Advanced Linear Algebra

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1 Fundamental Concepts of Linear Algebra

1.1 Row Echelon Form

As studied in Linear Algebra, we can solve linear systems by applying the following elementary row operations to any matrix $\mathbf{A}$.

Type I. Exchange any two rows.

Type II. Multiply any row by a constant.

Type III. Add a multiple of one row to another row.

This allows us to reduce $\mathbf{A}$ into **row echelon form** such that the entries below the main diagonal are zero:

$$\mathbf{R}_{\text{ref}} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{mn} \end{bmatrix}$$

1.2 Elementary Matrix

Mathematically, we can represent these row operations as a matrix which is left multiplied to $\mathbf{A}$.

Definition 1.1 (Elementary matrix)**.** An elementary matrix $\mathbf{E}_i$ is constructed by applying a row operation to the elementary matrix $\mathbf{I}_m$. Consider a 3 by 4 matrix $\mathbf{A}$; a common first elementary row operation might be

$$r_2 \leftarrow r_2 - \frac{a_{21}}{a_{11}} r_1$$

which when applied to $\mathbf{I}_3$ yields

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the 1 subscript simply indicates the first of many elementary row operations. Left multiplying this to an arbitrary $\mathbf{A}$ gives

$$\begin{aligned} \mathbf{E}_1\mathbf{A} &= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{13}a_{21}}{a_{11}} & a_{24} - \frac{a_{14}a_{21}}{a_{11}} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \end{aligned}$$

which has the desired result of eliminating the first column of the second row.

1.3 Reduced Row Echelon Form

As there are infinitely many ways to reduce a matrix to row echelon form, we typically reduce $\mathbf{R}_{\text{ref}}$ further into **reduced row echelon form** which is a unique reduction for every $\mathbf{A}$.

This matrix $\mathbf{R}_{\text{rref}}$ (or simply $\mathbf{R}$) generally requires $m \times n$ elementary row operations and is only useful for theoretical analysis. In reduced row echelon form, any entries in the same column as a pivot must be 0, and each pivot is 1.

1.4 Elimination Matrix

The elementary matrices involved in row reduction can be expressed as a single matrix containing every each row operation.

$$\begin{aligned} \mathbf{E}_9\mathbf{E}_8 \dots \mathbf{E}_2\mathbf{E}_1\mathbf{A} &= \mathbf{E}\mathbf{A} \\ &= \mathbf{R} \end{aligned}$$

1.5 Linear Systems

Given the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ we can augment $\mathbf{A}$ with $\boldsymbol{b}$ to draw conclusions about the solutions.

If we left multiply the elimination matrix $\mathbf{E}$ to $\left[\mathbf{A} \mid \boldsymbol{b}\right]$ we can apply the same operations to $\boldsymbol{b}$.

$$\begin{aligned} \mathbf{E}\left[\mathbf{A} \mid \boldsymbol{b}\right] &= \left[\mathbf{E}\mathbf{A} \mid \mathbf{E}\boldsymbol{b}\right] \\ &= \left[\mathbf{R} \mid \boldsymbol{z}\right] \end{aligned}$$

Therefore

$$\mathbf{R}\boldsymbol{x} = \boldsymbol{z}$$

After reducing the matrix $\mathbf{A}$ to $\mathbf{R}$, we can summarise certain characteristics about $\mathbf{A}$.

1.5.1 Basic and Free Variables

Identifying the pivots in $\mathbf{R}$ allows us to determine the dimensions of various subspaces of $\mathbf{A}$.

Definition 1.2 (Basic variables)**.** The columns that a pivot corresponds to are known as basic variables (or leading variables).

Definition 1.3 (Free variables)**.** Any columns not corresponding to any pivots are known as free variables (or parameters). Consequently, any variables that are not basic variables are free variables.

When using backward substitution to solve $\mathbf{R}\boldsymbol{x} = \mathbf{z}$, we assign new variables to any free variables to indicate that they are parameters to the system.

1.5.2 Singular Matrices

An n by n square matrix $\mathbf{A}$ is singular if its associated reduced matrix $\mathbf{R}$ has fewer than n basic variables. It follows that a singular matrix also has a determinant of 0 (as the product of the diagonal is 0) which means it is also noninvertible.

1.6 The Four Fundamental Subspaces

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 2 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{E} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that x_1 and x_3 are basic variables, whereas x_2, x_4 and x_5 are free variables.

1.6.1 Row space

The rows containing pivots form the basis vectors for the rowspace of $\mathbf{A}$, denoted $\mathcal{C}\left(\mathbf{A}^\top\right)$.

$$\mathcal{C}\left(\mathbf{A}^\top\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}\right\}$$

Note that the row vectors here are represented as column vectors to allows us to conveniently compare these vectors with other spaces.

1.6.2 Null space

The span of vectors that satisfy the homogeneous system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, form the null space of $\mathbf{A}$, denoted $\mathcal{N}\left(\mathbf{A}\right)$.

$$\mathcal{N}\left(\mathbf{A}\right) = \operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

1.6.3 Column space

By considering a general vector $\boldsymbol{b}$, we can construct the column space of $\mathbf{A}$, denoted $\mathcal{C}\left(\mathbf{A}\right)$. This is done by augmenting $\mathbf{A}$ with $\boldsymbol{b}$, and appling the same elimination matrix $\mathbf{E}$.

$$\mathbf{E}\left[\mathbf{A} \mid \boldsymbol{b}\right] = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -2 & -1 & b_1 - 3b_2 \\ 0 & 0 & 1 & 2 & 2 & b_2 \\ 0 & 0 & 0 & 0 & 0 & b_3 \end{array}\right]$$

here we determine any constraints required to make $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ consistent, resulting in

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$$

This vector will guarantee a consistent solution for any b_1 and b_2, with $b_3 = 0$.

Rewriting $\boldsymbol{b}$ in terms of its two parameters b_1 and b_2, we can construct the basis vectors for the column space of $\mathbf{A}$.

$$\mathcal{C}(\mathbf{A}) = \operatorname{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$$

Note that we could have also calculated the column space by finding the rowspace of $\mathbf{A}^\top$.

1.6.4 Left-Null Space

Just as we found the null space for $\mathbf{A}$, we can find a null space for $\mathbf{A}^\top$. This is known as the left-null space, denoted $\mathcal{N}(\mathbf{A}^\top)$.

$$\mathcal{N}(\mathbf{A}^\top) = \operatorname{span}\left\{ \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

These four spaces form the fundamental subspaces for any matrix $\mathbf{A}$.

1.6.5 Dimensions of Subspaces

For the matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$:

Definition 1.4 (Rank)**.** The dimension of the row space is called the **rank** of a matrix.

$$\operatorname{rank}(\mathbf{A}) = \dim\left(\mathcal{C}(\mathbf{A}^\top)\right) = r$$

To determine the rank, we can count the number of basic variables in $\mathbf{R}$.

Note that the $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^\top)$.

Definition 1.5 (Nullity)**.** The dimension of the null space is called the **nullity** of the matrix.

$$\operatorname{null}(\mathbf{A}) = \dim\left(\mathcal{N}(\mathbf{A})\right)$$

Definition 1.6 (Left nullity)**.** The dimension of the left null space is called the **left nullity** of the matrix.

$$\operatorname{null}(\mathbf{A}^\top) = \dim\left(\mathcal{N}(\mathbf{A}^\top)\right)$$

Theorem 1.6.1 (Rank-nullity theorem)**.** *The dimension of the domain of* $\mathbf{A}$*,* $\mathbb{R}^n$*, is given by the sum of the dimensions of the row space and null space of* $\mathbf{A}$*.*

$$\begin{aligned}\dim\left(\mathcal{C}(\mathbf{A}^\top)\right) + \dim\left(\mathcal{N}(\mathbf{A})\right) &= \dim\left(\mathbb{R}^n\right)\\ \operatorname{rank}(\mathbf{A}) + \operatorname{null}(\mathbf{A}) &= n\\ r + \operatorname{null}(\mathbf{A}) &= n\end{aligned}$$

Therefore

$$\operatorname{null}(\mathbf{A}) = n - r$$

Corollary 1.6.1.1 (Rank-nullity theorem for the transpose)**.** *The dimension of the codomain of* $\mathbf{A}$*,* $\mathbb{R}^m$*, is given by the sum of the dimensions of the column space and left-null space of* $\mathbf{A}$*.*

$$\begin{aligned}\dim\left(\mathcal{C}\left(\mathbf{A}\right)\right)+\dim\left(\mathcal{N}\left(\mathbf{A}^\top\right)\right)&=\dim\left(\mathbb{R}^m\right)\\ \operatorname{rank}\left(\mathbf{A}^\top\right)+\operatorname{null}\left(\mathbf{A}^\top\right)&=m\\ r+\operatorname{null}\left(\mathbf{A}^\top\right)&=m\end{aligned}$$

Therefore

$$\operatorname{null}\left(\mathbf{A}^\top\right)=m-r$$

Theorem 1.6.2 (Orthogonality of subspaces)**.** *The row space and null space are orthogonal complements in* $\mathbb{R}^n$*.*

$$\mathcal{C}\left(\mathbf{A}^\top\right)^\perp=\mathcal{N}\left(\mathbf{A}\right)$$

Similarly, the column space and left-null space are orthogonal complements in $\mathbb{R}^m$*.*

$$\mathcal{C}\left(\mathbf{A}\right)^\perp=\mathcal{N}\left(\mathbf{A}^\top\right)$$

1.7 Consistency of a Linear System

Given a matrix $\mathbf{A}\in\mathbb{R}^{m\times n}$, the linear system $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$, with any vector $\boldsymbol{b}$ can be described as follows:

1. Consistent with unique solution:

$$\operatorname{rank}\left(\mathbf{A}\right)=\operatorname{rank}\left(\left[\mathbf{A}\;\middle|\;\boldsymbol{b}\right]\right)$$

2. Consistent with infinitely many solutions:

$$\operatorname{rank}\left(\mathbf{A}\right)=\operatorname{rank}\left(\left[\mathbf{A}\;\middle|\;\boldsymbol{b}\right]\right)$$

3. Inconsistent with no solutions:

$$\operatorname{rank}\left(\mathbf{A}\right)\neq\operatorname{rank}\left(\left[\mathbf{A}\;\middle|\;\boldsymbol{b}\right]\right)$$

However, if $\boldsymbol{b}\in\mathcal{C}\left(\mathbf{A}\right)$, the system must be consistent.

1.8 General Solution to a Linear System

Given $\boldsymbol{b}\in\mathcal{C}\left(\mathbf{A}\right)$, the general solution to a system can be expressed as

$$\boldsymbol{x}_g=\boldsymbol{x}_p+\boldsymbol{x}_n$$

where $\boldsymbol{x}_p$ is a particular solution obtained by backward substitution, and $\boldsymbol{x}_n$ represents the linear combination of all null space basis vectors.

1.9 Minimum Norm Solution

In general, the particular solution may contain a linear combination of null space basis vectors requiring $\boldsymbol{x}_p\in\mathbb{R}^n$.

If we consider the solution vector $\boldsymbol{x}_r\in\mathcal{C}\left(\mathbf{A}^\top\right)$, then this vector will be the minimum norm solution to $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$.