

A **vector space**  $V$  is closed under vector addition and scalar multiplication:  $\mathbf{u} + \mathbf{v} \in V$  and  $k\mathbf{u} \in V$ .

A **subset**  $W$  of a vector space  $V$  is called a **subspace** of  $V$  if  $W$  is itself a vector space. The intersection of subspaces is also a subspace of  $V$ .

$S$  is **linearly independent (LI)** if  $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n = \mathbf{0}$  has  $k_i = 0$ .  $S$  forms a **basis** for  $V$  if  $S$  spans  $V$  and  $S$  is LI. For  $\mathbf{A} \in \mathbb{R}^{m \times n}$ :  $r = \text{rank } \mathbf{A} = \text{rank } \mathbf{A}^\top = \dim(\mathcal{C}(\mathbf{A})) = \dim(\mathcal{C}(\mathbf{A}^\top))$ .  $\text{null}(\mathbf{A}) = \dim(\mathcal{N}(\mathbf{A})) = n - r$  and  $\text{null}(\mathbf{A}^\top) = \dim(\mathcal{N}(\mathbf{A}^\top)) = m - r$ .

The **four fundamental subspaces**:  $(\mathcal{C}(\mathbf{A}))^\perp = \mathcal{N}(\mathbf{A}^\top)$  and  $(\mathcal{C}(\mathbf{A}^\top))^\perp = \mathcal{N}(\mathbf{A})$ .

**Row equivalent** matrices have the same row space and null space.

The subspaces  $U$  and  $W$  of a vector space  $V$  are **orthogonal subspaces** iff  $\forall \mathbf{u} \in U : \forall \mathbf{w} \in W : \mathbf{u}^\top \mathbf{w} = 0$ .  $\mathbf{v}^\top \mathbf{v} = \|\mathbf{v}\|^2$ .

The **orthogonal complement** of  $U$ :  $U^\perp = \{\forall \mathbf{u} \in U : \mathbf{v} \in V : \mathbf{v}^\top \mathbf{u} = 0\}$  and  $(U^\perp)^\perp = U$ .  $\dim U + \dim U^\perp = \dim V$ .

**Projections**:  $\text{proj}_{\mathbf{a}} \mathbf{b} = \mathbf{a}x = \mathbf{a} \frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}}$ .  $\forall \mathbf{w} \in W : \mathbf{w} \neq \mathbf{p} : \text{proj}_W \mathbf{b} = \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b} : \|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{w}\|$ .

$\det(\mathbf{A}) = \sum_{j=1}^n a_{ij} C_{ij} = \sum_{i=1}^n a_{ij} C_{ij}$ , where  $C_{ij} = (-1)^{i+j} M_{ij}$ .  $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj}(\mathbf{A})$ , where  $\text{adj}(\mathbf{A}) = \mathbf{C}^\top$ .

**Linear transformations**:  $T : V \rightarrow W \iff \forall \mathbf{u}, \mathbf{v} \in V : \forall k \in \mathbb{R} : T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \wedge T(k\mathbf{u}) = kT(\mathbf{u})$ .

**Rotations**:  $\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$ .  $\mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$ .  $\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . Anticlockwise

rotations looking down from the positive direction of the axis of rotation. In 2-d:  $\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ .

**Shears**:  $\mathbf{S}_x = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .  $\mathbf{S}_y = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ .  $\mathbf{S}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$ . Where the standard basis vector in the subscripted axis

maps to itself. Think about where the standard basis vectors maps.

**Reflections**:  $\mathbf{M}_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ .  $\mathbf{M}_{xz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .  $\mathbf{M}_{yz} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . Where vectors are reflected across the

plane formed by the subscripts of  $\mathbf{M}$ .

**2-d Reflections** about  $y = mx + c$ , where  $\theta = \arctan(m)$ :

$$\begin{aligned} T(\mathbf{v}) &= \mathbf{R}\mathbf{M}_{xz}\mathbf{R}^{-1} \left( \mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^{-1} \left( \mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \\ &= \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \left( \mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \end{aligned}$$

**Invariant (IV) subspaces**: For  $T : V \rightarrow V$ .  $\mathcal{V}$  is IV if  $T(\mathcal{V}) \subseteq \mathcal{V} \iff \forall \mathbf{v} \in \mathcal{V} \implies T(\mathbf{v}) \in \mathcal{V}$ . **Trivial IV subspaces**:  $V$ ,  $\text{im}(T) = T(V) = \{T(\mathbf{v}) : \mathbf{v} \in V\} \subset W$ ,  $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$ ,  $\{\mathbf{0}\}$ , and any linear combination of IVs.

**Eigenspace** (1-d IV subspace):  $\mathcal{V} = \{\mathbf{q} \in \mathcal{V} : \exists \lambda \in \mathbb{C} : T(\mathbf{q}) = \lambda \mathbf{q}\}$  where  $\lambda_i$  are the eigenvalues of  $\mathbf{A}$  and  $\mathbf{q}_i$  are the eigenvectors of  $\mathbf{A}$ .  $(\mathbf{A} - \lambda \mathbf{1}) \mathbf{q} = \mathbf{0}$ . If  $\mathbf{A}$  is invertible:  $\det(\mathbf{A} - \lambda \mathbf{1}) = 0$ . **Characteristic polynomial**:  $p_n(\lambda) = \det(\mathbf{A}_n - \lambda \mathbf{1}_n)$ ; in 2-d:  $p_2(\lambda) = \lambda^2 - \text{Tr}(\mathbf{A})\lambda + \det(\mathbf{A})$ .  $\text{Tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$  and  $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$ .

**Similarity transformation**:  $\mathbf{A} \rightarrow \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q}$ . If  $\mathbf{q}_i$  is LI, then  $\mathbf{A}$  is **diagonalisable**:  $\mathbf{A} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q}$ .

Where  $\mathbf{A} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$  and  $\mathbf{Q} = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n \\ | & | & & | \end{bmatrix}$ .

If  $\mathbf{A}$  is diagonalisable,  $\forall k \in \mathbb{N}_0 : \mathbf{A}^k = \mathbf{Q} \mathbf{A}^k \mathbf{Q}^{-1}$ . The eigenvalues of  $\mathbf{A}^k$  are the eigenvalues of  $\mathbf{A}$  to the  $k$ -th power:  $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$ . The eigenvectors of  $\mathbf{A}^k$  equal the eigenvectors of  $\mathbf{A}$ .

The **ordinary differential equation (ODE)**:  $x' = ax$ , has the solution:  $x(t) = c_1 e^{at}$ .  $c_1$  is determined through initial conditions.

The **system of differential equations**:  $\begin{cases} x'_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x'_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ x'_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{cases} \iff \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\iff \mathbf{x}' = \mathbf{A}\mathbf{x}$  can be solved using  $\mathbf{x} = \mathbf{Q}\mathbf{u}$ , where  $\mathbf{Q}$  is the matrix that diagonalises  $\mathbf{A}$ , and  $\mathbf{u}$  is the solution to  $\mathbf{u}' = \mathbf{A}\mathbf{u}$ , where  $\mathbf{A}$  is the diagonal similarity transformation of  $\mathbf{A}$ .

If  $\mathbf{A}$  is diagonalisable, then for  $\mathbf{x}' = \mathbf{A}\mathbf{x}$ :  $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{q}_1 + c_2 e^{\lambda_2 t} \mathbf{q}_2 + \dots + c_n e^{\lambda_n t} \mathbf{q}_n$ .

For the **higher-order linear differential equation**  $x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n x = 0$ , define:  $x_1 = x, x_2 = x', \dots, x_n =$

$x^{(n-1)}$ , and let:  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ . Then solve the ODE:  $\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  using diagonalisation.

**Norm of a vector:**  $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$ .

**Unit vector:**  $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ .

**Dot product:**  $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n = \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta)$ .

**Cross product:**  $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \|\mathbf{v}\| \|\mathbf{w}\| \sin(\theta) \hat{\mathbf{n}}$ .

**2-d Inverse:**  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

**Vector space axioms:**

**Closure under addition:**  $\mathbf{u} + \mathbf{v} \in V$

**Commutativity of vector addition:**  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

**Associativity of vector addition:**  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

**Additive identity:**  $\mathbf{u} + \mathbf{0} = \mathbf{u}$

**Additive inverse:**  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

**Closure under scalar multiplication:**  $k\mathbf{u} \in V$

**Distributivity of vector addition:**  $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

**Distributivity of scalar addition:**  $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

**Associativity of scalar multiplication:**  $k(m\mathbf{u}) = (km)\mathbf{u}$

**Scalar multiplication identity:**  $1\mathbf{u} = \mathbf{u}$

Subspaces of  $\mathbb{R}^2$ :  $\{\mathbf{0}\}$ , lines through the origin, and  $\mathbb{R}^2$ .

Subspaces of  $\mathbb{R}^3$ :  $\{\mathbf{0}\}$ , lines through the origin, planes through the origin, and  $\mathbb{R}^3$ .

Subspaces of  $M_{nn}$ : Upper triangular matrices, lower triangular matrices, diagonal matrices, and  $M_{nn}$ .

**Determinant properties:**

$\det(\mathbf{1}) = 1$ .

Exchanging two rows of a matrix reverses the sign of its determinant.

Determinants are multilinear, so that  $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$  and  $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ .

If  $\mathbf{A}$  has two equal rows, then  $\det(\mathbf{A}) = 0$ .

Adding a scalar multiple of one row to another does not change the determinant of a matrix.

If  $\mathbf{A}$  has a row of zeros, then  $\det(\mathbf{A}) = 0$ .

If  $\mathbf{A}$  is triangular, then  $\det(\mathbf{A}) = \prod_{i=1}^n a_{ii}$ .

If  $\mathbf{A}$  is singular, then  $\det(\mathbf{A}) = 0$ .

$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$ .

$\det(\mathbf{A}^\top) = \det(\mathbf{A})$

**Matrix Identities:**

$\mathbf{A}(\mathbf{BC}) = \mathbf{AB} + \mathbf{AC}$

$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$

$(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$

If  $\mathbf{A}$  and  $\mathbf{B}$  are both invertible:

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$

$(\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$