

Vector Spaces

A **vector space** V is closed under vector addition and scalar multiplication:

$$\mathbf{u} + \mathbf{v} \in V \text{ and } k\mathbf{u} \in V.$$

A *subset* W of a vector space V is called a **subspace** of V if W is itself a vector space. The intersection of subspaces is also a subspace of V .

S is **linearly independent (LI)** if

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n = \mathbf{0}$$

has $k_i = 0$.

S forms a **basis** for V if S spans V and S is LI.

Fundamental Subspaces

For $\mathbf{A} \in \mathbb{R}^{m \times n}$:

- $r = \text{rank}(\mathbf{A}) = \dim(\mathcal{C}(\mathbf{A}))$
- $r = \text{rank}(\mathbf{A}^\top) = \dim(\mathcal{C}(\mathbf{A}^\top))$
- $n - r = \text{null}(\mathbf{A}) = \dim(\mathcal{N}(\mathbf{A}))$
- $m - r = \text{null}(\mathbf{A}^\top) = \dim(\mathcal{N}(\mathbf{A}^\top))$

Row equivalent matrices have the same row space and null space.

Orthogonality

The subspaces U and W of a vector space V are **orthogonal subspaces** iff

$$\forall \mathbf{u} \in U : \forall \mathbf{w} \in W : \mathbf{u}^\top \mathbf{w} = 0.$$

- $\mathbf{v}^\top \mathbf{v} = \|\mathbf{v}\|^2$

The **orthogonal complement** of U :

$$U^\perp = \{\forall \mathbf{u} \in U : \mathbf{v} \in V : \mathbf{v}^\top \mathbf{u} = 0\}$$

- $(U^\perp)^\perp = U$
- $\dim U + \dim U^\perp = \dim V$
- $(\mathcal{C}(\mathbf{A}))^\perp = \mathcal{N}(\mathbf{A}^\top)$
- $(\mathcal{C}(\mathbf{A}^\top))^\perp = \mathcal{N}(\mathbf{A})$

Projections:

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \mathbf{a}x = \mathbf{a} \frac{\mathbf{a}^\top \mathbf{b}}{\mathbf{a}^\top \mathbf{a}}.$$

$$\forall \mathbf{w} \in W : \mathbf{w} \neq \mathbf{p} :$$

$$\text{proj}_W \mathbf{b} = \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b} : \|\mathbf{b} - \mathbf{p}\| < \|\mathbf{b} - \mathbf{w}\|.$$

Determinants

$$\det(\mathbf{A}) = \sum_{j=1}^n a_{ij} C_{ij} = \sum_{i=1}^n a_{ij} C_{ij}$$

where $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj}(\mathbf{A})$$

where $\text{adj}(\mathbf{A}) = \mathbf{C}^\top$.

Differential Equations

The **ordinary differential equation (ODE)**: $x' = ax$, has the solution: $x(t) = c_1 e^{at}$. c_1 is determined through initial conditions.

Linear Maps

Linear transformations:

$$T : V \rightarrow W \iff \forall \mathbf{u}, \mathbf{v} \in V : \forall k \in \mathbb{R} :$$

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \wedge T(k\mathbf{u}) = kT(\mathbf{u})$$

Rotations: Anticlockwise looking down from the positive direction of the axis of rotation.

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Shears:

$$\mathbf{S}_x = \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{S}_y = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{S}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$$

where the standard basis vector in the subscripted axis maps to itself. Think about where the standard basis vectors maps.

Reflections:

$$\mathbf{M}_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \mathbf{M}_{xz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M}_{yz} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where vectors are reflected across the plane formed by the subscripts of \mathbf{M} .

2d reflections about $y = mx + c$, where $\theta = \arctan(m)$:

$$\begin{aligned} T(\mathbf{v}) &= \mathbf{R} \mathbf{M}_{xz} \mathbf{R}^{-1} \left(\mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^{-1} \left(\mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \\ &= \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \left(\mathbf{v} - \begin{bmatrix} 0 \\ c \end{bmatrix} \right) + \begin{bmatrix} 0 \\ c \end{bmatrix} \end{aligned}$$

Invariant Subspaces

Invariant (IV) subspaces:

For $T : V \rightarrow V$, \mathcal{V} is IV if

$$T(\mathcal{V}) \subseteq \mathcal{V} \iff \forall \mathbf{v} \in \mathcal{V} \implies T(\mathbf{v}) \in \mathcal{V}.$$

Trivial IV subspaces:

- V
- $\text{im}(T) \equiv T(V) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$
- $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$
- $\{\mathbf{0}\}$
- linear combination of IV subspaces

Eigenspaces (1d IV subspace):

$$\mathcal{V} = \{\forall \mathbf{q} \in \mathcal{V} : \exists \lambda \in \mathbb{C} : T(\mathbf{q}) = \lambda \mathbf{q}\}$$

If \mathbf{A} is diagonalisable:

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n \\ | & | & & | \end{bmatrix}$$

$$\forall k \in \mathbb{N}_0 : \mathbf{A}^k = \mathbf{Q} \mathbf{A}^k \mathbf{Q}^{-1}$$

The eigenvalues of \mathbf{A}^k are the eigenvalues of \mathbf{A} to the k -th power: $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$. The eigenvectors of \mathbf{A}^k equal the eigenvectors of \mathbf{A} .

where λ_i are the eigenvalues of \mathbf{A} and \mathbf{q}_i are the eigenvectors of \mathbf{A} , and they satisfy $(\mathbf{A} - \lambda \mathbf{I}) \mathbf{q} = \mathbf{0}$.

If \mathbf{A} is invertible: $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.

Characteristic polynomial:

$$p_n(\lambda) = \det(\mathbf{A}_n - \lambda \mathbf{I}_n)$$

In 2d:

$$p_2(\lambda) = \lambda^2 - \text{Tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

$$\text{Tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i \quad \text{and} \quad \det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$$

Similarity transformation:

$$\mathbf{A} \rightarrow \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q}$$

If \mathbf{q}_i are LI, then \mathbf{A} is **diagonalisable**:

$$\mathbf{A} = \mathbf{Q}^{-1} \mathbf{A} \mathbf{Q} \iff \mathbf{A} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^{-1}$$

The **system of differential equations**:

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{cases}$$

$$\Leftrightarrow \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Leftrightarrow \mathbf{x}' = \mathbf{A}\mathbf{x}$$

can be solved using $\mathbf{x} = \mathbf{Q}\mathbf{u}$ where \mathbf{Q} is the matrix that diagonalises \mathbf{A} and \mathbf{u} is the solution to $\mathbf{u}' = \mathbf{\Lambda}\mathbf{u}$ where $\mathbf{\Lambda}$ is the diagonal similarity transformation of \mathbf{A} .

If \mathbf{A} is diagonalisable, then for $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{q}_1 + c_2 e^{\lambda_2 t} \mathbf{q}_2 + \dots + c_n e^{\lambda_n t} \mathbf{q}_n$$

For the **higher-order linear differential equation**:

$$x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n x = 0$$

define

$$x_1 = x, x_2 = x', \dots, x_n = x^{(n-1)}$$

and let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then solve the following ODE using diagonalisation:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Vector Operations

Norm of a vector:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Unit vector:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Dot product:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= v_1 w_1 + v_2 w_2 + \dots + v_n w_n \\ &= \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta) \end{aligned}$$

Cross product:

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \|\mathbf{v}\| \|\mathbf{w}\| \sin(\theta) \hat{\mathbf{n}} \end{aligned}$$

2-d Inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Vector Space Axioms

Closure under addition:

$$\mathbf{u} + \mathbf{v} \in V$$

Commutativity of vector addition:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Associativity of vector addition:

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

Additive identity:

$$\mathbf{u} + \mathbf{0} = \mathbf{u}$$

Additive inverse:

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

Closure under scalar multiplication:

$$k\mathbf{u} \in V$$

Distributivity of vector addition:

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$$

Distributivity of scalar addition:

$$(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$$

Associativity of scalar multiplication:

$$k(m\mathbf{u}) = (km)\mathbf{u}$$

Scalar multiplication identity:

$$1\mathbf{u} = \mathbf{u}$$

Subspaces

Subspaces of \mathbb{R}^2 : $\{\mathbf{0}\}$, lines through the origin, and \mathbb{R}^2 .

Subspaces of \mathbb{R}^3 : $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and \mathbb{R}^3 .

Subspaces of M_{nn} : Upper triangular matrices, lower triangular matrices, diagonal matrices, and M_{nn} .

Determinant Properties

$$1. \det(\mathbf{1}) = 1$$

$$2. \text{Exchanging two rows of a matrix reverses the sign of its determinant}$$

$$3. \text{Determinants are multilinear, so that}$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

and

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$4. \text{If } \mathbf{A} \text{ has two equal rows, then } \det(\mathbf{A}) = 0$$

$$5. \text{Adding a scalar multiple of one row to another does not change the determinant of a matrix}$$

$$6. \text{If } \mathbf{A} \text{ has a row of zeros, then } \det(\mathbf{A}) = 0$$

$$7. \text{If } \mathbf{A} \text{ is triangular, then } \det(\mathbf{A}) = \prod_{i=1}^n a_{ii}$$

$$8. \text{If } \mathbf{A} \text{ is singular, then } \det(\mathbf{A}) = 0$$

$$9. \det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$$

$$10. \det(\mathbf{A}^\top) = \det(\mathbf{A})$$

Matrix Identities

$$1. \mathbf{A}(\mathbf{BC}) = \mathbf{AB} + \mathbf{AC}$$

$$2. (\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$$

$$3. (\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$$

$$4. \text{If } \mathbf{A} \text{ and } \mathbf{B} \text{ are both invertible:}$$

$$(a) (\mathbf{AB})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$$

$$(b) (\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$$