Lecture notes on Analysis and Geometry on Manifolds

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Lecturer
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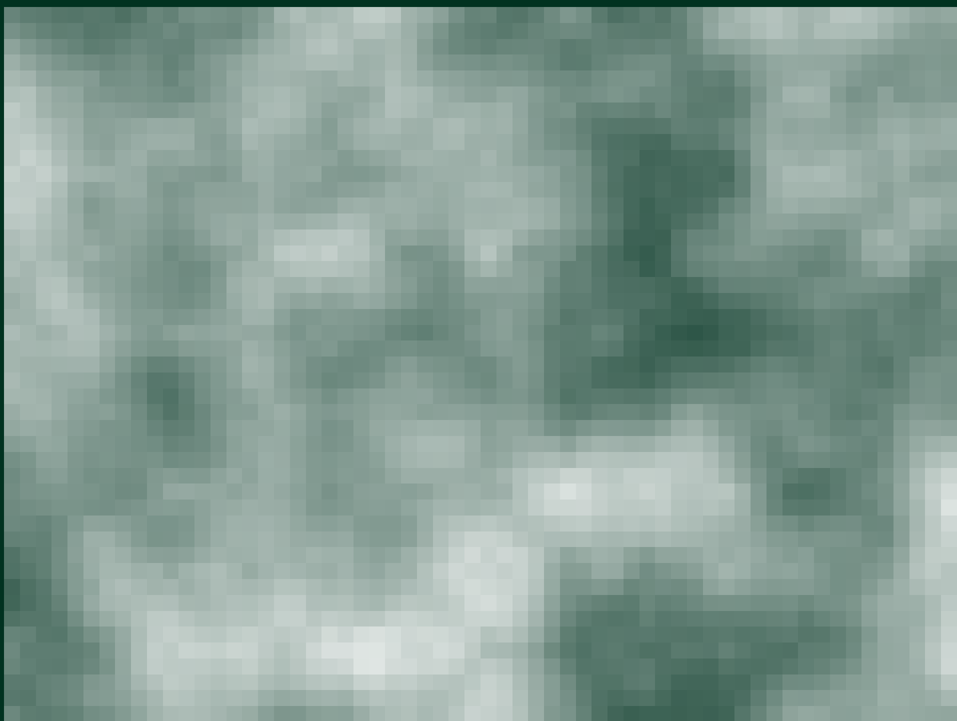

University of Bonn
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Contents

Chapter 0:
Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Basis: Basis
- Website: https://www.math.uni-bonn.de/~lcote/V3D3_2024.html
- Time slot(s): **Tuesday: 14-16** Nussallee Anatomie B and **Friday: 12-14** GHS
- Exams: Tuesday **11.02.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10 and Friday **21.03.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10
- Deadlines: **Friday before noon**

0.1 Organization

Start of lecture 01 (08.10.2024)

- Four exercise classes, in the break come to the front and sign up.
- First homework is due this Friday
- Exercise sheets are due on Fridays, every week electronically (groups, at most 2)
- No published lecture notes by him!
- 5 Minute break right before the full hour
- Friday after class for questions

0.2 Course overview

He assumes we already know about

- Analysis on $\mathbb{R}^n$
- Basic point set topology

For this class: **smooth manifolds**

- Intersection between analysis and topology
- Exiting: Connections between those two point of views

Main topics:

Topic 00: Topological manifolds

Topic 01: Basic theory of smooth manifolds

Topic 02: Vector fields on smooth manifolds

Topic 03: Tensor calculus and Stokes' theorem

Topic 04: Lie groups, symplectic and Riemannian geometry

Chapter 1: Topological manifolds

1.1 Some point set topology

Some (set theoretical) conventions for the whole course:

- $A \subset B$ means A subset (not necessarily proper!) of B, i.e. $\subset = \subseteq$
- A **neighborhood** of some point $p \in X$ means *an open set $U \subset X$ containing p*
- Given $p = (p_1, \ldots, p_n) \in \mathbb{R}^n, r > 0$, $B_r^n(p) \coloneqq \{(x_1, \ldots, x_n) \mid \sum_{x_i - p_i}^2 < r^2\}$. Often while $B_s = B_s^n(0) \subset \mathbb{R}^n$

1.1.1 Locally Euclidean spaces

Definition. *A topological space X is called* ***locally Euclidean of dimension*** $n \geq 0$*, if every point of X is contained in a neighborhood homeomorphic to some open subset of $\mathbb{R}^n$.*

Remark. *When we speak of a topological space as being* ***locally Euclidean****. The dimension is fixed and implicit.*

Definition. *Assume that X is locally Euclidean. A* ***chart*** *is a pair U, ϕ, where $U \subset X$, $\phi : U \to \mathbb{R}^n$ is a homeomorphism into its image. Given $p \in X$, we say that U, ϕ is* ***centered at*** p *if $p \in U$ and $\phi(p) = 0 \in \mathbb{R}^n$*

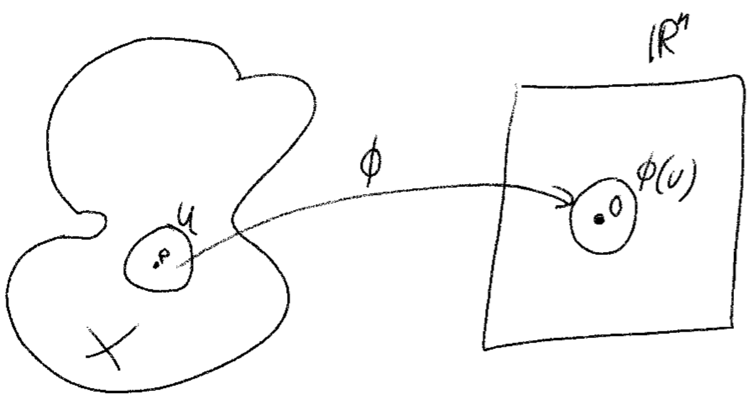

Figure 1.1: Sketch 1.01

Lemma 1.1. *The following are equivalent (TFAE):*

- *X is locally Euclidean*
- *For any $p \in X$, there is a chart U, ϕ centered at p with image $\phi(U) = B_1$*

- *For any $p \in X$, there is a chart U, ϕ centered at p with image $\phi(U) = \mathbb{R}^n$*

Proof. 2. and 3. are equivalent, since $B_1 \simeq \mathbb{R}^n$ are homeomorphic ($B_1^n \ni x \mapsto \frac{x}{1-\|x\|}$)
2. $\implies$ 1. is tautological
1. $\implies$ 2. given $p \in X$, since X is locally Euclidean, there exists **some** chart U, ϕ, $p \in U$.
$psi : U \to \mathbb{R}^n$, homeo onto its image $psi(U) = O \subset \mathbb{R}^n$. By translativity $\mathbb{R}^n \ni x \mapsto x - \psi(p)$, one can assume $\psi(p) = 0 \in \mathbb{R}^n$. By scaling $\mathbb{R}^n(x \mapsto \lambda x, \lambda > 0)$, can assume $B_1 \subset \psi(U)$. Let $U' = \psi^{-1}(B_1)$, then (U, ψ) as claimed. □

1.1.2 Hausdorff spaces

Definition. *A topological space X is called Hausdorff, if given any $p_1 \neq p_2$,$p_1, p_2 \in X$, there exist neighborhoods $p_1 \in U_1, p_2 \in U_2$ s.t. $U_1 \cap U_2 = \emptyset$.*

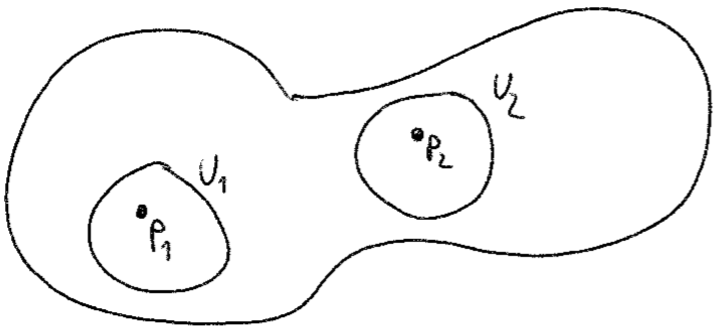

Figure 1.2: Sketch 1.02

Example.
- $\mathbb{R}^n$
- *CW complexes*
- *most reasonable spaces*

Example (Not Hausdorff)**.** $X = \{0, 1\}$, *open subsets* $\emptyset, \{0\}, \{0, 1\}$

Remark. *X is homeomorphic to $\mathbb{R}/\mathbb{R}^*$ (quotient topology), $R^*, (s, x \mapsto sx)$*

Lemma 1.2. *Let X be Hausdorff.*

(a) point sets $\{x\}$ are closed

(b) convergent sequences have unique limits. ($x_n \to p, x_n \to q \implies p = q$)

(c) compact sets are closed

Proof. (c) $\implies$ (a)
For (c): Let $K \subset X$ be compact. Want to show K^c is open. Pick $p \in K^c$. For each $q \in K$, we can choose $U_q \ni q, U_p \ni p : U_q \cap U_p = \emptyset$ Since K is compact, it can be covered by $U_{q_1}, \ldots, U_{q_l}$. Then $\bigcap_{i=1}^{l} U_{q_i}$ is oen and contains p, disjoint, then $\bigcup_{i=1}^{l} U_{q_i} \supset K$.

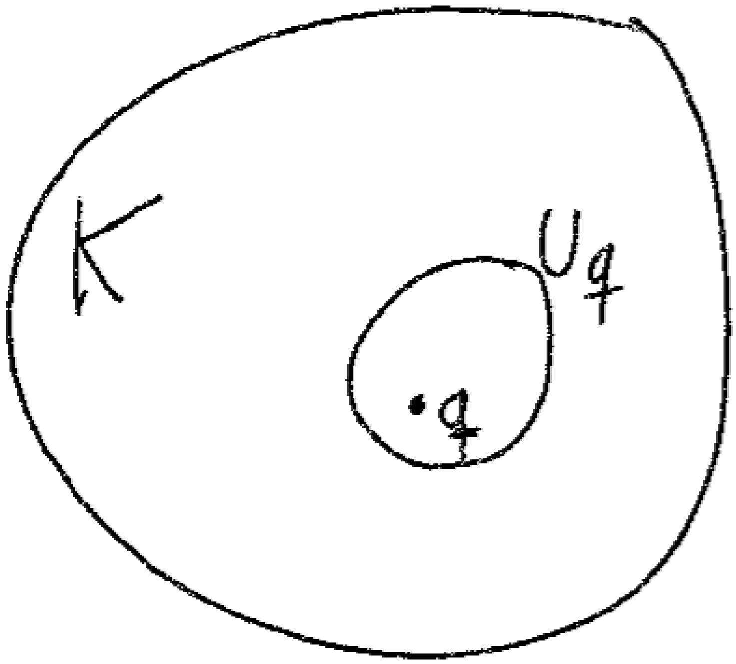

Figure 1.3: Sketch 1.03

(b) Suppose for contradiction that $x_i \to p, x_i \to q$ and $p \neq q$. Since X is Hausdorff, $\exists U \ni p, O \ni q, U \cap O = \emptyset$. But for $N >> 0 x_i \in U, x_i \in O \forall i > N$ $\square$

1.2 Basis and covers

Let X be a topological space.

Definition. *A collection $\mathcal{B}$ of subsets of X is called a **basis(base)** for X, if for any $p \in X$ and any neighborhood $U \ni p$, there exists an element $\mathcal{U} \in \mathcal{B}$ s.t. $p \in \mathcal{U} \subset U$.*

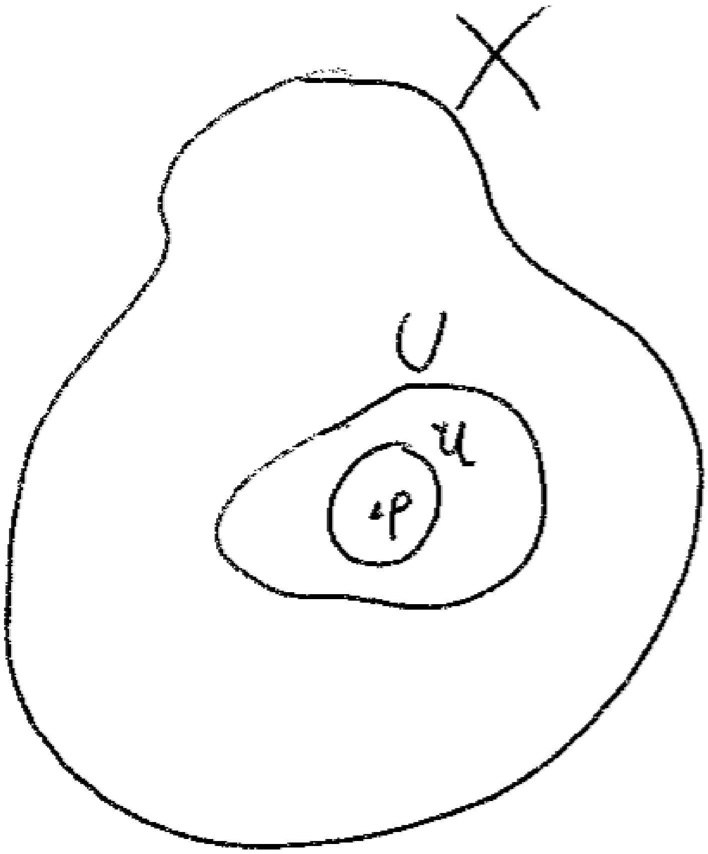

Figure 1.4: Sketch 1.04

Lemma 1.3. *$\mathcal{B}$ is a basis for $X \iff$ every open set of X is a union of elements of $\mathcal{B}$.*

Proof. Trivial. ☐

Definition. *A topological space X is **second-countable** if it admits a countable basis.*

Example.
- $\mathbb{R}^n$, $\mathcal{B} = \{B^n_s(p) \mid s \in \mathbb{Q}_+, p = (p_1, \dots, p_n) \in \mathbb{Q}^n \subset \mathbb{R}^n\}$

Lemma 1.4. *The property of being second-countable is closed under*

(a) subspaces

(b) finite unions

(c) countable products

Remark. *The property of being second-countable is not closed under arbitrary quotients $q : A \to A/B$. An obvious sufficient conditions is for q to be an open map. (Since it is a pushforward)*

When constructing manifolds via quotients, check that it is still second-coutable!

Lemma 1.5. *If X is second countable, then any open cover of X admits a countable subcover.*

Proof. Let $\mathcal{B}$ be a countable basis for X. Let $\mathcal{C}$ be an open cover. Let $\tilde{\mathcal{B}} \subset \mathcal{B}$ be the collection of basis elements U, which are contained in some $\mathcal{U} \in \mathcal{C}$. Observe (key!) $\tilde{\mathcal{B}}$ is a cover of X. For each $U \in \tilde{\mathcal{B}}$, choose $\mathcal{U}_U \in \mathcal{C}$ such that $U \subset \mathcal{U}_U$. Then $\{\mathcal{U}_U\}$ is a countable subcover of $\mathcal{C}$. ☐

Definition. *Let X be a topological space. An **exhaustion of X by compact subsets** is a sequence $\{K_i\}_{i\in\mathbb{N}}$, where $K_i \subset X$ compact and $K_i \subset int(K_{i+1})$ and $\bigcup_{i=1}^{\infty} K_i = X$.*

Recall given $A \subset X$. $\operatorname{int}(A) \coloneqq \{x \in A \mid x \text{ in a neighborhood } U \subset A\}$.

Lemma 1.6. *If X is locally Euclidean, Hausdorff and second countable. Then X admits an exhaustion by compact subsets.*

Proof. Since X is locally Euclidean, admits a basis $\mathcal{B}$ of open subsets having compact closure.

That is take the close of $B_{\frac{1}{2}} \subset \mathbb{R}^n$

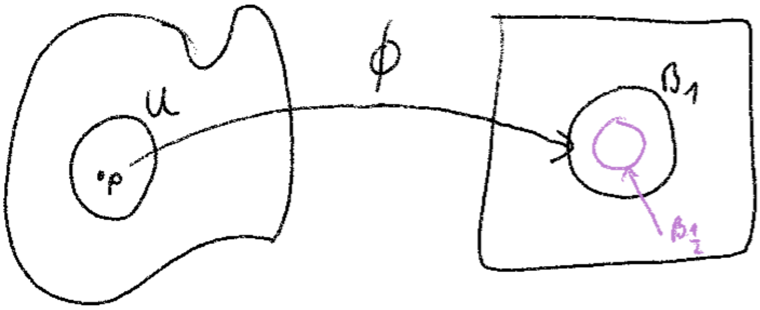

Figure 1.5: Sketch 1.05

By Lemma 1.5, one can extract a countable subcover $\{U_i\}_{i=1}^{\infty}$. Set $K_1 = \overline{U_1}$. Assume that we already constructed $K_1, \dots, K_k$ such that $U_j \subset K_j$ and $K_{j-1} \subset \operatorname{int}(K_j), j \geq 2$. Since K_k is compact and $K_k \subset X = \bigcup_{i=1}^{\infty} U_i$, then there exists some m_k such that $K_k \subset X = \bigcup_{i=1}^{m_k} U_i$ by compactness. Might as well assume that $m_k \geq k$. Set

$$K_{k+1} = \overline{\bigcup_{i=1}^{m_k} U_i} = \bigcup_{i=1}^{m_k} \overline{U_i}.$$

By construction K_{k+1} is compact, $K_k \subset \operatorname{int}(K_{k+1})$. We get $\{K_j\}_{j=1}^{\infty}$, $U_j \subset K_j$ (because $m_j \geq j$) $\implies \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} K_i$ $\square$

List of Lectures