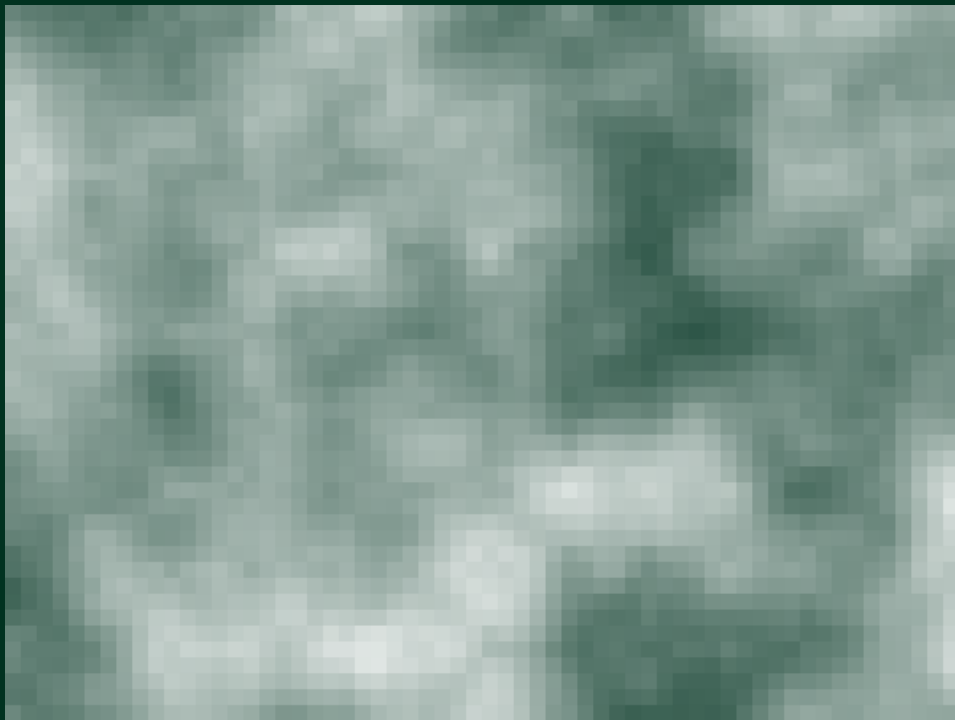

Lecture notes on Analysis and Geometry on Manifolds

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University of Bonn
Winter semester 2024/2025
Last update: October 10, 2024

Contents

- Chapter 0 Manuel’s notes 2
 - 0.1 Organization 2
 - 0.2 Course overview 2
- Chapter 1 Topological manifolds 4
 - 1.1 Some point set topology 4
 - 1.1.1 Locally Euclidean spaces 4
 - 1.1.2 Hausdorff spaces 5
 - 1.2 Basis and covers 6
- List of Lectures 9

Chapter 0:

Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Basis: Basis
- Website: https://www.math.uni-bonn.de/~lcote/V3D3_2024.html
- Time slot(s): **Tuesday: 14-16** Nussallee Anatomie B and **Friday: 12-14** GHS
- Exams: Tuesday **11.02.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10 and Friday **21.03.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10
- Deadlines: **Friday before noon**

0.1 Organization

- Four exercise classes, in the break come to the front and sign up.
- First homework is due this Friday
- Exercise sheets are due on Fridays, every week electronically (groups, at most 2)
- No published lecture notes by him!
- 5 Minute break right before the full hour
- Friday after class for questions

Start of lecture 01
(08.10.2024)

0.2 Course overview

He assumes we already know about

- Analysis on \mathbb{R}^n
- Basic point set topology

For this class: **smooth manifolds**

- Intersection between analysis and topology
- Exiting: Connections between those two point of views

Main topics:

Topic 00: Topological manifolds

Topic 01: Basic theory of smooth manifolds

Topic 02: Vector fields on smooth manifolds

Topic 03: Tensor calculus and Stokes' theorem

Topic 04: Lie groups, symplectic and Riemannian geometry

Chapter 1:

Topological manifolds

1.1 Some point set topology

Some (set theoretical) conventions for the whole course:

- $A \subset B$ means A subset (not necessarily proper!) of B , i.e. $\subset = \subseteq$
- A **neighborhood** of some point $p \in X$ means *an open set $U \subset X$ containing p*
- Given $p = (p_1, \dots, p_n) \in \mathbb{R}^n, r > 0$, $B_r^n(p) := \{(x_1, \dots, x_n) \mid \sum_{i=1}^n (x_i - p_i)^2 < r^2\}$. Often while $B_s = B_s^n(0) \subset \mathbb{R}^n$

1.1.1 Locally Euclidean spaces

Definition. A topological space X is called **locally Euclidean of dimension $n \geq 0$** , if every point of X is contained in a neighborhood homeomorphic to some open subset of \mathbb{R}^n .

Remark. When we speak of a topological space as being **locally Euclidean**. The dimension is fixed and implicit.

Definition. Assume that X is locally Euclidean. A **chart** is a pair U, ϕ , where $U \subset X$, $\phi : U \rightarrow \mathbb{R}^n$ is a homeomorphism into its image. Given $p \in X$, we say that U, ϕ is **centered at p** if $p \in U$ and $\phi(p) = 0 \in \mathbb{R}^n$

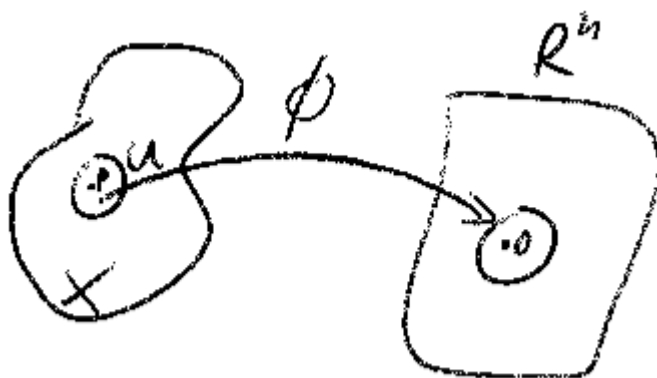


Figure 1.1: Sketch 1.01

Lemma 1.1. The following are equivalent (TFAE):

- X is locally Euclidean
- For any $p \in X$, there is a chart U, ϕ centered at p with image $\phi(U) = B_1$
- For any $p \in X$, there is a chart U, ϕ centered at p with image $\phi(U) = \mathbb{R}^n$

Proof. 2. and 3. are equivalent, since $B_1 \simeq \mathbb{R}^n$ are homeomorphic ($B_1 \ni x \mapsto \frac{x}{1-\|x\|}$)

2. \implies 1. is tautological

1. \implies 2. given $p \in X$, since X is locally Euclidean, there exists **some** chart $U, \phi, p \in U$.

$\psi : U \rightarrow \mathbb{R}^n$, homeo onto its image $\psi(U) = O \subset \mathbb{R}^n$. By translativity $\mathbb{R}^n \ni x \mapsto x - \psi(p)$, one can assume $\psi(p) = 0 \in \mathbb{R}^n$. By scaling $\mathbb{R}^n(x \mapsto \lambda x, \lambda > 0)$, can assume $B_1 \subset \psi(U)$. Let

$U' = \psi^{-1}(B_1)$, then (U, ψ) as claimed. \square

1.1.2 Hausdorff spaces

Definition. A topological space X is called Hausdorff, if given any $p_1 \neq p_2, p_1, p_2 \in X$, there exist neighborhoods $p_1 \in U_1, p_2 \in U_2$ s.t. $U_1 \cap U_2 = \emptyset$.

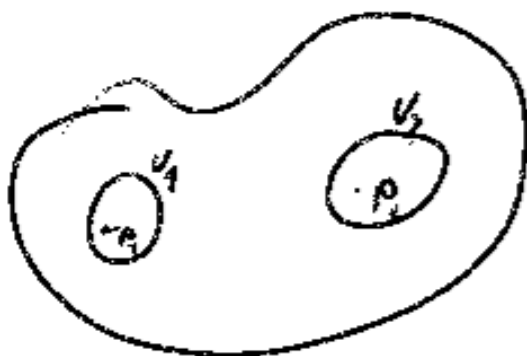


Figure 1.2: Sketch 1.02

Example. $\bullet \mathbb{R}^n$

- \bullet CW complexes
- \bullet most reasonable spaces

Example (Not Hausdorff). $X = \{0, 1\}$, open subsets $\emptyset, \{0\}, \{0, 1\}$

Remark. X is homeomorphic to \mathbb{R}/\mathbb{R}^* (quotient topology), $\mathbb{R}^*, (s, x \mapsto sx)$

Lemma 1.2. Let X be Hausdorff.

- (a) point sets $\{x\}$ are closed
- (b) convergent sequences have unique limits. $(x_n \rightarrow p, x_n \rightarrow q \implies p = q)$
- (c) compact sets are closed

Proof. (c) \implies (a)

For (c): Let $K \subset X$ be compact. Want to show K^c is open. Pick $p \in K^c$. For each $q \in K$, we can choose $U_q \ni q, U_p \ni p : U_q \cap U_p = \emptyset$. Since K is compact, it can be covered by U_{q_1}, \dots, U_{q_l} . Then $\bigcap_{i=1}^l U_{q_i}$ is open and contains p , disjoint, then $\bigcup_{i=1}^l U_{q_i} \supset K$.

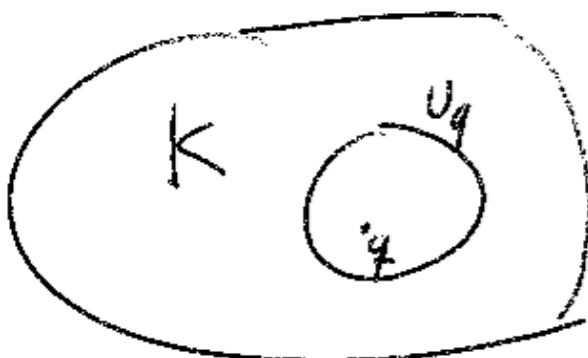


Figure 1.3: Sketch 1.03

(b) Suppose for contradiction that $x_i \rightarrow p, x_i \rightarrow q$ and $p \neq q$. Since X is Hausdorff, $\exists U \ni p, O \ni q, U \cap O = \emptyset$. But for $N \gg 0, x_i \in U, x_i \in O \forall i > N$

□

1.2 Basis and covers

Let X be a topological space.

Definition. A collection \mathcal{B} of subsets of X is called a **basis(base)** for X , if for any $p \in X$ and any neighborhood $U \ni p$, there exists an element $\mathcal{U} \in \mathcal{B}$ s.t. $p \in \mathcal{U} \subset U$.



Figure 1.4: Sketch 1.04

Lemma 1.3. \mathcal{B} is a basis for $X \iff$ every open set of X is a union of elements of \mathcal{B} .

Proof. Trivial. □

Definition. A topological space X is second-countable if it admits a countable basis.

Example. $\bullet \mathbb{R}^n, \mathcal{B} = \{B_s^n(p) \mid s \in \mathbb{Q}_+, p = (p_1, \dots, p_n) \in \mathbb{Q}^n \subset \mathbb{R}^n\}$

Lemma 1.4. The property of being second-countable is closed under

- (a) subspaces
- (b) finite unions
- (c) countable products

Remark. The property of being second-countable is not closed under arbitrary quotients $q : A \rightarrow A/B$. An obvious sufficient conditions is for q to be an open map. (Since it is a pushforward)

Lemma 1.5. If X is second countable, then any open cover of X admits a countable subcover.

Proof. Let \mathcal{B} be a countable basis for X . Let \mathcal{C} be an open cover. Let $\tilde{\mathcal{B}} \subset \mathcal{B}$ be the collection of basis elements U , which are contained in some $\mathcal{U} \in \mathcal{C}$. Observe (key!) $\tilde{\mathcal{B}}$ is a cover of X . For each $U \in \tilde{\mathcal{B}}$, choose $\mathcal{U}_U \in \mathcal{C}$ such that $U \subset \mathcal{U}_U$. Then $\{\mathcal{U}_U\}$ is a countable subcover of \mathcal{C} . □

When constructing manifolds via quotients, check that it is still second-countable!

Definition. Let X be a topological space. An exhaustion of X by compact subsets is a sequence $\{K_i\}_{i \in \mathbb{N}}$, where $K_i \subset X$ compact and $K_i \subset \text{int}(K_{i+1})$ and $\bigcup_{i=1}^{\infty} K_i = X$.

Recall given $A \subset X$. $\text{int}(A) := \{x \in A \mid x \text{ in a neighborhood } U \subset A\}$.

Lemma 1.6. If X is locally Euclidean, Hausdorff and second countable. Then X admits an exhaustion by compact subsets.

Proof. Since X is locally Euclidean, admits a basis \mathcal{B} of open subsets having compact closure.

That is take the close of $B_{\frac{1}{2}} \subset \mathbb{R}^n$

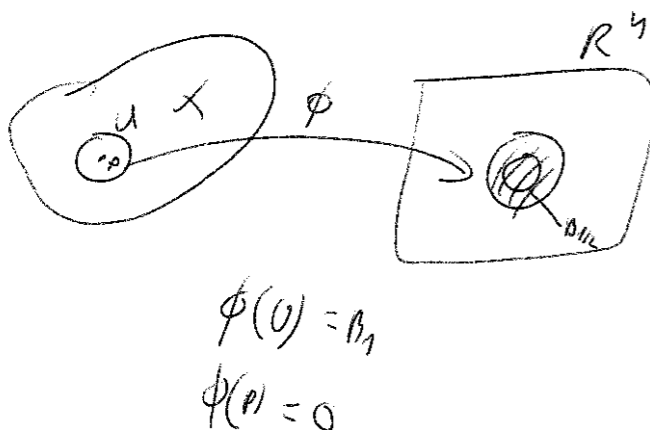


Figure 1.5: Sketch 1.05

By Lemma 1.5, one can extract a countable subcover $\{U_i\}_{i=1}^{\infty}$. Set $K_1 = \overline{U_1}$. Assume that we already constructed K_1, \dots, K_k such that $U_j \subset K_j$ and $K_{j-1} \subset \text{int}(K_j), j \geq 2$. Since K_k is compact and $K_k \subset X = \bigcup_{i=1}^{\infty} U_i$, then there exists some m_k such that $K_k \subset X = \bigcup_{i=1}^{m_k} U_i$ by compactness. Might as well assume that $m_k \geq k$. Set

$$K_{k+1} = \overline{\bigcup_{i=1}^{m_k} U_i} = \bigcup_{i=1}^{m_k} \overline{U_i}.$$

By construction K_{k+1} is compact, $K_k \subset \text{int}(K_{k+1})$. We get $\{K_j\}_{j=1}^{\infty}, U_j \subset K_j$ (because $m_j \geq j$)
 $\implies \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} K_i$ □

List of Lectures

- Lecture 01: Introduction