

# Trigonometry

This work is licensed under a Creative Commons  
“Attribution-NonCommercial-ShareAlike 4.0 International” license.

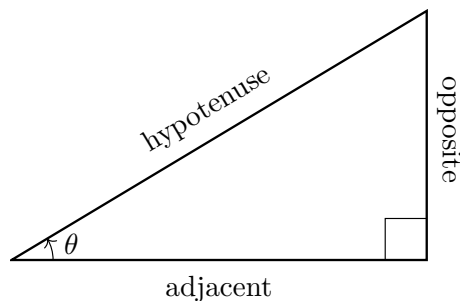


# Contents

<b>Contents</b>	<b>1</b>
<b>1 Definitions</b>	<b>2</b>
1.1 Trigonometric Functions . . . . .	2
1.2 Inverse Trigonometric Functions . . . . .	2
1.3 Properties as a Real Function . . . . .	2
1.4 Symmetry . . . . .	3
1.5 Periodicity . . . . .	3
<b>2 Trigonometric Identities</b>	<b>3</b>
2.1 Pythagorean Identities . . . . .	3
2.2 Angle Sum Identities . . . . .	3
2.3 Double-Angle Identities . . . . .	4
2.4 Power Reducing Identities . . . . .	4
2.5 Half Angle Identities . . . . .	4
2.6 Werner Identities . . . . .	4
2.7 Prosthaphaeresis Identities . . . . .	5
2.8 Inverse Reciprocal Identities . . . . .	5
<b>3 Geometric Identities</b>	<b>5</b>
3.1 Area of a Triangle . . . . .	5
3.2 Sine Rule . . . . .	5
3.3 Cosine Rule . . . . .	6
3.4 Tangent Rule . . . . .	6
3.5 Mollweide's Identity . . . . .	6
3.6 Newton's Identity . . . . .	6

# 1 Definitions

## 1.1 Trigonometric Functions



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$$

## 1.2 Inverse Trigonometric Functions

$$y = \arccos(x) \iff x = \cos(y)$$

$$y = \operatorname{arccsc}(x) \iff x = \csc(y)$$

$$y = \arcsin(x) \iff x = \sin(y)$$

$$y = \operatorname{arcsec}(x) \iff x = \sec(y)$$

$$y = \arctan(x) \iff x = \tan(y)$$

$$y = \operatorname{arccot}(x) \iff x = \cot(y)$$

## 1.3 Properties as a Real Function

Let  $n \in \mathbb{Z}$  be a constant.

Function	Period	Parity	Domain	Range
$\sin(x)$	$2\pi$	odd	$\mathbb{R}$	$[-1, 1]$
$\cos(x)$	$2\pi$	even	$\mathbb{R}$	$[-1, 1]$
$\tan(x)$	$\pi$	odd	$\mathbb{R} \setminus \left\{ \left(n + \frac{1}{2}\right)\pi \right\}$	$\mathbb{R}$
$\cot(x)$	$\pi$	odd	$\mathbb{R} \setminus \{n\pi\}$	$\mathbb{R}$
$\sec(x)$	$2\pi$	even	$\mathbb{R} \setminus \left\{ \left(n + \frac{1}{2}\right)\pi \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\csc(x)$	$2\pi$	odd	$\mathbb{R} \setminus \{n\pi\}$	$(-\infty, -1] \cup [1, \infty)$

Function	Parity	Domain	Range
$\arcsin(x)$	odd	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos(x)$	–	$[-1, 1]$	$[0, \pi]$
$\arctan(x)$	odd	$\mathbb{R}$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\operatorname{arccot}(x)$	odd	$\mathbb{R}$	$[0, \pi]$
$\operatorname{arcsec}(x)$	–	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] \setminus \{\frac{\pi}{2}\}$
$\operatorname{arccsc}(x)$	odd	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

## 1.4 Symmetry

$$\begin{aligned}
 \sin(-x) &= -\sin(x) & \csc(-x) &= -\csc(x) \\
 \cos(-x) &= \cos(x) & \sec(-x) &= \sec(x) \\
 \tan(-x) &= -\tan(x) & \cot(-x) &= -\cot(x)
 \end{aligned}$$

## 1.5 Periodicity

Let  $n \in \mathbb{Z}$  be a constant.

$$\begin{aligned}
 \sin(x + 2\pi n) &= \sin(x) & \csc(x + 2\pi n) &= \csc(x) \\
 \cos(x + 2\pi n) &= \cos(x) & \sec(x + 2\pi n) &= \sec(x) \\
 \tan(x + \pi n) &= \tan(x) & \cot(x + \pi n) &= \cot(x)
 \end{aligned}$$

# 2 Trigonometric Identities

## 2.1 Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

Dividing by either the sine or cosine function gives:

$$\begin{aligned}
 \tan^2(x) + 1 &= \sec^2(x) \\
 1 + \cot^2(x) &= \csc^2(x)
 \end{aligned}$$

## 2.2 Angle Sum Identities

$$\begin{aligned}
 \sin(x \pm y) &= \sin(x) \cos(y) \pm \cos(x) \sin(y) & \csc(x \pm y) &= \frac{1}{\sin(x) \cos(y) \pm \cos(x) \sin(y)} \\
 \cos(x \pm y) &= \cos(x) \cos(y) \mp \sin(x) \sin(y) & \sec(x \pm y) &= \frac{1}{\cos(x) \cos(y) \mp \sin(x) \sin(y)} \\
 \tan(x \pm y) &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x) \tan(y)} & \cot(x \pm y) &= \frac{\cot(x) \cot(y) \mp 1}{\cot(x) \pm \cot(y)}
 \end{aligned}$$

### 2.3 Double-Angle Identities

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$$\csc(2x) = \frac{\sec(x) \csc(x)}{2}$$

$$\sec(2x) = \frac{\sec^2(x) \csc^2(x)}{\csc^2(x) - \sec^2(x)}$$

$$\cot(2x) = \frac{\cot^2(x) - 1}{2 \cot(x)}$$

### 2.4 Power Reducing Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$\csc^2(x) = \frac{2}{1 - \cos(2x)}$$

$$\sec^2(x) = \frac{2}{1 + \cos(2x)}$$

$$\cot^2(x) = \frac{1 + \cos(2x)}{1 - \cos(2x)}$$

### 2.5 Half Angle Identities

$$\sin\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor} \sqrt{\frac{1 - \cos(x)}{2}}$$

$$\cos\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x+\pi}{2\pi} \rfloor} \sqrt{\frac{1 + \cos(x)}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos(x)}{\sin(x)}$$

$$\csc\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor} \sqrt{\frac{2 \sec(x)}{\sec(x) - 1}}$$

$$\sec\left(\frac{x}{2}\right) = (-1)^{\lfloor \frac{x+\pi}{2\pi} \rfloor} \sqrt{\frac{2 \sec(x)}{\sec(x) + 1}}$$

$$\cot\left(\frac{x}{2}\right) = \frac{\sin(x)}{1 - \cos(x)}$$

### 2.6 Werner Identities

$$2 \sin(x) \sin(y) = \cos(x - y) - \cos(x + y)$$

$$2 \cos(x) \cos(y) = \cos(x - y) + \cos(x + y)$$

$$2 \sin(x) \cos(y) = \sin(x - y) + \sin(x + y)$$

$$-2 \cos(x) \sin(y) = \sin(x - y) - \sin(x + y)$$

## 2.7 Prosthaphaeresis Identities

$$\sin(x) + \sin(y) = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) - \sin(y) = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos(x) - \cos(y) = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

## 2.8 Inverse Reciprocal Identities

$$\arcsin\left(\frac{1}{x}\right) = \operatorname{arccsc}(x)$$

$$\operatorname{arccsc}\left(\frac{1}{x}\right) = \arcsin(x)$$

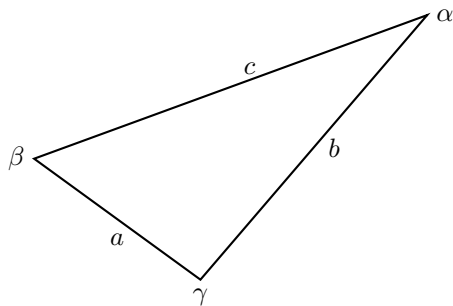
$$\arccos\left(\frac{1}{x}\right) = \operatorname{arcsec}(x)$$

$$\operatorname{arcsec}\left(\frac{1}{x}\right) = \arccos(x)$$

$$\arctan\left(\frac{1}{x}\right) = \operatorname{arccot}(x)$$

$$\operatorname{arccot}\left(\frac{1}{x}\right) = \arctan(x)$$

## 3 Geometric Identities



### 3.1 Area of a Triangle

$$A = \frac{1}{2}ab \sin(\gamma)$$

### 3.2 Sine Rule

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

**3.3 Cosine Rule**

$$a^2 = b^2 + c^2 - 2bc \cos(\alpha)$$

**3.4 Tangent Rule**

$$\frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)} = \frac{a-b}{a+b}$$

**3.5 Mollweide's Identity**

$$\frac{b-c}{a} = \frac{\sin\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$$

**3.6 Newton's Identity**

$$\frac{b+c}{a} = \frac{\cos\left(\frac{\beta-\gamma}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$