Calculus and Differential Equations

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Contents

1 Integration Techniques

1.1 Table of Derivatives

Let $f(x)$ be a function, and $a \in \mathbb{R}$ be a constant.

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
x^a	ax^{a-1}
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
a^x	$\ln(a)a^x$
$\mathbf{e}^x$	$\mathbf{e}^x$
$\log_a x,\ a \in \mathbb{R}\setminus\{0\}$	$\frac{1}{a\ln x}$
$\ln x$	$\frac{1}{x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
a	0
x	1
$a_1 u(x) \pm a_2 v(x)$	$a_1 \frac{\mathrm{d}u}{\mathrm{d}x} \pm a_2 \frac{\mathrm{d}v}{\mathrm{d}x}$
$u(x)v(x)$	$\frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}$
$\frac{u(x)}{v(x)}$	$\frac{\frac{\mathrm{d}u}{\mathrm{d}x}v - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v(x)^2}$
$u(v(x))$	$\frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}x}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\csc^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\csc(ax)$	$-a\csc(ax)\cot(ax)$
$\arcsin(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\frac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\frac{a}{1+a^2x^2}$
$\operatorname{arccot}(ax)$	$-\frac{a}{1+a^2x^2}$
$\operatorname{arcsec}(ax)$	$\frac{1}{x\sqrt{a^2x^2-1}}$
$\operatorname{arccsc}(ax)$	$-\frac{1}{x\sqrt{a^2x^2-1}}$

f	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$\sinh(ax)$	$a\cosh(ax)$
$\cosh(ax)$	$a\sinh(ax)$
$\tanh(ax)$	$a\,\operatorname{sech}^2(ax)$
$\coth(ax)$	$-a\,\operatorname{csch}^2(ax)$
$\operatorname{sech}(ax)$	$-a\,\operatorname{sech}(ax)\tan(ax)$
$\operatorname{csch}(ax)$	$-a\,\operatorname{csch}(ax)\cot(ax)$
$\operatorname{arcsinh}(ax)$	$\frac{a}{\sqrt{1+a^2x^2}}$
$\operatorname{arccosh}(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\operatorname{arctanh}(ax)$	$\frac{a}{1-a^2x^2}$
$\operatorname{arccoth}(ax)$	$\frac{a}{1-a^2x^2}$
$\operatorname{arcsech}(ax)$	$-\frac{1}{a(1+ax)\sqrt{\frac{1-ax}{1+ax}}}$
$\operatorname{arccsch}(ax)$	$-\frac{1}{ax^2\sqrt{1+\frac{1}{a^2x^2}}}$

Table 1: Derivatives of Elementary Functions

1.2 Trigonometric Identities

1.2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x) + 1 = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$

1.2.2 Double-Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \csc(2x) = \frac{\sec(x)\csc(x)}{2}$$
$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad \sec(2x) = \frac{\sec^2(x)\csc^2(x)}{\csc^2(x) - \sec^2(x)}$$
$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} \qquad \cot(2x) = \frac{\cot^2(x) - 1}{2\cot(x)}$$

1.2.3 Power Reducing Identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \csc^2(x) = \frac{2}{1-\cos(2x)}$$
$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sec^2(x) = \frac{2}{1+\cos(2x)}$$
$$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)} \qquad \cot^2(x) = \frac{1+\cos(2x)}{1-\cos(2x)}$$

1.3 Partial Fractions

Definition 1.3.1 (Partial Fraction Decomposition)**. Partial fraction decomposition** is a method where a rational function $\frac{P(x)}{Q(x)}$ is rewritten as a sum of fraction.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k,\ k \in \mathbb{N}$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{A}{ax^2+bx+c}$
$(ax^2+bx+c)^k,\ k \in \mathbb{N}$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$

Table 2: Partial Fraction Forms

1.4 Integration by Parts

Theorem 1.4.1.

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

1.5 Integration by Substitution

Theorem 1.5.1.

$$\int f(g\,(x)) \frac{\mathrm{d}g(x)}{\mathrm{d}x} \, \mathrm{d}x = \int f(u) \, \mathrm{d}u \,, \text{ where } u = g(x)$$

1.6 Trigonometric Substitutions

Form	**Substitution**	**Result**	**Domain**
$\left(a^2 - b^2x^2\right)^n$	$x = \dfrac{a}{b} \sin(\theta)$	$a^2 \cos^2(\theta)$	$\theta \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$
$\left(a^2 + b^2x^2\right)^n$	$x = \dfrac{a}{b} \tan(\theta)$	$a^2 \sec^2(\theta)$	$\theta \in \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$
$\left(b^2x^2 - a^2\right)^n$	$x = \dfrac{a}{b} \sec(\theta)$	$a^2 \tan^2(\theta)$	$\theta \in \left[0,\ \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\ \pi\right]$

Table 3: Trigonometric substitutions for various forms.

2 Limits, Continuity and Differentiability

2.1 Limits

Theorem 2.1.1 (Limits). *$\lim_{x \to x_0} f(x)$ exists if and only if $\lim_{x \to {x_0}^+} f(x)$ and $\lim_{x \to {x_0}^-} f(x)$ exist and are equal.*

Definition 2.1.1 (Finite limits using the ε-δ definition).

$$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in I : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

Theorem 2.1.2 (L'Hôpital's Rule). *For two differentiable functions $f(x)$ and $g(x)$. If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$, or $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \pm\infty$, then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.*

2.2 Continuity

Theorem 2.2.1 (Continuity at a Point). *$f(x)$ is continuous at c iff $\lim_{x \to c} f(x) = f(c)$.*

Theorem 2.2.2 (Continuity over an Interval). *$f(x)$ is continuous on I if $f(x)$ is continuous for all $x \in I$.*

- *$f(x)$ is continuous on $I : (a,\, b)$ if it is continuous for all $x \in I$.*
- *$f(x)$ is continuous on $I : [a,\, b]$ if it is continuous for all $x \in I$, but only right continuous at a and left continuous at b.*

If $f(x)$ is continuous on $(-\infty,\, \infty)$, $f(x)$ is continuous everywhere.

Theorem 2.2.3 (Intermediate Value Theorem). *If $f(x)$ is continuous on $I : [a,\, b]$ and c is any number between $f(a)$ and $f(b)$, inclusive, then there exists an $x \in I$ such that $f(x) = c$.*

2.3 Differentiability

Theorem 2.3.1 (Differentiability). *$f(x)$ is differentiable at $x = x_0$ iff*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, it defines the derivative

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x = x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Theorem 2.3.2. *$f(x)$ is differentiable on I if $f(x)$ is differentiable for all $x_0 \in I$.*

Theorem 2.3.3. *Differentiability implies continuity.*

Theorem 2.3.4 (Mean Value Theorem). *If $f(x)$ is continuous on $I : [a,\, b]$ and differentiable on I, then there exists a point $c \in I$ such that*

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x = c} = \frac{f(b) - f(a)}{b - a}$$

3 Definite Integrals

Theorem 3.0.1. *If $f(x)$ is continuous on an interval $I : [a,\, b]$, then the net signed area A between the graph of $f(x)$ and the interval I is*

$$A = \int_a^b f(x)\,\mathrm{d}x$$

Properties of Definite Integrals

Theorem 3.0.2. *Suppose that $f(x)$ and $g(x)$ are continuous on the interval I, with $a,\, b,\, c \in I$ and $k \in \mathbb{R}$ then*

a) $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0.$

b) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$

c) $\displaystyle\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x.$

d) $\displaystyle\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x.$

e) $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$

3.1 Riemann Sums

Theorem 3.1.1. *Let A be the area under $f(x)$ on the interval $[a,\, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

where n is the number of rectangles, x_k is the centre of the rectangle k, and Δx_k is the width of the rectangle k. If every rectangle has the same width, then

$$\forall k : \Delta x_k = \frac{b-a}{n}$$

3.2 Fundamental Theorem of Calculus

The fundamental theorem of calculus provides a logical connection between infinite series (definite integrals) and antiderivatives (indefinite integrals).

Theorem 3.2.1 (The Fundamental Theorem of Calculus: Part 1)**.** *If $f(x)$ is continuous on $[a,\, b]$ and F is any antiderivative of f on $[a,\, b]$ then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Equivalently

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x} F(x)\,\mathrm{d}x = F(b) - F(a) \equiv F(x)\big|_a^b$$

Theorem 3.2.2 (The Fundamental Theorem of Calculus: Part 2)**.** *If $f(x)$ is continuous on I then it has an antiderivative on I. In particular, if $a \in I$, then the function F defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is an antiderivative of $f(x)$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) = f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

Theorem 3.2.3. *Differentiation and integration are inverse operations.*

3.3 Taylor and Maclaurin Polynomials

Theorem 3.3.1 (Taylor Polynomials). *If $f(x)$ is a n differentiable function at x_0, then the nth degree Taylor polynomial for $f(x)$ near x_0, is given by*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Theorem 3.3.2 (Maclaurin Polynomials). *Evaluating a Taylor polynomial near 0, gives the nth degree Maclaurin polynomial for $f(x)$*

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Theorem 3.3.3 (Error in Approximation). *Let $R_n(x)$ denote the difference between $f(x)$ and its nth Taylor polynomial, that is*

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = \frac{f^{(n+1)}(s)}{(n+1)!} (x - x_0)^{n+1}$$

where s is between x_0 and x.

4 Taylor and Maclaurin Series

4.1 Infinite Series

Definition 4.1.1 (Taylor Series)**.** If $f(x)$ has derivatives of all orders at x_0, then the Taylor series for $f(x)$ about $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Definition 4.1.2 (Maclaurin Series)**.** If a Taylor series is centred on $x_0 = 0$, it is called a Maclaurin series, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 4.1.3 (Power Series)**.** Both Taylor and Maclaurin series are examples of **power series**, which are defined as follows

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

4.2 Convergence

Theorem 4.2.1 (Convergence of a Taylor Series)**.** *The equality*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

holds at a point x iff

$$\lim_{n\to\infty} \left[f(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \right] = 0$$
$$\lim_{n\to\infty} R_n(x) = 0$$

Definition 4.2.1 (Interval of Convergence)**.** The interval of convergence for a power series is the set of x values for which that series converges.

Definition 4.2.2 (Radius of Convergence)**.** The radius of convergence R is a nonnegative real number or ∞ such that a power series converges if

$$|x - a| < R$$

and diverges if

$$|x - a| > R$$

The behaviour of the power series on the boundary, that is, where $|x - a| = R$, can be determined by substituting $x = R + a$ for the upper boundary, and $x = -R + a$ for the lower boundary.

4.3 Convergence Tests

For any power series of the form $\sum_{i=i_0}^{\infty} a_i$.

Alternating Series

Conditions $a_i = (-1)^i b_i$ or $a_i = (-1)^{i+1} b_i$. $b_i > 0$.

$$\text{Is } b_{i+1} \leqslant b_i \ \&\ \lim_{i\to\infty} b_i = 0? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \text{Inconclusive} \end{cases}$$

Ratio Test

$$\text{Is } \lim_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right| < 1? \begin{cases} \text{YES} & \sum a_i \text{ Converges} \\ \text{NO} & \sum a_i \text{ Diverges} \end{cases}$$

The ratio test is inconclusive if $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = 1$.

4.4 Table of Maclaurin Series

Function	**Series**	**Interval of Convergence**
$\mathbf{e}^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$-\infty < x < \infty$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$

Table 4: Maclaurin Series of Common Functions

5 Multivariable Calculus

5.1 Multivariable Functions

Definition 5.1.1. A function is multivariable if its domain consists of several variables. In the reals, these functions are defined

$$f : \mathbb{R}^n \to \mathbb{R}$$

5.2 Level Curves

Definition 5.2.1. Level curves or *contour curves* of a function of two variables is a curve along which the function has constant value.

$$L_c(f) = \{(x,\, y) : f(x,\, y) = c\}$$

The level curves of a function can be determined by substituting $z = c$, and solving for y.

5.3 Limits and Continuity

Definition 5.3.1 (Finite Limit of Multivariable Functions using the ε-δ Definition)**.**

$$\begin{aligned}
&\lim_{(x_1,\, \dots,\, x_n)\to(c_1,\, \dots,\, c_n)} f(x_1,\, \dots,\, x_n) = L \\
&\iff \forall\varepsilon > 0 : \exists\delta > 0 : \forall(x_1,\, \dots,\, x_n) \in I : \\
&0 < |x_1 - c_1,\, \dots,\, x_n - c_n| < \delta \implies |f(x_1,\, \dots,\, x_n) - L| < \varepsilon
\end{aligned}$$

Theorem 5.3.1 (Limits along Smooth Curves)**.** *If $f(x,\, y) \to L$ as $(x,\, y) \to (x_0,\, y_0)$, then $\lim_{(x,\, y)\to(x_0,\, y_0)} = L$ along any smooth curve.*

Theorem 5.3.2 (Existence of a Limit)**.** *If the limit of $f(x,\, y)$ changes along different smooth curves, then $\lim_{(x,\, y)\to(x_0,\, y_0)}$ does not exist.*

Theorem 5.3.3 (Continuity of Multivariable Functions)**.** *A function $f(x_1,\, \dots,\, x_n)$ is continuous at $(c_1,\, \dots,\, c_n)$ iff*

$$\lim_{(x_1,\, \dots,\, x_n)\to(c_1,\, \dots,\, c_n)} f(x_1,\, \dots,\, x_n) = f(c_1,\, \dots,\, c_n)$$

Recognising continuous functions:

- A sum, difference or product of continuous functions is continuos.
- A quotient of continuous functions is continuos expect where the denominator is zero.
- A composition of continuous functions is continuos.

5.4 Partial Derivatives

Definition 5.4.1 (Partial Differentiation)**.** The partial derivative of a multivariable function is its derivative with respect to one of those variables, while the others are held constant.

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1,\, \dots,\, x_{i-1},\, x_i + h,\, x_{i+1},\, \dots,\, x_n) - f(x_1,\, \dots,\, x_n)}{h}$$

5.5 The Gradient Vector

Definition 5.5.1. Let ∇, pronounced "del", denote the vector differential operator defined as follows

$$\nabla = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{bmatrix}$$

5.6 Multivariable Chain Rule

Theorem 5.6.1 (Multivariable Chain Rule)**.** *Let $f = f(\boldsymbol{x}(t_1, \ldots, t_n))$ be the composition of f with $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$, then the partial derivative of f with respect to t_i is given by*

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \partial_{t_i} \boldsymbol{x}$$

6 Double and Triple Integrals

7 Vector-Valued Functions

8 First-Order Differential Equations

9 Second-Order Differential Equations