Exercises 5 solutions

T1. $A = U \Sigma V^T$

(a) $2A = 2U\Sigma V^T$

$= U(2\Sigma)V^T$

$= \hat{U} \tilde{\Sigma} \tilde{V}^T$

(b) $A^T = (U\Sigma V^T)^T$

$= V \Sigma^T U^T$

$= \hat{U} \hat{\Sigma} \hat{V}^T$

(c) $A^{-1} = (U\Sigma V^T)^{-1}$

$= (V^T)^{-1} \Sigma^{-1} U^{-1}$

$= V \Sigma^{-1} U^T$

But then to keep to convention,

set $V = [\underline{v}_n | \cdots | \underline{v}_1]$, $U = [\underline{u}_n | \cdots | \underline{u}_1]$

and $\Sigma = \mathrm{diag}\left(\frac{1}{\sigma_n}, \ldots, \frac{1}{\sigma_1}\right)$

T2. $A = U\Sigma V^T$

$A \cdot A = U\Sigma (V^T U)\Sigma V^T$?

$A = VDV^{-1}$

$A \cdot A = VD(\underbrace{V^{-1}V}_{I})DV^{-1}$

$= VD^2V^{-1}$

T3. Let $W \in \mathbb{R}^{m \times 1}$

i.e. a vector $\underline{w}$

$w^T w$ $\qquad$ $w w^T$

$W^T W$ gives V

$W W^T$ gives U

$$W^T W = \underline{w}^T \underline{w} = \|\underline{w}\|^2$$

"Diagonalise" the 1×1 matrix $\|\underline{w}\|^2$:

$$\underbrace{\left[\|\underline{w}\|^2\right]}_{W^T W}\underbrace{\left[\,1\,\right]}_{V} = \underbrace{\left[\,1\,\right]}_{V}\underbrace{\left[\|\underline{w}\|^2\right]}_{D}$$

$$\Sigma = D^{\frac{1}{2}} = \|\underline{w}\|$$

$$W = U \Sigma V^T$$

$$\underbrace{\underline{w}}_{m\times 1} = \underbrace{\hat{\underline{w}}}_{m\times 1} \cdot \underbrace{\|\underline{w}\|}_{1\times 1} \cdot \underbrace{1}_{1\times 1}, \quad \hat{\underline{w}} = \frac{\underline{w}}{\|\underline{w}\|}$$

$$W^+ = V \Sigma^+ U^T$$

$$= 1 \cdot \frac{1}{\|\underline{w}\|} \cdot \hat{\underline{w}}^T$$

$$= \frac{\hat{\underline{w}}^T}{\|\underline{w}\|}$$

n.b. $$W^+ W = \frac{\hat{\underline{w}}^T}{\|\underline{w}\|} \cdot \underline{w}$$

$$= \frac{1}{\|\underline{w}\|^2} \underline{w}^T \underline{w}$$

$$= 1$$

T4. $(AB)^+ = B^+ A^+$??

False. Choose $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $(AB)^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$

$A^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$, $B^+ = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

$B^+ A^+ = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & 0 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} = (AB)^+$