

Calculus and Differential Equations

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1 Integration Techniques

1.1 Table of Derivatives

Let $f(x)$ be a function, and $a \in \mathbb{R}$ be a constant.

f	$\frac{df}{dx}$	f	$\frac{df}{dx}$
x^a	ax^{a-1}	a	0
\sqrt{x}	$\frac{1}{2\sqrt{x}}$	x	1
a^x	$\ln(a)a^x$	$a_1u(x) \pm a_2v(x)$	$a_1\frac{du}{dx} \pm a_2\frac{dv}{dx}$
e^x	e^x	$u(x)v(x)$	$\frac{du}{dx}v + u\frac{dv}{dx}$
$\log_a x, a \in \mathbb{R} \setminus \{0\}$	$\frac{1}{a \ln x}$	$\frac{u(x)}{v(x)}$	$\frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v(x)^2}$
$\ln x$	$\frac{1}{x}$	$u(v(x))$	$\frac{du}{dv}\frac{dv}{dx}$

f	$\frac{df}{dx}$	f	$\frac{df}{dx}$
$\sin(ax)$	$a \cos(ax)$	$\arcsin(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\cos(ax)$	$-a \sin(ax)$	$\arccos(ax)$	$-\frac{a}{\sqrt{1-a^2x^2}}$
$\tan(ax)$	$a \sec^2(ax)$	$\arctan(ax)$	$\frac{a}{1+a^2x^2}$
$\cot(ax)$	$-a \csc^2(ax)$	$\operatorname{arccot}(ax)$	$-\frac{a}{1+a^2x^2}$
$\sec(ax)$	$a \sec(ax) \tan(ax)$	$\operatorname{arcsec}(ax)$	$\frac{1}{x\sqrt{a^2x^2-1}}$
$\csc(ax)$	$-a \csc(ax) \cot(ax)$	$\operatorname{arccsc}(ax)$	$-\frac{1}{x\sqrt{a^2x^2-1}}$

f	$\frac{df}{dx}$	f	$\frac{df}{dx}$	f	$\frac{df}{dx}$
$\sinh(ax)$	$a \cosh(ax)$	$\operatorname{arcsinh}(ax)$	$\frac{a}{\sqrt{1+a^2x^2}}$	$\operatorname{arccoth}(ax)$	$\frac{a}{1-a^2x^2}$
$\cosh(ax)$	$a \sinh(ax)$	$\operatorname{arccosh}(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$	$\operatorname{arcsech}(ax)$	$-\frac{1}{a(1+ax)\sqrt{\frac{1-ax}{1+ax}}}$
$\tanh(ax)$	$a \operatorname{sech}^2(ax)$	$\operatorname{arctanh}(ax)$	$\frac{a}{1-a^2x^2}$	$\operatorname{arccsch}(ax)$	$-\frac{1}{ax^2\sqrt{1+\frac{1}{a^2x^2}}}$
$\coth(ax)$	$-a \operatorname{csch}^2(ax)$				
$\operatorname{sech}(ax)$	$-a \operatorname{sech}(ax) \tanh(ax)$				
$\operatorname{csch}(ax)$	$-a \operatorname{csch}(ax) \cot(ax)$				

Table 1: Derivatives of Elementary Functions

1.2 Trigonometric Identities

1.2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x) + 1 = \sec^2(x)$$

$$1 + \cot^2(x) = \csc^2(x)$$

1.2.2 Double-Angle Identities

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\csc(2x) = \frac{\sec(x) \csc(x)}{2}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\sec(2x) = \frac{\sec^2(x) \csc^2(x)}{\csc^2(x) - \sec^2(x)}$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\cot(2x) = \frac{\cot^2(x) - 1}{2 \cot(x)}$$

1.2.3 Power Reducing Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\csc^2(x) = \frac{2}{1 - \cos(2x)}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\sec^2(x) = \frac{2}{1 + \cos(2x)}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$\cot^2(x) = \frac{1 + \cos(2x)}{1 - \cos(2x)}$$

1.3 Partial Fractions

Definition 1.3.1 (Partial Fraction Decomposition). **Partial fraction decomposition** is a method where a rational function $\frac{P(x)}{Q(x)}$ is rewritten as a sum of fraction.

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^k, k \in \mathbb{N}$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_k}{(ax + b)^k}$
$ax^2 + bx + c$	$\frac{A}{ax^2 + bx + c}$
$(ax^2 + bx + c)^k, k \in \mathbb{N}$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_k}{(ax + b)^k}$

Table 2: Partial Fraction Forms

1.4 Integration by Parts

Theorem 1.4.1.

$$\int u \, dv = uv - \int v \, du$$

1.5 Integration by Substitution

Theorem 1.5.1.

$$\int f(g(x)) \frac{dg(x)}{dx} dx = \int f(u) du, \text{ where } u = g(x)$$

1.6 Trigonometric Substitutions

Form	Substitution	Result	Domain
$(a^2 - b^2x^2)^n$	$x = \frac{a}{b} \sin(\theta)$	$a^2 \cos^2(\theta)$	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
$(a^2 + b^2x^2)^n$	$x = \frac{a}{b} \tan(\theta)$	$a^2 \sec^2(\theta)$	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$
$(b^2x^2 - a^2)^n$	$x = \frac{a}{b} \sec(\theta)$	$a^2 \tan^2(\theta)$	$\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

Table 3: Trigonometric substitutions for various forms.

2 Limits, Continuity and Differentiability

2.1 Limits

Theorem 2.1.1 (Limits). $\lim_{x \rightarrow x_0} f(x)$ exists if and only if $\lim_{x \rightarrow x_0^+} f(x)$ and $\lim_{x \rightarrow x_0^-} f(x)$ exist and are equal.

Definition 2.1.1 (Finite limits using the ε - δ definition).

$$\lim_{x \rightarrow x_0} f(x) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

Theorem 2.1.2 (L'Hôpital's Rule). For two differentiable functions $f(x)$ and $g(x)$. If $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$, or $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = \pm\infty$, then $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$.

2.2 Continuity

Theorem 2.2.1 (Continuity at a Point). $f(x)$ is continuous at c iff $\lim_{x \rightarrow c} f(x) = f(c)$.

Theorem 2.2.2 (Continuity over an Interval). $f(x)$ is continuous on I if $f(x)$ is continuous for all $x \in I$.

- $f(x)$ is continuous on $I : (a, b)$ if it is continuous for all $x \in I$.
- $f(x)$ is continuous on $I : [a, b]$ if it is continuous for all $x \in I$, but only right continuous at a and left continuous at b .

If $f(x)$ is continuous on $(-\infty, \infty)$, $f(x)$ is continuous everywhere.

Theorem 2.2.3 (Intermediate Value Theorem). If $f(x)$ is continuous on $I : [a, b]$ and c is any number between $f(a)$ and $f(b)$, inclusive, then there exists an $x \in I$ such that $f(x) = c$.

2.3 Differentiability

Theorem 2.3.1 (Differentiability). $f(x)$ is differentiable at $x = x_0$ iff

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, it defines the derivative

$$\left. \frac{df}{dx} \right|_{x=x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Theorem 2.3.2. $f(x)$ is differentiable on I if $f(x)$ is differentiable for all $x_0 \in I$.

Theorem 2.3.3. Differentiability implies continuity.

Theorem 2.3.4 (Mean Value Theorem). If $f(x)$ is continuous on $I : [a, b]$ and differentiable on I , then there exists a point $c \in I$ such that

$$\left. \frac{df}{dx} \right|_{x=c} = \frac{f(b) - f(a)}{b - a}$$

3 Definite Integrals

Theorem 3.0.1. *If $f(x)$ is continuous on an interval $I : [a, b]$, then the net signed area A between the graph of $f(x)$ and the interval I is*

$$A = \int_a^b f(x) \, dx$$

Properties of Definite Integrals

Theorem 3.0.2. *Suppose that $f(x)$ and $g(x)$ are continuous on the interval I , with $a, b, c \in I$ and $k \in \mathbb{R}$ then*

- a) $\int_a^a f(x) \, dx = 0.$
- b) $\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.$
- c) $\int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx.$
- d) $\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx.$
- e) $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$

3.1 Riemann Sums

Theorem 3.1.1. *Let A be the area under $f(x)$ on the interval $[a, b]$, then*

$$\int_a^b f(x) \, dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k) \Delta x_k$$

where n is the number of rectangles, x_k is the centre of the rectangle k , and Δx_k is the width of the rectangle k . If every rectangle has the same width, then

$$\forall k : \Delta x_k = \frac{b-a}{n}$$

3.2 Fundamental Theorem of Calculus

The fundamental theorem of calculus provides a logical connection between infinite series (definite integrals) and antiderivatives (indefinite integrals).

Theorem 3.2.1 (The Fundamental Theorem of Calculus: Part 1). *If $f(x)$ is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$ then*

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Equivalently

$$\int_a^b \frac{d}{dx} F(x) \, dx = F(b) - F(a) \equiv F(x) \Big|_a^b$$

Theorem 3.2.2 (The Fundamental Theorem of Calculus: Part 2). *If $f(x)$ is continuous on I then it has an antiderivative on I . In particular, if $a \in I$, then the function F defined by*

$$F(x) = \int_a^x f(t) \, dt$$

is an antiderivative of $f(x)$. That is,

$$\frac{d}{dx} F(x) = f(x) \equiv \frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$

Theorem 3.2.3. *Differentiation and integration are inverse operations.*

3.3 Taylor and Maclaurin Polynomials

Theorem 3.3.1 (Taylor Polynomials). *If $f(x)$ is a n differentiable function at x_0 , then the n th degree Taylor polynomial for $f(x)$ near x_0 , is given by*

$$f(x) \approx p_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Theorem 3.3.2 (Maclaurin Polynomials). *Evaluating a Taylor polynomial near 0, gives the n th degree Maclaurin polynomial for $f(x)$*

$$f(x) \approx p_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

Theorem 3.3.3 (Error in Approximation). *Let $R_n(x)$ denote the difference between $f(x)$ and its n th Taylor polynomial, that is*

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = \frac{f^{(n+1)}(s)}{(n+1)!} (x - x_0)^{n+1}$$

where s is between x_0 and x .

4 Taylor and Maclaurin Series

4.1 Infinite Series

Definition 4.1.1 (Taylor Series). If $f(x)$ has derivatives of all orders at x_0 , then the Taylor series for $f(x)$ about $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Definition 4.1.2 (Maclaurin Series). If a Taylor series is centred on $x_0 = 0$, it is called a Maclaurin series, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 4.1.3 (Power Series). Both Taylor and Maclaurin series are examples of **power series**, which are defined as follows

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

4.2 Convergence

Theorem 4.2.1 (Convergence of a Taylor Series). *The equality*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

holds at a point x iff

$$\lim_{n \rightarrow \infty} \left[f(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \right] = 0$$

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

Definition 4.2.1 (Interval of Convergence). The interval of convergence for a power series is the set of x values for which that series converges.

Definition 4.2.2 (Radius of Convergence). The radius of convergence R is a nonnegative real number or ∞ such that a power series converges if

$$|x - a| < R$$

and diverges if

$$|x - a| > R$$

The behaviour of the power series on the boundary, that is, where $|x - a| = R$, can be determined by substituting $x = R + a$ for the upper boundary, and $x = -R + a$ for the lower boundary.

4.3 Convergence Tests

For any power series of the form $\sum_{i=i_0}^{\infty} a_i$.

Alternating Series

Conditions $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$\text{Is } b_{i+1} \leq b_i \text{ \& } \lim_{i \rightarrow \infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

Ratio Test

$$\text{Is } \lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

The ratio test is inconclusive if $\lim_{i \rightarrow \infty} \frac{a_{i+1}}{a_i} = 1$.

4.4 Table of Maclaurin Series

Function	Series	Interval of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$-\infty < x < \infty$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

Table 4: Maclaurin Series of Common Functions

5 Multivariable Calculus

5.1 Multivariable Functions

Definition 5.1.1. A function is multivariable if its domain consists of several variables. In the reals, these functions are defined

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

5.2 Level Curves

Definition 5.2.1. Level curves or *contour curves* of a function of two variables is a curve along which the function has constant value.

$$L_c(f) = \{(x, y) : f(x, y) = c\}$$

The level curves of a function can be determined by substituting $z = c$, and solving for y .

5.3 Limits and Continuity

Definition 5.3.1 (Finite Limit of Multivariable Functions using the ε - δ Definition).

$$\begin{aligned} \lim_{(x_1, \dots, x_n) \rightarrow (c_1, \dots, c_n)} f(x_1, \dots, x_n) &= L \\ \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall (x_1, \dots, x_n) \in I : \\ 0 < |x_1 - c_1, \dots, x_n - c_n| < \delta &\implies |f(x_1, \dots, x_n) - L| < \varepsilon \end{aligned}$$

Theorem 5.3.1 (Limits along Smooth Curves). *If $f(x, y) \rightarrow L$ as $(x, y) \rightarrow (x_0, y_0)$, then $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L$ along any smooth curve.*

Theorem 5.3.2 (Existence of a Limit). *If the limit of $f(x, y)$ changes along different smooth curves, then $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$ does not exist.*

Theorem 5.3.3 (Continuity of Multivariable Functions). *A function $f(x_1, \dots, x_n)$ is continuous at (c_1, \dots, c_n) iff*

$$\lim_{(x_1, \dots, x_n) \rightarrow (c_1, \dots, c_n)} f(x_1, \dots, x_n) = f(c_1, \dots, c_n)$$

Recognising continuous functions:

- A sum, difference or product of continuous functions is continuous.
- A quotient of continuous functions is continuous except where the denominator is zero.
- A composition of continuous functions is continuous.

5.4 Partial Derivatives

Definition 5.4.1 (Partial Differentiation). The partial derivative of a multivariable function is its derivative with respect to one of those variables, while the others are held constant.

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

5.5 The Gradient Vector

Definition 5.5.1. Let ∇ , pronounced “del”, denote the vector differential operator defined as follows

$$\nabla = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{bmatrix}$$

5.6 Multivariable Chain Rule

Definition 5.6.1. Let $f = f(x(t_1, \dots, t_n))$ be the composition of f with $\mathbf{x} = [x_1 \ \dots \ x_n]$, then the partial derivative of f with respect to t_i is given by

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \partial_{t_i} \mathbf{x}$$

5.7 Directional Derivatives

Definition 5.7.1. The directional derivative $\nabla_{\mathbf{u}} f$ is the rate at which the function f changes in the direction \mathbf{u} .

$$\begin{aligned} \nabla_{\mathbf{u}} f &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} \\ &= \nabla f \cdot \mathbf{u} \end{aligned}$$

where the slope is given by $\|\nabla_{\mathbf{u}} f\|$.

Remark. The directional derivative of f can be denoted in several ways:

$$\nabla_{\mathbf{u}} f = D_{\mathbf{u}} f = \partial_{\mathbf{u}} f = \frac{\partial f}{\partial \mathbf{u}}$$

Theorem 5.7.1 (Direction of Greatest Ascent). *The direction of greatest ascent is given by*

$$\max_{\|\mathbf{u}\|=1} \nabla_{\mathbf{u}} f = \nabla f$$

where the slope is given by $\|\nabla f\|$.

Theorem 5.7.2 (Direction of Greatest Descent). *The direction of greatest descent is given by*

$$\min_{\|\mathbf{u}\|=1} \nabla_{\mathbf{u}} f = -\nabla f$$

where the slope is given by $-\|\nabla f\|$.

Proof. Given that \mathbf{u} is a unit vector, the dot product definition gives

$$\begin{aligned} \nabla_{\mathbf{u}} f &= \nabla f \cdot \mathbf{u} \\ &= \|\nabla f\| \|\mathbf{u}\| \cos(\theta) \\ &= \|\nabla f\| \cos(\theta) \end{aligned} \tag{1}$$

Equation 1 is maximised when $\cos(\theta)$ is maximised. Thus the maximum slope is given by

$$\max \nabla_u f = \|\nabla f\|$$

and the direction of greatest ascent is given by

$$\mathbf{u} = \nabla f$$

□

Theorem 5.7.3 (The Gradient is Normal to the Level Curves of f). *If $\nabla f = 0$, then ∇f is normal to the level curves of f at any point (c_1, \dots, c_n) .*

5.8 Higher-Order Partial Derivatives

Definition 5.8.1. Higher-order partial derivatives can be denoted using three different notation. The following table shows the mixed partial derivative of $f(x, y)$ w.r.t. x then y .

Leibniz	Euler	Legendre
$\frac{\partial^2 f}{\partial y \partial x}$	$\partial_{xy} f$	f_{xy}

Table 5: Mixed Partial Derivative Notation

For partial derivatives w.r.t. the same variable, a superscript can be used in Euler notation.

Leibniz	Euler	Legendre
$\frac{\partial^2 f}{\partial x^2}$	$\partial_x^2 f$	f_{xx}

Table 6: Second-Order Partial Derivative Notation

5.9 Hessian Matrix

Definition 5.9.1. Let the Hessian matrix \mathbf{H} be the matrix of second-order partial derivative operators defined as shown below

$$\mathbf{H} = \begin{bmatrix} \partial_{x_1}^2 & \cdots & \partial_{x_n x_1} \\ \vdots & \ddots & \vdots \\ \partial_{x_1 x_n} & \cdots & \partial_{x_n}^2 \end{bmatrix}$$

5.10 Critical Points

6 Double and Triple Integrals

6.1 Double Integrals

Theorem 6.1.1. Divide the rectangular region of R into n rectangles with sides parallel to the coordinate axes. Discard rectangles which contain any points outside of R . Choose an arbitrary point in each remaining rectangle. The area of the k th remaining rectangle is ΔA_k . The arbitrary point in the k th remaining rectangle is (x_k^*, y_k^*) . The Riemann sum is

$$\iint_R f(x, y) \, dA = \sum_{k=1}^{\infty} f(x_k^*, y_k^*) \Delta A_k$$

Properties of Double Integrals

Theorem 6.1.2. Suppose that $f(x, y)$ and $g(x, y)$ are continuous on R and R can be subdivided into R_1 and R_2 then

- a) $\iint_R kf(x, y) \, dA = k \iint_R f(x, y) \, dA.$
- b) $\iint_R (f(x, y) \pm g(x, y)) \, dA = \iint_R f(x, y) \, dA \pm \iint_R g(x, y) \, dA.$
- c) $\iint_R f(x, y) \, dA = \iint_{R_1} f(x, y) \, dA + \iint_{R_2} f(x, y) \, dA.$

6.2 Properties of Triple Integrals

6.3 Triple Integrals

Definition 6.3.1. A triple integral of a function is the net signed volume defined over a finite closed solid region G in an xyz coordinate system.

Theorem 6.3.1. Divide the bounding box of G into n boxes with sides parallel to the coordinate planes. Discard boxes which contain any points outside of G . Choose an arbitrary point in each remaining box. The volume of the k th remaining box is ΔV_k . The arbitrary point in the k th remaining box is (x_k^*, y_k^*, z_k^*) . The Riemann sum is

$$\iiint_G f(x, y, z) \, dV = \sum_{k=1}^{\infty} f(x_k^*, y_k^*, z_k^*) \Delta V_k$$

Properties of Triple Integrals

Theorem 6.3.2. Suppose that $f(x, y, z)$ and $g(x, y, z)$ are continuous on G and G can be subdivided into G_1 and G_2 then

a)
$$\iiint_G k f(x, y, z) \, dV = k \iiint_G f(x, y, z) \, dV.$$

b)
$$\iiint_G (f(x, y, z) \pm g(x, y, z)) \, dV = \iiint_G f(x, y, z) \, dV \pm \iiint_G g(x, y, z) \, dV.$$

c)
$$\iiint_G f(x, y, z) \, dV = \iiint_{G_1} f(x, y, z) \, dV + \iiint_{G_2} f(x, y, z) \, dV.$$

7 Vector-Valued Functions

8 First-Order Differential Equations

9 Second-Order Differential Equations