Curves of Intersection

Choose one of the variables as the parameter, and express the remaining variables in terms of that parameter.

Arc Length

$$S = \int_a^b \|\mathbf{r}'(t)\| \, dt$$

Ordinary Differential Equations

Order: highest derivative in DE.

Autonomous DE: does not depend explicitly on the independent variable.

Qualitative Analysis

$$\frac{dy}{dt} = f(y)$$

A fixed point is the value of y for which $f(y) = 0$.

Stability of Fixed Points

Given a positive/negative perturbation from a fixed point, that point is

Stable: if both tend toward FP

Unstable: if both tend away from FP

Semi-Stable: if one tends toward FP, and another tends away from FP

Directly Integrable ODEs

For $\frac{dy}{dx} = f(x)$:

$$y(x) = \int f(x) \, dx.$$

Separable ODEs

For $\frac{dy}{dx} = p(x)q(y)$:

$$\int \frac{1}{q(y)} \frac{dy}{dx} \, dx = \int p(x) \, dx.$$

Linear ODEs

For $\frac{dy}{dx} + p(x)y = q(x)$, use the *integrating factor*: $I(x) = e^{\int p(x)dx}$, so that

$$y(x) = \frac{1}{I(x)} \int I(x)q(x) \, dx.$$

Exact ODEs

$P(x, y) + Q(x, y)\frac{dy}{dx} = 0$ has the solution $\Psi(x, y) = c$ iff it is exact, namely, when $P_y = Q_x$, where $P = \Psi_x$ and $Q = \Psi_y$. Then

$$\Psi(x, y) = \int P(x, y) \, dx + f(y)$$

$$\Psi(x, y) = \int Q(x, y) \, dy + g(x)$$

and $f(y)$ and $g(x)$ can be determined by solving these equations simultaneously.

Second-Order ODEs

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = F(x)$$

Initial Values

$$y(x_0) = y_0 \quad y'(x_0) = y_1$$

Boundary Values

$$y(x_0) = y_0 \quad y(x_1) = y_1$$

Reduction of Order

$$y_2(x) = v(x)y_1(x)$$

$v(x)$ can be determined by substituting y_2 into the ODE, using $w(x) = v'(x)$.

General Solution

$$y(x) = y_H(x) + y_P(x)$$

Homogeneous Solution

$$y_H(x) = e^{\lambda x}$$

Real Distinct Roots

$$y_H(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Real Repeated Roots

$$y_H(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

Complex Conjugate Roots

Given $\lambda = \alpha \pm \beta i$:

$$y_H(x) = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Particular Solution

| $F(x)$ | $y_P(x)$ |
|--------------------------------------|---|
| constant | A |
| polynomial degree n | $\sum_{i=0}^n A_i x^i$ |
| e^{kx} | $A e^{kx}$ |
| $\cos(\omega x)$ or $\sin(\omega x)$ | $A_0 \cos(\omega x) + A_1 \sin(\omega x)$ |

If $F(x)$ is a combination of the above, $y_P(x)$ should be too. If the choice would be linearly dependent to $y_H(x)$, multiply by x until it isn't.

Substitute y_P into the nonhomogeneous ODE, and solve the undetermined coefficients.

Spring and Mass Systems

$$my'' + \gamma y' + ky = f(t)$$

Newton's Law: $F = my''$

Spring force: $F_s = -ky$

Damping force: $F_d = -\gamma y'$

m : mass k : spring constant

γ : damping $f(t)$: external force

Electrical Circuits

The sum of voltages around a loop equals 0.

$$v(t) - iR - L \frac{di}{dt} - \frac{q}{C} = 0$$

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = v(t)$$

where $i = \frac{dq}{dt}$.

R : resistance C : capacitance

L : inductance $v(t)$: voltage supply