

Trigonometry

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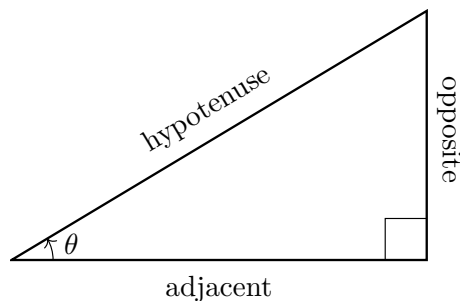


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1 Definitions

1.1 Trigonometric Functions



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$$

1.2 Inverse Trigonometric Functions

$$y = \arccos(x) \iff x = \cos(y)$$

$$y = \operatorname{arccsc}(x) \iff x = \csc(y)$$

$$y = \arcsin(x) \iff x = \sin(y)$$

$$y = \operatorname{arcsec}(x) \iff x = \sec(y)$$

$$y = \arctan(x) \iff x = \tan(y)$$

$$y = \operatorname{arccot}(x) \iff x = \cot(y)$$

1.3 Properties as a Real Function

Let $n \in \mathbb{Z}$ be a constant.

Function	Period	Parity	Domain	Range
$\sin(x)$	2π	odd	\mathbb{R}	$[-1, 1]$
$\cos(x)$	2π	even	\mathbb{R}	$[-1, 1]$
$\tan(x)$	π	odd	$\mathbb{R} \setminus \left\{ \left(n + \frac{1}{2}\right)\pi \right\}$	\mathbb{R}
$\cot(x)$	π	odd	$\mathbb{R} \setminus \{n\pi\}$	\mathbb{R}
$\sec(x)$	2π	even	$\mathbb{R} \setminus \left\{ \left(n + \frac{1}{2}\right)\pi \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\csc(x)$	2π	odd	$\mathbb{R} \setminus \{n\pi\}$	$(-\infty, -1] \cup [1, \infty)$

Function	Parity	Domain	Range
$\arcsin(x)$	odd	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos(x)$	–	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$	odd	\mathbb{R}	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\operatorname{arccot}(x)$	odd	\mathbb{R}	$[0, \pi]$
$\operatorname{arcsec}(x)$	–	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] \setminus \{\frac{\pi}{2}\}$
$\operatorname{arccsc}(x)$	odd	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

1.4 Symmetry

$$\sin(-x) = -\sin(x)$$

$$\cos(-x) = \cos(x)$$

$$\tan(-x) = -\tan(x)$$

$$\csc(-x) = -\csc(x)$$

$$\sec(-x) = \sec(x)$$

$$\cot(-x) = -\cot(x)$$

1.5 Periodicity

Let $n \in \mathbb{Z}$ be a constant.

$$\sin(x + 2\pi n) = \sin(x)$$

$$\cos(x + 2\pi n) = \cos(x)$$

$$\tan(x + \pi n) = \tan(x)$$

$$\csc(x + 2\pi n) = \csc(x)$$

$$\sec(x + 2\pi n) = \sec(x)$$

$$\cot(x + \pi n) = \cot(x)$$

2 Trigonometric Identities

2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\tan^2(x) + 1 = \sec^2(x)$$

$$1 + \cot^2(x) = \csc^2(x)$$

2.2 Angle Sum Identities

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \quad \csc(x \pm y) = \frac{1}{\sin(x)\cos(y) \pm \cos(x)\sin(y)}$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \quad \sec(x \pm y) = \frac{1}{\cos(x)\cos(y) \mp \sin(x)\sin(y)}$$

$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} \quad \cot(x \pm y) = \frac{\cot(x)\cot(y) \mp 1}{\cot(x) \pm \cot(y)}$$

2.3 Double-Angle Identities

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\csc(2x) = \frac{\sec(x) \csc(x)}{2}$$

$$\sec(2x) = \frac{\sec^2(x) \csc^2(x)}{\csc^2(x) - \sec^2(x)}$$

$$\cot(2x) = \frac{\cot^2(x) - 1}{2 \cot(x)}$$

2.4 Power Reducing Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$\csc^2(x) = \frac{2}{1 - \cos(2x)}$$

$$\sec^2(x) = \frac{2}{1 + \cos(2x)}$$

$$\cot^2(x) = \frac{1 + \cos(2x)}{1 - \cos(2x)}$$

2.5 Half Angle Identities

$$\sin\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor} \sqrt{\frac{1 - \cos(x)}{2}}$$

$$\cos\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x+\pi}{2\pi} \rfloor} \sqrt{\frac{1 + \cos(x)}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos(x)}{\sin(x)}$$

$$\csc\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor} \sqrt{\frac{2 \sec(x)}{\sec(x) - 1}}$$

$$\sec\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x+\pi}{2\pi} \rfloor} \sqrt{\frac{2 \sec(x)}{\sec(x) + 1}}$$

$$\cot\left(\frac{x}{2}\right) = \frac{\sin(x)}{1 - \cos(x)}$$

2.6 Werner Identities

$$2 \sin(x) \sin(y) = \cos(x - y) - \cos(x + y)$$

$$2 \cos(x) \cos(y) = \cos(x - y) + \cos(x + y)$$

$$2 \sin(x) \cos(y) = \sin(x - y) + \sin(x + y)$$

$$-2 \cos(x) \sin(y) = \sin(x - y) - \sin(x + y)$$

2.7 Prosthaphaeresis Identities

$$\sin(x) + \sin(y) = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) - \sin(y) = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos(x) - \cos(y) = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

2.8 Inverse Reciprocal Identities

$$\arcsin\left(\frac{1}{x}\right) = \operatorname{arccsc}(x)$$

$$\operatorname{arccsc}\left(\frac{1}{x}\right) = \arcsin(x)$$

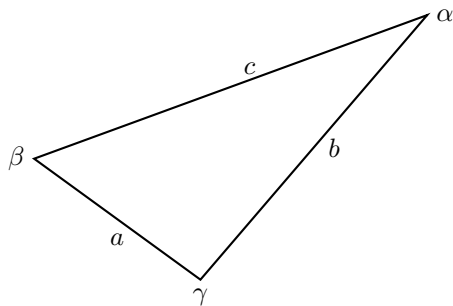
$$\arccos\left(\frac{1}{x}\right) = \operatorname{arcsec}(x)$$

$$\operatorname{arcsec}\left(\frac{1}{x}\right) = \arccos(x)$$

$$\arctan\left(\frac{1}{x}\right) = \operatorname{arccot}(x)$$

$$\operatorname{arccot}\left(\frac{1}{x}\right) = \arctan(x)$$

3 Geometric Identities



3.1 Area of a Triangle

$$A = \frac{1}{2}ab \sin(\gamma)$$

3.2 Sine Rule

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

3.3 Cosine Rule

$$a^2 = b^2 + c^2 - 2bc \cos(\alpha)$$

3.4 Tangent Rule

$$\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} = \frac{a-b}{a+b}$$

3.5 Mollweide's Identity

$$\frac{b-c}{a} = \frac{\sin\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$$

3.6 Newton's Identity

$$\frac{b+c}{a} = \frac{\cos\left(\frac{\beta-\gamma}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$