# Calculus and Differential Equations

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# Contents

# 1 Integration Techniques

## 1.1 Table of Derivatives

Let $f(x)$ be a function, and $a \in \mathbb{R}$ be a constant.

| $f$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $x^a$ | $ax^{a-1}$ |
| $\sqrt{x}$ | $\dfrac{1}{2\sqrt{x}}$ |
| $a^x$ | $\ln(a)a^x$ |
| $\mathbf{e}^x$ | $\mathbf{e}^x$ |
| $\log_a x,\ a \in \mathbb{R}\setminus\{0\}$ | $\dfrac{1}{a\ln x}$ |
| $\ln x$ | $\dfrac{1}{x}$ |

| $f$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $a$ | $0$ |
| $x$ | $1$ |
| $a_1u(x) \pm a_2v(x)$ | $a_1\dfrac{\mathrm{d}u}{\mathrm{d}x} \pm a_2\dfrac{\mathrm{d}v}{\mathrm{d}x}$ |
| $u(x)v(x)$ | $\dfrac{\mathrm{d}u}{\mathrm{d}x}v + u\dfrac{\mathrm{d}v}{\mathrm{d}x}$ |
| $\dfrac{u(x)}{v(x)}$ | $\dfrac{\frac{\mathrm{d}u}{\mathrm{d}x}v - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v(x)^2}$ |
| $u(v(x))$ | $\dfrac{\mathrm{d}u}{\mathrm{d}v}\dfrac{\mathrm{d}v}{\mathrm{d}x}$ |

| $f$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $\sin(ax)$ | $a\cos(ax)$ |
| $\cos(ax)$ | $-a\sin(ax)$ |
| $\tan(ax)$ | $a\sec^2(ax)$ |
| $\cot(ax)$ | $-a\csc^2(ax)$ |
| $\sec(ax)$ | $a\sec(ax)\tan(ax)$ |
| $\csc(ax)$ | $-a\csc(ax)\cot(ax)$ |

| $f$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $\arcsin(ax)$ | $\dfrac{a}{\sqrt{1-a^2x^2}}$ |
| $\arccos(ax)$ | $-\dfrac{a}{\sqrt{1-a^2x^2}}$ |
| $\arctan(ax)$ | $\dfrac{a}{1+a^2x^2}$ |
| $\text{arccot}(ax)$ | $-\dfrac{a}{1+a^2x^2}$ |
| $\text{arcsec}(ax)$ | $\dfrac{1}{x\sqrt{a^2x^2-1}}$ |
| $\text{arccsc}(ax)$ | $-\dfrac{1}{x\sqrt{a^2x^2-1}}$ |

| $f$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $\sinh(ax)$ | $a\cosh(ax)$ |
| $\cosh(ax)$ | $a\sinh(ax)$ |
| $\tanh(ax)$ | $a\,\mathrm{sech}^2(ax)$ |
| $\coth(ax)$ | $-a\,\mathrm{csch}^2(ax)$ |
| $\mathrm{sech}(ax)$ | $-a\,\mathrm{sech}(ax)\tan(ax)$ |
| $\mathrm{csch}(ax)$ | $-a\,\mathrm{csch}(ax)\cot(ax)$ |

| $f$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $\mathrm{arcsinh}(ax)$ | $\frac{a}{\sqrt{1+a^2x^2}}$ |
| $\mathrm{arccosh}(ax)$ | $\frac{a}{\sqrt{1-a^2x^2}}$ |
| $\mathrm{arctanh}(ax)$ | $\frac{a}{1-a^2x^2}$ |

| $f$ | $\frac{\mathrm{d}f}{\mathrm{d}x}$ |
|---|---|
| $\mathrm{arccoth}(ax)$ | $\frac{a}{1-a^2x^2}$ |
| $\mathrm{arcsech}(ax)$ | $-\frac{1}{a(1+ax)\sqrt{\frac{1-ax}{1+ax}}}$ |
| $\mathrm{arccsch}(ax)$ | $-\frac{1}{ax^2\sqrt{1+\frac{1}{a^2x^2}}}$ |

Table 1: Derivatives of Elementary Functions

## 1.2 Trigonometric Identities

### 1.2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x) + 1 = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$

### 1.2.2 Double-Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \csc(2x) = \frac{\sec(x)\csc(x)}{2}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad \sec(2x) = \frac{\sec^2(x)\csc^2(x)}{\csc^2(x) - \sec^2(x)}$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} \qquad \cot(2x) = \frac{\cot^2(x) - 1}{2\cot(x)}$$

### 1.2.3 Power Reducing Identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \csc^2(x) = \frac{2}{1-\cos(2x)}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sec^2(x) = \frac{2}{1+\cos(2x)}$$

$$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)} \qquad \cot^2(x) = \frac{1+\cos(2x)}{1-\cos(2x)}$$

## 1.3 Partial Fractions

**Definition 1.3.1** (Partial Fraction Decomposition)**. Partial fraction decomposition** is a method where a rational function $\frac{P(x)}{Q(x)}$ is rewritten as a sum of fraction.

| **Factor in denominator** | **Term in partial fraction decomposition** |
|---|---|
| $ax+b$ | $\frac{A}{ax+b}$ |
| $(ax+b)^k,\ k\in\mathbb{N}$ | $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$ |
| $ax^2+bx+c$ | $\frac{A}{ax^2+bx+c}$ |
| $(ax^2+bx+c)^k,\ k\in\mathbb{N}$ | $\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$ |

Table 2: Partial Fraction Forms

## 1.4 Integration by Parts

**Theorem 1.4.1.**

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

## 1.5 Integration by Substitution

**Theorem 1.5.1.**

$$\int f(g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x = \int f(u)\,\mathrm{d}u\,,\ \textit{where}\ u=g(x)$$

## 1.6 Trigonometric Substitutions

| **Form** | **Substitution** | **Result** | **Domain** |
|---|---|---|---|
| $(a^2-b^2x^2)^n$ | $x=\frac{a}{b}\sin(\theta)$ | $a^2\cos^2(\theta)$ | $\theta\in[-\frac{\pi}{2},\ \frac{\pi}{2}]$ |
| $(a^2+b^2x^2)^n$ | $x=\frac{a}{b}\tan(\theta)$ | $a^2\sec^2(\theta)$ | $\theta\in(-\frac{\pi}{2},\ \frac{\pi}{2})$ |
| $(b^2x^2-a^2)^n$ | $x=\frac{a}{b}\sec(\theta)$ | $a^2\tan^2(\theta)$ | $\theta\in[0,\ \frac{\pi}{2})\cup(\frac{\pi}{2},\ \pi]$ |

Table 3: Trigonometric substitutions for various forms.

# 2 Limits, Continuity and Differentiability

## 2.1 Limits

**Theorem 2.1.1** (Limits). $\lim\limits_{x \to x_0} f(x)$ *exists if and only if* $\lim\limits_{x \to {x_0}^+} f(x)$ *and* $\lim\limits_{x \to {x_0}^-} f(x)$ *exist and are equal.*

**Definition 2.1.1** (Finite limits using the $\varepsilon$-$\delta$ definition).

$$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

**Theorem 2.1.2** (L'Hôpital's Rule). *For two differentiable functions* $f(x)$ *and* $g(x)$*. If* $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$*, or* $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = \pm\infty$*, then* $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.

## 2.2 Continuity

**Theorem 2.2.1** (Continuity at a Point). $f(x)$ *is continuous at* $c$ *iff* $\lim\limits_{x \to c} f(x) = f(c)$.

**Theorem 2.2.2** (Continuity over an Interval). $f(x)$ *is continuous on* $I$ *if* $f(x)$ *is continuous for all* $x \in I$.

- $f(x)$ *is continuous on* $I : (a,\ b)$ *if it is continuous for all* $x \in I$.
- $f(x)$ *is continuous on* $I : [a,\ b]$ *if it is continuous for all* $x \in I$*, but only right continuous at* $a$ *and left continuous at* $b$.

*If* $f(x)$ *is continuous on* $(-\infty,\ \infty)$*,* $f(x)$ *is continuous everywhere.*

**Theorem 2.2.3** (Intermediate Value Theorem). *If* $f(x)$ *is continuous on* $I : [a,\ b]$ *and* $c$ *is any number between* $f(a)$ *and* $f(b)$*, inclusive, then there exists an* $x \in I$ *such that* $f(x) = c$.

## 2.3 Differentiability

**Theorem 2.3.1** (Differentiability). $f(x)$ *is differentiable at* $x = x_0$ *iff*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

*exists. When this limit exists, it defines the derivative*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

**Theorem 2.3.2.** $f(x)$ *is differentiable on* $I$ *if* $f(x)$ *is differentiable for all* $x_0 \in I$.

**Theorem 2.3.3.** *Differentiability implies continuity.*

**Theorem 2.3.4** (Mean Value Theorem). *If* $f(x)$ *is continuous on* $I : [a,\ b]$ *and differentiable on* $I$*, then there exists a point* $c \in I$ *such that*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=c} = \frac{f(b) - f(a)}{b - a}$$

# 3 Definite Integrals

**Theorem 3.0.1.** *If $f(x)$ is continuous on an interval $I : [a, b]$, then the net signed area $A$ between the graph of $f(x)$ and the interval $I$ is*

$$A = \int_a^b f(x)\,\mathrm{d}x$$

## Properties of Definite Integrals

**Theorem 3.0.2.** *Suppose that $f(x)$ and $g(x)$ are continuous on the interval $I$, with $a, b, c \in I$ and $k \in \mathbb{R}$ then*

a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0.$

b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$

c) $\displaystyle\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x.$

d) $\displaystyle\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x.$

e) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$

## 3.1 Riemann Sums

**Theorem 3.1.1.** *Let $A$ be the area under $f(x)$ on the interval $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

*where $n$ is the number of rectangles, $x_k$ is the centre of the rectangle $k$, and $\Delta x_k$ is the width of the rectangle $k$. If every rectangle has the same width, then*

$$\forall k : \Delta x_k = \frac{b-a}{n}$$

## 3.2 Fundamental Theorem of Calculus

The fundamental theorem of calculus provides a logical connection between infinite series (definite integrals) and antiderivatives (indefinite integrals).

**Theorem 3.2.1** (The Fundamental Theorem of Calculus: Part 1)**.** *If $f(x)$ is continuous on $[a,\, b]$ and $F$ is any antiderivative of $f$ on $[a,\, b]$ then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

*Equivalently*

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x} F(x)\,\mathrm{d}x = F(b) - F(a) \equiv F(x)\big|_a^b$$

**Theorem 3.2.2** (The Fundamental Theorem of Calculus: Part 2)**.** *If $f(x)$ is continuous on $I$ then it has an antiderivative on $I$. In particular, if $a \in I$, then the function $F$ defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

*is an antiderivative of $f(x)$. That is,*

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

**Theorem 3.2.3.** *Differentiation and integration are inverse operations.*

## 3.3 Taylor and Maclaurin Polynomials

**Theorem 3.3.1** (Taylor Polynomials)**.** *If $f(x)$ is a $n$ differentiable function at $x_0$, then the $n$th degree Taylor polynomial for $f(x)$ near $x_0$, is given by*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

**Theorem 3.3.2** (Maclaurin Polynomials)**.** *Evaluating a Taylor polynomial near 0, gives the $n$th degree Maclaurin polynomial for $f(x)$*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

**Theorem 3.3.3** (Error in Approximation)**.** *Let $R_n(x)$ denote the difference between $f(x)$ and its $n$th Taylor polynomial, that is*

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = \frac{f^{(n+1)}(s)}{(n+1)!} (x - x_0)^{n+1}$$

*where $s$ is between $x_0$ and $x$.*

# 4 Taylor and Maclaurin Series

## 4.1 Infinite Series

**Definition 4.1.1** (Taylor Series)**.** If $f(x)$ has derivatives of all orders at $x_0$, then the Taylor series for $f(x)$ about $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

**Definition 4.1.2** (Maclaurin Series)**.** If a Taylor series is centred on $x_0 = 0$, it is called a Maclaurin series, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

**Definition 4.1.3** (Power Series)**.** Both Taylor and Maclaurin series are examples of **power series**, which are defined as follows

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

## 4.2 Convergence

**Theorem 4.2.1** (Convergence of a Taylor Series)**.** *The equality*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

*holds at a point $x$ iff*

$$\lim_{n\to\infty} \left[ f(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \right] = 0$$
$$\lim_{n\to\infty} R_n(x) = 0$$

**Definition 4.2.1** (Interval of Convergence)**.** The interval of convergence for a power series is the set of $x$ values for which that series converges.

**Definition 4.2.2** (Radius of Convergence)**.** The radius of convergence $R$ is a nonnegative real number or $\infty$ such that a power series converges if

$$|x - a| < R$$

and diverges if

$$|x - a| > R$$

The behaviour of the power series on the boundary, that is, where $|x - a| = R$, can be determined by substituting $x = R + a$ for the upper boundary, and $x = -R + a$ for the lower boundary.

## 4.3 Convergence Tests

For any power series of the form $\sum_{i=i_0}^{\infty} a_i$.

### Alternating Series

**Conditions** $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$\text{Is } b_{i+1} \leqslant b_i \ \& \ \lim_{i\to\infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

### Ratio Test

$$\text{Is } \lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

The ratio test is inconclusive if $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = 1$.

## 4.4 Table of Maclaurin Series

| **Function** | **Series** | **Interval of Convergence** |
|---|---|---|
| $\mathbf{e}^x$ | $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ | $-\infty < x < \infty$ |
| $\sin(x)$ | $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ | $-\infty < x < \infty$ |
| $\cos(x)$ | $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ | $-\infty < x < \infty$ |
| $\frac{1}{1-x}$ | $\sum_{n=0}^{\infty} x^n$ | $-1 < x < 1$ |

Table 4: Maclaurin Series of Common Functions

# 5 Multivariable Calculus

## 5.1 Multivariable Functions

**Definition 5.1.1.** A function is multivariable if its domain consists of several variables. In the reals, these functions are defined

$$f : \mathbb{R}^n \to \mathbb{R}$$

## 5.2 Level Curves

**Definition 5.2.1.** Level curves or *contour curves* of a function of two variables is a curve along which the function has constant value.

$$L_c\,(f) = \{(x,\, y) : f\,(x,\, y) = c\}$$

The level curves of a function can be determined by substituting $z = c$, and solving for $y$.

## 5.3 Limits and Continuity

**Definition 5.3.1** (Finite Limit of Multivariable Functions using the $\varepsilon$-$\delta$ Definition)**.**

$$\begin{gathered}\lim_{(x_1,\, \dots,\, x_n) \to (c_1,\, \dots,\, c_n)} f(x_1,\, \dots,\, x_n) = L \\ \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall (x_1,\, \dots,\, x_n) \in I : \\ 0 < |x_1 - c_1,\, \dots,\, x_n - c_n| < \delta \implies |f(x_1,\, \dots,\, x_n) - L| < \varepsilon\end{gathered}$$

**Theorem 5.3.1** (Limits along Smooth Curves)**.** *If $f(x,\, y) \to L$ as $(x,\, y) \to (x_0,\, y_0)$, then $\lim\limits_{(x,\, y) \to (x_0,\, y_0)} = L$ along any smooth curve.*

**Theorem 5.3.2** (Existence of a Limit)**.** *If the limit of $f(x,\, y)$ changes along different smooth curves, then $\lim\limits_{(x,\, y) \to (x_0,\, y_0)}$ does not exist.*

**Theorem 5.3.3** (Continuity of Multivariable Functions)**.** *A function $f(x_1,\, \dots,\, x_n)$ is continuous at $(c_1,\, \dots,\, c_n)$ iff*

$$\lim_{(x_1,\, \dots,\, x_n) \to (c_1,\, \dots,\, c_n)} f(x_1,\, \dots,\, x_n) = f(c_1,\, \dots,\, c_n)$$

Recognising continuous functions:

- A sum, difference or product of continuous functions is continuos.
- A quotient of continuous functions is continuos expect where the denominator is zero.
- A composition of continuous functions is continuos.

## 5.4 Partial Derivatives

**Definition 5.4.1** (Partial Differentiation)**.** The partial derivative of a multivariable function is its derivative with respect to one of those variables, while the others are held constant.

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1,\, \dots,\, x_{i-1},\, x_i + h,\, x_{i+1},\, \dots,\, x_n) - f(x_1,\, \dots,\, x_n)}{h}$$

## 5.5 The Gradient Vector

**Definition 5.5.1.** Let $\nabla$, pronounced "del", denote the vector differential operator defined as follows

$$\nabla = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{bmatrix}$$

## 5.6 Multivariable Chain Rule

**Theorem 5.6.1** (Multivariable Chain Rule)**.** *Let $f = f(\boldsymbol{x}(t_1, \dots, t_n))$ be the composition of $f$ with $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$, then the partial derivative of $f$ with respect to $t_i$ is given by*

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \partial_{t_i} \boldsymbol{x}$$

# 6 Double and Triple Integrals

# 7 Vector-Valued Functions

# 8 First-Order Differential Equations

# 9 Second-Order Differential Equations