Lecture notes on Analysis and Geometry on Manifolds

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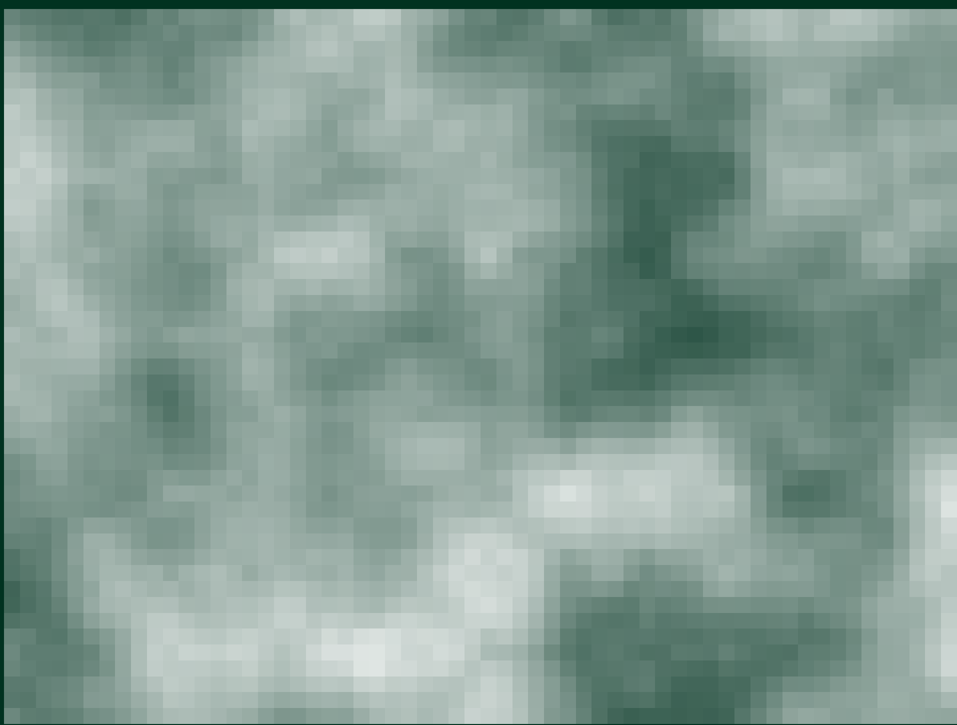

University of Bonn
Winter semester 2024/2025
Last update: October 14, 2024

Contents

Chapter 0: Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Basis: Basis
- Website: https://www.math.uni-bonn.de/~lcote/V3D3_2024.html
- Time slot(s): **Tuesday: 14-16** Nussallee Anatomie B and **Friday: 12-14** GHS
- Exams: Tuesday **11.02.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10 and Friday **21.03.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10
- Deadlines: **Friday before noon**

0.1 Organization

Start of lecture 01 (08.10.2024)

- Four exercise classes, in the break come to the front and sign up.
- First homework is due this Friday
- Exercise sheets are due on Fridays, every week electronically (groups, at most 2)
- No published lecture notes by him!
- 5 Minute break right before the full hour
- Friday after class for questions

0.2 Course overview

He assumes we already know about

- Analysis on $\mathbb{R}^n$
- Basic point set topology

For this class: **smooth manifolds**

- Intersection between analysis and topology
- Exiting: Connections between those two point of views

Main topics:

Topic 00: Topological manifolds

Topic 01: Basic theory of smooth manifolds

Topic 02: Vector fields on smooth manifolds

Topic 03: Tensor calculus and Stokes' theorem

Topic 04: Lie groups, symplectic and Riemannian geometry

Chapter 1: Topological manifolds

1.1 Some point set topology

Some (set theoretical) conventions for the whole course:

- $A \subset B$ means A subset (not necessarily proper!) of B, i.e. $\subset = \subseteq$
- A **neighborhood** of some point $p \in X$ means *an open set* $U \subset X$ *containing* p
- Given $p = (p_1, \ldots, p_n) \in \mathbb{R}^n, r > 0$, $B^n_r(p) := \{(x_1, \ldots, x_n) \mid \sum^2_{x_i - p_i} < r^2\}$. Often while $B_s = B^n_s(0) \subset \mathbb{R}^n$

1.1.1 Locally Euclidean spaces

Definition. *A topological space X is called **locally Euclidean of dimension** $n \geq 0$, if every point of X is contained in a neighborhood homeomorphic to some open subset of $\mathbb{R}^n$.*

Remark. *When we speak of a topological space as being **locally Euclidean**. The dimension is fixed and implicit.*

Definition. *Assume that X is locally Euclidean. A **chart** is a pair U, ϕ, where $U \subset X$, $\phi : U \to \mathbb{R}^n$ is a homeomorphism into its image. Given $p \in X$, we say that U, ϕ is **centered at** p if $p \in U$ and $\phi(p) = 0 \in \mathbb{R}^n$*

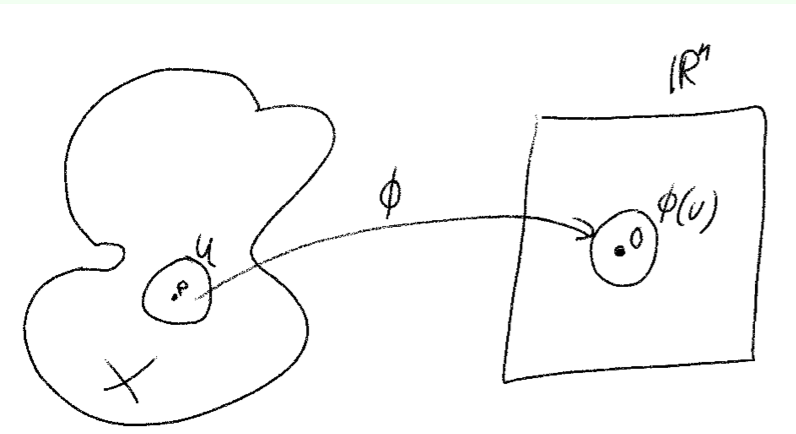

Figure 1.1: Sketch 1.01

Lemma 1.1. *The following are equivalent (TFAE):*

- *X is locally Euclidean*
- *For any $p \in X$, there is a chart U, ϕ centered at p with image $\phi(U) = B_1$*

- *For any $p \in X$, there is a chart U, ϕ centered at p with image $\phi(U) = \mathbb{R}^n$*

Proof. 2. and 3. are equivalent, since $B_1 \simeq \mathbb{R}^n$ are homeomorphic ($B_1^n \ni x \mapsto \frac{x}{1-\|x\|}$)
2. $\implies$ 1. is tautological
1. $\implies$ 2. given $p \in X$, since X is locally Euclidean, there exists **some** chart U, ϕ, $p \in U$.
$psi : U \to \mathbb{R}^n$, homeo onto its image $psi(U) = O \subset \mathbb{R}^n$. By translativity $\mathbb{R}^n \ni x \mapsto x - \psi(p)$, one can assume $\psi(p) = 0 \in \mathbb{R}^n$. By scaling $\mathbb{R}^n(x \mapsto \lambda x, \lambda > 0)$, can assume $B_1 \subset \psi(U)$. Let $U' = \psi^{-1}(B_1)$, then (U, ψ) as claimed. ☐

1.1.2 Hausdorff spaces

Definition. *A topological space X is called Hausdorff, if given any $p_1 \neq p_2$,$p_1, p_2 \in X$, there exist neighborhoods $p_1 \in U_1, p_2 \in U_2$ s.t. $U_1 \cap U_2 = \emptyset$.*

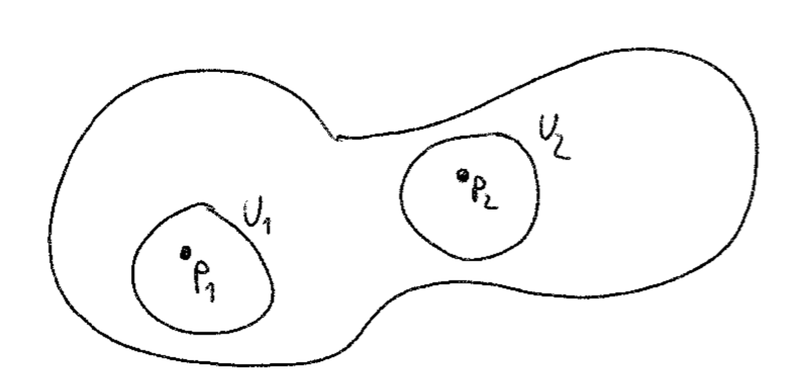

Figure 1.2: Sketch 1.02

Example.
- $\mathbb{R}^n$
- *CW complexes*
- *most reasonable spaces*

Example (Not Hausdorff)**.** *$X = \{0, 1\}$, open subsets $\emptyset, \{0\}, \{0, 1\}$*

Remark. *X is homeomorphic to $\mathbb{R}/\mathbb{R}^*$ (quotient topology), $R^*, (s, x \mapsto sx)$*

Lemma 1.2. *Let X be Hausdorff.*

(a) point sets $\{x\}$ are closed

(b) convergent sequences have unique limits. ($x_n \to p, x_n \to q \implies p = q$)

(c) compact sets are closed

Proof. (c) $\implies$ (a)
For (c): Let $K \subset X$ be compact. Want to show K^c is open. Pick $p \in K^c$. For each $q \in K$, we can choose $U_q \ni q, U_p \ni p : U_q \cap U_p = \emptyset$ Since K is compact, it can be covered by $U_{q_1}, \ldots, U_{q_l}$. Then $\bigcap_{i=1}^{l} U_{q_i}$ is oen and contains p, disjoint, then $\bigcup_{i=1}^{l} U_{q_i} \supset K$.

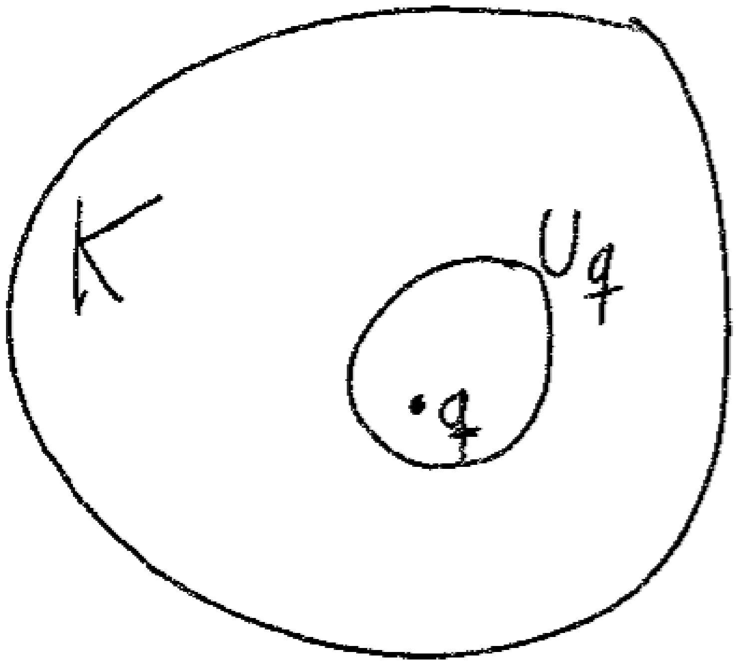

Figure 1.3: Sketch 1.03

(b) Suppose for contradiction that $x_i \to p, x_i \to q$ and $p \neq q$. Since X is Hausdorff, $\exists U \ni p, O \ni q, U \cap O = \emptyset$. But for $N >> 0 x_i \in U, x_i \in O \forall i > N$ $\square$

1.1.3 Basis and covers

Let X be a topological space.

Definition. *A collection $\mathcal{B}$ of subsets of X is called a* ***basis(base)*** *for X, if for any $p \in X$ and any neighborhood $U \ni p$, there exists an element $\mathcal{U} \in \mathcal{B}$ s.t. $p \in \mathcal{U} \subset U$.*

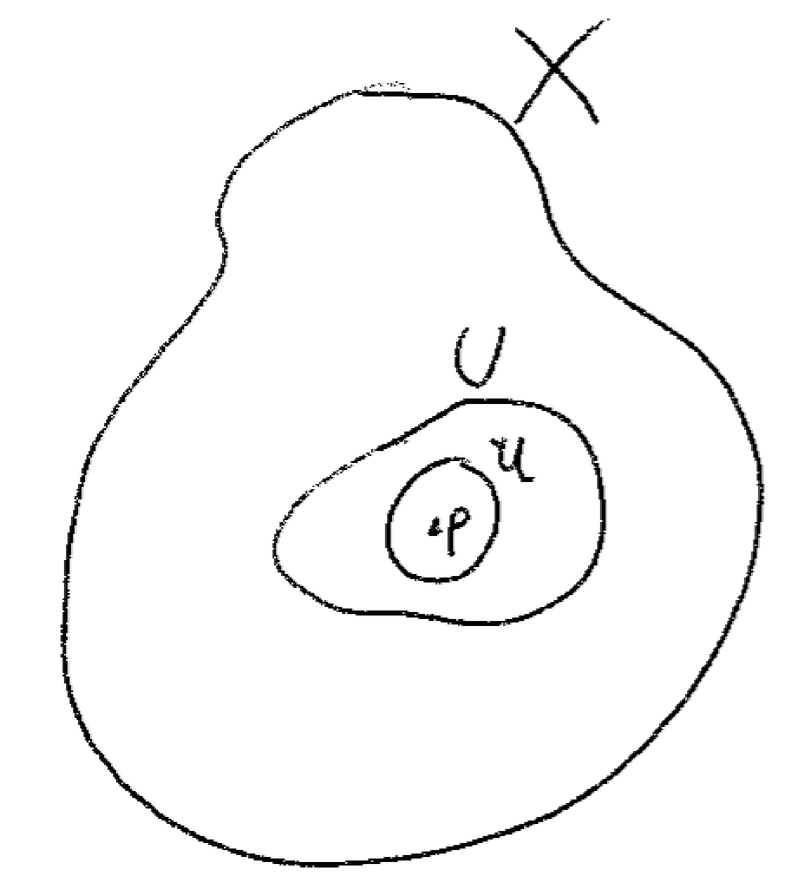

Figure 1.4: Sketch 1.04

Lemma 1.3. *$\mathcal{B}$ is a basis for $X \iff$ every open set of X is a union of elements of $\mathcal{B}$.*

Proof. Trivial. □

Definition. *A topological space X is* ***second-countable*** *if it admits a countable basis.*

Example.
- $\mathbb{R}^n$, $\mathcal{B} = \{B^n_s(p) \mid s \in \mathbb{Q}_+, p = (p_1, \ldots, p_n) \in \mathbb{Q}^n \subset \mathbb{R}^n\}$

Lemma 1.4. *The property of being second-countable is closed under*

(a) subspaces

(b) countable disjoint unions

(c) countable products

Remark. *The property of being second-countable is not closed under arbitrary quotients $q : A \to A/B$. An obvious sufficient conditions is for q to be an open map. (Since it is a pushforward)*

When constructing manifolds via quotients, check that it is still second-coutable!

Lemma 1.5. *If X is second countable, then any open cover of X admits a countable subcover.*

Proof. Let $\mathcal{B}$ be a countable basis for X. Let $\mathcal{C}$ be an open cover. Let $\tilde{\mathcal{B}} \subset \mathcal{B}$ be the collection of basis elements U, which are contained in some $\mathcal{U} \in \mathcal{C}$. Observe (key!) $\tilde{\mathcal{B}}$ is a cover of X. For each $U \in \tilde{\mathcal{B}}$, choose $\mathcal{U}_U \in \mathcal{C}$ such that $U \subset \mathcal{U}_U$. Then $\{\mathcal{U}_U\}$ is a countable subcover of $\mathcal{C}$. □

Definition. *Let X be a topological space. An* ***exhaustion of X by compact subsets*** *is a sequence $\{K_i\}_{i\in\mathbb{N}}$, where $K_i \subset X$ compact and $K_i \subset int(K_{i+1})$ and $\bigcup_{i=1}^{\infty} K_i = X$.*

Recall given $A \subset X$. $\mathrm{int}(A) \coloneqq \{x \in A \mid x \text{ in a neighborhood } U \subset A\}$.

Lemma 1.6. *If X is locally Euclidean, Hausdorff[a] and second countable. Then X admits an exhaustion by compact subsets.*

[a]not needed

Proof. Since X is locally Euclidean, admits a basis $\mathcal{B}$ of open subsets having compact closure.

That is take the close of $B_{\frac{1}{2}} \subset \mathbb{R}^n$

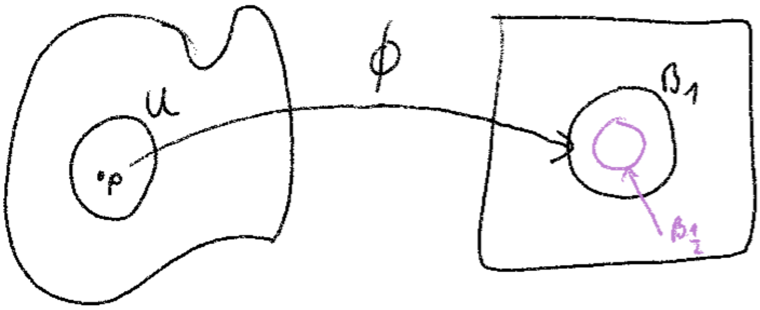

Figure 1.5: Sketch 1.05

By Lemma 1.5, one can extract a countable subcover $\{U_i\}_{i=1}^{\infty}$. Set $K_1 = \overline{U_1}$. Assume that we already constructed $K_1, \ldots, K_k$ such that $U_j \subset K_j$ and $K_{j-1} \subset \operatorname{int}(K_j), j \geq 2$. Since K_k is compact and $K_k \subset X = \bigcup_{i=1}^{\infty} U_i$, then there exists some m_k such that $K_k \subset X = \bigcup_{i=1}^{m_k} U_i$ by compactness. Might as well assume that $m_k \geq k$. Set

$$K_{k+1} = \overline{\bigcup_{i=1}^{m_k} U_i} = \bigcup_{i=1}^{m_k} \overline{U_i}.$$

By construction K_{k+1} is compact, $K_k \subset \operatorname{int}(K_{k+1})$. We get $\{K_j\}_{j=1}^{\infty}$, $U_j \subset K_j$ (because $m_j \geq j$) $\implies \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} K_i$ □

Start of lecture 02 (11.10.2024)

Some remarks and apologies

- Homework 01, apology for paracompactness (if this happens again, feel free to complain)
- Lemma 1.4 should be countable disjoint unions (already corrected in my notes)
- In lemma 1.6, Hausdorff assumption not needed (already corrected in my notes)
- For questions about exercises, email Koen von der Dungen or your TA[1] directly

Definition. *Let X be a topological space. Let $\mathcal{C}$ be a collection of subsets of X. We say that $\mathcal{C}$ is* ***locally finite*** *if for every $x \in X$ there exists a neighborhood $U \ni x$ such that the intersection of U with all but finitely many elements of $\mathcal{C}$ is empty.*

Example (Example for local finiteness)**.** *Take $X = \mathbb{R}$, $\mathcal{C} = \{(i-1, i+1)\}_{i \in \mathbb{Z}}$.*

[1]tutor

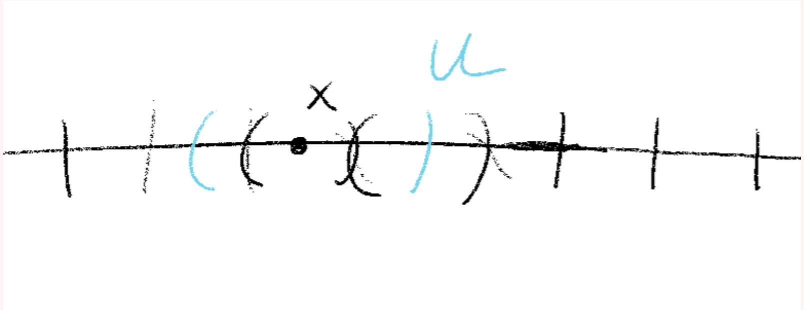

Figure 1.6: Sketch 1.06

Example (Non-example for local finiteness). $X = \mathbb{R}$, $\mathcal{C} = (q-1, q+1)_{q \in \mathbb{Q}}$

Definition. *Let X be a topological space. Let $\mathcal{C}$ be a cover of X. A cover $\mathcal{C}'$ of X is called a **refinement of $\mathcal{C}$**, if for all elements $U \in \mathcal{C}'$, there exists such $V \in \mathcal{C}$: $U \subset V$.*

Example (Example of Refinement). *In the proof of lemma 1.5, we showed that any open cover admits a refinement by basis elements.*

Definition. *A topological space X is called **paracompact** if every open cover admits a locally finite refinement.*

Whats up with the word **para**compact? It's like compact, but weaker! It is necessary that it only admits a locally finite refinement!

Lemma 1.7. *Let X be Hausdorff and suppose that X admits an exhaustion by compact subsets. Then X is paracompact.In fact, we will show that given any basis $\mathcal{B}$ of X, any open cover admits a locally finite refinement by elements of $\mathcal{B}$.*

Proof. By assumption, $\{K_i\}_{i \in \mathbb{N}}$, K_i compact, $K_i \subset \operatorname{int}(K_{i+1}), \bigcup_{i=1}^{\infty} K_i = X$. Let, for $j \in \mathbb{Z} : V_j = K_{j+1} \setminus \int(K_j)$ if $j \leq 0 : K_j = \emptyset$[2].

Careful! There are many definitions of exhaustion by compact sets ...

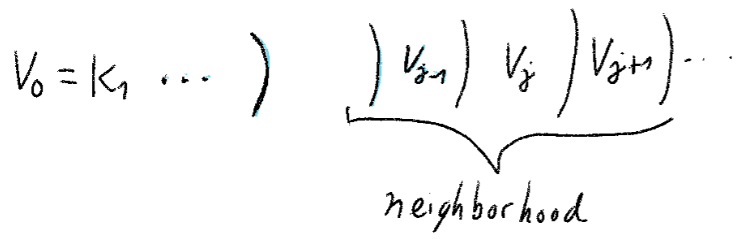

Figure 1.7: Sketch 1.07

Notice:

- V_j is compact, since we take the intersection of a compact set and a closed set. $(\operatorname{int}(K_j)^c$ is closed)
- $\bigcup_{j \in \mathbb{Z}} V_J = X$, since $\bigcup_{j \leq n} = \bigcup_{j \leq n+1} K_j = K_{j+1}$
- The compact sets V_j are intersecting (along their boundary?)
  $V_j \cap V_{j-1} = \partial K_j := K_j \setminus \operatorname{int}(K_j)$

Evidently $\{U_\alpha \cap \operatorname{int}(K_{j+1}) \cap \operatorname{int}(K_{j-1})^c\}_{\alpha \in \mathcal{A}}$ covers $V_j = K_{j+1} - \setminus K_{j-1}^c$, where the $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is an open cover. Since $\mathcal{B}$ is a basis, we can find a refinement of this cover by basis elements. Since V_j are compact, we can extract a finite subcover $\{V_l^j\}_{l=1,\ldots,k_j}$. Let's consider: $\{V_l^j\}_{j \in Z, l=1,\ldots,k_j}$. This subcover works, i.e.

Here we use Hausdorffness

[2] He writes $-$ for $\setminus$

- obviously a cover, since the V_j cover X, obviously a refinement of $\{U_\alpha\}$
- locally finite: given $x \in X, x \in V_j$, hence $x \in \text{int}(K_{K_{j+2}}) \cap K_{j-1}^c =: U$. If $U \cap V_l^k$, then we must have $j-2 \leq k \leq j+2$. But $\{V_l^k\}_{j-2\leq k\leq j+2}$ is finite. □

Corollary 1.8. *If X is locally Euclidean, Hausdorff and second countable $\implies$ X is paracompact.*

Proof. By lemma 1.6 (exhaustion by compact subsets) and lemma 1.7 $\implies$ paracompact. □

Corollary 1.8'. *Let X be Euclidean amd Hausdorff. Then X is second countable iff X has countably many components and X is paracompact.*

Remark. *There are different definitions of manifolds. They differ in either forcing second countability or paracompactness. This lemma shows that there only is a difference if there are uncountably many components.*

Proof. Corollary 1.8 and the bonus homework problem from sheet 01. □

Remark. *Basis elements are open.*

1.2 Topological manifolds

Definition. *A **topological n-manifold** M is a topological space with the following properties:*

(i) M is locally Euclidean (of dimension n)

(ii) M is Hausdorff

(iii) M is second countable

Morally we only really need condition (i). Why do we need the others? For (ii) you will not get a useful theory without it, while (iii) can be replaced by paracompactness (see corollary 1.8').

Definition. *Let Man^0 be the category of topological manifolds with*

1. *objects: topological manifolds*
2. *morphisms: continuous functions*

Remark. *Man^0 full subcategory of Top.*

Remark. *By definition, $M, N \in Man^0$, then M, N are isomorphic iff M, N are homeomorphic.*

1.2.1 Examples of topological manifolds

Example (Spaces isomorphic to R^n)**.** *$\mathbb{R}^n, n \geq 0$ More generally, if V a finite dimensional $\mathbb{R}$-vector space, then V is a topological n-manifold.*

Example. *Any open subset of $\mathbb{R}^n$*

Example (Graphs)**.** *Let $U \subseteq \mathbb{R}^n$ open, let $f : U \to \mathbb{R}^n$ be a continuous function. We set*

$$M := graph(f) := \{(x, y) \in U \times \mathbb{R}^n \mid y = f(x)\}.$$

Then M is a manifold. The map $M \to U$ by $(x, y) \mapsto U$ gives a global chart.

Example (Spheres)**.** *Let $S^n := \{x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$. Then S^n is a manifold. We define charts*

$$\phi_i^\pm : U_i^\pm = \{(x_0, \ldots, x_n) \in S^n \mid \pm x_i > 0\} \to B_1^n(0)$$

by $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, \hat{x}_i, \ldots, x_n) := (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

Here we no longer have a global chart (for topological reasons)

Figure 1.8: Sketch 1.08

Example (spheres'). *Let* $C^n := \partial([-1,1]^{n+1}) = [-1,1]^{n+1} \setminus int([-1,1]^{n+1})$. *Homework:* $C^n \simeq S^n$ *(homeomorphic)*

Example (n-torus). *Let* $\Pi^n := \mathbb{R}^n/\mathbb{Z}^n$ *with the quotient topology. Then this is a manifold (exercise).*

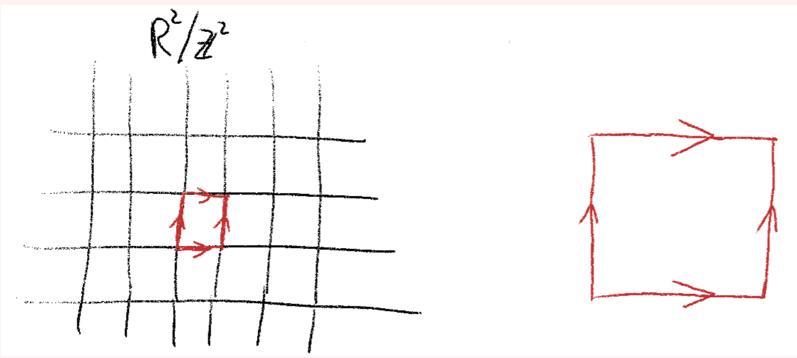

Figure 1.9: Sketch 1.09

Example ($\mathbb{RP}^n := S^n/\{x \sim -x\}$). $\mathbb{RP}^n$ *are also manifolds (called the real projective spaces).*

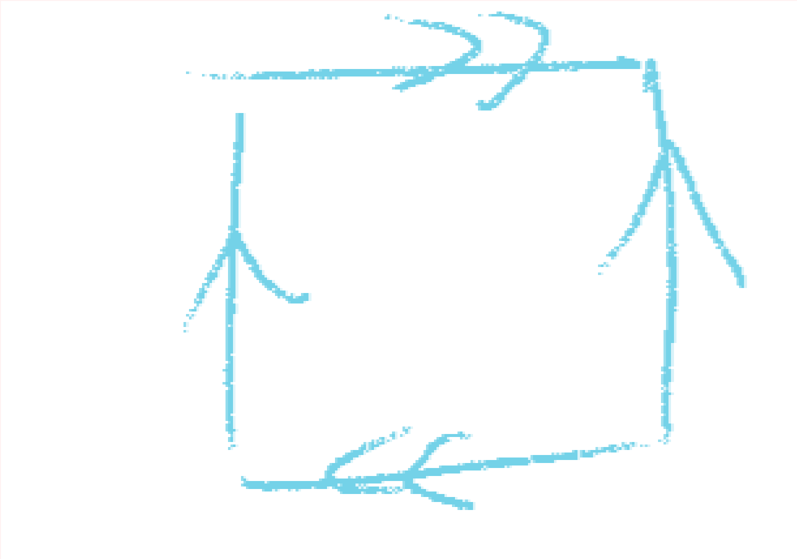

Figure 1.10: Sketch 1.10

Example (Klein bottle)**.**

Remark. *$\mathbb{RP}^2$ or generally $\mathbb{RP}^{2n}$ and the Klein bottle are not orientable.*

1.2.2 Brief interlude: Why do we need Hausdorffness?

- Back in the day (Riemann)
- There is no hope to classify even 1d locally Euclidean, second-countable NOT Hausdorff spaces (See the line with two origins)
- With Hausdorff: Only 1d manifolds are $\mathbb{R}, S^1$ (see website)

Why do we need second countability?

- Subspaces of $\mathbb{R}^n$ are second countable
- We want partitions of unity (paracompactness suffices for that)

1.2.3 Manifolds with boundary

Let $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

Definition. *A **manifold with boundary** is a topological space with the following properties:*

(i) Every point has a neighborhood homeomorphic to an open subset of $\mathbb{H}^n$

(ii) Hausdorff

(iii) second countable

Clearly every manifold is also a manifold with boundary.

Example. *$\mathbb{H}^n$ is a manifold with boundary, but not a manifold. Since for points on the boundary, there are no neighborhoods homeomorphic to Euclidean space.*

Example. $S^n \cap \mathcal{H}^{n+1}, S^n \subset \mathbb{R}^{n+1}$, $[a, b], [0, \infty)$

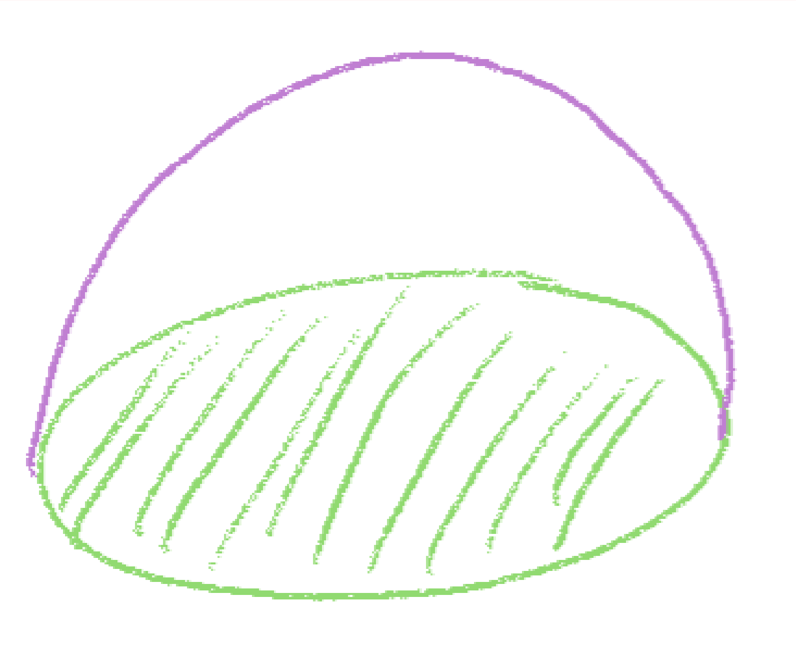

Figure 1.11: Sketch 1.12

Definition. *If M manifold with boundary, we say x is a **boundary point**, if $x \in M \setminus int(M)$ (i.e. it has no neighborhood homeomorphic to Euclidean space?), otherwise x is an iterior point. We let $\partial M := \{boundary\ points\}$.*

List of Lectures