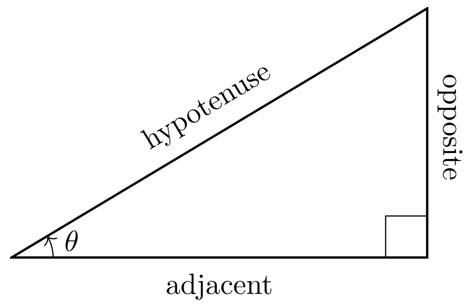
Trigonometry

Contents

1 Definitions

1.1 Trigonometric Functions



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$$

1.2 Inverse Trigonometric Functions

$$y = \arccos(x) \iff x = \cos(y) \qquad y = \operatorname{arccsc}(x) \iff x = \csc(y)$$

$$y = \arcsin(x) \iff x = \sin(y) \qquad y = \operatorname{arcsec}(x) \iff x = \sec(y)$$

$$y = \arctan(x) \iff x = \tan(y) \qquad y = \operatorname{arccot}(x) \iff x = \cot(y)$$

1.3 Properties as a Real Function

Let $n \in \mathbb{Z}$ be a constant.

Function	Period	Parity	Domain	Range
$\sin(x)$	2π	odd	$\mathbb{R}$	$[-1,\ 1]$
$\cos(x)$	2π	even	$\mathbb{R}$	$[-1,\ 1]$
$\tan(x)$	π	odd	$\mathbb{R}\backslash\left\{\left(n+\frac{1}{2}\right)\pi\right\}$	$\mathbb{R}$
$\cot(x)$	π	odd	$\mathbb{R}\backslash\left\{n\pi\right\}$	$\mathbb{R}$
$\sec(x)$	2π	even	$\mathbb{R}\backslash\left\{\left(n+\frac{1}{2}\right)\pi\right\}$	$(-\infty,\ -1] \cup [1,\ \infty)$
$\csc(x)$	2π	odd	$\mathbb{R}\backslash\left\{n\pi\right\}$	$(-\infty,\ -1] \cup [1,\ \infty)$

Function	Parity	Domain	Range
$\arcsin(x)$	odd	$[-1,\ 1]$	$[-\frac{\pi}{2},\ \frac{\pi}{2}]$
$\arccos(x)$	–	$[-1,\ 1]$	$[0,\ \pi]$
$\arctan(x)$	odd	$\mathbb{R}$	$[-\frac{\pi}{2},\ \frac{\pi}{2}]$
$\text{arccot}(x)$	odd	$\mathbb{R}$	$[0,\ \pi]$
$\text{arcsec}(x)$	–	$(-\infty,\ -1] \cup [1,\ \infty)$	$[0,\ \pi] \setminus \{\frac{\pi}{2}\}$
$\text{arccsc}(x)$	odd	$(-\infty,\ -1] \cup [1,\ \infty)$	$[-\frac{\pi}{2},\ \frac{\pi}{2}] \setminus \{0\}$

1.4 Symmetry

$$\begin{aligned} \sin(-x) &= -\sin(x) & \csc(-x) &= -\csc(x) \\ \cos(-x) &= \cos(x) & \sec(-x) &= \sec(x) \\ \tan(-x) &= -\tan(x) & \cot(-x) &= -\cot(x) \end{aligned}$$

1.5 Periodicity

Let $n \in \mathbb{Z}$ be a constant.

$$\begin{aligned} \sin(x+2\pi n) &= \sin(x) & \csc(x+2\pi n) &= \csc(x) \\ \cos(x+2\pi n) &= \cos(x) & \sec(x+2\pi n) &= \sec(x) \\ \tan(x+\pi n) &= \tan(x) & \cot(x+\pi n) &= \cot(x) \end{aligned}$$

2 Trigonometric Identities

2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\begin{aligned} \tan^2(x) + 1 &= \sec^2(x) \\ 1 + \cot^2(x) &= \csc^2(x) \end{aligned}$$

2.2 Angle Sum Identities

$$\begin{aligned} \sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y) & \csc(x \pm y) &= \frac{1}{\sin(x)\cos(y) \pm \cos(x)\sin(y)} \\ \cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y) & \sec(x \pm y) &= \frac{1}{\cos(x)\cos(y) \mp \sin(x)\sin(y)} \\ \tan(x \pm y) &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} & \cot(x \pm y) &= \frac{\cot(x)\cot(y) \mp 1}{\cot(x) \pm \cot(y)} \end{aligned}$$

2.3 Double-Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \csc(2x) = \frac{\sec(x)\csc(x)}{2}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \qquad \sec(2x) = \frac{\sec^2(x)\csc^2(x)}{\csc^2(x) - \sec^2(x)}$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)} \qquad \cot(2x) = \frac{\cot^2(x) - 1}{2\cot(x)}$$

2.4 Power Reducing Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \qquad \csc^2(x) = \frac{2}{1 - \cos(2x)}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \qquad \sec^2(x) = \frac{2}{1 + \cos(2x)}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)} \qquad \cot^2(x) = \frac{1 + \cos(2x)}{1 - \cos(2x)}$$

2.5 Half Angle Identities

$$\sin\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor}\sqrt{\frac{1 - \cos(x)}{2}} \qquad \csc\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor}\sqrt{\frac{2\sec(x)}{\sec(x) - 1}}$$

$$\cos\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x+\pi}{2\pi} \rfloor}\sqrt{\frac{1 + \cos(x)}{2}} \qquad \sec\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x+\pi}{2\pi} \rfloor}\sqrt{\frac{2\sec(x)}{\sec(x) + 1}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos(x)}{\sin(x)} \qquad \cot\left(\frac{x}{2}\right) = \frac{\sin(x)}{1 - \cos(x)}$$

2.6 Werner Identities

$$\begin{aligned}
2\sin(x)\sin(y) &= \cos(x - y) - \cos(x + y) \\
2\cos(x)\cos(y) &= \cos(x - y) + \cos(x + y) \\
2\sin(x)\cos(y) &= \sin(x - y) + \sin(x + y) \\
-2\cos(x)\sin(y) &= \sin(x - y) - \sin(x + y)
\end{aligned}$$

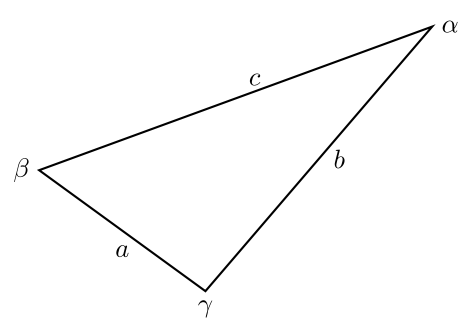
2.7 Prosthaphaeresis Identities

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

2.8 Inverse Reciprocal Identities

$$\arcsin\left(\frac{1}{x}\right) = \operatorname{arccsc}(x) \qquad \operatorname{arccsc}\left(\frac{1}{x}\right) = \arcsin(x)$$
$$\arccos\left(\frac{1}{x}\right) = \operatorname{arcsec}(x) \qquad \operatorname{arcsec}\left(\frac{1}{x}\right) = \arccos(x)$$
$$\arctan\left(\frac{1}{x}\right) = \operatorname{arccot}(x) \qquad \operatorname{arccot}\left(\frac{1}{x}\right) = \arctan(x)$$

3 Geometric Identities



3.1 Area of a Triangle

$$A = \frac{1}{2}ab\sin(\gamma)$$

3.2 Sine Rule

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

3.3 Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

3.4 Tangent Rule

$$\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} = \frac{a-b}{a+b}$$

3.5 Mollweide's Identity

$$\frac{b-c}{a} = \frac{\sin\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$$

3.6 Newton's Identity

$$\frac{b+c}{a} = \frac{\cos\left(\frac{\beta-\gamma}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$