Vector Spaces

A **vector space** V is closed under vector addition and scalar multiplication:

$$\boldsymbol{u}+\boldsymbol{v} \in V \text{ and } k\boldsymbol{u} \in V.$$

A *subset* W of a vector space V is called a **subspace** of V if W is itself a vector space. The intersection of subspaces is also a subspace of V.
S is **linearly independent (LI)** if

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_n\boldsymbol{v}_n = \mathbf{0}$$

has $k_i = 0$.
S forms a **basis** for V if S spans V and S is LI.

Fundamental Subspaces

For $\mathbf{A} \in \mathbb{R}^{m\times n}$:

- $r = \operatorname{rank}(\mathbf{A}) = \dim(\mathcal{C}(\mathbf{A}))$
- $r = \operatorname{rank}(\mathbf{A}^\top) = \dim(\mathcal{C}(\mathbf{A}^\top))$
- $n - r = \operatorname{null}(\mathbf{A}) = \dim(\mathcal{N}(\mathbf{A}))$
- $m - r = \operatorname{null}(\mathbf{A}^\top) = \dim(\mathcal{N}(\mathbf{A}^\top))$

Row equivalent matrices have the same row space and null space.

Orthogonality

The subspaces U and W of a vector space V are **orthogonal subspaces** iff

$$\forall \boldsymbol{u} \in U : \forall \boldsymbol{w} \in W : \boldsymbol{u}^\top\boldsymbol{w} = 0.$$

- $\boldsymbol{v}^\top\boldsymbol{v} = \|\boldsymbol{v}\|^2$

The **orthogonal complement** of U:

$$U^\perp = \{\forall \boldsymbol{u} \in U : \boldsymbol{v} \in V : \boldsymbol{v}^\top\boldsymbol{u} = 0\}$$

- $(U^\perp)^\perp = U$
- $\dim U + \dim U^\perp = \dim V$
- $(\mathcal{C}(\mathbf{A}))^\perp = \mathcal{N}(\mathbf{A}^\top)$
- $(\mathcal{C}(\mathbf{A}^\top))^\perp = \mathcal{N}(\mathbf{A})$

Projections:

$$\operatorname{proj}_{\boldsymbol{a}} \boldsymbol{b} = \boldsymbol{a}x = \boldsymbol{a}\frac{\boldsymbol{a}^\top\boldsymbol{b}}{\boldsymbol{a}^\top\boldsymbol{a}}.$$

$$\forall \boldsymbol{w} \in W : \boldsymbol{w} \neq \boldsymbol{p} :$$

$$\operatorname{proj}_W \boldsymbol{b} = \mathbf{A}\hat{\boldsymbol{x}} = \mathbf{A}\left(\mathbf{A}^\top\mathbf{A}\right)^{-1}\mathbf{A}^\top\boldsymbol{b} :$$

$$\|\boldsymbol{b}-\boldsymbol{p}\| < \|\boldsymbol{b}-\boldsymbol{w}\|.$$

Determinants

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{ij}\mathbf{C}_{ij} = \sum_{i=1}^{n} a_{ij}\mathbf{C}_{ij}$$

where $\mathbf{C}_{ij} = (-1)^{i+j}\mathbf{M}_{ij}$.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\operatorname{adj}(\mathbf{A})$$

where $\operatorname{adj}(\mathbf{A}) = \mathbf{C}^\top$.

Linear Maps

Linear transformations:

$$T : V \to W \iff \forall \boldsymbol{u},\, \boldsymbol{v} \in V : \forall k \in \mathbb{R} :$$
$$T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}) \wedge T(k\boldsymbol{u}) = kT(\boldsymbol{u})$$

Rotations: Anticlockwise looking down from the positive direction of the axis of rotation.

$$\mathbf{R}_x = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{bmatrix} \quad \mathbf{R}_y = \begin{bmatrix}\cos(\theta) & 0 & \sin(\theta)\\ 0 & 1 & 0\\ -\sin(\theta) & 0 & \cos(\theta)\end{bmatrix}$$

$$\mathbf{R}_z = \begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix} \quad \mathbf{R} = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}.$$

Shears:

$$\mathbf{S}_x = \begin{bmatrix}1 & a & b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \mathbf{S}_y = \begin{bmatrix}1 & 0 & 0\\ a & 1 & b\\ 0 & 0 & 1\end{bmatrix} \quad \mathbf{S}_z = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & b & 1\end{bmatrix}$$

where the standard basis vector in the subscripted axis maps to itself. Think about where the standard basis vectors maps.

Reflections:

$$\mathbf{M}_{xy} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix} \quad \mathbf{M}_{xz} = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \mathbf{M}_{yz} = \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

where vectors are reflected across the plane formed by the subscripts of $\mathbf{M}$.
2d reflections about $y = mx + c$, where $\theta = \arctan(m)$:

$$\begin{aligned}
T(\boldsymbol{v}) &= \mathbf{R}\mathbf{M}_{xz}\mathbf{R}^{-1}\left(\boldsymbol{v} - \begin{bmatrix}0\\ c\end{bmatrix}\right) + \begin{bmatrix}0\\ c\end{bmatrix}\\
&= \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}^{-1}\left(\boldsymbol{v} - \begin{bmatrix}0\\ c\end{bmatrix}\right) + \begin{bmatrix}0\\ c\end{bmatrix}\\
&= \frac{1}{1+m^2}\begin{bmatrix}1-m^2 & 2m\\ 2m & m^2-1\end{bmatrix}\left(\boldsymbol{v} - \begin{bmatrix}0\\ c\end{bmatrix}\right) + \begin{bmatrix}0\\ c\end{bmatrix}
\end{aligned}$$

Invariant Subspaces

Invariant (IV) subspaces:
For $T : V \to V$, $\mathcal{V}$ is IV if

$$T(\mathcal{V}) \subseteq \mathcal{V} \iff \forall \boldsymbol{v} \in \mathcal{V} \implies T(\boldsymbol{v}) \in \mathcal{V}.$$

Trivial IV subspaces:

1. V
2. $\operatorname{im}(T) \equiv T(V) = \{T(\boldsymbol{v}) : \boldsymbol{v} \in V\}$
3. $\ker(T) = \{\boldsymbol{v} \in V : T(\boldsymbol{v}) = \mathbf{0}\}$
4. $\{\mathbf{0}\}$
5. linear combination of IV subspaces

Eigenspaces (1d IV subspace):

$$\mathcal{V} = \{\forall \boldsymbol{q} \in \mathcal{V} : \exists \lambda \in \mathbb{C} : T(\boldsymbol{q}) = \lambda\boldsymbol{q}\}$$

where λ_i are the eigenvalues of $\mathbf{A}$ and $\boldsymbol{q}_i$ are the eigenvectors of $\mathbf{A}$, and they satisfy $(\mathbf{A} - \lambda\mathbf{I})\,\boldsymbol{q} = \mathbf{0}$.
If $\mathbf{A}$ is invertible: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
Characteristic polynomial:

$$p_n(\lambda) = \det(\mathbf{A}_n - \lambda\mathbf{I}_n)$$

In 2d:

$$p_2(\lambda) = \lambda^2 - \operatorname{Tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

$$\operatorname{Tr}(\mathbf{A}) = \sum_{i=1}^{n}\lambda_i \quad \text{and} \quad \det(\mathbf{A}) = \prod_{i=1}^{n}\lambda_i$$

Similarity transformation:

$$\mathbf{A} \to \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$$

If $\boldsymbol{q}_i$ are LI, then $\mathbf{A}$ is **diagonalisable**:

$$\mathbf{\Lambda} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} \iff \mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$$

If $\mathbf{A}$ is diagonalisable:

$$\mathbf{\Lambda} = \begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix}| & | & & |\\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots & \boldsymbol{q}_n\\ | & | & & |\end{bmatrix}$$

$$\forall k \in \mathbb{N}_0 : \mathbf{A}^k = \mathbf{Q}\mathbf{\Lambda}^k\mathbf{Q}^{-1}$$

The eigenvalues of $\mathbf{A}^k$ are the eigenvalues of $\mathbf{A}$ to the k-th power: $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$. The eigenvectors of $\mathbf{A}^k$ equal the eigenvectors of $\mathbf{A}$.

Differential Equations

The **ordinary differential equation (ODE)**: $x' = ax$, has the solution: $x(t) = c_1 e^{at}$. c_1 is determined through initial conditions.

The **system of differential equations**:

$$\begin{cases} x_1' &= a_{11}x_1 &+ a_{12}x_2 &+ \cdots &+ a_{1n}x_n \\ x_2' &= a_{21}x_1 &+ a_{22}x_2 &+ \cdots &+ a_{2n}x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_n' &= a_{n1}x_1 &+ a_{n2}x_2 &+ \cdots &+ a_{nn}x_n \end{cases}$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\iff \boldsymbol{x}' = \mathbf{A}\boldsymbol{x}$

can be solved using $\boldsymbol{x} = \mathbf{Q}\boldsymbol{u}$ where $\mathbf{Q}$ is the matrix that diagonalises $\mathbf{A}$ and $\boldsymbol{u}$ is the solution to $\boldsymbol{u}' = \boldsymbol{\Lambda}\boldsymbol{u}$ where $\boldsymbol{\Lambda}$ is the diagonal similarity transformation of $\mathbf{A}$.

If $\mathbf{A}$ is diagonalisable, then for $\boldsymbol{x}' = \mathbf{A}\boldsymbol{x}$:

$$\boldsymbol{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{q}_1 + c_2 e^{\lambda_2 t}\boldsymbol{q}_2 + \cdots + c_n e^{\lambda_n t}\boldsymbol{q}_n$$

For the **higher-order linear differential equation**:

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}x' + a_n x = 0$$

define

$$x_1 = x,\ x_2 = x',\ \ldots,\ x_n = x^{(n-1)}$$

and let

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}.$$

Then solve the following ODE using diagonalisation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Vector Operations

Norm of a vector:

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Unit vector:

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

Dot product:

$$\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{w} &= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \\ &= \|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos(\theta) \end{aligned}$$

Cross product:

$$\begin{aligned} \boldsymbol{v}\times\boldsymbol{w} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \|\boldsymbol{v}\|\|\boldsymbol{w}\|\sin(\theta)\hat{\boldsymbol{n}} \end{aligned}$$

2-d Inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Vector Space Axioms

Closure under addition:

$$\boldsymbol{u} + \boldsymbol{v} \in V$$

Commutativity of vector addition:

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$$

Associativity of vector addition:

$$\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$$

Additive identity:

$$\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$$

Additive inverse:

$$\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$$

Closure under scalar multiplication:

$$k\boldsymbol{u} \in V$$

Distributivity of vector addition:

$$k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$$

Distributivity of scalar addition:

$$(k+m)\boldsymbol{u} = k\boldsymbol{u} + m\boldsymbol{u}$$

Associativity of scalar multiplication:

$$k(m\boldsymbol{u}) = (km)\boldsymbol{u}$$

Scalar multiplication identity:

$$1\boldsymbol{u} = \boldsymbol{u}$$

Subspaces

Subspaces of $\mathbb{R}^2$: $\{\mathbf{0}\}$, lines through the origin, and $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^3$: $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$.

Subspaces of $\mathbf{M}_{nn}$: Upper triangular matrices, lower triangular matrices, diagonal matrices, and $\mathbf{M}_{nn}$.

Determinant Properties

1. $\det(\mathbf{I}) = 1$
2. Exchanging two rows of a matrix reverses the sign of its determinant
3. Determinants are multilinear, so that
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$
and
$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
4. If $\mathbf{A}$ has two equal rows, then $\det(\mathbf{A}) = 0$
5. Adding a scalar multiple of one row to another does not change the determinant of a matrix
6. If $\mathbf{A}$ has a row of zeros, then $\det(\mathbf{A}) = 0$
7. If $\mathbf{A}$ is triangular, then $\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii}$
8. If $\mathbf{A}$ is singular, then $\det(\mathbf{A}) = 0$
9. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
10. $\det(\mathbf{A}^\top) = \det(\mathbf{A})$

Matrix Identities

1. $\mathbf{A}(\mathbf{BC}) = \mathbf{AB} + \mathbf{AC}$
2. $(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$
3. $(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top$
4. If $\mathbf{A}$ and $\mathbf{B}$ are both invertible:
   (a) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
   (b) $(\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$