Advanced Linear Algebra

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Contents

1 Fundamental Concepts of Linear Algebra

1.1 Row Echelon Form

As studied in Linear Algebra, we can solve linear systems by applying the following elementary row operations to any matrix $\mathbf{A}$.

Type I. Exchange any two rows.

Type II. Multiply any row by a constant.

Type III. Add a multiple of one row to another row.

This allows us to reduce $\mathbf{A}$ into **row echelon form** such that the entries below the main diagonal are zero:

$$\mathbf{R}_{\text{ref}} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{mn} \end{bmatrix}$$

1.2 Elementary Matrix

Mathematically, we can represent these row operations as a matrix which is left multiplied to $\mathbf{A}$.

Definition 1.1 (Elementary matrix)**.** An elementary matrix $\mathbf{E}_i$ is constructed by applying a row operation to the elementary matrix $\mathbf{I}_m$. Consider a 3 by 4 matrix $\mathbf{A}$; a common first elementary row operation might be

$$r_2 \leftarrow r_2 - \frac{a_{21}}{a_{11}} r_1$$

which when applied to $\mathbf{I}_3$ yields

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the 1 subscript simply indicates the first of many elementary row operations. Left multiplying this to an arbitrary $\mathbf{A}$ gives

$$\begin{aligned} \mathbf{E}_1\mathbf{A} &= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{13}a_{21}}{a_{11}} & a_{24} - \frac{a_{14}a_{21}}{a_{11}} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \end{aligned}$$

which has the desired result of eliminating the first column of the second row.

1.3 Reduced Row Echelon Form

As there are infinitely many ways to reduce a matrix to row echelon form, we typically reduce $\mathbf{R}_{\text{ref}}$ further into **reduced row echelon form** which is a unique reduction for every $\mathbf{A}$.

This matrix $\mathbf{R}_{\text{rref}}$ (or simply $\mathbf{R}$) generally requires $m \times n$ elementary row operations and is only useful for theoretical analysis. In reduced row echelon form, any entries in the same column as a pivot must be 0, and each pivot is 1.

1.4 Elimination Matrix

The elementary matrices involved in row reduction can be expressed as a single matrix containing every each row operation.

$$\begin{aligned}\mathbf{E}_9\mathbf{E}_8 \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A} &= \mathbf{E}\mathbf{A} \\ &= \mathbf{R}\end{aligned}$$

1.5 Linear Systems

Given the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ we can augment $\mathbf{A}$ with $\boldsymbol{b}$ to draw conclusions about the solutions.

If we left multiply the elimination matrix $\mathbf{E}$ to $[\mathbf{A} \mid \boldsymbol{b}]$ we can apply the same operations to $\boldsymbol{b}$.

$$\begin{aligned}\mathbf{E}\left[\mathbf{A} \mid \boldsymbol{b}\right] &= \left[\mathbf{E}\mathbf{A} \mid \mathbf{E}\boldsymbol{b}\right] \\ &= \left[\mathbf{R} \mid \boldsymbol{z}\right]\end{aligned}$$

Therefore

$$\mathbf{R}\boldsymbol{x} = \boldsymbol{z}$$

1.6 Consistency of a Linear System

After reducing the matrix $\mathbf{A}$ to $\mathbf{R}$, we can summarise certain characteristics about $\mathbf{A}$.

1.6.1 Basic and Free Variables

Identifying the pivots in $\mathbf{R}$ allows us to determine the dimensions of various subspaces of $\mathbf{A}$.

Definition 1.2 (Basic variables)**.** The columns that a pivot corresponds to are known as basic variables (or leading variables).

Definition 1.3 (Free variables)**.** Any columns not corresponding to any pivots are known as free variables (or parameters). Consequently, any variables that are not basic variables are free variables.

In the following example, x_1, x_3, and x_4 are basic variables, whereas x_2 and x_5 are free variables.

$$\begin{bmatrix}1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8\end{bmatrix}$$

When using backward substitution to solve $\mathbf{R}\boldsymbol{x} = \mathbf{z}$, we assign new variables to any free variables to indicate that they are parameters to the system.

1.6.2 Singular Matrices

After reducing $\mathbf{A}$ to $\mathbf{R}$ we can consider its