

Linear Algebra

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1 Euclidean Vector Spaces

1.1 Vectors

Definition 1.1.1. An n -dimensional **vector** is an ordered list of n numbers.

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

Theorem 1.1.1. \mathbb{R}^n is the set of all ordered n -tuples of real numbers.

$$\mathbb{R}^n = \{(v_1, v_2, \dots, v_n) : v_1, v_2, \dots, v_n \in \mathbb{R} : n \in \mathbb{N}\}$$

Notation:

1. Component form: $\mathbf{v} = \langle v_1, v_2 \rangle = (v_1, v_2) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
2. Unit vector form: $\mathbf{v} = v_1 \hat{\mathbf{i}} + v_2 \hat{\mathbf{j}}$, where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are basis vectors along the x and y axes respectively.
3. Denotation: $\mathbf{v} = v = \vec{v}$

1.2 Position and Displacement Vectors

Definition 1.2.1. The displacement vector \overrightarrow{AB} from \mathbf{a} to \mathbf{b} can be defined as $\mathbf{b} - \mathbf{a}$.

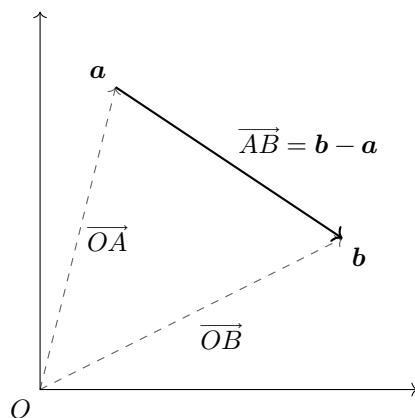


Figure 1: Displacement vector between two points.

1.3 Vector Addition

Definition 1.3.1. **Vector addition** is performed by adding the corresponding components of two vectors of the same dimension.

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

1.4 Scalar Multiplication

Definition 1.4.1. **Scalar multiplication** is performed by multiplying each element of the vector by the scalar.

$$a\mathbf{v} = \begin{bmatrix} av_1 \\ \vdots \\ av_n \end{bmatrix}$$

1.5 Norm of a Vector

Definition 1.5.1. The **norm** of a vector \mathbf{v} , denoted by $\|\mathbf{v}\|$, is the *length* or *magnitude* of \mathbf{v} .

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

1.6 The Unit Vector

Definition 1.6.1. A **unit vector** is a vector, denoted $\hat{\mathbf{v}}$, that has a length of 1 in the direction of \mathbf{v} .

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

1.7 The Dot Product

Definition 1.7.1. The **dot product** is a function that associates each pair of vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ a real number $\mathbf{v} \cdot \mathbf{w}$.

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \\ &= \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta) \end{aligned}$$

where θ is the angle between \mathbf{v} and \mathbf{w} .

Theorem 1.7.1. If $\mathbf{v} \cdot \mathbf{w} = 0$ then \mathbf{v} and \mathbf{w} are orthogonal.

1.8 The Cross Product

Definition 1.8.1. The **cross product** is a function that associates each ordered pair of vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ a vector $\mathbf{v} \times \mathbf{w} \in \mathbb{R}^3$.

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \|\mathbf{v}\| \|\mathbf{w}\| \sin(\theta) \hat{\mathbf{n}} \end{aligned}$$

where $\hat{\mathbf{n}}$ is the normal vector given by the right-hand rule.

2 Vector Identities

Theorem 2.0.1. *Commutativity of vector addition.*

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

Theorem 2.0.2.

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$$

Theorem 2.0.3. *Commutativity of dot products.*

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

Theorem 2.0.4. *Distributivity of dot products over vector addition.*

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Theorem 2.0.5. *Associativity of dot products over scalar multiplication.*

$$(r\mathbf{a}) \cdot \mathbf{b} = r(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (r\mathbf{b})$$

Theorem 2.0.6. *Bilinearity of dot products.*

$$\mathbf{a} \cdot (r\mathbf{b} + \mathbf{c}) = r(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$$

Theorem 2.0.7.

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

Theorem 2.0.8. *Anticommutativity of cross products.*

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

Theorem 2.0.9. *Distributivity of cross products over vector addition.*

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Theorem 2.0.10. *Associativity of cross products over scalar multiplication.*

$$(r\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (r\mathbf{b}) = r(\mathbf{a} \times \mathbf{b})$$

Theorem 2.0.11.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

Theorem 2.0.12.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

3.6 Elementary Row Operations

Definition 3.6.1. A linear system can be solved using the following **elementary row operations**:

1. **scalar multiplication**: multiplying any row by a constant
2. **row addition**: adding a multiple of one row to another
3. **row exchange**: exchanging any two rows

3.7 Pivots

Definition 3.7.1. The first non-zero entry of the row in a matrix is called the **pivot** of the row.

Theorem 3.7.1. *If a row apart from the first has a pivot, then this pivot must be to the right of the pivot in the preceding row.*

3.8 Gaussian Elimination

Definition 3.8.1. Gaussian elimination is a method for solving linear systems. These systems can be solved by composing the augmented matrix of a system, and performing elementary row operations, to put the matrix in the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{mn} \end{bmatrix}$$

3.9 Row-Echelon Form

Definition 3.9.1. A matrix that has undergone Gaussian elimination is in **row-echelon form** if the pivots of the augmented matrix are all 1.

$$\begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & 1 & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

3.10 Gauss-Jordan Elimination

Definition 3.10.1. Gauss-Jordan elimination extends Gaussian elimination so that the entries in a column containing a pivot are zeros, and the pivots are all 1. This new augmented matrix is then in **reduced row-echelon form**.

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

3.11 Solutions to Linear Systems

Definition 3.11.1. A **consistent system** of equations has at least one solution, and an **inconsistent system** has no solution.

4 Matrices

Definition 4.0.1. A **matrix** is an array of numbers arranged into *rows* and *columns*, and can be used to represent a linear transformation.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

4.1 Matrix Addition

Definition 4.1.1. **Matrix addition** is performed by adding the corresponding components of two matrices of the same dimension.

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

4.2 Scalar Multiplication

Definition 4.2.1. **Scalar multiplication** is performed by multiplying each element of a matrix by a scalar.

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

4.3 Matrix Multiplication

Definition 4.3.1. **Matrix multiplication** is performed by multiplying each row in the first matrix by the columns of the second matrix.

$$AB = C$$

$$\begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ - & \vdots & - \\ - & \mathbf{a}_m & - \end{bmatrix} \begin{bmatrix} \left| \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right| & \left| \begin{array}{c} b_2 \\ b_2 \\ \vdots \\ b_n \end{array} \right| & \cdots & \left| \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right| \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \mathbf{b}_1 & \mathbf{a}_1 \mathbf{b}_2 & \cdots & \mathbf{a}_1 \mathbf{b}_n \\ \mathbf{a}_2 \mathbf{b}_1 & \mathbf{a}_2 \mathbf{b}_2 & \cdots & \mathbf{a}_2 \mathbf{b}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_m \mathbf{b}_1 & \mathbf{a}_m \mathbf{b}_2 & \cdots & \mathbf{a}_m \mathbf{b}_n \end{bmatrix}$$

Theorem 4.3.1. A matrix product is defined if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

4.4 The Identity Matrix

Definition 4.4.1. The **identity matrix** is the simplest nontrivial **diagonal matrix**, denoted $\mathbf{1}$, such that

$$\mathbf{1}A = A$$

written explicitly as

$$\mathbb{1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

4.5 The Inverse Matrix

Definition 4.5.1. The **inverse** of a **square matrix** is a matrix \mathbf{A}^{-1} , such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbb{1}$$

Theorem 4.5.1. *The inverse of a 2×2 matrix is given by*

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Theorem 4.5.2. *The inverse of an $n \times n$ matrix can be determined by solving $[\mathbf{A} \mid \mathbb{1}]$.*

4.6 The Diagonal Matrix

Definition 4.6.1. A **diagonal** matrix, denoted $\text{diag}(d_{11}, d_{22}, \dots, d_{nn})$, is an $n \times n$ matrix \mathbf{D} in which entries outside the main diagonal are all zero.

$$\mathbf{D} = \text{diag}(d_{11}, d_{22}, \dots, d_{nn}) = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

4.7 Matrix Transpose

Definition 4.7.1. The **transpose** of a matrix, denoted by \mathbf{A}^\top , is obtained by replacing all a_{ij} elements with a_{ji} , so that the matrix \mathbf{A} is flipped over its main diagonal.

4.8 Matrix Trace

Definition 4.8.1. The **trace** of an $n \times n$ matrix \mathbf{A} , denoted $\text{Tr}(\mathbf{A})$, is defined as

$$\text{Tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$$

5 General Vector Spaces

5.1 Real Vector Spaces

Definition 5.1.1. A **vector space** is a set that is closed under vector addition and scalar multiplication.

Theorem 5.1.1. *If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and all scalars k and m , then V is a **vector space**, and the objects in V are vectors.*

Axiom 1 (Closure under addition).

$$\mathbf{u} + \mathbf{v} \in V$$

Axiom 2 (Commutativity of vector addition).

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Axiom 3 (Associativity of vector addition).

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

Axiom 4 (Additive identity).

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

Axiom 5 (Additive inverse).

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

Axiom 6 (Closure under scalar multiplication).

$$k\mathbf{u} \in V$$

Axiom 7 (Distributivity of vector addition).

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$$

Axiom 8 (Distributivity of scalar addition).

$$(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$$

Axiom 9 (Associativity of scalar multiplication).

$$k(m\mathbf{u}) = (km)\mathbf{u}$$

Axiom 10 (Scalar multiplication identity).

$$1\mathbf{u} = \mathbf{u}$$

To identify that a set with two operations is a vector space:

1. Identify the set V of objects that will become vectors.
2. Identify the addition and scalar multiplication operations on V .

3. Verify Axioms 1 and 6.
4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

Theorem 5.1.2. *Let V be a vector space. If $\mathbf{v} \in V$, and k is a scalar.*

1. $0\mathbf{v} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{v} = -\mathbf{v}$
4. If $k\mathbf{v} = \mathbf{0}$, then $k = 0$ or $\mathbf{v} = \mathbf{0}$

5.2 Subspaces

Definition 5.2.1. A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication operations defined on V .

Theorem 5.2.1. *Let W be a subspace of the vector space V , then the following axioms must be satisfied.*

1. **Axiom 1:** Closure under addition
2. **Axiom 6:** Closure under scalar multiplication

Theorem 5.2.2. *Every vector space has at least two subspaces, itself and its zero subspace.*

Theorem 5.2.3. *Subspaces of \mathbb{R}^2 .*

1. $\{\mathbf{0}\}$
2. Lines through the origin
3. \mathbb{R}^2

Theorem 5.2.4. *Subspaces of \mathbb{R}^3 .*

1. $\{\mathbf{0}\}$
2. Lines through the origin
3. Planes through the origin
4. \mathbb{R}^3

Theorem 5.2.5. *Subspaces of M_{nn} .*

1. Upper triangular matrices
2. Lower triangular matrices
3. Diagonal matrices
4. M_{nn}

5.3 Spanning Sets

Definition 5.3.1. If the vector \mathbf{w} is in a vector space V , then \mathbf{w} is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$, if \mathbf{w} can be expressed in the form

$$\mathbf{w} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n$$

Theorem 5.3.1. If $S = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is a nonempty set of vectors in a vector space V , then the set W of all possible linear combinations of the vectors in S is a subspace of V . The subspace W is called the subspace of V **spanned** by S and the vectors in S **span** W . If a vector in S can be expressed as the linear combination of any vectors in S then the set is **linearly dependent**.

5.4 Linear Independence

Definition 5.4.1. If S is a set of two or more vectors in a vector space V , then S is **linearly independent** if no vector in S can be expressed as a linear combination of the others.

Theorem 5.4.1. A set S is linearly independent if and only if there is one solution to the equation

$$k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n = \mathbf{0}$$

where the coefficients satisfying this equation are $k_1 = 0, k_2 = 0, \dots, k_n = 0$.

5.5 Basis Vectors

Definition 5.5.1. If S is a set of vectors in a vector space V , then S is called a **basis** for V if

1. S spans V .
2. S is linearly independent.

5.6 Dimension

Definition 5.6.1. The **dimension** of a finite-dimensional vector space V , denoted $\dim(V)$, is the number of vectors in a basis for V .

Theorem 5.6.1. The zero vector space is defined to have dimension zero.

6 Fundamental Subspaces

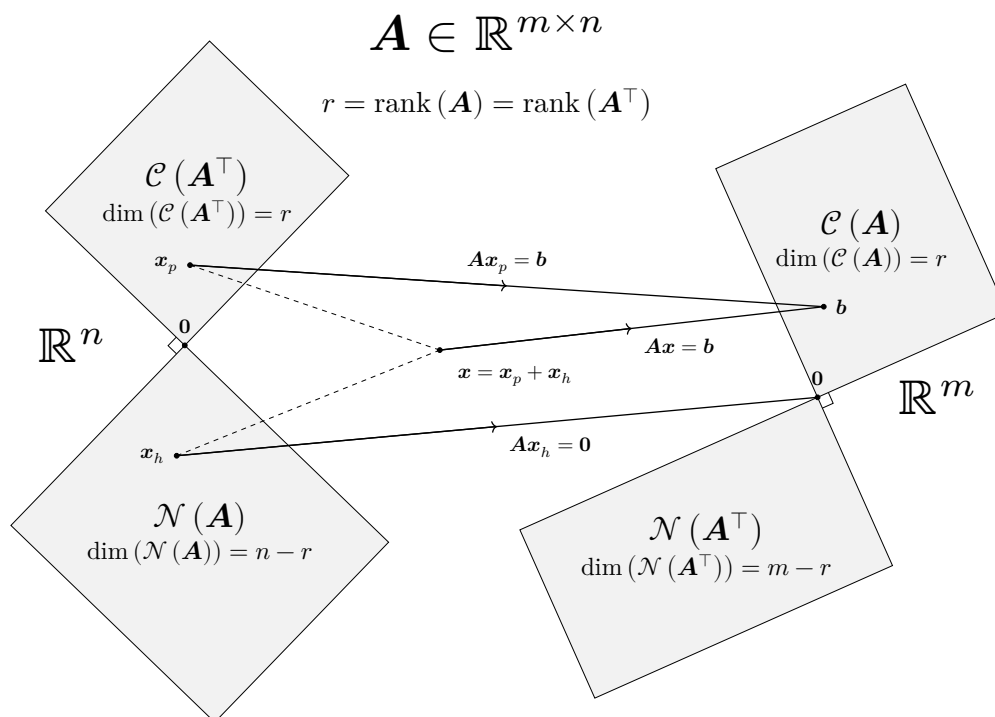


Figure 2: The Four Fundamental Subspaces of a Matrix.

6.1 The Four Fundamental Subspaces of a Matrix

Definition 6.1.1. If $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an $m \times n$ matrix, then:

1. The subspace spanned by the *column vectors* of \mathbf{A} , is the **column space** of \mathbf{A} , denoted $\mathcal{C}(\mathbf{A})$.
2. The subspace spanned by the *row vectors* of \mathbf{A} , is the **row space** of \mathbf{A} , denoted $\mathcal{C}(\mathbf{A}^\top)$.
3. The subspace spanned by the *solution space* of the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, is the **null space** of \mathbf{A} , denoted $\mathcal{N}(\mathbf{A})$.
4. The subspace spanned by the *solution space* of the equation $\mathbf{A}^\top \mathbf{y} = \mathbf{0}$ (or $\mathbf{y}^\top \mathbf{A} = \mathbf{0}$), is the **left null space** of \mathbf{A} , denoted $\mathcal{N}(\mathbf{A}^\top)$.

6.2 The General Solution of a System of Equations

Theorem 6.2.1. The *general solution* to a matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, can be given by adding the *particular* and *homogeneous* solutions, where the particular solution is the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, or

$\mathcal{C}(\mathbf{A}^\top)$, and the homogeneous solution is the solution to $\mathbf{A}\mathbf{x} = \mathbf{0}$, or $\mathcal{N}(\mathbf{A})$.

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$$

6.3 Row Equivalence

Definition 6.3.1. Two matrices are **row equivalent** if each can be obtained from the other by elementary row operations. These matrices have the same row space and null space.

6.4 Rank

Definition 6.4.1. The **rank** of a matrix, denoted by $\text{rank}(\mathbf{A})$, is given by $\dim(\mathcal{C}(\mathbf{A}))$.

Theorem 6.4.1. *The column space and row space have the same dimension so that*

$$\text{rank}(\mathbf{A}) = \dim(\mathcal{C}(\mathbf{A})) = \dim(\mathcal{C}(\mathbf{A}^\top))$$

6.5 Nullity

Definition 6.5.1. The **nullity** of a matrix, denoted by $\text{null}(\mathbf{A})$, is given by $\dim(\mathcal{N}(\mathbf{A}))$.

7 Orthogonality

Definition 7.0.1. Two vectors are **orthogonal** if the following holds.

$$\mathbf{v} \cdot \mathbf{w} = 0 \iff \mathbf{v}^\top \mathbf{w} = 0$$

Theorem 7.0.1. $\mathbf{0}$ is orthogonal to every vector in V .

Theorem 7.0.2. $\mathbf{0}$ is the only vector in V , that is orthogonal to itself.

Theorem 7.0.3.

$$\|\mathbf{v}\|^2 = \mathbf{v}^\top \mathbf{v}$$

Theorem 7.0.4.

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}^\top \mathbf{v}}$$

7.1 Orthogonal Subspaces

Definition 7.1.1. Two subspaces U and W of a vector space V , are **orthogonal subspaces** iff every vector in U is orthogonal to every vector in W .

$$\forall \mathbf{u} \in U : \forall \mathbf{w} \in W : \mathbf{u}^\top \mathbf{w} = 0$$

7.2 Orthogonal Complements

Definition 7.2.1. If U is a subspace of V , then the **orthogonal complement** of U , denoted U^\perp , is the set of all vectors in V that are orthogonal to every vector in U .

$$U^\perp = \{\forall \mathbf{u} \in U : \mathbf{v} \in V : \mathbf{v}^\top \mathbf{u} = 0\}$$

Theorem 7.2.1.

$$(U^\perp)^\perp = U$$

Theorem 7.2.2.

$$\dim U + \dim U^\perp = \dim V$$

7.3 Vector Projections

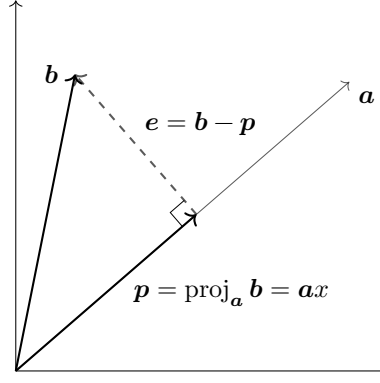


Figure 3: Vector Projection of \mathbf{b} onto \mathbf{a} .

Definition 7.3.1. Let the **vector projection** of \mathbf{b} onto \mathbf{a} , denoted as $\text{proj}_{\mathbf{a}} \mathbf{b}$, be the orthogonal projection of \mathbf{b} in the direction of \mathbf{a} , that minimises the error vector: $\mathbf{e} = \mathbf{b} - \mathbf{p}$.

Theorem 7.3.1. The projection of \mathbf{b} onto \mathbf{a} is given by

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \mathbf{a}x = \mathbf{a} (\mathbf{a}^\top \mathbf{a})^{-1} \mathbf{a}^\top \mathbf{b}$$

alternatively

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \mathbf{a}x = \mathbf{a} \frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}} = \mathbf{a} \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$$

Proof.

As \mathbf{p} lies on line through \mathbf{a} , $\mathbf{p} = \mathbf{a}x$, so that $\mathbf{e} = \mathbf{b} - \mathbf{a}x$. As \mathbf{e} is orthogonal to \mathbf{a} , we can construct the following relationship.

$$\begin{aligned} \mathbf{a}^\top \mathbf{e} &= 0 \\ \mathbf{a}^\top (\mathbf{b} - \mathbf{a}x) &= 0 \\ \mathbf{a}^\top \mathbf{b} - \mathbf{a}^\top \mathbf{a}x &= 0 \\ \mathbf{a}^\top \mathbf{a}x &= \mathbf{a}^\top \mathbf{b} \\ x &= (\mathbf{a}^\top \mathbf{a})^{-1} \mathbf{a}^\top \mathbf{b} = \frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}} \end{aligned}$$

□

7.4 Projection onto a Subspace

Theorem 7.4.1. Let W be a subspace of the vector space V such that if $\mathbf{b} \in V$, then $\mathbf{p} = \text{proj}_W \mathbf{b}$ is the **best approximation** of \mathbf{b} on W , so that

$$\|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{w}\|$$

for all $\mathbf{w} \in W$, where $\mathbf{w} \neq \mathbf{p}$.

Theorem 7.4.2. *The projection of \mathbf{b} onto the vector space W is given by*

$$\text{proj}_W \mathbf{b} = \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$$

Proof.

As $\mathbf{p} \in W$, \mathbf{p} can be represented as the linear combination of the basis vectors \mathbf{a}_i that span W .

$$\begin{aligned} \mathbf{p} &= \hat{x}_1 \mathbf{a}_1 + \hat{x}_2 \mathbf{a}_2 + \cdots + \hat{x}_n \mathbf{a}_n \\ &= \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix} \\ &= \mathbf{A}\hat{\mathbf{x}} \end{aligned}$$

Consider the error vector $\mathbf{e} = \mathbf{b} - \mathbf{p}$. As \mathbf{e} is orthogonal to W , it will also be orthogonal to the vectors that span W . Therefore

$$\begin{cases} \mathbf{a}_1^\top (\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}) = 0 \\ \mathbf{a}_2^\top (\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}) = 0 \\ \vdots \\ \mathbf{a}_n^\top (\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}) = 0 \end{cases}$$

which gives the following equation

$$\mathbf{A}^\top (\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}) = \mathbf{0}$$

where we solve for $\hat{\mathbf{x}}$

$$\begin{aligned} \mathbf{A}^\top \mathbf{b} - \mathbf{A}^\top \mathbf{A}\hat{\mathbf{x}} &= \mathbf{0} \\ \mathbf{A}^\top \mathbf{A}\hat{\mathbf{x}} &= \mathbf{A}^\top \mathbf{b} \\ \hat{\mathbf{x}} &= (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b} \end{aligned}$$

□

7.5 Least Squares

Theorem 7.5.1. *Suppose $\mathbf{A}\mathbf{x} = \mathbf{b}$ is an *inconsistent* linear system. The **least squares** solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by the orthogonal projection $\text{proj}_{\mathcal{C}(\mathbf{A})} \mathbf{b}$.*

8 Linear Maps

8.1 Matrix Transformations

Definition 8.1.1. A **matrix transformation** $T_{\mathbf{A}} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a mapping of the form

$$T_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$. As this transformation is linear, the following linearity properties hold.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(k\mathbf{u}) = kT(\mathbf{u})$

8.2 General Linear Transformations

Theorem 8.2.1. *If $T : V \rightarrow W$ is a mapping between two vector spaces V and W , then T is the **linear transformation** from V to W , and the following properties hold.*

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(k\mathbf{u}) = kT(\mathbf{u})$

Theorem 8.2.2. *When $V = W$, the linear map is called a **linear operator**.*

8.3 Subspaces of Linear Transformations

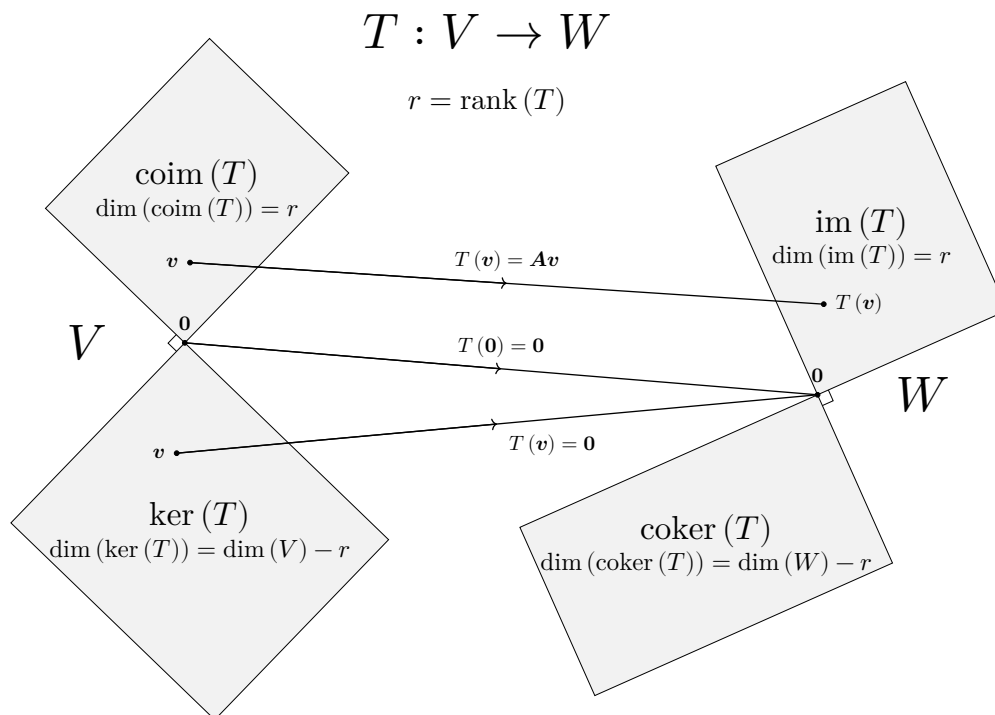


Figure 4: Subspaces of a Linear Transformation.

Definition 8.3.1. If $T : V \rightarrow W$ is a linear transformation between two vector spaces V and W , then:

1. The vector space V is the **domain** of T .
2. The vector space W is the **codomain** of T .
3. The **image** (or **range**) of T is the set of vectors the linear transformation maps to.

$$\text{im}(T) = T(V) = \{T(v) : v \in V\} \subset W$$

4. The **kernel** of T is the set of vectors that map to the zero vector.

$$\text{ker}(T) = \{v \in V : T(v) = 0\}$$

8.4 Constructing a Transformation Matrix

Theorem 8.4.1. *The standard matrix for a linear transformation is given by the formula:*

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are the *standard basis vectors* for \mathbb{R}^n .

9 Determinants

9.1 Properties of Determinants

1. $\det(\mathbb{1}) = 1$.
2. Exchanging two rows of a matrix reverses the sign of its determinant.
3. Determinants are multilinear, so that

$$(a) \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$(b) \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

4. If \mathbf{A} has two equal rows, then $\det(\mathbf{A}) = 0$.
5. Adding a scalar multiple of one row to another does not change the determinant of a matrix.
6. If \mathbf{A} has a row of zeros, then $\det(\mathbf{A}) = 0$.
7. If \mathbf{A} is triangular, then $\det(\mathbf{A}) = \prod_{i=1}^n a_{ii}$.
8. If \mathbf{A} is singular, then $\det(\mathbf{A}) = 0$.
9. $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$.
10. $\det(\mathbf{A}^\top) = \det(\mathbf{A})$.

9.2 Matrix Minors

Definition 9.2.1. The **minor** of a_{ij} in \mathbf{A} , denoted M_{ij} , is the determinant of the submatrix formed by deleting the i th row and j th column of \mathbf{A} .

9.3 Matrix Cofactors

Definition 9.3.1. The **cofactor** of a_{ij} in \mathbf{A} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

9.4 The Determinant of a Matrix

Theorem 9.4.1. The determinant of an $n \times n$ matrix \mathbf{A} is given by

$$\det(\mathbf{A}) = \sum_{j=1}^n a_{ij} C_{ij} = \sum_{i=1}^n a_{ij} C_{ij}$$

where a_{ij} is the entry in the i th row and j th column of \mathbf{A} .

9.5 The Cofactor Matrix

Definition 9.5.1. The **cofactor matrix** of an $n \times n$ matrix \mathbf{A} , denoted \mathbf{C} , is defined as the matrix of the cofactors of \mathbf{A} .

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

9.6 The Adjugate of a Matrix

Definition 9.6.1. The **adjugate** (or *classical adjoint*) of a square matrix \mathbf{A} , denoted $\text{adj}(\mathbf{A})$, is the transpose of its cofactor matrix.

$$\text{adj}(\mathbf{A}) = \mathbf{C}^\top$$

9.7 The Inverse of a Matrix

Theorem 9.7.1. The *inverse* of a nonsingular matrix \mathbf{A} is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj}(\mathbf{A})$$

10 Invariant Subspaces

Definition 10.0.1. Consider the subspace \mathcal{V} of the linear mapping $T : V \rightarrow V$ from a vector space V to itself, then \mathcal{V} is an **invariant subspace** of T if

$$T(\mathcal{V}) \subseteq \mathcal{V}$$

Theorem 10.0.1. If \mathcal{V} is an invariant subspace of a linear mapping $T : V \rightarrow V$ from a vector space V to itself, then

$$\forall \mathbf{v} \in \mathcal{V} \implies T(\mathbf{v}) \in \mathcal{V}$$

10.1 Trivial Invariant Subspaces

1. V .
2. $\{\mathbf{0}\}$.
3. $\ker(T)$.
4. $\text{im}(T)$.
5. Any linear combination of invariant subspaces.

10.2 Eigenspaces

Definition 10.2.1. If an invariant subspace is one-dimensional, then the subspace is called an **eigenspace** of the linear transformation.

Theorem 10.2.1. If \mathcal{V} is an eigenspace of the linear mapping $T : V \rightarrow V$, then

$$\mathcal{V} = \{\forall \mathbf{q} \in \mathcal{V} : \exists \lambda \in \mathbb{C} : T(\mathbf{q}) = \lambda \mathbf{q}\}$$

where λ is the **eigenvalue** associated with the **eigenvector** \mathbf{q} .

10.3 The Eigenvalue Problem

Theorem 10.3.1. The eigenvalues λ of an invertible square matrix \mathbf{A} , are the solutions to

$$\det(\mathbf{A} - \lambda \mathbf{1}) = 0$$

Theorem 10.3.2. The eigenvectors associated with each eigenvalue, of an invertible square matrix \mathbf{A} , are the solutions to

$$(\mathbf{A} - \lambda \mathbf{1}) \mathbf{q} = \mathbf{0}$$

Proof.

The eigenvalues and associated eigenvectors of a square matrix \mathbf{A} , are the solutions to $\mathbf{A}\mathbf{q} = \lambda\mathbf{q}$.

$$\begin{aligned} \mathbf{A}\mathbf{q} &= \lambda\mathbf{q} \\ \mathbf{A}\mathbf{q} - \lambda\mathbf{q} &= \mathbf{0} \\ (\mathbf{A} - \lambda \mathbf{1}) \mathbf{q} &= \mathbf{0} \end{aligned}$$

The linear system $(\mathbf{A} - \lambda \mathbf{1}) \mathbf{q} = \mathbf{0}$ has a nontrivial solution iff $\mathbf{A} - \lambda \mathbf{1}$ is singular. □

10.4 Properties of Eigenvalues

Theorem 10.4.1.

$$\operatorname{Tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$$

Theorem 10.4.2.

$$\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$$

11 Eigen Decomposition

11.1 Similarity Transformations

Definition 11.1.1. A **similarity transformation** is a linear mapping of the form

$$A \rightarrow Q^{-1}AQ$$

in which the matrices A and Q are $n \times n$ invertible matrices. Here we say, “ A is similar to $Q^{-1}AQ$ ”.

11.2 Matrix Diagonalisation

Definition 11.2.1. The matrix A is a **diagonalisable** matrix if it is similar to a diagonal matrix. That is, there exists an invertible matrix Q , and diagonal matrix Λ , such that

$$\Lambda = Q^{-1}AQ$$

Theorem 11.2.1. Let A be an $n \times n$ matrix with n linearly independent eigenvectors, then A is diagonalisable if $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and Q is a matrix composed of the eigenvectors of A . Explicitly,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{bmatrix}$$

where $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ are the eigenvectors of A .

Proof.

Let $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$, be linearly independent eigenvectors of A , and $\lambda_1, \lambda_2, \dots, \lambda_n$, the associated eigenvalues. By definition of an eigenspace, we have

$$\begin{cases} A\mathbf{q}_1 = \lambda_1\mathbf{q}_1 \\ A\mathbf{q}_2 = \lambda_2\mathbf{q}_2 \\ \vdots \\ A\mathbf{q}_n = \lambda_n\mathbf{q}_n \end{cases}$$

which we can rewrite as

$$\begin{aligned} AQ &= Q\Lambda \\ \Lambda &= Q^{-1}AQ \end{aligned} \tag{1}$$

by rearranging Equation 1, we have A in terms of its eigenvalues and eigenvectors.

$$A = Q\Lambda Q^{-1}$$

□

11.3 Powers of a Matrix

Theorem 11.3.1. *Let \mathbf{A} be a diagonalisable matrix, then for all $k \in \mathbb{N}_0$*

$$\mathbf{A}^k = \mathbf{Q} \mathbf{\Lambda}^k \mathbf{Q}^{-1}$$

Proof.

$$\begin{aligned} \mathbf{A}^k &= (\mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1})^k \\ &= \underbrace{(\mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1}) (\mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1}) \dots (\mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1})}_{k \text{ times}} \\ &= \underbrace{\mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1} \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1} \dots \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1}}_{k \text{ times}} \\ &= \underbrace{\mathbf{Q} \cancel{\mathbf{\Lambda} \mathbf{Q}^{-1} \mathbf{Q}} \mathbf{\Lambda} \cancel{\mathbf{Q}^{-1} \mathbf{Q}} \dots \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1}}_{k \text{ times}} \\ &= \mathbf{Q} \underbrace{\mathbf{\Lambda} \mathbf{\Lambda} \dots \mathbf{\Lambda}}_{k \text{ times}} \mathbf{Q}^{-1} \\ &= \mathbf{Q} \mathbf{\Lambda}^k \mathbf{Q}^{-1} \end{aligned}$$

□

Theorem 11.3.2. *The eigenvalues of \mathbf{A}^k , $\forall k \in \mathbb{N}$ are $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$.*

Theorem 11.3.3. *The eigenvectors of \mathbf{A} are equal to the eigenvectors of \mathbf{A}^k .*

12 System of Differential Equations

12.1 First-Order Differential Equations

Definition 12.1.1. A **first-order differential equation** is a differential equation where the highest derivative is of order one.

$$x' = ax$$

Theorem 12.1.1. *The general solution to a first-order linear differential equation is of the form*

$$x(t) = c_1 e^{at}$$

where c_1 is an arbitrary constant.

12.2 First-Order System of Differential Equations

Definition 12.2.1. A **first-order system of differential equations** is of the form

$$\begin{cases} x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases}$$

where $x_1 = x_1(t), x_2 = x_2(t), \dots, x_n = x_n(t)$ are the functions to be determined. In matrix form, the system can be written as

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

12.3 Solution using Diagonalisation

Theorem 12.3.1. *The first-order system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be solved using the following substitution*

$$\mathbf{x} = \mathbf{Q}\mathbf{u}$$

where \mathbf{u} is a vector to be determined, and \mathbf{Q} is the matrix that diagonalises \mathbf{A} . \mathbf{u} is determined by solving

$$\mathbf{u}' = \mathbf{\Lambda}\mathbf{u}$$

where $\mathbf{\Lambda}$ is the diagonal similarity transformation of \mathbf{A} . This substitution uncouples the system of differential equations so that each equation can be solved as a first-order differential equation.

Proof.

$$\begin{aligned} \mathbf{x}' &= \mathbf{A}\mathbf{x} \\ (\mathbf{Q}\mathbf{u})' &= (\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1})(\mathbf{Q}\mathbf{u}) \end{aligned}$$

$$\begin{aligned}
Qu' &= Q\Lambda Q^{-1}Qu \\
\cancel{Q}u' &= \cancel{Q}\Lambda\cancel{Q}^{-1}\cancel{Q}u \\
u' &= \Lambda u
\end{aligned}$$

□

Theorem 12.3.2. *If \mathbf{A} is a diagonalisable matrix, then the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be expressed as*

$$\mathbf{x}(t) = \mathbf{c}_1 \mathbf{e}^{\lambda_1 t} \mathbf{q}_1 + \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \mathbf{q}_2 + \cdots + \mathbf{c}_n \mathbf{e}^{\lambda_n t} \mathbf{q}_n$$

12.4 Principle of Superposition

Theorem 12.4.1. *If x_1 and x_2 are two solutions to a linear differential equation, then*

$$x = \mathbf{c}_1 x_1 + \mathbf{c}_2 x_2$$

is also a solution to the differential equation.

12.5 Higher-Order Differential Equations

Theorem 12.5.1. *A **higher-order linear differential equation** can be solved by first converting it to a first-order linear system. Consider the n th-order differential equation*

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1} x' + a_n x = 0$$

We then define

$$\begin{aligned}
x_1 &= x \\
x_2 &= x' \\
&\vdots \\
x_n &= x^{(n-1)}
\end{aligned}$$

Let $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]^\top$. Then the first-order linear system of differential equations can be written as

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

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