Lecture notes on Analysis and Geometry on Manifolds

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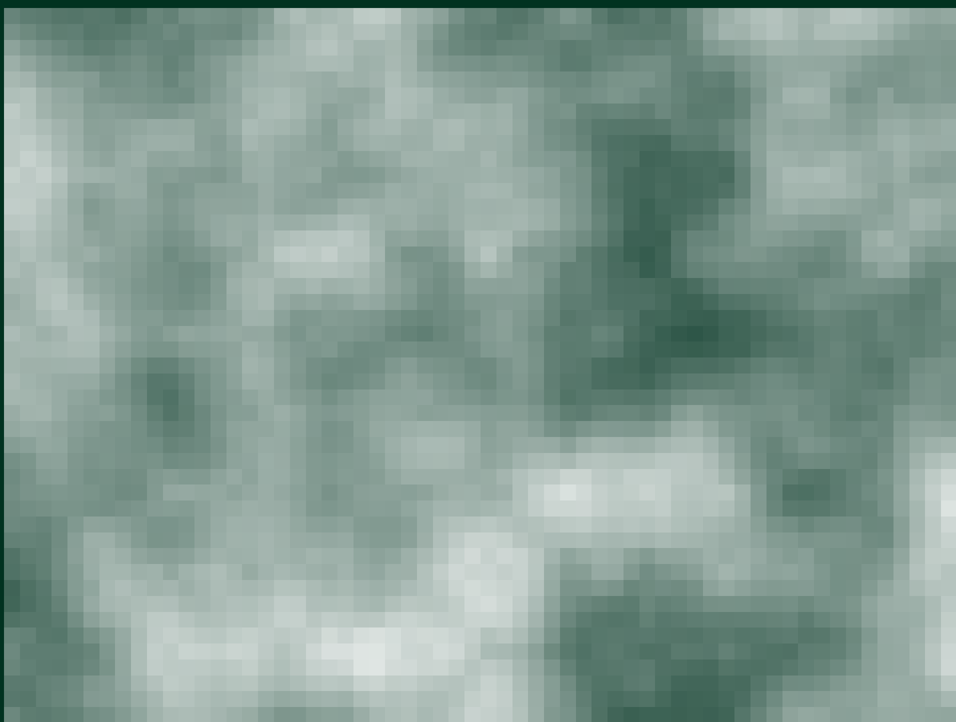

University of Bonn
Winter semester 2024/2025
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Contents

Chapter 0: Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Basis: Basis
- Website: https://www.math.uni-bonn.de/~lcote/V3D3_2024.html
- Time slot(s): **Tuesday: 14-16** Nussallee Anatomie B and **Friday: 12-14** GHS
- Exams: Tuesday **11.02.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10 and Friday **21.03.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10
- Deadlines: **Friday before noon**

0.1 Organization

Start of lecture 01 (08.10.2024)

- Four exercise classes, in the break come to the front and sign up.
- First homework is due this Friday
- Exercise sheets are due on Fridays, every week electronically (groups, at most 2)
- No published lecture notes by him!
- 5 Minute break right before the full hour
- Friday after class for questions

0.2 Course overview

He assumes we already know about

- Analysis on $\mathbb{R}^n$
- Basic point set topology

For this class: **smooth manifolds**

- Intersection between analysis and topology
- Exiting: Connections between those two point of views

Main topics:

Topic 00: Topological manifolds

Topic 01: Basic theory of smooth manifolds

Topic 02: Vector fields on smooth manifolds

Topic 03: Tensor calculus and Stokes' theorem

Topic 04: Lie groups, symplectic and Riemannian geometry

Chapter 1: Topological manifolds

1.1 Some point set topology

Some (set theoretical) conventions for the whole course:

- $A \subset B$ means A subset (not necessarily proper!) of B, i.e. $\subset = \subseteq$
- A **neighborhood** of some point $p \in X$ means *an open set $U \subset X$ containing p*
- Given $p = (p_1, \dots, p_n) \in \mathbb{R}^n, r > 0$, $B_r^n(p) := \{(x_1, \dots, x_n) \mid \sum_{x_i - p_i}^2 < r^2\}$. Often while $B_s = B_s^n(0) \subset \mathbb{R}^n$

1.1.1 Locally Euclidean spaces

Definition. *A topological space X is called* ***locally Euclidean of dimension*** $n \geq 0$*, if every point of X is contained in a neighborhood homeomorphic to some open subset of $\mathbb{R}^n$.*

Remark. *When we speak of a topological space as being* ***locally Euclidean****. The dimension is fixed and implicit.*

Definition. *Assume that X is locally Euclidean. A* ***chart*** *is a pair U, ϕ, where $U \subset X$, $\phi : U \to \mathbb{R}^n$ is a homeomorphism into its image. Given $p \in X$, we say that U, ϕ is* ***centered at*** p *if $p \in U$ and $\phi(p) = 0 \in \mathbb{R}^n$*

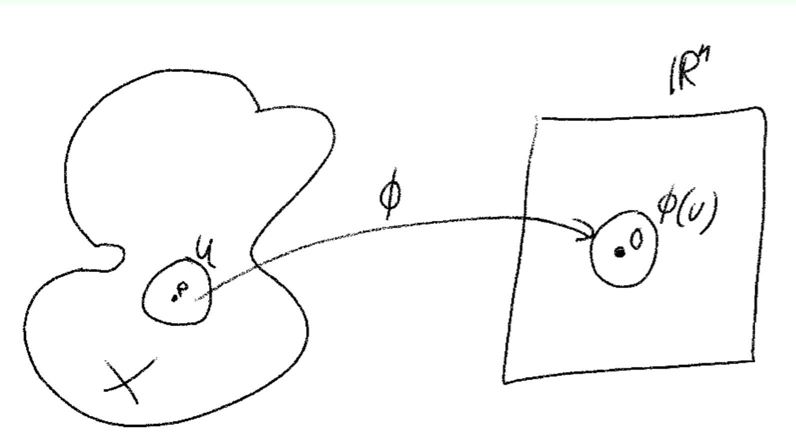

Figure 1.1: Sketch 1.01

Lemma 1.1. *The following are equivalent (TFAE):*

- *X is locally Euclidean*
- *For any $p \in X$, there is a chart U, ϕ centered at p with image $\phi(U) = B_1$*

- *For any $p \in X$, there is a chart U, ϕ centered at p with image $\phi(U) = \mathbb{R}^n$*

Proof. 2. and 3. are equivalent, since $B_1 \simeq \mathbb{R}^n$ are homeomorphic ($B_1^n \ni x \mapsto \frac{x}{1-\|x\|}$)
2. $\implies$ 1. is tautological
1. $\implies$ 2. given $p \in X$, since X is locally Euclidean, there exists **some** chart U, ϕ, $p \in U$.
$psi : U \to \mathbb{R}^n$, homeo onto its image $psi(U) = O \subset \mathbb{R}^n$. By translativity $\mathbb{R}^n \ni x \mapsto x - \psi(p)$, one can assume $\psi(p) = 0 \in \mathbb{R}^n$. By scaling $\mathbb{R}^n (x \mapsto \lambda x, \lambda > 0)$, can assume $B_1 \subset \psi(U)$. Let $U' = \psi^{-1}(B_1)$, then (U, ψ) as claimed. □

1.1.2 Hausdorff spaces

Definition. *A topological space X is called Hausdorff, if given any $p_1 \neq p_2$,$p_1, p_2 \in X$, there exist neighborhoods $p_1 \in U_1, p_2 \in U_2$ s.t. $U_1 \cap U_2 = \emptyset$.*

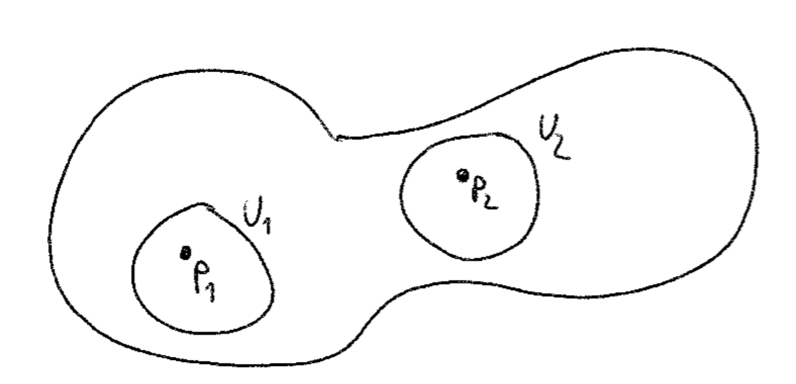

Figure 1.2: Sketch 1.02

Example.
- $\mathbb{R}^n$
- *CW complexes*
- *most reasonable spaces*

Example (Not Hausdorff)**.** $X = \{0, 1\}$, *open subsets* $\emptyset, \{0\}, \{0, 1\}$

Remark. *X is homeomorphic to $\mathbb{R}/\mathbb{R}^*$ (quotient topology), $R^*, (s, x \mapsto sx)$*

Lemma 1.2. *Let X be Hausdorff.*

(a) point sets $\{x\}$ are closed

(b) convergent sequences have unique limits. ($x_n \to p, x_n \to q \implies p = q$)

(c) compact sets are closed

Proof. (c) $\implies$ (a)
For (c): Let $K \subset X$ be compact. Want to show K^c is open. Pick $p \in K^c$. For each $q \in K$, we can choose $U_q \ni q, U_p \ni p : U_q \cap U_p = \emptyset$ Since K is compact, it can be covered by $U_{q_1}, \ldots, U_{q_l}$. Then $\bigcap_{i=1}^{l} U_{q_i}$ is oen and contains p, disjoint, then $\bigcup_{i=1}^{l} U_{q_i} \supset K$.

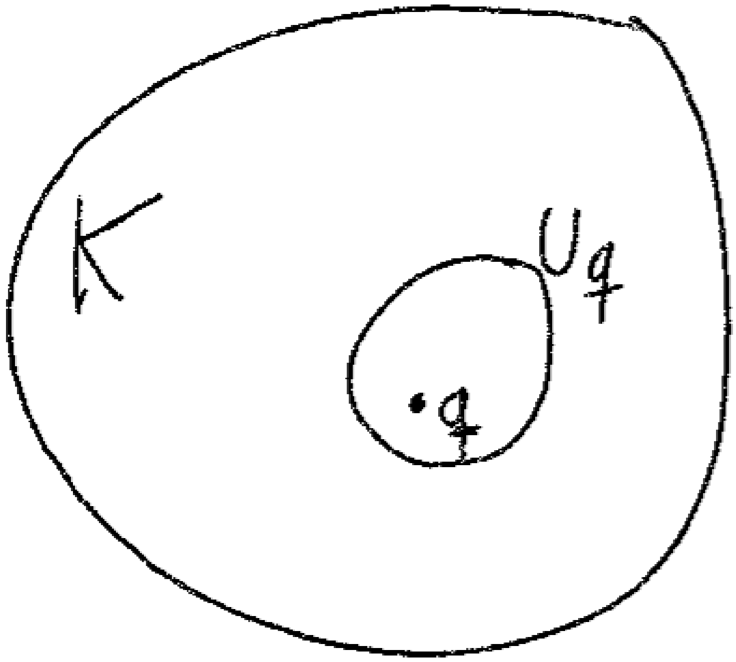

Figure 1.3: Sketch 1.03

(b) Suppose for contradiction that $x_i \to p, x_i \to q$ and $p \neq q$. Since X is Hausdorff, $\exists U \ni p, O \ni q, U \cap O = \emptyset$. But for $N >> 0 x_i \in U, x_i \in O \forall i > N$ $\square$

1.1.3 Basis and covers

Let X be a topological space.

Definition. *A collection $\mathcal{B}$ of subsets of X is called a **basis(base)** for X, if for any $p \in X$ and any neighborhood $U \ni p$, there exists an element $\mathcal{U} \in \mathcal{B}$ s.t. $p \in \mathcal{U} \subset U$.*

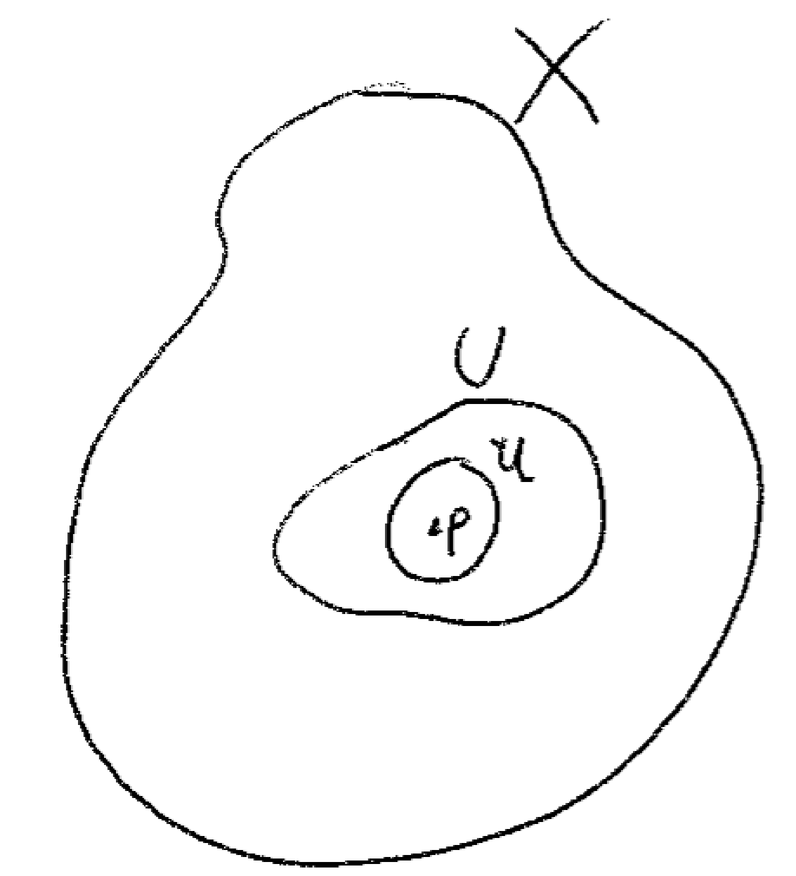

Figure 1.4: Sketch 1.04

Lemma 1.3. *$\mathcal{B}$ is a basis for $X \iff$ every open set of X is a union of elements of $\mathcal{B}$.*

Proof. Trivial. □

Definition. *A topological space X is **second-countable** if it admits a countable basis.*

Example.
- $\mathbb{R}^n$, $\mathcal{B} = \{B^n_s(p) \mid s \in \mathbb{Q}_+, p = (p_1, \ldots, p_n) \in \mathbb{Q}^n \subset \mathbb{R}^n\}$

Lemma 1.4. *The property of being second-countable is closed under*

(a) subspaces

(b) countable disjoint unions

(c) countable products

Remark. *The property of being second-countable is not closed under arbitrary quotients $q : A \to A/B$. An obvious sufficient conditions is for q to be an open map. (Since it is a pushforward)*

When constructing manifolds via quotients, check that it is still second-coutable!

Lemma 1.5. *If X is second countable, then any open cover of X admits a countable subcover.*

Proof. Let $\mathcal{B}$ be a countable basis for X. Let $\mathcal{C}$ be an open cover. Let $\tilde{\mathcal{B}} \subset \mathcal{B}$ be the collection of basis elements U, which are contained in some $\mathcal{U} \in \mathcal{C}$. Observe (key!) $\tilde{\mathcal{B}}$ is a cover of X. For each $U \in \tilde{\mathcal{B}}$, choose $\mathcal{U}_U \in \mathcal{C}$ such that $U \subset \mathcal{U}_U$. Then $\{\mathcal{U}_U\}$ is a countable subcover of $\mathcal{C}$. □

Definition. *Let X be a topological space. An **exhaustion of X by compact subsets** is a sequence $\{K_i\}_{i \in \mathbb{N}}$, where $K_i \subset X$ compact and $K_i \subset int(K_{i+1})$ and $\bigcup_{i=1}^{\infty} K_i = X$.*

Recall given $A \subset X$. $\operatorname{int}(A) \coloneqq \{x \in A \mid x \text{ in a neighborhood } U \subset A\}$.

Lemma 1.6. *If X is locally Euclidean, Hausdorff[a] and second countable. Then X admits an exhaustion by compact subsets.*

[a] not needed

Proof. Since X is locally Euclidean, admits a basis $\mathcal{B}$ of open subsets having compact closure.

That is take the close of $B_{\frac{1}{2}} \subset \mathbb{R}^n$

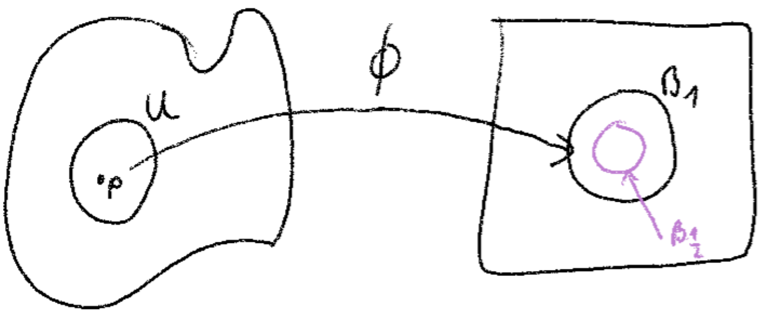

Figure 1.5: Sketch 1.05

By Lemma 1.5, one can extract a countable subcover $\{U_i\}_{i=1}^{\infty}$. Set $K_1 = \overline{U_1}$. Assume that we already constructed $K_1, \dots, K_k$ such that $U_j \subset K_j$ and $K_{j-1} \subset \operatorname{int}(K_j), j \geq 2$. Since K_k is compact and $K_k \subset X = \bigcup_{i=1}^{\infty} U_i$, then there exists some m_k such that $K_k \subset X = \bigcup_{i=1}^{m_k} U_i$ by compactness. Might as well assume that $m_k \geq k$. Set

$$K_{k+1} = \overline{\bigcup_{i=1}^{m_k} U_i} = \bigcup_{i=1}^{m_k} \overline{U_i}.$$

By construction K_{k+1} is compact, $K_k \subset \operatorname{int}(K_{k+1})$. We get $\{K_j\}_{j=1}^{\infty}$, $U_j \subset K_j$ (because $m_j \geq j$) $\implies \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} K_i$ □

Start of lecture 02 (11.10.2024)

Some remarks and apologies

- Homework 01, apology for paracompactness (if this happens again, feel free to complain)
- Lemma 1.4 should be countable disjoint unions (already corrected in my notes)
- In lemma 1.6, Hausdorff assumption not needed (already corrected in my notes)
- For questions about exercises, email Koen von der Dungen or your TA[1] directly

Definition. *Let X be a topological space. Let $\mathcal{C}$ be a collection of subsets of X. We say that $\mathcal{C}$ is **locally finite** if for every $x \in X$ there exists a neighborhood $U \ni x$ such that the intersection of U with all but finitely many elements of $\mathcal{C}$ is empty.*

Example (Example for local finiteness)**.** *Take $X = \mathbb{R}$, $\mathcal{C} = \{(i-1, i+1)\}_{i \in \mathbb{Z}}$.*

[1] tutor

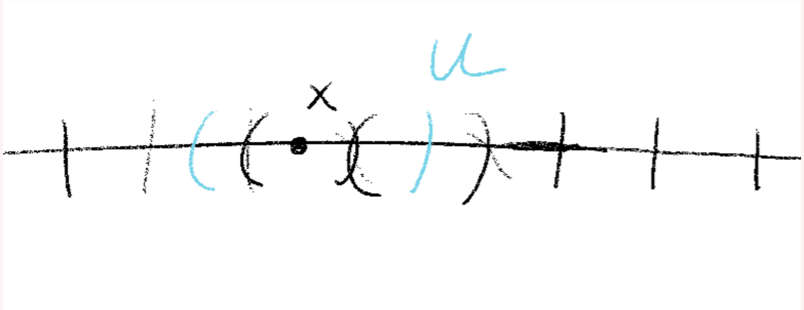

Figure 1.6: Sketch 1.06

Example (Non-example for local finiteness). $X = \mathbb{R}$, $\mathcal{C} = (q-1, q+1)_{q \in \mathbb{Q}}$

Definition. *Let X be a topological space. Let $\mathcal{C}$ be a cover of X. A cover $\mathcal{C}'$ of X is called a **refinement of $\mathcal{C}$**, if for all elements $U \in \mathcal{C}'$, there exists such $V \in \mathcal{C}$: $U \subset V$.*

Example (Example of Refinement). *In the proof of lemma 1.5, we showed that any open cover admits a refinement by basis elements.*

Definition. *A topological space X is called **paracompact** if every open cover admits a locally finite refinement.*

Whats up with the word **para**compact? It's like compact, but weaker! It is necessary that it only admits a locally finite refinement!

Lemma 1.7. *Let X be Hausdorff and suppose that X admits an exhaustion by compact subsets. Then X is paracompact.In fact, we will show that given any basis $\mathcal{B}$ of X, any open cover admits a locally finite refinement by elements of $\mathcal{B}$.*

Proof. By assumption, $\{K_i\}_{i \in \mathbb{N}}$, K_i compact, $K_i \subset \text{int}(K_{i+1}), \bigcup_{i=1}^{\infty} K_i = X$. Let, for $j \in \mathbb{Z} : V_j = K_{j+1} \setminus \int(K_j)$ if $j \leq 0 : K_j = \emptyset$[2].

Careful! There are many definitions of exhaustion by compact sets ...

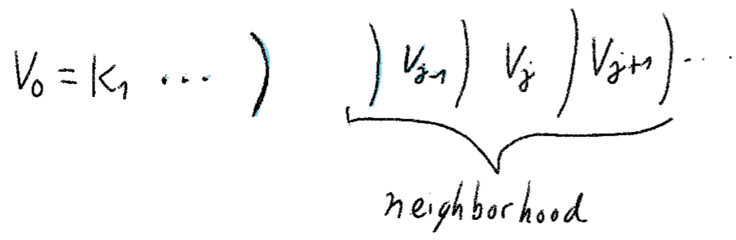

Figure 1.7: Sketch 1.07

Notice:

- V_j is compact, since we take the intersection of a compact set and a closed set. $(\text{int}(K_j)^c$ is closed)
- $\bigcup_{j \in \mathbb{Z}} V_J = X$, since $\bigcup_{j \leq n} = \bigcup_{j \leq n+1} K_j = K_{j+1}$
- The compact sets V_j are intersecting (along their boundary?) $V_j \cap V_{j-1} = \partial K_j := K_j \setminus \text{int}(K_j)$

Evidently $\{U_\alpha \cap \text{int}(K_{j+1}) \cap \text{int}(K_{j-1})^c\}_{\alpha \in \mathcal{A}}$ covers $V_j = K_{j+1} - \setminus K_{j-1}^c$, where the $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is an open cover. Since $\mathcal{B}$ is a basis, we can find a refinement of this cover by basis elements. Since V_j are compact, we can extract a finite subcover $\{V_l^j\}_{l=1,\ldots,k_j}$. Let's consider: $\{V_l^j\}_{j \in Z, l=1,\ldots,k_j}$. This subcover works, i.e.

Here we use Hausdorffness

[2] He writes $-$ for $\setminus$

- obviously a cover, since the V_j cover X, obviously a refinement of $\{U_\alpha\}$
- locally finite: given $x \in X, x \in V_j$, hence $x \in \text{int}(K_{K_{j+2}}) \cap K_{j-1}^c =: U$. If $U \cap V_l^k$, then we must have $j-2 \leq k \leq j+2$. But $\{V_l^k\}_{j-2 \leq k \leq j+2}$ is finite. □

Corollary 1.8. *If X is locally Euclidean, Hausdorff and second countable $\implies$ X is paracompact.*

Proof. By lemma 1.6 (exhaustion by compact subsets) and lemma 1.7 $\implies$ paracompact. □

Corollary 1.8'. *Let X be Euclidean amd Hausdorff. Then X is second countable iff X has countably many components and X is paracompact.*

Remark. *There are different definitions of manifolds. They differ in either forcing second countability or paracompactness. This lemma shows that there only is a difference if there are uncountably many components.*

Proof. Corollary 1.8 and the bonus homework problem from sheet 01. □

Remark. *Basis elements are open.*

1.2 Topological manifolds

Definition. *A **topological n-manifold** M is a topological space with the following properties:*

(i) M is locally Euclidean (of dimension n)

(ii) M is Hausdorff

(iii) M is second countable

Morally we only really need condition (i). Why do we need the others? For (ii) you will not get a useful theory without it, while (iii) can be replaced by paracompactness (see corollary 1.8').

Definition. *Let Man^0 be the category of topological manifolds with*

1. *objects: topological manifolds*
2. *morphisms: continuous functions*

Remark. *Man^0 full subcategory of Top.*

Remark. *By definition, $M, N \in Man^0$, then M, N are isomorphic iff M, N are homeomorphic.*

1.2.1 Examples of topological manifolds

Example (Spaces isomorphic to R^n)**.** $\mathbb{R}^n, n \geq 0$ *More generally, if V a finite dimensional $\mathbb{R}$-vector space, then V is a topological n-manifold.*

Example. *Any open subset of $\mathbb{R}^n$*

Example (Graphs)**.** *Let $U \subseteq \mathbb{R}^n$ open, let $f : U \to \mathbb{R}^n$ be a continuous function. We set*

$$M := \mathit{graph}(f) := \{(x, y) \in U \times \mathbb{R}^n \mid y = f(x)\}.$$

Then M is a manifold. The map $M \to U$ by $(x, y) \mapsto U$ gives a global chart.

Example (Spheres)**.** *Let $S^n := \{x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$. Then S^n is a manifold. We define charts*

$$\phi_i^\pm : U_i^\pm = \{(x_0, \ldots, x_n) \in S^n \mid \pm x_i > 0\} \to B_1^n(0)$$

by $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, \hat{x}_i, \ldots, x_n) := (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

Here we no longer have a global chart (for topological reasons)

Figure 1.8: Sketch 1.08

Example (spheres'). *Let $C^n := \partial([-1,1]^{n+1}) = [-1,1]^{n+1} \setminus int([-1,1]^{n+1})$. Homework: $C^n \simeq S^n$ (homeomorphic)*

Example (n-torus). *Let $\Pi^n := \mathbb{R}^n/\mathbb{Z}^n$ with the quotient topology. Then this is a manifold (exercise).*

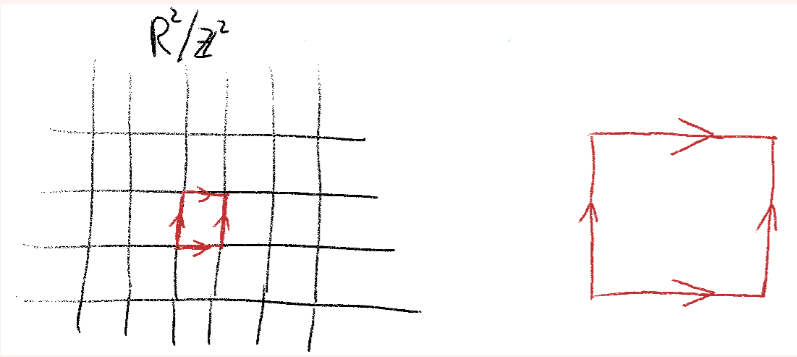

Figure 1.9: Sketch 1.09

Example ($\mathbb{RP}^n := S^n/\{x \sim -x\}$). *$\mathbb{RP}^n$ are also manifolds (called the real projective spaces).*

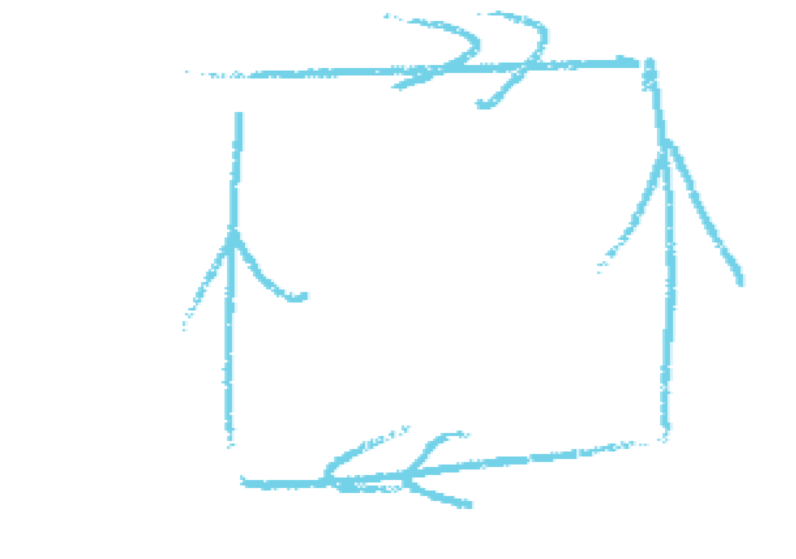

Figure 1.10: Sketch 1.10

Example (Klein bottle).

Remark. $\mathbb{RP}^2$ *or generally* $\mathbb{RP}^{2n}$ *and the Klein bottle are not orientable.*

1.2.2 Brief interlude: Why do we need Hausdorffness?

- Back in the day (Riemann)
- There is no hope to classify even 1d locally Euclidean, second-countable NOT Hausdorff spaces (See the line with two origins)
- With Hausdorff: Only 1d manifolds are $\mathbb{R}, S^1$ (see website)

Why do we need second countability?

- Subspaces of $\mathbb{R}^n$ are second countable
- We want partitions of unity (paracompactness suffices for that)

1.2.3 Manifolds with boundary

Let $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

Definition. *A* ***manifold with boundary*** *is a topological space with the following properties:*

(i) Every point has a neighborhood homeomorphic to an open subset of $\mathbb{H}^n$

(ii) Hausdorff

(iii) second countable

Clearly every manifold is also a manifold with boundary.

Example. $\mathbb{H}^n$ *is a manifold with boundary, but not a manifold. Since for points on the boundary, there are no neighborhoods homeomorphic to Euclidean space.*

Example. $S^n \cap \mathcal{H}^{n+1}, S^n \subset \mathbb{R}^{n+1}$, $[a, b], [0, \infty)$

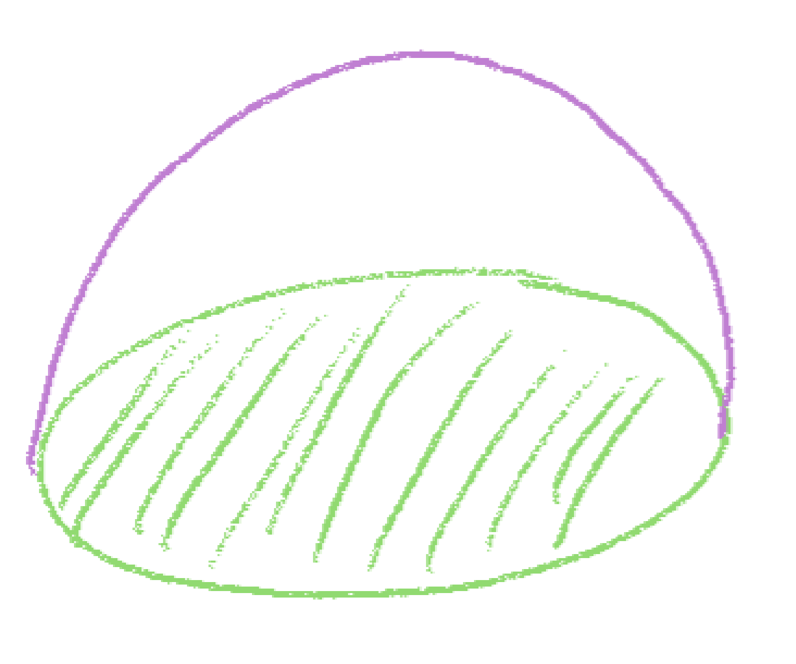

Figure 1.11: Sketch 1.12

Definition. *If M manifold with boundary, we say x is a* ***boundary point****, if $x \in M \setminus int(M)$ (i.e. it has no neighborhood homeomorphic to Euclidean space?), otherwise x is an iterior point. We let $\partial M := \{boundary\ points\}$.*

Addendum to the last lecture:

Start of lecture 03 (15.10.2024)

- $\mathbb{H}^n \not\simeq \mathbb{R}^n$ If you remove a point from $\mathbb{H}^n$ on the boundary, it is still contractable, while removing an point from $\mathbb{R}^n$ yields an uncontractable space.
- Given M a manifold with ∂, a point $p \in M$ is an **interior point**, if it admits a neighborhood homeomorphic to $\mathbb{R}^n$. Otherwise p is a boundary point. (Already specified in my notes)

Remark. *Most of what he says in the course can be generalized to manifolds with boundary (unless it makes no sense). Those results are only stated (and proofed) for manifolds. it might be a good exercise to go through the notes and generalize the statements to manifolds with boundary.*

1.2.4 Elementary topological properties of topological manifolds

Not proved here, but we are welcome to use

- A manifold is connected iff it is path connected
- For manifolds, all forms of compactness (ordinary compactness (every open cover has a finite subcover), limit point compactness, sequential compactness) are equivalent
- All manifolds are metrizable (Urysohn metrization theorem + second countable $\implies$ metrizable)
- Any manifold is homotopy equivalent to a countable CW complex (Milner?) $\pi_k(M)$ are countable

The first two point were proven on the first sheet. The last two use countability

1.3 Classification of topological manifolds (proofs are not examable)

1.3.1 Classification of 1-dimensional manifolds

Theorem 1.9. *Any connected one dimensional manifold is homeomorphic to*

- $\mathbb{R}^1$ *or*
- $\mathbb{H}^1$

Proof. See Course website: proof in the style of an exercise with hints. □

Remark. *If you allow a boundary, then you also have* $[0,1],[0,1)$.

1.3.2 Classification of 2-dimensional manifolds

2-dimensional manifolds are often called surfaces

- $S^2 = \{(x_0,x_1,x_2) \in \mathbb{R}^3 \mid \sum_{i=0}^{2} x_i^2 = 1\}$
- $\Pi^2 := \mathbb{R}^2/\mathbb{Z}^2$
- $\mathbb{RP}^2 = S^2/\{x \sim -x\}$

Construction(Connected sum of surfaces): Let M_1, M_2 be surfaces (i.e. 2-dimensional manifolds). Choose charts $M_i \supset U_i \xrightarrow{\phi_i} B_1 \subset \mathbb{R}^2$. Let $M_i^\circ = M_i \setminus \phi_i^{-1}(B_{\frac{1}{2}})$. Let $M_1 \# M_2 := M_1^\circ \perp M_2^\circ / \sim$, where $X \in M_1^\circ \sim y \in M_2^\circ$ if $x \in \phi_1^{-1}(\partial \overline{B_{\frac{1}{2}}})$ and $y = (\phi_2^{-1} \circ \phi_1)(x)$
Fact:s

- If M_1, M_2 are connected, then $M_1 \# M_2$ is well defined up to homeomorphism.
- The operation of connected sum is also well defined for connected n-manifolds
- (for the future) The operation of connected sum also works in the smooth category.

Theorem 1.10 (Classification of surfaces)**.** *Every compact, connected surface is homeomorphic to one of the following manifolds:*

- S^2
- $\underbrace{\Pi^2 \# \dots \# \Pi^2}_{k \text{ times}}$
- $\underbrace{\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2}_{l \text{ times}}$ *(non-orientable)*

Remark. *Surfaces are classified by the following invariants:*

(a) orientability

(b) Euler characteristic

For later: This classification also works in the smooth category.

1.3.3 Classification of high dimensional manifolds (not examable at all)

Poincaré conjecture (now theorem of G. Perelman (2003), W. Thurston (1980s)): Any compact connected 3 dimensional manifold which is **simply connected** is homeomorphic to S^3. This paper is all about PDEs and Ricci flows.
Generalized Poincaré conjecture: Any n-manifold, which is homotopy equivalent to S^n is homeomorphic to S^n. This is true in all dimensions. for $n \geq 5$ Smale in the 1960s, for $n = 4$ Freedsen in the 1980s.
Unlike in dimension 1,2,3 the classification of $n \geq 3$-dimensional manifolds is complicated and not complete.

Example. *Any finitely presented group arises as the fundamental group of a compact connected* 4*-manifold. Which is provably to hard?*

Chapter 2: Smooth manifolds

2.1 Basic theory

2.1.1 Charts and atlases

Definition. *Given $U \subset \mathbb{R}^n$ open, a function $f : U \to \mathbb{R}^m, f = (f_1, \dots, f_m)$ is called* ***smooth*** *(or* $\mathbb{C}^\infty$ *or* ***infinitely differentiable****), if the* ***component functions*** *f_i admit all partial derivatives of all orders and all these partial derivatives are continuous.*

In other words f smooth: $\iff \forall 1 \leq i \leq m, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \partial_\alpha f := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f$ exists.

Remark. *Given $k \geq 0$, we can similarly say hat f is* ***k-times continuously differentiable*** *and write ($f \in$) and write $f \in C^k(U, \mathbb{R}^m)$, if for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \sum \alpha_i \leq k$ $\partial_x^\alpha f_i$ is continuous for all i.*

Definition. *Let M be a topological manifold. We say that two charts (U_1, ϕ_1), (U_2, ϕ_2) are* ***smoothly compatible*** *if the map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is smooth. We call $\phi_2 \circ \phi_1^{-1}$ a* ***transition function****.*

Definition. *Let M be a topological manifold. An* ***(smooth) atlas*** *$\mathcal{A}$ of M is a collection of charts $\{U_\alpha, \phi_\alpha\}_{\alpha \in \mathcal{A}}$ such that*

- *the $\{U_\alpha\}$ cover M*
- *the charts are pairwise smoothly compatible (i.e. for all $\alpha, \beta \in \mathcal{A}(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ are smoothly compatible).*

Definition. *We say that two atlases $\mathcal{A}, \mathcal{A}'$ (on a fixed topological manifold) are* ***equivalent****, if their union $\mathcal{A} \cup \mathcal{A}'$ is still an atlas.*

Fact(Sheet 03): This defines an equivalence relation.

Definition. *A* ***smooth manifold*** *$M = (M, [\mathcal{A}])$ consists of the following data:*

(i) a topological manifolds M

(ii) an equivalence class of smooth atlases

Remark.
- *typically, we will designate smooth manifolds by a capital letter, e.g. M. But we always mean $(M, [\mathcal{A}])$.* ***Note*** *being a smooth manifolds is* ***extra*** *structure on a topological space, while being a topological manifold is a property*
- *Using Zorn's lemma, it can be shown that any atlas is contained in a* ***unique maximal atlas****. Uniqueness here does not use Zorn's lemma, only existence needs that! Equally well define a smooth manifold to be a topological manifold and a maximal atlas.*
- *$\forall 0 \leq k \leq \infty$, we can define the notion of a C^k-atlas, simply by requiring that the transition*

Typically we are given an atlas, since the maximal atlases have uncountably mani charts, which is why we work with equivalence classes, rather than maximal atlases *functions are C^k functions. This yields the definition of C^k-Manifolds. Two extreme cases: C^0-manifold (topological manifolds) and C^∞-manifolds. Any $k \geq 1$ is not more interesting than C^∞!*

Correction

Start of lecture 04 (18.10.2024)

- Definition of smoothness was corrected, s.t. derivatives are continuous. (Corrected in my notes)

Remark. *Fine in dimension $n = 1$, but necessary for $n > 1$. See website for a counterexample.*

2.1.2 First examples of smooth manifolds

Example (Example 1: The cannoical smooth manifold)**.** $\mathbb{R}^n, n \geq 0$ *is* ***canonically*** *a smooth manifold. The* ***canonical atlas*** *is induced by the topological chart $U = \mathbb{R}^n, \phi : U \overset{id}{\to} \mathbb{R}^n$.*

Example (Example 2: Another canonical smooth manifold)**.** *Let V be a finite dimensional real vector space . Then V is canonically a smooth manifold. Pick a vector space basis $\mathcal{B}$. This basis induces a homeomorphism $\phi_\mathcal{B} : V \to \mathbb{R}^n$. If we had picked another basis $\mathcal{B}'$, then then the transition map $\phi_{\mathcal{B}'} \circ \phi_\mathcal{B}^{-1} \in GL(n, \mathbb{R})$. Hence $\phi_{\mathcal{B}'} \circ \phi_\mathcal{B}^{-1}$ is smooth.*

Example (Example 3: Spheres)**.** *We have $S^n_c := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = c^2\}$ for $c > 0$. Let $\phi_i^\pm : \underbrace{U_i^\pm}_{:=\{(x_0,\dots,x_n)\in S^n_c \mid \pm x_i > 0\}} \to B^n_c$. Then $\phi_j^\pm \circ (\phi_i^{pm})^{-1}(y_1, \dots, y_n) = \phi_j^\pm\left(y_1, \dots, \pm\sqrt{c^2 - \sum y_i}, \dots, y_n\right)$, where $(y_1, \dots, y_n) \in B^n_c$.*

$$= \begin{cases} (y_1, \dots, y_n) & i = j \\ (y_1, \dots, \sqrt{c^2 - \sum y_k}, \dots, \hat{y_j}, \dots, y_n) & j > i \\ (y_1, \dots, \hat{y_{j+1}}, \dots, \sqrt{c^2 - \sum y_k}, \dots, y_n) & j < i \end{cases} \tag{1}$$

We conclude $\{U_i^\pm, \phi_i^\pm\}$ is a smooth atlas.

Example (Example 4: Level sets)**.** *Let $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. Fix $c \in \mathbb{R}$. Recall that the set $\Phi^{-1}(c) = \{x \in \mathbb{R}^{n+1} \mid \Phi(x) = c\}$ is called a* ***level set*** *of value c.* ***Suppose*** *that, $\forall p \in \Phi^{-1}(c) : D\underbrace{\Phi(p)}_{=(\partial_{x_0}\Phi(p),\dots,\partial_{x_n}\Phi(p))} \neq 0$. This means that $\exists 0 \leq i \leq n$ s.t. $\partial_{x_i}\Phi(c) \neq 0$. By the* ***implicit function theorem*** *(Lee, Theorem C.40, Course website), there exists a neighborhood U of p such that $U \cap \Phi^{-1}(p) = \{(x_0, \dots, f(x_0, \dots, \hat{x_i}, \dots, x_n), x_n)\}$.*
Let $M = \phi^{-1}(c)$. We define $\hat{\pi}_i : \mathbb{R}^{n+1} \to \mathbb{R}^n, (x_0, \dots, x_n) \mapsto (x_0, \dots, \hat{x_i}, \dots, x_n)$.

$$\{(U, \hat{\pi}_i) \mid U \subset M, \hat{\pi}_i\mid_U \text{ homeomorphism, } \partial_{x_i}\Phi \neq 0 \text{ on } U\}$$

Remains to check the formula:

$$\hat{\pi_j} \circ \hat{\pi_i}^{-1}(y_1, \dots, y_n) = \begin{cases} (y_1, \dots, f, \dots, \hat{y_j}, \dots, y_n) & j > i \\ (y_1, \dots, \hat{y_{j+1}}, \dots, f, \dots, y_n) & i < j \\ (y_1, \dots, y_n) & i = j \end{cases}$$

Remark. *The condition $D\Phi \neq 0$ is very explicit! It is very easy to generate lots of manifolds. For example: $\Phi(x) = \sum \lambda_i x_i^2$*

Example (Example 5: Subset of smooth manifold)**.** *Let M be a smooth manifold. Then $U \subset M$ open, is also a smooth manifold. (Take charts of M and intersect / restrict each chart)*

Example (Example 6: Product of manifolds)**.** *Let M, N be smooth manifolds. Then $M \times N$ is also a smooth manifolds. Take as charts*

$$\{(U \times V, (\phi, \psi)) \mid (U, \phi), (V, \psi) \text{ charts of M,N respectively}\}$$

This takes care of the torus!

Example (Example 7:)**.** *Let's consider $\mathbb{R}$. We define a chart $\mathbb{R} \to \mathbb{R}, x \mapsto x^3$. Observe that*

$$M = (U = \mathbb{R}, U \xrightarrow{id} \mathbb{R})$$

and

$$N = (U = \mathbb{R}, U \xrightarrow{x \mapsto x^3} \mathbb{R})$$

are smooth manifolds, which are different! Since the transition functions between them are not smooth:
Indeed $id \circ (x \mapsto x^3)^{-1} = (x \mapsto x^{\frac{1}{3}})$, which is not smooth!

*This is one to pay **attention** to!*

2.1.3 Smooth maps

Definition. *Let M be a smooth manifold. A map $f : M \to \mathbb{R}^m$ is said to be **smooth**, if for all $p \in M$, there exists a chart (U, ϕ) containing p, such that*

$$f \circ \phi^{-1} : \underbrace{\phi(U)}_{\subset \mathbb{R}^n} \to \mathbb{R}^m$$

is smooth.

Definition. *Let M, N be manifolds. We say $f : M \to N$ is **smooth** if, for all $p \in M$ there exists charts (U, ϕ) with $p \in U \subset M$ and (V, ψ) with $V \subset N$ such that:*

- $V \supset f(U)$
- $\psi \circ f \circ \phi^{-1} : \underbrace{\phi(U)}_{\subset \mathbb{R}^n} \to \mathbb{R}^m$ *is smooth*

manifolds = smooth manifolds as always (unless otherwise stated)

Reality check.

Lemma 2.1. *Smooth maps are continuous.*

Proof. Enough to show that $\forall p \in M$, there exists a neighborhood of p on which $f : M \to N$ is continuous, for f smooth. By definition $\exists (U, \phi), p \in U, (V, \psi), V \subset N$ s.t.
$\psi \circ f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ smooth.
Observe $f = \psi^{-1} \circ (\psi \circ f \circ \phi^{-1}) \circ \phi$ on U. □

Lemma 2.2. *$f : M \to N$ is smooth if and only if each $p \in M$ has a neighborhood U suhc that $f\mid_U$ is smooth.*

Proof. Sheet 03. □

Lemma 2.3 (Properties of smooth maps)**.** *(i) Any constant map $c : M \to N$ is smooth*[a]

(ii) The identity map $id : M \to M$ is smooth

(iii) If $U \circ M$ open, then the inclusion $i : U \to M$ is smooth

(iv) Compositions of smooth functions are smooth

[a]Since it sends M to a point in N

Proof. Sheet 03. □

Definition. *Let M, N be manifolds. A **diffeomorphism** $f : M \to N$ is a smooth map, which is bijective and admits a smooth inverse.*

In particular, diffeomorphisms are homeomorphism!

Example. *$f : \mathbb{R} \to \mathbb{R}, x \mapsto x + 3$ is a diffeomorphism with inverse $x \mapsto x - 3$.*

Example. *Let $A \in GL(n, \mathbb{R})$. Define a map*

$$f_A : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax.$$

This is a diffeomorphism (smooth, since linear) with inverse $f_A^{-1} = f_{A^{-1}}$.

Example. *Let $S_c^n := \{(x_0, \dots, x_n) \mid \sum_{i=0}^n x_i^2 = c^2\} \subset \mathbb{R}^{n+1}$. Given $d > c > 0$, we define a diffeomorphism.*

$$S_c^n \to S_d^n, (x_0, \dots, x_n) \mapsto \frac{d}{c}(x_0, \dots, x_n).$$

Example. *$M = (\mathbb{R}, id), N = (\mathbb{R}, x \mapsto x^3)$. The map $M \to N, x \mapsto x^{\frac{1}{3}}$ is a diffeomorphism. Indeed,*

$$(x \mapsto x^3) \circ (x \mapsto x^{\frac{1}{3}}) \circ id^{-1} = id$$

2.1.4 The category of smooth manifolds

Definition. *Let Man^∞ be the category of smooth manifolds. The objects are the smooth manifolds. The morphisms are the smooth maps.*

Exercise: M, N objects in Man^∞ are isomorphic if and only if they are diffeomorphic.
Observe that there is a forgetful functor: $\mathrm{Man}^\infty \to \mathrm{Man}^0$ by $(M, [\mathcal{A}]) \to M$ and $f : M \to N \mapsto f$.
In general:

- not full
- not essentially surjective

Remark (Hierarchy of categories).
- *for $k = 0, \dots, \infty$, we can consider the category Man^k with objects C^k-Manifolds, and morphisms C^k-maps. for $k \leq l$ there is a forgetful functor $Man^l \to Man^k$*
- *if $k \geq 1$,then the forgetful functor $Man^\infty \to Man^k$ is essentially surjective. This is different from the C^0 case. For this reason, we mainly focus on Man^0, Man^∞. This is a theorem by Whitney*
- *there are other interesting categories: $Man^{\text{Real-analytic}}, Man^{\text{Cplx-analytic}}, \dots$, which both come with a forgetful functor to Man^∞*

Remark (Classification of manifolds (not examinable)).
- *all topological manifolds of dimension ≤ 3 admit a unique smooth structure*
- *S^7, as a topological manifold, admits 15 pairwise non-diffeomorphic smooth structures. These are called **exotic spheres**. They also exist in higher dimensions (Milan-Kervaire?)*
- *$\mathbb{R}^4$ admits uncountably many pairwise non-diffeomorphic smooth structures (Taubes 1980s)*
- *Open problem(**Smooth 4 dimensional Poincaré conjecture**): Prove or disprove: any smooth 4-manifold, which is homeomorphic to S^4 is diffeomorphic to S^4. Most experts believe this is false!*

List of Lectures