Partial Differential Equations

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Contents

1 Fourier Series

Definition 1.1 (Fourier series expansion)**.** The **Fourier series expansion** of f represents f by a periodic function on an interval, using trigonometric (sine and cosine) terms.
Suppose a function $f(x)$ is defined on an interval $[-L, L]$, then the Fourier series expansion of f is given by:

$$f_F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

so that $f = f_F$ on $[-L, L]$. Here we cannot be certain about the equailty $f = f_F$ for all x as f_F is perdioc and the convergence properties of the infinite sum are not known.

Before attempting to determine the coefficients a_n and b_n for $n \geq 1$, we must first evaluate certain useful integral relationships involving trigonometric functions.

1.1 Integral Relationships

1.1.1 Sine and Cosine

For $n \in \mathbb{N}$:

$$\begin{aligned}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x &= \frac{L}{n\pi}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \\ &= \frac{L}{n\pi}\left[\sin(n\pi) - \sin(-n\pi)\right] \\ &= \frac{L}{n\pi}\left[0 - 0\right] \\ &= 0. \end{aligned} \tag{2}$$

$$\begin{aligned}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x &= -\frac{L}{n\pi}\left[\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \\ &= \frac{L}{n\pi}\left[\cos(n\pi) - \cos(-n\pi)\right] \\ &= \frac{L}{n\pi}\left[1 - 1\right] \\ &= 0. \end{aligned} \tag{3}$$

1.1.2 Combinations of Sine and Cosine

Recall the Werner formulas:

$$\begin{aligned}2\cos(\alpha)\cos(\beta) &= \cos(\alpha - \beta) + \cos(\alpha + \beta) \\ 2\sin(\alpha)\sin(\beta) &= \cos(\alpha - \beta) - \cos(\alpha + \beta) \\ 2\sin(\alpha)\cos(\beta) &= \sin(\alpha - \beta) + \sin(\alpha + \beta)\end{aligned}$$

For $n,\ m \in \mathbb{N}$,

Product of two cosine functions:

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x=\frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n-m)\,\pi x}{L}\right)\mathrm{d}x+\frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n+m)\,\pi x}{L}\right)\mathrm{d}x$$

As $n+m\in\mathbb{N}$, the second integral term will evaluate to 0 due to Equation 2. For the first integral term, $n-m\in\mathbb{N}$, except when $n=m$ which results in $\cos(0)=1$. Hence

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x=\begin{cases}0, & n\neq m\\ L, & n=m\end{cases}$$

Product of two sine functions:

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x=\frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n-m)\,\pi x}{L}\right)\mathrm{d}x-\frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n+m)\,\pi x}{L}\right)\mathrm{d}x$$

By the same argument,

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x=\begin{cases}0, & n\neq m\\ L, & n=m\end{cases}$$

Product of sine and cosine functions:

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x=\frac{1}{2}\int_{-L}^{L}\sin\left(\frac{(n-m)\,\pi x}{L}\right)\mathrm{d}x+\frac{1}{2}\int_{-L}^{L}\sin\left(\frac{(n+m)\,\pi x}{L}\right)\mathrm{d}x$$

Similary, $n+m\in\mathbb{N}$ results in 0 for the second integral term, $n-m\in\mathbb{N}$ also results in 0 for the first term, and when $n=m$, as $\sin(0)=0$, the first term is always 0. Therefore

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x=0$$

1.2 Coefficients of the Fourier Series

For a_0 consider integrating Equation 1 from $-L$ to L.

$$\begin{aligned}\int_{-L}^{L}f(x)\,\mathrm{d}x&=\int_{-L}^{L}a_0\,\mathrm{d}x+\sum_{n=1}^{\infty}a_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x+\sum_{n=1}^{\infty}b_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\\ \int_{-L}^{L}f(x)\,\mathrm{d}x&=2a_0L\\ a_0&=\frac{1}{2L}\int_{-L}^{L}f(x)\,\mathrm{d}x\end{aligned}$$

so that a_0 represents the average value of f on $[-L,L]$.

For coefficients a_m, multiply the equation by $\cos\left(\frac{m\pi x}{L}\right)$ before integrating.

$$\begin{aligned} f(x)\cos\left(\frac{m\pi x}{L}\right) &= a_0\cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) \\ &\quad + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) \end{aligned}$$

$$\begin{aligned} \int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x &= a_0 \cancelto{0}{\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x} + \sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x \\ &\quad + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)}\mathrm{d}x \\ \int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x &= a_m L \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x \end{aligned}$$

For coefficients b_m, multiply the equation by $\sin\left(\frac{m\pi x}{L}\right)$ before integrating.

$$\begin{aligned} f(x)\sin\left(\frac{m\pi x}{L}\right) &= a_0\sin\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \\ &\quad + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \end{aligned}$$

$$\begin{aligned} \int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x &= a_0 \cancelto{0}{\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x} + \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)}\mathrm{d}x \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x \\ \int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x &= b_m L \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x \end{aligned}$$

To summarise,

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x \end{aligned}$$

for $m \in \mathbb{N}$.

Definition 1.2 (Piecewise smooth)**.** A function $f : [a, b] \to \mathbb{R}$, is **piecewise smooth** if each component f_i of f has a <u>bounded derivative</u> f_i' which is <u>continuous everywhere</u> in $[a, b]$, except at a finite number of points at which left- and right-sided derivatives exist.

Theorem 1.2.1 (Convergence of piecewise smooth functions)**.** *If f is a periodic piecewise smooth function on $[-L, L]$, f_F will converge to*

$$f_F(x) = \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon) + f(x-\epsilon)}{2}$$

that is, $f = f_F$, except at discontinuities, where f_F is equal to the point halfway between the left- and right-hand limits.

Corollary 1.2.1.1 (Dirichlet conditions)**.** *The Dirichlet conditions provide sufficient conditions for a real-valued function f to be equal to its Fourier series f_F on $[-L, L]$, at each point where f is continuous.*
The conditions are:

1. *f has a finite number of maxima and minima over $[-L, L]$.*
2. *f has a finite number of discontinuities, in each of which the derivative f' exists and does not change sign.*
3. *$\int_{-L}^{L} |f(x)| \, \mathrm{d}x$ exists.*

Definition 1.3 (Gibbs phenomenon)**.** If f_F does not converge to f at discontiniuties x_i, then the f_F converges non-uniformally. For Fourier series this property is known as the *Gibbs phenomenon.*

Note 1.2.1. When f is non-periodic, f_F converges to the periodic extension of f. The endpoints may converge non-uniformally, corresponding to jump disontinuities in the periodic extension of f.

1.3 Sine and Cosine Series

Definition 1.4 (Odd function)**.** f is an *odd* function if it satisfies

$$f(-x) = -f(x)$$

Definition 1.5 (Even function)**.** f is an *even* function if it satisfies

$$f(-x) = f(x)$$

If f is an odd function on $[-L, L]$, then the coefficients corresponding to the cosine terms will be zero. The Fourier series simplifies to

$$f_F = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$.

Likewise if f is an even function on $[-L, L]$, then the coefficients corresponding to the sine terms will be zero. The Fourier series simplifies to

$$f_F = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x$ and $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$.
These special cases are known as the sine and cosine series expansions respectively, resulting in the **odd** or **even** periodic extension of f.