Calculus and Differential Equations

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Contents

1 Integration Techniques

1.1 Derivative Table

Let $f(x)$ be a function, and $a \in \mathbb{R}$ be a constant.

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
x^a	ax^{a-1}
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
a^x	$\ln(a)a^x$
$\mathbf{e}^x$	$\mathbf{e}^x$
$\log_a x,\ a \in \mathbb{R}\setminus\{0\}$	$\frac{1}{a\ln x}$
$\ln x$	$\frac{1}{x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
a	0
x	1
$a_1 u(x) \pm a_2 v(x)$	$a_1 \frac{\mathrm{d}u}{\mathrm{d}x} \pm a_2 \frac{\mathrm{d}v}{\mathrm{d}x}$
$u(x)v(x)$	$\frac{\mathrm{d}u}{\mathrm{d}x} v + u \frac{\mathrm{d}v}{\mathrm{d}x}$
$\frac{u(x)}{v(x)}$	$\frac{\frac{\mathrm{d}u}{\mathrm{d}x} v - u \frac{\mathrm{d}v}{\mathrm{d}x}}{v(x)^2}$
$u(v(x))$	$\frac{\mathrm{d}u}{\mathrm{d}v} \frac{\mathrm{d}v}{\mathrm{d}x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\csc^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\csc(ax)$	$-a\csc(ax)\cot(ax)$
$\arcsin(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\frac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\frac{a}{1+a^2x^2}$
$\operatorname{arccot}(ax)$	$-\frac{a}{1+a^2x^2}$
$\operatorname{arcsec}(ax)$	$\frac{1}{x\sqrt{a^2x^2-1}}$
$\operatorname{arccsc}(ax)$	$-\frac{1}{x\sqrt{a^2x^2-1}}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sinh(ax)$	$a\cosh(ax)$
$\cosh(ax)$	$a\sinh(ax)$
$\tanh(ax)$	$a\,\operatorname{sech}^2(ax)$
$\coth(ax)$	$-a\,\operatorname{csch}^2(ax)$
$\operatorname{sech}(ax)$	$-a\,\operatorname{sech}(ax)\tan(ax)$
$\operatorname{csch}(ax)$	$-a\,\operatorname{csch}(ax)\cot(ax)$
$\operatorname{arcsinh}(ax)$	$\frac{a}{\sqrt{1+a^2x^2}}$
$\operatorname{arccosh}(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\operatorname{arctanh}(ax)$	$\frac{a}{1-a^2x^2}$
$\operatorname{arccoth}(ax)$	$\frac{a}{1-a^2x^2}$
$\operatorname{arcsech}(ax)$	$-\frac{1}{a(1+ax)\sqrt{\frac{1-ax}{1+ax}}}$
$\operatorname{arccsch}(ax)$	$-\frac{1}{ax^2\sqrt{1+\frac{1}{a^2x^2}}}$

Table 1: Derivatives of Elementary Functions

1.2 Partial Fractions

Definition 1.2.1 (Partial Fraction Decomposition)**. Partial fraction decomposition** is a method where a rational function $\frac{P(x)}{Q(x)}$ is rewritten as a sum of fraction.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k},\ k\in\mathbb{N}$
ax^2+bx+c	$\frac{A}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k},\ k\in\mathbb{N}$

Table 2: Partial Fraction Forms

1.3 Integration by Parts

Theorem 1.3.1.

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)v(x)\right) &= \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)+u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\\
u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)v(x)\right)-\frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\\
\int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x &= \int \frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)v(x)\right)\mathrm{d}x-\int \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\,\mathrm{d}x\\
\int u(x)\,\mathrm{d}v(x) &= u(x)v(x)-\int v(x)\,\mathrm{d}u(x)
\end{aligned}$$

□

1.4 Integration by Substitution

Theorem 1.4.1.

$$\int f(g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\,,\ \textit{where } u=g(x)$$

1.5 Trigonometric Substitutions

Form	Substitution	Result	Domain
$\left(a^2 - b^2x^2\right)^n$	$x = \dfrac{a}{b}\sin(\theta)$	$a^2\cos^2(\theta)$	$\theta \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$
$\left(a^2 + b^2x^2\right)^n$	$x = \dfrac{a}{b}\tan(\theta)$	$a^2\sec^2(\theta)$	$\theta \in \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$
$\left(b^2x^2 - a^2\right)^n$	$x = \dfrac{a}{b}\sec(\theta)$	$a^2\tan^2(\theta)$	$\theta \in \left[0,\ \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\ \pi\right]$

Table 3: Trigonometric substitutions for various forms.

2 Limits, Continuity and Differentiability

2.1 Limits

Theorem 2.1.1 (Limits)**.** $\lim_{x\to x_0} f(x)$ *exists if and only if* $\lim_{x\to {x_0}^+} f(x)$ *and* $\lim_{x\to {x_0}^-} f(x)$ *exist and are equal.*
For $f : S \to T$*,*

$$I \subseteq S : \exists L \in I : \lim_{x\to x_0} f(x) = L \iff \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = L$$

Theorem 2.1.2 (L'Hôpital's Rule)**.** *For two differentiable functions* $f(x)$ *and* $g(x)$*. If* $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$*, or* $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \pm\infty$*, then* $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ *(as long as the limit exists, or diverges to* $\pm\infty$*).*

2.2 Continuity

Theorem 2.2.1 (Continuity at a Point)**.** $f(x)$ *is continuous at* c *iff* $\lim_{x\to c} f(x) = f(c)$*.*

Theorem 2.2.2 (Continuity over an Interval)**.** $f(x)$ *is continuous on* I *if* $f(x)$ *is continuous for all* $x \in I$*.*

- $f(x)$ *is continuous on* $I : (a,\, b)$ *if it is continuous for all* $x \in I$*.*
- $f(x)$ *is continuous on* $I : [a,\, b]$ *if it is continuous for all* $x \in I$*, but only right continuous at* a *and left continuous at* b*.*

If $f(x)$ *is continuous on* $(-\infty,\, \infty)$*,* $f(x)$ *is continuous everywhere.*

Theorem 2.2.3 (Intermediate Value Theorem)**.** *If* $f(x)$ *is continuous on* $I : [a,\, b]$ *and* c *is any number between* $f(a)$ *and* $f(b)$*, inclusive, then there exists an* $x \in I$ *such that* $f(x) = c$*.*

2.3 Differentiability

Theorem 2.3.1 (Differentiability)**.** $f(x)$ *is differentiable at* $x = x_0$ *iff*

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, it defines the derivative

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Theorem 2.3.2. $f(x)$ *is differentiable on* I *if* $f(x)$ *is differentiable for all* $x_0 \in I$*.*

Theorem 2.3.3. *Differentiability implies continuity.*

Theorem 2.3.4 (Mean Value Theorem)**.** *If* $f(x)$ *is continuous on* $I : [a,\, b]$ *and differentiable on* I*, then there exists a point* $x_0 \in I$ *such that*

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{f(b) - f(a)}{b - a}$$

3 Definite Integrals

Theorem 3.0.1. *If $f(x)$ is continuous on an interval $I : [a, b]$, then the net signed area A between the graph of $f(x)$ and the interval I is*

$$A = \int_a^b f(x)\,\mathrm{d}x$$

Properties of Definite Integrals

Theorem 3.0.2. *Suppose that $f(x)$ and $g(x)$ are continuous on the interval I, with $a, b, c \in I$ and $k \in \mathbb{R}$ then*

a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0.$

b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$

c) $\displaystyle\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x.$

d) $\displaystyle\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x.$

e) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$

3.1 Riemann Sums

Theorem 3.1.1. *Let A be the area under $f(x)$ on the interval $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

where n is the number of rectangles, x_k is the centre of the rectangle k, and Δx_k is the width of the rectangle k. If every rectangle has the same width, then

$$\forall k : \Delta x_k = \frac{b-a}{n}$$

3.2 Fundamental Theorem of Calculus

The fundamental theorem of calculus provides a logical connection between infinite series (definite integrals) and antiderivatives (indefinite integrals).

Theorem 3.2.1 (The Fundamental Theorem of Calculus: Part 1)**.** *If $f(x)$ is continuous on $[a,\, b]$ and F is any antiderivative of f on $[a,\, b]$ then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Equivalently

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x} F(x)\,\mathrm{d}x = F(b) - F(a) \equiv \left. F(x)\right|_a^b$$

Theorem 3.2.2 (The Fundamental Theorem of Calculus: Part 2)**.** *If $f(x)$ is continuous on I then it has an antiderivative on I. In particular, if $a \in I$, then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is an antiderivative of $f(x)$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x) \iff \frac{\mathrm{d}}{\mathrm{d}x} = f(x)$$

Theorem 3.2.3. *Differentiation and integration are inverse operations.*

3.3 Taylor and Maclaurin Polynomials

Theorem 3.3.1 (Taylor Polynomials). *If $f(x)$ is a n differentiable function at x_0, then the nth degree Taylor polynomial for $f(x)$ near x_0, is given by*

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Theorem 3.3.2 (Maclaurin Polynomials). *Evaluating a Taylor polynomial near 0, gives the nth degree Maclaurin polynomial for $f(x)$*

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Theorem 3.3.3 (Error in Approximation). *Let $R_n(x)$ denote the difference between $f(x)$ and its nth Taylor polynomial, that is*

$$R_n(x) = f(x) - P_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

4 Taylor and Maclaurin Series

4.1 Infinite Series

Definition 4.1.1 (Taylor Series)**.** If $f(x)$ has derivatives of all orders at x_0, then the Taylor series for $f(x)$ about $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Definition 4.1.2 (Maclaurin Series)**.** If a Taylor series is centred on $x_0 = 0$, it is called a Maclaurin series, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 4.1.3 (Power Series)**.** Both Taylor and Maclaurin series are examples of **power series** defined as follows

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

4.2 Convergence Tests

For any infinite series of the form $\displaystyle\sum_{i=i_0}^{\infty} a_i$.

Alternating Series

Conditions $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$\text{Is } b_{i+1} \leqslant b_i \ \& \ \lim_{i\to\infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

Ratio Test

Conditions $\forall i : a_i > 0$ and $\displaystyle\lim_{i\to\infty} \frac{a_{i+1}}{a_i} \neq 1$.

$$\text{Is } \lim_{i\to\infty} \frac{a_{i+1}}{a_i} < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

$$\text{Is } \lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges Absolutely} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

5 Multivariable Calculus

Definition 5.0.1. A function is multivariable if its domain consists of several variables. In the reals, these functions are defined

$$f : \mathbb{R}^n \to \mathbb{R}$$

6 Double and Triple Integrals

7 Vector-Valued Functions

8 First-Order Differential Equations

9 Second-Order Differential Equations