Derivative Rules

$f(x)$	$f'(x)$
$u(x)v(x)$	$u'v + uv'$
$\frac{u(x)}{v(x)}$	$\frac{u'v-uv'}{v^2}$
$u(v(x))$	$\frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\csc^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\csc(ax)$	$-a\csc(ax)\cot(ax)$

Trigonometric Identities

$$1 = \sin^2(x) + \cos^2(x)$$
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$
$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

Partial Fraction Decomposition

$$ax+b \to \frac{A}{ax+b}$$
$$(ax+b)^k \to \frac{A_1}{ax+b} + \cdots + \frac{A_k}{(ax+b)^k}$$
$$ax^2+bx+c \to \frac{A}{ax^2+bx+c}$$
$$(ax^2+bx+c)^k \to \frac{A_1x+B_1}{ax^2+bx+c}$$
$$+\frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

Integration Techniques

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$
$$\int f(g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

where $u = g(x)$.

Trigonometric Substitutions

Form	Substitution
$a^2 - b^2x^2$	$x = \frac{a}{b}\sin(\theta)$
$a^2 + b^2x^2$	$x = \frac{a}{b}\tan(\theta)$
$b^2x^2 - a^2$	$x = \frac{a}{b}\sec(\theta)$

L'Hôpital's Rule

If $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$ or $\pm\infty$, then $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = \lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$.

Continuity

$f(x)$ continuous at c iff $\lim\limits_{x\to c} f(x) = f(c)$.
$f(x)$ is continuous on $I : (a,\, b)$ if it is continuous for all $x \in I$.
$f(x)$ is continuous on $I : [a,\, b]$ if it is continuous for all $x \in I$, but only right continuous at a and left continuous at b.

Intermediate Value Theorem

If $f(x)$ is continuous on $I : [a,\, b]$ and $f(a) \le c \le f(b)$, then $\exists x \in I : f(x) = c$.

Differentiability

$f(x)$ is differentiable at $x = x_0$ iff

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

exists. This defines the derivative

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

Differentiability implies continuity.

Mean Value Theorem

If $f(x)$ is continuous and differentiable on $I : [a,\, b]$, then

$$\exists c \in I : f'(c) = \frac{f(b)-f(a)}{b-a}$$

Definite Integrals

$$A = \int_a^b f(x)\,\mathrm{d}x$$

Fundamental Theorem of Calculus

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}F(x)\,\mathrm{d}x = F(b) - F(a)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

Taylor Polynomials

$$f(x) \approx p_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Maclaurin Series: $x_0 = 0$.
Power Series: $\sum_{n=0}^{\infty} c_n (x-x_0)^n$

For a series of the form $\sum\limits_{i=i_0}^{\infty} a_i$:

Alternating Series Test

Given $a_i = (-1)^i b_i$ and $b_i > 0$.
If $b_{i+1} \leqslant b_i$ & $\lim_{i\to\infty} b_i = 0$, then convergent, else inconclusive.

Ratio Test

If $\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| < 1$, then convergent, else divergent. The ratio test is inconclusive if $\lim\limits_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right| = 1$.

Function	Series Term	Conv.
e^x	$\frac{x^n}{n!}$	all x
$\sin(x)$	$(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	all x
$\cos(x)$	$(-1)^n\frac{x^{2n}}{(2n)!}$	all x
$\frac{1}{1-x}$	x^n	$(-1,\, 1)$
$\frac{1}{1+x^2}$	$(-1)^n x^{2n}$	$(-1,\, 1)$
$\ln(1+x)$	$(-1)^{n+1}\frac{x^n}{n}$	$(-1,\, 1]$

Multivariable Functions

$$f : \mathbb{R}^n \to \mathbb{R}$$

Level Curves

$$L_c(f) = \{(x,\, y) : f(x,\, y) = c\}$$

If $f(x,\, y) \to L$ as $(x,\, y) \to (x_0,\, y_0)$, then $\lim\limits_{(x,\, y)\to(x_0,\, y_0)} = L$ along any smooth curve.
The limit does not exist if L changes along different smooth curves.
Partial Derivatives: w.r.t one variable, others held constant.
Gradient: $\boldsymbol{\nabla} = \langle \partial_{x_1},\, \partial_{x_2},\, \ldots,\, \partial_{x_n}\rangle$

Multivariable Chain Rule

For $f = f(\boldsymbol{x}(t_1,\, \ldots,\, t_n))$ with $\boldsymbol{x} = [x_1 \;\cdots\; x_m]$

$$\frac{\partial f}{\partial t_i} = \boldsymbol{\nabla} f \cdot \partial_{t_i}\boldsymbol{x}$$

Directional Derivative

$$\boldsymbol{\nabla}_{\boldsymbol{u}} f = \boldsymbol{\nabla} f \cdot \boldsymbol{u}$$

where the slope is given by $\|\boldsymbol{\nabla}_{\boldsymbol{u}} f\|$.

Critical Points

$(x_0,\, y_0)$ is a critical point if $\boldsymbol{\nabla} f(x_0,\, y_0) = 0$ or if $\boldsymbol{\nabla} f(x_0,\, y_0)$ is undefined.

Classification of Critical Points

$$D = f_{xx}f_{yy} - (f_{xy})^2$$

$D > 0$ and $f_{xx} < 0$: local maxima

$D > 0$ and $f_{xx} > 0$: local minima

$D < 0$: saddle point

$D = 0$: inconclusive

Double Integrals

The volume of the solid enclosed between the surface $z = f(x,\, y)$ and the region Ω is defined by

$$V = \iint_\Omega f(x,\, y)\,\mathrm{d}A$$

If Ω is a region bounded by $a \le x \le b$ and $c \le y \le d$, then

$$\iint_\Omega f(x,\, y)\,\mathrm{d}A = \int_c^d\int_a^b f(x,\, y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_a^b\int_c^d f(x,\, y)\,\mathrm{d}y\,\mathrm{d}x$$

Type I Regions

Bounded left & right by:

$x = a$ and $x = b$

Bounded below & above by:

$y = g_1(x)$ and $y = g_2(x)$

where $g_1(x) \le g_2(x)$ for $a \le x \le b$:

$$\iint_\Omega f(x,\, y)\,\mathrm{d}A = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,\, y)\,\mathrm{d}y\,\mathrm{d}x$$

Type II Regions

Bounded left & right by:

$x = h_1(y)$ and $x = h_2(y)$

Bounded below & above by:

$$y = c \text{ and } y = d$$

where $h_1(y) \le h_2(y)$ for $c \le y \le d$, then

$$\iint\limits_{\Omega} f(x,\,y)\,\mathrm{d}A = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,\,y)\,\mathrm{d}x\,\mathrm{d}y$$

To integrate, solve the inner integrals first.

The **domain** of $\mathbf{r}(t)$ is the intersection of the domains of its components.

The **orientation** of $\mathbf{r}(t)$ is the direction of motion along the curve as the value of the parameter increases.

Limits, **derivatives** and **integrals** are all *component-wise*. Each component has its <u>own</u> constant of integration.

Parametric Line

$$\mathbf{l}(t) = \boldsymbol{P}_0 + t\boldsymbol{v}$$

where $\mathbf{l}(t)$ passes through $\boldsymbol{P}_0$, and is parallel to $\boldsymbol{v}$.

Tangent Lines

If $\mathbf{r}(t)$ is differentiable at t_0 and $\mathbf{r}'(t_0) \neq \mathbf{0}$

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

Curves of Intersection

Choose one of the variables as the parameter, and express the remaining variables in terms of that parameter.

Arc Length

$$S = \int_a^b \|\mathbf{r}'(t)\|\,\mathrm{d}t$$

Ordinary Differential Equations

Order: highest derivative in DE.
Autonomous DE: does not depend explicitly on the independent variable.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

A fixed point is the value of y for which $f(y) = 0$.

Stability of Fixed Points

Given a positive/negative perturbation from a fixed point, that point is

Stable: if both tend toward FP
Unstable: if both tend away from FP
Semi-Stable: if one tends toward FP, and another tends away from FP

Directly Integrable ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$

$$y(x) = \int f(x)\,\mathrm{d}x$$

Separable ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)q(y)$

$$\int \frac{1}{q(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int p(x)\,\mathrm{d}x$$

Linear ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$, use the *integrating factor*: $I(x) = e^{\int p(x)\mathrm{d}x}$, so that

$$y(x) = \frac{1}{I(x)}\int I(x)q(x)\,\mathrm{d}x$$

Exact ODEs

$P(x,\,y) + Q(x,\,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ has the solution $\Psi(x,\,y) = c$ iff it is exact, namely when $P_y = Q_x$ where $P = \Psi_x$ and $Q = \Psi_y$. Then

$$\Psi(x,\,y) = \int P(x,\,y)\,\mathrm{d}x + f(y)$$

$$\Psi(x,\,y) = \int Q(x,\,y)\,\mathrm{d}y + g(x)$$

and $f(y)$ and $g(x)$ can be determined by solving these equations simultaneously.

Second-Order ODEs

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = F(x)$$

Homogeneous: $F(x) = 0$
Nonhomogeneous: $F(x) \neq 0$

Initial Values

$$y(x_0) = y_0 \quad y'(x_0) = y_1$$

Boundary Values

$$y(x_0) = y_0 \quad y(x_1) = y_1$$

Reduction of Order

$$y_2(x) = v\,(x)\,y_1(x)$$

$v(x)$ can be determined by substituting y_2 into the ODE, using $w(x) = v'(x)$.

Homogeneous ODEs

$$y_H(x) = e^{\lambda x}$$

Real Distinct Roots

$$y_H(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Real Repeated Roots

$$y_H(x) = c_1 e^{\lambda x} + c_2 t e^{\lambda x}$$

Complex Conjugate Roots

Given $\lambda = \alpha \pm \beta i$:

$$y_H(x) = e^{\alpha x}(c_1 \cos{(\beta x)} + c_2 \sin{(\beta x)})$$

Nonhomogeneous ODEs

$$y(x) = y_H(x) + y_P(x).$$

Method of Undetermined Coefficients

See table below. Substitute y_P into the nonhomogeneous ODE, and solve the undetermined coefficients.

Spring and Mass Systems

$$my'' + \gamma y' + ky = f(t)$$

Newton's Law: $F = my''$
Spring force: $F_s = -ky$
Damping force: $F_d = -\gamma y'$

m: mass $\quad$ k: spring constant
γ: damping $\quad$ $f(t)$: external force

Electrical Circuits

The sum of voltages around a loop equals 0.

$$v(t) - iR - L\frac{\mathrm{d}i}{\mathrm{d}t} - \frac{q}{C} = 0$$

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = v(t)$$

where $i = \frac{\mathrm{d}q}{\mathrm{d}t}$.

Voltage drop across various elements:

$$v_R = iR$$

$$v_C = q/C$$

$$v_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

R: resistance $\quad$ C: capacitance
L: inductance $\quad$ $v(t)$: voltage supply

$F(x)$	$y_P(x)$
a constant	A
a polynomial of degree n	$\sum_{i=0}^{n} A_i x^i$
e^{kx}	Ae^{kx}
$\cos{(\omega x)}$ or $\sin{(\omega x)}$	$A_0 \cos{(\omega x)} + A_1 \sin{(\omega x)}$
a combination of the above	a combination of the above
linearly dependent to $y_H(x)$	multiply $y_P(x)$ by x until linearly independent