Derivative Rules

$f(x)$	$f'(x)$
$u(x)v(x)$	$u'v+uv'$
$\frac{u(x)}{v(x)}$	$\frac{u'v-uv'}{v^2}$
$u(v(x))$	$u'(v(x))v'(x)$
x^n	nx^{n-1}
$\ln(u(x))$	$\frac{u'(x)}{u(x)}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\csc^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\csc(ax)$	$-a\csc(ax)\cot(ax)$

Trigonometric Identities

$$1=\sin^2(x)+\cos^2(x)$$
$$\sin(2x)=2\sin(x)\cos(x)$$
$$\cos(2x)=\cos^2(x)-\sin^2(x)$$
$$\sin^2(x)=\frac{1-\cos(2x)}{2}$$
$$\cos^2(x)=\frac{1+\cos(2x)}{2}$$

Partial Fraction Decomposition

$$\frac{p(x)}{(x+a)(x+b)}=\frac{A}{x+a}+\frac{B}{x+b}$$

Integration Techniques

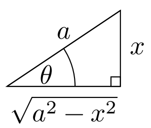
$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x=uv-\int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$
$$\int f(g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x=\int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

where $u=g(x)$.

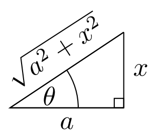
Trigonometric Substitutions

Substitute the appropriate value in and find that the value under the root is now square. Calculate $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ to change the variable. Use substitution to find the new limits.

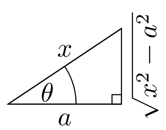
Sine ($a^2-b^2x^2$, $x=\frac{a}{b}\sin(\theta)$)



Tangent ($a^2+b^2x^2$, $x=\frac{a}{b}\tan(\theta)$)



Secant ($b^2x^2-a^2$, $x=\frac{a}{b}\sec(\theta)$)



L'Hôpital's Rule

If $\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=0$ or $\pm\infty$, then $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$.

Continuity

$f(x)$ continuous at c iff $\lim_{x\to c} f(x)=f(c)$.
$f(x)$ is continuous on $I:(a,\,b)$ if it is continuous for all $x\in I$.
$f(x)$ is continuous on $I:[a,\,b]$ if it is continuous for all $x\in I$, but only right continuous at a and left continuous at b.

Intermediate Value Theorem

If $f(x)$ is continuous on $I:[a,\,b]$ and $f(a)\le c\le f(b)$, then $\exists x\in I: f(x)=c$.

Differentiability

$f(x)$ is differentiable at $x=x_0$ iff

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

exists. This defines the derivative

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

Differentiability implies continuity.

Mean Value Theorem

If $f(x)$ is continuous and differentiable on $I:[a,\,b]$, then

$$\exists c\in I: f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Definite Integrals

$$A=\int_a^b f(x)\,\mathrm{d}x$$

Fundamental Theorem of Calculus

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}F(x)\,\mathrm{d}x=F(b)-F(a)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t=f(x)$$

Taylor Polynomials / Series

$$f(x)\approx p_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

The Taylor Series is taking the limit as n goes to infinity of p_n.
Maclaurin Series: $x_0=0$.

Common Maclaurin Series

Function	Series Term	Conv.
e^x	$\frac{x^n}{n!}$	all x
$\sin(x)$	$(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	all x
$\cos(x)$	$(-1)^n\frac{x^{2n}}{(2n)!}$	all x
$\frac{1}{1-x}$	x^n	$(-1,\,1)$
$\frac{1}{1+x^2}$	$(-1)^n x^{2n}$	$(-1,\,1)$
$\ln(1+x)$	$(-1)^{n+1}\frac{x^n}{n}$	$(-1,\,1]$

Power Series: $\sum_{n=0}^{\infty} c_n(x-x_0)^n$

Series Tests

For a series of the form $\sum_{i=i_0}^{\infty} a_i$:

Alternating Series Test

Given $a_i=(-1)^i b_i$ and $b_i>0$.
If $b_{i+1}\leqslant b_i$ & $\lim_{i\to\infty} b_i=0$, then convergent, else inconclusive.

Ratio Test

Given $\rho=\lim_{i\to\infty}\left|\frac{a_{i+1}}{a_i}\right|$:

$\rho<1$: convergent
$\rho>1$: divergent
$\rho=1$: inconclusive

Multivariable Functions

Partial Derivatives are done w.r.t one variable, other variables unchanged.
The **Gradient** is $\boldsymbol{\nabla} f=\langle\frac{\partial f}{\partial x},\,\frac{\partial f}{\partial y},\,\ldots\rangle$

Level Curves

$$L_c(f)=\{(x,\,y): f(x,\,y)=c\}$$

If $f(x,\,y)\to L$ as $(x,\,y)\to(x_0,\,y_0)$, then $\lim_{(x,\,y)\to(x_0,\,y_0)}=L$ along any smooth curve.
The limit does not exist if L changes along different smooth curves.

Multivariable Chain Rule

For $\frac{\partial f}{\partial s}$, create a tree from f to the variable s. On each edge, where the node above is a and below is b, each edge is $\frac{\partial a}{\partial b}$. Traverse each path between f and s and multiply the edges. Add each path together. (i.e. for $f(x,y),x(s,t),y(s,t)$, $\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$)

Directional Derivatives

$$\mathbf{D}_{\boldsymbol{u}} f=\boldsymbol{\nabla}_{\boldsymbol{u}} f=\boldsymbol{\nabla} f\cdot\boldsymbol{u}$$

where $\boldsymbol{u}$ is a unit vector and the slope is given by $\|\boldsymbol{\nabla}_{\boldsymbol{u}} f\|$. If $\boldsymbol{\nabla}_{\boldsymbol{u}} f=0$, $\boldsymbol{u}$ is tangent to the level curve at $\boldsymbol{x}_0$.

$$\max_{\|\boldsymbol{u}\|=1}\boldsymbol{\nabla}_{\boldsymbol{u}} f=\boldsymbol{\nabla} f$$

If $\boldsymbol{\nabla} f\neq 0$, $\boldsymbol{\nabla} f$ is a normal vector to the level curve at $\boldsymbol{x}_0$.

Critical Points

$(x_0,\,y_0)$ is a critical point if $\boldsymbol{\nabla} f(x_0,\,y_0)=\mathbf{0}$ or if $\boldsymbol{\nabla} f(x_0,\,y_0)$ is undefined.

Classification of Critical Points

$$D=f_{xx}f_{yy}-\left(f_{xy}\right)^2$$

$D>0$ **and** $f_{xx}<0$**:** local maxima

$D>0$ **and** $f_{xx}>0$**:** local minima

$D<0$**:** saddle point

$D=0$**:** inconclusive

Double Integrals

The volume of the solid enclosed between the surface $z=f(x,\,y)$ and the region Ω

is defined by

$$V = \iint_{\Omega} f(x,\, y)\,\mathrm{d}A\,.$$

If Ω is a region bounded by $a \leq x \leq b$ and $c \leq y \leq d$, then

$$\iint_{\Omega} f(x,\, y)\,\mathrm{d}A = \int_c^d \int_a^b f(x,\, y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_a^b \int_c^d f(x,\, y)\,\mathrm{d}y\,\mathrm{d}x$$

Type I Regions

$$\iint_{\Omega} f(x,\, y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,\, y)\,\mathrm{d}y\,\mathrm{d}x$$

Bounded left & right by:

$$x = a \text{ and } x = b$$

Bounded below & above by:

$$y = g_1(x) \text{ and } y = g_2(x)$$

where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$:

Type II Regions

$$\iint_{\Omega} f(x,\, y)\,\mathrm{d}A = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,\, y)\,\mathrm{d}x\,\mathrm{d}y$$

Bounded left & right by:

$$x = h_1(y) \text{ and } x = h_2(y)$$

Bounded below & above by:

$$y = c \text{ and } y = d$$

where $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$.
To integrate, solve the inner integrals first.

Vector Valued Functions

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$$

The **domain** of $\mathbf{r}(t)$ is the intersection of the domains of its components.
The **orientation** of $\mathbf{r}(t)$ is the direction of motion along the curve as the value of the parameter increases.
Limits, **derivatives** and **integrals** are all *component-wise*. Each component has its <u>own</u> constant of integration.

Parametric Lines

$$\mathbf{l}(t) = \boldsymbol{P}_0 + t\boldsymbol{v}$$

where $\mathbf{l}(t)$ passes through $\boldsymbol{P}_0$, and is parallel to $\boldsymbol{v}$.

Tangent Lines

If $\mathbf{r}(t)$ is differentiable at t_0 and $\mathbf{r}'(t_0) \neq \mathbf{0}$

$$\mathbf{l}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0).$$

Curves of Intersection

Choose one of the variables as the parameter, and express the remaining variables in terms of that parameter.

Arc Length

$$S = \int_a^b \|\mathbf{r}'(t)\|\,\mathrm{d}t$$

Ordinary Differential Equations

Order: highest derivative in DE.
Autonomous DE: does not depend explicitly on the independent variable.

Qualitative Analysis

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

A fixed point is the value of y for which $f(y) = 0$.

Stability of Fixed Points

Given a positive/negative perturbation from a fixed point, that point is
Stable: if both tend toward FP
Unstable: if both tend away from FP
Semi-Stable: if one tends toward FP, and another tends away from FP

Directly Integrable ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$:

$$y(x) = \int f(x)\,\mathrm{d}x\,.$$

Separable ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)q(y)$:

$$\int \frac{1}{q(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int p(x)\,\mathrm{d}x\,.$$

Linear ODEs

For $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$, use the *integrating factor*: $I(x) = e^{\int p(x)\mathrm{d}x}$, so that

$$y(x) = \frac{1}{I(x)}\int I(x)q(x)\,\mathrm{d}x\,.$$

Exact ODEs

$P(x,\, y) + Q(x,\, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ has the solution $\Psi(x,\, y) = c$ iff it is exact, namely, when $P_y = Q_x$, where $P = \Psi_x$ and $Q = \Psi_y$. Then

$$\Psi(x,\, y) = \int P(x,\, y)\,\mathrm{d}x + f(y)$$
$$\Psi(x,\, y) = \int Q(x,\, y)\,\mathrm{d}y + g(x)$$

and $f(y)$ and $g(x)$ can be determined by solving these equations simultaneously.

Second-Order ODEs

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = F(x)$$

Initial Values

$$y(x_0) = y_0 \quad y'(x_0) = y_1$$

Boundary Values

$$y(x_0) = y_0 \quad y(x_1) = y_1$$

Reduction of Order

$$y_2(x) = v\,(x)\,y_1(x)$$

$v(x)$ can be determined by substituting y_2 into the ODE, using $w(x) = v'(x)$.

General Solution

$$y(x) = y_H(x) + y_P(x)$$

Homogeneous Solution

$$y_H(x) = e^{\lambda x}$$

Real Distinct Roots

$$y_H(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Real Repeated Roots

$$y_H(x) = c_1 e^{\lambda x} + c_2 t e^{\lambda x}$$

Complex Conjugate Roots

Given $\lambda = \alpha \pm \beta i$:

$$y_H(x) = e^{\alpha x}(c_1 \cos{(\beta x)} + c_2 \sin{(\beta x)})$$

Particular Solution

$F(x)$	$y_P(x)$
constant	A
polynomial degree n	$\sum_{i=0}^{n} A_i x^i$
e^{kx}	Ae^{kx}
$\cos{(\omega x)}$ or $\sin{(\omega x)}$	$A_0 \cos{(\omega x)}$ $+A_1 \sin{(\omega x)}$

If $F(x)$ is a combination of the above, $y_P(x)$ should be too. If the choice would be linearly dependent to $y_H(x)$, multiply by x until it isn't.
Substitute y_P into the nonhomogeneous ODE, and solve the undetermined coefficients.

Spring and Mass Systems

$$my'' + \gamma y' + ky = f(t)$$

Newton's Law: $F = my''$
Spring force: $F_s = -ky$
Damping force: $F_d = -\gamma y'$

m : mass $\quad k$: spring constant
γ : damping $\quad f(t)$: external force

Electrical Circuits

The sum of voltages around a loop equals 0.

$$v(t) - iR - L\frac{\mathrm{d}i}{\mathrm{d}t} - \frac{q}{C} = 0$$
$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = v(t)$$

where $i = \dfrac{\mathrm{d}q}{\mathrm{d}t}$.

R : resistance $\quad C$: capacitance
L : inductance $\quad v(t)$: voltage supply