

A **vector space** V is closed under vector addition and scalar multiplication: $\mathbf{u} + \mathbf{v} \in V$ and $k\mathbf{u} \in V$.

A **subset** W of a vector space V is called a **subspace** of V if W is itself a vector space. The intersection of subspaces is also a subspace of V .

S is **linearly independent (LI)** if $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n = \mathbf{0}$ has $k_i = 0$. S forms a **basis** for V if S spans V and S is LI. For $\mathbf{A} \in \mathbb{R}^{m \times n}$: $r = \text{rank } \mathbf{A} = \text{rank } \mathbf{A}^\top = \dim(\mathcal{C}(\mathbf{A})) = \dim(\mathcal{C}(\mathbf{A}^\top))$. $\text{null}(\mathbf{A}) = \dim(\mathcal{N}(\mathbf{A})) = n - r$ and $\text{null}(\mathbf{A}^\top) = \dim(\mathcal{N}(\mathbf{A}^\top)) = m - r$.

The **four fundamental subspaces**: $(\mathcal{C}(\mathbf{A}))^\perp = \mathcal{N}(\mathbf{A}^\top)$ and $(\mathcal{C}(\mathbf{A}^\top))^\perp = \mathcal{N}(\mathbf{A})$.

Row equivalent matrices have the same row space and null space.

The subspaces U and W of a vector space V are **orthogonal subspaces** iff $\forall \mathbf{u} \in U : \forall \mathbf{w} \in W : \mathbf{u}^\top \mathbf{w} = 0$. $\mathbf{v}^\top \mathbf{v} = \|\mathbf{v}\|^2$.

The **orthogonal complement** of U : $U^\perp = \{\forall \mathbf{u} \in U : \mathbf{v} \in V : \mathbf{v}^\top \mathbf{u} = 0\}$ and $(U^\perp)^\perp = U$. $\dim U + \dim U^\perp = \dim V$.

Projections: $\text{proj}_{\mathbf{a}} \mathbf{b} = \mathbf{a}x = \mathbf{a} \frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}}$. $\forall \mathbf{w} \in W : \mathbf{w} \neq \mathbf{p} : \text{proj}_W \mathbf{b} = \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b} : \|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{w}\|$.

$\det(\mathbf{A}) = \sum_{j=1}^n a_{ij} C_{ij} = \sum_{i=1}^n a_{ij} C_{ij}$, where $C_{ij} = (-1)^{i+j} M_{ij}$. $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj}(\mathbf{A})$, where $\text{adj}(\mathbf{A}) = \mathbf{C}^\top$.

Linear transformations: $T : V \rightarrow W \iff \forall \mathbf{u}, \mathbf{v} \in V : \forall k \in \mathbb{R} : T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \wedge T(k\mathbf{u}) = kT(\mathbf{u})$.

Rotations: $\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$. $\mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$. $\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Anticlockwise

rotations looking down from the positive direction of the axis of rotation. In 2-d: $\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

Shears: $\mathbf{S}_x = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $\mathbf{S}_y = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$. $\mathbf{S}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$. Where the standard basis vector in the subscripted axis

maps to itself. Think about where the standard basis vectors maps.

Reflections: $\mathbf{M}_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. $\mathbf{M}_{xz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $\mathbf{M}_{yz} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Where vectors are reflected across the

plane formed by the subscripts of \mathbf{M} .

2-d Reflections about $y = mx + c$, where $\theta = \arctan(m)$:

$$\begin{aligned} T(\mathbf{v}) &= \mathbf{R} \mathbf{M}_{xz} \mathbf{R}^{-1} \left(\mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^{-1} \left(\mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \\ &= \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \left(\mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \end{aligned}$$

Invariant (IV) subspaces: For $T : V \rightarrow V$. \mathcal{V} is IV if $T(\mathcal{V}) \subseteq \mathcal{V} \iff \forall \mathbf{v} \in \mathcal{V} \implies T(\mathbf{v}) \in \mathcal{V}$. **Trivial IV subspaces**: V , $\text{im}(T) = T(V) = \{T(\mathbf{v}) : \mathbf{v} \in V\} \subset W$, $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$, $\{\mathbf{0}\}$, and any linear combination of IVs.

Eigenspace (1-d IV subspace): $\mathcal{V} = \{\mathbf{q} \in \mathcal{V} : \exists \lambda \in \mathbb{C} : T(\mathbf{q}) = \lambda \mathbf{q}\}$ where λ_i are the eigenvalues of \mathbf{A} and \mathbf{q}_i are the eigenvectors of \mathbf{A} . $(\mathbf{A} - \lambda \mathbf{1}) \mathbf{q} = \mathbf{0}$. If \mathbf{A} is invertible: $\det(\mathbf{A} - \lambda \mathbf{1}) = 0$. **Characteristic polynomial**: $p_n(\lambda) = \det(\mathbf{A}_n - \lambda \mathbf{1}_n)$; in 2-d: $p_2(\lambda) = \lambda^2 - \text{Tr}(\mathbf{A})\lambda + \det(\mathbf{A})$. $\text{Tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$ and $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.

Similarity transformation: $\mathbf{A} \rightarrow \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q}$. If \mathbf{q}_i is LI, then \mathbf{A} is **diagonalisable**: $\mathbf{A} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q}$.

Where $\mathbf{A} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n \\ | & | & & | \end{bmatrix}$.

If \mathbf{A} is diagonalisable, $\forall k \in \mathbb{N}_0 : \mathbf{A}^k = \mathbf{Q} \mathbf{A}^k \mathbf{Q}^{-1}$. The eigenvalues of \mathbf{A}^k are the eigenvalues of \mathbf{A} to the k -th power: $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$. The eigenvectors of \mathbf{A}^k equal the eigenvectors of \mathbf{A} .

The **ordinary differential equation (ODE)**: $x' = ax$, has the solution: $x(t) = c_1 e^{at}$. c_1 is determined through initial conditions.

The **system of differential equations**: $\begin{cases} x'_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x'_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ x'_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{cases} \iff \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\iff \mathbf{x}' = \mathbf{A} \mathbf{x}$ can be solved using $\mathbf{x} = \mathbf{Q} \mathbf{u}$, where \mathbf{Q} is the matrix that diagonalises \mathbf{A} , and \mathbf{u} is the solution to $\mathbf{u}' = \mathbf{A} \mathbf{u}$, where \mathbf{A} is the diagonal similarity transformation of \mathbf{A} .

If \mathbf{A} is diagonalisable, then for $\mathbf{x}' = \mathbf{A} \mathbf{x}$: $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{q}_1 + c_2 e^{\lambda_2 t} \mathbf{q}_2 + \dots + c_n e^{\lambda_n t} \mathbf{q}_n$.

For the **higher-order linear differential equation** $x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n x = 0$, define: $x_1 = x, x_2 = x', \dots, x_n =$

$x^{(n-1)}$, and let: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Then solve the ODE: $\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ using diagonalisation.

Norm of a vector: $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.

Unit vector: $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$.

Dot product: $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n = \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta)$.

Cross product: $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \|\mathbf{v}\| \|\mathbf{w}\| \sin(\theta) \hat{\mathbf{n}}$.

2-d Inverse: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Vector space axioms:

Closure under addition: $\mathbf{u} + \mathbf{v} \in V$

Commutativity of vector addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

Associativity of vector addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

Additive identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$

Additive inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Closure under scalar multiplication: $k\mathbf{u} \in V$

Distributivity of vector addition: $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

Distributivity of scalar addition: $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

Associativity of scalar multiplication: $k(m\mathbf{u}) = (km)\mathbf{u}$

Scalar multiplication identity: $1\mathbf{u} = \mathbf{u}$

Subspaces of \mathbb{R}^2 : $\{\mathbf{0}\}$, lines through the origin, and \mathbb{R}^2 .

Subspaces of \mathbb{R}^3 : $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and \mathbb{R}^3 .

Subspaces of M_{nn} : Upper triangular matrices, lower triangular matrices, diagonal matrices, and M_{nn} .

Determinant properties:

$\det(\mathbf{I}) = 1$.

Exchanging two rows of a matrix reverses the sign of its determinant.

Determinants are multilinear, so that $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ and $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

If \mathbf{A} has two equal rows, then $\det(\mathbf{A}) = 0$.

Adding a scalar multiple of one row to another does not change the determinant of a matrix.

If \mathbf{A} has a row of zeros, then $\det(\mathbf{A}) = 0$.

If \mathbf{A} is triangular, then $\det(\mathbf{A}) = \prod_{i=1}^n a_{ii}$.

If \mathbf{A} is singular, then $\det(\mathbf{A}) = 0$.

$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$.

$\det(\mathbf{A}^\top) = \det(\mathbf{A})$

Matrix Identities:

$\mathbf{A}(\mathbf{BC}) = \mathbf{AB} + \mathbf{AC}$

$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$

$(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$

If \mathbf{A} and \mathbf{B} are both invertible:

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$

$(\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$