Calculus and Differential Equations

Queensland University of Technology

Dr Vivien Challis

2021, Semester 2

Tarang Janawalkar

Contents

1 Integration Techniques

1.1 Table of Derivatives

Let $f(x)$ be a function, and $a \in \mathbb{R}$ be a constant.

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
x^a	ax^{a-1}
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$
a^x	$\ln(a)a^x$
e^x	e^x
$\log_a x,\ a \in \mathbb{R}\setminus\{0\}$	$\dfrac{1}{a\ln x}$
$\ln x$	$\dfrac{1}{x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
a	0
x	1
$a_1 u(x) \pm a_2 v(x)$	$a_1 \dfrac{\mathrm{d}u}{\mathrm{d}x} \pm a_2 \dfrac{\mathrm{d}v}{\mathrm{d}x}$
$u(x)v(x)$	$\dfrac{\mathrm{d}u}{\mathrm{d}x} v + u \dfrac{\mathrm{d}v}{\mathrm{d}x}$
$\dfrac{u(x)}{v(x)}$	$\dfrac{\frac{\mathrm{d}u}{\mathrm{d}x} v - u \frac{\mathrm{d}v}{\mathrm{d}x}}{v(x)^2}$
$u(v(x))$	$\dfrac{\mathrm{d}u}{\mathrm{d}v} \dfrac{\mathrm{d}v}{\mathrm{d}x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\csc^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\csc(ax)$	$-a\csc(ax)\cot(ax)$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\arcsin(ax)$	$\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\dfrac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\dfrac{a}{1+a^2x^2}$
$\operatorname{arccot}(ax)$	$-\dfrac{a}{1+a^2x^2}$
$\operatorname{arcsec}(ax)$	$\dfrac{1}{x\sqrt{a^2x^2-1}}$
$\operatorname{arccsc}(ax)$	$-\dfrac{1}{x\sqrt{a^2x^2-1}}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sinh(ax)$	$a\cosh(ax)$
$\cosh(ax)$	$a\sinh(ax)$
$\tanh(ax)$	$a\,\mathrm{sech}^2(ax)$
$\coth(ax)$	$-a\,\mathrm{csch}^2(ax)$
$\mathrm{sech}(ax)$	$-a\,\mathrm{sech}(ax)\tan(ax)$
$\mathrm{csch}(ax)$	$-a\,\mathrm{csch}(ax)\cot(ax)$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\operatorname{arcsinh}(ax)$	$\frac{a}{\sqrt{1+a^2x^2}}$
$\operatorname{arccosh}(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\operatorname{arctanh}(ax)$	$\frac{a}{1-a^2x^2}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\operatorname{arccoth}(ax)$	$\frac{a}{1-a^2x^2}$
$\operatorname{arcsech}(ax)$	$-\frac{1}{a(1+ax)\sqrt{\frac{1-ax}{1+ax}}}$
$\operatorname{arccsch}(ax)$	$-\frac{1}{ax^2\sqrt{1+\frac{1}{a^2x^2}}}$

Table 1: Derivatives of Elementary Functions

1.2 Trigonometric Identities

1.2.1 Pythagorean Identities

$$\sin^2(x)+\cos^2(x)=1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x)+1=\sec^2(x)$$
$$1+\cot^2(x)=\csc^2(x)$$

1.2.2 Double-Angle Identities

$$\sin(2x)=2\sin(x)\cos(x) \qquad \csc(2x)=\frac{\sec(x)\csc(x)}{2}$$

$$\cos(2x)=\cos^2(x)-\sin^2(x) \qquad \sec(2x)=\frac{\sec^2(x)\csc^2(x)}{\csc^2(x)-\sec^2(x)}$$

$$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)} \qquad \cot(2x)=\frac{\cot^2(x)-1}{2\cot(x)}$$

1.2.3 Power Reducing Identities

$$\sin^2(x)=\frac{1-\cos(2x)}{2} \qquad \csc^2(x)=\frac{2}{1-\cos(2x)}$$

$$\cos^2(x)=\frac{1+\cos(2x)}{2} \qquad \sec^2(x)=\frac{2}{1+\cos(2x)}$$

$$\tan^2(x)=\frac{1-\cos(2x)}{1+\cos(2x)} \qquad \cot^2(x)=\frac{1+\cos(2x)}{1-\cos(2x)}$$

1.3 Partial Fractions

Definition 1.3.1 (Partial Fraction Decomposition)**. Partial fraction decomposition** is a method where a rational function $\dfrac{P(x)}{Q(x)}$ is rewritten as a sum of fraction.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k,\ k\in\mathbb{N}$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{A}{ax^2+bx+c}$
$(ax^2+bx+c)^k,\ k\in\mathbb{N}$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$

Table 2: Partial Fraction Forms

1.4 Integration by Parts

Theorem 1.4.1.

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

1.5 Integration by Substitution

Theorem 1.5.1.

$$\int f(g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\,,\ \textit{where}\ u=g(x)$$

1.6 Trigonometric Substitutions

Form	Substitution	Result	Domain
$(a^2-b^2x^2)^n$	$x=\dfrac{a}{b}\sin(\theta)$	$a^2\cos^2(\theta)$	$\theta\in[-\frac{\pi}{2},\ \frac{\pi}{2}]$
$(a^2+b^2x^2)^n$	$x=\dfrac{a}{b}\tan(\theta)$	$a^2\sec^2(\theta)$	$\theta\in(-\frac{\pi}{2},\ \frac{\pi}{2})$
$(b^2x^2-a^2)^n$	$x=\dfrac{a}{b}\sec(\theta)$	$a^2\tan^2(\theta)$	$\theta\in[0,\ \frac{\pi}{2})\cup(\frac{\pi}{2},\ \pi]$

Table 3: Trigonometric substitutions for various forms.

2 Limits, Continuity and Differentiability

2.1 Limits

Theorem 2.1.1 (Limits). *$\lim_{x \to x_0} f(x)$ exists if and only if $\lim_{x \to {x_0}^+} f(x)$ and $\lim_{x \to {x_0}^-} f(x)$ exist and are equal.*

Definition 2.1.1 (Finite limits using the ε-δ definition).

$$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

Theorem 2.1.2 (L'Hôpital's Rule). *For two differentiable functions $f(x)$ and $g(x)$. If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$, or $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \pm\infty$, then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.*

2.2 Continuity

Theorem 2.2.1 (Continuity at a Point). *$f(x)$ is continuous at c iff $\lim_{x \to c} f(x) = f(c)$.*

Theorem 2.2.2 (Continuity over an Interval). *$f(x)$ is continuous on I if $f(x)$ is continuous for all $x \in I$.*

- *$f(x)$ is continuous on $I : (a,\, b)$ if it is continuous for all $x \in I$.*
- *$f(x)$ is continuous on $I : [a,\, b]$ if it is continuous for all $x \in I$, but only right continuous at a and left continuous at b.*

If $f(x)$ is continuous on $(-\infty,\, \infty)$, $f(x)$ is continuous everywhere.

Theorem 2.2.3 (Intermediate Value Theorem). *If $f(x)$ is continuous on $I : [a,\, b]$ and c is any number between $f(a)$ and $f(b)$, inclusive, then there exists an $x \in I$ such that $f(x) = c$.*

2.3 Differentiability

Theorem 2.3.1 (Differentiability). *$f(x)$ is differentiable at $x = x_0$ iff*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, it defines the derivative

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x = x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Theorem 2.3.2. *$f(x)$ is differentiable on I if $f(x)$ is differentiable for all $x_0 \in I$.*

Theorem 2.3.3. *Differentiability implies continuity.*

Theorem 2.3.4 (Mean Value Theorem). *If $f(x)$ is continuous on $I : [a,\, b]$ and differentiable on I, then there exists a point $c \in I$ such that*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x = c} = \frac{f(b) - f(a)}{b - a}$$

3 Definite Integrals

Theorem 3.0.1. *If $f(x)$ is continuous on an interval $I : [a,\, b]$, then the net signed area A between the graph of $f(x)$ and the interval I is*

$$A = \int_a^b f(x)\,\mathrm{d}x$$

Properties of Definite Integrals

Theorem 3.0.2. *Suppose that $f(x)$ and $g(x)$ are continuous on the interval I, with $a,\, b,\, c \in I$ and $k \in \mathbb{R}$ then*

a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0.$

b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$

c) $\displaystyle\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x.$

d) $\displaystyle\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x.$

e) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$

3.1 Riemann Sums

Theorem 3.1.1. *Let A be the area under $f(x)$ on the interval $[a,\, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

where n is the number of rectangles, x_k is the centre of the rectangle k, and Δx_k is the width of the rectangle k. If every rectangle has the same width, then

$$\forall k : \Delta x_k = \frac{b-a}{n}$$

3.2 Fundamental Theorem of Calculus

The fundamental theorem of calculus provides a logical connection between infinite series (definite integrals) and antiderivatives (indefinite integrals).

Theorem 3.2.1 (The Fundamental Theorem of Calculus: Part 1)**.** *If $f(x)$ is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$ then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Equivalently

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x} F(x)\,\mathrm{d}x = F(b) - F(a) \equiv F(x)\big|_a^b$$

Theorem 3.2.2 (The Fundamental Theorem of Calculus: Part 2)**.** *If $f(x)$ is continuous on I then it has an antiderivative on I. In particular, if $a \in I$, then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is an antiderivative of $f(x)$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

Theorem 3.2.3. *Differentiation and integration are inverse operations.*

3.3 Taylor and Maclaurin Polynomials

Theorem 3.3.1 (Taylor Polynomials). *If $f(x)$ is a n differentiable function at x_0, then the nth degree Taylor polynomial for $f(x)$ near x_0, is given by*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Theorem 3.3.2 (Maclaurin Polynomials). *Evaluating a Taylor polynomial near 0, gives the nth degree Maclaurin polynomial for $f(x)$*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Theorem 3.3.3 (Error in Approximation). *Let $R_n(x)$ denote the difference between $f(x)$ and its nth Taylor polynomial, that is*

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = \frac{f^{(n+1)}(s)}{(n+1)!} (x - x_0)^{n+1}$$

where s is between x_0 and x.

4 Taylor and Maclaurin Series

4.1 Infinite Series

Definition 4.1.1 (Taylor Series)**.** If $f(x)$ has derivatives of all orders at x_0, then the Taylor series for $f(x)$ about $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Definition 4.1.2 (Maclaurin Series)**.** If a Taylor series is centred on $x_0 = 0$, it is called a Maclaurin series, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 4.1.3 (Power Series)**.** Both Taylor and Maclaurin series are examples of **power series**, which are defined as follows

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

4.2 Convergence

Theorem 4.2.1 (Convergence of a Taylor Series)**.** *The equality*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

holds at a point x iff

$$\begin{aligned} \lim_{n\to\infty} \left[f(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \right] &= 0 \\ \lim_{n\to\infty} R_n(x) &= 0 \end{aligned}$$

Definition 4.2.1 (Interval of Convergence)**.** The interval of convergence for a power series is the set of x values for which that series converges.

Definition 4.2.2 (Radius of Convergence)**.** The radius of convergence R is a nonnegative real number or ∞ such that a power series converges if

$$|x - a| < R$$

and diverges if

$$|x - a| > R$$

The behaviour of the power series on the boundary, that is, where $|x - a| = R$, can be determined by substituting $x = R + a$ for the upper boundary, and $x = -R + a$ for the lower boundary.

4.3 Convergence Tests

For any power series of the form $\sum_{i=i_0}^{\infty} a_i$.

Alternating Series

Conditions $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$\text{Is } b_{i+1} \leqslant b_i \ \& \ \lim_{i\to\infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

Ratio Test

$$\text{Is } \lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

The ratio test is inconclusive if $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = 1$.

4.4 Table of Maclaurin Series

Function	**Series**	**Interval of Convergence**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$-\infty < x < \infty$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

Table 4: Maclaurin Series of Common Functions

5 Multivariable Calculus

5.1 Multivariable Functions

Definition 5.1.1. A function is multivariable if its domain consists of several variables. In the reals, these functions are defined

$$f : \mathbb{R}^n \to \mathbb{R}$$

5.2 Level Curves

Definition 5.2.1. Level curves or *contour curves* of a function of two variables is a curve along which the function has constant value.

$$L_c(f) = \{(x,\, y) : f(x,\, y) = c\}$$

The level curves of a function can be determined by substituting $z = c$, and solving for y.

5.3 Limits and Continuity

Definition 5.3.1 (Finite Limit of Multivariable Functions using the ε-δ Definition)**.**

$$\begin{gathered}\lim_{(x_1,\,\dots,\,x_n)\to(c_1,\,\dots,\,c_n)} f(x_1,\,\dots,\,x_n) = L \\ \iff \forall\varepsilon > 0 : \exists\delta > 0 : \forall(x_1,\,\dots,\,x_n) \in I : \\ 0 < |x_1 - c_1,\,\dots,\,x_n - c_n| < \delta \implies |f(x_1,\,\dots,\,x_n) - L| < \varepsilon\end{gathered}$$

Theorem 5.3.1 (Limits along Smooth Curves)**.** *If $f(x,\,y) \to L$ as $(x,\,y) \to (x_0,\,y_0)$, then $\lim_{(x,\,y)\to(x_0,\,y_0)} = L$ along any smooth curve.*

Theorem 5.3.2 (Existence of a Limit)**.** *If the limit of $f(x,\,y)$ changes along different smooth curves, then $\lim_{(x,\,y)\to(x_0,\,y_0)}$ does not exist.*

Theorem 5.3.3 (Continuity of Multivariable Functions)**.** *A function $f(x_1,\,\dots,\,x_n)$ is continuous at $(c_1,\,\dots,\,c_n)$ iff*

$$\lim_{(x_1,\,\dots,\,x_n)\to(c_1,\,\dots,\,c_n)} f(x_1,\,\dots,\,x_n) = f(c_1,\,\dots,\,c_n)$$

Recognising continuous functions:

- A sum, difference or product of continuous functions is continuous.
- A quotient of continuous functions is continuous expect where the denominator is zero.
- A composition of continuous functions is continuous.

5.4 Partial Derivatives

Definition 5.4.1 (Partial Differentiation)**.** The partial derivative of a multivariable function is its derivative with respect to one of those variables, while the others are held constant.

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1,\,\dots,\,x_{i-1},\,x_i + h,\,x_{i+1},\,\dots,\,x_n) - f(x_1,\,\dots,\,x_n)}{h}$$

5.5 The Gradient Vector

Definition 5.5.1. Let $\boldsymbol{\nabla}$, pronounced "del", denote the vector differential operator defined as follows

$$\boldsymbol{\nabla} = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{bmatrix}$$

5.6 Multivariable Chain Rule

Definition 5.6.1. Let $f = f(\boldsymbol{x}(t_1, \dots, t_n))$ be the composition of f with $\boldsymbol{x} = [x_1 \quad \cdots \quad x_m]$, then the partial derivative of f with respect to t_i is given by

$$\frac{\partial f}{\partial t_i} = \boldsymbol{\nabla} f \cdot \partial_{t_i} \boldsymbol{x}$$

5.7 Directional Derivatives

Definition 5.7.1. The directional derivative $\boldsymbol{\nabla}_{\boldsymbol{u}} f$ is the rate at which the function f changes in the direction $\boldsymbol{u}$.

$$\begin{aligned} \boldsymbol{\nabla}_{\boldsymbol{u}} f &= \lim_{h \to 0} \frac{f(\boldsymbol{x} + h\boldsymbol{u}) - f(\boldsymbol{x})}{h} \\ &= \boldsymbol{\nabla} f \cdot \boldsymbol{u} \end{aligned}$$

where the slope is given by $\|\boldsymbol{\nabla}_{\boldsymbol{u}} f\|$.

Remark. *The directional derivative of f can be denoted in several ways:*

$$\boldsymbol{\nabla}_{\boldsymbol{u}} f = D_{\boldsymbol{u}} f = \partial_{\boldsymbol{u}} f = \frac{\partial f}{\partial \boldsymbol{u}}$$

Theorem 5.7.1 (Direction of Greatest Ascent)**.** *The direction of greatest ascent is given by*

$$\max_{\|\boldsymbol{u}\|=1} \boldsymbol{\nabla}_{\boldsymbol{u}} f = \boldsymbol{\nabla} f$$

where the slope is given by $\|\boldsymbol{\nabla} f\|$.

Theorem 5.7.2 (Direction of Greatest Descent)**.** *The direction of greatest descent is given by*

$$\min_{\|\boldsymbol{u}\|=1} \boldsymbol{\nabla}_{\boldsymbol{u}} f = -\boldsymbol{\nabla} f$$

where the slope is given by $-\|\boldsymbol{\nabla} f\|$.

Proof. Given that $\boldsymbol{u}$ is a unit vector, the dot product definition gives

$$\begin{aligned} \boldsymbol{\nabla}_{\boldsymbol{u}} f &= \boldsymbol{\nabla} f \cdot \boldsymbol{u} \\ &= \|\boldsymbol{\nabla} f\| \|\boldsymbol{u}\| \cos(\theta) \\ &= \|\boldsymbol{\nabla} f\| \cos(\theta) \qquad (1) \end{aligned}$$

Equation 1 is maximised when $\cos(\theta)$ is maximised. Thus the maximum slope is given by

$$\max \boldsymbol{\nabla}_{\boldsymbol{u}} f = \|\boldsymbol{\nabla} f\|$$

and the direction of greatest ascent is given by

$$\boldsymbol{u} = \boldsymbol{\nabla} f$$

□

Theorem 5.7.3 (The Gradient is Normal to the Level Curves of f)**.** *If $\boldsymbol{\nabla} f = 0$, then $\boldsymbol{\nabla} f$ is normal to the level curves of f at any point $(c_1, \ldots, c_n)$.*

5.8 Higher-Order Partial Derivatives

Definition 5.8.1. Higher-order partial derivatives can be denoted using three different notation. The following table shows the mixed partial derivative of $f(x, y)$ w.r.t. x then y.

Leibniz	**Euler**	**Legendre**
$\dfrac{\partial^2 f}{\partial y \partial x}$	$\partial_{xy} f$	f_{xy}

Table 5: Mixed Partial Derivative Notation

For partial derivatives w.r.t. the same variable, a superscript can be used in Euler notation.

Leibniz	**Euler**	**Legendre**
$\dfrac{\partial^2 f}{\partial x^2}$	$\partial_x^2 f$	f_{xx}

Table 6: Second-Order Partial Derivative Notation

5.9 Hessian Matrix

Definition 5.9.1. Let the Hessian matrix $\mathbf{H}$ be the matrix of second-order partial derivative operators defined as shown below

$$\mathbf{H} = \begin{bmatrix} \partial_{x_1}^2 & \cdots & \partial_{x_n x_1} \\ \vdots & \ddots & \vdots \\ \partial_{x_1 x_n} & \cdots & \partial_{x_n}^2 \end{bmatrix}$$

5.10 Critical Points

6 Double and Triple Integrals

6.1 Double Integrals

Theorem 6.1.1. *Divide the rectangular region of R into n rectangles with sides parallel to the coordinate axes. Discard rectangles which contain any points outside of R. Choose an arbitrary point in each remaining rectangle. The area of the kth remaining rectangle is ΔA_k. The arbitrary point in the kth remaining rectangle is $({x_k}^*,\ {y_k}^*)$. The Riemann sum is*

$$\iint\limits_R f(x,\ y)\,\mathrm{d}A = \sum_{k=1}^{\infty} f({x_k}^*,\ {y_k}^*)\Delta A_k$$

Properties of Double Integrals

Theorem 6.1.2. *Suppose that $f(x,\ y)$ and $g(x,\ y)$ are continuous on R and R can be subdivided into R_1 and R_2 then*

a) $\displaystyle\iint\limits_R kf(x,\ y)\,\mathrm{d}A = k\iint\limits_R f(x,\ y)\,\mathrm{d}A.$

b) $\displaystyle\iint\limits_R (f(x,\ y) + g(x,\ y))\,\mathrm{d}A = \iint\limits_R f(x,\ y)\,\mathrm{d}A + \iint\limits_R g(x,\ y)\,\mathrm{d}A.$

c) $\displaystyle\iint\limits_R f(x,\ y)\,\mathrm{d}A = \iint\limits_{R_1} f(x,\ y)\,\mathrm{d}A + \iint\limits_{R_2} f(x,\ y)\,\mathrm{d}A.$

6.2 Triple Integrals

Definition 6.2.1. A triple integral is of a function is the net signed volume defined over a finite closed solid region G in an xyz coordinate system.

Theorem 6.2.1. *Divide the bounding box of G into n boxes with sides parallel to the coordinate planes. Discard boxes which contain any points outside of G. Choose an arbitrary point in each remaining box. The volume of the kth remaining box is ΔV_k. The arbitrary point in the kth remaining box is $({x_k}^*,\ {y_k}^*,\ {z_k}^*)$. The Riemann sum is*

$$\iiint\limits_G f(x,\ y,\ z)\,\mathrm{d}V = \sum_{k=1}^{\infty} f({x_k}^*,\ {y_k}^*,\ {z_k}^*)\Delta V_k$$

Properties of Triple Integrals

Theorem 6.2.2. *Suppose that $f(x, y, z)$ and $g(x, y, z)$ are continuous on G and G can be subdivided into G_1 and G_2 then*

a) $\displaystyle\iiint_G kf(x, y, z)\,\mathrm{d}V = k\iiint_G f(x, y, z)\,\mathrm{d}V.$

b) $\displaystyle\iiint_G (f(x, y, z) + g(x, y, z))\,\mathrm{d}V = \iiint_G f(x, y, z)\,\mathrm{d}V + \iiint_G g(x, y, z)\,\mathrm{d}V.$

c) $\displaystyle\iiint_G f(x, y, z)\,\mathrm{d}V = \iiint_{G_1} f(x, y, z)\,\mathrm{d}V + \iiint_{G_2} f(x, y, z)\,\mathrm{d}V.$

7 Vector-Valued Functions

Definition 7.0.1. A Vector-Valued Function or VVF is some function with domain $\mathbb{R}$ and codomain $\mathbb{R}^n$. For example, $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ is a VVF with $\mathbf{r}(t) = \langle\, x(t),\ y(t),\ (z(t)\,\rangle$ where $x,\ y,\ z : \mathbb{R} \to \mathbb{R}$.

Theorem 7.0.1. *The domain of $\mathbf{r}(t)$ is the intersection of the domains of its components.*

Definition 7.0.2 (Orientation)**.** The orientation of $\mathbf{r}(t)$ is the direction of increasing parameter (t).

Theorem 7.0.2 (Limits of VVFs)**.** *The limit of a VFF is the vector of the limits of its components. E.g.* $\lim_{x\to a} \mathbf{r}(t) = \langle\, \lim_{x\to a} x(t),\ \lim_{x\to a} y(t),\ \lim_{x\to a} z(t)\,\rangle$

Theorem 7.0.3 (Derivatives of VVFs)**.** *The derivative of a VFF is the vector of the derivatives of its components. For example,* $\mathbf{r}'(t) = \langle\, x'(t),\ y'(t),\ z'(t)\,\rangle$

Theorem 7.0.4 (Integration of VVFs)**.** *The integral of a VFF is the vector of the integrals of its components. E.g.* $\int \mathbf{r}(t)\,\mathrm{d}t = \langle\, \int x(t)\,\mathrm{d}t\,,\ \int y(t)\,\mathrm{d}t\,,\ \int z(t)\,\mathrm{d}t\,\rangle$

Remark. *When integrating a VVF, each component has its own constant of integration.*

Definition 7.0.3 (VVF of a Line)**.** A line can be determined by a point which it passes through, $\boldsymbol{P}_0$ and a vector the line is parallel to, $\boldsymbol{v}$. The VVF of a line can be written as $\mathbf{r}(t) = \boldsymbol{P}_0 + t\boldsymbol{v}$.

Definition 7.0.4 (Tangent Line of a VVF)**.** When some VVF $\mathbf{r}(t)$ is differentiable at t_0 and $\mathbf{r}'(t_0) \neq 0$, the tangent line of $\mathbf{r}(t)$ is $\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$.

Definition 7.0.5 (Arc Length of a VVF)**.** The arc length S of some VVF $\mathbf{r}(t)$ is the distance along $\mathbf{r}(t)$ between two points where $t = a$ and $t = b$ and the curve is smooth (defined and nonzero). The arc length is given by

$$S = \int_a^b ||\,\mathbf{r}'(t)\,||\,\mathrm{d}t$$

8 First-Order Differential Equations

9 Second-Order Differential Equations