A **vector space** V is closed under vector addition and scalar multiplication: $\boldsymbol{u}+\boldsymbol{v}\in V$ and $k\boldsymbol{u}\in V$.
A *subset* W of a vector space V is called a **subspace** of V if W is itself a vector space. The intersection of subspaces is also a subspace of V.
S is **linearly independent (LI)** if $k_1\boldsymbol{v}_1+k_2\boldsymbol{v}_2+\cdots+k_n\boldsymbol{v}_n=\boldsymbol{0}$ has $k_i=0$. S forms a **basis** for V if S spans V and S is LI.
For $\boldsymbol{A}\in\mathbb{R}^{m\times n}$: $r=\operatorname{rank}\boldsymbol{A}=\operatorname{rank}\boldsymbol{A}^\top=\dim\left(\mathcal{C}\left(\boldsymbol{A}\right)\right)=\dim\left(\mathcal{C}\left(\boldsymbol{A}^\top\right)\right)$. $\operatorname{null}\left(\boldsymbol{A}\right)=\dim\left(\mathcal{N}\left(\boldsymbol{A}\right)\right)=n-r$ and $\operatorname{null}\left(\boldsymbol{A}^\top\right)=\dim\left(\mathcal{N}\left(\boldsymbol{A}^\top\right)\right)=m-r$.
The **four fundamental subspaces**: $\left(\mathcal{C}\left(\boldsymbol{A}\right)\right)^\perp=\mathcal{N}\left(\boldsymbol{A}^\top\right)$ and $\left(\mathcal{C}\left(\boldsymbol{A}^\top\right)\right)^\perp=\mathcal{N}\left(\boldsymbol{A}\right)$.
Row equivalent matrices have the same row space and null space.
The subspaces U and W of a vector space V are **orthogonal subspaces** iff $\forall\boldsymbol{u}\in U:\forall\boldsymbol{w}\in W:\boldsymbol{u}^\top\boldsymbol{w}=0$. $\boldsymbol{v}^\top\boldsymbol{v}=\|\boldsymbol{v}\|^2$.
The **orthogonal complement** of U: $U^\perp=\left\{\forall\boldsymbol{u}\in U:\boldsymbol{v}\in V:\boldsymbol{v}^\top\boldsymbol{u}=0\right\}$ and $\left(U^\perp\right)^\perp=U$. $\dim U+\dim U^\perp=\dim V$.
Projections: $\operatorname{proj}_{\boldsymbol{a}}\boldsymbol{b}=\boldsymbol{a}x=\boldsymbol{a}\frac{\boldsymbol{a}^\top\boldsymbol{b}}{\boldsymbol{a}^\top\boldsymbol{a}}$. $\forall\boldsymbol{w}\in W:\boldsymbol{w}\neq\boldsymbol{p}:\operatorname{proj}_W\boldsymbol{b}=\boldsymbol{A}\hat{\boldsymbol{x}}=\boldsymbol{A}\left(\boldsymbol{A}^\top\boldsymbol{A}\right)^{-1}\boldsymbol{A}^\top\boldsymbol{b}:\|\boldsymbol{b}-\boldsymbol{p}\|<\|\boldsymbol{b}-\boldsymbol{w}\|$.
$\det\left(\boldsymbol{A}\right)=\sum_{j=1}^n a_{ij}C_{ij}=\sum_{i=1}^n a_{ij}C_{ij}$, where $C_{ij}=(-1)^{i+j}M_{ij}$. $\boldsymbol{A}^{-1}=\frac{1}{\det\boldsymbol{A}}\operatorname{adj}\left(\boldsymbol{A}\right)$, where $\operatorname{adj}\left(\boldsymbol{A}\right)=C^\top$.
Linear transformations: $T:V\to W\iff\forall\boldsymbol{u},\,\boldsymbol{v}\in V:\forall k\in\mathbb{R}:T\left(\boldsymbol{u}+\boldsymbol{v}\right)=T\left(\boldsymbol{u}\right)+T\left(\boldsymbol{v}\right)\wedge T\left(k\boldsymbol{u}\right)=kT\left(\boldsymbol{u}\right)$.
Rotations: $\boldsymbol{R}_x=\begin{bmatrix}1&0&0\\0&\cos(\theta)&-\sin(\theta)\\0&\sin(\theta)&\cos(\theta)\end{bmatrix}$. $\boldsymbol{R}_y=\begin{bmatrix}\cos(\theta)&0&\sin(\theta)\\0&1&0\\-\sin(\theta)&0&\cos(\theta)\end{bmatrix}$. $\boldsymbol{R}_z=\begin{bmatrix}\cos(\theta)&-\sin(\theta)&0\\\sin(\theta)&\cos(\theta)&0\\0&0&1\end{bmatrix}$. Anticlockwise rotations looking down from the positive direction of the axis of rotation. In 2-d: $\boldsymbol{R}=\begin{bmatrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{bmatrix}$.
Shears: $\boldsymbol{S}_x=\begin{bmatrix}1&a&b\\0&1&0\\0&0&1\end{bmatrix}$. $\boldsymbol{S}_y=\begin{bmatrix}1&0&0\\a&1&b\\0&0&1\end{bmatrix}$. $\boldsymbol{S}_z=\begin{bmatrix}1&0&0\\0&1&0\\a&b&1\end{bmatrix}$. Where the standard basis vector in the subscripted axis maps to itself. Think about where the standard basis vectors maps.
Reflections: $\boldsymbol{M}_{xy}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$. $\boldsymbol{M}_{xz}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$. $\boldsymbol{M}_{yz}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. Where vectors are reflected across the plane formed by the subscripts of $\boldsymbol{M}$.
2-d Reflections about $y=mx+c$, where $\theta=\arctan\left(m\right)$:

$$\begin{aligned}T\left(\boldsymbol{v}\right)&=\boldsymbol{R}\boldsymbol{M}_{xz}\boldsymbol{R}^{-1}\left(\boldsymbol{v}-\begin{bmatrix}0\\c\end{bmatrix}\right)+\begin{bmatrix}0\\c\end{bmatrix}\\&=\begin{bmatrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{bmatrix}^{-1}\left(\boldsymbol{v}-\begin{bmatrix}0\\c\end{bmatrix}\right)+\begin{bmatrix}0\\c\end{bmatrix}\\&=\frac{1}{1+m^2}\begin{bmatrix}1-m^2&2m\\2m&m^2-1\end{bmatrix}\left(\boldsymbol{v}-\begin{bmatrix}0\\c\end{bmatrix}\right)+\begin{bmatrix}0\\c\end{bmatrix}\end{aligned}$$

Invariant (IV) subspaces: For $T:V\to V$. $\mathcal{V}$ is IV if $T\left(\mathcal{V}\right)\subseteq\mathcal{V}\iff\forall\boldsymbol{v}\in\mathcal{V}\implies T\left(\boldsymbol{v}\right)\in\mathcal{V}$. **Trivial IV subspaces**: V, $\operatorname{im}\left(T\right)=T(V)=\left\{T\left(\boldsymbol{v}\right):\boldsymbol{v}\in V\right\}\subset W$, $\ker\left(T\right)=\left\{\boldsymbol{v}\in V:T\left(\boldsymbol{v}\right)=\boldsymbol{0}\right\}$, $\{\boldsymbol{0}\}$, and any linear combination of IVs.
Eigenspace (1-d IV subspace): $\mathcal{V}=\left\{\forall\boldsymbol{q}\in\mathcal{V}:\exists\lambda\in\mathbb{C}:T\left(\boldsymbol{q}\right)=\lambda\boldsymbol{q}\right\}$ where λ_i are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{q}_i$ are the eigenvectors of $\boldsymbol{A}$. $\left(\boldsymbol{A}-\lambda\mathbb{1}\right)\boldsymbol{q}=\boldsymbol{0}$. If $\boldsymbol{A}$ is invertible: $\det\left(\boldsymbol{A}-\lambda\mathbb{1}\right)=\boldsymbol{0}$. **Characteristic polynomial**: $p_n\left(\lambda\right)=\det\left(\boldsymbol{A}_n-\lambda\mathbb{1}_n\right)$; in 2-d: $p_2\left(\lambda\right)=\lambda^2-\operatorname{Tr}\left(\boldsymbol{A}\right)\lambda+\det\left(\boldsymbol{A}\right)$. $\operatorname{Tr}\left(\boldsymbol{A}\right)=\sum_{i=1}^n\lambda_i$ and $\det\left(\boldsymbol{A}\right)=\prod_{i=1}^n\lambda_i$.
Similarity transformation: $\boldsymbol{A}\to\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$. If $\boldsymbol{q}_i$ is LI, then $\boldsymbol{A}$ is **diagonalisable**: $\boldsymbol{\Lambda}=\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$.
Where $\boldsymbol{\Lambda}=\begin{bmatrix}\lambda_1&&&\\&\lambda_2&&\\&&\ddots&\\&&&\lambda_n\end{bmatrix}$ and $\boldsymbol{Q}=\begin{bmatrix}|&|&&|\\\boldsymbol{q}_1&\boldsymbol{q}_2&\cdots&\boldsymbol{q}_n\\|&|&&|\end{bmatrix}$.
If $\boldsymbol{A}$ is diagonalisable, $\forall k\in\mathbb{N}_0:\boldsymbol{A}^k=\boldsymbol{Q}\boldsymbol{\Lambda}^k\boldsymbol{Q}^{-1}$. The eigenvalues of $\boldsymbol{A}^k$ are the eigenvalues of $\boldsymbol{A}$ to the k-th power: $\lambda_1^k,\,\lambda_2^k,\,\dots,\,\lambda_n^k$. The eigenvectors of $\boldsymbol{A}^k$ equal the eigenvectors of $\boldsymbol{A}$.
The **ordinary differential equation (ODE)**: $x'=ax$, has the solution: $x(t)=c_1e^{at}$. c_1 is determined through initial conditions.
The **system of differential equations**: $\begin{cases}x_1'=a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\\x_2'=a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\\\vdots\quad\vdots\qquad\vdots\qquad\qquad\vdots\\x_n'=a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n\end{cases}\iff\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$
$\iff\boldsymbol{x}'=\boldsymbol{A}\boldsymbol{x}$ can be solved using $\boldsymbol{x}=\boldsymbol{Q}\boldsymbol{u}$, where $\boldsymbol{Q}$ is the matrix that diagonalises $\boldsymbol{A}$, and $\boldsymbol{u}$ is the solution to $\boldsymbol{u}'=\boldsymbol{\Lambda}\boldsymbol{u}$, where $\boldsymbol{\Lambda}$ is the diagonal similarity transformation of $\boldsymbol{A}$.
If $\boldsymbol{A}$ is diagonalisable, then for $\boldsymbol{x}'=\boldsymbol{A}\boldsymbol{x}$: $\boldsymbol{x}(t)=c_1e^{\lambda_1t}\boldsymbol{q}_1+c_2e^{\lambda_2t}\boldsymbol{q}_2+\cdots+c_ne^{\lambda_nt}\boldsymbol{q}_n$.
For the **higher-order linear differential equation** $x^{(n)}+a_1x^{(n-1)}+\cdots+a_{n-1}x'+a_nx=0$, define: $x_1=x$, $x_2=x'$, $\dots$, $x_n=x^{(n-1)}$, and let: $\boldsymbol{x}=\begin{bmatrix}x_1\\x_2\\\cdots\\x_n\end{bmatrix}$. Then solve the ODE: $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\-a_n&-a_{n-1}&-a_{n-2}&\cdots&-a_1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$ using diagonalisation.

Norm of a vector: $\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.
Unit vector: $\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$.
Dot product: $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos(\theta)$.
Cross product: $\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin(\theta) \hat{\boldsymbol{n}}$.
2-d Inverse: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
Vector space axioms:
- **Closure under addition**: $\boldsymbol{u} + \boldsymbol{v} \in V$
- **Commutativity of vector addition**: $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
- **Associativity of vector addition**: $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
- **Additive identity**: $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$
- **Additive inverse**: $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$
- **Closure under scalar multiplication**: $k\boldsymbol{u} \in V$
- **Distributivity of vector addition**: $k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$
- **Distributivity of scalar addition**: $(k + m)\boldsymbol{u} = k\boldsymbol{u} + m\boldsymbol{u}$
- **Associativity of scalar multiplication**: $k(m\boldsymbol{u}) = (km)\boldsymbol{u}$
- **Scalar multiplication identity**: $1\boldsymbol{u} = \boldsymbol{u}$

Subspaces of $\mathbb{R}^2$: $\{\boldsymbol{0}\}$, lines through the origin, and $\mathbb{R}^2$.
Subspaces of $\mathbb{R}^3$: $\{\boldsymbol{0}\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$.
Subspaces of $\boldsymbol{M}_{nn}$: Upper triangular matrices, lower triangular matrices, diagonal matrices, and $\boldsymbol{M}_{nn}$.
Determinant properties:
- $\det(\mathbb{1}) = 1$.
- Exchanging two rows of a matrix reverses the sign of its determinant.
- Determinants are multilinear, so that $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ and $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.
- If $\boldsymbol{A}$ has two equal rows, then $\det(\boldsymbol{A}) = 0$.
- Adding a scalar multiple of one row to another does not change the determinant of a matrix.
- If $\boldsymbol{A}$ has a row of zeros, then $\det(\boldsymbol{A}) = 0$.
- If $\boldsymbol{A}$ is triangular, then $\det(\boldsymbol{A}) = \prod_{i=1}^{n} a_{ii}$.
- If $\boldsymbol{A}$ is singular, then $\det(\boldsymbol{A}) = 0$.
- $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \det(\boldsymbol{B})$.
- $\det(\boldsymbol{A}^\top) = \det(\boldsymbol{A})$

Matrix Identities:
- $\boldsymbol{A}(\boldsymbol{BC}) = \boldsymbol{AB} + \boldsymbol{AC}$
- $(\boldsymbol{A} + \boldsymbol{B})^\top = \boldsymbol{A}^\top + \boldsymbol{B}^\top$
- $(\boldsymbol{AB})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top$
- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both invertible:
- $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1} \boldsymbol{A}^{-1}$
- $(\boldsymbol{A}^{-1})^\top = (\boldsymbol{A}^\top)^{-1}$