

# Calculus and Differential Equations

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# 1 Integration Techniques

## 1.1 Table of Derivatives

Let  $f(x)$  be a function, and  $a \in \mathbb{R}$  be a constant.

$f$	$\frac{df}{dx}$	$f$	$\frac{df}{dx}$
$x^a$	$ax^{a-1}$	$a$	0
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$x$	1
$a^x$	$\ln(a)a^x$	$a_1u(x) \pm a_2v(x)$	$a_1\frac{du}{dx} \pm a_2\frac{dv}{dx}$
$e^x$	$e^x$	$u(x)v(x)$	$\frac{du}{dx}v + u\frac{dv}{dx}$
$\log_a x, a \in \mathbb{R} \setminus \{0\}$	$\frac{1}{a \ln x}$	$\frac{u(x)}{v(x)}$	$\frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v(x)^2}$
$\ln x$	$\frac{1}{x}$	$u(v(x))$	$\frac{du}{dv}\frac{dv}{dx}$

$f$	$\frac{df}{dx}$	$f$	$\frac{df}{dx}$
$\sin(ax)$	$a \cos(ax)$	$\arcsin(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\cos(ax)$	$-a \sin(ax)$	$\arccos(ax)$	$-\frac{a}{\sqrt{1-a^2x^2}}$
$\tan(ax)$	$a \sec^2(ax)$	$\arctan(ax)$	$\frac{a}{1+a^2x^2}$
$\cot(ax)$	$-a \csc^2(ax)$	$\operatorname{arccot}(ax)$	$-\frac{a}{1+a^2x^2}$
$\sec(ax)$	$a \sec(ax) \tan(ax)$	$\operatorname{arcsec}(ax)$	$\frac{1}{x\sqrt{a^2x^2-1}}$
$\csc(ax)$	$-a \csc(ax) \cot(ax)$	$\operatorname{arccsc}(ax)$	$-\frac{1}{x\sqrt{a^2x^2-1}}$

$f$	$\frac{df}{dx}$	$f$	$\frac{df}{dx}$	$f$	$\frac{df}{dx}$
$\sinh(ax)$	$a \cosh(ax)$	$\operatorname{arcsinh}(ax)$	$\frac{a}{\sqrt{1+a^2x^2}}$	$\operatorname{arccoth}(ax)$	$\frac{a}{1-a^2x^2}$
$\cosh(ax)$	$a \sinh(ax)$	$\operatorname{arccosh}(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$	$\operatorname{arcsech}(ax)$	$-\frac{1}{a(1+ax)\sqrt{\frac{1-ax}{1+ax}}}$
$\tanh(ax)$	$a \operatorname{sech}^2(ax)$	$\operatorname{arctanh}(ax)$	$\frac{a}{1-a^2x^2}$	$\operatorname{arccsch}(ax)$	$-\frac{1}{ax^2\sqrt{1+\frac{1}{a^2x^2}}}$
$\coth(ax)$	$-a \operatorname{csch}^2(ax)$				
$\operatorname{sech}(ax)$	$-a \operatorname{sech}(ax) \tanh(ax)$				
$\operatorname{csch}(ax)$	$-a \operatorname{csch}(ax) \cot(ax)$				

Table 1: Derivatives of Elementary Functions

## 1.2 Trigonometric Identities

### 1.2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x) + 1 = \sec^2(x)$$

$$1 + \cot^2(x) = \csc^2(x)$$

### 1.2.2 Double-Angle Identities

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\csc(2x) = \frac{\sec(x) \csc(x)}{2}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\sec(2x) = \frac{\sec^2(x) \csc^2(x)}{\csc^2(x) - \sec^2(x)}$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\cot(2x) = \frac{\cot^2(x) - 1}{2 \cot(x)}$$

### 1.2.3 Power Reducing Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\csc^2(x) = \frac{2}{1 - \cos(2x)}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\sec^2(x) = \frac{2}{1 + \cos(2x)}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$\cot^2(x) = \frac{1 + \cos(2x)}{1 - \cos(2x)}$$

### 1.3 Partial Fractions

**Definition 1.3.1** (Partial Fraction Decomposition). **Partial fraction decomposition** is a method where a rational function  $\frac{P(x)}{Q(x)}$  is rewritten as a sum of fraction.

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^k, k \in \mathbb{N}$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_k}{(ax + b)^k}$
$ax^2 + bx + c$	$\frac{A}{ax^2 + bx + c}$
$(ax^2 + bx + c)^k, k \in \mathbb{N}$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_k}{(ax + b)^k}$

Table 2: Partial Fraction Forms

### 1.4 Integration by Parts

**Theorem 1.4.1.**

$$\int u \, dv = uv - \int v \, du$$

### 1.5 Integration by Substitution

**Theorem 1.5.1.**

$$\int f(g(x)) \frac{dg(x)}{dx} dx = \int f(u) du, \text{ where } u = g(x)$$

### 1.6 Trigonometric Substitutions

Form	Substitution	Result	Domain
$(a^2 - b^2x^2)^n$	$x = \frac{a}{b} \sin(\theta)$	$a^2 \cos^2(\theta)$	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
$(a^2 + b^2x^2)^n$	$x = \frac{a}{b} \tan(\theta)$	$a^2 \sec^2(\theta)$	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$
$(b^2x^2 - a^2)^n$	$x = \frac{a}{b} \sec(\theta)$	$a^2 \tan^2(\theta)$	$\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

Table 3: Trigonometric substitutions for various forms.

## 2 Limits, Continuity and Differentiability

### 2.1 Limits

**Theorem 2.1.1** (Limits).  $\lim_{x \rightarrow x_0} f(x)$  exists if and only if  $\lim_{x \rightarrow x_0^+} f(x)$  and  $\lim_{x \rightarrow x_0^-} f(x)$  exist and are equal.

**Definition 2.1.1** (Finite limits using the  $\varepsilon$ - $\delta$  definition).

$$\lim_{x \rightarrow x_0} f(x) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

**Theorem 2.1.2** (L'Hôpital's Rule). For two differentiable functions  $f(x)$  and  $g(x)$ . If  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$ , or  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = \pm\infty$ , then  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ .

### 2.2 Continuity

**Theorem 2.2.1** (Continuity at a Point).  $f(x)$  is continuous at  $c$  iff  $\lim_{x \rightarrow c} f(x) = f(c)$ .

**Theorem 2.2.2** (Continuity over an Interval).  $f(x)$  is continuous on  $I$  if  $f(x)$  is continuous for all  $x \in I$ .

- $f(x)$  is continuous on  $I : (a, b)$  if it is continuous for all  $x \in I$ .
- $f(x)$  is continuous on  $I : [a, b]$  if it is continuous for all  $x \in I$ , but only right continuous at  $a$  and left continuous at  $b$ .

If  $f(x)$  is continuous on  $(-\infty, \infty)$ ,  $f(x)$  is continuous everywhere.

**Theorem 2.2.3** (Intermediate Value Theorem). If  $f(x)$  is continuous on  $I : [a, b]$  and  $c$  is any number between  $f(a)$  and  $f(b)$ , inclusive, then there exists an  $x \in I$  such that  $f(x) = c$ .

### 2.3 Differentiability

**Theorem 2.3.1** (Differentiability).  $f(x)$  is differentiable at  $x = x_0$  iff

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, it defines the derivative

$$\left. \frac{df}{dx} \right|_{x=x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

**Theorem 2.3.2.**  $f(x)$  is differentiable on  $I$  if  $f(x)$  is differentiable for all  $x_0 \in I$ .

**Theorem 2.3.3.** Differentiability implies continuity.

**Theorem 2.3.4** (Mean Value Theorem). If  $f(x)$  is continuous on  $I : [a, b]$  and differentiable on  $I$ , then there exists a point  $c \in I$  such that

$$\left. \frac{df}{dx} \right|_{x=c} = \frac{f(b) - f(a)}{b - a}$$

### 3 Definite Integrals

**Theorem 3.0.1.** *If  $f(x)$  is continuous on an interval  $I : [a, b]$ , then the net signed area  $A$  between the graph of  $f(x)$  and the interval  $I$  is*

$$A = \int_a^b f(x) \, dx$$

#### Properties of Definite Integrals

**Theorem 3.0.2.** *Suppose that  $f(x)$  and  $g(x)$  are continuous on the interval  $I$ , with  $a, b, c \in I$  and  $k \in \mathbb{R}$  then*

- a)  $\int_a^a f(x) \, dx = 0.$
- b)  $\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.$
- c)  $\int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx.$
- d)  $\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$
- e)  $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$

#### 3.1 Riemann Sums

**Theorem 3.1.1.** *Let  $A$  be the area under  $f(x)$  on the interval  $[a, b]$ , then*

$$\int_a^b f(x) \, dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k) \Delta x_k$$

where  $n$  is the number of rectangles,  $x_k$  is the centre of the rectangle  $k$ , and  $\Delta x_k$  is the width of the rectangle  $k$ . If every rectangle has the same width, then

$$\forall k : \Delta x_k = \frac{b-a}{n}$$

#### 3.2 Fundamental Theorem of Calculus

The fundamental theorem of calculus provides a logical connection between infinite series (definite integrals) and antiderivatives (indefinite integrals).



**Theorem 3.2.1** (The Fundamental Theorem of Calculus: Part 1). *If  $f(x)$  is continuous on  $[a, b]$  and  $F$  is any antiderivative of  $f$  on  $[a, b]$  then*

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

*Equivalently*

$$\int_a^b \frac{d}{dx} F(x) \, dx = F(b) - F(a) \equiv F(x) \Big|_a^b$$

**Theorem 3.2.2** (The Fundamental Theorem of Calculus: Part 2). *If  $f(x)$  is continuous on  $I$  then it has an antiderivative on  $I$ . In particular, if  $a \in I$ , then the function  $F$  defined by*

$$F(x) = \int_a^x f(t) \, dt$$

*is an antiderivative of  $f(x)$ . That is,*

$$\frac{d}{dx} F(x) = f(x) \equiv \frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$

**Theorem 3.2.3.** *Differentiation and integration are inverse operations.*

### 3.3 Taylor and Maclaurin Polynomials

**Theorem 3.3.1** (Taylor Polynomials). *If  $f(x)$  is a  $n$  differentiable function at  $x_0$ , then the  $n$ th degree Taylor polynomial for  $f(x)$  near  $x_0$ , is given by*

$$f(x) \approx p_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

**Theorem 3.3.2** (Maclaurin Polynomials). *Evaluating a Taylor polynomial near 0, gives the  $n$ th degree Maclaurin polynomial for  $f(x)$*

$$f(x) \approx p_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

**Theorem 3.3.3** (Error in Approximation). *Let  $R_n(x)$  denote the difference between  $f(x)$  and its  $n$ th Taylor polynomial, that is*

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = \frac{f^{(n+1)}(s)}{(n+1)!} (x - x_0)^{n+1}$$

where  $s$  is between  $x_0$  and  $x$ .

## 4 Taylor and Maclaurin Series

### 4.1 Infinite Series

**Definition 4.1.1** (Taylor Series). If  $f(x)$  has derivatives of all orders at  $x_0$ , then the Taylor series for  $f(x)$  about  $x = x_0$  is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

**Definition 4.1.2** (Maclaurin Series). If a Taylor series is centred on  $x_0 = 0$ , it is called a Maclaurin series, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

**Definition 4.1.3** (Power Series). Both Taylor and Maclaurin series are examples of **power series**, which are defined as follows

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

### 4.2 Convergence

**Theorem 4.2.1** (Convergence of a Taylor Series). *The equality*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

*holds at a point  $x$  iff*

$$\lim_{n \rightarrow \infty} \left[ f(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \right] = 0$$

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

**Definition 4.2.1** (Interval of Convergence). The interval of convergence for a power series is the set of  $x$  values for which that series converges.

**Definition 4.2.2** (Radius of Convergence). The radius of convergence  $R$  is a nonnegative real number or  $\infty$  such that a power series converges if

$$|x - a| < R$$

and diverges if

$$|x - a| > R$$

The behaviour of the power series on the boundary, that is, where  $|x - a| = R$ , can be determined by substituting  $x = R + a$  for the upper boundary, and  $x = -R + a$  for the lower boundary.

### 4.3 Convergence Tests

For any power series of the form  $\sum_{i=i_0}^{\infty} a_i$ .

#### Alternating Series

**Conditions**  $a_i = (-1)^i b_i$  or  $a_i = (-1)^{i+1} b_i$ .  $b_i > 0$ .

$$\text{Is } b_{i+1} \leq b_i \text{ \& } \lim_{i \rightarrow \infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

#### Ratio Test

$$\text{Is } \lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

The ratio test is inconclusive if  $\lim_{i \rightarrow \infty} \frac{a_{i+1}}{a_i} = 1$ .

### 4.4 Table of Maclaurin Series

Function	Series	Interval of Convergence
$e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$-\infty < x < \infty$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

Table 4: Maclaurin Series of Common Functions

## 5 Multivariable Calculus

### 5.1 Multivariable Functions

**Definition 5.1.1.** A function is multivariable if its domain consists of several variables. In the reals, these functions are defined

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

### 5.2 Level Curves

**Definition 5.2.1.** Level curves or *contour curves* of a function of two variables is a curve along which the function has constant value.

$$L_c(f) = \{(x, y) : f(x, y) = c\}$$

The level curves of a function can be determined by substituting  $z = c$ , and solving for  $y$ .

### 5.3 Limits and Continuity

**Definition 5.3.1** (Finite Limit of Multivariable Functions using the  $\varepsilon$ - $\delta$  Definition).

$$\lim_{(x_1, \dots, x_n) \rightarrow (c_1, \dots, c_n)} f(x_1, \dots, x_n) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall (x_1, \dots, x_n) \in I : \\ 0 < |x_1 - c_1, \dots, x_n - c_n| < \delta \implies |f(x_1, \dots, x_n) - L| < \varepsilon$$

**Theorem 5.3.1** (Limits along Smooth Curves). *If  $f(x, y) \rightarrow L$  as  $(x, y) \rightarrow (x_0, y_0)$ , then  $\lim_{(x, y) \rightarrow (x_0, y_0)} = L$  along any smooth curve.*

**Theorem 5.3.2** (Existence of a Limit). *If the limit of  $f(x, y)$  changes along different smooth curves, then  $\lim_{(x, y) \rightarrow (x_0, y_0)}$  does not exist.*

**Theorem 5.3.3** (Continuity of Multivariable Functions). *A function  $f(x_1, \dots, x_n)$  is continuous at  $(c_1, \dots, c_n)$  iff*

$$\lim_{(x_1, \dots, x_n) \rightarrow (c_1, \dots, c_n)} f(x_1, \dots, x_n) = f(c_1, \dots, c_n)$$

Recognising continuous functions:

- A sum, difference or product of continuous functions is continuous.
- A quotient of continuous functions is continuous except where the denominator is zero.
- A composition of continuous functions is continuous.

### 5.4 Partial Derivatives

**Definition 5.4.1** (Partial Differentiation). The partial derivative of a multivariable function is its derivative with respect to one of those variables, while the others are held constant.

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

## 5.5 The Gradient Vector

**Definition 5.5.1.** Let  $\nabla$ , pronounced “del”, denote the vector differential operator defined as follows

$$\nabla = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{bmatrix}$$

## 5.6 Multivariable Chain Rule

**Definition 5.6.1.** Let  $f = f(\mathbf{x}(t_1, \dots, t_n))$  be the composition of  $f$  with  $\mathbf{x} = [x_1 \ \dots \ x_n]$ , then the partial derivative of  $f$  with respect to  $t_i$  is given by

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \partial_{t_i} \mathbf{x}$$

## 5.7 Directional Derivatives

**Definition 5.7.1.** The directional derivative  $\nabla_{\mathbf{u}} f$  is the rate at which the function  $f$  changes in the direction  $\mathbf{u}$ .

$$\begin{aligned} \nabla_{\mathbf{u}} f &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} \\ &= \nabla f \cdot \mathbf{u} \end{aligned}$$

where the slope is given by  $\|\nabla_{\mathbf{u}} f\|$ .

**Remark.** The directional derivative of  $f$  can be denoted in several ways:

$$\nabla_{\mathbf{u}} f = D_{\mathbf{u}} f = \partial_{\mathbf{u}} f = \frac{\partial f}{\partial \mathbf{u}}$$

**Theorem 5.7.1** (Direction of Greatest Ascent). *The direction of greatest ascent is given by*

$$\max_{\|\mathbf{u}\|=1} \nabla_{\mathbf{u}} f = \nabla f$$

where the slope is given by  $\|\nabla f\|$ .

**Theorem 5.7.2** (Direction of Greatest Descent). *The direction of greatest descent is given by*

$$\min_{\|\mathbf{u}\|=1} \nabla_{\mathbf{u}} f = -\nabla f$$

where the slope is given by  $-\|\nabla f\|$ .

*Proof.* Given that  $\mathbf{u}$  is a unit vector, the dot product definition gives

$$\begin{aligned} \nabla_{\mathbf{u}} f &= \nabla f \cdot \mathbf{u} \\ &= \|\nabla f\| \|\mathbf{u}\| \cos(\theta) \\ &= \|\nabla f\| \cos(\theta) \end{aligned} \tag{1}$$

Equation 1 is maximised when  $\cos(\theta)$  is maximised. Thus the maximum slope is given by

$$\max \nabla_{\mathbf{u}} f = \|\nabla f\|$$

and the direction of greatest ascent is given by

$$\mathbf{u} = \nabla f$$

□

**Theorem 5.7.3.** *If  $\nabla f = 0$ , then  $\nabla f$  is normal to the level curves of  $f$  at any point  $(c_1, \dots, c_n)$ .*

## 5.8 Higher-Order Partial Derivatives

**Definition 5.8.1.** Higher-order partial derivatives can be denoted using three different notation. The following table shows the mixed partial derivative of  $f(x, y)$  w.r.t.  $x$  then  $y$ .

Leibniz	Euler	Legendre
$\frac{\partial^2 f}{\partial y \partial x}$	$\partial_{xy} f$	$f_{xy}$

Table 5: Mixed Partial Derivative Notation

For partial derivatives w.r.t. the same variable, a superscript can be used in Euler notation.

Leibniz	Euler	Legendre
$\frac{\partial^2 f}{\partial x^2}$	$\partial_x^2 f$	$f_{xx}$

Table 6: Second-Order Partial Derivative Notation

## 5.9 Hessian Matrix

**Definition 5.9.1.** Let the Hessian matrix  $\mathbf{H}$  be the matrix of second-order partial derivative operators defined as shown below

$$\mathbf{H} = \begin{bmatrix} \partial_{x_1}^2 & \cdots & \partial_{x_n x_1} \\ \vdots & \ddots & \vdots \\ \partial_{x_1 x_n} & \cdots & \partial_{x_n}^2 \end{bmatrix}$$

## 5.10 Critical Points

## 6 Double and Triple Integrals

### 6.1 Volume under a Two Variable Function

**Definition 6.1.1.** If  $f$  is a function of two variables that is continuous and nonnegative on a region  $\Omega$  in the  $xy$ -plane, then the volume of the solid enclosed between the surface  $z = f(x, y)$  and the region  $\Omega$  is defined by

$$V = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k \quad (1)$$

*Proof.* Using lines parallel to the coordinate axes, the region  $\Omega$  can be divided into  $n$  rectangles, where any rectangles outside  $\Omega$  are discarded. The area of the  $k$ th remaining rectangle at the arbitrary point  $(x_k^*, y_k^*)$  is given by  $\Delta A_k$ . Thus the product  $f(x_k^*, y_k^*) \Delta A_k$  is the volume of the  $k$ th rectangular parallelepiped, and the sum of all  $n$  volumes over the region  $\Omega$  approximate the volume  $V$  of the entire solid.  $\square$

### 6.2 Double Integral

**Definition 6.2.1.** By extension of the definite integral of a one variable function expressed in Theorem 3.1.1, the sums in Equation 1 are also called Riemann sums, and the limit is denoted as

$$\iint_{\Omega} f(x, y) \, dA = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k$$

#### Properties of Double Integrals

**Theorem 6.2.1.** Suppose that  $f(x, y)$  and  $g(x, y)$  are continuous on  $\Omega$ , and  $\Omega$  can be subdivided into  $\Omega_1$  and  $\Omega_2$ , then

- a)  $\iint_{\Omega} k f(x, y) \, dA = k \iint_{\Omega} f(x, y) \, dA.$
- b)  $\iint_{\Omega} (f(x, y) + g(x, y)) \, dA = \iint_{\Omega} f(x, y) \, dA + \iint_{\Omega} g(x, y) \, dA.$
- c)  $\iint_{\Omega} f(x, y) \, dA = \iint_{\Omega_1} f(x, y) \, dA + \iint_{\Omega_2} f(x, y) \, dA.$

### 6.3 Triple Integrals

**Definition 6.3.1.** A triple integral of a function is the net signed volume defined over a finite closed solid region  $G$  in an  $xyz$  coordinate system.



**Theorem 6.3.1.** *Divide the bounding box of  $G$  into  $n$  boxes with sides parallel to the coordinate planes. Discard boxes which contain any points outside of  $G$ . Choose an arbitrary point in each remaining box. The volume of the  $k$ th remaining box is  $\Delta V_k$ . The arbitrary point in the  $k$ th remaining box is  $(x_k^*, y_k^*, z_k^*)$ . The Riemann sum is*

$$\iiint_G f(x, y, z) \, dV = \sum_{k=1}^{\infty} f(x_k^*, y_k^*, z_k^*) \Delta V_k$$

### Properties of Triple Integrals

**Theorem 6.3.2.** *Suppose that  $f(x, y, z)$  and  $g(x, y, z)$  are continuous on  $G$  and  $G$  can be subdivided into  $G_1$  and  $G_2$  then*

- a)  $\iiint_G k f(x, y, z) \, dV = k \iiint_G f(x, y, z) \, dV.$
- b)  $\iiint_G (f(x, y, z) + g(x, y, z)) \, dV = \iiint_G f(x, y, z) \, dV + \iiint_G g(x, y, z) \, dV.$
- c)  $\iiint_G f(x, y, z) \, dV = \iiint_{G_1} f(x, y, z) \, dV + \iiint_{G_2} f(x, y, z) \, dV.$

## 7 Vector-Valued Functions

**Definition 7.0.1.** A vector-valued function (VVF) is some function with domain  $\mathbb{R}$  and codomain  $\mathbb{R}^n$ . For example,

$$\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3 : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

is a vector-valued function where  $x, y, z : \mathbb{R} \rightarrow \mathbb{R}$ .

**Theorem 7.0.1.** *The domain of  $\mathbf{r}(t)$  is the intersection of the domains of its components.*

**Definition 7.0.2** (Orientation). The orientation of  $\mathbf{r}(t)$  is the direction of motion along the curve as the value of the parameter increases.

### 7.1 Limits and Continuity

**Theorem 7.1.1** (Limits of VVFs). *The limit of a VVF is the vector of the limits of its components.*

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \right\rangle$$

**Theorem 7.1.2** (Continuity of VVFs). *The VVF  $\mathbf{r}(t)$  is continuous at  $t = a$  iff*

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a).$$

*This follows that a VVF is continuous if each of its components are also continuous.*

### 7.2 Calculus with VVFs

**Theorem 7.2.1** (Derivatives of VVFs). *The derivative of a VVF is the vector of the derivatives of its components.*

$$\frac{d}{dt} \mathbf{r}(t) = \left\langle \frac{d}{dt} x(t), \frac{d}{dt} y(t), \frac{d}{dt} z(t) \right\rangle$$

**Theorem 7.2.2** (Integration of VVFs). *The integral of a VVF is the vector of the integrals of its components.*

$$\int \mathbf{r}(t) dt = \left\langle \int x(t) dt, \int y(t) dt, \int z(t) dt \right\rangle$$

**Remark.** *When integrating a VVF, each component has its own constant of integration.*

### 7.3 Parametrising Lines with VVFs

**Definition 7.3.1** (Equation for a Line). A line can be expressed as

$$\mathbf{l}(t) = \mathbf{P}_0 + t\mathbf{v}$$

where the line  $\mathbf{l}(t)$  passes through the point  $\mathbf{P}_0$ , and is parallel to the vector  $\mathbf{v}$ .

**Definition 7.3.2** (Tangent Lines). If a VVF  $\mathbf{r}(t)$  is differentiable at  $t_0$  and  $\mathbf{r}'(t_0) \neq \mathbf{0}$ , the tangent line at  $t = t_0$  is given by

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0).$$

**Remark.** *Higher-order approximations can be determined using Taylor's formula.*

## 7.4 Applications of VVFs

**Theorem 7.4.1** (Curve of Intersection). *A VVF can be used to determine the curve of intersection between two surfaces. The method is to choose one of the variables (commonly the first) as the parameter, and express the remaining variables in terms of that parameter.*

*If the intersection is bounded between two points, the domain can be calculated using the component which was parametrised.*

*For example, the curve of intersection between*

$$y = 2x - 4 \quad \text{and} \quad z = 3x - 1$$

*between the points*

$$\mathbf{P}_1 = (2, 0, 7) \quad \text{and} \quad \mathbf{P}_2 = (3, 2, 10)$$

*is given by*

$$\mathbf{r}(t) = \langle t, 2t - 4, 3t - 1 \rangle : 2 \leq t \leq 3.$$

**Definition 7.4.1** (Arc Length). The arc length  $S$  of a smooth continuous VVF  $\mathbf{r}(t)$ , is the distance along  $\mathbf{r}(t)$  between  $t = a$  and  $t = b$ , defined by

$$S = \int_a^b \left\| \frac{d}{dt} \mathbf{r}(t) \right\| dt$$

## 8 First-Order Differential Equations

## 9 Second-Order Differential Equations