Events and Probability

Event

Set of outcomes from an experiment.

Sample Space

Set of all possible outcomes Ω.

Intersection

Outcomes occur in both A and B

$$A \cap B \qquad \text{or} \qquad AB$$

Disjoint

No common outcomes, $AB = \varnothing$

$$\Pr(AB) = \Pr(A \mid B) = 0$$

Union

Set of outcomes in either A or B

$$A \cup B$$

Complement

Set of all outcomes not in A, but in Ω

$$A\overline{A} = \varnothing$$
$$A \cup \overline{A} = \Omega$$

Subset

A is a (non-strict) subset of B if all elements in A are also in B — $A \subset B$.

$$AB = A \qquad \text{and} \qquad A \cup B = B$$

$$\forall A : A \subset \Omega \land \varnothing \subset A$$

$$\Pr(A) \leq \Pr(B)$$
$$\Pr(B \mid A) = 1$$
$$\Pr(A \mid B) = \frac{\Pr(A)}{\Pr(B)}$$

Identities

$$A(BC) = (AB)C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A(B \cup C) = AB \cup AC$$
$$A \cup BC = (A \cup B)(A \cup C)$$

Probability

Measure of the likeliness of an event occurring

$$\Pr(A) \qquad \text{or} \qquad \mathrm{P}(A)$$

$$0 \leq \Pr(A) \leq 1$$

where a probability of 0 never happens, and 1 always happens.

$$\Pr(\Omega) = 1$$
$$\Pr\left(\overline{A}\right) = 1 - \Pr(A)$$

Multiplication Rule

For independent events A and B

$$\Pr(AB) = \Pr(A)\Pr(B).$$

For dependent events A and B

$$\Pr(AB) = \Pr(A \mid B)\Pr(B)$$

Addition Rule

For independent A and B

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB).$$

If $AB = \varnothing$, then $\Pr(AB) = 0$, so that $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

De Morgan's Laws

$$\overline{A \cup B} = \overline{A}\;\overline{B}$$
$$\overline{AB} = \overline{A} \cup \overline{B}.$$

$$\Pr(A \cup B) = 1 - \Pr\left(\overline{A}\;\overline{B}\right)$$
$$\Pr(AB) = 1 - \Pr\left(\overline{A} \cup \overline{B}\right)$$

Bayes' Theorem

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}$$

Random Variables

Measurable variable whose value holds some uncertainty. An event is when a random variable assumes a certain value or range of values.

Probability Distribution

The probability distribution of a random variable X is a function that links all outcomes $x \in \Omega$ to the probability that they will occur $\Pr(X = x)$.

Probability Mass Function

$$\Pr(X = x) = p_x$$

Probability Density Function

$$\Pr(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)\,\mathrm{d}x$$

Cumulative Distribution Function

Probability that a random variable is less than or equal to a particular realisation x.
$F(x)$ is a valid CDF if:

1. F is monotonically increasing and continuous
2. $\lim_{x\to-\infty} F(x) = 0$
3. $\lim_{x\to\infty} F(x) = 1$

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x} f(u)\,\mathrm{d}u = f(x)$$

Complementary CDF (Survival Function)

$$\Pr(X > x) = 1 - \Pr(X \leq x) = 1 - F(x)$$

p-Quantiles

$$F(x) = \int_{-\infty}^{x} f(u)\,\mathrm{d}u = p$$

Special Quantiles

$$\begin{aligned} \text{Lower quartile } q_1\text{:} \quad & p = \frac{1}{4} \\ \text{Median } m\text{:} \quad & p = \frac{1}{2} \\ \text{Upper quartile } q_2\text{:} \quad & p = \frac{3}{4} \\ \text{Interquartile range IQR:} \quad & q_2 - q_1 \end{aligned}$$

Quantile Function

$$x = F^{-1}(p) = Q(p)$$

Expectation (Mean)

Expected value given an infinite number of observations. For $a < c < b$:

$$\mathrm{E}(X) = -\int_a^c F(x)\,\mathrm{d}x + \int_c^b (1 - F(x))\,\mathrm{d}x + c$$

Variance

Measure of spread of the distribution (average squared distance of each value from the mean).

$$\mathrm{Var}(X) = \sigma^2 = \mathrm{E}(X^2) - \mathrm{E}(X)^2$$

Standard Deviation

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

Distribution	**Restrictions**	**PMF**	**CDF**	$\mathrm{E}(X)$	$\mathrm{Var}(X)$
$X \sim \mathrm{Uniform}(a,\ b)$	$x \in \{a, \dots, b\}$	$\frac{1}{b-a+1}$	$\frac{x-a+1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
$X \sim \mathrm{Bernoulli}(p)$	$p \in [0,1], x \in \{0,1\}$	$p^x(1-p)^{1-x}$	$1-p$	p	$p(1-p)$
$X \sim \mathrm{Binomial}(n,\ p)$	$x \in \{0, \dots, n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	$\sum_{u=0}^{x}\binom{n}{u}p^u(1-p)^{n-u}$	np	$np(1-p)$
$N \sim \mathrm{Poisson}(\lambda)$	$n \geq 0$	$\frac{\lambda^n e^{-\lambda}}{n!}$	$e^{-\lambda}\sum_{u=0}^{n}\frac{\lambda^u}{u!}$	λ	λ

Table 1: Discrete probability distributions.

Distribution	**Restrictions**	**PDF**	**CDF**	$\mathrm{E}(X)$	$\mathrm{Var}(X)$
$X \sim \mathrm{Uniform}(a,\ b)$	$a < x < b$	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$T \sim \mathrm{Exp}(\eta)$	$t > 0$	$\eta e^{-\eta t}$	$1-e^{-\eta t}$	$1/\eta$	$1/\eta$
$X \sim \mathrm{N}(\mu,\ \sigma^2)$	$x \in \{0, \dots, n\}$	$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)$	μ	σ^2

Table 2: Continuous probability distributions.

	Discrete	**Continuous**
Valid probabilities	$0 \leq p_x \leq 1$	$f(x) \geq 0$
Cumulative probability	$\sum_{u \leq x} p_u$	$\int_{-\infty}^{x} f(u)\,\mathrm{d}u$
$\mathrm{E}(X)$	$\sum_{\Omega} x p_x$	$\int_{\Omega} x f(x)\,\mathrm{d}x$
$\mathrm{E}(g(X))$	$\sum_{\Omega} g(x)\, p_x$	$\int_{\Omega} g(x)\, f(x)\,\mathrm{d}x$
$\mathrm{Var}(X)$	$\sum_{\Omega} (x-\mu)^2\, p_x$	$\int_{\Omega} (x-\mu)^2\, f(x)\,\mathrm{d}x$

Table 3: Probability rules for univariate X.