# Lecture notes on Analysis and Geometry on Manifolds

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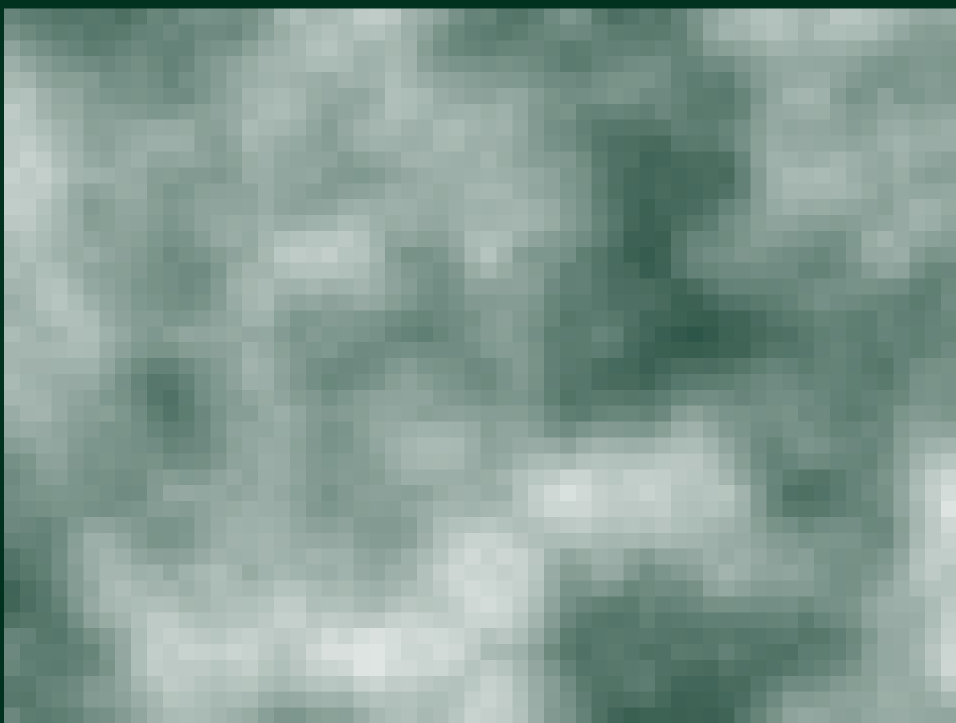

University of Bonn
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# Contents

# Chapter 0: Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Basis: Basis
- Website: https://www.math.uni-bonn.de/~lcote/V3D3_2024.html
- Time slot(s): **Tuesday: 14-16** Nussallee Anatomie B and **Friday: 12-14** GHS
- Exams: Tuesday **11.02.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10 and Friday **21.03.2025, 9-11**, Großer Hörsaal, Wegelerstraße 10
- Deadlines: **Friday before noon**

## 0.1 Organization

Start of lecture 01 (08.10.2024)

- Four exercise classes, in the break come to the front and sign up.
- First homework is due this Friday
- Exercise sheets are due on Fridays, every week electronically (groups, at most 2)
- No published lecture notes by him!
- 5 Minute break right before the full hour
- Friday after class for questions

## 0.2 Course overview

He assumes we already know about

- Analysis on $\mathbb{R}^n$
- Basic point set topology

For this class: **smooth manifolds** based on [2]

I would also recommend [5] and the notes of Gabriel Ong[3], which are also based on this course

- Intersection between analysis and topology
- Exiting: Connections between those two point of views

**Main topics:**

Topic 00: Topological manifolds

Topic 01: Basic theory of smooth manifolds

Topic 02: Vector fields on smooth manifolds

Topic 03: Tensor calculus and Stokes' theorem

Topic 04: Lie groups, symplectic and Riemannian geometry

# Chapter 1: Topological manifolds

## 1.1 Some point set topology

Some (set theoretical) conventions for the whole course:

- $A \subset B$ means A subset (not necessarily proper!) of B, i.e. $\subset = \subseteq$
- A **neighborhood** of some point $p \in X$ means *an open set $U \subset X$ containing $p$*
- Given $p = (p_1, \dots, p_n) \in \mathbb{R}^n, r > 0$, $B_r^n(p) := \{(x_1, \dots, x_n) \mid \sum_{x_i - p_i}^2 < r^2\}$. Often while $B_s = B_s^n(0) \subset \mathbb{R}^n$

### 1.1.1 Locally Euclidean spaces

**Definition.** *A topological space $X$ is called* ***locally Euclidean of dimension*** *$n \geq 0$, if every point of $X$ is contained in a neighborhood homeomorphic to some open subset of $\mathbb{R}^n$.*

**Remark.** *When we speak of a topological space as being* ***locally Euclidean****. The dimension is fixed and implicit.*

**Definition.** *Assume that $X$ is locally Euclidean. A* ***chart*** *is a pair $U, \phi$, where $U \subset X$, $\phi : U \to \mathbb{R}^n$ is a homeomorphism into its image. Given $p \in X$, we say that $U, \phi$ is* ***centered at*** *$p$ if $p \in U$ and $\phi(p) = 0 \in \mathbb{R}^n$*

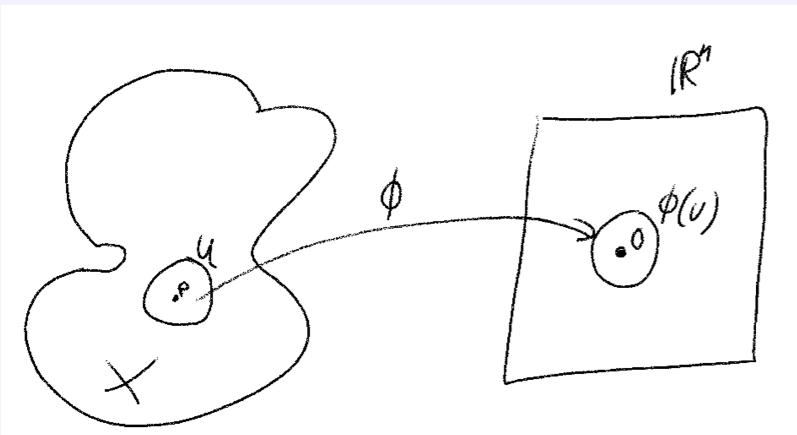


Figure 1.1: Sketch 1.01

**Lemma 1.1.** *The following are equivalent (TFAE):*

- *$X$ is locally Euclidean*
- *For any $p \in X$, there is a chart $U, \phi$ centered at $p$ with image $\phi(U) = B_1$*

- *For any $p \in X$, there is a chart $U, \phi$ centered at $p$ with image $\phi(U) = \mathbb{R}^n$*

*Proof.* 2. and 3. are equivalent, since $B_1 \simeq \mathbb{R}^n$ are homeomorphic ($B_1^n \ni x \mapsto \frac{x}{1-\|x\|}$)
2. $\implies$ 1. is tautological
1. $\implies$ 2. given $p \in X$, since $X$ is locally Euclidean, there exists **some** chart $U, \psi$, $p \in U$.
$\psi : U \to \mathbb{R}^n$, homeo onto its image $\psi(U) = O \subset \mathbb{R}^n$. By translativity $\mathbb{R}^n \ni x \mapsto x - \psi(p)$, one can assume $\psi(p) = 0 \in \mathbb{R}^n$. By scaling $\mathbb{R}^n(x \mapsto \lambda x, \lambda > 0)$, can assume $B_1 \subset \psi(U)$. Let $U' = \psi^{-1}(B_1)$, then $(U, \psi)$ as claimed. ☐

### 1.1.2 Hausdorff spaces

**Definition.** *A topological space $X$ is called Hausdorff, if given any $p_1 \neq p_2$,$p_1, p_2 \in X$, there exist neighborhoods $p_1 \in U_1, p_2 \in U_2$ s.t. $U_1 \cap U_2 = \emptyset$.*

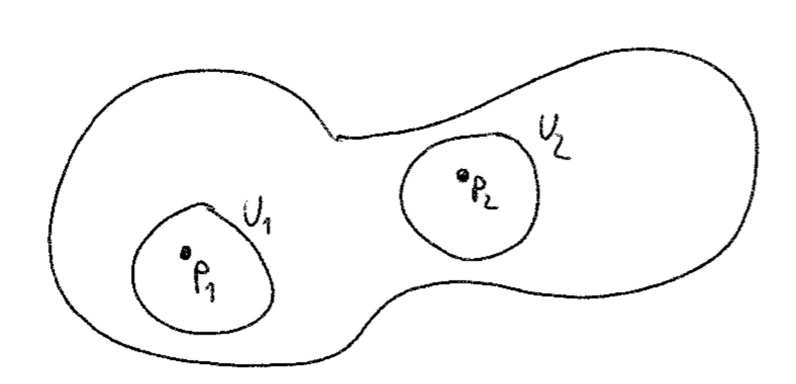


Figure 1.2: Sketch 1.02

**Example.**
- $\mathbb{R}^n$
- *CW complexes*
- *most reasonable spaces*

**Example** (Not Hausdorff)**.** $X = \{0, 1\}$, *open subsets* $\emptyset, \{0\}, \{0, 1\}$

**Remark.** *$X$ is homeomorphic to $\mathbb{R}/\mathbb{R}^*$ (quotient topology), $R^*, (s, x \mapsto sx)$*

**Lemma 1.2.** *Let $X$ be Hausdorff.*

*(a) point sets $\{x\}$ are closed*

*(b) convergent sequences have unique limits. ($x_n \to p, x_n \to q \implies p = q$)*

*(c) compact sets are closed*

*Proof.* (c) $\implies$ (a)
For (c): Let $K \subset X$ be compact. Want to show $K^c$ is open. Pick $p \in K^c$. For each $q \in K$, we can choose $U_q \ni q, U_p \ni p : U_q \cap U_p = \emptyset$ Since K is compact, it can be covered by $U_{q_1}, \ldots, U_{q_l}$. Then $\bigcap_{i=1}^l U_{q_i}$ is oen and contains $p$, disjoint, then $\bigcup_{i=1}^l U_{q_i} \supset K$.

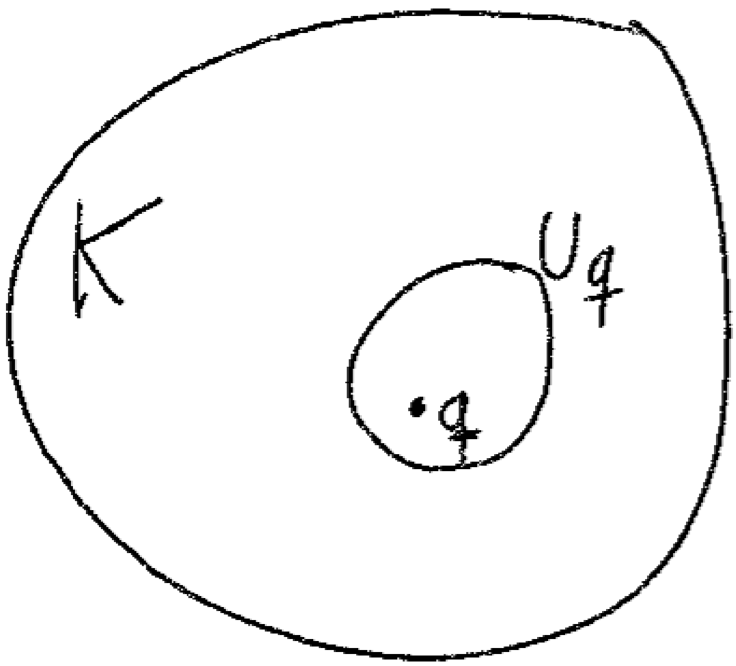


Figure 1.3: Sketch 1.03

(b) Suppose for contradiction that $x_i \to p, x_i \to q$ and $p \neq q$. Since $X$ is Hausdorff, $\exists U \ni p, O \ni q, U \cap O = \emptyset$. But for $N >> 0 x_i \in U, x_i \in O \forall i > N$ □

### 1.1.3 Basis and covers

Let $X$ be a topological space.

**Definition.** *A collection $\mathcal{B}$ of subsets of $X$ is called a **basis(base)** for $X$, if for any $p \in X$ and any neighborhood $U \ni p$, there exists an element $\mathcal{U} \in \mathcal{B}$ s.t. $p \in \mathcal{U} \subset U$.*

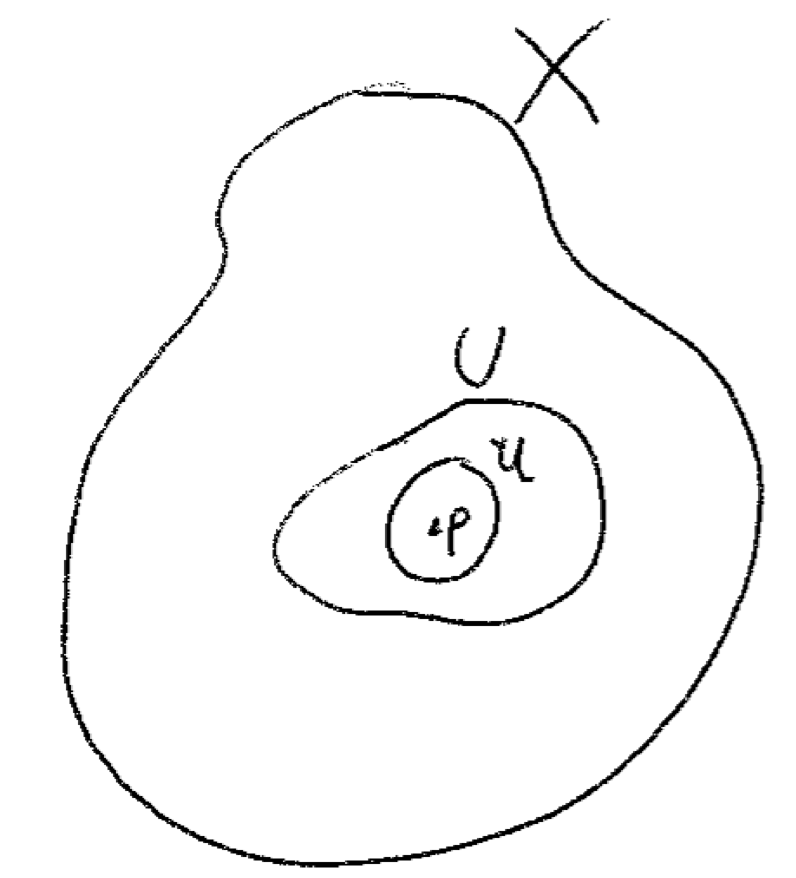


Figure 1.4: Sketch 1.04

**Lemma 1.3.** *$\mathcal{B}$ is a basis for $X \iff$ every open set of $X$ is a union of elements of $\mathcal{B}$.*

*Proof.* Trivial. □

**Definition.** *A topological space $X$ is* ***second-countable*** *if it admits a countable basis.*

**Example.**
- $\mathbb{R}^n$, $\mathcal{B} = \{B^n_s(p) \mid s \in \mathbb{Q}_+, p = (p_1, \ldots, p_n) \in \mathbb{Q}^n \subset \mathbb{R}^n\}$

**Lemma 1.4.** *The property of being second-countable is closed under*

*(a) subspaces*

*(b) countable disjoint unions*

*(c) countable products*

**Remark.** *The property of being second-countable is not closed under arbitrary quotients $q : A \to A/B$. An obvious sufficient conditions is for $q$ to be an open map. (Since it is a pushforward)*

*When constructing manifolds via quotients, check that it is still second-coutable!*

**Lemma 1.5.** *If $X$ is second countable, then any open cover of $X$ admits a countable subcover.*

*Proof.* Let $\mathcal{B}$ be a countable basis for $X$. Let $\mathcal{C}$ be an open cover. Let $\tilde{\mathcal{B}} \subset \mathcal{B}$ be the collection of basis elements $U$, which are contained in some $\mathcal{U} \in \mathcal{C}$. Observe (key!) $\tilde{\mathcal{B}}$ is a cover of $X$. For each $U \in \tilde{\mathcal{B}}$, choose $\mathcal{U}_U \in \mathcal{C}$ such that $U \subset \mathcal{U}_U$. Then $\{\mathcal{U}_U\}$ is a countable subcover of $\mathcal{C}$. □

**Definition.** *Let $X$ be a topological space. An* ***exhaustion of $X$ by compact subsets*** *is a sequence $\{K_i\}_{i\in\mathbb{N}}$, where $K_i \subset X$ compact and $K_i \subset int(K_{i+1})$ and $\bigcup_{i=1}^{\infty} K_i = X$.*

Recall given $A \subset X$. $\mathrm{int}(A) \coloneqq \{x \in A \mid x \text{ in a neighborhood } U \subset A\}$.

**Lemma 1.6.** *If $X$ is locally Euclidean, Hausdorff[a] and second countable. Then $X$ admits an exhaustion by compact subsets.*

[a] not needed

*Proof.* Since $X$ is locally Euclidean, admits a basis $\mathcal{B}$ of open subsets having compact closure.

That is take the close of $B_{\frac{1}{2}} \subset \mathbb{R}^n$

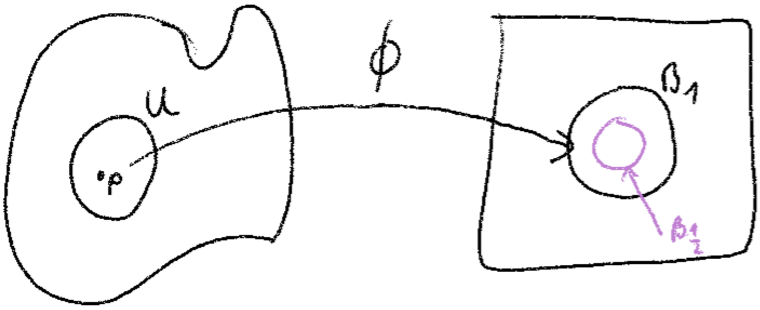


Figure 1.5: Sketch 1.05

By Lemma 1.5, one can extract a countable subcover $\{U_i\}_{i=1}^{\infty}$. Set $K_1 = \overline{U_1}$. Assume that we already constructed $K_1, \ldots, K_k$ such that $U_j \subset K_j$ and $K_{j-1} \subset \operatorname{int}(K_j), j \geq 2$. Since $K_k$ is compact and $K_k \subset X = \bigcup_{i=1}^{\infty} U_i$, then there exists some $m_k$ such that $K_k \subset X = \bigcup_{i=1}^{m_k} U_i$ by compactness. Might as well assume that $m_k \geq k$. Set

$$K_{k+1} = \overline{\bigcup_{i=1}^{m_k} U_i} = \bigcup_{i=1}^{m_k} \overline{U_i}.$$

By construction $K_{k+1}$ is compact, $K_k \subset \operatorname{int}(K_{k+1})$. We get $\{K_j\}_{j=1}^{\infty}$, $U_j \subset K_j$ (because $m_j \geq j$) $\implies \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} K_i$ □

Start of lecture 02 (11.10.2024)

**Definition.** *Let $X$ be a topological space. Let $\mathcal{C}$ be a collection of subsets of $X$. We say that $\mathcal{C}$ is **locally finite** if for every $x \in X$ there exists a neighborhood $U \ni x$ such that the intersection of $U$ with all but finitely many elements of $\mathcal{C}$ is empty.*

**Example** (Example for local finiteness)**.** *Take $X = \mathbb{R}$, $\mathcal{C} = \{(i-1, i+1)\}_{i \in \mathbb{Z}}$.*

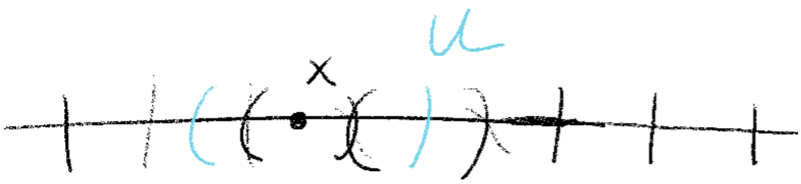


Figure 1.6: Sketch 1.06

**Example** (Non-example for local finiteness)**.** $X = \mathbb{R}$, $\mathcal{C} = (q-1, q+1)_{q \in \mathbb{Q}}$

**Definition.** *Let $X$ be a topological space. Let $\mathcal{C}$ be a cover of $X$. A cover $\mathcal{C}'$ of $X$ is called a **refinement of $\mathcal{C}$**, if for all elements $U \in \mathcal{C}'$, there exists such $V \in \mathcal{C}$: $U \subset V$.*

**Example** (Example of Refinement)**.** *In the proof of lemma 1.5, we showed that any open cover admits a refinement by basis elements.*

**Definition.** *A topological space $X$ is called* ***paracompact*** *if every open cover admits a locally finite refinement.*

Whats up with the word **para**compact? It's like compact, but weaker! It is necessary that it only admits a locally finite refinement!

**Lemma 1.7.** *Let $X$ be Hausdorff and suppose that $X$ admits an exhaustion by compact subsets. Then $X$ is paracompact.In fact, we will show that given any basis $\mathcal{B}$ of $X$, any open cover admits a locally finite refinement by elements of $\mathcal{B}$.*

*Proof.* By assumption, $\{K_i\}_{i\in\mathbb{N}}$, $K_i$ compact, $K_i \subset \mathrm{int}(K_{i+1}), \bigcup_{i=1}^{\infty} K_i = X$. Let, for $j \in \mathbb{Z} : V_j = K_{j+1} \setminus \int(K_j)$ if $j \leq 0 : K_j = \emptyset$[1].

Careful! There are many definitions of exhaustion by compact sets ...

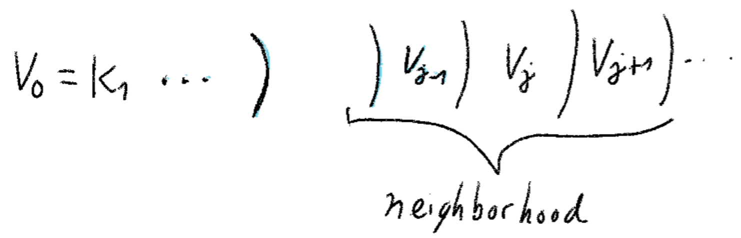


Figure 1.7: Sketch 1.07

Notice:

- $V_j$ is compact, since we take the intersection of a compact set and a closed set. $(\mathrm{int}(K_j)^c$ is closed)
- $\bigcup_{j\in\mathbb{Z}} V_J = X$, since $\bigcup_{j\leq n} = \bigcup_{j\leq n+1} K_j = K_{j+1}$
- The compact sets $V_j$ are intersecting (along their boundary?)
  $V_j \cap V_{j-1} = \partial K_j := K_j \setminus \mathrm{int}(K_j)$

Evidently $\{U_\alpha \cap \mathrm{int}(K_{j+1}) \cap \mathrm{int}(K_{j-1})^c\}_{\alpha\in\mathcal{A}}$ covers $V_j = K_{j+1} - \setminus K_{j-1}^c$, where the $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover. Since $\mathcal{B}$ is a basis, we can find a refinement of this cover by basis elements. Since $V_j$ are compact, we can extract a finite subcover $\{V_l^j\}_{l=1,\ldots,k_j}$. Let's consider: $\{V_l^j\}_{j\in Z, l=1,\ldots,k_j}$. This subcover works, i.e.

Here we use Hausdorffness

- obviously a cover, since the $V_j$ cover $X$, obviously a refinement of $\{U_\alpha\}$
- locally finite: given $x \in X, x \in V_j$, hence $x \in \mathrm{int}(K_{K_{j+2}}) \cap K_{j-1}^c =: U$. If $U \cap V_l^k$, then we must have $j-2 \leq k \leq j+2$. But $\{V_l^k\}_{j-2\leq k\leq j+2}$ is finite. □

**Corollary 1.8.** *If $X$ is locally Euclidean, Hausdorff and second countable $\implies$ $X$ is paracompact.*

*Proof.* By lemma 1.6 (exhaustion by compact subsets) and lemma 1.7 $\implies$ paracompact. □

**Corollary 1.8'.** *Let $X$ be Euclidean amd Hausdorff. Then $X$ is second countable iff $X$ has countably many components and $X$ is paracompact.*

**Remark.** *There are different definitions of manifolds. They differ in either forcing second countability or paracompactness. This lemma shows that there only is a difference if there are uncountably many components.*

*Proof.* Corollary 1.8 and the bonus homework problem from sheet 01. □

**Remark.** *Basis elements are open.*

## 1.2 Topological manifolds

[1]He writes – for \

**Definition.** *A* ***topological $n$-manifold*** *$M$ is a topological space with the following properties:*

*(i) $M$ is locally Euclidean (of dimension $n$)*

*(ii) $M$ is Hausdorff*

*(iii) $M$ is second countable*

Morally we only really need condition (i). Why do we need the others? For (ii) you will not get a useful theory without it, while (iii) can be replaced by paracompactness (see corollary 1.8').

**Definition.** *Let $Man^0$ be the category of topological manifolds with*

1. *objects: topological manifolds*
2. *morphisms: continuous functions*

**Remark.** *$Man^0$ full subcategory of Top.*

**Remark.** *By definition, $M, N \in Man^0$, then $M, N$ are isomorphic iff $M, N$ are homeomorphic.*

### 1.2.1 Examples of topological manifolds

**Example** (Spaces isomorphic to $R^n$)**.** $\mathbb{R}^n, n \geq 0$ *More generally, if $V$ a finite dimensional $\mathbb{R}$-vector space, then $V$ is a topological $n$-manifold.*

**Example.** *Any open subset of $\mathbb{R}^n$*

**Example** (Graphs)**.** *Let $U \subseteq \mathbb{R}^n$ open, let $f : U \to \mathbb{R}^n$ be a continuous function. We set*

$$M := graph(f) := \{(x, y) \in U \times \mathbb{R}^n \mid y = f(x)\}.$$

*Then $M$ is a manifold. The map $M \to U$ by $(x, y) \mapsto U$ gives a global chart.*

**Example** (Spheres)**.** *Let $S^n := \{x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$. Then $S^n$ is a manifold. We define charts*

*Here we no longer have a global chart (for topological reasons)*

$$\phi_i^{\pm} : U_i^{\pm} = \{(x_0, \ldots, x_n) \in S^n \mid \pm x_i > 0\} \to B_1^n(0)$$

*by $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, \hat{x}_i, \ldots, x_n) := (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$*

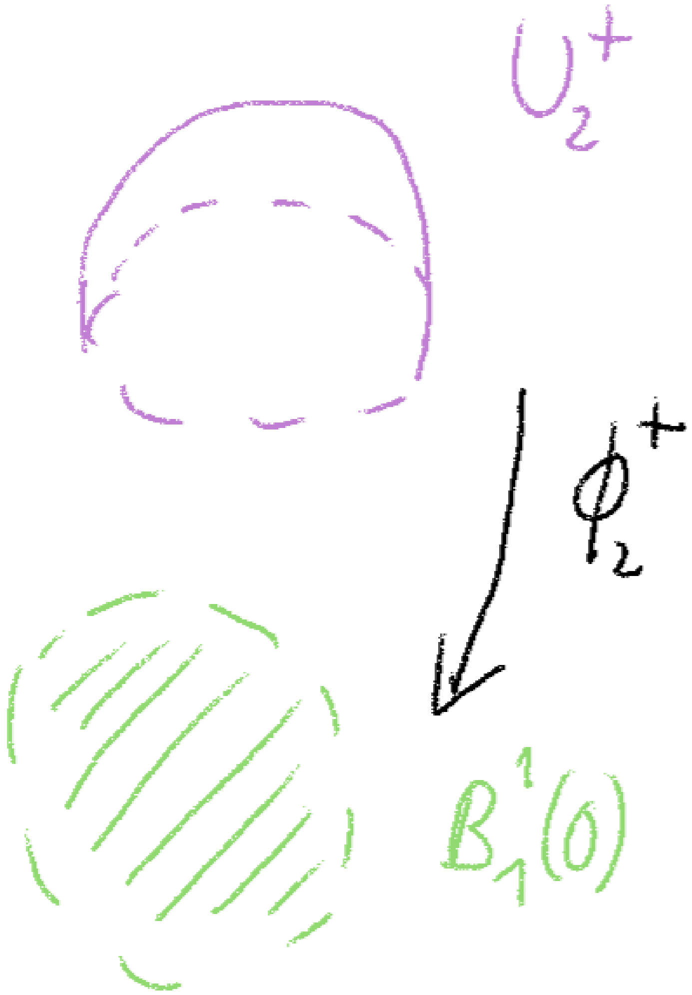


Figure 1.8: Sketch 1.08

**Example** (spheres'). *Let* $C^n := \partial([-1,1]^{n+1}) = [-1,1]^{n+1} \setminus int([-1,1]^{n+1})$. *Homework:* $C^n \simeq S^n$ *(homeomorphic)*

**Example** ($n$-torus). *Let* $\Pi^n := \mathbb{R}^n/\mathbb{Z}^n$ *with the quotient topology. Then this is a manifold (exercise).*

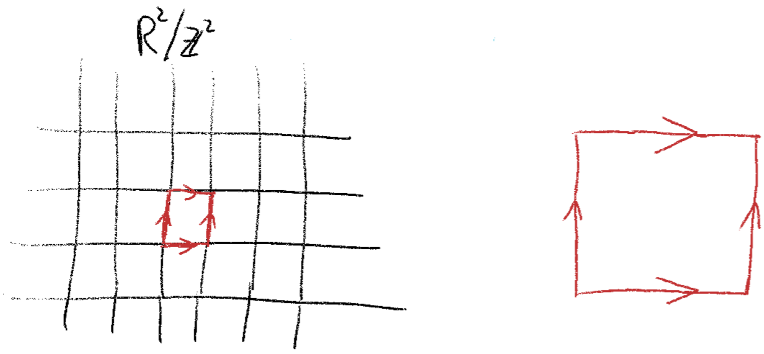


Figure 1.9: Sketch 1.09

**Example** ($\mathbb{RP}^n := S^n/\{x \sim -x\}$). $\mathbb{RP}^n$ *are also manifolds (called the real projective spaces).*

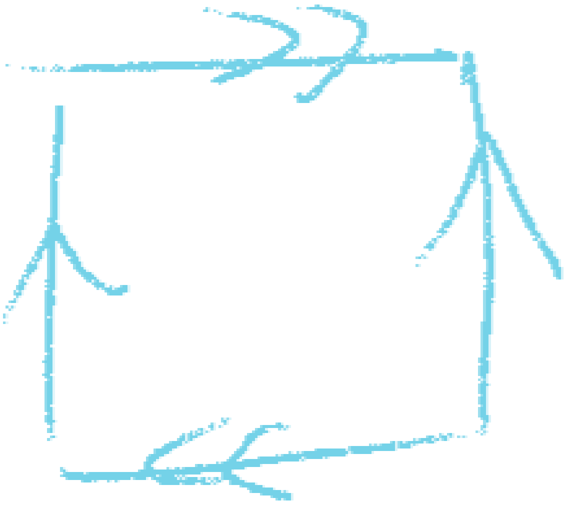

Figure 1.10: Sketch 1.10

**Example** (Klein bottle).

**Remark.** $\mathbb{RP}^2$ *or generally* $\mathbb{RP}^{2n}$ *and the Klein bottle are not orientable.*

### 1.2.2 Brief interlude: Why do we need Hausdorffness?

- Back in the day (Riemann)
- There is no hope to classify even 1d locally Euclidean, second-countable NOT Hausdorff spaces (See the line with two origins)
- With Hausdorff: Only 1d manifolds are $\mathbb{R}, S^1$ (see website)

Why do we need second countability?

- Subspaces of $\mathbb{R}^n$ are second countable
- We want partitions of unity (paracompactness suffices for that)

### 1.2.3 Manifolds with boundary

Let $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

**Definition.** *A* ***manifold with boundary*** *is a topological space with the following properties:*

*(i) Every point has a neighborhood homeomorphic to an open subset of* $\mathbb{H}^n$

*(ii) Hausdorff*

*(iii) second countable*

Clearly every manifold is also a manifold with boundary.

**Example.** $\mathbb{H}^n$ *is a manifold with boundary, but not a manifold. Since for points on the boundary, there are no neighborhoods homeomorphic to Euclidean space.*

**Example.** $S^n \cap \mathcal{H}^{n+1}, S^n \subset \mathbb{R}^{n+1}$, $[a,b], [0,\infty)$

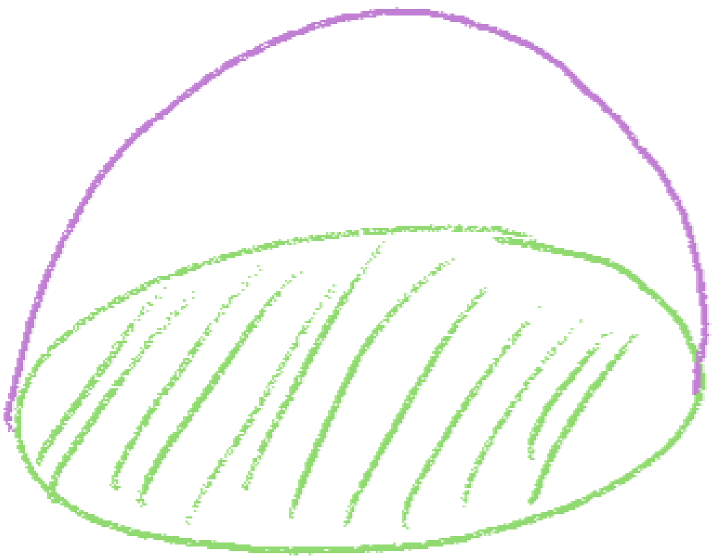

Figure 1.11: Sketch 1.12

**Definition.** *If $M$ manifold with boundary, we say $x$ is a* ***boundary point****, if $x \in M \setminus int(M)$ (i.e. it has no neighborhood homeomorphic to Euclidean space?), otherwise $x$ is an iterior point. We let $\partial M := \{boundary\ points\}$.*

Start of lecture 03 (15.10.2024)

**Remark.** *Most of what he says in the course can be generalized to manifolds with boundary (unless it makes no sense). Those results are only stated (and proofed) for manifolds. it might be a good exercise to go through the notes and generalize the statements to manifolds with boundary.*

### 1.2.4 Elementary topological properties of topological manifolds

Not proved here, but we are welcome to use

- A manifold is connected iff it is path connected
- For manifolds, all forms of compactness (ordinary compactness (every open cover has a finite subcover), limit point compactness, sequential compactness) are equivalent
- All manifolds are metrizable (Urysohn metrization theorem + second countable $\implies$ metrizable)
- Any manifold is homotopy equivalent to a countable CW complex (Milner?) $\pi_k(M)$ are countable

The first two point were proven on the first sheet. The last two use countability

## 1.3 Classification of topological manifolds (proofs are not examinable)

### 1.3.1 Classification of 1-dimensional manifolds

**Theorem 1.9.** *Any connected one dimensional manifold is homeomorphic to*

- $\mathbb{R}^1$ *or*
- $\mathbb{H}^1$

*Proof.* See Course website: [1] in the form of a take-home exam □

**Remark.** *If you allow a boundary, then you also have* $[0,1], [0,1)$.

### 1.3.2 Classification of 2-dimensional manifolds

2-dimensional manifolds are often called surfaces

- $S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid \sum_{i=0}^{2} x_i^2 = 1\}$
- $\Pi^2 \coloneqq \mathbb{R}^2/\mathbb{Z}^2$
- $\mathbb{RP}^2 = S^2/\{x \sim -x\}$

**Construction**(Connected sum of surfaces):

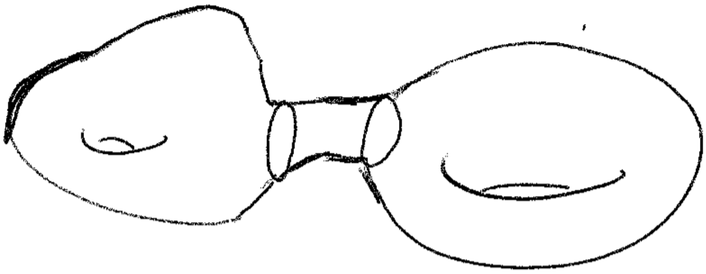

Figure 1.12: Sketch 1.14

Let $M_1, M_2$ be surfaces (i.e. 2-dimensional manifolds). Choose charts $M_i \supset U_i \xrightarrow{\phi_i} B_1 \subset \mathbb{R}^2$. Let $\mathring{M}_i = M_i \setminus \phi_i^{-1}(B_{\frac{1}{2}})$. Let $M_1 \# M_2 \coloneqq \mathring{M}_1 \perp \mathring{M}_2/\sim$, where $X \in \mathring{M}_1 \sim y \in \mathring{M}_2$ if $x \in \phi_1^{-1}(\partial \overline{B_{\frac{1}{2}}})$ and $y = (\phi_2^{-1} \circ \phi_1)(x)$
**Facts:**

- If $M_1, M_2$ are connected, then $M_1 \# M_2$ is well defined up to homeomorphism.
- The operation of connected sum is also well defined for connected $n$-manifolds
- (for the future) The operation of connected sum also works in the smooth category.

**Theorem 1.10** (Classification of surfaces)**.** *Every compact, connected surface is homeomorphic to one of the following manifolds:*

- $S^2$
- $\underbrace{\Pi^2 \# \ldots \# \Pi^2}_{k \text{ times}}$
- $\underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{l \text{ times}}$ *(non-orientable)*

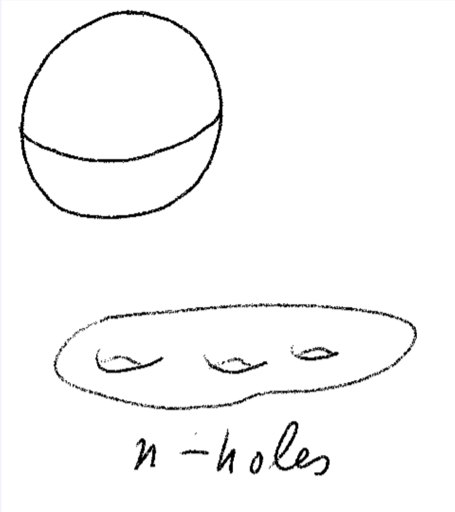


Figure 1.13: Sketch 1.15

**Remark.** *Surfaces are classified by the following invariants:*

*(a) orientability*

*(b) Euler characteristic*

*For later: This classification also works in the smooth category.*

### 1.3.3 Classification of high dimensional manifolds (not examinable at all)

**Poincaré conjecture** (now theorem of G. Perelman (2003), W. Thurston (1980s)): Any compact connected 3 dimensional manifold which is **simply connected** is homeomorphic to $S^3$. This paper is all about PDEs and Ricci flows.
**Generalized Poincaré conjecture**: Any $n$-manifold, which is homotopy equivalent to $S^n$ is homeomorphic to $S^n$. This is true in all dimensions. for $n \geq 5$ Smale in the 1960s, for $n = 4$ Freedsen in the 1980s.
Unlike in dimension 1,2,3 the classification of $n \geq 3$-dimensional manifolds is complicated and not complete.

**Example.** *Any finitely presented group arises as the fundamental group of a compact connected 4-manifold(Which is provably too hard).*

# Chapter 2:
# Smooth manifolds

## 2.1 Basic theory

### 2.1.1 Charts and atlases

**Definition.** *Given $U \subset \mathbb{R}^n$ open, a function $f : U \to \mathbb{R}^m, f = (f_1, \dots, f_m)$ is called* ***smooth*** *(or* $\mathbb{C}^\infty$ *or* ***infinitely differentiable****), if the* ***component functions*** *$f_i$ admit all partial derivatives of all orders and all these partial derivatives are continuous.*

In other words $f$ smooth: $\iff \forall 1 \leq i \leq m, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \partial_\alpha f \coloneqq \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f$ exists.

**Remark.** *Given $k \geq 0$, we can similarly say hat $f$ is* ***$k$-times continuously differentiable*** *and write ($f \in$) and write $f \in C^k(U, \mathbb{R}^m)$, if for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \sum \alpha_i \leq k \; \partial_x^\alpha f_i$ is continuous for all $i$.*

**Definition.** *Let $M$ be a topological manifold. We say that two charts $(U_1, \phi_1)$, $(U_2, \phi_2)$ are* ***smoothly compatible*** *if the map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is smooth. We call $\phi_2 \circ \phi_1^{-1}$ a* ***transition function****.*

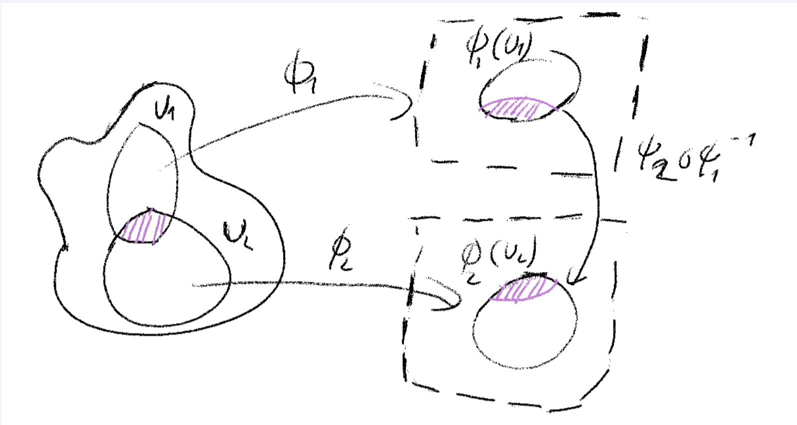


Figure 2.1: Sketch 2.01

**Definition.** *Let $M$ be a topological manifold. An* ***(smooth) atlas*** *$\mathcal{A}$ of $M$ is a collection of charts $\{U_\alpha, \phi_\alpha\}_{\alpha \in \mathcal{A}}$ such that*

- *the $\{U_\alpha\}$ cover $M$*
- *the charts are pairwise smoothly compatible (i.e. for all $\alpha, \beta \in \mathcal{A}(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ are smoothly compatible).*

**Definition.** *We say that two atlases $\mathcal{A}, \mathcal{A}'$ (on a fixed topological manifold) are* ***equivalent****, if their union $\mathcal{A} \cup \mathcal{A}'$ is still an atlas.*

**Fact(Sheet 03):** This defines an equivalence relation.

**Definition.** *A* ***smooth manifold*** *$M = (M, [\mathcal{A}])$ consists of the following data:*

*(i) a topological manifolds $M$*

*(ii) an equivalence class of smooth atlases*

**Remark.**

- *typically, we will designate smooth manifolds by a capital letter, e.g. $M$. But we always mean $(M, [\mathcal{A}])$.* ***Note*** *being a smooth manifolds is* ***extra*** *structure on a topological space, while being a topological manifold is a property*

- *Using Zorn's lemma, it can be shown that any atlas is contained in a* ***unique maximal atlas****. Uniqueness here does not use Zorn's lemma, only existence needs that! Equally well define a smooth manifold to be a topological manifold and a maximal atlas.*

- *$\forall 0 \leq k \leq \infty$, we can define the notion of a $C^k$-atlas, simply by requiring that the transition functions are $C^k$ functions. This yields the definition of $C^k$-Manifolds. Two extreme cases: $C^0$-manifold (topological manifolds) and $C^\infty$-manifolds. Any $k \geq 1$ is not more interesting than $C^\infty$!*

*Typically we are given an atlas, since the maximal atlases have uncountably mani charts, which is why we work with equivalence classes, rather than maximal atlases.*

Start of lecture 04 (18.10.2024)

### 2.1.2 First examples of smooth manifolds

**Example** (Example 1: The cannoical smooth manifold)**.** *$\mathbb{R}^n, n \geq 0$ is* ***canonically*** *a smooth manifold. The* ***canonical atlas*** *is induced by the topological chart $U = \mathbb{R}^n, \phi : U \xrightarrow{id} \mathbb{R}^n$.*

**Example** (Example 2: Another canonical smooth manifold)**.** *Let $V$ be a finite dimensional real vector space . Then $V$ is canonically a smooth manifold. Pick a vector space basis $\mathcal{B}$. This basis induces a homeomorphism $\phi_{\mathcal{B}} : V \to \mathbb{R}^n$. If we had picked another basis $\mathcal{B}'$, then then the transition map $\phi_{\mathcal{B}'} \circ \phi_{\mathcal{B}}^{-1} \in GL(n, \mathbb{R})$. Hence $\phi_{\mathcal{B}'} \circ \phi_{\mathcal{B}}^{-1}$ is smooth.*

**Example** (Example 3: Spheres)**.** *We have $S^n_c := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = c^2\}$ for $c > 0$. Let $\phi_i^\pm : \underbrace{U_i^\pm}_{:=\{(x_0,\ldots,x_n) \in S^n_c \mid \pm x_i > 0\}} \to B^n_c$. Then*

$\phi_j^\pm \circ (\phi_i^{pm})^{-1}(y_1, \ldots, y_n) = \phi_j^\pm \left(y_1, \ldots, \pm\sqrt{c^2 - \sum y_i}, \ldots, y_n\right)$, *where $(y_1, \ldots, y_n) \in B^n_c$.*

$$= \begin{cases} (y_1, \ldots, y_n) & i = j \\ (y_1, \ldots, \sqrt{c^2 - \sum y_k}, \ldots, \hat{y_j}, \ldots, y_n) & j > i \\ (y_1, \ldots, \hat{y_{j+1}}, \ldots, \sqrt{c^2 - \sum y_k}, \ldots, y_n) & j < i \end{cases} \tag{1}$$

*We conclude $\{U_i^\pm, \phi_i^\pm\}$ is a smooth atlas.*

**Example** (Example 4: Level sets)**.** *Let $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. Fix $c \in \mathbb{R}$. Recall that the set $\Phi^{-1}(c) = \{x \in \mathbb{R}^{n+1} \mid \Phi(x) = c\}$ is called a* ***level set*** *of value $c$.* ***Suppose*** *that, $\forall p \in \Phi^{-1}(c) : D \underbrace{\Phi(p)}_{=(\partial_{x_0}\Phi(p), \ldots, \partial_{x_n}\Phi(p))} \neq 0$. This means that $\exists 0 \leq i \leq n$ s.t. $\partial_{x_i}\Phi(c) \neq 0$. By the* ***implicit function theorem*** *(Lee, Theorem C.40, Course website), there exists a neighborhood $U$ of $p$ such that $U \cap \Phi^{-1}(p) = \{(x_0, \ldots, f(x_0, \ldots, \hat{x_i}, \ldots, x_n), x_n)\}$.*

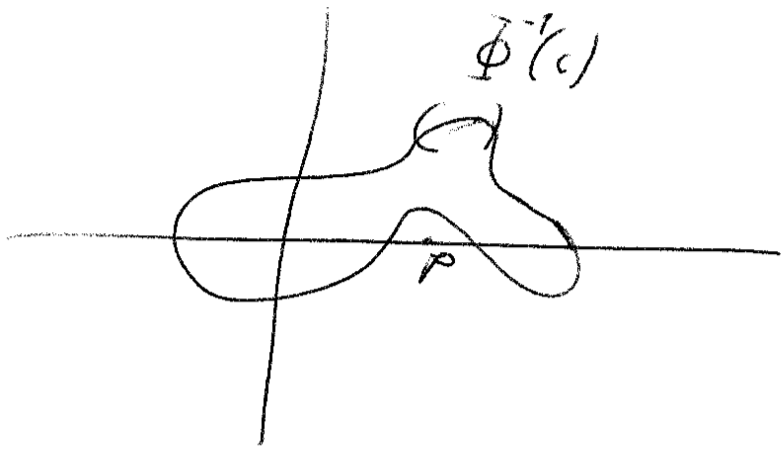


Figure 2.2: Sketch 2.02

*Let $M = \phi^{-1}(c)$. We define $\hat{\pi}_i : \mathbb{R}^{n+1} \to \mathbb{R}^n, (x_0, \dots, x_n) \mapsto (x_0, \dots, \hat{x_i}, \dots, x_n)$.*

$$\{(U, \hat{\pi}_i) \mid U \subset M, \hat{\pi}_i \mid_U \text{ homeomorphism, } \partial_{x_i}\Phi \neq 0 \text{ on } U\}$$

*Remains to check the formula:*

$$\hat{\pi_j} \circ \hat{\pi_i}^{-1}(y_1, \dots, y_n) = \begin{cases} (y_1, \dots, f, \dots, \hat{y_j}, \dots, y_n) & j > i \\ (y_1, \dots, \hat{y_{j+1}}, \dots, f, \dots, y_n) & i < j \\ (y_1, \dots, y_n) & i = j \end{cases}$$

**Remark.** *The condition $D\Phi \neq 0$ is very explicit! It is very easy to generate lots of manifolds. For example: $\Phi(x) = \sum \lambda_i x_i^2$*

**Example** (Example 5: Subset of smooth manifold). *Let $M$ be a smooth manifold. Then $U \subset M$ open, is also a smooth manifold. (Take charts of $M$ and intersect / restrict each chart)*

**Example** (Example 6: Product of manifolds). *Let $M, N$ be smooth manifolds. Then $M \times N$ is also a smooth manifolds. Take as charts*

*This takes care of the torus!*

$$\{(U \times V, (\phi, \psi)) \mid (U, \phi), (V, \psi) \text{ charts of M,N respectively}\}$$

**Example** (Example 7: ). *Let's consider $\mathbb{R}$. We define a chart $\mathbb{R} \to \mathbb{R}, x \mapsto x^3$. Observe that*

*This is one to pay **attention** to!*

$$M = (U = \mathbb{R}, U \overset{id}{\to} \mathbb{R})$$

*and*

$$N = (U = \mathbb{R}, U \overset{x \mapsto x^3}{\to} \mathbb{R})$$

*are smooth manifolds, which are different! Since the transition functions between them are not smooth:*
*Indeed $id \circ (x \mapsto x^3)^{-1} = (x \mapsto x^{\frac{1}{3}})$, which is not smooth!*

### 2.1.3 Smooth maps

**Definition.** *Let $M$ be a smooth manifold. A map $f : M \to \mathbb{R}^m$ is said to be **smooth**, if for all $p \in M$, there exists a chart $(U, \phi)$ containing $p$, such that*

$$f \circ \phi^{-1} : \underbrace{\phi(U)}_{\subset \mathbb{R}^n} \to \mathbb{R}^m$$

*is smooth.*

**Definition.** *Let $M, N$ be manifolds. We say $f : M \to N$ is* ***smooth*** *if, for all $p \in M$ there exists charts $(U, \phi)$ with $p \in U \subset M$ and $(V, \psi)$ with $V \subset N$ such that:*

- $V \supset f(U)$
- $\psi \circ f \circ \phi^{-1} : \underbrace{\phi(U)}_{\subset \mathbb{R}^n} \to \mathbb{R}^m$ *is smooth*

*manifolds = smooth manifolds as always (unless otherwise stated)*

Reality check.

**Lemma 2.1.** *Smooth maps are continuous.*

*Proof.* Enough to show that $\forall p \in M$, there exists a neighborhood of $p$ on which $f : M \to N$ is continuous, for $f$ smooth. By definition $\exists (U, \phi), p \in U, (V, \psi), V \subset N$ s.t.
$\psi \circ f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ smooth.
Observe $f = \psi^{-1} \circ (\psi \circ f \circ \phi^{-1}) \circ \phi$ on $U$. □

**Lemma 2.2.** *$f : M \to N$ is smooth if and only if each $p \in M$ has a neighborhood $U$ such that $f|_U$ is smooth.*

*Proof.* Sheet 03. □

**Lemma 2.3** (Properties of smooth maps)**.** *(i) Any constant map $c : M \to N$ is smooth*[a]

*(ii) The identity map $id : M \to M$ is smooth*

*(iii) If $U \circ M$ open, then the inclusion $i : U \hookrightarrow M$ is smooth*

*(iv) Compositions of smooth functions are smooth*

[a] Since it sends $M$ to a point in $N$

*Proof.* Sheet 03. □

**Definition.** *Let $M, N$ be manifolds. A* ***diffeomorphism*** *$f : M \to N$ is a smooth map, which is bijective and admits a smooth inverse.*

*In particular, diffeomorphisms are homeomorphism!*

**Example.** *$f : \mathbb{R} \to \mathbb{R}, x \mapsto x + 3$ is a diffeomorphism with inverse $x \mapsto x - 3$.*

**Example.** *Let $A \in GL(n, \mathbb{R})$. Define a map*

$$f_A : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax.$$

*This is a diffeomorphism (smooth, since linear) with inverse $f_A^{-1} = f_{A^{-1}}$.*

**Example.** *Let $S^n_c := \{(x_0, \dots, x_n) \mid \sum_{i=0}^n x_i^2 = c^2\} \subset \mathbb{R}^{n+1}$. Given $d > c > 0$, we define a diffeomorphism.*

$$S^n_c \to S^n_d, (x_0, \dots, x_n) \mapsto \frac{d}{c}(x_0, \dots, x_n).$$

**Example.** *$M = (\mathbb{R}, id), N = (\mathbb{R}, x \mapsto x^3)$. The map $M \to N, x \mapsto x^{\frac{1}{3}}$ is a diffeomorphism. Indeed,*

$$(x \mapsto x^3) \circ (x \mapsto x^{\frac{1}{3}}) \circ id^{-1} = id$$

### 2.1.4 The category of smooth manifolds

**Definition.** *Let $Man^\infty$ be the category of smooth manifolds. The objects are the smooth manifolds. The morphisms are the smooth maps.*

Exercise: $M, N$ objects in $\mathrm{Man}^\infty$ are isomorphic if and only if they are diffeomorphic.
Observe that there is a forgetful functor: $\mathrm{Man}^\infty \to \mathrm{Man}^0$ by $(M, [\mathcal{A}]) \to M$ and $f : M \to N \mapsto f$.
In general:

- not full
- not essentially surjective

**Remark** (Hierarchy of categories). • *for $k = 0, \dots, \infty$, we can consider the category $Man^k$ with objects $C^k$-Manifolds, and morphisms $C^k$-maps. for $k \leq l$ there is a forgetful functor $Man^l \to Man^k$*

- *if $k \geq 1$,then the forgetful functor $Man^\infty \to Man^k$ is essentially surjective. This is different from the $C^0$ case. For this reason, we mainly focus on $Man^0, Man^\infty$. This is a theorem by Whitney*
- *there are other interesting categories: $Man^{\text{Real-analytic}}, Man^{\text{Cplx-analytic}}, \dots$, which both come with a forgetful functor to $Man^\infty$*

**Remark** (Classification of manifolds (not examinable)). • *all topological manifolds of dimension $\leq 3$ admit a unique smooth structure*

- *$S^7$, as a topological manifold, admits 15 pairwise non-diffeomorphic smooth structures. These are called **exotic spheres**. They also exist in higher dimensions (Milan-Kervaire?)*
- *$\mathbb{R}^4$ admits uncountably many pairwise non-diffeomorphic smooth structures (Taubes 1980s)*
- *Open problem(**Smooth 4 dimensional Poincaré conjecture**): Prove or disprove: any smooth 4-manifold, which is homeomorphic to $S^4$ is diffeomorphic to $S^4$. Most experts believe this is false!*

Start of lecture 05 (22.10.2024)

### 2.1.5 Smooth manifolds with boundary

**Definition.** *A function $f : \mathbb{H}^n \supset U \to \mathbb{R}^k$ is **smooth** if every $p \in U$ admits an open neighborhood $p \in U_p \subset \mathbb{R}^n$ on which $f$ extends to a smooth function. (i.e. there exists $\tilde{f}_p : U_p \to \mathbb{R}^k, \tilde{f}_p$ smooth and $\tilde{f}_p |_{\mathbb{H}^n \cap U} = f$)*

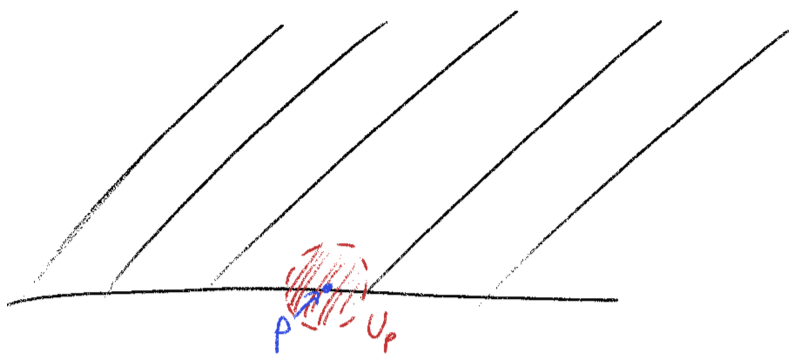


Figure 2.3: Sketch 2.03

**Example.** $n = 1, \mathbb{H}^1 = [0, \infty), f(x) = x^2$

**Example** (Non-Example). $n = 1, \mathbb{H}^1 = [0, \infty), f(x) = \sqrt{x}$ *has no smooth extension to* 0*, since the derivative goes to* $\infty$.

Give a topological manifold with boundary, we can define unproblematically the notions of

- smoothly compatible charts: $(U, \phi) : M \to \mathbb{H}^n$, $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap u_\beta) \to \mathbb{H}^n$
- smooth atlases

**Definition.** *A smooth manifold with boundary $M = (M, [\mathcal{A}])$ is the data of*

- *a topological manifold with boundary*

- *an equivalence class of atlases*

**Remark.** *Every smooth manifold is a smooth manifold with boundary. This is an enlargement of $Man^\infty$.*

*Similarly we cna generalise even more to manifolds with corners ...*

## 2.2 Partitions of unity

### 2.2.1 Preparatory lemmas

This section is technical, but also very important!

**Lemma 2.4.** *The function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

*is smooth.*

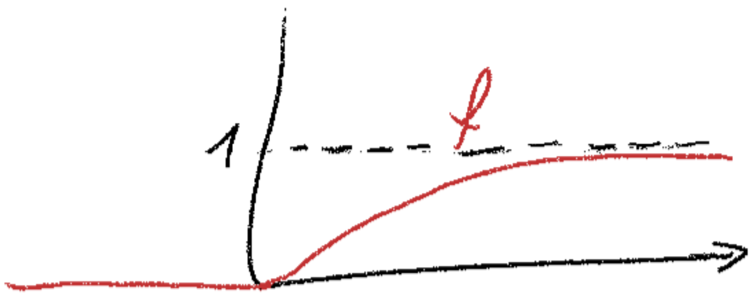


Figure 2.4: Sketch 2.04

*Proof.* It is enough to proof, that $f$ has well defined derivatives of all orders, since $f$ is a function on $\mathbb{R}$.
$f^0 = f$, for $k \geq 1$, assume

1. $f^{(k-1)}$ exists
2. $f^{(k-1)}\mid_{(-\infty,0]} = 0$
3. $f^{(k-1)}\mid_{(-\infty,0]}(t) = P_{k-1}(\frac{1}{t})e^{-\frac{1}{t}}$ for some polynomial $P_{(k-1)}$.

Clearly this holds for $k = 1$.
We have

$$\begin{aligned} \lim_{t\to 0^+} \frac{f^{(k-1)}(t) - f^{(k-1)}(0)}{t} &= \lim_{t\to 0^+} \frac{f^{(k-1)}(t)}{t} \\ &= \lim_{t\to 0^+} P_{(k-1)}(\frac{1}{t})\frac{1}{t}e^{-\frac{1}{t}} \\ &= \lim_{x\to\infty} P_{(k-1)}(x)\cdot x \cdot e^{-x} = 0 \end{aligned}$$

Therefore $f^{(k-1)}$ is differentiable at the origin, the derivative $f^{(k-1)'}(0) = 0.$ and $f^{(k-1)}\mid_{(-\infty,0]} = 0$. Therefore $f^{(k-1)}$ is differentiable. Therefore we only have to check 3., which only takes place on $\mathbb{R}_+$!
Finally
$f^{(k-1)}\mid_{(0,\infty)}(t) = P_{(k-1)}(\frac{1}{t})e^{-\frac{1}{t}} \implies P'_{(k-1)}(\frac{1}{t})\left(\frac{-1}{t^2}e^{-\frac{1}{t}} + P_{(k-1)}(\frac{1}{t})e^{-\frac{1}{t}}\right) =: P_{(k)}(\frac{1}{t})e^{-\frac{1}{t}}$. □

**Lemma 2.5.** *Fix real numbers $r_1 < r_2$. Then there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that*

1. *$h \equiv 1$ on$(-\infty, r_1]$*

2. $0 < h < 1$ *on* $r_1, r_2$

3. $h \equiv 0$ *on* $[r_2, \infty)$

*Proof.* $h(t) := \frac{f(s_2-t)}{f(s_2-t)+f(t-s_1)}$, since the denominator never goes to 0. □

**Lemma 2.6** (Existence of **cutoff functions**). *Given* $0 < r_1 < r_2$*, there exists a smooth function* $H : \mathbb{R}^n \to \mathbb{R}$ *such that*

1. $H \equiv 1$ *on* $\overline{B_{r_1}}$

2. $0 < H < 1$ *on* $B_{r_2} \setminus \overline{B_{r_1}}$

3. $H \equiv 0$ *on* $\mathbb{R}^n \setminus B_{r_2}$

*Proof.* Set $H(x) := h(|x|)$, where $h$ is defined as in lemma 2.5. (Recall: $|x| := \sqrt{x_1^2 + \cdots + x_n^2}$). Then $H$ is smooth, since it is a composition of smooth functions on $\mathbb{R}^n \setminus \overline{B_{r_1}}$ and constant on $\overline{B_{r_1}}$. □

### 2.2.2 Partitions of unity

**Definition.** *Given a topological space* $X$ *and a function* $f : X \to \mathbb{R}$*, the* ***support*** *of* $f$ *is the set*

$$supp(f) := \overline{\{x \in X \mid f(x) \neq 0\}} \subset X$$

**Example.** *If* $f : \mathbb{R} \to \mathbb{R}$ *has the form* $f(x) = a_0 + a_1 x + \ldots, a_n x^n \implies supp(f) = \mathbb{R}$*. In fact, by Taylor's theorem, if* $f$ *analytic, then* $supp(f)$ *either* $\mathbb{R}$ *or* $\emptyset$*. In contrast, the function* $h : \mathbb{R} \to \mathbb{R}$ *defined in lemma 2.5 has support* $(-\infty, r_2] \subsetneq \mathbb{R}$.

**Definition.** *Let* $M$ *be a smooth manifold. Let* $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ *be an open cover. A* ***partition of unity subordinate to the cover*** *is the data of a collection of smooth functions* $\{\psi_\alpha\}_{\alpha \in \mathcal{A}}, \psi_\alpha : M \to \mathbb{R}$ *such that*

*(1)* $0 < \psi_\alpha < 1$

*(2)* $supp(\psi_\alpha) \subset U_\alpha$

*(3)* $\{supp(\psi_\alpha)\}_{\alpha \in \mathcal{A}}$ *is locally finite*

*(4)* $\sum_{\alpha \in \mathcal{A}} \psi_\alpha \equiv 1$

**Remark.** *There is an analogous notion in the category Top,*$Man^0$*,* $Man^k$*, etc.,...*

**Example.** $M = \mathbb{R}$, $U_1 = (-\infty, r_2 + 1), U_2 = (r_1 - 1, \infty)$*, where* $r_1 < r_2$ *as in lemma 2.5. Similarly let* $h$ *as in lemma 2.5. and set* $\psi_1 = h, \psi_2 = 1 - h$

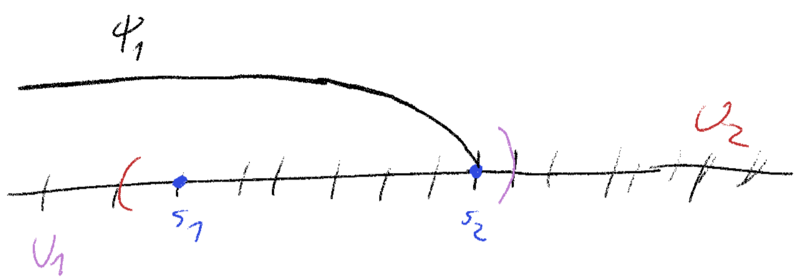


Figure 2.5: Sketch 2.05

**Theorem 2.7** (Existence of partitions of unity). *Let* $M$ *be a smooth manifold. Let* $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ *be an open cover. Then there exists a partition of unity subordinate to this cover.*

**Remark.** *The same theorem works in Top, $Man^0, Man^k$. It will not work in $Man^{Analytic}$, $Man^{Cplx\text{-}Analytic}$, Varieties $/\mathbb{C}$.*

*Proof.* **Step 1: Construction of the $V_i$** An open supset $U \subset M$ is called a **regular coordinate ball** if there exists $\overline{U} \subset \tilde{U}, (\tilde{U}, \tilde{\phi})$ a chart such that $\tilde{\phi}(U) = B_{r_1}, \tilde{\phi}(\tilde{U}) = B_{r_2}$.

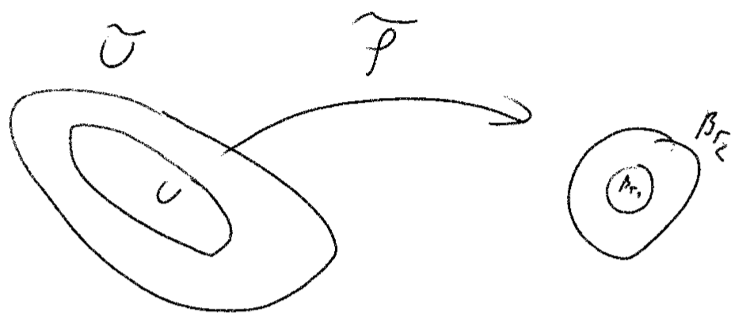

Figure 2.6: Sketch 2.06

By lemma 1.6 $M$ admits an exhaustion by compact sets. By lemma 1.7, given any basis, any open cover, one can find a locally finite, countable basis refinement of this cover by basis elements.
Claim: {regular coordinate balls whose closure is contained in some $U_\alpha$} basis of $M$

The claim is easy to verify

These tree points imply that $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ admits a countable, locally finite refinement by regular coordinate balls $\{V_i\}_{i \in I}$.
By sheet 2, exercise 1 (a) $\{\overline{V_i}\}$ is still locally finite.
**Step 2: Construction of the $f_i$** For each $V_i \exists V_i \supset \tilde{V}_i, \tilde{\phi}_i : \tilde{V}_i \to \mathbb{R}^n$ such that $\tilde{\psi}_i(V_i) = B_{r_1^i}, \tilde{\psi}_i(\tilde{V}_i) = B_{r_2^i}$ with $0 < r_1^i < r_2^i$, $\tilde{V}_i \subset U_\alpha$ for some $\alpha$. Using lemma 2.6, let $H_i : \mathbb{R}^n \to \mathbb{R}$ be a cutoff function, i.e. $H_i \mid_{B_{r_1}} > 0, H = 0$ on $\mathbb{R} \setminus B_{r_1^i}$. Let us set

$$f_i : M \to \mathbb{R}, f_i = \begin{cases} H_i \circ \tilde{\phi}_i & \text{on } \tilde{V}_i \\ 0 & M \setminus \overline{V_i} \end{cases}$$

Finging $\tilde{V}_i$ s.t. $\tilde{V}_i \subset U_\alpha$ is the reason we considered regular coordinate balls whose **closure is contained** in some $U_\alpha$

**Step 3: Construction of the $g_i$** Let us set $f = \sum_{i \in I} f_i$. This is well defined by local finiteness of the $\overline{V_i}$ Note also that $f > 0$.. We set $g_i = f_i / f$. Then clearly we have $0 \leq g_i \leq 1, \sum_{i \in I} g_i \equiv 1$
**Step 4: Reindexing and conformation** Since $\tilde{V}_i \subset U_\alpha$, for some $\alpha$, we can choose for each $i \in I, \alpha(i) \in \mathcal{A}$ s.t. $V_i \in U_{\alpha(i)}$. Let us set

Here the empty sum is 0

$$\psi_\alpha := \sum_{i \mid \alpha = \alpha(i)} g_i$$

Observe for **(2)**:

$$\operatorname{supp}(\psi_\alpha) = \overline{\bigcup_{\alpha(i)=\alpha} V_i} \overset{\text{Exercise 2.1}}{=} \bigcup_{\alpha(i)\alpha} \overline{V_i} \subset U_\alpha$$

We still have $0 \leq \psi_\alpha \leq 1$, which is **(1)**
and $\operatorname{supp}(\psi_\alpha)$ are locally finite: for each $p \in M$, since $\{\overline{V_i}\}$ locally finite, there exists a neighborhood $U_p$ of $p$ which only intersects finitely many of the $\{\overline{V_i}\}$, call them $V_1, \ldots, V_k$. Then the only $\psi_\alpha$ which have a chance of being non-zero must satisfy $\alpha \in \{\alpha(1), \ldots, \alpha(k)\}$ (this is **(3)**).
Lastly

$$\sum_{\alpha \in \mathcal{A}} \psi_\alpha = \sum_\alpha \left( \sum_{i : \alpha = \alpha(i)} g_i \right) = \sum_{i \in I} g_i \equiv 1,$$

which confirms **(4)**.

□

Start of lecture 06 (25.10.2024)

### 2.2.3 Applications of partitions of unity

**Definition.** *Let $X$ be a topological space. Let $A \subset X$ be closed, $U \subset X$, $A \subset U$ be open. A* ***bump function for** $A$* *supported in $U$ is a function*

$$\phi : X \to \mathbb{R}$$

*such that $\psi\mid_A \equiv 1, supp(\phi) \subset U$.*

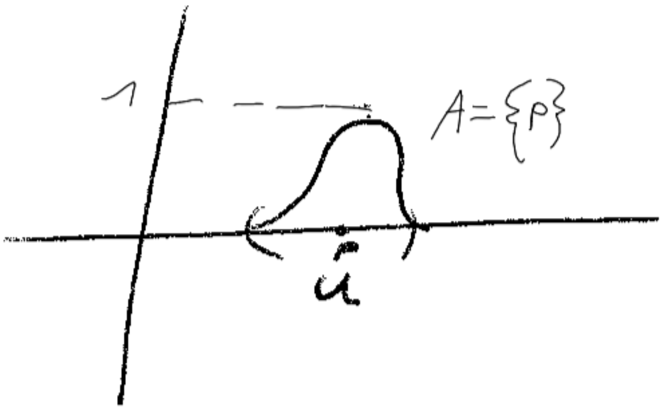


Figure 2.7: Sketch 2.07

**Proposition 2.8.** *Let $M$ be a smooth manifold. Fix $A \subset M$ closed, $U \subset M, A \subset U \subset M$ open. Then there exists a smooth bump function for $A$ supported in $U$*

*Proof.* Let $V = M \setminus A$. Then $\{U, V\}$ is a covering and by theorem 2.7, there exist $\{\Psi_U, \Psi_V\}$ partitions of unity subordinate to this cover. Now $\Psi_U$ does the job. □

**Definition.** *Let $M, N$ be smooth manifolds. Let $A \subset M$ be closed. We say that $f : A \to N$ is smooth if it admits a smooth extension in a neighborhood of each point $p \in A$.*

*I.e. for any $p \in A$ there exists $U_p \ni p$, a smooth function $\tilde{f}_p : U_p \to N$ s.t. $\tilde{f}_p\mid_{U_p \cap A} = f\mid_{U_p \cap A}$*

**Proposition 2.9.** *Let $M$ be a smooth manifold. Let $A \subset M$ be closed and $f : A \to \mathbb{R}^k, k \geq 0$ be smooth. Then for any open $U \subset M, A \subset U$, there exists $\tilde{f} : M \to \mathbb{R}^k$, such that $\tilde{f}\mid_A = f$ and $supp(\tilde{f}) \subset U$*

**Remark.** *This would be* ***false*** *if we replaced $\mathbb{R}^k$ by an arbitrary smooth manifold $N$. E.g. take $\mathbb{R}^2 \hookleftarrow A = S^1 \overset{f=id}{\to} S^1$*

*Proof.* For each $p \in A$, choose a neighborhood $p \in W_p \subset U$, $\tilde{f}_p : W_p \to \mathbb{R}^k$ smooth extension of $f_{|A \cap W_p}$. Then observe that $\{W_p\}_{p\in A} \cup (M - A)$ forms an open cover of $M$. $\{\psi_p\}_{p\in A} \cup \psi_0$ be a partition of unity subordinate to the cover. Now we set $\tilde{f} = \sum_{p\in A} \psi_p \tilde{f}_p$. By local finiteness $\tilde{f}$ is smooth. Also

We maybe need $\overline{W_p} \subset U$? Prob. not?

$$\tilde{f}_{|_A} = \sum_{p\in A} \psi_{p|_A} \underbrace{\tilde{f}_{p|_A}}_{=f}$$
$$= f \sum_{p\in A} \psi_{p|_A} = f_{|_A} \cdot 1 = f_{|_A}.$$ □

**Definition.** *Let $X$ be a topological space. An* ***exhaustion function*** *$f : X \to \mathbb{R}$ is a continuous function such that $\forall C \in \mathbb{R}, f^{-1}(-\infty, c]$ is compact.*

*If $X$ is compact every $f$ is an exhaustion function ...*

**Example.** $X = \mathbb{R}, f : \mathbb{R} \to \mathbb{R}, x \overset{f}{\mapsto} x^2$

**Example** (NON-EXAMPLE)**.** $X = \mathbb{R}, f(x) = x$

**Proposition 2.10.** *Every smooth manifold admits a smooth exhaustion function.*

*Proof.* Pick a countable partition of unity $\{U_i\}_{i\in N_+}$ by open subsets having compact closure[1].
Let $\{\Psi_i\}_{i\in\mathbb{N}_+}$ be a subordinate partition of unity. Let $f \coloneqq \sum_{i\in\mathbb{N}_+} i\psi_i$.
Observe that for any $c\in\mathbb{R}$, $c<N\in\mathbb{N}$ that

$$f^{-1}(-\infty,c]\subset f^{-1}(-\infty,c]\subset\bigcup_{i=1}^{N}\overline{U_i}$$

Why $f^{-1}(-\infty,c]\subset\bigcup_{i=1}^{N}\overline{U_i}$? Let $q\notin\bigcup_{i=1}^{N}\overline{U_i}$. Then

$$\begin{aligned} f(q) &= \underbrace{\sum_{i=1}^{N} i\psi_i(q)}_{=0} + \sum_{i=N+1}^{\infty} i\psi_i(q) \\ &\geq (N+1)\sum_{i=N+1}^{\infty}\psi_i(q) = (N+1)\underbrace{\sum_{i=1}^{\infty}\psi_i(q)}_{=1} \\ &= N+1 \end{aligned}$$

□

**Proposition 2.11.** *Let $M$ be a smooth manifold. Let $A\subset M$ be a closed subset. Then there exists a smooth function*

$$f: M\to\mathbb{R}, f^{-1}(0)=A$$

*In fact, the prove shows one can assume $f\geq 0$*

E.g. take $M=\mathbb{R}, A=$ Cantor set, shows that this is non-trivial.

*Proof.* Assume $M=\mathbb{R}^n$ (general case: Sheet 04).
Choose a countable cover of $\mathbb{R}^n\setminus A$ by balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ with $r_i<1$. By Lemma 2.6 there exists a cutoff function

$$H:\mathbb{R}^n\to\mathbb{R}$$

s.t. $H\equiv 1$ on $\overline{B_{\frac{1}{2}}(0)}$ and $0<H<1$ on $B_1(0)\setminus\overline{B_{\frac{1}{2}}(0)}$ and $H\equiv 0$ on $R^n\setminus B_1(0)$. For each $i\geq 1$ let $C_i\gg 1$ be large enough so that

$$C_i>\sup\{\partial_x^{\alpha}H \mid \alpha=\overbrace{(\alpha_1,\dots,\alpha_n)}^{\in\mathbb{N}^n}, |\alpha|\leq i\}$$

Let

$$f\coloneqq\sum_{i=1}^{\infty}\frac{r_i^i}{2^ic_i}H\left(\frac{x-x_i}{r_i}\right).$$

We need to argue that $f$ is smooth. Observe that, since $r_i<1$ $\frac{r_i}{2^ic_i}H\left(\frac{x-x_i}{r_i}\right)\leq\frac{1}{2^i}$ It follows from Analysis 2 that $f$ is continuous. To prove that $f$ is smooth assume for $k\geq 1$ that all partial of order $k<1$ exist and are continuous. If $|\alpha|=k$, then

$$\partial^{\alpha}\frac{r_i^i}{2^iC_i}H\left(\frac{x-x_i}{r_i}\right)=\frac{r_i^{i-k}}{2^iC_i}\partial^{\alpha}H\left(\frac{x-x_i}{r_i}\right)$$

If $i>k$, then

$$\left|\frac{r_i^{i-k}}{2^iC_i}\partial^{\alpha}H\left(\frac{x-x_i}{r_i}\right)\right|<\frac{1}{2^i}$$

Again follows by Analysis 2 that $\partial^{\alpha}f$ exists and equals $\sum\partial^{\alpha}\left(\frac{r_i^i}{2^iC_i}H\left(\frac{x-x_i}{r_i}\right)\right)$. □

[1] Like in the proof of 2.7

# Chapter 3: Tangent Vectors

## 3.1 Motivation

Consider the following pictures

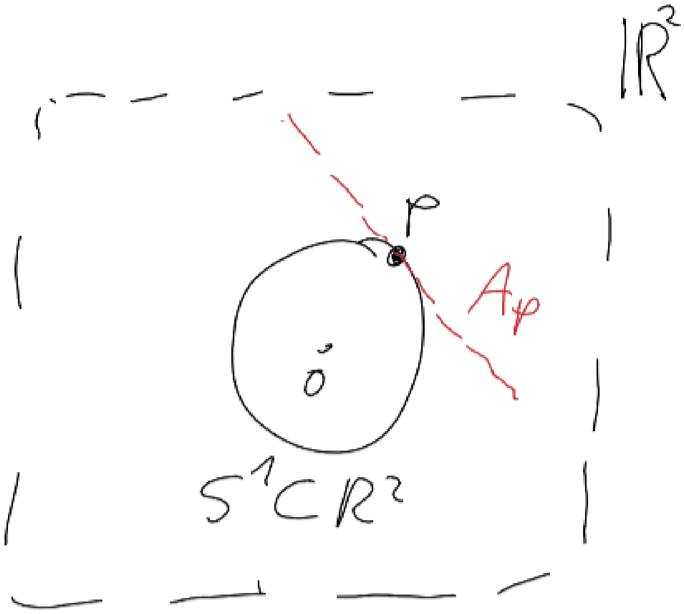


Figure 3.1: Sketch 3.01

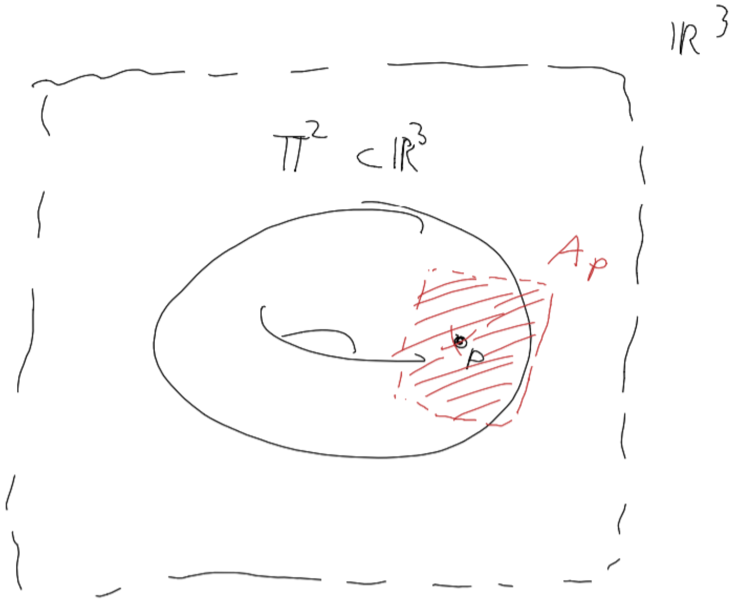


Figure 3.2: Sketch 3.02

$A_p$ the affine hyperplane tangent to $S^1(\Pi^2)$ at the point $p$. Let $T_pM := A_p - p \subset \mathbb{R}^{n+1}$. This is a vector subspace of $\mathbb{R}^{n+1}$. It is called the **tangent space of $M$ at $p$**. Consider

$$TM = \coprod_{p \in M} T_pM,$$

called the **tangent bundle**. Observe that there is a map

Think of $\pi$ as a map of $p, T_pM$

$$\pi : TM \to M$$

by

$$x \in T_pM \mapsto p$$

the data $TM \xrightarrow{\pi}$ forms a **vector bundle**.
**Problems with this approach:**

- not very intrinsic (depends on $R^{n+1} \dots$)
- need to prove that manifolds can always embedded into $R^N$

This is really the picture / intuition we should have, but we will construct it in a different way.

## 3.2 Two (equivalent) theories of tangent vectors

### 3.2.1 Definition via equivalence classes of smooth curves

Let $M$ be a smooth manifold. Fix $p \in M$.

I could not quite make out what he called this chapter, so I named it according to [4]

**Definition.** *The* ***tangent space*** *of $M$ at $p$ denoted by* $T_pM$ *is the set of equivalence classes of smooth curves gamma* $: [-\epsilon, \epsilon] \to M, \gamma(0) = p$ *with* $\gamma_1 \sim \gamma_2 \iff$ *for any smooth function $f$ defined near*[a] *$p$, we have* $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$. *Here the* $\epsilon > 0$ *is any positive real number, which depends on* $\gamma$.

[a] in a neighborhood of

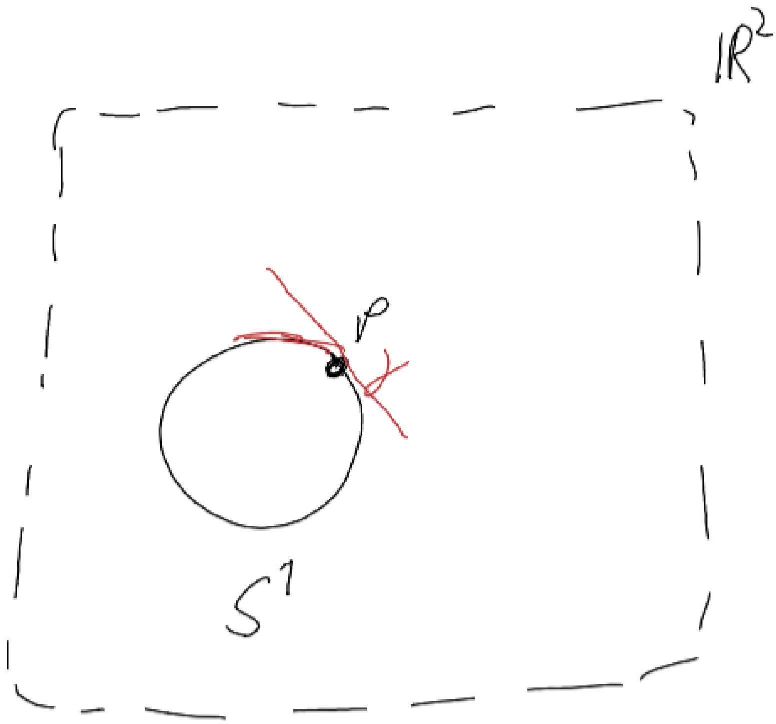


Figure 3.3: Sketch 3.03

**Definition.** *Given a smooth map $F : M \to N$, let*

$$dF_p : T_pM \to T_{F(p)}N$$

*be given by*

$$[\gamma] \mapsto [F \circ \gamma].$$

*This map $dF_p$ is called the* ***differential of $F$ at $p$****.*

*This is clearly well defined*

**Remark.** *The map is also called the* ***tangent map of $M$ at $p$*** *and the* ***total derivative****. It is also denoted by*

$$DF_p, TF_p, \nabla F_p, F'_p, DF(p), TF(p), \ldots$$

**Lemma 3.1** (Fundamentality of the differential)**.** *Let $F^1 : M_1 \to M_2$, $F^2 : M_2 \to M_3$ smooth. Then:*

*(i)* $dF^2_{F^1(p)} \circ dF^1_p = d(F^2 \circ F^1)_p$

*(ii) If $F : M \to M$ is the identity, then $dF_p = id$*

*Proof.* Exercise. □

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**Lemma 3.2.** *Let $\gamma : [-\epsilon, \epsilon] \to \mathbb{R}^n$ and $\sigma : (-\delta, \delta) \to \mathbb{R}^n$ with $\gamma(0) = \sigma(0) = p \in \mathbb{R}^n$. Then* $\gamma \sim \sigma \iff \underbrace{\gamma'(0)}_{(\gamma_1'(0),\ldots\gamma_n'(0))\in\mathbb{R}^n} = \sigma'(0)$

*Proof.* By abusive notation, we denote by $x_i$ the map $\mathbb{R}^n \to \mathbb{R}, (x_1, \ldots, x_n) \mapsto x_i$.. If $\gamma \sim \sigma$, then ${\gamma^i}'(0) = (x_i \circ \gamma)'(0) \overset{\text{Def.}}{=} (x_i \circ \sigma)'(0) = {\sigma^i}'(0) \implies \gamma'(0) = \sigma'(0)$.

$x^i$ might be better (in the sense of the dual space), but $x_i$ is used in practice

Conversely, suppose $\sigma'(0) = \gamma'(0)$. Given any $f$ smooth defined near $p$, we have

$$\begin{aligned}(f \circ \gamma)'(0) &= (\partial_{x_1} f(p), \dots, \partial_{x_n} f(p)) \cdot (\gamma^{1'}(0), \dots, \gamma^{n'}(0)) \\ &= (\partial_{x_1} f(p), \dots, \partial_{x_n} f(p)) \cdot (\sigma^{1'}(0), \dots, \sigma^{n'}(0)) \\ &= (f \circ \sigma')(0).\end{aligned}$$

□

**Corollary 3.3.** *Let $V$ be a finite dimensional $\mathbb{R}$ vector space. Then, for any $p \in V$, the* ***canonical*** *map*

$$\begin{aligned}V &\to T_pV \\ w &\mapsto [t \mapsto p + tw]\end{aligned}$$

*is a bijection.*

*Proof.* If $V = \mathbb{R}^n$, then this is immediate from lemma 3.2. In general pick a basis to define an isomorphism[1] $F : V \to \mathbb{R}^n$. Then the following diagram commutes:

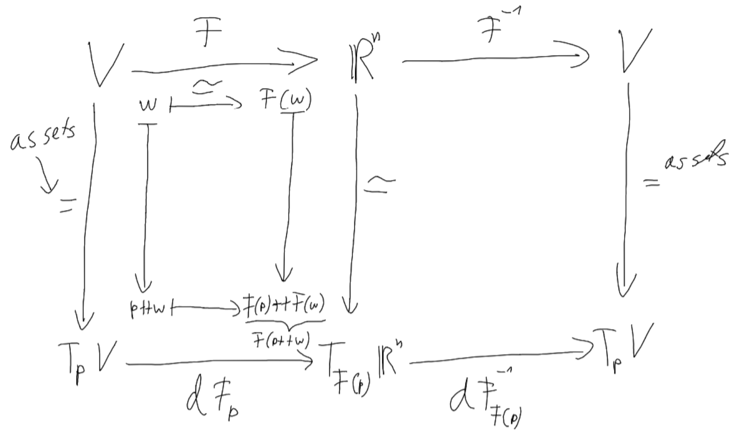


Figure 3.4: Sketch 3.04

using lemma 3.1. □

### 3.2.2 Definition via derivations

**Definition.** *Let $M$ be a smooth manifold. A* ***derivation at*** *$p \in M$ is a linear map*

$$\nu : C^\infty(M) \to \mathbb{R}$$

*satisfying the property*

$$\nu(fg) = f(p)\nu(g) + \nu(f)g(p), \tag{1}$$

*which is also called the* ***Leibniz rule****.*

**Remark.** *Here $C^\infty(M)$ is the set of smooth functions from $f : M \to \mathbb{R}$. It is naturally an $\mathbb{R}$-vector space. Similarly we have $C^0(M)$ the space of continuous functions $f : M \to \mathbb{R}$ and $C^k(M)$ the space of $k$-times differentiable function $f : M \to \mathbb{R}$.*

**Definition.** *The set of derivations at $p$ shall be also called the* ***tangent space*** *of $M$ at $p$, denoted by $T_pM$.*

**Lemma 3.4.** *$T_pM$ is naturally a vector subspace of $C^\infty(M)^\vee$*

*$C^\infty(M)^\vee$ denotes the dual space of $C^\infty(M)$*

[1] In particular a diffeomorphism

*Proof.* Given derivations $\nu_1, \nu_2 \in T_pM$ we must show that $a\nu_1 + \nu_2$ is still an element of $T_pM \forall a \in \mathbb{R}$. We compute we compute

$$\begin{aligned}(a\nu_1 + \nu_2)(fg) &= a\nu_1(fg) + \nu_2(fg) = a[\nu_1(f)g(p) + f(p)\nu_1(g)] + [\nu_2(f)g(p) + f(p)\nu_2(g)] \\ &= f(p)[a\nu_1 + \nu_2] + [a\nu_1 + \nu_2](f)g(p)\end{aligned}$$

□

**Definition.** *Given a smooth map $F : M \to N$, we let $dF_p : T_pM \to T_{F(p)}N$ be the map*

$$\nu \mapsto dF_p(\nu) := C^\infty(N) \ni f \mapsto \nu(f \circ F)$$

**Lemma 3.5.** *(i) the previous definition gives a derivation*

*(ii) $dF^2_{F^1(p)} \circ dF^1_p = d(F^2 \circ F^1)_p$*

*(iii) If $F : M \to M$ is the identity, then $dF_p = id$*

*By (ii) and (iii) $d$ is a Functor*

**Lemma 3.6.** *Let $\nu$ be a derivation at $p \in M$. Then*

*(a) $f \equiv C$, then $\nu(f) = 0$. That is $\nu$ annihilates constant functions.*

*(b) if $f(p) = g(p) = 0$, then $\nu(fg) = 0$*

*Proof.* (a): Since $\nu$ is linear, it is enough to prove $\nu(f) = 0$ for $f \equiv 1$. But then

$$\nu(f) = \nu(f^2) = f(p)\nu(f) + \nu(f)f(p) = 2\nu(f).$$

(b) is obvious by the Leibniz rule (1). □

**Lemma 3.7.** *Let $V$ be a finite dimensional vector space over $\mathbb{R}$. A derivation $\nu \in T_pV$ is entirely determined by its action on any dual basis $\{\xi^1, \dots, \xi^n\}$..*

*This should remind us of lemma 3.2*

*Proof.* Fix a basis ($\{e_1, \dots, e_n\}$) to identify $V \equiv \mathbb{R}^n$. It is enough to show that $\nu(f) = 0$ if $\{\partial_{x_1} f(p), \dots, \partial_{x_n} f(p)\}$ all vanish(Indeed, consider $f \to f - \sum_{k=1}^n \partial_{x_i} f(p)\xi_i$). By Taylor's formula (Appendix C.15, Lee), we have

with $\xi_i : \mathbb{R}^n \to \mathbb{R}$ as before

$$f(x) = \underbrace{f(p)}_{\text{constant}} + \underbrace{\sum_{i=1}^n \partial_{x_i} f(p)(x_i - p_i)}_{=0} + \sum_{i,j=1}^n \underbrace{(x_i - p_i)}_{\text{constant at } p} \underbrace{(x_j - p_j) \int_0^1 (1-t)\partial_{x_i x_j} f(p + t(x-p))dt}_{\text{constant at } p}.$$

Then by lemma 3.6 $\nu(f) = 0$. □

**Corollary 3.8.** *The canonical map $V \to T_pV$, $p \in V$ defined by*

$$w \mapsto (C^\infty(V) \ni f \mapsto \frac{d}{dt}_{|t=0} f(p + tw))$$

*is an isomorphism of vector spaces.*

*This should remind us of corollary 3.3*

*Proof.* We define

These are really canonically equal! No choice needed

$$\begin{aligned}&T_pV \to V \\ &\nu \mapsto \sum_{i=1}^n \nu(\xi^i)e_i, \xi^i : V \to \mathbb{R}\end{aligned}$$

By lemma 3.7 this map is injective and hence $\dim T_pV \le \dim V$. So it is enough to show that $V \mapsto T_pV$ is also injective. Suppose for contradiction that $V \ni w \neq 0$, that maps to the zero derivation.

$$\begin{aligned}&0 = \frac{d}{dt}_{|t=0} f(p + tw) \forall f \\ \implies\ &0 = \frac{d}{dt} w^\vee(p + tw) = \frac{d}{dt}_{|t=0} t = 1\end{aligned}$$

□

### 3.2.3 Both definitions agree

**Temporary notation:** Let $T_pM^{(1)}, dF_p^{(1)}, \ldots,$ be those objects defined in section 3.2.1 and $T_pM^{(2)}, dF_p^{(2)}, \ldots,$ the analogous objects defined in 3.2.2
**Key observation:** There is a **canonical** map, for any $p \in M$,

$$K_P : T_pM^{(1)} \to T_pM^{(2)}$$
$$\gamma \mapsto (C^\infty(M) \ni f \mapsto (f \circ \gamma)'(0)).$$

Note that this commutes with $dF^{(i)}$, i.e. $dF^{(2)} \circ K_p = K_{F^{(1)}(p)} \circ dF_p^{(1)}$ (exercise).

**Proposition 3.9.** *$K_p$ is a bijection.*

*Proof.* Choose a chat $(U, \varphi), p \in U$. Then we have a map

$$U \to \varphi(U) \subset R^n$$

$$\begin{tikzcd}
T_pM^{(1)} \arrow[r, "d\varphi_p^{(1)}", "="'] \arrow[d, "K_p"] & T_{\varphi(p)}\varphi(U)^{(2)} \arrow[d, "K_p"] \\
T_pM^{(2)} \arrow[r, "d\varphi_p^{(2)}"] & T_{\varphi(p)}\varphi(U)^{(2)}
\end{tikzcd}$$

Figure 3.5: Sketch 3.05

Finally, we have

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\text{Cor. 3.3}", "="'] \arrow[dr, "\text{Cor 3.8}", "="'] & T_{\varphi(p)}\varphi(U)^{(1)} = T_p\mathbb{R}^{n\,(1)} \arrow[d, "K_{\varphi(p)}"] \\
& T_{\varphi(p)}\varphi(U)^{(2)} = T_p\mathbb{R}^{n\,(2)}
\end{tikzcd}$$

Figure 3.6: Sketch 3.06

□

## 3.3 Coordinates

**Definition.** *(1) Given a point $p \in \mathbb{R}^n$ let $(\partial_{x_i})_p \in T_p\mathbb{R}^n$ be the vector represented by the curve $t \mapsto p + t\underbrace{(0, \ldots, 1, \ldots, 0)}_{e_i}$.*

*(2) Given $p \in M$, we shall abusenotation by writing $(\partial_{x_i})_p \coloneqq d\varphi^{-1}_{\varphi_p}(\partial x_i)_p$ for some chart $((U, \phi))$*

**Remark.** 1. *Various authors also write $\partial_{x_i}(p)$*

2. *$\{(\partial_{x_1})_p, \ldots, (\partial_{x_n})_p\}$ form a basis for $T_pM$, by construction*

3. *$\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ very much depend on the chart $(U, \varphi)$*

Suppose now that $F : M \to N$ smooth map. Let $(U, \varphi), (V, \psi)$ be charts, $F(U) \subset V$. Let $\hat{p} \coloneqq \phi(p) \in \mathbb{R}^m$. Then we have

$$\begin{tikzcd}
M \supset U \arrow[r, "U"] \arrow[d, "\varphi"] & V \subset N \arrow[d, "\psi"] \\
\{(x_1, \ldots, x_m)\} = \mathbb{R}^m \supset \varphi(U) \arrow[r, dashed, "\hat{F}"] & \psi(V) \subset \mathbb{R}^n = \{(y_1, \ldots, y_n)\}
\end{tikzcd}$$

Figure 3.7: Sketch 3.07

where $\hat{F} = \psi \circ F \circ \varphi^{-1}$.
Note that $d\hat{F}_{\hat{p}} : T_{\hat{p}}\mathbb{R}^m \to T_{\hat{F}(\hat{p})}\mathbb{R}^n$ is a linear map. We want to find an expression ofr the matrix $d\hat{F}_{\hat{p}}$ w.r.t the basis $\{\partial_{x_1}, \ldots, \partial_{x_m}\}$ and $\{\partial_{y_1}, \ldots, \partial_{y_k}\}$.
Well, by definition

$$\begin{aligned} d\hat{F}_{\hat{p}}((\partial_{x_i})_{\hat{p}}) &\coloneqq [\hat{F}(\hat{p} + (0, \ldots, 1, 0, \ldots 0))] \\ &= \sum_{j=1}^{n} \partial_{x_i} F^j(\hat{p})(\partial_{y_j})_{\hat{F}(\hat{p})} \end{aligned}$$

and therefore

$$d\hat{F}_{\hat{p}} = \begin{pmatrix} \partial_{x_1}\hat{F}^1(\hat{p}) & \cdots & \partial_{x_m}\hat{F}^1(\hat{p}) \\ \vdots & & \vdots \\ \partial_{x_1}\hat{F}^n(\hat{p}) & \cdots & \partial_{x_m}\hat{F}^n(\hat{p}) \end{pmatrix}.$$

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**Remark.** *By abuse of notation we often write $F \equiv \hat{F}, p \equiv \hat{p}, \partial_{x_i} f \equiv \partial_{x_i}\hat{F}, dF_p \equiv d\hat{F}_{\hat{p}}$*

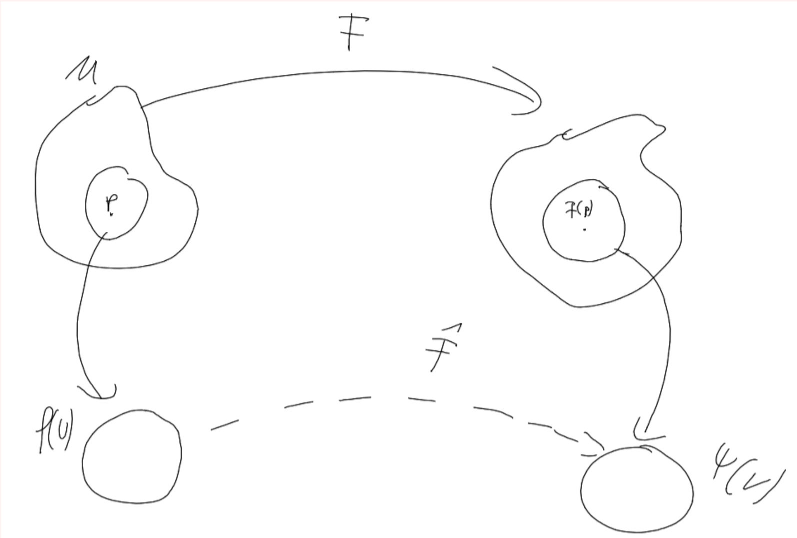


Figure 3.8: Sketch 3.08

**Remark.** $d\hat{F}_{\cdot} : \underbrace{x}_{\in \phi(U) \subset \mathbb{R}^m} \mapsto d\hat{F}_x \in Mat(n \times m) \equiv \mathbb{R}^{n \times m}$. *This it clearly a smooth map.*

## 3.4 The tangent bundle

**Definition.** *Given a smooth manifold $M$, let $TM \coloneqq \coprod_{p \in M} T_pM$. We write elements of $TM$ as pairs $(p, v)$, where $v \in T_pM$. Note that we have a map*

$$\pi : TM \to M, (p, v) \mapsto p.$$

**Remark** (Added by Manuel, was an answer to my question)**.** *For $p \in M$ the preimage of $p$ under $\pi$ is called a **fiber**. He also highlighted, the condition that $\pi^{-1}(p)$ is a vector space (namely $T_pM$), which seems to be important in our context, but not generally required for fibers.*

A priori, TM is just a set. We will exhibit natural smooth manifold structure.
**Special case:** $M \subset U\mathbb{R}^n$. Then

$$TU \coloneqq \coprod_{p \in U} T_pU \equiv U \times \mathbb{R}^n$$
$$(t \mapsto p + tv) \mapsto (p, v)$$

Remember that this is a canonical identification!

**General construction** Given a smooth chart $(U, \phi)$ for a smooth manifold $M$, we have a map $d\phi$

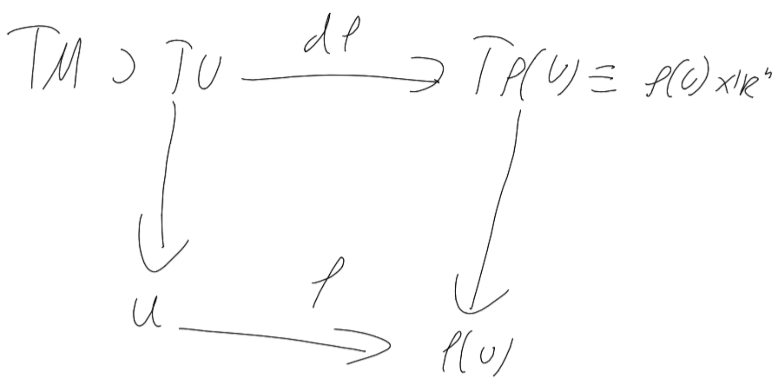


Figure 3.9: Sketch 3.09

where
$$d\phi(p,v) := (\phi(p), d\phi_p(v)).$$
Define a subset $S \subset$ TM to be open, if, for any chart $U, \phi$, $d\phi(S \cap TU)$ open in $T\phi(U) \equiv \phi(U) \times \mathbb{R}^n$.

This is a pullback

**Lemma 3.10.** *This prescription defines a topological space on TM. Moreover, TM is a topological manifold.*

*Proof.* Omitted. Check transition maps
$$d\psi \circ d\phi^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$
is it an elementary, but tedious proof. □

**Remark.** *Alternatively define the same topology on TM by taking the basis the union over all charts $(U, \phi)$ in your atlas of $\{d\phi^{-1}(V) \mid V \subset T(\psi(U))$ open$\}$.*

To make TM into a **smooth** manifold, we take as our atlas the set $\{(TU, d\phi)\}_{(U,\phi)}$, where $(U, \phi)$ runs over the smooth charts of $M$.

**Lemma 3.11.** *this is a smooth atlas.*

*Proof.* Fix charts $(U, \phi), (V, \psi)$. Then the transition functions take the form

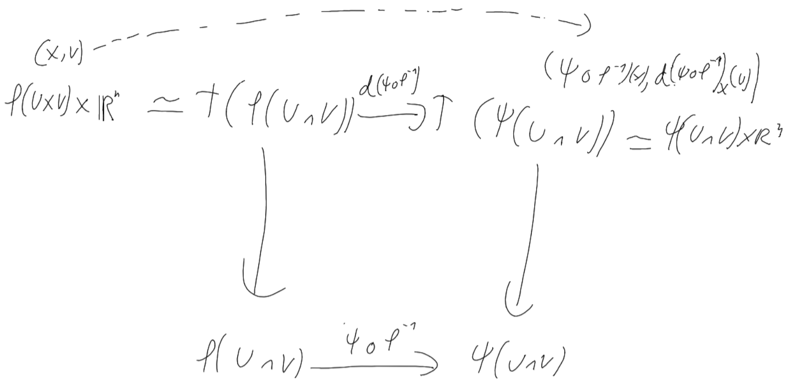

Figure 3.10: Sketch 3.11

Check if both components are smooth:

- The first component $x \mapsto \psi \circ \phi^{-1}(x)$ is smooth, since $M$ is a smooth manifold an $(U, \phi), (V, \psi)$ are smooth
- For the second component can be fractured as follows:
$$(x, v) \mapsto (d(\underbrace{\psi \circ \phi_x^{-1}}_{\in \mathrm{Man}(n \times n) \equiv \mathbb{R}^{2n}}, v)) \mapsto d(\psi \circ \phi^{-1})_x v$$

Since we can write the second component as a concatination of maps, it is smooth

**Exercise:** the map $\mathrm{Mat}(m \times n) \times \mathrm{Mat}(n \times p) \to \mathrm{Mat}(m \times p)$ by $A, B \mapsto AB$ is smooth. □

**Remark.** *We will see later that $(\phi : TM \to M)$ forms a vector bundle. It can be shown that given $F : M \to N$ the map $dF : TM \to TN, (p, v) \mapsto (F(p), dF_p(v))$ is smooth..*
*In fact, we have*

*This is the exact same computation as in the proof of lemma 3.11*

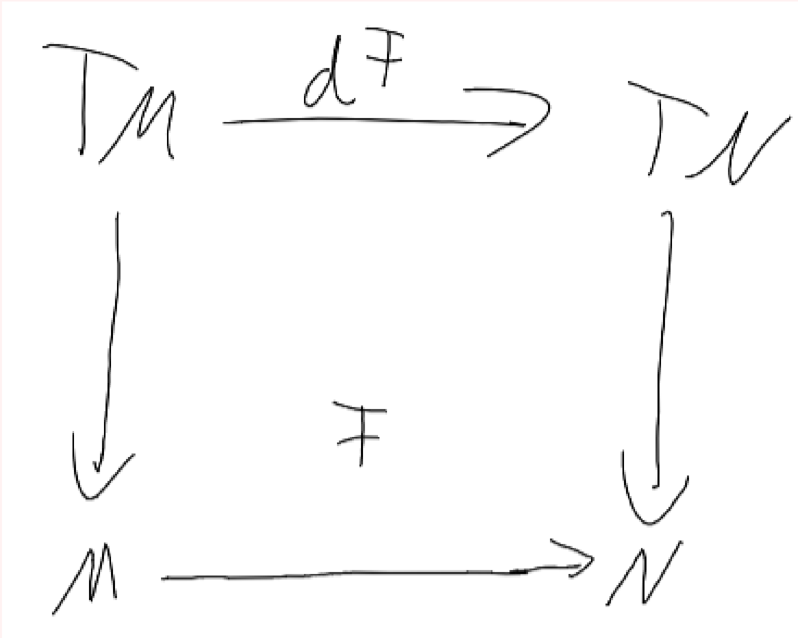


Figure 3.11: Sketch 3.12

*commutes. This can be restated as follows: There is a functor* $Man^{\infty} \to$ *Smooth vector bundles by*

$$M \mapsto (\pi : TM \to M)$$

$$F : M \to N \mapsto dF : TM \to TN$$

# Chapter 4: Submersions, immersions and embeddings

## 4.1 Basic definitions

**Definition.** *Let $F : M \to N$ be smooth. The* ***rank*** *of $F$ at $p \in M$ is the rank of the linear map $dF_p : T_pM \to T_{F(p)}M$.*

Smooth maps, which have full rank (highest possible rank, i.e. $\text{rank}F = \max(m, n)$) are particularly important:

**Definition.** *Let $F : M^m \to N^n$ be smooth. We say*

- *$F$ is a* ***submersion*** *if $dF_p$ is surjective, for all $p \in M$ $(m \geq n)$*
- *$F$ is an* ***immersion*** *if $dF_p$ is injective, for all $p \in M$ $(m \leq n)$*

*$M^m, N^n$ means $M, N$ are $m, n$ dimensional manifolds*

**Lemma 4.1.** *Given $(m, n) \in \mathbb{N}_+ \times \mathbb{N}_+$, let $Mat(m \times n) \equiv \mathbb{R}^{m\times n}$. The subset $Mat(m \times n)^{full\ rank} := \{A \in Mat(m \times n) \mid A \text{ has full rank}\}$ is open in $Mat(m \times n)$.*

*Proof.* Fix $M \in \text{Mat}(m \times n)^{\text{full rank}}$. Without loss of generality $m \leq n$, otherwise apply $\text{Mat}(m \times n) \to \text{Mat}(n \times m), A \mapsto A^T$. By definition there exists a submatrix $M'$, obtained by deleting $n - m$ columns, which is invertible. Now the map

$M$ is fixed and $F$ depends on $M$, but it does not matter here!

$$\text{Mat}(m \times n) \overset{F:M \mapsto M'}{\to} \text{Mat}(m \times m) \overset{\det(\cdot)}{\to} \mathbb{R}$$

is continuous, since booth the forgetful $F$ and det is smooth.

$$M \in (\det \circ F)^{-1}(\underbrace{\mathbb{R} \setminus \{0\}}_{\text{open}}) \subset \text{Mat}(m \times n)^{\text{full rank}}$$

since $M$ was arbitrary this completes the proof.

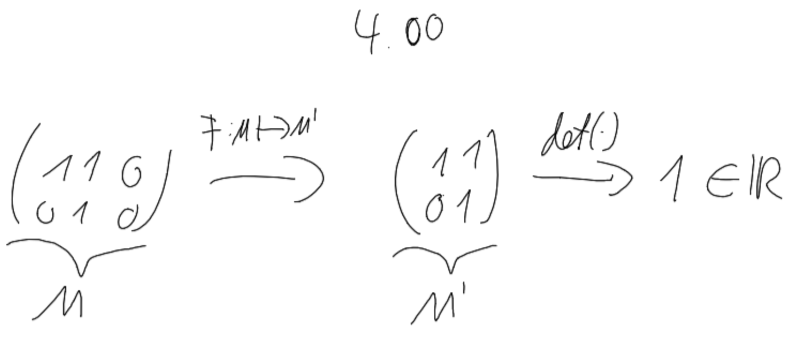


Figure 4.1: Sketch 4.00

□

**Lemma 4.2.** *Fix $F : M^m \to N^n, p \in M$.*

1. *If $dF_p$ is injective, then there exists a neighborhood of $p$ on which $dF.$ is injective.*
2. *If $dF_p$ is surjective, then there exists a neighborhood of $p$ on which $dF.$ is surjective.*

*The property of full rank is stable under small pertubation!*

*Proof.* This is a local statement. We can therefore assume that $M, N$ are open subsets of $\mathbb{R}^m, \mathbb{R}^n$ respectively. Then

$$dF_{(\cdot)} : M \to \mathrm{Mat}(n \times m)$$

is smooth, hence continuous. By assumption $dF_p \in \mathrm{Mat}(n \times m)^{\text{full rank}}$. But $\mathrm{Mat}(n \times m)^{\text{full rank}}$, so the preimage is open (by lemma 4.1) and contains $p$. □

**Remark.**

*important: contains both a definition and a counterexample!*

1. *If $F : M \to N$ is both an immersion and a submersion, then we say that $F$ is a* ***local diffeomorphism****. We will see (by the rank theorem 4.3) that $F$ is a local diffeomorphism $\iff \forall p \in M \exists p \in U : F_{|_U}$ is a diffeomorphism.*
2. *be warned. local diffeomorphism need not be global:*

$$\begin{array}{rcl} \mathbb{R}^2 \equiv \mathbb{C} \supset & S^1 = \{|z| = 1\} & \to S^1 \\ (x, y) \mapsto x + iy & z \longmapsto & z^2 \end{array}$$

**Definition.** *An immersion is an* ***embedding*** *if it is a homeomorphism onto its image with the subspace topology.*

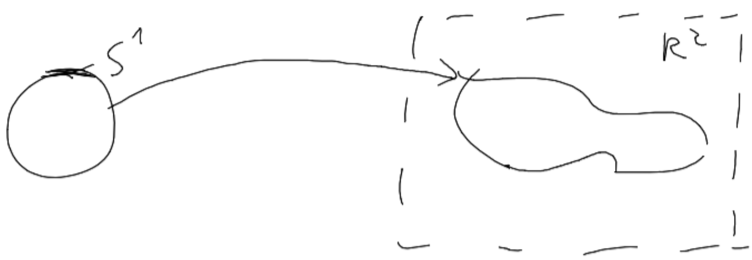


Figure 4.2: Sketch 4.01

**Example.** *Another example:*

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{1+n} \mid \sum x_i^2 = 1\} \subset \mathbb{R}^{1+n}$$

*with*

$$i : S^n \hookrightarrow \mathbb{R}^{1+n}$$

***Non-examples***


Figure 4.3: Sketch 4.02

*parametrized by*

$$t \mapsto (\sin t, \sin 2t)$$

*and*

$$\mathbb{R} \mapsto \mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$$
$$t \mapsto (t, \alpha t),\ \alpha \in \mathbb{R} \setminus \mathbb{Q}$$

*Can show*[1] *that the image is dense. It is an immersion, but no a homeomorphism!*

Start of lecture 09 (08.11.2024)

## 4.2 The rank theorem

**Theorem 4.3** (rank theorem)**.** *Let $F : M^m \to N^n$ be a smooth map of constant rank $r$. For each $p \in M$, there exist charts $(U, \varphi) : p \in U$ and $(V, \psi) : F(U) \subset V$, such that*

$$\hat{F} := \psi \circ F \circ \varphi^{-1} :$$

Figure 4.4: Sketch 4.03

$\hat{F}$ takes the form

$$\hat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$$

This is arguably the most important result of the first half of the course. There is a lot of results in [2], what is actually useful? Implied answer: Rank theorem

**Remark.** *By lemma 4.2, if $F$ has full rank at $p \in M$, then*

- *if $m = r \geq n$, then $F$ is an submersion near $p$ and*

$$\hat{F}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n)$$

- *$m = r \leq n$, then $F$ is an immersion near $p$, and*

$$\hat{F}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$$

- *$m = n \implies$ up to the diffeomorphism, $\hat{F}$ is just the identity*

*Up to diffeomorphism there is only one map of constant, full, rank*

**Remark.** *This theorem is a non-linear generalization of the following linear algebra fact: $L : V^m \to W^n$, then there are linear maps $\varphi : V^m \xrightarrow{\simeq} \mathbb{R}^m, \psi : W^n \xrightarrow{\simeq} \mathbb{R}^n$, such that $\hat{L} := \psi \circ L \circ \phi^{-1}$ takes the form*

$$\hat{L}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0),$$

*where $r = rank(L)$.*

[1] not obvious, non-examable

*Proof of theorem 4.3.* **Step 0:** We might as well assume that $M = U \subset \mathbb{R}^m, N = V \subset \mathbb{R}^n$, since we only make a local statement up to diffeomorphism. We may also assume, up to reordering the coordinates, that the matrix $(\partial_{x_i} F^j(p))_{1\leq i,j\leq r}$ is **invertible** for $p \in U$. We label our coordinates: see [2]

source coordinates in $U$

$$(x_1, \ldots, x_r, y_1, \ldots, y_{m-r})$$

Tarkget coordiantes

$$(v_1, \ldots, v_r, \ldots, w_1, \ldots, w_{n-r})$$

and wlog $F(0,0) = (0,0)$.
We write $F(x,y) = \underbrace{Q(x,y)}_{v\text{-coordinates}}, \underbrace{R(x,y)}_{w\text{-coordinates}}$. Notice that $(\partial_{x_i} Q^j)$ is non-singular.

**Step 1:** Define $\varphi : U \to \mathbb{R}^m, \varphi(x,y) = (Q(x,y), y)$. Then

$$d\varphi_{(0,0)} = \begin{pmatrix} \overbrace{\partial_{x_i} Q^j}^{\in \mathrm{Mat}(r\times r)} & \partial_{y_i} Q^j \\ 0 & \underbrace{1}_{\in \mathrm{Mat}((n-r)\times(n-r))} \end{pmatrix}$$

$\implies$ by the inverse function theorem, there exist connected neighborhoods
$U_0 \subset U, \tilde{U_0} \subset\in \mathrm{Mat}((n-r)\times(n-r))\varphi(U)$, such that $\varphi_{|U_0} : U_0 \to \tilde{U_0}$. We may as well assume that $\tilde{U_0}$ is a cube, i.e. $(-\epsilon, \epsilon)^n$.

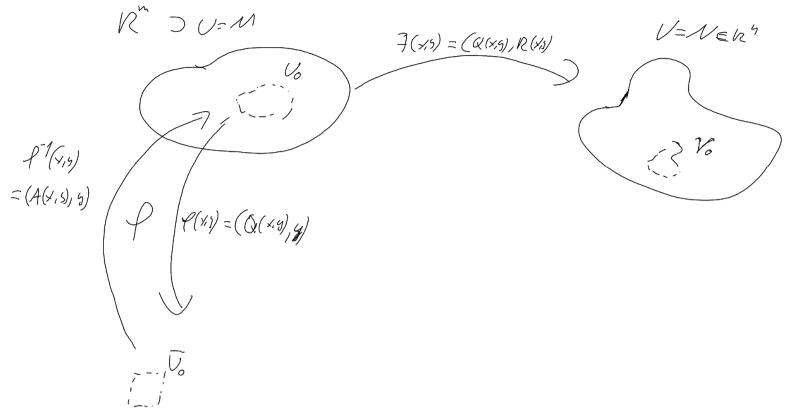


Figure 4.5: Sketch 4.04

While $\varphi^{-1}(x,y) = (A(x,y), B(x,y))$, for some $A : \tilde{U_0} \to \mathbb{R}^r$, $B : \tilde{U_0} \to \mathbb{R}^{m-r}$. We compute

$$(x,y) = \varphi\circ\varphi^{-1}(x,y) = \varphi\left(A(x,y), B(x,y)\right) = (Q(A(x,y), B(x,y)), B(x,y)) \implies \substack{x=Q(A(x,y),B(x,y)) \\ y=B(x,y)}$$

Hence $\varphi^{-1}(x,y) = (A(x,y), y)$.

**Step 2:** Observe that

$$F\circ\varphi^{-1}(x,y) = (Q(\varphi^{-1}(x,y)), R(\varphi^{-1}(x,y))) = (x, \tilde{R}(x,y)),$$

where

$$\tilde{R}(x,y) = R(\varphi^{-1}(x,y)).$$

Then

$$d(F\circ\varphi^{-1}) = \begin{pmatrix} \overbrace{1}^{\in \mathrm{Mat}(r\times r)} & 0 \\ \partial_{x_i}\tilde{R}(x,y)^j & \underbrace{\partial_{y_i}\tilde{R}^j}_{\in \mathrm{Mat}((m-r)\times(m-r))} \end{pmatrix}$$

But the rank of $d(F\circ\varphi^{-1})$ is $r$, because $\varphi^{-1}$ is a diffeomorphism and $F$ has rank $r$

- Since $1_{r\times r}$ has rank $r$, we must have $\partial_{y_i}\tilde{R} \equiv 0$

We write $S(x) \coloneqq \tilde{R}(x, y)$, we now have

$$F \circ \varphi^{-1}(x, y) = (x, S(x)) \tag{1}$$

.

to make clear $\tilde{R}$ does not really depend on $y$

**Step 3:** Recall

$$F : U \to V \subset \mathbb{R}^n$$
$$F(0, 0) = (0, 0)$$

Let $V_0 \subset V$ be defined as follows:

$$V_0 \coloneqq \{(v, w) \in V \mid (v, 0) \in \tilde{U}_0\}$$

By (1), $F \circ \varphi^{-1}(\tilde{U}_0) \subset V_0$. Hence $F(U_0) \subset V_0$. Set $\psi : V_0 \to \mathbb{R}^n, \psi(v, w) = (v, w - S(v))$. Clearly $\psi$ is a diffeomorphism, since

$$(v, w) \mapsto (v, w + S(w))$$

is am inverse. $\implies$ $(V_0, \psi)$ is a smooth chart.

$S(v)$ makes perfect sense, since both $x, v$ have $r$ entries

$$\hat{F} \coloneqq \psi \circ F \circ \phi^{-1} = \Psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0)$$ □

**Remark.** *This is one theorem you should really not forget! If you continue to think about Manifolds in your life, this is really useful! Do not remember the proof, remember the statement!*

# Chapter 5: Submanifolds

## 5.1 Basic definitions

**Definition.** *Let $M$ be a topological manifolds. A subset $S \subset M$ is a* ***topological submanifold****, if $S$ is a topological manifold with the subspace topology.*

**Example.** $S^n = \{(x_0, \ldots, x_n) \mid \sum x_i^2 = 1\} \subset \mathbb{R}^{1+n}$

**Example** (Non-example)**.** $\{(x, y) \mid x = 0 \vee y = 0\} \subset \mathbb{R}^2$*, since this is not a manifold (see sheet 01).*

**Definition.** *Let $M$ be a smooth manifold. A topological submanifold $S \subset M$ is a* ***smooth submanifold****, if it is equipped with a smooth structure, s.t. the embedding $i : S \hookrightarrow M$ is smooth.*

**Example.** *If $M$ is a smooth manifold and $U \subset M$ open, then $U \subset M$ is a smooth manifold.*

*With the restricted smooth structure of $M$*

**Remark.** *Some authors (including Lee's textbook) use the term* ***embedded submanifold*** *to distinguish from* ***immersed submanifolds****. For use "submanifolds"≡"embedded submanifold".*

**Lemma 5.1.** *Suppose that $f : M \to N$ smooth embedding. Let $S := F(M) \subset N$. Then $S$ admits a unique smooth structure making it a smooth submanifold, with the property that $f$ is a diffeomorphism onto its image.*

*Proof.* By definition of $f$ being an embedding, $f$ is a homeomorphism onto it's image, with the subspace topology. $\implies S$ is a topological manifold.
We define a smooth atlas on $S$ by taking $\{(f(U), \varphi \circ f^{-1})\}$, as $(U, \varphi)$ ranges over the set of charts for $M$.
Clearly $f$ is a diffeomorphism, since $\varphi \circ f \circ f^{-1} \circ \psi^{-1}$, for $(U, \varphi), (V, \psi)$ smooth charts, this follows from the fact that $(U, \varphi), (V, \psi)$ are smoothly compatible on $M$.
This is the only smooth at with the property that $f$ is a diffeomorphism, if $\mathcal{B}$ is an other such atlas, then the fact that $f$ is a diffeomorphism for $(S, \mathcal{B}) \iff (S, \mathcal{A})$ compatible.
Finally

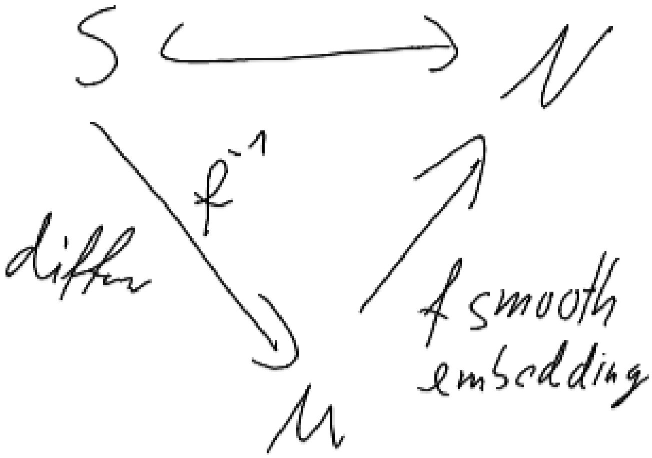


Figure 5.1: Sketch 5.01

so $i$ is a smooth embedding. □

**Definition.** *A embedded submanifold $S$ is called* ***properly embedded****, if the inclusion map $i \hookrightarrow N$ is proper (i.e. the preimage of a compact set is compact).*

**Example.** $S^n \hookrightarrow R^{n+1}$ *properly embedded.*

**Example** (Non-example)**.** $S^n \setminus \{pt\} \subset \mathbb{R}^{n+1}$

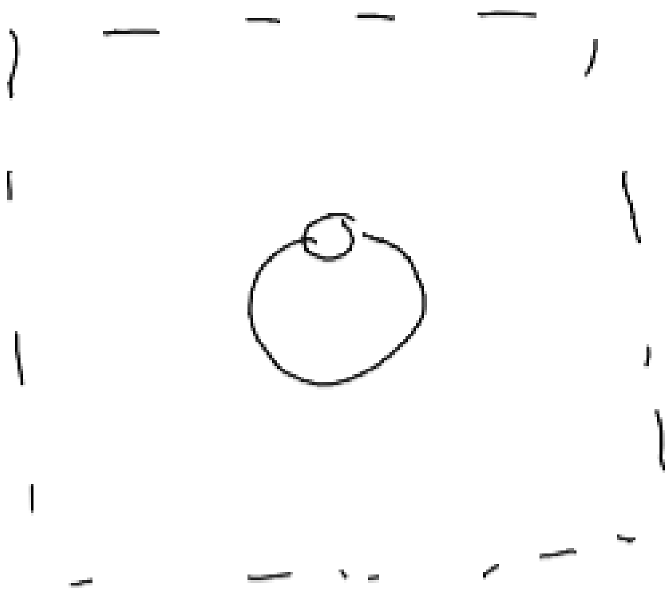

Figure 5.2: Sketch 5.02

**Lemma 5.2.** *A topological submanifold $S \subset N$ is properly embedded iff $S$ is closed.*

*Proof.* Exercise. □

Elementary exercise in point set topology

## 5.2 The “slice lemma”

**Theorem 5.3** (Slice lemma[a]). *(a) Suppose $S^k \subset M^n$ is a submanifold of codimension $n-k$. Then, for all $p \in S$, there exists a chart $(V, \psi), p \in U \subset N$, such that*

*this is also a definition of codimension:* $\dim M - \dim S$

$$\psi(V \cap S) = \{(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \psi(V) \mid x_{k+1} = c_{k+1}, \dots, x_n = c_n\}.$$

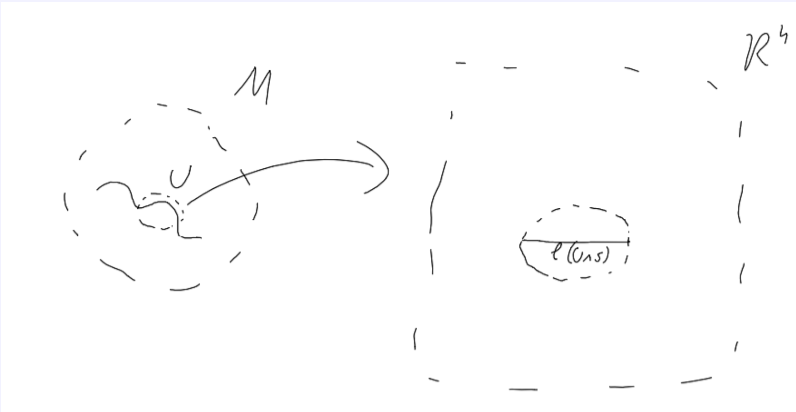


Figure 5.3: Sketch 5.03

*Start of lecture 10 (12.11.2024)*

*The converse of (a)*

*(b) Suppose that $S \subset N$ is a subset with the property that, for all $p \in S$, there exists a slice chart $(V, \psi), p \in V \subset N$, such that*

$$\psi(V \cap S) = \{(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \psi(V) \mid x_{k+1} = c_{k+1}, \dots, x_n = c_n\},$$

*then $S$ admits a smooth manifold structure making it a smooth submanifold of $N$.*

[a] Lee [2] calls it a theorem

**Remark.**
- *We get an equivalent theorem by requiring $c_{k+1} = \dots = c_n = 0$*
- *Part (b) of theorem 5.3 tells us, that being a smooth submanifold $S \subset N$ of ambient smooth manifold $N$ is a property property of the subset. It suffices to check, pointwise, the local property described above!*

*Proof.* **(a):** By assumption $S \hookrightarrow N$ is an immersion. By theorem 4.3 (rank theorem), there exists charts $(U, \varphi), (V, \psi)$ such that $i(U) \subset V$ and

Locally, all immersions look the same

$$\hat{i} = \psi \circ i \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$
$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$$

Up to shrinking $\psi$ (restricting the image of $\varphi$), we find that

$$\psi(V \cap S) = \{(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \mid x_{k+1} = \dots = x_n = 0\}$$

**Warning:** What can go wrong here? Consider

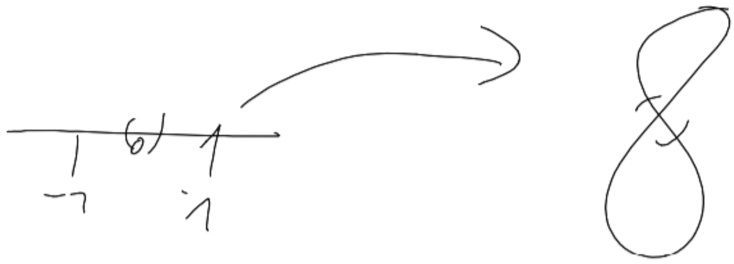

Figure 5.4: Sketch 5.04

Show that there is no more stuff in the set!
**(b):** We have to check that the local charts given form an atlas. Which is almost a tautology and quite tedious, as we can use $\{S \cap V, \psi_{|S}\}$ as the atlas. $\square$

**Remark** (+Exercise)**.** *In section 2.1.2, example 4, we considered $\Phi : \mathbb{R}^{1+n} \to \mathbb{R}$. We assumed $d\Phi$ is nonzero on the set $\Phi^{-1}(0) \subset \mathbb{R}^{1+n}$. Under this assumption, we proved that $\Phi^{-1}(0)$ is a naturally smooth manifold. Using theorem 5.3 (or by hand) $\Phi^{-1}(0)$ is a smooth submanifold.*

A priori, $S \subset N$ could admit multiple smooth structures making it a submanifold. We know seek to show that this is not the case.

**Lemma 5.4.** *Let $S \subset N$ be a submanifold. If $F : M \to N$ is a smooth map which factors through $S \hookrightarrow N$ as a continuous map, then $F$ is smooth as a map $M \to S$.*

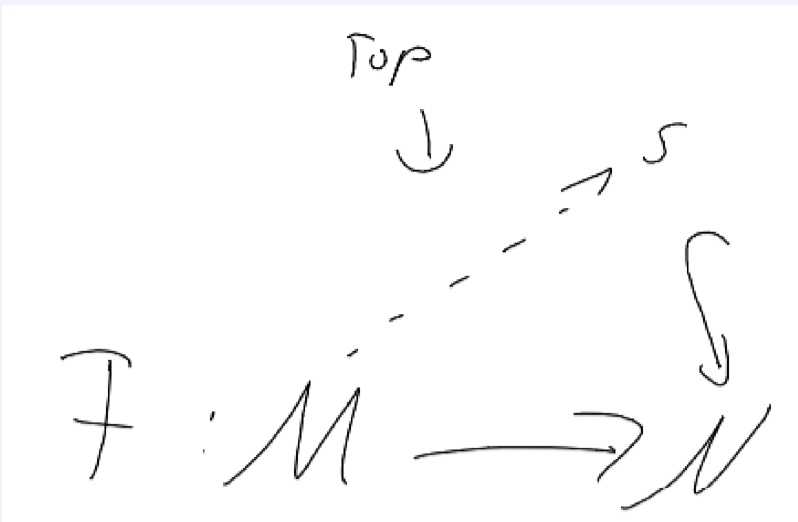


Figure 5.5: Sketch 5.05

*Proof.* By theorem 5.3, there exists $U \subset S \hookrightarrow N \supset V$

More by the proof of the theorem ...

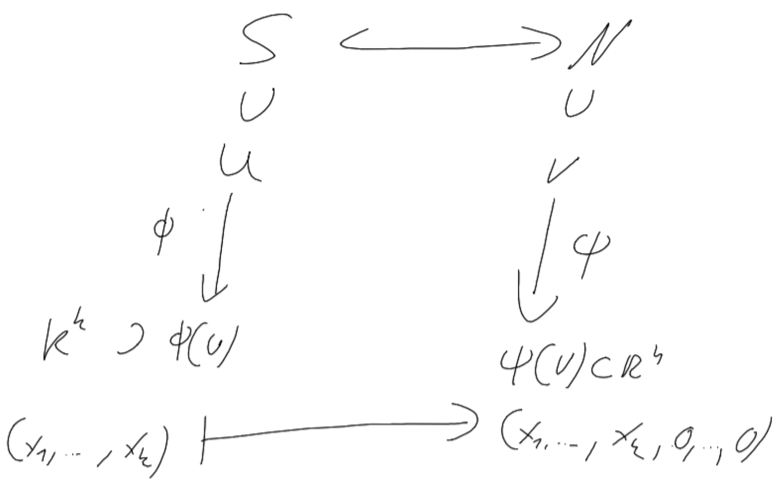

Figure 5.6: Sketch 5.06

Let us call $\overset{\vee}{F}\colon M \to S, \overset{\vee}{F}(x) = F(x)$. Since $\overset{\vee}{F}$ is continuous, $\overset{\vee}{F}^{-1}(U) \subset M$ open. So, we can write, for $(W, U)$, $W \subset \overset{\vee}{F}^{-1}(U)$

$$\begin{tikzcd}
F^{-1}(U) \arrow[r, "F^{\vee}"] \arrow[d, "\cup"] & S \arrow[d, "\cup"] \\
W \arrow[r] & U
\end{tikzcd}$$

$$(x_1, \dots, x_m) \longrightarrow (F^{\vee 1}(x_1, \dots, x_m), \dots, F^{\vee k}(x_1, \dots, x_m))$$

Figure 5.7: Sketch 5.07

were, a priori, $\overset{\vee}{F}^i$ are continuous.
Concatenating the two diagrams, we find that
$F(x_1, \dots, x_m) = i\circ \overset{\vee}{F}(x_1, \dots, x_m) = (\overset{\vee}{F}^1(x_1, \dots, x_m), \dots, \overset{\vee}{F}^k(x_1, \dots, x_m), 0, \dots, 0)$. But then each $\overset{\vee}{F}^i$ has to be smooth and therefore $\overset{\vee}{F}$ is smooth. $\square$

**Lemma 5.5.** *Let $S \subset M$ be a subset satisfying the conditions of theorem 5.3 (b), then the smooth structure produced by the theorem is the* ***unique*** *smooth structure, such that $S \hookrightarrow M$ is a smooth submanifold.*

*Proof.* Let $\tilde{S}$ be a copy of $S$, but endowed with some possibly different smooth structure s.t. $\tilde{S} \hookrightarrow M$ is an embedding.
$\tilde{S} \hookrightarrow M$ factors through $S$, so $\tilde{S} \overset{\text{id}}{\to} S$ smooth. Similarly $S \overset{\text{id}}{\to} \tilde{S}$ smooth. $\square$

Ergo it is a smooth submanifold of $M$.

This uses lemma 5.4

## 5.3 The (weak) Whitney embedding theorem

**Theorem 5.6** (Whitney)**.** *Every compact $n$-dimensional smooth manifold admits an embedding into $\mathbb{R}^N$ for $N \gg 1$ large enough.*

**Remark.** *Later (probably this month), we will remove the compactness assumption and also argue that one can take $N = 2n + 1$.*
*Whitney proofed that one can take $N = 2n$.*

*Don't sue him, if he is off by one :)*

**Added remark.** *This is a very philosophically pleasing statement, since we recover our intuition of embedded manifold from the abstract theory. It is also true, that there is only one embedding (up to isotopy).*

*Proof of theorem 5.6.* Fix a finite cover of $M$ $\{B_1, \dots, B_k\}, B_i \subset M$ open. We may as well assume that there exist charts $(B_i', \phi_i), \overline{B_i} \subset B_i', \varphi_i(B_i') = B_1(0) \subset \mathbb{R}^m$.
Let $\rho_i : M \to \mathbb{R}$ be a cutoff function for $(\overline{B}_i \subset B_i')$, i.e. $\rho_{i|_{B_i}} =\equiv 1, \mathrm{supp}(\rho_i) \subset B_i', 0 \leq \rho_i \leq 1$. The existence of the $\rho_i$ follows from proposition 2.8.
We now define

Notice the $k$ comes from compactness, i.e. we have no control over it, as it its non-constructive

$$\begin{aligned} F : M &\to \mathbb{R}^{mk+k} \\ p &\mapsto (\rho_1(p) \underbrace{\varphi_1(p)}_{\in \mathbb{R}^m}, \dots, \rho_k(p)\varphi_k(p), \rho_1(p), \dots, \rho_k(p)) \end{aligned}$$

We will now see that $F$ is an embedding. First, we will argue $F$ is an injective immersion.
If $F(p) = F(q) \implies \rho_i(p) = \rho_i(q) \forall i = 1, \dots, k$. Let $i_0$ be such that $p \in B_{i_0}$. Then $\rho_{i_0}(p) = 1 = \rho_{i_0}(q) \implies q \in \mathrm{supp}(\rho_{i_0}) \subset B_{i_0}'$. But now $\underbrace{\varphi_{i_0}(p)}_{\in \mathbb{R}^m} = \rho_{i_0}(p)\varphi_{i_0}(p) = \rho_{i_0}(q)\varphi_{i_0}(q) = \varphi_{i_0}(q)$. Hence $p, q \in B_{i_0}' \implies p = q$.
**$F$ is an immersion:** Choose $p \in M$. Then p $p \in B_{i_0}$, for some $i_0$. Hence $\rho_{i_0} \equiv 1$ for some neighborhood of $p$.

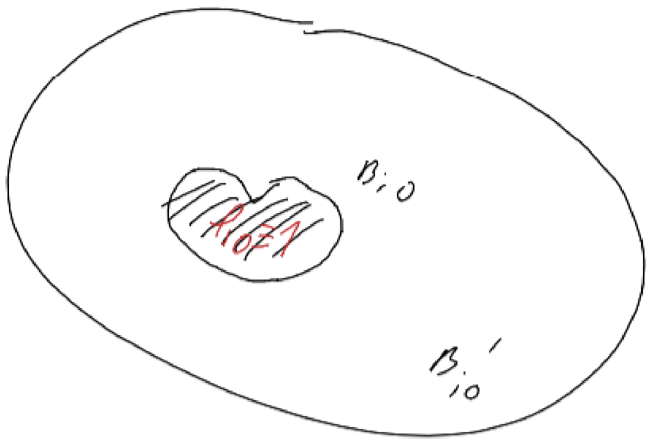

Figure 5.8: Sketch 5.08

Hence $d(\rho_{i_0}\varphi_{i_0}) = \underbrace{d\rho_{i_0}}_{\text{invertible } m \times m}$ near $p \implies dF$ is injective near $p$, but $p$ was arbitrary.
Finally, since $M$ is compact, the theorem follows from the following lemma 5.7.

Kind of cheating ...

I.e. it is enough to show that $F^{-1} : F(M) \to M$ is continuous, i.e. $F : M \to F(M)$ is a closed map. But since $M$ is compact, $F$ is proper $\overset{\text{lemma 5.7}}{\implies}$ $F$ closed.
Please add to your notes: (Said in lecture 11)

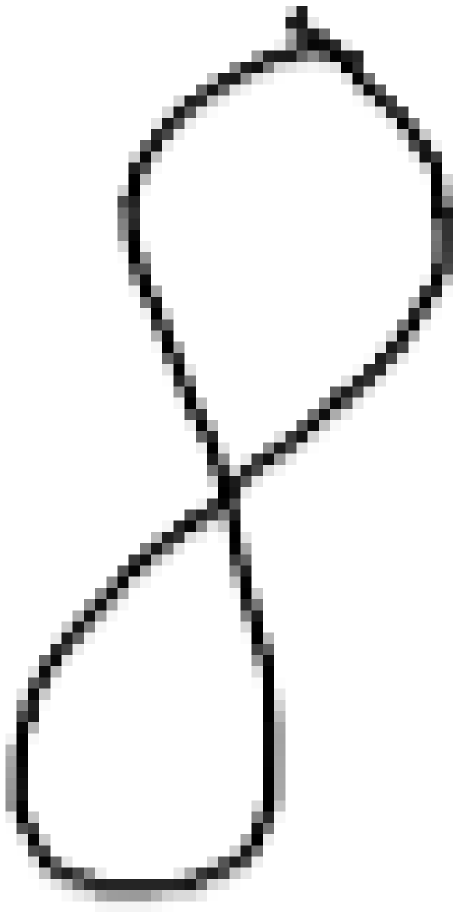

Figure 5.9: Sketch 6.04

Since $S \hookrightarrow N$ is an embedding, $i(U)$ is open in the subspace topology, so there exists $W \subset N$ such that $i(U) = S \cap W$.

$\square$

**Lemma 5.7** (Lee Appendix A: 57)**.** *Let $X$ be a topological space. Let $Y$ be locally compact (e.g. a topological manifold), then any proper continuous map is closed.*

# Chapter 6: Transversality

## 6.1 Basic definition

### 6.1.1 Motivation

Let $l_1, l_2 \subset \mathbb{R}^2$ be (linear) lines. We will say that $l_1, l_2$ are **transverse**, if $\underbrace{T_0 l_1}_{l_1} \oplus \underbrace{T_1 l_2}_{l_2} = T_0\mathbb{R}^2 \equiv \mathbb{R}^2$

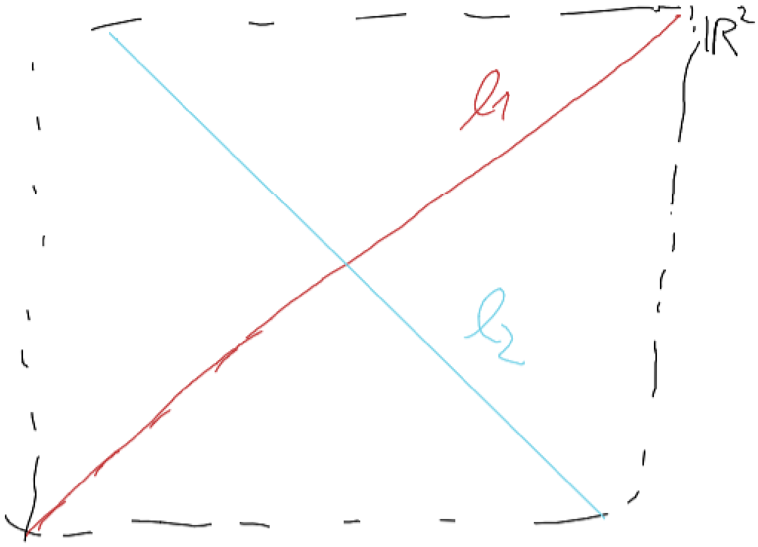


Figure 6.1: Sketch 6.01

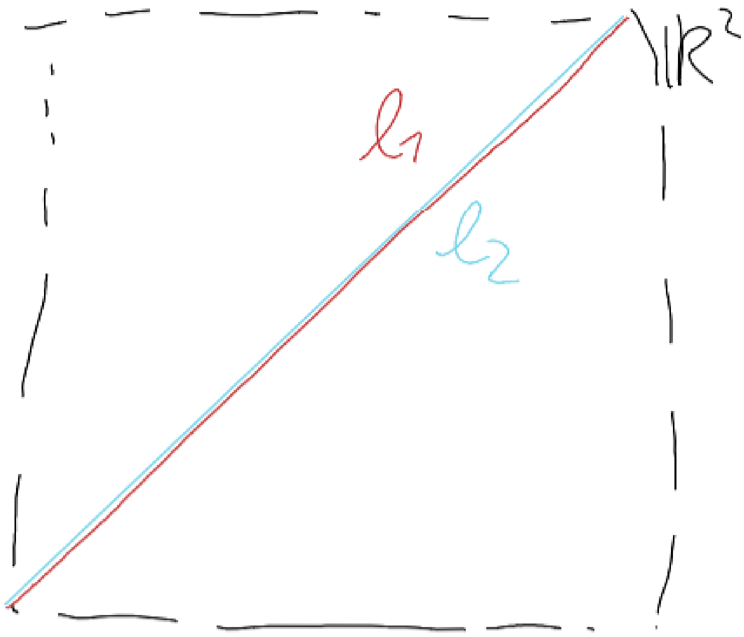


Figure 6.2: Sketch 6.02

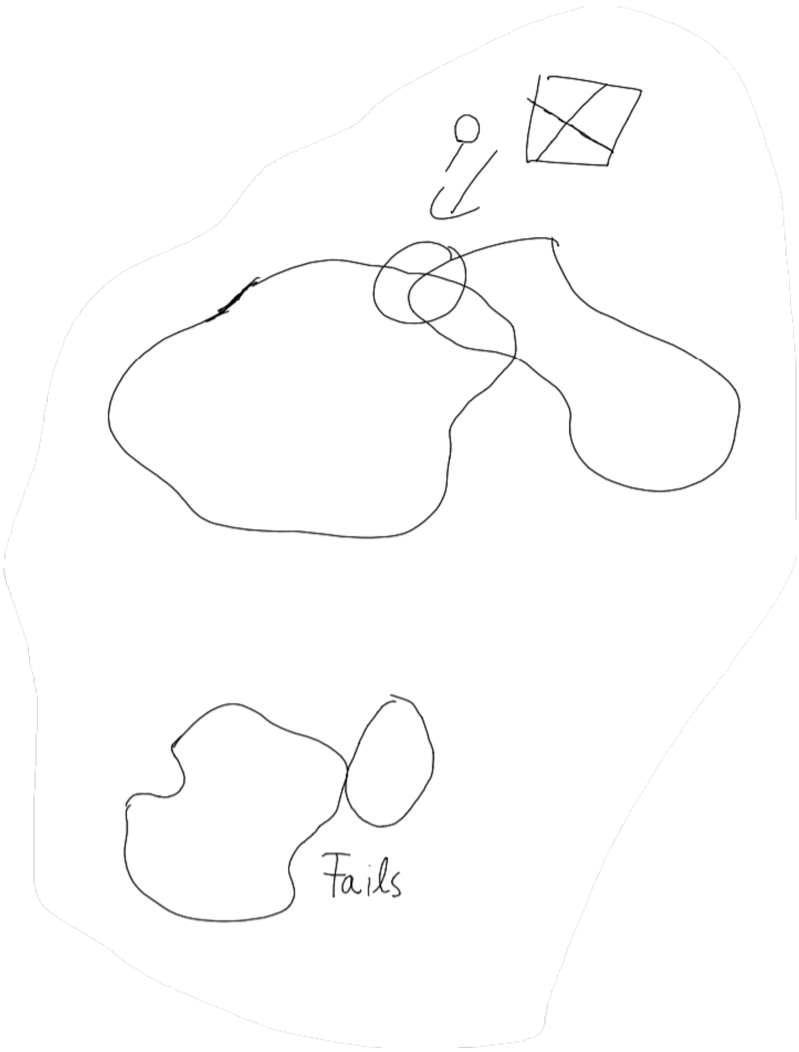


Figure 6.3: Sketch 6.03

**Observations:**

1. transversality is stable (slight changes to the lines don't change transversality)
2. transversality is generic(for pretty much any lines $l_1, l_2$ they are transverse)

Similarly to being full rank

One goal: Develop non-linear theory of transversality. I.e. replace $l_1, l_2 \subset \mathbb{R}^2$ by manifolds.
Both of the above observations will still be true.

**Announcement** On Tuesday, November 26, there will be a course evaluation.

Start of lecture 11 (15.11.2024)

- Please show up that day!
- Bring a phone / computer

### 6.1.2 Transversality for submanifolds

Let $M$ be a smooth manifold.

**Definition.** *We say that a pair of submanifolds $K, L \subset M$ are **transverse** at $p \in K \cap L$ if*

$$T_pK + T_pL = T_pM.$$

*We say that $K, L$ are **transverse** and write $K \pitchfork L$.*

*Here the sum is a gain the span of both of them*

**Remark.** *In the literature, we also see "transversal", "transversally intersecting".*

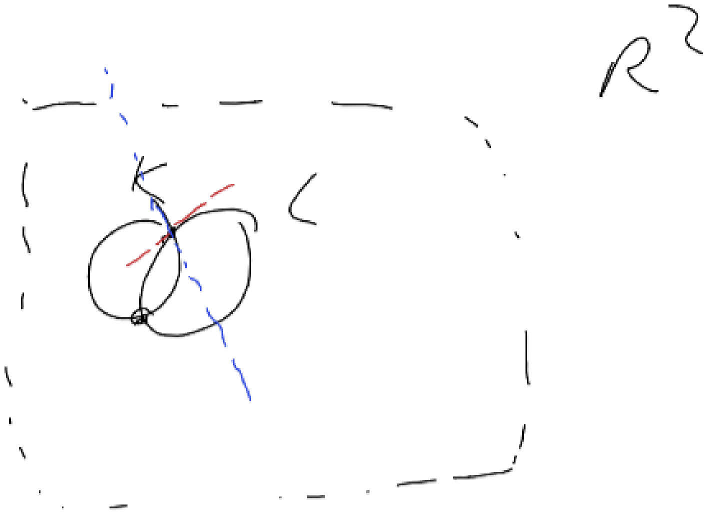


Figure 6.4: Sketch 6.05

**Example.** *$K, L$ are transverse.*


Figure 6.5: Sketch 6.06

$T_pK = T_PL$, *transversality fails.*

**Lemma 6.1.** *Let $K^k, L^l$ be submanifolds of $M$. If $K, L$ are transversal, then $K \cap L \subset M$ is a submanifold.*

*Key lemma for transversality*

**Remark.** *In general, if $S, T$ are submanifolds of $N$, then $S \cap T$ need not be a topological submanifold. For example: $f : \mathbb{R}^2 \to R$*

$$f(x, y) = x^2 - y^2.$$

*Let $g : \mathbb{R}^2 \to \mathbb{R}$*

$$g(x, y) = 0.$$

*Let* $S = \{(x,y,z) \mid z = f(x,y)\} \subset \mathbb{R}^{2+1}$ *and* $T = \{(x,y,z) \mid z = g(x,y)\} \subset \mathbb{R}^{2+1}$. *But*

$$S \cap T = \{(x,y,z) \mid z = 0, x^2 - y^2 = 0\}$$

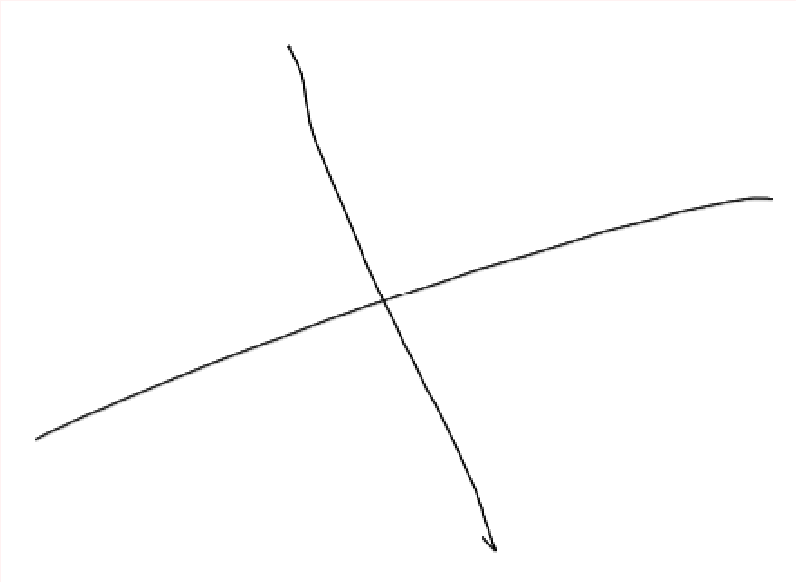

Figure 6.6: Sketch 6.07

*Look at the derivative at 0 ....*

*Proof.* This is a local question, e.g. by theorem 5.3. So we may as well assume that $M = U \subset \mathbb{R}^n$. We can also assume that $0 \in U$.
It is enough to check that $K \cap L$ smooth submanifold in a neighborhood of $p = 0$. By rank theorem (4.3), we may assume (after possibly further shrinking $U \ni 0$) that $K = f^{-1}(0), f : U \to \mathbb{R}^{n-k}$, $L = g^{-1}(0), g : U \to \mathbb{R}^{n-l}$ where $f, g$ have full rank.

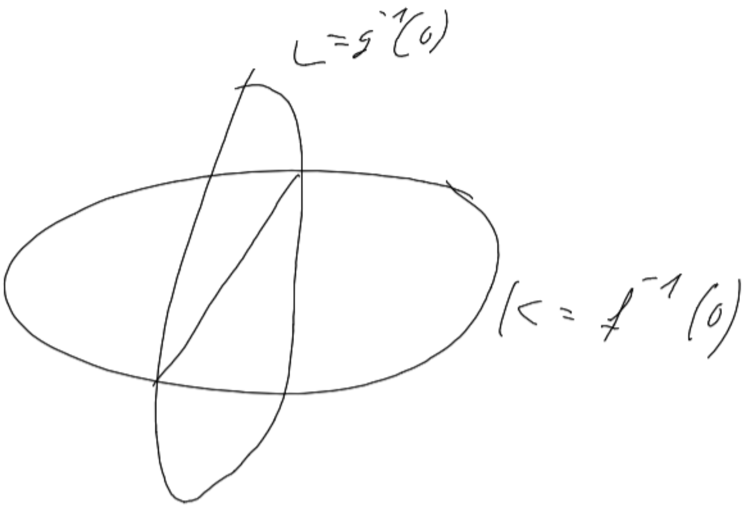


Figure 6.7: Sketch 6.08

Now we consider $H = (f, g) : U \to \mathbb{R}^{n-k} \oplus \mathbb{R}^{n-l}$. It is enough to prove that $dH_0$ is surjective (by the rank theorem). Note that $H^{-1}(0) = f^{-1}(0) \cap g^{-1}(0) = K \cap L$.
To see surjectivity of $dH_0$, we consider the exact sequences:

$$\begin{tikzcd}
T_0L + T_0K \arrow[r] \arrow[d, "\wr"', "transversality"] & T_0L/(T_0L \cap T_0K) \oplus T_0K/(T_0L \cap T_0K) \arrow[d, "{(df_0, dg_0)}"] \\
\mathbb{R}^m \equiv T_0U \arrow[r, "{dH_0=(df_0,dg_0)}"] & R^{n-k} \oplus \mathbb{R}^{m-l}
\end{tikzcd}$$

Figure 6.8: Sketch 6.09

The horizontal map $T_0L + T_0K \to T_L/(T_0L \cap T_0K) \oplus T_K/(T_0L \cap T_0K)$ sends $v + w$ to $(v, w)$. This is well defined, because if $v + w = v' + w' \implies v - v' = w - w' \in T_0L \cap T_0K$. (Equivalently, this map is just quotient by $T_0L \cap T_0K$)
Clearly the R.H vertical arrow is injective: the kernel of $df_0 = T_0K$, so $(df_0)_{|T_0L/(T_0L \cap T_0K)}$ and similarly for $dg_0$. To prove the R.H. vertical arrow is an isomorphism, do a dimension count: Exact sequence

$$0 \longrightarrow T_0K \cap T_0L \xrightarrow{v \mapsto (v,v)} T_0K + T_0L \xrightarrow{(u,w) \mapsto u-w} T_0U \equiv \mathbb{R}^n \longrightarrow 0$$

$\implies \dim(T_0K \cap T_0L) + n = k + l \implies \dim(T_0L/_{(T_0K \cap T_0L)}) = l - (k + l - n) = n - k$ and $\dim(T_0K/_{(T_0K \cap T_0L)}) = k - (k + l - n) = n - l$. We conclude that the R.H. vertical arrow is an isomorphism. □

**Remark.** *We have*

$$\begin{tikzcd}
T_0L \cap T_0K \arrow[r, hook] \arrow[d, "\wr"] & T_0L + T_0K \arrow[d, "\wr"] \\
\ker(dH_0) \arrow[r, hook] & T_0U \equiv \mathbb{R}^3
\end{tikzcd}$$

Figure 6.9: Sketch 6.10

*where the left vertical arrow is an isomorphism, due to the five lemma or diagram chasing. Hence* $\ker(dH_0) = T_0L \cap T_0K = T_0(L \cap K)$.

### 6.1.3 Transversality of maps

**Definition.** *Let*

$$\begin{tikzcd}
 & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

Figure 6.10: Sketch 6.11

*be a diagram in Top (the category of topological spaces). We let* $X \times_Z Y := \{(x, y) \mid f(x) = g(y)\} \subset X \times Y$*, endowed with the subspace topology. We call* $X \times_Z Y$ *the* ***fiber product (of the diagram)****.*

**Remark** (for enthusiasts only)**.** *It can be shown that given any topological space* $W \in$ *Top and maps*

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}$$

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd}$$

Figure 6.11: Sketch 6.12

*there exists a unique map $W \to X{}^{\times}_{z}Y$ commutes.(Universal property)*

Lots of categories admit fiber products! This is a good property for categories to have.
**Bad news:** The (not-full) subcategory $\mathrm{Man}^\infty \subset \mathrm{Top}$ does not admit fiber products (nor does $\mathrm{Man}^0 \subset \mathrm{Top}$).

**Example** (Non-example)**.** $Z = \mathbb{R}^{2+1}, X = graph(x^2 - y^2), Y = graph(0)$.

**Definition.** *Let*

$$\begin{tikzcd}
& Y \arrow[d, "g"] \\
X \arrow[r, hook, "f"] & Z
\end{tikzcd}$$

Figure 6.12: Sketch 6.13

*be a diagram in $Man^\infty$. We say that $f, g$ are **transverse** at $z = f(x) = g(y)$ if*

$$im\ df_x + im\ dg_y = T_z Z.$$

*We say that $f, g$ are **transverse** and say $f \pitchfork g$ if this holds for all such $z$.*

**Remark.** *Transversality for maps generalizes transversality for submanifolds. Take the diagram*

$$\begin{tikzcd}
& Y \arrow[d, hook] \\
X \arrow[r, hook] & Z
\end{tikzcd}$$

Figure 6.13: Sketch 6.14

**Proposition 6.2.** *If $f \pitchfork g$, then $X \times_Z Y \xrightarrow{i} X \times Y$ is a smooth embedding.*

The diagonal arrows are the obvious projections

$$\begin{tikzcd}
& Y & \\
X \times_Z Y \arrow[ur] \arrow[rr, hook] & & X \times Y \arrow[ul] \\
& X \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 6.14: Sketch 6.15

*Proof.* Some observations:

- exercise sheet 07: $X \times_Z X \times Y$ is proper.
- similarly to the proof of theorem 5.6, it is enough to prove that $i$ is an injective immersion. By definition $i$ is injective. Therefore we need to check that $i$ is smooth and the differential is injective.

Consider

$$\begin{tikzcd}
\Delta := (X,Y,Z,Z) \arrow[d, hook] & \\
X \times Y \times Z \times Z \arrow[r, "\pi"] & X \times Y \\
W = \operatorname{graph}(f,g) \arrow[u, hook] &
\end{tikzcd}$$

Figure 6.15: Sketch 6.16

where

$$\operatorname{graph}(f,g) =:= \{(x,y,x_2,z_1,z_2) \mid z_1 = f(x), z_2 = g(y)\}.$$

Then

$$W \cap \Delta = \{(x,y,z_1,z_2) \mid z_1 = z_2 = f(x) = g(y)\} = X \times_Z Y.$$

We have:

$$\begin{tikzcd}
W \cap \Delta \arrow[rr, hook, "j"] \arrow[d, "\alpha"', "\wr"] \arrow[dr, hook, "i"] & & X \times X \times Z \times Z \arrow[dl, "\pi"] \\
X \times_Z Y \arrow[r, hook] & X \times Y &
\end{tikzcd}$$

Figure 6.16: Sketch 6.17

$\alpha$ is clearly bijective and continuous. It is elementary that $\alpha$ is a closed map. That means we have to check the limit points. $W \cap \Delta$ is closed, i.e. contains the same limit points… Therefore $\alpha$ is a homeomorphism.

By lemma 6.1, if we can show that $W \pitchfork \Delta$, then $W \cap \Delta \overset{j}{\hookrightarrow} X \times Y \times Z \times Z$ is smooth embedding. Hence $i := \pi \circ j$ smooth. Let us now check that $W \pitchfork \Delta$ at some arbitrary point $p = (x,y,z,z) \in W \cap \Delta \subset X \times Y \times Z \times Z$. Note that $z = f(x) = g(y)$. We have

$$T_pW = \{(v, w, df_x(v), dg_y(w))\}$$

and

$$T_p\Delta = \{v', w', u, u\},$$

where $v, v' \in T_xX, w, w' \in T_yY, u \in T_zZ$. We need to check: $T_pW + T_p\Delta = T_p(X \times Y \times Z \times Z)$. We must show that for an arbitrary $(a,b,c,d) \in T_p(X \times Y \times Z \times Z) = T_xX \oplus T_yY \oplus T_zZ \oplus T_zZ$. We must solve:

Start of lecture 12 (19.11.2024)

$$\begin{aligned}
a &= v + v' \\
b &= w + w' \\
c &= u + df_x(v) \\
d &= u + dg_y(w)
\end{aligned}$$

for some $\underbrace{(v, w, df_x(v), dg_y(w))}_{\in T_pW}, (v', w', u, u) \in T_p\Delta$. The above is equivalent to

we solve this, since we want to show $f \pitchfork g \iff \forall z \in X \times_Z Y :$ $\operatorname{im} df + \operatorname{im} dg = T_zZ$

$$\begin{aligned}
c - d &= df_x(v) - dg_y(w) \in T_zZ && \text{By assumption there exists } v, w \text{ s.t. equation holds} \\
c + d &= 2u + df_x(v) + dg_y(w) && \text{can solve by picking suitable } u \\
a - v &= v' && \\
b - w &= w' && \text{choose } v', w' \text{ s.t. this holds}
\end{aligned}$$

Follows from Lemma 6.1 that

$$\begin{tikzcd} \Delta \cap W \arrow[r, hook] \arrow[rr, bend right, "i"] & X \times Y \times Z \times Z \arrow[r, "\pi"] & X \times Y \end{tikzcd}$$

is a smooth submanifold. Finally, $d_i$ is injective. This is clear, because $T_pW \longrightarrow T_{(x,y)}X \times Y$ is injective. Indeed, if $(v, w, df_x(v), dg_y(w)) \mapsto 0$, then $(v, w) = (0, 0)$, but then $(v, w, df_x(v), dg_y(w)) = (0, 0, 0, 0)$. $\square$

## 6.2 Sard's theorem

### 6.2.1 Measure theory on manifolds

**Definition.** *A subset $S \subset \mathbb{R}^n$ has* ***measure zero*** *if, for any $\epsilon > 0$, there exists a family $\{C_i\}_{i=1}^{\infty}$ of rectangles:*

$$\mathbb{R}^n \supset C_i = (a_1^i - \epsilon_1^i, a_1^i + \epsilon_1^i) \times \cdots \times (a_n^i - \epsilon_n^i, a_n^i + \epsilon_n^i),$$

*where $(a_1^i, \ldots, a_n^i) \in \mathbb{R}^n$ and $(\epsilon_1^i, \ldots, \epsilon_n^i) \in \mathbb{R}^n_{>0}$, s.t.*

$$S \subset \bigcup_{i=1}^{\infty} C_i \wedge \sum_{i=1}^{\infty} vol(C_i) < \epsilon.$$

**Remark.** *We would get an equivalent definition, if we replaced rectangles with balls, cubes, parallellograms,....*

**Example.** *Suppose $S \subset \mathbb{R}^1$ and $|S| < \infty$, clearly $S$ now has measure zero:*
*$a_i \in S \implies (a_i - \epsilon, a_i + \epsilon)$ has finite volume $2n\epsilon$.*

- *$S \subset \mathbb{R}$, $S$ countable. Then $S$ has measure zero: Take $(a_i - \frac{\epsilon}{2^i}, a_i + \frac{\epsilon}{2^i})$*

**Lemma 6.3.** *(i) If $A \subset B \subset \mathbb{R}^n$ and $B$ has measure zero, then $A$ has measure zero.*

*(ii) if $A \subset \mathbb{R}^n$ is a countable union of measure zero subsets, then $A$ also has zero measure.*

*Proof.* Emitted. $\square$

**Lemma 6.4.** *Let $A \subset \mathbb{R}^n$ be compact. Suppose that for all $c \in \mathbb{R} : A \cap \{c\} \times \mathbb{R}^{n-1}$ has $(n-1)$-dimensional measure zero. Then $A$ has $n$-dimensional measure zero.*

This is misleading ...

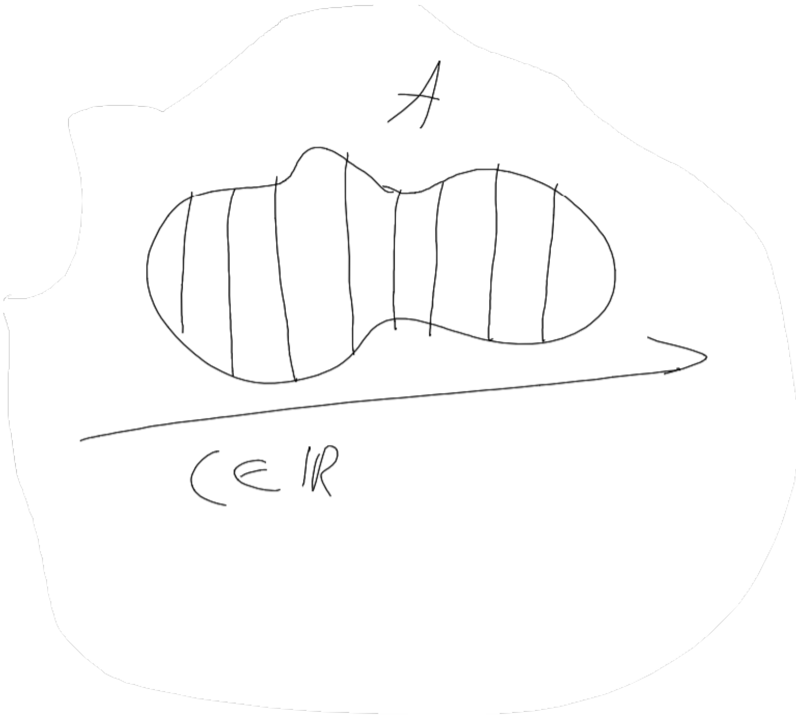


Figure 6.17: Sketch 6.18

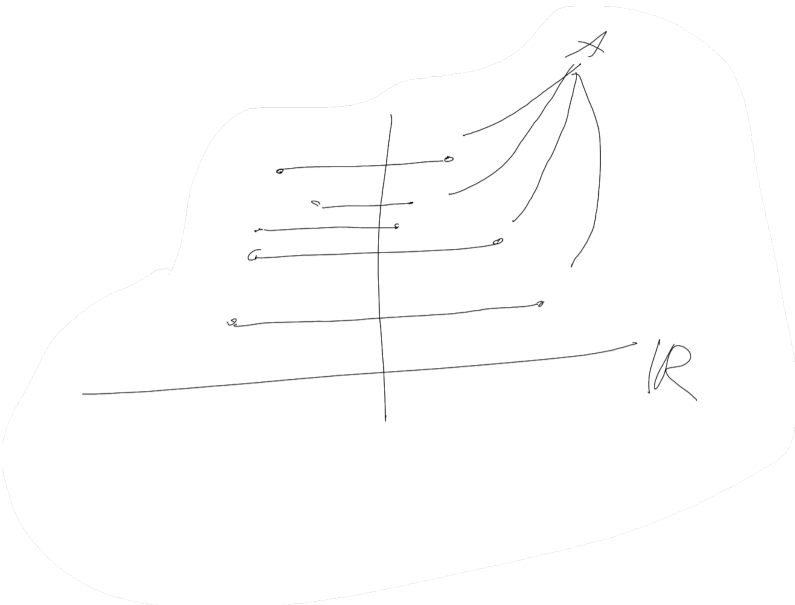


Figure 6.18: Sketch 6.19

*Proof.* Choose $a < b, a, b \in \mathbb{R}$ so that $A \subset (a, b) \times \mathbb{R}^{n-1}$. Let $A_c := \{x \in \mathbb{R}^{n-1} : (c, x) \in A\}$. Fix $\epsilon > 0$. By assumption, we can cover $A_c$ by a union of rectangles $U_c := \bigcup_{i=1}^{\infty} C_c^i$ such that $\sum_{i=1}^{\infty} \text{vol}(C_i) < \epsilon$.
By compactness there exists an open interval $J_c$ such that $A \cap J_c \times \mathbb{R}^{k-1} \subset J_c \times U_c$.

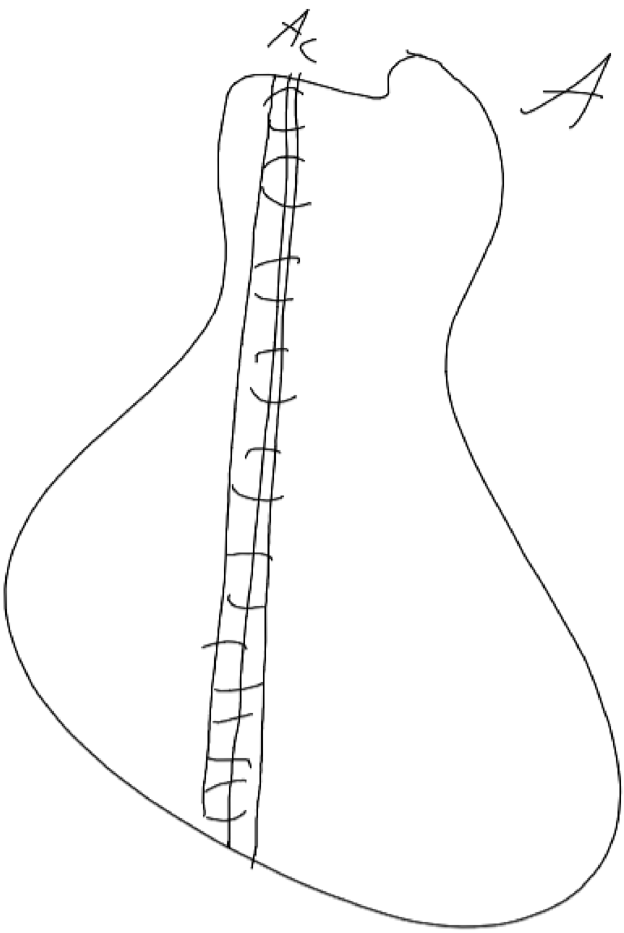


Figure 6.19: Sketch 6.20

Otherwise, there exists a sequence $(c_i, x_i), c_i \to c \wedge x_i \notin U_c \wedge (c_i, x_i) \in A$. By compactness one can extract a convergent subsequence $\to (c, x) \in A$, $x \in U_c^C$. This is impossible, since $A \subset U_c$. By compactness of $[a, b]$, there is a finite sequence $a = c_1 < c_2 < \cdots < c_l = b$ such that

Since $A$ compact

$$\bigcup_{i=1}^{l} J_{c_i} \text{ covers } [a, b].$$

We can freely assume up to deleting certain $J_{c_i}$ that $\sum \mathrm{vol}(J_{c_i}) < 2|b-a|$. Finally: $A \subset \bigcup_{i=1}^{l} J_{c_i} \times U_{c_i}$. But

He writes $|J_{c_i}| \ldots$

$$\mathrm{vol}(J_{c_i} \times U_{c_i}) \leq \sum |J_{c_i}| \times |U_{c_i}| \leq \epsilon \sum |J_{c_i}| = 2\epsilon|b-a|$$

$\square$

**Corollary 6.5.** *Let $f : \underbrace{A}_{\subset \mathbb{R}^n} \to \mathbb{R}^1$, where $A$ is a countable union of compact subsets (e.g. $A$ could be open or closed) and $f$ continuous. Then the graph of $f$:*

$$\mathit{graph}(f) = \{(x, y) \mid y = f(x)\}$$

*has measure zero as a subset of $\mathbb{R}^{n+1}$.*

$\mathbb{R}^n \times \{c\} \subset \mathbb{R}^{n+1}$ *is a measure zero set*

*Proof.* Assume $A$ compact. Argue by induction. If $n = 0$, trivial. Assume that the result holds for $\leq n+1$. Observe that $\forall c \in \mathbb{R}$, $\mathrm{graph}(f) \cap \{c\} \times \mathbb{R}^{(n-1)+1} = \mathrm{graph}\left(f_{|_{\{c\} \times \mathbb{R}^{n-1}}}\right)$, which by induction has measure zero. Hence follows from Lemma 6.4. For general $A$, write $A = \bigcup_{i=1}^{\infty} K_i$. Then

Holds by Lemma 6.3

$$\text{graph}(f) = \bigcup_{i=1}^{\infty} \underbrace{\text{graph}\left(f_{|\{K_i\}}\right)}_{\text{measure zero}}$$

□

**Lemma 6.6.** *Let $A \subset \mathbb{R}^n$, let $F : A \to \mathbb{R}^n$ be smooth. If $A$ has measure zero, so does $F(A)$.*

**Remark.** *Smoothness is important. The lemma would be false if we only assume $F$ to be continuous. Example: $F$ the cantor function. Smoothness is way to strong. Absolutely continuous functions are the correct class.*

*Proof of lemma 6.6.* By definition, for any $p \in A$, $F$ extends to a smooth map on a neighborhood of $p$. Up to shrinking this neighborhood $U_p$ that $F$ extends to $\overline{U_p}$. We can also assume that $U_p$ is a ball. Note that $A \subset \bigcup_{p \in A} U_p$. By lemma 1.5, we can extract a countable subcover. Hence it is enough to prove that $F(A) \cap U_p$ has measure zero for all $p \in A$.
By Taylor's theorem, $F$ is uniformly continuous on $\overline{U_p}$, and we have

Here we use smoothness

$$|F(x) - F(y)| < Q|x - y| \tag{1}$$

for all $x, y \in \overline{U_p}$. Fix $\delta > 0$. Since $A \cap \overline{U_p}$ has measure zero, can cover $A \cup \overline{U_p}$ by a countable union of balls $C_i$, such that $\sum_i \text{vol}(C_i) < \delta$. By (1),

$$\text{diam}(F(\overline{U_p} \cap C_i)) < Q' \text{diam}(C_i)$$

, where $Q' \leq 100Q$. $\implies$ $F(A \cap \overline{U_p})$ is contained in a countable union of balls $D_i$ of diameter $\leq Q' \text{diam}(C_i)$. Hence

$$\sum_{i=1}^{\infty} \text{vol}(D_i) < Q' \sum \text{vol}(C_i) < 100^n Q' \delta.$$

□

**Definition.** *Let $M$ be a manifold. A subset $S \subset M$ has measure zero, if, for all charts $(U, \phi)$, $\varphi(S \cap U)$ has measure zero in $\mathbb{R}^n$.*

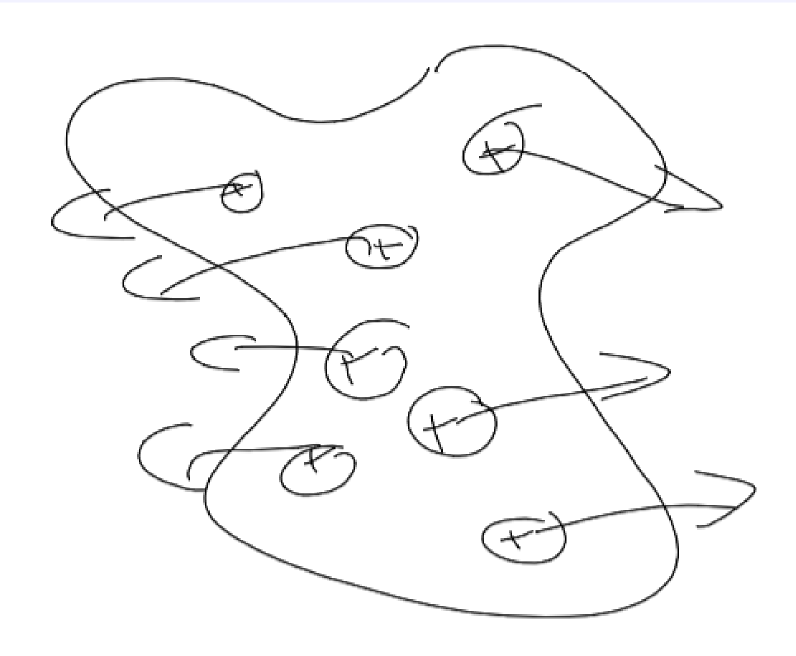

Figure 6.20: Sketch 6.21

**Lemma 6.7.** *If $\mathcal{A} \subset \mathcal{A}'$ is an inclusion of atlases. Then $S$ has measure zero w.r.t. $\mathcal{A} \iff S$ has measure zero w.r.t. $\mathcal{A}'$.*

*Proof.* Assume that $S$ has measure zero w.r.t. $\mathcal{A}$, i.e. $\phi_\alpha(S \cap U_\alpha)$ has measure zero for all charts $(U_\alpha, \phi_\alpha)$. Assume $(V, \psi)$ is a chart for $\mathcal{A}'$. Then

$$\begin{aligned} \psi(S \cap V) &= \psi\left(\bigcup_{\alpha \in \mathcal{A}} (U_\alpha \cap S) \cap V\right) \\ &= \psi\left(\bigcup_{\alpha \in \mathcal{A}} (U_\alpha \cap S \cap V)\right) \\ &= \psi\left(\bigcup_{\alpha \in \mathcal{A}} \phi_\alpha^{-1} \circ \phi_\alpha(U_\alpha \cap S \cap V)\right) \\ &= \bigcup_{\alpha \in \mathcal{A}} \underbrace{\psi \circ \phi_\alpha^{-1} \left(\phi_\alpha(U_\alpha \cap S \cap V)\right)}_{\text{measure by smoothness}} \end{aligned}$$

because the $U_\alpha$ form a cover of $M$, up to replacing $\mathcal{A}$ by a countable cover. □

Start of lecture 13 (22.11.2024)

**Lemma 6.8.** *Let $M, N$ be smooth manifolds and $f : M \to N$ a smooth map. If $A \subset M$ has measure zero, then $F(A) \subset N$ also has measure zero.*

*Compare lemma 6.6*

*Proof.* Fix $\{(U_\alpha, \varphi_\alpha)\}$ a countable atlas for $M$. We need to show that given any chart $(V, \psi)$ on $N$, $\psi(V \cap F(A))$ has measure zero. We may as well assume that $F(A) \subset V$ (otherwise replace $A$ with $F^{-1}(A) \cap V$). Observe that $\psi(F(A))$ is the countable union of these sets $\psi(F(\phi_i^{-1}(\phi_i(A \cap U_i))))$:

$$\psi(F(A)) = \bigcup_i \psi(F(\phi_i^{-1}(\phi_i(A \cap U_i)))).$$

But $\phi_i(A \cap U)$ has measure zero and $\psi \circ F \circ \phi_i^{-1}$ is a smooth function, which is applied to a subset of measure zero of $\mathbb{R}^n$. Therefore the set has measure zero by lemma 6.6 along with the fact that countable unions of measure zeros subsets have measure zero (lemma 6.3). □

### 6.2.2 Sard's theorem

**Definition.** *Let $F : M \to N$ be smooth. Given a point $x \in M$, we say that $x$ is a* ***critical point*** *of $F$, if the differential $dF_x : T_xM \to T_{F(x)}N$ fails to be surjective. Otherwise we say that $x$ is a* ***regular point****.*
*A point $y \in N$ is a* ***critical value*** *if $F^{-1}(y)$ contains a critical point. Otherwise we say $y$ is a* ***regular value****.*

*This coincides with the analyis 1 definition*

**Remark.** *If $F^{-1}(y) = \emptyset \implies y$ is a regular value, but not the image of a regular point!*

**Theorem 6.9** (Sard)**.** *Let $M, N$ be smooth manifolds. Let $F : M \to N$ be a smooth map. Then the set of critical values of $F \subset N$ has measure zero.*

*Very important theorem*

**Example.** *$M \to \mathbb{R}, M \ni x \mapsto 0 \in \mathbb{R}$. Here the set of* ***critical points*** *has full measure (since it is $M$), But the set of* ***critical values*** *is $\{0\} \subset \mathbb{R}$.*

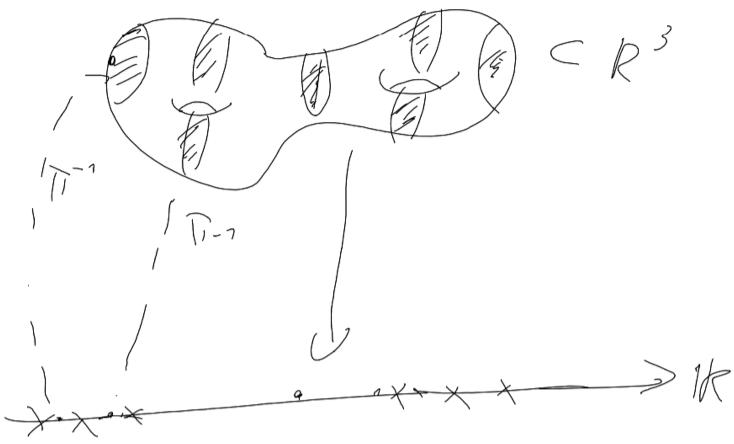


Figure 6.21: Sketch 6.22

**Example.** *Morse theory: study the topology of manifolds by studying functions on them.*

**Corollary 6.10.** *Let $F : M^m \to N^n$ be a smooth map. If $m < n$, then $im(F) \subset N$ has measure zero.*

*Proof.* Clear for dimensional reasons. ☐

**Corollary 6.11.** *Let $M^m \subset \mathbb{R}^N$ be a submanifold. Write $\mathbb{R}^{N-1} = \{(x_1, \ldots, x_{N-1}, 0)\} \subset \mathbb{R}^N$. Given $v \in \mathbb{R}^N - \mathbb{R}^{N-1}$, we set $\pi_v : \mathbb{R}^N \to \mathbb{R}^{N-1}$ to be the projection with kernel $\mathbb{R} \cdot v$.*

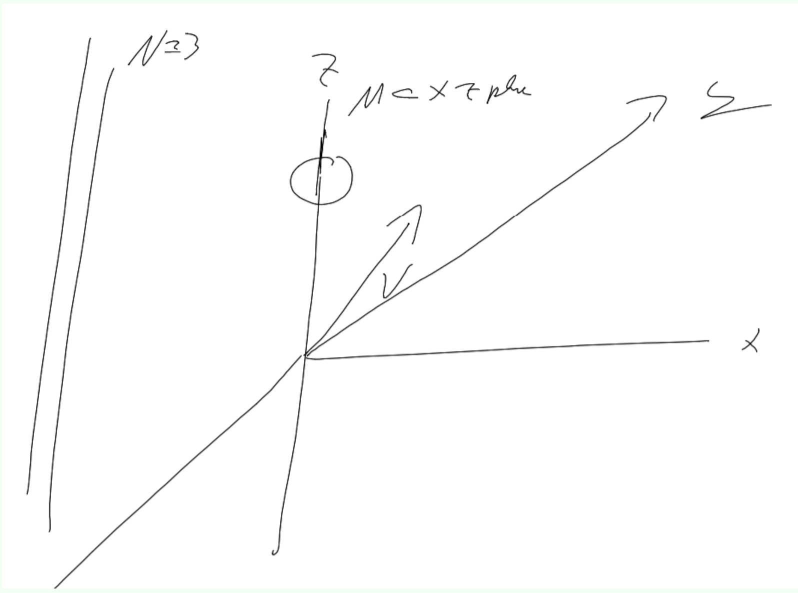


Figure 6.22: Sketch 6.23

*Assume that $N > 2m + 1$. Then the set of $v \in \mathbb{R}^N \setminus \mathbb{R}^{N-1}$ such that $\pi_{v|_M} : M \to \mathbb{R}^{N-1}$ is an injective immersion is non-empty. It is, in fact, dense*[a].

[a] In $\mathbb{RP}^{N-1}, \mathbb{R}^N$

**Example.** *Take $v = (0, 0, 1)$. Then $\pi_{(0,0,1)}(M = S^1) = [-1, 1]$, and not injective.*

**Example.** *$v = (1, 1, 1), \pi_v(M = S^1) = S^1$, up to scaling of the axis.*

*Proof.* Firstly, $\pi_{v|_M}$ is injective $iff$ for all $p \in M$, $(p + tv)_{t \in \mathbb{R}} \cap M = \{p\}$.

Secondly, $\pi_{v|_M}$ is an immersion $\iff$ for all $p \in M$, $T_pM \cap \ker d\pi_v = 0$[1] $\iff$ $\overbrace{T_pM}^{\subset \mathbb{R}^N}$ does not contain $v$.
Let $\Delta \subset M \times M$ be the diagonal (i.e. $\Delta = \{(x,x) \mid x \in M\} \subset M \times M$). Let $0_M \subset TM$ be the **zero section**:

$$0_M := \{(p,0) \in TM\} \subset TM$$

where

$$TM = \bigcup_{p \in M} T_pM.$$

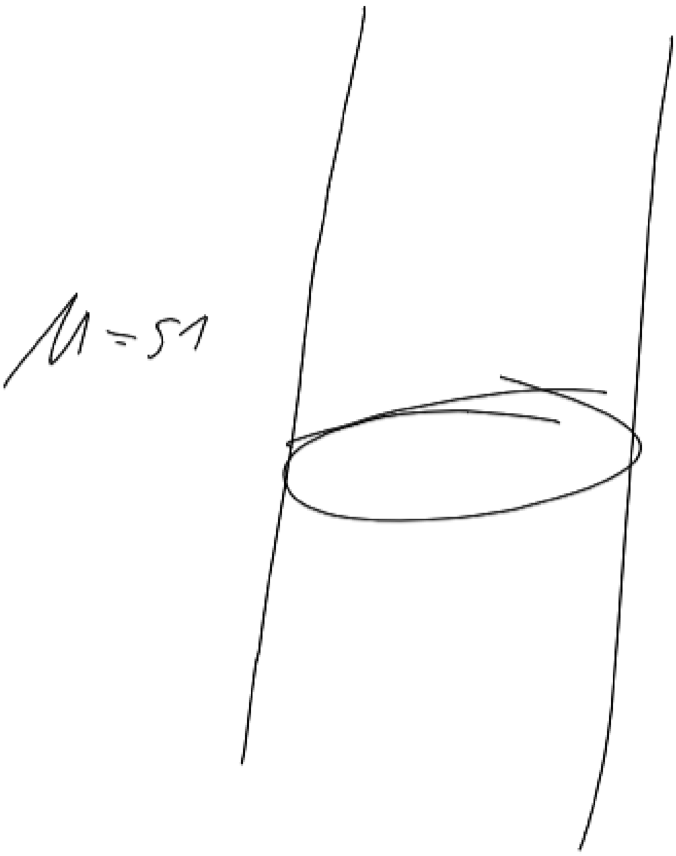


Figure 6.23: Sketch 6.24

Define

$$\begin{aligned}\alpha :& M \times M \setminus \Delta \to \mathbb{RP}^{N-1}\\ &(p,q) \mapsto [p-q]\\ \beta :& TM \setminus O_M \to \mathbb{RP}^{N-1}\\ &(p,w) \mapsto [w]\end{aligned}$$

It is easy to check that $\alpha, \beta$ are smooth. Check $\alpha$

$$(p,q) \mapsto \underbrace{p-q}_{\in \mathbb{R}^N - \{0\}} \mapsto \underbrace{[p-q]}_{\in \mathbb{R}^N \setminus \{0\}/\mathbb{R}^\times} \equiv \mathbb{RP}^{N-1}.$$

[1] i.e. the zero vector space

Note that $N-1 > 2m$, and dimension of $M \times M \setminus \Delta$ and $TM - O_M$ is $2m$. It follows by corollary 6.10 that $\operatorname{im}(\alpha) \cup \operatorname{im}(\beta) \subset \mathbb{RP}^{N-1}$ has measure zero. Finally the conclusion follows from sheet 08. □

To see that quotient maps are smooth is a good exercise for the exam

**Corollary 6.12** (Strong Whitney embedding)**.** *Suppose that $M^m$ (compact) manifold. Then $M$ admits an embedding into $\mathbb{R}^{2m+1}$.*

**Remark.** *Compactness is not a necessary assumption. But our prof. assumes compactness. If we use this, we don't have to use compactness.*

*Proof.* By theorem 5.6, $M$ admits an embedding into $\mathbb{R}^N$, $N \gg 1$. If $N > 2m+1$, then by corollary 6.11 there exists $v \in \mathbb{R}^N \setminus \mathbb{R}^{N-1}$, such that $\pi_{v|_M} : M \to \mathbb{R}^{N-1}$ is an injective immersion.

By repeatedly applying corollary 6.11, we get an injective immersion from $M \overset{i}{\hookrightarrow} \mathbb{R}^{2m+1}$. As in the proof of theorem 5.6, $i$ must be an embedding, because $M$ is compact. □

**Corollary 6.13.** *Let $I, X, Y, Z$ be manifolds. Let $f : X \times I \to Z, g : Y \to Z$ be smooth maps. Suppose that $f \pitchfork g$. Then for almost all $s \in I$*

$$f_s(\cdot) = f(\cdot, s) \pitchfork g.$$

*i.e. away from a set of measure zero*

**Remark.** *Let $f_0 : X \to \mathbb{R}^n, g : Y \to \mathbb{R}^n$ be any maps. Consider*

$$f : X \times \mathbb{R}^n \to \mathbb{R}, (x, s) \mapsto f_0(x) + s$$

*. Then $f$ is clearly a submersion. Hence $f \pitchfork \psi \implies f_s \pitchfork g$ for almost all $s$.*

*Proof.* By assumption and proposition 6.2:

$$\begin{tikzcd} W = (X \times I) \times_Z Y \arrow[r, hook, "\text{smooth embedding}"] \arrow[dr, "\pi"'] & (X \times I) \times Y \arrow[d] \\ & I \end{tikzcd}$$

We will show that if $s \in I$ is a regular value of $\pi$, then $f_S \pitchfork g$. This implies the corollary by Sard's theorem 6.9.
So suppose that $s \in I$ is a regular value. Then either

(i) $s \notin \operatorname{im}(\pi)$. In this cae $\operatorname{im}(f_s) \cap g = \emptyset$

(ii) $s \in \operatorname{im}(\pi)$. In this case $d_\pi$ is surjective on $\pi^{-1}$.

Let's assume case (ii). Suppose that $z = f_s(x) = g(y)$. Since $f \pitchfork g$, we have

$$\operatorname{im} df_x + \operatorname{im} dg_y = T_z Z.$$

FOr any $a \in T_z Z$, there exists a pair $b = (w, e) \subset T_{x,s}(X \times I) = T_x X \oplus T_s I$, such that $df_{(x,s)}(w, e) - a \in \operatorname{im} dg_y$.
Since $d\pi$ is surjective, there exist an element $(w', e, c') \in T_{(x,s,y)} W = T_{(x,s,y)}(X \times I) \times_Z Y$. But now

$$(df_s)_x(w - w') - a = df_{(x,s)}((w, e) - (w', e)) - a = \underbrace{df(\overbrace{b}^{=(w,e)}) - a}_{\in \operatorname{im} dg_y} - \underbrace{\overbrace{df(w', e)}^{dg_y(c')}}_{\in \operatorname{im} dg_y}$$

$$\begin{tikzcd} (X \times I) \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \times I \arrow[r] & Z \end{tikzcd}$$

$\implies (df_s)_x(w - w') - a \in \operatorname{im} dg_y$.

□

In this lecture we will try to prove theorem 6.9 using three intermediate lemmas.

Start of lecture 14 (26.11.2024)

**Notation (auxiliaray):** Consider $U \subset \mathbb{R}^m$ open, $F : U \to \mathbb{R}^n$. We let $C \subset U$ be the set of critical points of $F$. More generally, for $k \geq 1$ we let

$$C \supset C_k \coloneqq \{x \in C \mid \forall 1 \leq i \leq k : \text{All ith partial derivatives of } F \text{ vanish at } x\}.$$

Clearly $C \supset C_1 \subset C_2 \subset \ldots$. Note also $C, C_k$ are all closed.

Because being zero is a closed condition

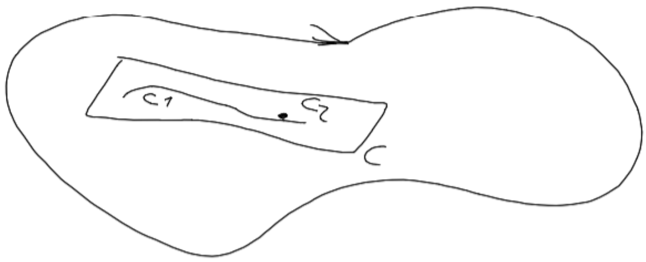


Figure 6.24: Sketch 6.26

**Lemma 6.14.** *Suppose that $k > \frac{m}{n} - 1$. Then $F(C_k)$ has measure zero.*

*Proof.* For each $a \in U$, there exists a closed cube $a \in E \subset U$. By second countability we can cover $C_k$ by countability many such cubes. Hence it is enough to prove, for arbitrary such $E$ that $F(C_k \cap E)$ has measure zero. Now fix $a \in C_k$ and cube $E \ni a$. Also fix $A > \sup_{y \in E} |\partial_x^\alpha F(y)|$ for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m, |\alpha| \leq k+1$.
Let $L > 0$ be the side length of $E$. Let $K \gg 1$ be a natural number. We now subdivide $E$ by $K^m$ cubes of side length $L/K$. Let $E_1, \ldots, E_{K^m}$ be an enumeration of these subcubes.

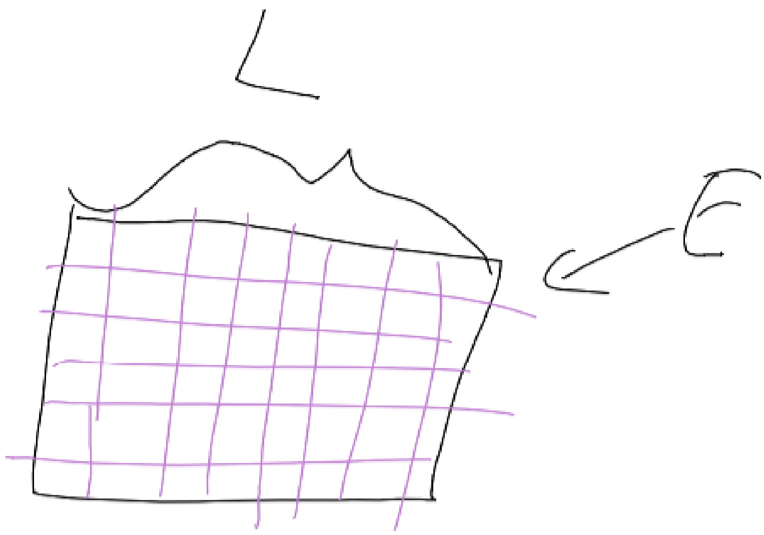


Figure 6.25: Sketch 6.27

Since $a \in E$ there exists some $i_0$ such that $a \in E_{i_0}$. Since $a \in C_k$, Taylor's theorem finishes the following inequality:

$$|F(x) - F(a)| \leq A'|x-a|^{k+1}$$

for all $x \in E_{i_0}$, where $A'$ depends only on $A$.

$\implies F(E_{i_0})$ is contained in a ball centered at $F(a)$ of radius $A'(L/K)^{k+1}$. Now

$$F(C_k \cap E) = \bigcup_{i \mid C_k \cap E_i \neq \emptyset} F(C_k \cap E_i).$$

But each $F(C_k \cap E_i)$ is contained in a union of balls of volume $\leq \Lambda \left[A'(L/K)^{k+1}\right]^n$. Therefore at most $K^m$ cubes $E_i$ which intersect $C_k$ non-emptily. Hence $F(C_k \cap E)$ is contained in a union of balls of total volume at most

$$\Lambda {A'}^n K^m \left[(L/K)^{k+1}\right]^n = \lambda {A'}^n L^{(k+1)n} K^{m-(k+1)n}.$$

Since $k > \frac{m}{n} - 1$ the exponent of $K$ is negative and hence increasing $K$ forces the equation above to go to zero. □

Restricting to the cubes which intersect $C_k$ is a very important step, because otherwise we can't Taylor and have no control over the measure!

**Lemma 6.15.** *Assume that Sard's theorem holds for domains of dimension $< m$. Then $F(C \setminus C_1)$ has measure zero.*

*Proof.* Since $C_1$ is closed in $U$, we can assume after replacing $U$ by $U \setminus C_1$, that $C_1 = \emptyset$. Then we just prove that $F(C)$, under that assumption, has measure zero.
Fix $a \in C$. By assumption that $C_1 = \emptyset$. Up to reordering coordinates in the source and in the target, we can assume that $\partial_{x_1} F^1(a) \neq 0$. Set $\begin{cases} u(x) = F^1(x) \\ v^i(x) = x_i & 2 \leq i \leq m \end{cases}$. By the inverse function theorem $(u, v) = (u, v^1, \ldots, v^m)$ forms a coordinate system in some neighborhood $V_a$ of $a$. Since the transition matrix is

$$\begin{bmatrix} \partial_{x_1} F^1 & \star & \star & \star \\ 0 & & 1 & \end{bmatrix}.$$

Bird's eye view: Change coordinates, consider modified functions

We can assume that $(u, v)$ extend to $\overline{V_a}$. With respect to these new coordinates $(u, v)$, we have $F(u, v) = (u, F^2(u, v), \ldots, F^n(u, v))$. So we have: $dF(u, v) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \star & & & \\ \vdots & & \frac{\partial F^i}{\partial v^j} & \\ \star & & & \end{bmatrix}$ where $2 \leq i \leq n, 2 \leq j \leq m$. Therefore $C \cap \overline{V_a}$ is precisely the set of points such that $\operatorname{rank}\left(\frac{\partial F^i}{\partial v^j}\right) < n - 1$.
Note that $F(C \cap \overline{V_a})$ is compact. By lemma 6.4, if we can show that $F(C\cap) \cap \{y^1 = d\}$ has measure zero, then $F(C \cap \overline{V_a})$ has measure zero. Since $C$ is covered by countably many such $V_a$ (by second countability), we could conclude that $F(C)$ has measure zero.
For $d \in \mathbb{R}$, $B_d \coloneqq \{v \mid (d, v) \in \overline{V_\alpha}\} \subset \mathbb{R}^{n-1}$. Set $F_d(v) \coloneqq (F^2(d, v), \ldots, F^n(d, v))$. Since $F(d, v) = (d, F_d(v))$, we have that the critical values of $F_{|\overline{V_a}}$ that lie in $\{y^1 = d\}$ are precisely the points $(d, q)$, where $q$ are critical values of $F_d$. By assumption that Sard's theorem 6.9 holds for dimension $< m$, since the domain of $F_d$ has dimension $m - 1 < m$, $\{\text{critical values of } F_d\} = \{y_1 = d\} \cap F(C \cap \overline{V_a})$ has measure zero. □

**Lemma 6.16.** *Assume that Sard's theorem holds for domains of dimension $< m$. For all $k \geq 1$, $F(C_k \setminus \{F(C_{k+1})\})$ has measure zero.*

*Proof.* As in the proof of the previous lemma 6.15, we can assume $C_{k+1} = \emptyset$. We will prove under that assumption that $F(C_k)$ has measure zero.
Let $a \in C_k$ be arbitrary. Let $\sigma : U \to \mathbb{R}$ be a k-th partial derivative of $F$, with the property that $\sigma$ has at least one non-vanishing partial at $a$. I.e. $\begin{cases} \sigma = \partial_x^\alpha F & |\alpha| = k \\ \partial_{x_i} \sigma(a) \neq 0 & \forall i \end{cases}$. Let $V_a$ be a neighborhood of $a$ consisting of regular points of $\sigma$. Let $\Sigma \coloneqq \{\sigma^{-1}(0)\} \cap V$. Then $\Sigma$ is a smooth submanifold in $V_a$. By definition of $C_k$, we have $(C_k \cap V_a) \subset \sigma^{-1}(0) \cap V_a$.
Moreover $F(C_k \cap V_a)$ is contained in the set of critical values of $F_{|\Sigma}$ (that's because $\partial_{x_i} F^j = 0 \implies dF_{|T\Sigma} \equiv 0$). But $\dim(\Sigma) = \dim(U) - 1 = m - 1$. Hence $\{$ critical values of $F_{|\Sigma}\}$ has measure zero, by assumption. □

*Proof of Sard's theorem 6.9.* We prove it by induction on the dimension of the source. If $F : M^m \to N^n$, and $m = 0$ the statement is true.
Let's assume that Sard's theorem has been proven for all manifolds $F : M^m \to N^n$, where $m < \tilde{m}$. We need to prove it for maps $\tilde{F} : \tilde{M} \to \tilde{N}$, where $\dim(\tilde{M}) = \tilde{m}$.

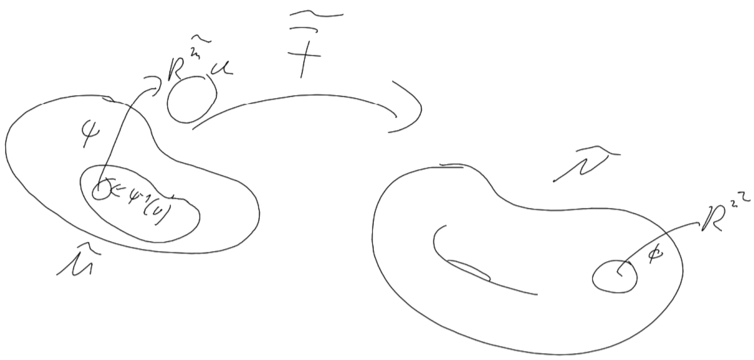

Figure 6.26: Sketch 6.28

By covering source and target by charts, we can assume

If the intersection is non-empty, we reall need lemma 6.14

- $\tilde{M} = U \subset \mathbb{R}^m$
- $\tilde{N} \subset \mathbb{R}^n$.

Apply lemma 6.14,6.15,6.16 to get the claim. $\square$

# Chapter 7: Vector fields

## 7.1 Basics

Start of lecture 15 (29.11.2024)

Let $M$ be a smooth manifold. Recall that we have

$$\pi : TM \to M$$

where $TM = \coprod_{p \in M} T_p M$. A typical point in $TM$ is $(p, \underbrace{v}_{\in T_p M})$ and

$$\pi((p, v)) = p.$$

**Definition.** *A (smooth)* ***vector field*** *$X$ on $M$ is a section of $\pi : TM \to M$. In other words*

1. *$X : M \to TM$ smooth map*
2. *$\pi \circ X = id$*

*We let $\mathcal{X}(M)$ be the set of vector fields on $M$.*

**Concretely:** To every point $p \in M$, we associate a vector

$$X(p) = X_{\in} T_p M.$$

Visually

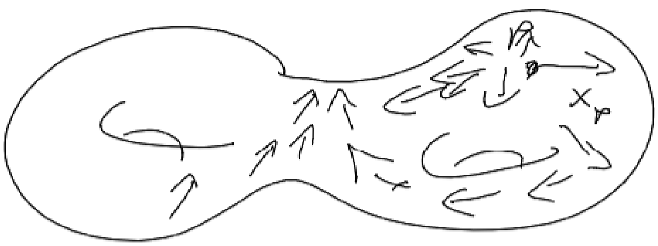


Figure 7.1: Sketch 7.01

Another picture:

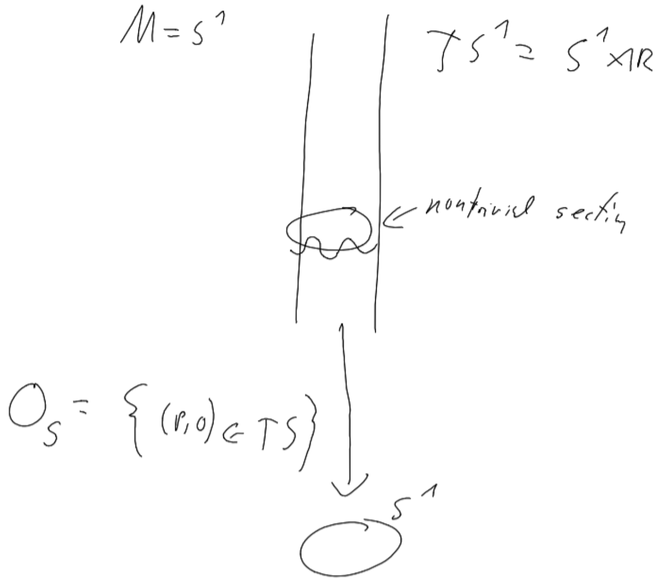


Figure 7.2: Sketch 7.02

**Lemma 7.1.** *Let $M$ be a smooth manifold.*

*(a) $\mathcal{X}(M)$ is a $\mathbb{R}$ vector space*

*(b) $\mathcal{X}(M)$ is a module over the ring $C^\infty(M)$ of smooth functions on $M$:*

$$(f, X) \mapsto fX$$

$$fX(p) = \underbrace{f(p)}_{\in\mathbb{R}}\underbrace{X(p)}_{\in T_pM}$$

*Proof.* Exercise. □

**Remark.** *In this class, we only consider smooth vector fields. If you drop the smoothness condition on the map $X : M \to TM$, you get a* ***rough vector field****.*

*We are not gonna study those, but it is useful to know they exist*

**Example.** *Recall that $T_p\mathbb{R}^n \equiv \mathbb{R}^n$, hence $T\mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$. A vector field $X \in \mathcal{X}(\mathbb{R}^n)$ is just a map*

$$X : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

$$p \longmapsto (p, v(p))$$

*Equivalently $X$ is a map $\mathbb{R}^n \to \mathbb{R}^n$, since the first coordinate is just fixed to be the identity. This agrees with the notion from Analysis 2.*

*A map $\mathbb{R}^n \ni p \mapsto (X^1(p), \dots, X^n(p))$*

**Remark.** *$T_p\mathbb{R}^n$ has a canonical basis $\{(\partial_{x_1})_p, \dots, (\partial_{x_n})_p\}$.*

*This identifies $(\partial_{x_i})_p \equiv (0, \dots, 0, \overbrace{1}^{i\text{th component}}, 0 \dots, 0)$. We can equivalently write a vector field on $\mathbb{R}^n$ as*

$$p \mapsto (X^1(p), \dots, X^n(p))$$

*or*

$$p \mapsto X^1(p)(\partial_{x_1})_p + \cdots + X^1(p)(\partial_{x_n})_p.$$

**Notation:** We write $\partial_{x_i} \in \mathcal{X}(\mathbb{R}^n)$ for the vector field

$$p \mapsto (\partial_{x_i})_p \in T_p\mathbb{R}^m \equiv (0, \dots, 0, \overbrace{1}^{i\text{th component}}, 0 \dots, 0)$$

In the literature another common notation for the same thing is $\frac{\partial}{\partial x_i}$.

**Example** (Vector field on $S^3$)**.** *Let* $M = \{(x_0,\dots,x_3) \mid \sum x_i^2 = 1\} \subset \mathbb{R}^{1+3}$. *Let* $X = x_0\partial_{x_1} - x_1\partial_{x_0} + x_2\partial_{x_3} - x_3\partial_{x_2} \in \mathcal{X}(\mathbb{R}^{1+3})$. *Observe that* $X \perp S^3 \iff \underbrace{X \cdot v}_{X_v \cdot v} = 0, v \in S^3$.

*Hence* $X \in \mathcal{X}(S^3)$.

$$\begin{tikzcd} TS^3 \arrow[r, hook] \arrow[d] & T\mathbb{R}^{1+3} \arrow[d, "\pi"', shift right] \\ S^3 \arrow[u, dotted, bend left, "X_{|_{S^3}}"] \arrow[r, hook] & \mathbb{R}^{1+3} \arrow[u, bend right] \end{tikzcd}$$

*where the map* $X_{|_{S^3}}$ *is implied by composition.*

**Example.** *For any* $M$ *smooth, the map* $p \mapsto 0 \in T_pM$ *is a vector field called the* ***zero section***.

*This is the only canonical vector field. "There is no 1"*

Let $F : M \to N$ be a smooth map. Let $X \in \mathcal{X}(M), Y \in \mathcal{X}(N)$.

**Definition.** *We say that* $X, Y$ *are* ***F-related*** *if the following diagram commutes*

$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d] & TN \arrow[d] \\ M \arrow[u, bend left, "X"] \arrow[r, "F"] & N \arrow[u, bend right, "Y"'] \end{tikzcd}$$

**Be warned!** Given $F : M \to N, X \in \mathcal{X}(M)$, there need not exist a $Y \in \mathcal{X}(N)$ s.t. $X, Y$ are $F$-related. Vector fields do not push forward.

They push back!

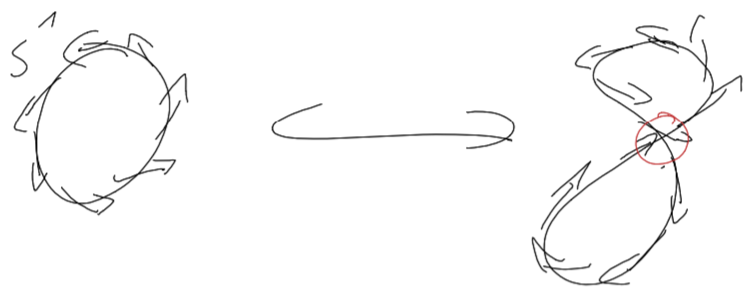


Figure 7.3: Sketch 7.04

**Definition.** *Let* $F : M \to N$ *be a diffeomorphism. Let* $X \in \mathcal{X}(M)$, *we define the* ***pushforward*** *of* $X$ $F_\star X \in \mathcal{X}(N)$ *by*

$$(F_\star X)_p = dF_{F^{-1}(p)}(X_{F^{-1}(p)}),$$

*i.e.:*

$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d] & TN \arrow[d] \\ M \arrow[u, bend left, "X"] & N \arrow[l, "F^{-1}"] \arrow[u, dotted, bend right, "F_\star X"'] \end{tikzcd}$$

**Lemma 7.2.** *Given* $F : M \to N$, $G : N \to P$ *diffeomorphisms,*

*(i)* $(G \circ F)_\star = G_\star \circ F_\star : \mathcal{X}(M) \to \mathcal{X}(P)$

*(ii) if* $F = id, M = N$, *then* $F_\star = id : \mathcal{X}(M) \to \mathcal{X}(M)$

*Proof.* Exercise. □

## 7.2 Vector fields as derivations

Recall: a tangent vector $V \in T_pM$, $p \in M$ can be viewed as a **derivation at** $p$, i.e.

$$V : C^\infty(M) \to \mathbb{R}, V(fg) = f(p)V(g) + V(f)g(p).$$

**Notation:** Let $X \in \mathcal{X}(M)$. Given a smooth function on $f \in C^\infty(M)$, we let $Xf$ be the map $M \ni p \mapsto X_p f \in \mathbb{R}$.

**Lemma 7.3.** *(i) If $X \in \mathcal{X}(M)$, i.e. $X$ is a smooth vector field, then $Xf$ is a smooth function*

*(ii) Suppose that $X : M \to TM$ is an **arbitrary** section (this is also known as a rough vector field). If $Xf$ is smooth for all $f \in C^\infty(M)$, then $X$ is a **smooth** vector field.*

*Proof.* Sheet 09. Hint: Test against coordinate functions. □

Check in $\mathbb{R}^n$

**Definition.** *An $\mathbb{R}$ linear map $X : C^\infty(M) \to C^\infty(M)$ is called a **derivation** if, for all $f, g \in C^\infty(M)$:*

$$X(fg) = f \cdot Xg + Xf \cdot g$$

*The $\cdot$ are multiplications of functions*

**Lemma 7.4.** *1. If $X \in \mathcal{X}(M)$, then the map*

$$C^\infty(M) \ni f \mapsto Xf \in C^\infty(M)$$

*is a derivation.*

*2. every derivation is of this form.*

Upshot of the lemma:

$$\mathcal{X}(M) \equiv \{\text{derivations } C^\infty(M) \to C^\infty(M)\}$$

just as we identified before

$$T_pM \equiv \{\text{derivations at } p\}.$$

*Proof.* (1) By definition, $\forall p \in M$, we have

All of this follows basically by applying point wise definitions

$$\begin{aligned} X(fg)(p) = X_p(fg) &= f(p)X_pg + X_pfg(p) \\ &= f(p)Xg(p) + Xf(p)g(p). \end{aligned}$$

Suppose that $\nu : C^\infty \to C^\infty$ is a derivation. Define a (possibly discontinuous) vector field $X$ by setting

$$X_pf = \underbrace{\nu f}_{\in C^\infty(M)}(p).$$

By lemma 7.3 (ii) $X$ is smooth, because $\nu f \in C^\infty$. □

**Definition.** *Let $X, Y \in \mathcal{X}(M)$. We let $[X, Y] \in \mathcal{X}(M)$ defined by the rule*

$$C^\infty(M) \ni f \mapsto XYf - YXf \in C^\infty(M). \tag{1}$$

*We call $[X, Y]$ the **Lie bracket** of $X, Y$.*

**Lemma 7.5.** *Equation 1 defines a derivation, hence $[X, Y]$ is a smooth vector field (by Lemma 7.4).*

*Proof.* For $f, g \in C^\infty(M)$:

$$\begin{aligned} [X, Y](f, g) &= XY(fg) - YX(fg) \\ &= X[f \cdot Yg + Yf \cdot g] - Y[f \cdot Xg + Xf \cdot g] \\ &= Xf \cdot Yg + f \cdot XYg + XYf \cdot g + Xf \cdot Xg \\ &\quad - Yf \cdot Xg - f \cdot YXg - YXf \cdot g - Xf \cdot Yg \\ &= f(XY - YX)(g) - g(XY - YX)f \\ &= f[X, Y]g + g[X, Y]f \end{aligned}$$

□

**Remark** (Properties of Lie bracket). *The Lie bracket* $[\cdot,\cdot] : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ *satisfies:*

*(i) bilinearity:*

$$\begin{aligned}[aX + bY, Z] = a[X,Z] + b(Y,Z)\\ [X, aY + bZ] = a[X,Y] + b(X,Z)\end{aligned}$$

*(ii) anti-symmetry*

$$[X,Y] = -[Y,X]$$

*(iii) Jacobi identity:*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]$$

*Thus* $(\mathcal{X}(M), [\cdot,\cdot])$ *is a Lie algebra (an* $\infty$*-dimensional one).*

**Warning:**

$$C^\infty(M) \ni f \mapsto XYf \in C^\infty(M)$$

does not define a vector field in general!

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**Lemma 7.6** (Naturality of Lie brackets). *Given* $F : M \to N$ *smooth,* $X_1, X_2 \in \mathcal{X}(M), Y_1, Y_2 \in \mathcal{X}(N)$*. Assume* $(X_i, Y_i)$ *are* $F$ *related for* $i = 1, 2$*. Then* $[X_1, X_2], [Y_1, Y_2]$ *are* $F$*-related.*

*Proof.* For $f \in C^\infty(N)$

$$X_1X_2(f \circ F) = X_1((Y_1 f) \circ F) = (Y_1Y_2 f) \circ F$$

SImilarly swapping the order of $X_1, X_2$. Hence

$$\begin{aligned}[X_1, X_2](f \circ F) &= (X_1X_2 - X_2X_1)(f \circ F)\\ &= (Y_1Y_2 - Y_2Y_1)(f) \circ F\\ &= [Y_1, Y_2](f) \circ F\end{aligned}$$

□

## 7.3 Coordinate vector fields

Let $M$ be a smooth manifold. Let $(U, \varphi)$ be a smooth chart.
Recall that we abuse notation by writing $x_i \equiv x_i \circ \varphi$, where $\underbrace{x_i}_{x^i}(x) = x_i = \pi_i(x)$. w.r.t the chart

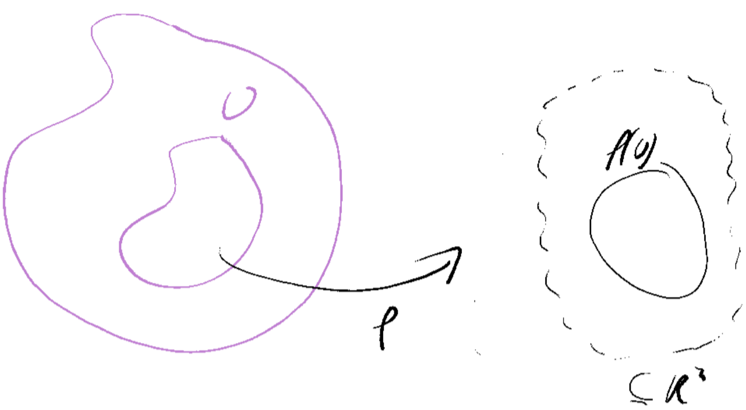


Figure 7.4: Sketch 7.05

If $p \in U$, we also have been writing $(\partial_{x_i})_p = d\varphi^{-1}_{\varphi(p)}((\partial_{x_i})_{\varphi(p)})$. Section of the tangent bundle, a vector field . . .

**Lemma 7.7.** *The map $U \ni p \mapsto (\partial_{x_i})_p \in T_pM$ is a smooth vector field on $U$, i.e. an element of $\mathcal{X}(U)$.*

*Proof.* Recall from last week $M = \mathbb{R}^n$m then the map

We often ommit the identity, i.e. the first $n$ entries of the following

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0, 1, 0 \dots, 0)$$

Then by section 7.1 ,we have this is true, when $M$ is an open subset of $\mathbb{R}^n$.
In general, $d\varphi^{-1}_{\varphi(p)}((\partial_{x_i})_{\varphi(p)}) = (\varphi)^{-1}_\star \partial_{x_i}$, where the pushfoward is

$$\varphi^{-1}_\star : \mathcal{X}(\varphi(u)) \to \mathcal{X}(U).$$

The lemma follows from the fact the fact that pushforwards of diffeomorphism send smooth vector fields to smooth vector fields. □

**Notation:** The vector field $U \ni p \mapsto (\partial_{x_i})_p$ shall be denoted by $\partial_{x_i}$. Other sources / authors write $\frac{\partial}{\partial x_i}$.

It is important to understand the difference between vector fields and tangent vectors, like the difference between functions and elements of the target of those functions

**Definition.** *Let $M$ be smooth of dimension $m$.*

*(i) Given a point $p \in M$, an $m$-tuple of vector fields $(X^1, \dots, X^m) \in \mathcal{X}(M)^m$ is called a* ***local frame at $p$****, if $(X^1_p, \dots, X^m_p) \in T_pM^m$*

*(ii) We say that $(X^1, \dots, X^m) \in \mathcal{X}(M)^m$ is a* ***global frame*** *if $(X^1_p, \dots, X^m_p)$ spans $T_pM$ for all $p \in M$.*

**Remark.** *We take $X^i \in \mathcal{X}(M)$ and not $\mathcal{X}(U)$ for local frames, since (similar to functions), we can always extend them!*

**Lemma 7.8.** *If $(X^1, \dots, X^m)$ is a local frame at $p \in M$, then there exists $p \in U \subset M$ s.t. $(X^1_{|_U}, \dots, X^m_{|_U}) \in \mathcal{X}(U)^m$ is a global frame on $U$.*

*This uses the fact that being full rank is an open condition*

*Proof.* Exercise using lemma 4.1. □

**Key example:** If $(U, \varphi)$ is a chart on $M$, then $(\partial_{x_1}, \dots, \partial_{x_m}) \in \mathcal{X}(M)^m$ form a global frame on $U$.

**Remark** (Warning)**.** *It is not the case that all frames are of this form, i.e. there exists frames $(X^1 \dots, X^m) \in \mathcal{X}(M)^m$ local frames at some $p \in M$, such that $(X^1_{|_U}, \dots, X^m_{|_U})$ is not a coordinate vector field for any chart $(V, \psi), V \subset U$. E.g. $[\partial_{x_i}, \partial_{x_j}] \equiv 0$.*

*It turns our the condition $[\partial_{x_i}, \partial_{x_j}] \equiv 0$ is necessary and sufficient*

## 7.4 Integral curves

Let $\gamma : (a, b) \to M$ be a smooth map (a curve). We write $\dot\gamma(t) = d\gamma_t(\partial_t) \in T_{\gamma(t)}M$

$$\begin{tikzcd} T\mathbb{R} \arrow[r, "d\gamma"] \arrow[d] & TM \arrow[d] \\ \mathbb{R} \arrow[u, bend left, "\partial_t"] \arrow[ur, dotted, "\dot\gamma(t)"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

In coordinates,
$gamma(t) = (\gamma^1(t), \dots, \gamma^n(t)) \in \mathbb{R}^n \implies \dot\gamma(t) = (\dot\gamma^1(t), \dots, \dot\gamma^n(t)) = \gamma^1(t)\partial_{x_1} + \dots + \gamma^n(t)\partial_{x_n}$,
where $\dot\gamma^i(t) = \frac{d}{dt}\gamma^i(t)$.

**Definition.** *Let $M$ be a manifold and let $V \in \mathcal{X}(M)$. An* ***integral courve*** *for $V$ is a curve $\gamma : (a, b) \to M$ such that*

$$\dot\gamma(t) = V_{\gamma(t)}.$$

*We typically assume $0 \in (a, b)$, we say that the* ***starting point*** *$\gamma$, is the point $\gamma(0) \in M$.*

**Example.** $M = \mathbb{R}^2, V = \partial_x = (1, 0)$.


Figure 7.5: Sketch 7.06

*The integral curves are precisely the curves*

$$(t \mapsto p + t(1, 0))$$

*where* $p \in \mathbb{R}^2$ *is the starting point.*

**Example.** $M = \mathbb{R}^2, V = x\partial_y - y\partial_x \equiv (-y, x)$-


Figure 7.6: Sketch 7.07

*Suppose that* $\gamma : \mathbb{R} \to \mathbb{R}^2$ *is an integral curve*

$$t \mapsto (\gamma^1(t), \gamma^2(t))$$

*Then we need*

$$\dot{\gamma}(t) = -\gamma^2(t)$$
$$\dot{\gamma}^2(t) = \gamma^1$$

*which is an ODE, with the following unique solution:*

$$\gamma^1(t) = a\cos t - bsin(t), \gamma^2(t) = a\sin t + b\cos(t).$$

*Hence*

$$\gamma(t) = (a\cos t - bsin(t), a\sin t + b\cos(t)),\ a, b \in \mathbb{R}$$

*integral curve with starting point* $(a, b)$.

**Proposition 7.9.** *Let* $M$ *be a smooth manifold. Let* $V \in \mathcal{X}(M)$.

*(a)* ***Existence:*** *Given any point* $p \in M$, *there exists an open interval* $0 \in J \subset \mathbb{R}$ *and an integral curve* $\gamma : J \to M$ *starting at* $p$

*(b)* ***Uniqueness:*** *If* $\sigma, \gamma : J \to M$ *starting at the same point* $p = \sigma(0) = \gamma(0)$, *then* $\sigma = \gamma$

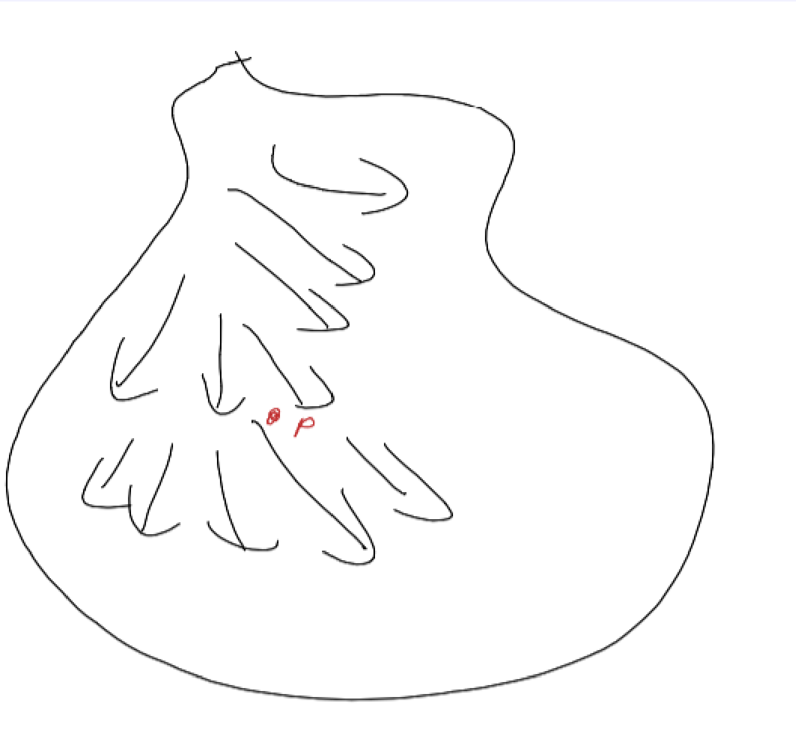


Figure 7.7: Sketch 7.08

**Remark.** *It follows from the proposition that,*$\forall p \in M$ *there is a largest interval* $0 \in J \subset \mathbb{R}$ *admitting an integral curve* $\gamma : J \to M$. *We call* $\gamma : J \to M$ *the* ***maximal integral curve***.

*This probably needs Zorn's lemma.*

*Proof of proposition 7.9.* (a): This is a local statement, hence we can assume open $M \subset \mathbb{R}^n$. Then we must solve

$$\dot{\gamma}(t) = V_{\gamma(t)},$$

$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$.

$$\begin{cases} \dot{\gamma}^1(t) & = V^1_{\gamma(t)} \\ \vdots & \vdots \\ \dot{\gamma}^n(t) & = V^n_{\gamma(t)} \end{cases}$$

This is a system of ordinary differential equations (ODEs). Hence by Theorem D.1 in the appendix of [2][1], the system admits a unique solution with $\gamma(0) = p \in M \subset \mathbb{R}^n$.

[1] On the course website

(b): Let $\mathcal{E} \subset J$ be a subset of points $t \in J$ such that $\sigma(t) = \gamma(t)$. Observe that $0 \in \mathcal{E}$ by assumption. Observe also that $\mathcal{E}$ is closed, since $\sigma, \gamma$ are continuous functions. Moreover $\mathcal{E}$ is open by the uniqueness part of theorem D1. $\implies \mathcal{E} = J$. $\square$

We need this, because the uniqueness part of the theorem is local, but our statement in (b) is not

**Lemma 7.10.** *If $F : M \to N, X \in \mathcal{X}(M), Y \in \mathcal{X}(N), X, Y$ are $F$ related, then $F$ takes integral curves of $X$ to integral curves of $Y$.*

*Proof.* Suppose $\gamma : J \to M$ integral curve of $X$.

$$(F \circ \gamma)'(t) = dF_{\gamma(t)} \dot{\gamma}(t) \overset{F\text{-related}}{=} Y_{F \circ \gamma(t)} \qquad \square$$

**Remark.** *There is also a converse. (exercise)*

**Definition.** *We say that a vector field $V \in \mathcal{X}(M)$ is **complete** if for all $p \in M$, the maximal integral curve starting at $p \in M$ is defined on $\mathbb{R}$.*

**Example** (Example of a non-complete vector field). $M = \mathbb{R} \setminus \{0\}, V = \partial_x$. *Pick $p = -1 \in \mathbb{R}$. Then the integral curve starting at $p$ is the map*

$$t \mapsto -1 + t.$$

*This is only defined on $(-\infty, 1)$.*

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**Example** (Another example of incompleteness). $M = \mathbb{R}, V = x^2 \partial_x$


Figure 7.8: Sketch 7.09

*Setting $p = 1$, let $x \mapsto \gamma(x)$ the maximal integral curve of $V$ starting at $p$.*

$$\implies \begin{cases} \gamma(0) & = p \\ \dot{\gamma}(t) & = V_{\gamma(t)} = \gamma(t)^2 (\partial_x)_t \overset{M=\mathbb{R}}{=} \gamma(t)^2 \end{cases}$$

*Therefore $\gamma(t) = \frac{1}{1-t}$, as $t \to 1$*

*There is a finite time blow-up at 1 of the ODE. We never reach $t = 1$. Incomplete integral curves leave all compact sets!*

## 7.5 Flows

Fix a smooth manifold $M$.

**Definition.** *A **flow domain** is an open set $\mathcal{D} \subset \mathbb{R} \times M$ such that, for all $p \in M$,*

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} \mid (t, p) \in \mathcal{D}\} \subset \mathbb{R}$$

*is an open interval and $0 \in \mathcal{D}^{(p)}$.*

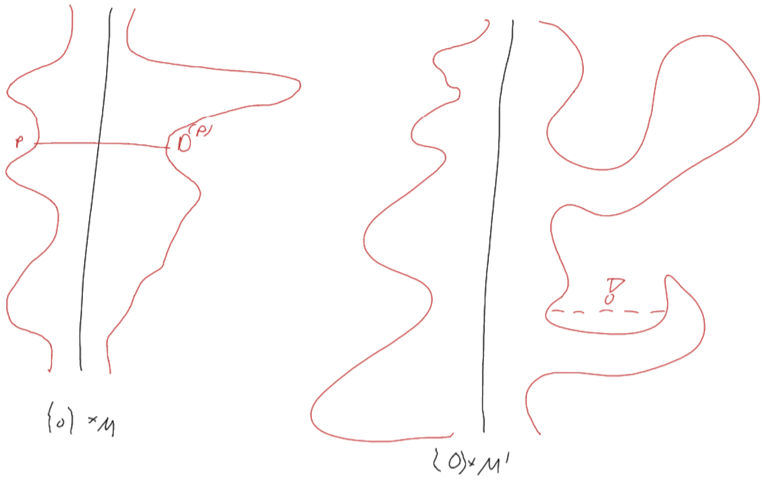


Figure 7.9: Sketch 7.10

**Example** (and non-example)**.**

**Definition.** *A* ***flow*** $\Psi : \mathcal{D} \to M$ *is a map such that*

*(i)* $\Psi((0, p)) = p$

*(ii)* $\Psi((t, \Psi(s, p))) = \Psi(s + t, p)$ *whenever this makes sense, i.e.* $s, s + t \in \mathcal{D}^{(p)}, t \in \mathcal{D}^{(p)}$

*One parameter subgroup property*

**Notation**: Given a flow $\Psi : \mathcal{D} \to M$, we write:

- if $(-\epsilon, \epsilon) \times M \subset \mathcal{D}, s \in (-\epsilon, \epsilon), \Psi_s = \Psi(s, \cdot) : M \to M$. This is a diffeomorphism, since $\Psi_{-s} \circ \Psi_s = \Psi_0 = \mathrm{id}$
- given $p \in M$: $\Phi^{(p)} = \Psi(\cdot, p) : \underbrace{\mathcal{D}^{(p)}}_{\subset \mathbb{R}} \to M$, which is a path

**Lemma 7.11.** *(a) If* $\Psi : \mathcal{D} \to M$ *is a flow, then* $p \mapsto \frac{d}{dt}_{|t=0} \underbrace{\Psi(t, p)}_{\Psi^{(p)}(t)} \in T_pM$ *defines a smooth vector field* $V^\Psi$ *on* $M$

*(b) For* $p \in M, \Psi^{(p)} : \mathcal{D}^{(p)} \to M$ *are integral curves of the vector* $V^\Psi$*.*

*Proof.* **(a):** Enough to show, for all smooth functions $f : M \to \mathbb{R}, V^\Psi f = (p \mapsto V_p^\Psi f)$ is also smooth (lemma 7.3). But $V^\Psi f(p) = V_p f = \frac{d}{dt}_{|t=0}(f \circ \Psi^{(p)}(t)) = \partial_t(f \circ \Psi)(0, p)$, $p \mapsto \partial_t(f \circ \Psi(0, p))$ is smooth.
**(b):** We need tho show that $V_{\Psi^{(p)}(t_0)} = \Psi^{(p)}(t_0), t_0 \in \mathcal{D}^{(p)}$. But for $f \in C^\infty(M), q = \Psi^{(p)}(t_0) = \Psi(t_0, p)$, we have

$$\begin{aligned}
V_q^\Psi f = \Psi^q(0) f &= \frac{d}{dt}_{|t=0} f(\Psi^{(q)}(t)) \\
&= \frac{d}{dt}_{|t=0} f(\Psi(t, \underbrace{\Psi(t_0, p)}_{=q})) \\
&= \frac{d}{dt}_{|t=0} f(\Psi(t + t_0, p)) = \frac{d}{dt}_{|t=0} f(\Psi^{(p)}(t_0 + t)) \\
&= \dot{\Psi}^{(p)}(t_0) f \subset T_{\Psi^{(p)}(t_0)}
\end{aligned}$$

□

**Theorem 7.12** (Theorem of flows)**.** *Let $V \in \mathcal{X}(M)$. Then there exists a unique flow $\Psi : \mathcal{D} \to M$ such that*

$$V^{\Psi} = V \wedge \Psi^{(p)} : \mathcal{D}^{(p)} \to M \text{ are maximal integral curves}$$

*The map AND the domain are unqiue*

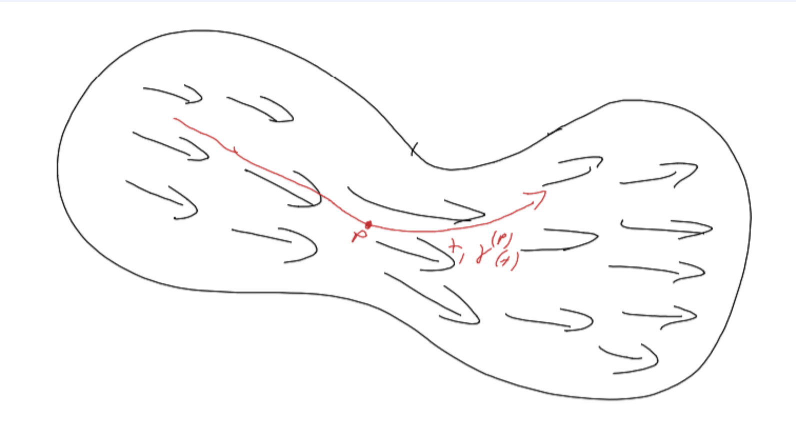


Figure 7.10: Sketch 7.11

*Sketch.* To define $\Psi$ send $(t, p) \mapsto \gamma^{(p)}(t)$, where $\gamma^{(p)}$ is the (unique) integral curve at $p$. □

## 7.6 The Lie derivative

**Trivia:** (Sophus Lie): Lie groups, Lie algebras, but also: (to) lie. There was a hearing in congress (NSF) about a grant of Lie groups.
Let $V = (V^1, \dots, V^n)$ be a vector field on $\mathbb{R}^n$. $V^i : \mathbb{R}^n \to \mathbb{R}$. The expression
$\partial_{x_i} V(x) = (\partial_{x_i} V^1, \dots, \partial_{x_i} V^n)$ the derivative of $V \in \mathcal{X}(\mathbb{R}^n)$ in the direction of $\partial_{x_i}$.
**However:** on a general manifold this does not make sense.

A lot of the proves are sketches, to increase speed


Figure 7.11: Sketch 7.12

$\gamma : (-\epsilon, \epsilon) \to M, \gamma(0) = p$. Want to define $\partial_{\gamma(0)} V$

$$\lim_{t \to 0} \frac{\overbrace{V_{\gamma(t)}}^{T_{\gamma(t)}M} - \overbrace{V_{\gamma(0)}}^{\in T_{\gamma(0)}M}}{t}$$

Two approaches to defining derivatives of vector fields

- introduce connections ($\pi : E \to B$) Choose extra data. A connection $\implies \nabla_{\gamma(0)}\sigma$ makes sense
- Lie derivatives: Only works on the tangent bundle ( a associated vector bundles)

**Definition.** *Let $V, W \in \mathcal{X}(M)$. We define at $p$*

$$\lim_{t\to 0}(\mathcal{L}_V W)_p := \frac{d}{dt}_{|t=0}\left(\frac{d({}^V\Psi_{-t})(W_{{}^V\Psi_t(p)}) - W_p}{t}\right)$$

Note $d\left({}^V\Psi_{-t}\right)_{{}^V\Psi_t(p)} : T_{{}^V\Psi_t(p)} \to T_p$

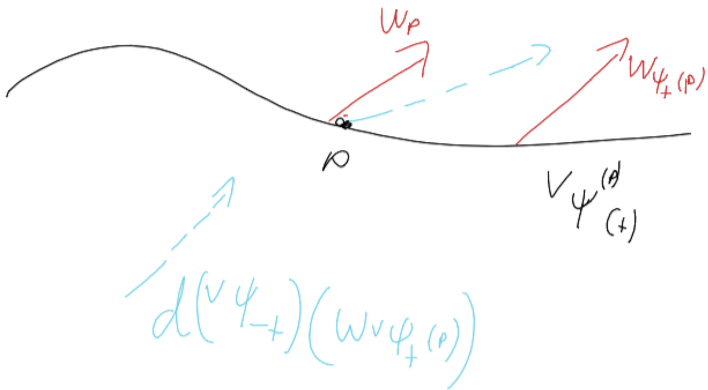


Figure 7.12: Sketch 7.13

**Lemma 7.13.** $\mathcal{L}_V W \in \mathcal{X}(M)$.

*Proof.* Let $(U, \varphi)$ be a chart with $p \in U$. Let $J_0 \subset \mathbb{R}$ be the interval containing 0, let $U_0 \subset U$
Assume $J_0, U_0$ are sufficiently small such that ${}^V\Psi_{|J_0 \times U_0}$ has image in $U$.
In $(U, \varphi)$ coordinates (i.e. $x_i \equiv x_i \circ \varphi$), we have

$$\Psi(t, x) = (\Psi^1(t, x), \ldots, \Psi^n(t, x)).$$

Then $(d\Psi_{-t})_{\Psi(t,x)} = \left(\partial_{x_i}\Psi^j(-t, x)\right)_{1\le i,j\le n}$. Hence
$d\Psi_{-t}W_{\Psi(t,x)} = \left(\partial_{x_i}\Psi^j(-t, x)\right)_{1\le i,j\le n}\underbrace{(W^1(\Psi(t, x)), W^n(\Psi(t, x)))}_{n\times 1}$. Hence

$$\frac{d}{dt}_{|t=0}\left(d\Psi_{-t}W_{\Psi(t,x)}\right) = \lim_{t\to 0}\left(\frac{d\Psi_{-t}W_{\Psi(t,x)} - W_{\Psi(0,x)}}{t}\right)$$ □

**Remark.** *If $F : M \to N$ diffeomorphism,then*

$$F_\star(\mathcal{L}_V W) = \mathcal{L}_{F_\star V} F_\star W$$

**Proposition 7.14.** $(\mathcal{L}_V W) = [V, W]$ *for $V, W \in \mathcal{X}(M)$.*

*In lecture 17 this was a lemma, but in lecture 18 this is a proposition*

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*Proof.* Enough to check that $(\mathcal{L}_V W)_p = [V, W]_p$ We are going to consider three cases: **1.** $V_p \neq 0$: According to Sheet 10, we can find a chart $U, \varphi$, $p \in U$ with the property that $\partial_{x_1} = V$ on $U$, where $(x_i)_{i=1}^n$ are the coordinate functions.

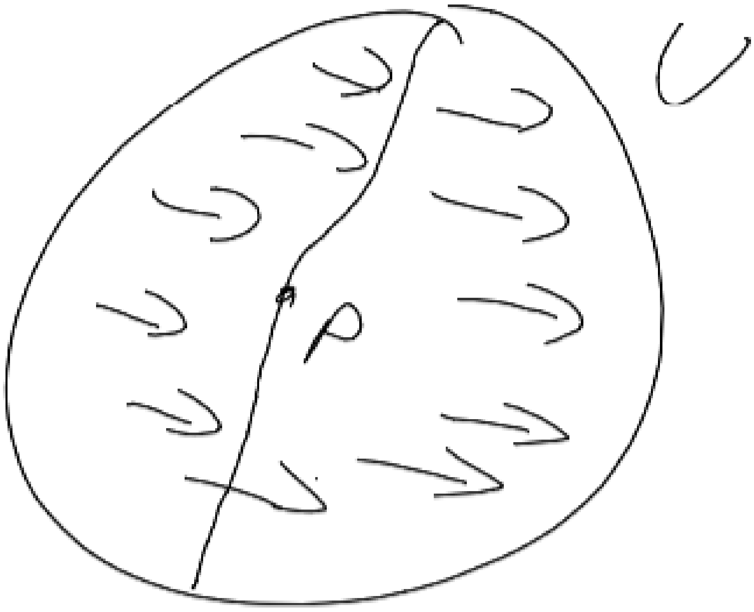


Figure 7.13: Sketch 7.14

Hence, we can now assume that $M = U \subset \mathbb{R}^n, V = \partial_{x_1}$.

Due to being invariant under diffeomorphisms, we can just pick nice coordinates

$$\implies {}^{\partial_{x_1}}\Psi_t(x) = (x_1 + t, x_2, \dots, x_n)$$

$$\implies d(\Psi_{-t})_{\Psi_t(x)}(W_{\Psi_t(x)}) = d(\Psi_{-t})\left(\overbrace{W^1(x_1+t, x_2, \dots, x_n), \dots, W^n(x_1+t, x_2, \dots, x_n)}^{\in T_{\Psi_t(x)} \equiv \mathbb{R}^n}\right)$$

$$= \overbrace{(W^1(x_1+t, x_2, \dots, x_n), \dots, W^n(x_1+t, x_2, \dots, x_n))}^{\in T_x M = \mathbb{R}^n}$$

$$\implies (\mathcal{L}_V W)_0 = \frac{d}{dt}_{|t=0}\left(d(\Psi_{-t})_{\Psi_t(0)}(W_{\Psi_t(0)})\right) = \frac{d}{dt}_{|t=0}\left(W^1(t, 0, \dots, 0), \dots, W^n(t, 0, \dots, 0)\right)$$

$$= (\partial_{x_1} W^1(0, \dots, 0), \dots, \partial_{x_1} W^n(0, \dots, 0)) = [\partial_{x_1} W]_0$$

**2.** $p \in \overline{\{q \mid V_q \neq 0\}}$. This implies

$$(\mathcal{L}_V W)_p = [V, W]_p$$

by continuity.

**3.** $p$ admits a neighborhood $\mathcal{O} \subset M$ s.t. $V_{|\mathcal{O}} \equiv 0$. Then

$$(\mathcal{L}_V W)_p = 0 = [V, W]_p.$$

This is an exercise. □

**A very important property:** We say that $V, W \in \mathcal{X}(M)$ **commute**, if $[V, W] = 0$. This is equivalent by 7.15 to ${}^V\Psi, {}^W\Psi$ commuting.

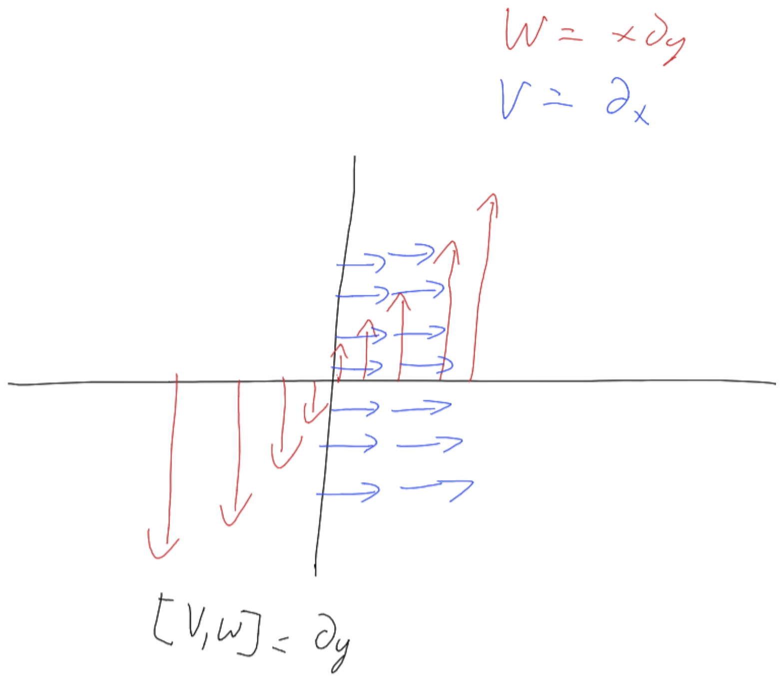


Figure 7.14: Sketch 7.15

**Theorem 7.15.** *The following are equivalent:*

*(i)* $[V, W] = \mathcal{L}_V W = 0$

*(i')* $[W, V] = \mathcal{L}_W V = 0$

*(ii)* ${}^V\Psi_t(\cdot)_\star W = W$

*(ii')* ${}^W\Psi_t(\cdot)_\star V = V$

*(iii)* ${}^V\Psi_s {}^W\Psi_t = {}^W \Psi_t {}^V\Psi_s$ *for all* $s, t$ *where this makes sense.*

*Proof sketch.* $(i) \iff (i')$: By definition.
$(i) \implies (ii)$: Need the identity from Sheet 10 (optional problem). Differentiate the expression $\Psi_{t\star}W$.
$(ii) \implies (iii)$: Need to check (essentially a tautology) that if the flow of $V$ preserves $W$, then it preserves the flow of $W$ as well.
$(iii) \implies (i)$: ${}^W\Psi^{({}^V\Psi_t(p))}(s) = {}^V\Psi_t({}^W\Psi^{(p)})(s)$

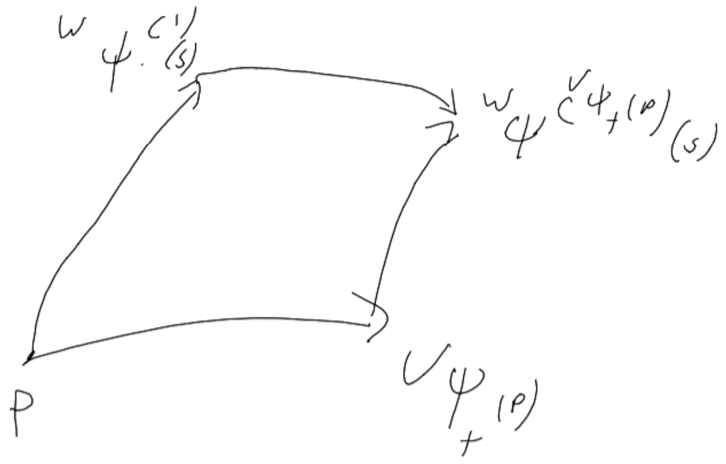

Figure 7.15: Sketch 7.16

$$\frac{d}{ds}_{|s=0} {}^{W}\Psi^{({}^{V}\Psi_t(p))}(s) = W_{\Psi_t(p)} = d({}^{V}\Psi_t)(W_p)$$

Apply $d(\Psi_{-t})(\cdot)$ to both sides of the equality, to get

$$d^{V}\Psi_{-t}\left(W_{\Psi_t(p)}\right) = W_p.$$

Finally, apply $\frac{d}{dt}_{|t=0}$ to get $(\mathcal{L}_V W = 0)$. ☐

**Theorem 7.16.** *Let $M$ be a smooth manifold. Let $(V^1, \dots, V^m) \in \mathcal{X}(M)^m$ be a local frame at $p$. Then, the following are equivalent:*

*(i) there exists a chart $(U, \psi), p \in U$ s.t. $V^i = \partial_{x_i}$, where $(x_i)$ are the coordinate functions*

*(ii) $[V^i, V^j] = 0$ near $p$ for all $i, j$.*

*This is a very good theorem to remember! Might very well be relevant to an exam problem*

*Proof idea.* $(i) \implies (ii)$ is obvious.
$(ii) \implies (i)$. Define a chart

$$\begin{aligned}(-\epsilon, \epsilon)^m &\to M \\ (t_1, \dots, t_n) &\mapsto \underbrace{\left(\left({}^{V^1}\Psi^{(p)}(t_1), \dots, {}^{V^m}\Psi^{(p)}(t_m)\right) = \right)}_{\text{bad notation}} {}^{V^1}\Psi_{t_1}\left(\dots {}^{V^m}\Psi_{t_m}(p)\right)\end{aligned}$$

☐

**Remark.** *Lecture 20 will start 5 minutes earlier!*

# Chapter 8: Vector bundles

## 8.1 Review of linear algebra

### 8.1.1 The category of vector spaces

Fix $\mathbb{K}$ a field (for this class we only care about $\mathbb{K} = \mathbb{R}$).
Let $\mathrm{vec}_{\mathbb{K}}$ be the category of finite-dimensional $\mathbb{K}$ vector spaces:

- Objects are (finite dimensional) $\mathbb{K}$ vector spaces
- Morphisms are linear maps.

The category $\mathrm{vec}_{\mathbb{K}}$ is **abelian**. In particular

- given $\psi : V \to W$, there exists
$$0 \to \ker \phi \hookrightarrow V \overset{\psi}{\to} W$$
$$V \overset{\psi}{\to} W \to W/\psi(V) \to 0$$
- $V, W$, can form $V \oplus W \equiv V \times W$

We can only form direct sums of finitely many vector spaces, since $\mathrm{vec}_{\mathbb{K}}$ only contain finite dimensional vector spaces

The category $\mathrm{vec}_{\mathbb{K}}$ is symmetric monoidal:

$$\begin{aligned} \mathrm{vec}_{\mathbb{K}} \times \mathrm{vec}_{\mathbb{K}} &\to \mathrm{vec}_{\mathbb{K}} \\ (V, W) &\mapsto V \otimes_{\mathbb{K}} W \end{aligned}$$

**Note:**
$$V \otimes W \simeq W \otimes V$$
symmetric and
$$V \otimes (W \otimes Z) \simeq (V \otimes W) \otimes Z.$$

**Note:** If $V^1, \dots, V^k$ are vector spaces and $\{e^i_j\}_{j=1}^{l}$ basis for $V^i$, then
$$\{e^1_{j_1} \otimes \cdots \otimes e^k_{j_k}\}$$
basis for $V^1 \otimes \cdots \otimes V^k$.
The category $\mathrm{vec}_{\mathbb{K}}$ admits an anti-involution:

$$\begin{gathered} (\cdot)^{\vee} : \mathrm{vec}_{\mathbb{K}} \to \mathrm{vec}_{\mathbb{K}}^{\mathrm{op}} \\ V \mapsto V^{\vee} \\ \psi \in \hom(V, W) \mapsto \psi^{\vee} \in \hom(W^{\vee}, V^{\vee}) \\ (\cdot)^{\vee\vee} \equiv \mathrm{id} \end{gathered}$$

These notions will have corresponding notions for vector bundles.

### 8.1.2 Tensor products

Recall that the **tensor product** of vector spaces is characterized by:
**Universal property:** If $\alpha : V^1 \times \cdots \times V^k \to W$ is a multi-linear map, then there exists

$$\begin{tikzcd} V^1 \times \cdots \times V^m \arrow[r, "\alpha"] \arrow[d, "\pi"] & W \\ V^1 \otimes \cdots \otimes V^m \arrow[ur, "\underline{\alpha} \equiv \alpha"] & \end{tikzcd}$$

**Definition.** *Given a vector space $V \in vec_{\mathbb{K}}$, a **tensor of type** $(k, l) \in \mathbb{N} \times \mathbb{N}$ is an element of*

$$T^{k,l}V := \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \otimes \underbrace{V^\vee \otimes \cdots \otimes V^\vee}_{l \text{ times}}$$

We write $T^k V \equiv T^{k,0}V$ and $T^l V^\vee \equiv T^{0,l}V$

**Remark.** *This is not the same as the decomposition into $(l, k)$ forms in complex linear algebra.*

A tensor $\alpha \in T^l V^\vee = (T^l V)^\vee$ defines a map $T^l V \to \mathbb{K}$, hence also a multi-linear map $V \times \cdots \times V \to \mathbb{K}$. We denote these maps by $\alpha$ by abuse of notation.

**Definition.** *A tensor $\alpha \in T^l V^\vee$ is*

- ***alternating**: if the induced map $\alpha : V^1 \times \cdots \times V^l \to \mathbb{K}$ satisfies*

$$\alpha(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_l) = (-1)^{j-i}\alpha(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_l).$$

- ***symmetric** if the induced map $\alpha : V^1 \times \cdots \times V^l \to \mathbb{K}$ satisfies*

$$\alpha(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_l) = \alpha(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_l).$$

*We let $\Lambda^l(V^\vee) \subset T^l V^\vee$ be the subspace of alternating tensors. We let $\Sigma^l V^\vee \subset T^l V^\vee$ be the subspace of symmetric tensors.*

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**Lemma 8.1.** ***(1):*** *The inclusion $\Lambda^k(V^\vee) \subset T^k(V^\vee)$ splits via the map*

$$\begin{aligned} T^k(V^\vee) &\to \Lambda^k(V^\vee) \\ \alpha &\mapsto alt(\alpha) := \frac{1}{k!} \sum_{\sigma \in S_k} (sgn(\sigma))^\sigma \alpha \end{aligned}$$

*where*

$$^\sigma\alpha(v_1, \ldots, v_k) = \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

***(2):*** *The inclusion $\Sigma^k(V^\vee) \subset T^k(V^\vee)$ also splits via the map*

$$\begin{aligned} T^k(V^\vee) &\to \Sigma^k(V^\vee) \\ \alpha &\mapsto sym(\alpha) := \frac{1}{k!} \sum_{\sigma \in S_k} {}^\sigma\alpha \end{aligned}$$

*Proof.* **(1):** We need to show that the composition

$$\Lambda^k(V^\vee) \xrightarrow{i} T^k(V^\vee) \xrightarrow{\text{alt}} \Lambda^k(V^\vee)$$

We should also check that the maps acutally land in the claimed target set

is the identity. This is the case because $(\text{sgn}(\sigma))^\sigma\alpha = \alpha$ if $\alpha \in \Lambda^k(V^\vee)$.
**(2)** is similar. ∎

## 8.2 Vector bundles

### 8.2.1 Basic definitions

**Definition.** *A (real, smooth)* ***vector bundle*** *is a triple* $(\pi, E, B)$ *where*

- $E, B$ *are manifolds*
- $\pi : E \to B$ *is a smooth map*
- $E_b = \pi^{-1}(b)$ *carries the structure of a real vector space (finite dimensional).*

*If we look at the fibres . . .*

*This data must satisfy the following* ***"local triviality"*** *condition:*
*Given any* $b \in B$*, there exists a neighborhood* $b \subset U$ *and a diffeomorphism* $\psi^{-1} : \pi^{-1}(U) \simeq U \times V$ *such that:*

*$V$ is some real, finite dimensional vector space*

*(i)* $\pi \circ \psi(x, v) = x$

*(ii)* $\psi_{|E_b} : E_b \xrightarrow{\simeq} \{b\} \times V$ *is an isomorphism of real vector spaces.*

$B$ *is called the* ***base****,* $\pi$ *the* ***projection*** *and* $E$ *is called the* ***total space****.*

**Remark.** *We assume in this course that* $\dim_{\mathbb{K}}(E_b)$ *is constant (this is automatic if* $B$ *is connected). Under this convention we can assume that* $V = \mathbb{R}^k$*, since every finite dimensional real vector space is non-cannoically isomorphic to* $\mathbb{R}^k$ *for some* $k \in \mathbb{N}$ *by composing* $\psi$ *with* $(id, \phi_V)$*, where* $\phi_V$ *is said isomorphism.*

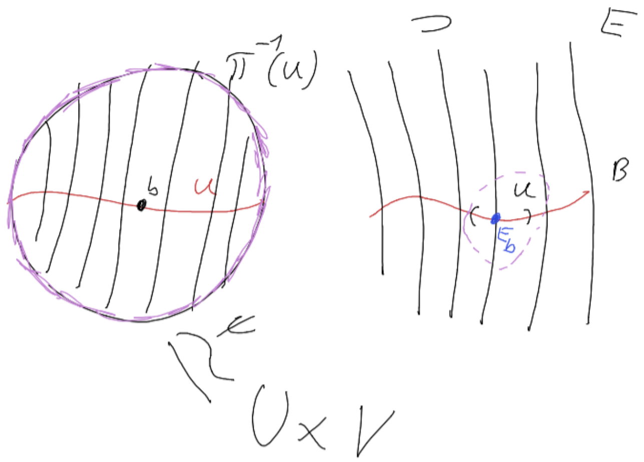


Figure 8.1: Sketch 8.01

In the homework

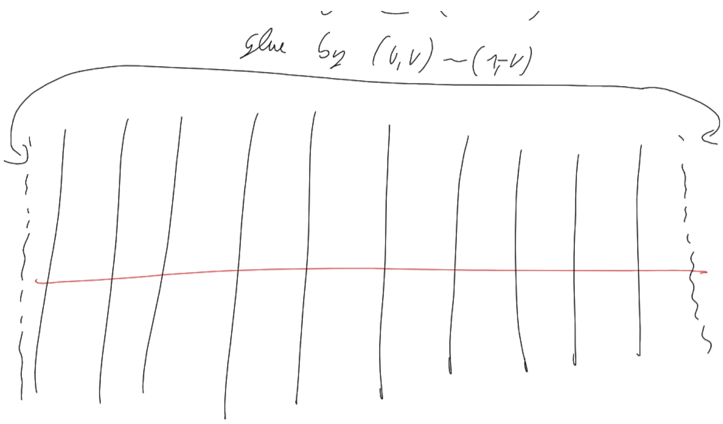


Figure 8.2: Sketch 8.02

Rigorously $E : [0,1] \times \mathbb{R}/(0,v) \sim (1,-v)$. Notice that in this case the local triviality really is local, i.e. we can't take $U = E$.

**Definition.** *A morphism of (smooth, real) vector bundles $(\pi, E, B) \to (\pi', E', B')$ is the data of smooth maps $F : E \to E', f : B \to B'$ such that*

*(i)*
$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$
*commutes.*

*(ii) $F_{|E_b} : E_b \to E'_{f(b)}$ is a linear map.*

**Remark.** *$f$ is determined by $F$ and the condition that $F$ sends fibers to fibers.*

*Recoverable through quitioning*

**Notation:** We write $E = (\pi, E, B), (\pi, E, B) \overset{(F,f)}{\to} (\pi', E', B')$ is written as $F : E \to E'$.

**Definition.** *Given a vector bundle $E$, a **sub-bundle** is a vector bundle $F$ over the same base, and a map $i : F \hookrightarrow E$ covering the identity, such that $F_b \overset{i}{\hookrightarrow} E_b$ is injective, i.e.*

$$\begin{tikzcd} F \arrow[r, hook, "i"] \arrow[d] & E \arrow[d, "\pi"] \\ B \arrow[r, "="] & B \end{tikzcd}$$

*and $F_b \hookrightarrow E_b$ is injective.*

**Definition.** *We let vectbund be the category whose objects are (smooth, real) vector bundles. We let vectbund$(B)$ be the subcategory on vector bundles over $B$ with morphisms covering the identity.*

**Remark.** *Can similarly set up a theory of $C^0, C^k, k \geq 1$ vector bundles over $\mathbb{R}, \mathbb{C}$. This gives rise to analogous categories. Other would therefore write vectbund$^\infty$ for what is here jut called vectbund.*

**Lemma 8.2** (Construction)**.** *Let $f : B \to C$ be a smooth map. Then there is a function $f^\star$ : vectbund$(C) \to$ vectbund$(B)$. This is called **pullback**. On objects, we have $f : \underbrace{E}_{\in vectbund(C)} \to f^\star(E) := B \times_C E \to B$.*

**Note:** Here we use the diagram:

$$\begin{tikzcd} f^\star(E) \arrow[r] \arrow[d, "\pi_B"] & E \arrow[d, "\pi"] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

*Proof.* Note that $\pi : E \to C$ is a submersion (always true for vector bundles, follows from the condition $\pi^{-1}(U) \simeq U \times V$). Hence $f \pitchfork \pi$ are transverse, and the fiber product exists (in the category $\mathrm{Man}^\infty$).

- $\pi_B^{-1}(b) = \pi^{-1}(f(b))$ endows $\pi_B^{-1}(b)$ with the structure of a vector space
- given $b \in B$ there exist $U \ni f(b)$, and $\psi : \pi^{-1}(U) \sim U \times V$. Hence $\pi_B^{-1}(U) \simeq \pi^{-1}(U) \times V$, which verifies local triviality.

There is more to check, but the rest is trivial. □

### 8.2.2 Examples of vector bundles

**Example.** *Let $B = \{x\}$. Then $vectbund(\{x\}) \simeq vect_{\mathbb{R}}$ by*

$$\{x\} \times V \xrightarrow{\pi} \{x\} \leftarrow\!\shortmid V$$

**Example.** *Let $B$ be our favorite smooth manifold. Then there is a unique map $f : B \to \{x\}$. Hence by lemma 8.2, we have a vector bundle $f^\star V$. In fact $f^\star V = B \times V \xrightarrow{\pi_B} B$ by $(b, v) \mapsto b$. Any vector bundle of the form $f^\star V$ is called **trivial**.*

*This is a small exercise*

**Remark.** *The Möbius bundle is not trivial: $M \xrightarrow{\pi} S^1$, if $M = S^1 \times \mathbb{R} \to S^1$ $(x, 1) \leftarrow\!\shortmid x$ , then we could define a section $\sigma : S^1 \to M, \pi \circ \sigma = id$ and $\sigma \neq 0$. Contradiction via the intermediate value theorem.*

**Example.** *Let $B$ be any manifold. Then $TB \xrightarrow{\pi} B$ is a vector bundle.*

*Given a manifold: is the vector bundle trivial? is a nice question to ask (in the exam?)*

**Exercise:** $TS^1 \simeq S^1 \times \mathbb{R}$
What about $TS^2$?

### 8.2.3 Vector bundles from gluing data

This gives a mechanism to produce many more examples.
**Construction**: Let $M$ be a manifold. Let

- $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a cover of $M$
- $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of vector spaces
- $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V_\alpha, V_\beta) \simeq \mathrm{GL}(\mathbb{R}^k) \subset \mathbb{R}^{k^2}$ be smooth maps satisfying

  $$\varphi_{\alpha\gamma} = \varphi_{\beta\gamma}\varphi_{\alpha\beta}$$

  on $U_\alpha \cap U_\beta \cap U_\gamma$ for all $\alpha, \beta, \gamma \in \mathcal{A}$. The equation is called **cocycle condition**.

We let

$$E := \coprod_{\alpha \in \mathcal{A}} U_\alpha \times V_\alpha / \sim \tag{1}$$

where $U_\alpha \times V_\alpha \ni (x, v) \sim (x, w) \in U_\beta \times V_\beta$ if $w = \varphi_{\alpha\beta}(x)(v)$.
We let $\pi : E \to M$ be the forgetful map

$$[(x, v)] \mapsto x.$$

**Lemma 8.3.** *(a) $\pi : E \to M$ is a vector bundle*

*(b) All vector bundles arise in this way.*

*Proof sketch.* **(a)** is obvious (once you believe $\sim$ is an equivalence relation)
**(b):** Let $\pi : E \to B$ be a vector bundle, cover $B$ by $\{U_\alpha\}$, $\psi : \pi^{-1}(U_\alpha) \to U_\alpha \times V_\alpha$ and let $\varphi_{\alpha,\beta} = \psi_\beta \circ \psi_\alpha^{-1}$. □

**Terminology:** A vector bundle $\pi : E \to B$ has to satisfy the local triviality condition: around any $p \in B \exists U \ni o \wedge \psi_U : \pi^{-1}(U) \simeq U \times V$, where $V$ is a vector field. $\psi : U$ is called a **local trivialization**. We call $E_b, \in B$ the **fibers**. $(U \cap \mathcal{V}) \times V_U \to (U \cap \mathcal{V}) \times V_{\mathcal{V}}$ by the transition function $\psi_{\mathcal{V}} \circ \psi_U^{-1}$

Start of lecture 20 (17.12.2024) and the $E_b$ trivially are vector spaces

$$(x, v) \mapsto (x, w) = (x, \psi_{\mathcal{V}} \circ \psi_U^{-1}(x, v)).$$

The transition function only really acts on the second component

**Remark.** *If the instructor had different taste, the above construction might as well have been the definition*

**Remark.**

- *Different choices of data $(\{U_\alpha\}, \{V_\alpha\}, \{\phi_{\alpha\beta}\})$ may give rise to the same vector bundle.*
- *however, with appropriate notion of equivalence, $\{\text{vector bundles}\} \equiv (\{U_\alpha\}, \{V_\alpha\}, \{\phi_{\alpha\beta}\})$. This can be made precise by defining a category of gluing data, which then becomes equivalent to the category of vector bundles.*

*For example, if you take more open sets, you still get the same structure*

## 8.3 Globalizing linear algebra constructions

**Theorem 8.4** (Omnibus theorem). *Any canonical linear algebra construction can be globalized to the category of smooth vector bundles.*
*More precisely: The category vectbund$_{\mathbb{R}}$ admits $\cdot \oplus \cdot, \cdot \otimes \cdot, (\cdot)^\vee, (\cdot)/(\cdot), Hom(\cdot, \cdot), \ldots$*

*Not really well defined ...*

*(i) the operations are compatible with pullback: i.e. given $f : B \to B'$,*

$$f^\star(E' \oplus F') = f^\star(E') \oplus f^\star(F').$$

*Also $f^\star(E' \otimes F') = f^\star(E') \otimes f^\star(F')$ for $E' \to B'$ and $F' \to B'$.*

*(ii) On vectbund$_{\mathbb{R}}(\{x\}) \simeq$ vect$_{\mathbb{R}}$, the operations coincide.*

**Remark** (Warning). *In contrast to vect$_{\mathbb{K}}$, the category vectbund$_{\mathbb{R}}$ is **not abelian** (nor is vectbund$(B)$ in general). To see what goes wrong:*
*Consider $(E = \mathbb{R} \times \mathbb{R}) \to \mathbb{R}$; $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$,*

*first component is the base*

$$(t, v) \mapsto (t, tv).$$

*Note: $\phi_t : \underbrace{E_t}_{=\mathbb{R}} \to E_t, v \mapsto tv$. If $t \neq 0$, then $\phi_t : E_t \to E_t$ is an isomorphism. If $t = 0, \phi_t \equiv 0$ (not an isomorphism).*
*So $ker(\phi)$ wants to be*

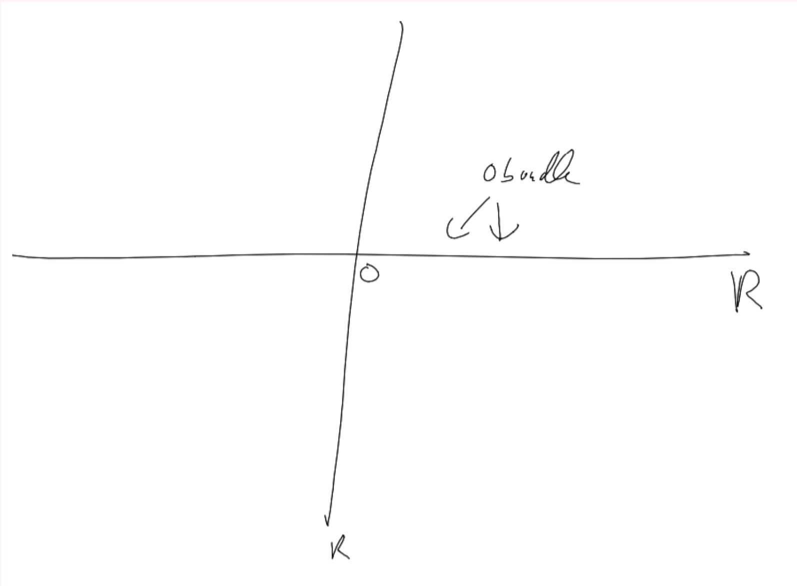


Figure 8.3: Sketch 8.03

*The fix is to embed vectbund$_{\mathbb{R}}$ $\subset$ sheaves$_{\mathbb{R}}$. This is the start of a long story ...*

**Remark** (Warning)**.** *In this category vectbund$_{\mathbb{R}}$ there is a difference between **sub-bundles** and **sub-objects**. We say that*

$$\begin{tikzcd} E \arrow[rr, hook, "i"] \arrow[dr] & & F \arrow[dl] \\ & B & \end{tikzcd}$$

*is a sub-bundle, if $i_b : E_b \to F_b$ is injective.*
*Meanwhile, in any category, a sub-object $C$ of $D$ is a map $C \to D$, which is a **monomorphism** (this means that for all $Z$,* $\begin{tikzcd} Z \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_1"'] & C \end{tikzcd}$ $i \circ f_1 = i \circ f_2 \implies f_1 = f_2$*).*
*We can only form quotients of sub-bundles. In fact $\phi$ from the previous example is a sub-object, but not a sub-bundle.*

The omnibus theorem allows us to produce new bundles from old:

**Example.**

$$\begin{tikzcd} & T^\star M \arrow[d, "="] & := U_{m\in M}(T_mM)^\vee \arrow[dl, "="] \\ TM \arrow[ur] & (TM)^\vee & \end{tikzcd}$$

*the cotangent bundle and*

$$\begin{tikzcd} & \Lambda^k T^\star M \\ TM \arrow[ur] \arrow[dr] & \\ & \Sigma^k T^\star M \end{tikzcd}$$

*which are the **k-th exterior power of the cotangent bundle** and the **k-th symmetric power of the cotangent bundle**.*

*Proof of theorem 8.4.*

Proof: These definitions are all natural, so how could they fail? (Prof. Côté)

In (slightly) more detail: Let $E^1, \dots, E^k$ be vector bundles over $B$. Let $\{U_\alpha\}$ be an open cover of $B$, which simultaneously trivializes all of the $E^i$. Then the $E^i$ cna be presented as follows:

- $\{V^i_\alpha\}_{\alpha\in\mathcal{A}}$
- $\{\phi^i_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V^i_\alpha V^i_\beta)\}$, such that $\phi_{\alpha\gamma} = \phi_{\beta\gamma}\phi_{\alpha\beta}$ (lemma 8.3).

Let $\mathfrak{F}(E^1 \dots, E^k) \in \mathrm{vectbund}_\mathbb{R}(B)$ defined by applying some combination of $\oplus, \otimes, \mathrm{Hom}(\cdot,\cdot), (\cdot)/(\cdot), \dots$ (any cannoical linear algebraic construction) fiberwise.
I.e. $\mathfrak{F}(E^1,\dots,E^k) \coloneqq \bigcup_{b\in B} \mathfrak{F}(E^1_b,\dots,E^k_b)$

the fibers are all disjoint

We can present $\mathfrak{F}(E^1,\dots,E^k)$ by

- $\{\mathfrak{F}(V^1_\alpha,\dots,V^k_\alpha)\}_{\alpha\in\mathcal{A}}$
- $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathfrak{F}(V^1_\alpha,\dots,V^k_\alpha), \mathfrak{F}(V^1_\beta,\dots,V^k_\beta))$

Must check: Smoothness of the $\phi^i_{\alpha\beta}$ implies smoothness of $\phi_{\alpha\beta}$. This is essentially because, of the fixing $V^i_\alpha \simeq \mathbb{R}^k$, then $\phi^i_{\alpha\beta}$ is just a matrix and $\phi_{\alpha\beta}(x)$ is obtained from $\{\phi^i_{\alpha\beta}\}$ by a sequence of matrix addition, multiplications, taking adjoints, ... all of which preserve smoothness of the coefficients. □

## 8.4 Sections of vector bundles

**Definition.** *Let $\pi : E \to B$ be a vector bundle. A **Section** $\sigma : B \to E$ is a smooth map, such that $\pi \circ \sigma = id$.*

We let $\Gamma(E)$ denote the space of sections of $E \to M$.

Here $M = B$ will be switched arbitrarily

**Lemma 8.5.** *(i) $\Gamma(E)$ is an $\mathbb{R}$-vector space*

*(ii) $\Gamma(E)$ is a module over $C^\infty(M,\mathbb{R}) = C^\infty(M)$*

*Proof.* Exercise. Same proof as for $TM$. □

These are very important definitions!

**Example.** *$E = TM \to M, \Gamma(E) = \mathcal{X}(M)$, elements are called **vector fields**.*

**Example.** *$E = T^\star M \to M, \Gamma(E) =: \Omega^1(M)$, elements are called **1-forms**.*

*1-forms are dual to vector fields*

**Example.** *$E = \Lambda^k T^\star M \to M$, $\Gamma(\Lambda^k T^s tar M) =: \Omega^k(M)$, elements are called **k-forms***

**Example.** *$E = M \times \mathbb{R}^k \to M, \Gamma(E) \equiv C^\infty(M,\mathbb{R}^k)$ He wrote:*

*A little silly, but good to see*

$$\Gamma(E) \ni \sigma \mapsto (x \mapsto (x, \sigma_x(x)))$$

*But erased it later, to be fixed. Essentially the point is that the second component describes the function.*

**Remark.** *$\Gamma(E)$ is an infinite dimensional vector space (unless $E = M \times \{x\}$).*

**Lemma 8.6.** *Let $\pi : E \to B$ be a vector bundle. Suppose that $A \subset B$ is closed. Let $A \subset U$ be open. Suppose that $\sigma : A \to E$ is a smooth section (i.e. around every $a \in A, \sigma$ extends to a smooth section in a neighborhood of $a$).*
*Then there exists $\tilde\sigma \in \Gamma(E)$ such that*

*(i) $\tilde\sigma_{|_A} = \sigma$*

*(ii) $supp(\tilde\sigma) \subset U$*

*Proof.* Sheet 12. We already have a lemma for the trivial case, then use partition of unity. □

Picking a section to a point, we can globally extend it to a smooth function, then we can easily connect it to the previous remark.

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**Definition.** *Let $f : M \to M'$ be a smooth map. Let $\pi : E' \to M'$ be a vector bundle over $M'$. Then there is a $\mathbb{R}$ linear map*

$$\Gamma(E') \to \Gamma(f^\star(E'))\sigma \qquad \mapsto \sigma \circ f = f^\star \sigma$$

*i.e.*

$$\begin{tikzcd} f^\star E' \arrow[r] \arrow[d] & E' \arrow[d, "\pi'"'] \\ M \arrow[r, "f"] \arrow[u, bend left, "\sigma\circ f"] & M' \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

*If $E' = T^\star M'$, then we have*

$$\begin{tikzcd} T^\star M \arrow[d] & f^\star(T^\star M') \arrow[l, "df^\vee"] \arrow[r] \arrow[d] & T^\star M' \arrow[d] \\ M \arrow[r, "="] \arrow[u, bend left, "df^\vee\circ\sigma\circ f"] & M \arrow[r, "f"] & M' \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

*where all vertical maps are projections. Here $df^\vee(p, v) = (p, df_p^\vee(v))$ for $p \in M, v \in f^\star(T^\star M')_p = T^\star_{f(p)} M'$ By abuse of notation we also write*

$$f^\star \sigma = df^\vee \circ \sigma \circ f \in \Gamma(T^\star M) = \Omega^1(M).$$

The **upshot** is that we can pullback 1-forms along smooth maps.
In contrast, we can not push forward vector fields along arbitrary smooth maps.
Concretely: if $\sigma \in \Omega^1(M')$, then $f^\star\sigma \in \Omega^1(M)$ is characterized by the formula

$$(f^\star\sigma)_p(\underbrace{v}_{T_pM}) = \sigma_{f(p)}(df_p(v)).$$

More generally, if $\sigma \in \Gamma(T^k T^\star M')$, $f^\star \in \gamma(T^k T^\star M)$ is characterized by the formula

$$(f^\star\sigma)_p(v_1, \dots, v_k) = \sigma_{f(p)}(df_p(v_1), \dots, df_p(v_k)).$$

We have already seen examples, but it is obvious, since we can't reverse the projection maps ...

This is the only thing that can make sense

**Definition.** *Let $\pi : E \to M$ be a vector bundle. Let $X^1, \dots, X^k \in \Gamma(E)$ (assuming that the fibers have dimension $k$). We say:*

*(1) $X^1, \dots, X^k$ is a **local frame** at $p \in M$, if span$(X^1(p), \dots, X^k(p)) = E_p$*

*(2) $X^1, \dots, X^k$ is a **global frame**, if span$(X^1(q), \dots, X^k(q)) = E_q, \forall q \in M$*

**Exercise:** If $X^1, \dots, X^k$ is a local / global frame for $E \xrightarrow{\pi} M$, then $(X^1)^\vee, \dots, (X^k)^\vee$ is a local / global frame for $E^\vee \to M$, hence $((X^i(p))^\vee)(X^j(p)) = \delta_{ij}$.

**Example.** *For $M = \mathbb{R}^n$ we have $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ a global frame and we write $dx_i = (\partial_{x_i})^\vee$.*

# Chapter 9: Riemannian metrics (a quick tour)

**Definition.** *Let $V$ be a (finite dimensional, $\mathbb{R}$-) vector space. An* ***inner product*** *is a bilinear map*

$$g \in \Sigma^2 V^\star : V \times V \to \mathbb{R}$$

*which is*

*(i)* ***symmetric***

*(ii)* ***positive definite****: $g(v,v) \geq 0, = 0 \iff v = 0$*

*We say that $(V,g)$ is an* ***inner product space****.*

**Definition.** *Let $M$ be a manifold. A* ***Riemannian metric*** *$g \in \Gamma(\Sigma^2 T^\star M)$ such that for all $p \in M, (T_pM, g_p(\cdot,\cdot))$ is an inner product space. A pair $(M,g)$ is called a* ***Riemannian manifold****.*

Note $g_p \in (\Sigma^2 T^\star M)_p = \Sigma^2(T_pM^\vee)$.

This is just notation I use, because everyone does (talking about the postion of the p, after the $T$!)

**Remark.** *If $E \to M$ is an vector bundle, a metric on $E$ is a section $g \in \Gamma(\Sigma^2 E^\vee)$ s.t. $(E_p, g_p(\cdot,\cdot))$ is an inner product space.*

Note that on $\mathbb{R}^n$ we have a frame $\{dx_1, \ldots, dx_n\}$ for $T^\star\mathbb{R}^n$. More generally, $T^2T^\star\mathbb{R}^n$ admits a global frame $\{dx_i \otimes dx_j\}_{1\leq i,j\leq n}$. Hence, any section $\sigma \in \Gamma(T^2T^\star\mathbb{R}^n)$ can be written as

$$\sigma = \sum_{i,j=1}^n \sigma_{ij} dx_i \otimes dx_j$$

for $\sigma_{ij} : \mathbb{R}^n \to \mathbb{R}$.
Then $\sigma$ is a metric iff:

(i) $\sigma_{ij} = \sigma_{ji}$, i.e. any symmetric matrix

(ii) $\sum_{i,j=1^n} \sigma_{ij} v^i v^j \geq 0, = 0 \iff v = 0$

in other words $\sigma \in \Gamma(T^2T^\star\mathbb{R}^n)$ defines a metric, iff $(\sigma_{ij}) : \mathbb{R}^n \to \mathrm{Mat}(n \times n)$ lands in the subspace of symmetric and positive definite matrices.

**Example.** *$g_0 \coloneqq \sum i = 1^n dx_i \otimes dx_i = \sum_{i,j=1}^n \delta_{ij} dx_i \otimes dx_j$. We call $g_0$ the* ***Euclidean metric****.*

**Definition.** *Let $(M,g)$ be a Riemannian manifold.*

*(1) Given $v \in T_pM$, the* ***length of $v$***

$$|v|_g = \sqrt{g_p(v,v)}$$

*(2) Given $v, w \in T_pM, v, w \neq 0$ the* ***angle between*** *$v, w$ is the unique $\theta \in [0, \pi]$*

$$\cos\theta = \frac{g_p(v,w)}{|v|_g |w|_g}$$

*(3) We say that $v, w \in T_pM$ are orthogonal if $g(v, w) = 0$.*

**Exercise:** If $M = \mathbb{R}^n, g = g_0$ the above definitions recovers the classical definitions.

**Definition.** *Given $\gamma : [a, b] \to M$, the* ***length*** *of $\gamma$ is given by*

$$L_g(\gamma) \int_a^b |\underbrace{\dot{\gamma(t)}}_{\in T_{\gamma(t)}M}|dt$$

*Given $p, q \in M$, we define*

$$d(p,q) = \inf\{L_g(\gamma) \mid \gamma : [a,b] \to M, p = \gamma(a), q = \gamma(b)\}.$$

**Lemma 9.1.** *$L_g(\gamma)$ does not depend on the parametrization. In other words if $\gamma : [c, d] \to [a, b], \gamma' > 0$ Then setting $\sigma := \gamma \circ \phi$. We have*

$$L_g(\sigma) = L_g(\gamma)$$

*Proof.*

$$\begin{aligned} L_g(\sigma) &= \int_c^d |\dot{\sigma}(t)|dt = \int_c^d |\dot{\gamma}(\sigma(\phi(t)))\dot{\phi}(t)|dt \\ &= \int_c^d |\dot{\gamma}|\dot{\phi}(t)dt = \int_a^b |\dot{\gamma}(s)|ds = L_g(\gamma), \end{aligned}$$

where the last equality is just a change of variables. □

**Fact:** $(M, d_g(\cdot, \cdot))$ is a metric space.
The infimum in the definition can not be a minimum, take $\mathbb{R}^2 \setminus \{0\}$ with the euclidean metric, the distance is not different from the distance in $\mathbb{R}$.

**Lemma 9.2.** *Let $M$ be a smooth manifold. Then $M$ admits a Riemannian metric.*

*Proof.* Choose a covering of $M$ $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ by coordiante charts $\varphi : U_\alpha \to \mathbb{R}^n$. Let $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ be a partition of unity subject to the cover. Let $g_\alpha = \varphi_\alpha^\star \in \Gamma(\Sigma^2 T^\star U_\alpha)$ and let $g = \sum_{\alpha\in\mathcal{A}} = \underbrace{\psi_\alpha g_\alpha}_{\in\Gamma(\Sigma^2T^\star M)}$. This makes sense by local finiteness. It is clear that $g$ is a metric:
It has to be symmetric, because the $g_\alpha$ are symmetric. To check that $g_p(v, v) > 0$ whenever $v \neq 0$ note that $\exists\alpha$ s.t. $\psi_\alpha(p) > 0 \implies g_\alpha(v, v) > 0$ whenever $v \neq 0, \psi_\alpha g_\alpha(v, v) \geq 0 \forall\alpha$ □

**Corollary 9.3.** *The space of Riemannian metrics on $M$ is convex and non-empty, hence contractable.*

*Proof.* Exercise. □

**Lemma 9.4.** *Let $(W, g)$ be an inner product space. If $V \overset{i}{\hookrightarrow} W$, then $g$ restricts to an inner product on $V$. We can write $g_{|V} = i^\star g(\cdot, \cdot), i^\star g(v, w) = g(i(v), i(w))$.*

*Proof.* Straightforward check, therefore omitted. □

**Corollary 9.5.** *Let $i : M \hookrightarrow N$ an immersion. If $g \in \Gamma(\Sigma^2 T^\star N)$ is a Riemannian metric, then $i^\star g \in \Gamma(\Sigma^2 T^\star M)$ is also a metric and we call $i^\star g$ the* ***induced metric****.*

*Proof.* By definition, $di : T_p^M \to T_{i(p)}N$ is injective. Now apply lemma 9.4. □

**Example.** *Suppose that $M \hookrightarrow \mathbb{R}^n$ embedded submanifold. Then $i^\star g_0$ is a metric on $M$.*

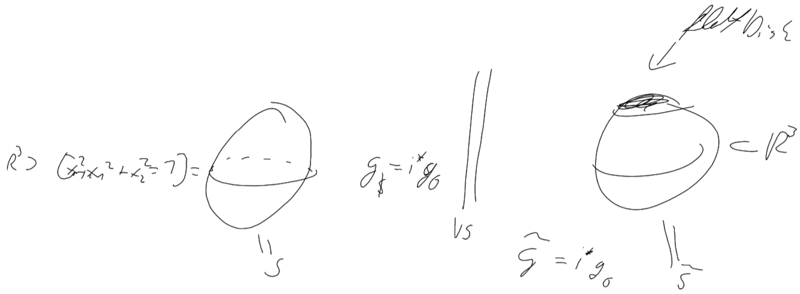

Figure 9.1: Sketch 9.1

*He writes $g_\mathbb{S} = i^\star g_0$. Question: does there exist a diffeomorphism $\varphi : S \to \tilde{S}$ s.t. $\varphi^\star \tilde{g} = g_\mathbb{S}$? The answer is no!*
*Need: Invariant of Riemannian manifold.*

*Everything after the fact that metrics exist is non-examable.*

Given $(M, g) \to R_g \in \Gamma(T^{3,1}TM)$ called the curvature tensor. We have

$$\varphi^\star R_g = R_{\varphi^\star g},$$

for $\varphi : M \to M'$ a diffeomorphism.
**Fact:** $R_{g_0} \equiv 0$ and $R_{g_\mathbb{S}} \iff \underbrace{s_g}_{\in C^\infty(S)} \equiv 1.$
Not only is there no global diffeomorphism, there is also no local diffeomorphism!

Start of lecture 22 (07.01.2025)

Exam: Same difficulty as the homework, slightly easier. He will not fail everyone.
How to study: do lots of problems. No notes, no anything.
No comments on true / false questions etc. The last week is non examable and that Tuesdays is canceled.
Change of notation:
If $V$ finite dimensional vector space, $V^\vee$ is the dual space. We now also write $V^\star = V^\vee$.

# Chapter 10: Differential forms

Recall from section 8.4 the following definition:

**Definition.** *Given $M$ a smooth manifold, $k \geq 0$, a* ***k-form*** *is a section $\alpha \in \Gamma(\Lambda^k T^\star M) \equiv= \Omega^k(M)$.*

Upshot: The spaces $\Omega^k(M)$ carry a great deal of information about $M$.

as $k$ varies

## 10.1 $k$-forms

### 10.1.1 Differentials of functions

**Definition.** *Let $f : M \to \mathbb{R}$. we let. $df \in \Omega^1(M)$, be defined by*

$$df_p(v) := v(f), v \in T_pM$$

*for all $p \in M$.*

*$f$ smooth, of course*

**Lemma 10.1.** *Fix $f : \mathbb{R}^n \to \mathbb{R}$. With respect to the canonical identification $T^\star\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$.*

$$df = (\underbrace{\partial_{x_1} f}_{\in C^\infty(\mathbb{R}^n,\mathbb{R})}, \dots, \partial_{x_n} f).$$

.

*Proof.* Test $df$ against coordiante vector fields $(\partial_{x_i})_p \equiv (0, \dots, 0, 1, 0, \dots, 0)$. Then

$$df_p((\partial_{x_i})_p) = (\partial_{x_i})_p(f) = (\partial_{x_i} f)_p \tag{1}$$

□

**Lemma 10.2.** *Given $\varphi : M \to N, f : N \to \mathbb{R}, \varphi^\star(df) = d(f \circ \varphi)$.*

*Proof.*

$$(\varphi^\star df)_p(v) = df_{C(p)}(d\varphi(v)) = d\varphi(v)(f) = v(f \circ \varphi) = d(f \circ \varphi)(v)$$ □

As a consequence: if $f : M \to \mathbb{R}$ smooth, then $df \in \Omega^1(M)$ is smooth. To see this, note that locally we can write $f = f \circ \varphi^{-1} \circ \phi$. Then

$$df = d((f \circ \varphi^{-1}) \circ \varphi) \overset{10.2}{\text{under}} \varphi^\star d(f \circ \varphi^{-1}),$$

which is smooth by 10.1.

**Remark.** *We have seen $(dx_i)_p = (\partial_{x_i})_p^\star$ Write $x_i \in C^\infty(\mathbb{R}^n, \mathbb{R})(x_1, \dots, x_n) \mapsto x_i$. Then $(dx_i)_p((\partial_{x_i})_p) = \delta_{ij}$. Hence $(dx_i)_p = (dx_i)_p$ (i.e. the notations agree).*

**Remark.** *Given $f : M \to \mathbb{R}$, we have*

$$\begin{tikzcd} TM \arrow[rr] \arrow[dr] & & f^*T\mathbb{R} \equiv M \times \mathbb{R} \arrow[dl] \\ & M & \end{tikzcd}$$

*by the map*

$$(p, v) \mapsto (p, df_p(v)).$$

*Again the notations are consistent: View $v$ as a derivation at $p$, write $x = id : \mathbb{R} \to \mathbb{R}$. Then*

$$df_p(v)(x) = v(x \circ f) = v(f).$$

*Hence*

$$\underbrace{df_p(v)}_{\in T_{f(p)}\mathbb{R}} = \underbrace{v(f)}_{\in\mathbb{R}} \cdot \underbrace{(\partial_x)_{f_p}}_{T_{f(p)}\mathbb{R}}.$$

### 10.1.2 Line integrals

**Definition.** *(1) Let $\omega \in \Omega^1([a, b])$. Then $\omega = f(t)dt$ uniquely, $f : [a, b] \to \mathbb{R}$. We let*

$$\int_a^b \omega = \int_a bf(t)dt.$$

*He does not write the dt ...*

*(2) Let $\omega \in \Omega^1(M)$. Let gamma : $[a, b] \to M$. We let*

$$\int_\gamma \omega := \int_a^b \underbrace{\gamma^\star\omega}_{\in\Omega^1([a,b])}.$$

**Lemma 10.3.** *If $\sigma : [c, d] \to [a, b], \sigma' > 0, \sigma(c) = a, \sigma(d) = b$. Then*

$$\int_c^d (\gamma \circ \sigma)^\star\omega = \int_a^b \gamma^\star\omega.$$

*Proof.* Observe that $(\gamma \circ \sigma)^\star = \sigma^\star(\gamma^\star)\omega$. So it is enough to prove that

$$\int_c^d \sigma^\star\eta = \int_a^b \eta$$

where $\eta = f(t)dt \in \Omega^1([a, b])$.
But

$$\begin{aligned}\int_c^d \sigma^\star\eta &= \int_c^d \eta(\sigma(t))\sigma'(t)dt \\ &= \int_a^b \eta(s)ds = \int_a^b \eta\end{aligned}$$ □

by the change of variables formula.

**Lemma 10.4.** *If $f : M \to \mathbb{R}$ and $\gamma : [a, b] \to M$, then*

$$\int_a^b \gamma^\star(df) = \int_a^b (f \circ \gamma)dt$$

*Proof.* Homework □

Hence, if $\gamma(0) = \gamma(1)$, $\int_\gamma df = 0$.

**Example.** *Let* $\omega = \frac{xdy - ydx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$. *Let* $\gamma : [0, 2\pi] \to \mathbb{R}^2 \setminus \{0\}, t \mapsto (\cos t, \sin t)$.

$$\begin{aligned}\int_\gamma \omega &= \int_0^{2\pi} \frac{\cos t d(\sin t) - \sin t d(\cos t)}{\cos^2 t + \sin^2 t} \\ &= \int_0^{2\pi} \cos t(\cos t dt) - \sin t(\sin t dt) = \int_0^{2\pi} (\cos^2 t + \sin^2 t) dt = 2\pi\end{aligned}$$

$\implies$ *$\omega$ is not the differential of some function* $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$

Second EXAM same difficulty and same content (i.e. everything that is examable in the first exam is examable in the second and vice verca)

**Definition.** *Let* $U \subset \mathbb{R}^n$ *be open We say that* $\omega \in \Omega^1(U)$ *is*

- ***exact***, *if omega* $= df$ *for some* $f : U \to \mathbb{R}$
- ***closed*** *of* $\partial_{x_i} w^j = \partial_{x_j} w^i$, *where* $\omega = \omega^1 dx_1 + \cdots + \omega^n dx_n, \omega^i : U \to \mathbb{R}$.

*This definition only makes sense locally ...*

Observe that exact $\implies$ closed. Because if $\omega = df \implies \omega = \sum_i \partial_{x_i} f dx_i$, but $\partial_{x_i}\partial_{x_j} f = \partial_{x_j}\partial_{x_i} f$.
**Question:** Does closed imply exact? **No!** Test that the previous example is closed, but not exact.

**Proposition 10.5** (Poincaré Lemma)**.** *Let* $\omega \in \Omega^1(B_1(0)), B_1(0) \subset \mathbb{R}^n$, *be a closed 1-form. Then* $\omega$ *is exact.*

*Very important, should remember for the exam*

**Remark.** *The failure of 1 forms to be exact is a measure of the topology of the domain?*

*Proof.* Assume $\omega \in \Omega^1(B_1(0))$ is closed. Define

$$f(x) := \int_{\gamma_x} \omega,$$

where $\gamma_x : [0,1] \to B_1(0) \ni x$, $t \mapsto tx$.
Write $\omega = \omega^1 dx_1 + \cdots + \omega^n dx_n$. We have

$$f(x) = \int_0^1 \omega_{\gamma_x(t)} \left(\dot{\gamma}_x(t)\right) dt = \int_0^1 \left(\sum_{i=1}^n \omega^i(tx) \cdot x_i\right) dt$$

$$\begin{aligned}\partial_{x_j} f(x) &= \partial_{x_j} \left(\int_0^1 \left(\sum_{i=1}^n \omega^i(tx) \cdot x_i\right) dt\right) \\ &= \int_0^1 \left(\sum_{i=1}^n \partial_{x_i}\omega^j(tx) \cdot t \cdot x_i + \omega^j(tx)\right) dt \\ &\overset{\omega \text{ closed}}{=} \int_0^1 \left(\left(\sum_{i=1}^n \partial_{x_j}\omega^j(tx) \cdot t \cdot x_i\right) + \omega^j(tx)\right) \\ &= \int_0^1 \frac{d}{dt}(t\omega^j(tx))dt = t\omega^j(x) \mid_{t=0}^{t=1} = \omega^j(x)\end{aligned}$$

□

**Remark.** *The same proof works if we replace the ball wit any star shaped domain.*

## 10.2 $k$-forms

### 10.2.1 More linear algebra

Let $V$ be a vector space. Recall from section 8.2.2. that $\Lambda^k V^\star \subset T^k V^\star$ is spanned by multilinear maps $\alpha : V \times \cdots \times V \to \mathbb{R}$ which are alternating. We have $\mathrm{Alt}^k : T^k(V \to \Lambda^k V)$. Let $\{\epsilon^1, \ldots, \epsilon^n\}$

be a basis for $V^\star$. Write $I = (i_1, \dots, i_k)$, where $1 \leq u_1, \dots, i_k, \leq n$. We let $\epsilon^I \in \Lambda^k(V^\star)$ be defined by

$$\epsilon^I(v_1, \dots, v_k) = \det \underbrace{\begin{pmatrix} \epsilon^{i_1}(v_1) & \dots & \epsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \epsilon^{i_k}(v_1) & \dots & \epsilon^{i_k}(v_k) \end{pmatrix}}_{\in \mathbb{R}^{k \times k}}.$$

Start of lecture 23 (10.01.2025)

**Example.** *Let $V = \mathbb{R}^3$, with the standard basis / dual basis. Let $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3), z = (z_1, z_2, z_3) \in \mathbb{R}^3$*

$$\epsilon^{123}(v, w, z) = \det \begin{pmatrix} v_1 & w_1 & z_1 \\ v_2 & w_2 & z_2 \\ v_3 & w_3 & z_3 \end{pmatrix}$$

**Lemma 10.6.** *Let $\{e_1, \dots, e_n\}$ be the basis for $V$, $\{\epsilon_1, \dots, \epsilon_n\}$ be the dual basis. Let $I = (i_1, \dots, i_k)$.*

*(a) If $I$ has a repeated index, then $\epsilon^I = 0$*

*(b) Let $J := I_\sigma, \sigma \in S_k$. Then*
$$\epsilon^J = (sgn(\sigma))\epsilon^I$$

*(c) Given $J = (j_1, \dots, j_k)$,*
$$\epsilon^I(e_{j_1}, \dots, e_{j_k}) = \delta^I_J,$$
*where $\delta^I_J := \det(\delta^{i_m}_{j_l})_{1 \leq l, m \leq k}$*

*Proof.* omitted / follows directly from the definitions. □

**Lemma 10.7.** *Fix basis $\{\epsilon_1, \dots, \epsilon_n\}$ for $V^\star$. Then, for $k \leq n$, the collection $\{\epsilon^I \mid I$ is an increasing multiindex of length $k\}$ forms a basis for $\Lambda^k V^\star$.*

E.g. $V = \mathbb{R}^3, k = 2, \{\epsilon^{12}, \epsilon^{23}, \epsilon^{23}\}$.

*Proof.* See sheet 11 exercise 3 for details. □

**Definition.** *Given $\omega \in \Lambda^k V^\star, \eta \in \Lambda^l V^\star$, let*

$$\omega \wedge \eta := \frac{(k+l)!}{k! l!}(\omega \otimes \eta) \in \Lambda^{k+l} V^\star.$$

*This operation is called the **wedge product** or the **exterior product**.*

**Lemma 10.8.** *Let $(\epsilon_1, \dots, \epsilon_n)$ be a basis for $V^\star$. Given $I = (i_1, \dots, i_k), J = (j_1, \dots, j_l)$, let $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$. Then*
$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}.$$

*Proof.* Homework of this week. The factor in the definition makes this statement true. Hint: Test against the dual basis and show they are equal. □

**Remark.** *We defined $\Lambda^k V^\star \subset T^k V^\star$, but we can also do*

$$\oplus_k T^k V^* / (v \otimes v).$$

*In this case we would not have the factor in the definition of the wedge product.*

**Proposition 10.9** (Properties of the wedge product)**.** *Suppose that $w, w', \eta, \eta', \xi \in \Lambda^\cdot(V^\star)$.*

*(a) $(a\omega + a'\omega') \wedge \eta = a\omega \wedge \eta + a'\omega' \wedge \eta$ and similar in the second component*

*(b) $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$*

*(c)* $\omega \wedge \eta = (-1)^{|\omega|\cdot|\eta|}\eta \wedge \omega$

*(d) given* $I = (i_1, \dots, i_k), \{\epsilon^1, \dots, \epsilon^k\}$ *Basis for* $V^\star$,

$$\epsilon^I = \epsilon^{i_1} \wedge \dots \wedge \epsilon^{i_k}$$

*(e) given* $w^1, \dots w^k \in V^\star, v_1, \dots, v_k \in V$ *we have*

$$\omega^i \wedge \dots \wedge \omega^k(v_1, \dots, v_k) = \det(\omega^i(v_j)).$$

*Proof.* **(a)** follows from multilinearity of $\otimes$
**(b)** Fix a basis $\{\epsilon^1, \dots, \epsilon^n\}$. Lemma 10.8 implies

$$(\epsilon^I \wedge \epsilon^J) \wedge \epsilon^K = \epsilon^{IJ} \wedge \epsilon^K = \epsilon^{IJK}$$
$$\epsilon^I \wedge \epsilon^{JK} = \epsilon^I \wedge (\epsilon^J \wedge \epsilon^K).$$

The generality follows from lemma 10.7 and multilinearity.
**(c)** Note that

$$\begin{aligned}\epsilon^I \wedge \epsilon^J &\overset{10.8}{=} \epsilon^{IJ} \\ &\overset{10.6\,(b)}{=} (-1)^{|I||J|} = (-1)^{|I||J|}\epsilon^J \wedge \epsilon^I\end{aligned}$$

where the general case follows from multilinearity.
**(d)** By lemma 10.8 and induction.
**(e)** If $w^i = \epsilon^i$, then this follows from (d) + the definition of $\epsilon^I, I = (1, \dots, k)$. The general case follows by multilinearity. ☐

**Definition.** *Given a vector space* $V$, *let* $\Lambda^\cdot V^\star := \oplus_{k\geq 0}\Lambda^K V^\star$, *which is just a vector space. We can make this into a (skew-) commutative algebra with respect to* $\wedge$. *This is usually called the* ***exterior algebra*** *of* $V$.

$\Lambda^0 V^\star = \mathbb{R}$

### 10.2.2 The algebra of differential forms

Let $M$ be a manifold. We have

$$\begin{aligned}\Lambda^k T_p^\star M \times \lambda^l T_p^\star M &\to \Lambda^{k+l}T_p^\star M \\ (\omega_p, \eta_p) &\mapsto \omega_p \wedge \eta_p.\end{aligned}$$

Since, by definition, $T^\star M = \coprod_{p\in M} T_p^\star M$, we get

$$\begin{aligned}\Gamma(\Lambda^k T^\star M) \times \Gamma(\Lambda^l T^\star M) &\to \Gamma(\Lambda^{k+l}T^\star M) \\ (\omega, \eta) &\mapsto \omega \wedge \eta\end{aligned}$$

**Notation:** $\Omega^k(M) := \Gamma(\lambda^k T^\star M)$.
We let $\Omega^\cdot(M) :=_{k\geq 0} \Omega^k(M)$ with multiplication $\wedge$. This is called the **algebra of differential forms on $M$**.
Exercise: Suppose that $\dim V = n$, then $\Lambda^k V^\star = 0$ , whenever $k > n$. This algebra is only supported up to degree $n$.

**Example.** *On* $M = \mathbb{R}$. $\Omega^0(\mathbb{R}^1) = \{$*smooth maps* $f : \mathbb{R} \to \mathbb{R}\}$, $\Omega^1(\mathbb{R}) = \{f(x)dx, f : \mathbb{R} \to \mathbb{R}\}$. *A typical element of* $\Omega^\cdot(\mathbb{R})$ *is* $f_0(x) + f_1(x)dx$.

**Example.** $M = \mathbb{R}^3, \Omega^2(\mathbb{R}^3) = \{f_{12}(x)dx_1 \wedge dx_2 + f_{13}(x)dx_1 \wedge dx_3 + f_{23}(x)dx_2 \wedge dx_3\}$

**Lemma 10.10** (Naturality of the wedge product)**.** *Let* $F : M \to N$. *Let* $F^\star : \Omega^k(N) \to \Omega^k(M)$ *be the pullback (section 8.4)*

*(a)* $F^\star : \Omega^k(N) \to \Omega^k(M)$ *is* $\mathbb{R}$ *linear*

(b) $F^\star(\omega \wedge \eta) = F^\star\omega \wedge F^\star\eta$

(c) *If* $(y_i)$ *are local coordinates in* $N$*, we have* $F^\star \sum_I \omega_I dy^i \wedge \dots \wedge y^{i_k} = \sum (\omega_I \circ F) dF^{i_1} \wedge \dots \wedge dF^{i_k}$

(d) *If* $\dim M = \dim N$*,* $(x_i)$ *local coordinates on* $U \subset M$*,* $(y_i)$ *local coordinates on* $V \subset M$*,* $U \subset F^{-1(V)}$*, then* $F^\star(vdy_1 \wedge \dots \wedge dy_n) = v \circ F \det(dF) dx^1 \wedge \dots \wedge x^n, v : V \to \mathbb{R}$.

*Proof.* **(a)** ok
**(b)** ok. Proof point-wise, follows from the definitions.
**(c)** Combine (b) with lemma 10.2 ($F^\star dy_i = d(y_i \circ F) = dF_i$).
**(d)** From (c), we have $F^\star v dy_1 \wedge \dots \wedge dy_n = (v \circ F) dF_1 \wedge \dots \wedge dF_n$. By proposition 10.9 (e), $(dF_1 \wedge \dots \wedge dF_n)(\partial_{x_1}, \dots, \partial_{x_n}) = \det(dF_i(\partial_{x_j})) = \det(\partial_{x_j} F_i) - \det(dF)$.

□

### 10.2.3 The exterior derivative

The **punchline**: There is a canonical $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$, called the **exterior derivative**. It holds $d \circ d = 0$

$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$ this is a chain complex.

**Definition.** $H^k_{dR}(M) \coloneqq \ker(d : \Omega^k(M) \to \Omega^{k+1}(M))/im(d : \Omega^{k-1}(M) \to \Omega^k(M))$

**Theorem 10.11** (deRham)**.** *There is a natural isomorphism*

$$H^{\cdot}_{dR}(M) \simeq H^{\cdot}_{sing}(M, \mathbb{R})$$

*for any smooth manifold* $M$*.*

*Non examable. But really nice. These objects come from different parts of mathematics*

There are four more lectures (including this one) that are examable!

Start of lecture 24 (14.01.2025)

**Recall** we had the following complex

$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$

**Notation:** If $\omega \in \Omega^k(U), U \subset \mathbb{R}^n$, we write

Just to save some time

$$\omega = \sum_{|I|=k}{}' \omega_I dx_I \coloneqq \sum_{\substack{|I|=k \\ I \text{ increasing}}} \omega_I dx_I.$$

**Definition.** *Let* $U \subset \mathbb{R}^n$ *or* $U \subset \mathbb{H}n$*. Let* $\omega = \left(\sum'_{|I|=k} \omega_I dx_I\right) \in \Omega^k(U)$*. Then, we let* $d(\omega) = d\left(\sum'_{|I|=k} \omega_I dx_I\right) \coloneqq \sum'_{|I|=k} d\omega_I \wedge dx_I$*. We call* $d$ *the* ***exterior derivative****.*

*Remember* $\mathbb{H}^n = \{x_1, \dots, x_n\} \subset \mathbb{R}^n \mid x_n \geq 0$

**Example.** $\omega \in \Omega^0(U) = C^\infty(U), d\omega = d\omega$*, where the latter is the definition from section 10.1.*

**Example.** $\omega \in \Omega^1(U), U \subset \mathbb{R}^n, \omega = \sum_{i=1}^n \omega_i dx_i$*, then*

$$d\omega = \sum_{i=1}^n \left(\sum_{j=1}^n \partial_{x_j} w_i dx_j\right) \wedge dx_i = \sum_{1 \leq i < j \leq n} (\partial_{x_i} w_j) - \partial_{x_j} w_i dx_i \wedge dx_j,$$

*since* $dx_i \wedge dx_j = dx_j \wedge dx_i$*.*

**Remark.** $\omega$ *closed (in the sense of section 10.1.2 ) is equivalent to* $d\omega = 0$

**Now to defining $d$ on manifolds (not just subsets of $\mathbb{R}^n$).**

**Proposition 10.12.** *Let* $U \subset \mathbb{R}^n/\mathbb{H}^n$ *open. Let* $d : \Omega^k(U) \to \Omega^{k+1}(U)$*. Then*

(a) $d$ *is* $\mathbb{R}$ *linear.*

*(b) (Leibniz rule) Given $\omega \in \Omega^k(U), \eta \in \Omega^l(U)$, we have*

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \wedge d\eta$$

*(c) $d^2 = 0$*

*(d) Suppose $f : \underbrace{U}_{\subset \mathbb{R}^n/\mathbb{H}^n} \to \underbrace{V}_{\in \mathbb{R}^n \mathbb{H}^m}, \omega \in \Omega^k(V)$. Then $dF^\star \omega = F^\star d\omega$.*

*Proof.* **(a)** is trivial.
**(b)**: As a first step, let us prove that for **any** multiindex $I$, we have that $d(u_I dx_I) = du_I \wedge dx_I$. If $I = (i_1, \dots, i_k)$ has repeating indices, then both sides are zero. If $I$ has no repeated indices, there exists some $\sigma \in S_k$ s.t. $J = I_\sigma$ is increasing and

$$d(u_I dx_I) = d(\mathrm{sgn}(\sigma) u_I dx_j) = (\mathrm{sgn}\sigma) d(U_i dx_J) = du_I \wedge \{(\mathrm{sgn}\sigma) dx_J\} = du_I \wedge dx_I.$$

We may assume $\omega = u dx_I, |I| = k$, increasing, $\eta = v dx_J, |J| = l$, increasing. Now

$$\begin{aligned}
d(\omega \wedge \eta) &= d(u dx_I \wedge v dx_J) \\
&= d(uv dx_I \wedge dx_J) \\
&= d(uv) \wedge dx_I \wedge dx_J \\
&= (v du + u dv) \wedge dx_I \wedge dx_j \\
&= v du \wedge dx_I \wedge dx_J + u dv \wedge dx_I \wedge dx_J \\
&= du \wedge dx_I (v dx_J) + (-1)^k u dx_I \wedge dv \wedge dx_J \\
&= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta
\end{aligned}$$

**(c)**: We can assume that $\omega = u dx_I = u dx_{i_1} \wedge \dots \wedge dx_{i_k}$.

$$d\omega = du \wedge dx_I$$

$$dd\omega \overset{(b)}{=} \underbrace{ddu}_{=0} \wedge dx_I - du \wedge ddx_I$$

$$ddu = d\left(\sum_{i=1}^n \partial_{x_i} u dx_i\right) = \sum_{i<j} \underbrace{(\partial_{x_i}\partial_{x_j} u - \partial_{x_j}\partial_{x_i} u)}_{=0} dx_i \wedge dx_j$$

$ddx_I$ is zero by $b$ and induction.
**(d)**: Can assume $\omega = u dx_{i_1} \wedge \dots \wedge dx_{i_k}$.

$$\begin{aligned}
F^\star d\omega &= F^\star d(u dx_{i_1} \wedge \dots \wedge dx_{i_k}) \\
&= F^\star (du \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}) \\
&= d(u \circ F) \wedge dF_{i_1} \wedge \dots \wedge dF_{i_k} \\
dF\omega &= d(u \circ F dF_{i_1} \wedge \dots \wedge dF_{i_k}) \\
&= d(u \circ F) \wedge dF_{i_1} \wedge \dots \wedge dF_{i_k}
\end{aligned}$$

□

In the exam, proving that two complicated things are equal, is as easy as fixing a point and then proving the statement at that point!

**Theorem 10.13.** *[Existence and uniqueness of the exterior derivative] Let $M$ be a smooth manifold (possibly with non-empty boundary). There* ***exists*** *a* ***unique*** *operator $d : \Omega^k(M) \to \Omega^{k+1}(M), k \geq 0$ satisfying the following properties:*

*Main theorem of this lecture*

*(i) $d$ is $\mathbb{R}$ linear*

*(ii)* *given* $\omega \in \Omega^k(M), \eta \in \Omega^l(M), d(\omega \wedge \eta) d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

*(iii)* $d^2 = 0$

*(iv)* $f \in \Omega^0(M) \equiv C^\infty(M), df$ *is the differential of* $f$*, as defined in section 10.1*

*Proof.* **Existence**: Fix $\omega \in \Omega^k(M)$. Given a chart $(U, \varphi)$, define

$$d\omega = \varphi^\star d(\varphi^{-1\star}\omega)$$

on $U$. This is well defined, since if $(V, \psi)$ another smooth chart, $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$, we have: By property (d) of 10.12

$$\begin{aligned} \varphi^{-1\star}\psi^\star d(\psi^{-1\star}\omega) &= (\psi \circ \varphi^{-1})^\star d(\psi^{-1\star}\omega) = d((\psi \circ \varphi^{-1})^\star \psi^{-1\star}\omega) \\ &= d(\varphi^{-1\star}\psi^\star\psi^{-1\star}\omega) = d(\varphi^{-1\star}\omega) \end{aligned}$$

$\implies$, by applying $\varphi^\star(\cdot)$,

$$\psi^\star(d\psi^{-1\star}\omega) = \varphi^\star d(\varphi^{-1\star}\omega).$$

This takes care of existence, since the properties follow from the case of subsets from $\mathbb{R}^n/\mathbb{H}^n$!
**Uniqueness:** Let $\tilde{d} := \Omega^k(M) \to \Omega^{k+1}(M)$ and satisfies the same properties. Let $\omega_1, \omega_2 \in \Omega^k(M)$. Suppose that $\omega_1, \omega_2$ agree on some non-empty open subset $U \subset M$. Fix $p \in U$. Then $\tilde{d}(w_-1\omega_2)_p = \tilde{d}(\omega_1)_p - \tilde{d}(\omega_2)_p$ Let $\phi$ be a bump function supported in $U$, s.t. $\phi \equiv 1$ near $p$. Then assuming $\omega$ vanishes in $U$, we have

$$0 = \tilde{d}(\underbrace{\phi\omega}_{\equiv 0}) = \underbrace{\tilde{d}\phi \wedge \omega}_{0 \text{ near } p} \quad \underbrace{\phi}_{\equiv 1 \text{ near } p} \quad \tilde{d}\omega$$

$\implies$ $\tilde{d}\omega$ vanishes near $p$.
Now let $\omega \in \Omega^k(M)$ be an arbitrary $k$ form. Fix $p \in M$ and a chart $(U, \varphi), p \in U$. Let us again let $\phi$ be a bump function supported on $U$ with $\phi \equiv 1$ near $p$. Then $\tilde{d}(\phi\omega)_p = (\tilde{d}\omega)_p$ (by what we just checked). Similarly $(d\phi\omega)_p = (d\omega)_p$. We claim that it is enough that $d\omega = \tilde{d}\omega$ under the assumption that supp$\omega \subset U$.
In local coordinates

$$\begin{aligned} \omega &= \sum{}' \omega_i dx_I = \sum{}' \omega_I \tilde{d}x_I \\ \tilde{d}\omega &= \sum{}' \tilde{d}(\omega_I dx_I) \\ &\overset{(ii)}{=} \sum{}' \tilde{d}\omega_I \wedge dx_I \overset{(iv)}{=} \sum{}' d\omega_i \wedge dx_i = \omega \end{aligned}$$

□

**Corollary 10.14** (naturality). *If* $F : M \to N, \omega \in \Omega^k(N)$*, then* $dF^\star\omega = F^\star d\omega$*.*

*Proof.* Let $(U, \varphi)$ be a chart on $M$, let $(V, \psi)$ a chart on $N$. Assume that $F(U) \subset V$. On $U$, we have

$$\begin{aligned} F^\star d\omega &= F^\star\psi^\star d(\psi^{-1\star}\omega) = \varphi^\star\varphi^{-1\star}F^\star\psi^\star d(\psi^{-1\star}\omega) \\ &= \varphi^\star(\psi \circ F \circ \varphi^{-1}) \star d(\psi^{-1\star}\omega) \\ &= \varphi^\star d(\psi \circ F \circ \varphi^{-1}) \star \psi^{-1\star}\omega \\ &= \varphi^\star d\varphi^{-1\star}F^\star\psi^\star\psi^{-1\star}\omega \\ &= \varphi^\star d(\varphi^{-1\star}(F^\star\omega)) = dF^\star\omega \end{aligned}$$

□

Start of lecture 25 (17.01.2025)

**Definition.** *Given a manifold* $M$*, we say that* $\omega \in \Omega^k(M)$ *is* ***closed*** *if* $d\omega = 0$ *and* ***exact*** *if there exists* $\eta \in \Omega^k(M)$ *such that* $d\eta = \omega$*.*

# Chapter 11: Integration theory

## 11.1 Orientations of vector spaces

Let $V$ be a real vector space of dimension $n \geq 0$. Then:

- $\Lambda^n V^\star$ is a 1 dimensional vector space
- $\Lambda^n V^\star \setminus \{0\}$, carries an action of $\mathbb{R}_+ := [0, \infty)$ by scaling $(\lambda, \omega) \mapsto \lambda\omega$. The quotient $\Lambda^n V^\star \setminus \{0\}/\mathbb{R}_+$ is a is a set with two elements, with discrete topology.
- Let $B = \{v_1, \dots, v_n\}, \tilde{B} = \{\tilde{v_1}, \dots, \tilde{v_n}\}$. We write $M_B^{\tilde{B}}$ for the transition matrix i.e.

$$v_i = \sum_{j=1}^{n} (M_B^{\tilde{B}})_i j \tilde{v}_j$$

**Exercise:** If $B^\star, \tilde{B}^\star$ are the dual basis, then $M_{B^\star}^{\tilde{B}^\star} = (M_B^{\tilde{B}})^\star$.

**Definition.** *Given $V$ as above, we write $B \sim \tilde{B} \iff \det(M_B^{\tilde{B}}) > 0$.*

*Only for this lecture*

Clearly this is an equivalence relation.

**Lemma 11.1.** *Let $V$ be a finite dimensional vector space. TFAE:*

*(i) an equivalence class $[B]$ of bases for $V$*

*(ii) an equivalence class $[B^\star]$ of bases for $V^\star$*

*(iii) an element of $\Lambda^{top} V^\star \setminus \{0\}$ .*

*We call any one of these equivalent pieces of data an* ***orientation****.*

$\Lambda^{top} V \setminus \{0\} = \Lambda^n V \setminus \{0\}$, *where $n = \dim V$. He also didn't use a star after $V$*

*Proof.* (*i*) $\iff$ (*ii*) is clear.
(ii) $\implies$ (iii). If $B = \{v_1, \dots, v_n\}$ basis of $V^\star$, then $v_1 \wedge \cdots \wedge v_n \in \Lambda^{\text{top}} V^\star \setminus \{0\}$. If $\tilde{B} = \{\tilde{v}_1, \dots, \tilde{v}_n\}$ is any other basis, then $\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_n = \det(M_B^{\tilde{B}}) v_1 \wedge \cdots \wedge v_n$. $\square$

**Definition** (Interior multiplication)**.** *Let $V$ be a finite-dimensional, $R$-vector space. Fix $v \in V, \omega \in \Lambda^k V^\star$. Then we let $\iota_v \omega(\cdot, \cdots) := \omega(v, \cdot, \dots, \cdot) \in \Lambda^{k-1} V^\star$*

**Lemma 11.2** (Induced orientation)**.** *Let $V$ be a finite-dimensional, $\mathbb{R}$-vector space. Let $j : H \hookrightarrow V$ be a $n-1$ dimensional subspace. Fix an orientation $\omega \in \Lambda^n V^\star \setminus \{0\}$. Fix $v \in V$ transverse to $H$.*

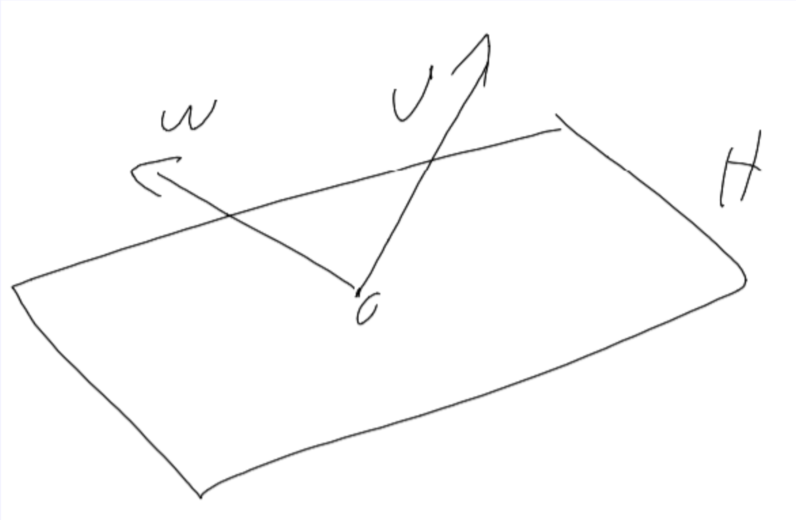


Figure 11.1: Sketch 11.1

*Then:*

*(i)* $\iota_v\omega_{|H} = j^\star(\iota_v\omega) \in \Lambda^{n-1}H^\star$ *is an orientation on* $H$ *it is non-zero!*

*(ii) the orientation from (i) only depends on the connected component of* $V \setminus H$ *in which* $v$ *is contained.*

*Proof.* **(i)** need to check, that $j^\star(\iota_v\omega) \neq 0$. Pick a basis $\{y_1, \ldots, y_{n-1}\}$ for $H$. Then

$$(j^\star(\iota_v\omega))(y_1, \ldots, y_{n-1}) = \omega(v, y_1, \ldots, y_{n-1}) \neq 0$$

since $v, y_1, \ldots, y_{n-1}$ is a basis for $V$.
**(ii)** If $v, w$ are in the same connected component, then $v = \lambda w \sum_{i=1}^n a_i y_i$, where $\lambda > 0, a_1, \ldots, a_{n-1}$. Then

$$\begin{aligned}
\iota_v\omega(y_1, \ldots, y_n) &= \omega(v, y_1, \ldots, y_{n-1}) \\
&= \omega(\lambda w + \sum_{i=1}^{n-1} a_i y_i, y_1, \ldots, y_{n-1}) \\
&= \omega(\lambda w, y_1, \ldots, y_{n-1}) + \underbrace{\omega(\sum_{i=1}^{n-1} a_i y_i, y_1, \ldots, y_{n-1})}_{=0} \\
&= \lambda\omega(w, y_1, \ldots, y_{n-1}) \\
&= \lambda\iota_w(y_1, \ldots, y_{n-1}) \\
\implies \iota_v\omega &= \lambda\iota_w\omega
\end{aligned}$$

□

## 11.2 Orientations of smooth manifolds

**Definition.** *Let* $M$ *be a smooth manifold. An* ***orientation*** *is a smooth section* $\sigma \in \Gamma^1(\Lambda^{top}T^\star M) = \Omega^{top}(M)$*, which is non-vanishing, modulo the equivalence relation* $\sigma \sim \sigma' \iff \exists f : M \to \mathbb{R}_+, \sigma' = f\sigma$.
*We call the data of a manifold + an orientation an* ***oriented manifold****.*

**Remark** (for topology enthusiast)**.** *The following definitions are equivalent to the one above:* *non examable, proved by approximation* *(i)* *a continuous $\sigma$ of $\Lambda^{top}T^\star M$, which is non-vanishing, modulo $\sigma \sim \sigma' \iff \exists f : M \to \mathbb{R}_+$, continuous, $\sigma' = f\sigma$.*

*(ii)* *A section of the fiber bundle $(\Lambda^{top}T^\star M \setminus 0_M)/\mathbb{R}_+ \xrightarrow{\pi} M \ni p\ \pi^{-1}(p) = (\Lambda^{top}T_p^\star M \setminus 0_M)/\mathbb{R}_+$*

**Example.** $\mathbb{R}^n, \omega_0 := dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(\mathbb{R}^n)$. *We call $\omega_0$ the* ***canonical*** *orientation.*

**Example** (Non-example / Warning)**.** *Not all manifolds admit an orientation! E.g. $\mathbb{RP}^{2n}$ are non-orientable, similarly the Möbius band.*

If we have a manifold, the $TM$ is also a manifold and it is always orientable!

**Lemma 11.3.** *Let $M$ be a manifold and let $j : H \hookrightarrow M$ be a codimension 1 submanifold. Fix an orientation $\omega$ on $M$. Let $V$ be a vector field along $H$, which is transverse to $H$, i.e. a section $V$ $(\underbrace{j^\star TM}_{\equiv TM_{|H}} \to H)$ s.t. $\sigma_p \perp T_pH, T_pH \subset T_pM$. Then*

*(i)* *$j^\star \iota_V \omega \in \Omega^{top}$ is an orientation*

*(ii)* *if $W$ is a vector field along $H$, transverse to $H$, such that $V_p, W_p$ lie in the same connected component of $T_pM \setminus T_pH$ for all $p \in H$. Then $j^\star(\iota_V\omega) = j^\star(\iota_W\omega)$.*

*We call this orientation the* ***induced orientation*** *(depends on $H \subset M, \omega, V$).*

*Proof.* Immediate corollary of lemma 11.2. □

**Definition.** *Let $M$ be a manifold with boundary. A vector $v \in T_pM, p \in \partial M \subset M$ is said to be* ***inward-pointing*** *if there exists a curve $\gamma : [0, \epsilon) \to M, \dot\gamma(0) = v \notin T_p\partial M$. We say $w \in T_pM$ is* ***outward pointing*** *if $-w$ is inward-pointing.*

**Observe:** Any positive linear combination of inward-pointing vectors is inward-pointing. If we have $a_1, \ldots, a_n \geq 0, \sum a_i = 1, v_1, \ldots, v_n$ inward-pointing at $p$, then so is $\sum_{a_i} v_i$.

**Lemma 11.4.** *Let $M$ be a manifold with non-empty boundary.*

*(i)* *There exists an inward pointing vector field (also outward-pointing vector fields)*

*(ii)* *If $\omega \in \Lambda^{top}(M)$ orientation on $M$, $Z$ any outward-pointing vector field, then $j^\star(\iota_Z\omega) \in \Omega^{top}$, $j : \partial M \hookrightarrow M$, is an orientation.*

*We call $j^\star(\iota_Z\omega) \in \Omega^{top}(\partial M)$ the* ***induced / Stokes orientation*** *on $\partial M$. It does not depend on the choice of outward pointing vector field.*

*Proof.* **(ii):** Immediate consequence of lemma 11.3.
**(i):** We seek $Z$ a section of $(\underbrace{j^\star TM}_{=\partial M \times_M TM} \to \partial M)$ s.t. $\forall p \in \partial M, Z_p \in T_p\partial M \subset T_pM$, $Z_p$ lies in the **canonical** component of $T_pM \setminus T_p\partial M$. Choose a covering of $\partial M$ by charts $(U_\alpha, \varphi_\alpha), \varphi_\alpha : U_\alpha \to \mathbb{H}^n$. Choose a subordinate partition of unity $\{\eta_\alpha\}_\alpha$.
Let $Z_0 \in \Gamma(\mathbb{H}^n), Z_0 = \partial_{x_n}$

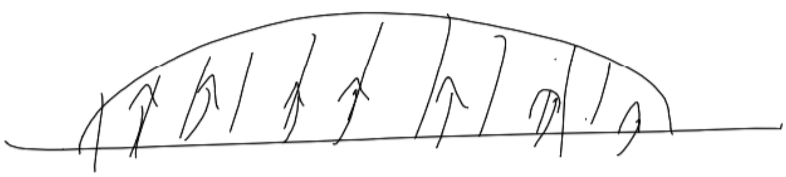

Figure 11.2: Sketch 11.2

Let $Z = \sum_\alpha \eta_\alpha(d\varphi_\alpha^{-1}(Z_0))$ this works because of the previous observation about positive combinations of inward pointing vectors. □

**Example.** *Let $B^{n+1}(1) \subset \mathbb{R}^{n+1}, \partial B^{n+1}(1) = S^n$. We have $Z = x_1\partial_{x_1} + \cdots + x_n\partial_{x_n}$. Clearly $\forall p \in S^n, Z_p \perp T_pS^n$*

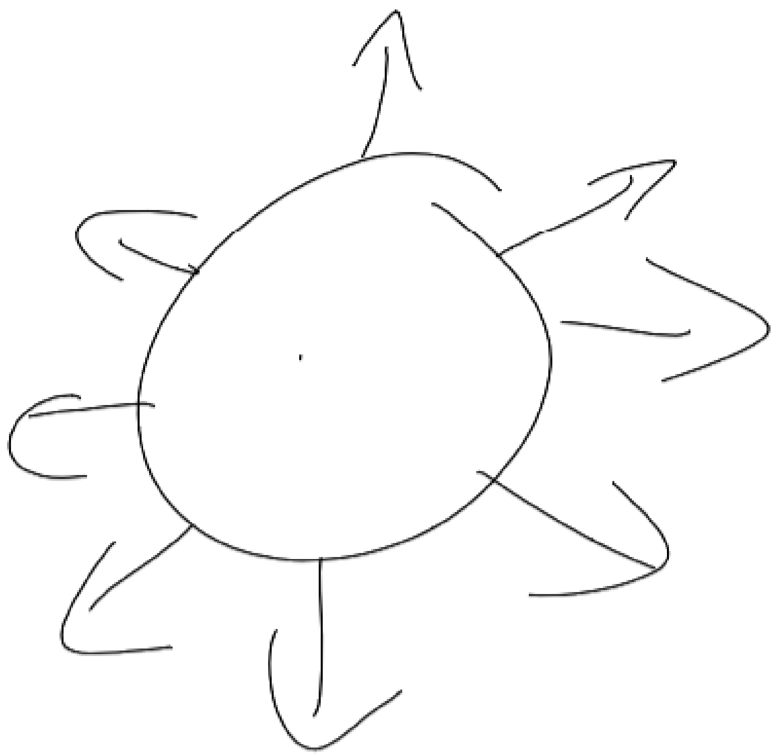

Figure 11.3: Sketch 11.3

$\implies$ *By lemma 11.4, $i_z\omega_0$ on $S^n$.*

**Exam announcement:**

Start of lecture 26 (21.01.2025)

- 1 Problem is drawn from homework (maybe slightly simplified)
- both exams are indented to be of similar difficulty
- Everything up to this week is relevant
- Prepare for the exam by doing old exams questions
- The exam consists of problems, not just repeating proves of the class, however the proof ideas might be useful!
- 2 hours
- Please look at the course webpage!

**Definition.** *Let $M$ be an oriented manifold. We say that a chart $(U, \varphi)$ is positively/ negatively oriented if, w.r.t to local coordinates $x_1, \ldots, x_n$, $\varphi^\star(dx_1, \ldots, dx_n) \in \Omega^{top}(\Omega)(U)$ agrees / disagrees with the orientation of $M$ (restricted to $U$).*

In this setting, any chart is either positively or negatively oriented! Let $r : \mathbb{R}^n \to \mathbb{R}^n, (x_1, \ldots, x_n) \mapsto (-x_1, \ldots, x_n)$. Then observe that $(U, \varphi)$ $\pm$ oriented iff $(U, r \circ \varphi)$ is $\mp$ oriented.
Recall a Riemannian manifold $(M, g)$ is a manifold $M$ + a section $g$ $\Gamma(T^{\otimes 2}T^\star M)$ such that $(T_pM, g_p)$ is an inner product space.
**Exercise:** Let $(M, g)$ be a Riemannian manifold. Given $p \in M$, there exists a local orthonormal frame at $p$ (i.e. $U \ni p, e_1, \ldots e_n \in \Gamma(TM_{|U}), g_q(e_i, e_j) = \delta_{ij} \forall q \in U$).

he would like us to do this exercise, it is insightful and important, but belongs to section9 *proof idea.* Apply the Gram-Schmitt process, all the operations are smooth. Take a local frame through this process and show it works. □

Might very well be important for the exam

For reasons which will become apparent later today, a nowhere-zero top-form $\omega \in \Omega^{\text{top}}(M)$ defines an orientation and is often called a **volume form**.

**Lemma 11.5.** *Let $(M,g)$ be a orientable Riemannian manifold. Then there exists a unique top form $\omega_g \in \Omega^{top}(M)$, with the property that $(\omega_g)_p(e_1,\dots,e_n)$, where $\{e_1,\dots,e_n\} = 1$ is an orthonormal basis at $p$.*

*Proof sketch.* Fix $p \in M$. Let $(\epsilon^1,\dots,\epsilon^n)$ be a local, orthonormal, co-frame. Define $\omega_g := \epsilon^1 \wedge \dots \wedge \epsilon^n$ near $p$. Suppose that $\tilde{\epsilon}^1,\dots,\tilde{\epsilon}^n$ is another orthonormal co-frame. Let $M_B^{\tilde{B}}$ be the transition matrix. We have $\epsilon^1 \wedge \dots \wedge \epsilon^n = \det(M_B^{\tilde{B}})\tilde{\epsilon}^1 \wedge \dots \wedge \tilde{\epsilon}^n$. Since $\{\epsilon\},\{\tilde{\epsilon}\}, M_B^{\tilde{B}} \in O(TM,g)$, i.e. $M_B^{\tilde{B}}(q) \in (T_qM, g_q)$. Therefore $\det(M_B^{\tilde{B}}) = \pm 1$. Choosing the correct orientation concludes the proof. □

## 11.3 Integration on manifolds

In Analysis 1-3, you developed the notion of the integral of function $f : \mathbb{R}^n \to \mathbb{R}$.

$$\int_{\mathbb{R}^n} f dx_1, \dots, f x_m$$

supp$(f)$ compact, $f$ measurable.
**Careful, this is not invariant under diffeomorphism!** This notion makes no sense in our category.

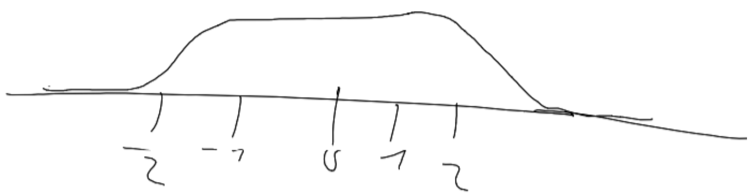


Figure 11.4: Sketch 11.4

**Example.** $\int f dx \approx 2$. *Consider $\phi : \mathbb{R} \to \mathbb{R}, x \mapsto x/100$*

$$\int_{\mathbb{R}} \phi^\star f = \int_{\mathbb{R}} (f \circ \phi) \approx 200.$$

**Solution:** Instead of integrating functions, we should be integrating top forms.
**Key lemma**: 10.10 (d) Recall
$\phi M \to N, \varphi^\star(f dy_1 \wedge \dots \wedge dy_n)(d\phi)dx_1,\dots,dx_n$. The change of variable is backed in the pullback of top forms.

**Definition.** *Let $U \subset \mathbb{R}^n/\mathbb{H}^n$. Let $\sigma = f(x_1,\dots,x_n)dx_1, \wedge \dots \wedge x_n \in \Omega^{top}(U)$ be a* ***compactly supported*** *top form. We define:*

$$\int_U sigma := \int_U f(x_1,\dots,x_n)dx_1,\dots,dx_n,$$

*where the right side is the usual integral from analysis 2/3.*

**Lemma 11.6.** *Let $U, V \subset \mathbb{R}^n/\mathbb{H}^n$. Let $\varphi : U \to V$ be a diffeomorphism. Given $\eta$ compactly supported, we have*

$$\int_U \varphi^\star \eta = \begin{cases} \int_V \eta & \text{if } \sigma \text{ orientated positively} \\ -\int_V \eta & \text{if } \sigma \text{ orientated negatively} \end{cases}$$

*Proof.*

$$\begin{aligned}
\int_U \varphi^\star \eta &= \int_U \varphi^\star (f d_y 1 \wedge \cdots \wedge dy_n) \\
&\overset{\text{Lemma 10.10 (d)}}{=} \int_U (f \circ \varphi)(\det d\varphi) dx_1 \wedge \ldots dx_n \\
&= \int_U (f \circ \varphi)(\det d\varphi) dx_1 \ldots dx_n \\
&= (\pm) \int_U (f \circ \varphi) |\det d\varphi| dx_1 \ldots dx_n \\
&= (\pm) \int_V f dy_1, \ldots, dy_n
\end{aligned}$$

by the change of variable theorem and

$$\begin{aligned}
(\pm) \int_V f dy_1, \ldots, dy_n &= \int_V f dy_1 \wedge \cdots \wedge dy_n \\
&= \int_V \eta
\end{aligned}$$

□

**Lemma 11.7** (Provisional definition, only for this lecture)**.** *Let $M$ be a smooth manifold, possibly with boundary Let $\omega \in \Omega^{top}(M)$. Suppose that:*

*(i) $supp(\omega)$ is compact*

*(ii) there exists a chart $(U, \varphi)$ such that $supp(\omega) \subset U$.*

*Then $\int_M \omega := \pm \int_{\varphi(U)} (\varphi^{-1})^\star \omega$ where we choose $\pm$ depending on wether $(U, \varphi)$ is positively or negative oriented. This expression does not depend on the choice of chart $(U, \varphi)$.*

*Proof.* Let $(V, \psi)$ be another such chart. For simplicity, assume that both are positively oriented. Then we have:

$$\begin{aligned}
\int_{\psi(V)} (\psi^{-1})^\star \omega &= \int_{\psi(V \cap V)} (\psi^{-1})^\star \omega \\
&= \int_{\varphi(U \cap V)} (\psi \circ \phi^{-1})^\star (\psi^{-1})^\star \omega \\
&= \int_{\varphi(U \cap V)} (\varphi^{-1})^\star \psi^\star (\psi^{-1})^\star \omega \\
&= \int_{\varphi(U)} (\varphi^{-1})^\star \omega
\end{aligned}$$

□

**Lemma 11.8.** *Let $M$ be an orientated, smooth $n$ manifold, possibly with boundary. Let $\omega \in \Omega^n(M)$. Suppose that:*

*(i) $supp\omega$ is compact*

*(ii) $\{U_i, \varphi_i\}_{i=1}^N$ such that $supp\omega \subset \bigcup_{i=1}^N U_i$*

*Unproblematic, since $supp\omega$ is compact*

*(iii) $\{\eta_i\}_{i=1}^N$ be a partition of unity subordinate to $\{U_i\}_{i=1}^N$.*

*We define*

$$\int_M \omega := \sum_{i=1}^N \int_M \eta_i \omega = \sum_{i=1}^N \int_{\varphi_i(U_i)} (\varphi^{-1})^\star \eta_i \omega$$

*This definition does not depend on the choice of $\{(U_i, \varphi_i)\}_i, \{\eta_i\}$*

*Please remember this for for the exam*

*Proof.* Let $\{(\tilde{U}_j, \tilde{\varphi}_j)\}_{j=1}^{\tilde{N}}, \{\tilde{\eta}_j\}_{j=1}^{\tilde{N}}$ be another such choice. Then observe that

$$\begin{aligned}\int_M \eta_i \omega &= \int_M \left(\sum_{j=1}^{\tilde{N}\tilde{\eta}_i}\right) \eta_i \omega \\ &= \sum_{j=1}^{\tilde{N}} \int_M (\tilde{\eta}_j \eta_i)\omega.\end{aligned}$$

Similarly

$$\int_m \tilde{\eta}_j \omega = \sum_{i=1}^{N} \int_M (\tilde{(\eta)}_j \eta_i)\omega.$$

Now

$$\begin{aligned}\int_M \omega &= \sum_{i=1}^{N} \int_M \eta_i \omega = \sum_{i=1}^{N} \sum_{j=1}^{\tilde{N}} \int_M (\tilde{\eta}_j \eta_i)\omega \\ &= \sum_{j=1}^{\tilde{N}} \sum_{i=1}^{N} \int_M (\tilde{\eta}_j \eta_i)\omega = \sum_{j=1}^{\tilde{N}} \int_M \tilde{\eta}_j \omega =: \int_M \omega\end{aligned}$$

□

**Remark.**
- *All of the familiar properties of the integral in $\mathbb{R}^n$ remain true, but are left as an exercise. E.g.*

$$\int_M a\omega + b\eta = \int_M a\omega + \int_M b\eta$$

- *You can do measure theory on manifold, but this is not what we have done today. Integration of smooth functions in $\mathbb{R}^n$ can be generalized in two ways*
  - *integration on manifolds*
  - *integrations of measurable functions in $\mathbb{R}^n$*

Klausureinsicht[1] for exam number one: SR0.011 at 12:30-13:30 on thursday 13.02.25. Last lecture he said it exam will be problem based, but he might ask us to prove a step of some proof of the lecture.

Start of lecture 27 (24.01.2025)

**Remark** (Integration on zero-dimensional manifolds)**.**

*This is important (in practice)*

- *By convention, if $V$ is a 0-dim. vector space, an orientation $o_V$ on $V$ is an element $o_V \in \{\pm 1\}$*
- *Hence if $M$ is a 0-dim manifold, an orientation is the assignment of $\pm 1$ to each element of $M$ ($\sigma \in \Omega^0(M) = C^\infty(M)$, nowhere zero)*
- *If $M$ is a zero-dimensional oriented manifold, $\eta \in \Omega^0(M) = C^\infty(M)$, compactly supported. Then*

$$\int_M \eta = \sum_{p \in M} (\pm)\eta(p)$$

*where $\pm$ depends on wether $p$ is oriented $\pm$.*

## 11.4 Stokes' theorem

**Theorem 11.9** (Stokes's theorem)**.** *Let $M$ be an oriented $n$-manifold with boundary. Let $\omega \in$*

[1] Revision / discussion of the exam grades

$\Omega^{n-1}(M)$ *be a compactly supported* $n-1$ *form. Then*

$$\int_M \underbrace{d\omega}_{\in\Omega^n(M)} = \int_{\partial M} \omega.$$

**Remark.**

- *$\partial M = \emptyset$ is allowed*
- *it is understood that $\partial M$ carries the induced boundary orientation from section 11.1 (outward pointing, stokes orientation)*
- *the expression $\int_{\partial M} \omega$ secretly means*

$$\int_{\partial M} j^\star \omega, j : \partial M \hookrightarrow M$$

**Lemma 11.10.** *Endow $\partial\mathbb{H}^n$ with the induced orientation. The map $\mathbb{R}^{n-1} \to \partial\mathbb{H}^n$, which sends*

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$$

*is orientation preserving if $n$ is even and orientation reversing if $n$ is odd.*

*Proof.* By definition, orientation on $\partial\mathbb{H}^n$
$i_{-\partial_{x_n}} w_0 = i_{(-\partial_{x_n})} dx_1 \wedge \dots \wedge dx_n = (-1)^n i_{\partial_{x_n}} dx_1 \wedge \dots \wedge dx_{n-1}$.

$$j^\star((-1)^n dx_1 \wedge \dots \wedge dx_{n-1}) = (-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$$ □

Proof of Stokes theorem, see [2] theorem 16.11.

*Proof when $M = \mathbb{H}^n$.* Since $\omega$ is compactly supported, $\mathrm{supp}\omega \subset \underbrace{[-R,R] \times \dots \times [-R,R]}_{n-1 \text{ times}} \times [0,R]$

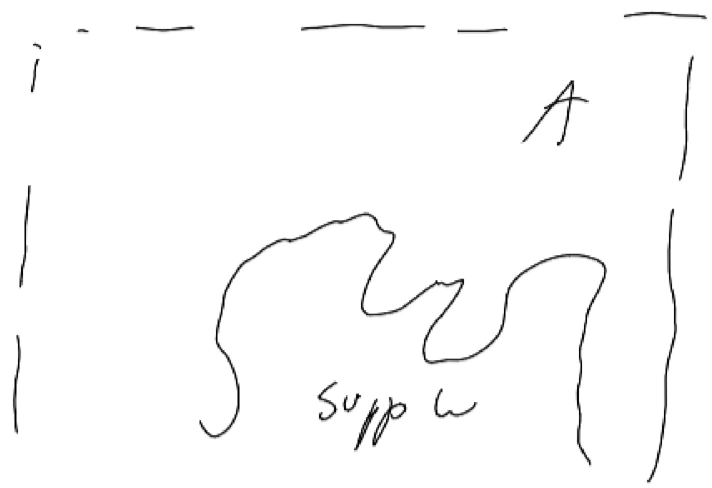


Figure 11.5: Sketch 11.5

We can write $\omega = \sum_{i=1}^n \omega_i dx_i \wedge \dots \wedge \hat{dx_i} \wedge \dots \wedge dx_n$. We can now compute

$\hat{dx_i}$ means we are omitting this term

$$\begin{aligned} d\omega &= \sum_{i=1}^n \sum_{j=1}^n \partial_{x_j} w_i dx_j \wedge dx_1 \cdots \wedge dx_i \cdots \wedge dx_n \\ &= \sum_{i=1}^n (-1)^{i-1} \partial_{x_i} w_i dx_1 \wedge \dots \wedge dx_n \end{aligned}$$

By definition

$$\begin{aligned}\int_{\mathbb{H}^n} d\omega = \int_A d\omega = \int_A \sum_{i=1}^{n} (-1)^{i-1} \partial_{x_i} w_i dx_1 \wedge \cdots \wedge dx_n \\ = \sum_{i=1}^{n} (-1)^{i-1} \int_A \partial_{x_i} w_i dx_1 \ldots dx_n = \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \partial_{x_i} w_i dx_1 \ldots dx_n = (\star)\end{aligned}$$

**Observe:**

$$\int_{\mathbb{R}}^R \partial_{x_i} \omega_i dx_i = \omega_i \mid_{-R}^R = 0$$

$$\begin{aligned}(\star) &= (-1)^{n-1} \int_0^R \cdots \int_{-R}^R \partial_{x_n} \omega_n dx_1 \ldots dx_n \\ &= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(x_1, \ldots, x_{n-1}, 0) dx_1 \ldots dx_n\end{aligned}$$

Meanwhile

$$\int_{\partial \mathbb{H}^n} \omega = \int_{\partial \mathbb{H}^n \cap A} \sum_{i=1}^{n} \omega_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$$

Note that, since $x_n \equiv 0\partial$ on $\mathbb{H}^n$, $dx_n \equiv 0$ on $\partial\mathbb{H}^n$

$$\implies \int_{\partial \mathbb{H}^n} \omega = \int_{A \cap \partial \mathbb{H}^n} \omega_n dx_1 \wedge \cdots \wedge dx_{n-1}$$

But now, by lemma 11.10, we have

$$\begin{aligned}\int_{A \cap \partial \mathbb{H}^n} \omega_n dx_1 \wedge \cdots \wedge dx_{n-1} &= (-1)^n \int_{[-R,R]^{n-1}} \omega_n dx_1 \wedge \cdots \wedge dx_{n-1} \\ &= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(x_1, \ldots, x_{n-1}, 0) dx_1 \ldots dx_{n-1}\end{aligned}$$

□

*Proof when $M = \mathbb{R}^n$.* As before $\omega = \sum_{i=1}^n \omega_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n$, $d\omega = \sum_{i=1}^n (-1)^{i-1} \partial_{x_i} \omega_i dx_1 \wedge \cdots \wedge dx_n$. Want:

$$\int_{\mathbb{R}^n} d\omega = \int_{\partial \mathbb{R}^n} = 0$$

$$\begin{aligned}\int_{\mathbb{R}^n} d\omega &= \int_{\mathbb{R}^n} \sum_{i=1}^{n} (-1)^{i-1} \partial_{x_i} \omega_i dx_1 \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_{-R}^R \partial_{x_i} \omega_i dx_1 \ldots dx_n = 0\end{aligned}$$

by the fundamental theorem of calculus. □

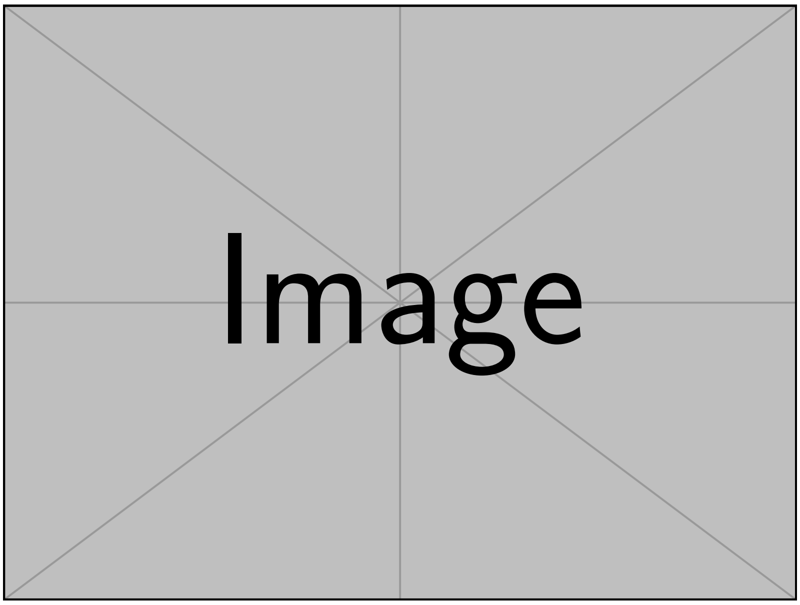


Figure 11.6: Sketch 11.6

*Proof for $M$ arbitrary, under the extra assumption that there exists a chart $(U, \varphi)$, s.t. supp$\omega \subset U$.* For simplicity assume that $(U, \varphi)$ is positively oriented. Then

$$\begin{aligned}
\int_M d\omega &:= \int_{\varphi(M)} (\varphi^{-1})^\star (d\omega) \\
&= \int_{\varphi(M)} d((\varphi^{-1})^\star \omega) \\
&= \int_{bH^n} d((\varphi^{-1})^\star \omega) \\
&= \int_{\partial \mathbb{H}^n} (\varphi^{-1})^\star \omega =: \int_{\partial M} \omega
\end{aligned}$$

□

*Proof of theorem 11.9 (General Case).* Let $\omega \in \Omega^{n-1}(M)$ with compact support. Let $(U_i, \varphi_i)_{i=1}^N$ be a finite set of charts, whose union covers the support of $\omega$. Let $\{\eta_i\}_{i=1}^N$ be a partition of unity subordinate to the cover. Then

$$\begin{aligned}
\int_{\partial M} \omega = \sum_{i=1}^N \int_{\partial M} (\underbrace{\eta_i \omega}_{\mathrm{supp}(\eta_i\omega) \subset U_i}) \\
&= \sum_{i=1}^N \int_M d(\eta_i \omega) \qquad = \sum_{i=1}^N \int (d\eta_i \wedge \omega + \eta_i d\omega) \\
&= \int_M sum_{i=1}^N d\eta_i \wedge \omega + \int_M (\sum_{i=1}^N \eta_i) d\omega \\
&= \int_M \underbrace{d(\sum_{i=1}^N \eta_i)}_{=0} \wedge \omega + \int_M d\omega \\
&= \int_M d\omega
\end{aligned}$$

□

Everything after now is not examable. Next Friday will be about next semester and the future :)

# Journal

- Lecture 01: Covering: Introduction, locally Euclidean, Hausdorff, second countable spaces, their covers and exhaustions by compact sets .
  Starting in 'Organization' on page 3 and ending in 'Basis and covers' on page 9. Spanning 6 pages
- Lecture 02: Covering: Local finiteness, refinements, paracompactness, introduction to topological manifolds and examples .
  Starting in 'Basis and covers' on page 9 and ending in 'Manifolds with boundary' on page 14. Spanning 5 pages
- Lecture 03: Covering: Topological properties of topological manifolds, classification of topological manifolds, introduction to smooth manifolds .
  Starting in 'Manifolds with boundary' on page 14 and ending in 'Charts and atlases' on page 18. Spanning 4 pages
- Lecture 04: Covering: Examples of smooth manifolds, smooth maps, the category of smooth manifolds, hierarchy of categories of manifolds .
  Starting in 'Charts and atlases' on page 18 and ending in 'The category of smooth manifolds' on page 21. Spanning 3 pages
- Lecture 05: Covering: Smooth manifolds with boundary, partitions of unity .
  Starting in 'The category of smooth manifolds' on page 21 and ending in 'Partitions of unity' on page 24. Spanning 3 pages
- Lecture 06: Covering: Applications of partitions of unity, motivation of tangent vectors, definition of tangent vectors via equivalence classes of smooth curves, definition of differentials, fundamentality of the differential .
  Starting in 'Partitions of unity' on page 24 and ending in 'Definition via equivalence classes of smooth curves' on page 29. Spanning 5 pages
- Lecture 07: Covering: Definition of tangent vectors via derivations, equivalence of both definitions, coordinates .
  Starting in 'Definition via equivalence classes of smooth curves' on page 29 and ending in 'Coordinates' on page 33. Spanning 4 pages
- Lecture 08: Covering: Coordinates (continued), tangent bundles, submersions, immersions and embeddings .
  Starting in 'Coordinates' on page 33 and ending in 'Basic definitions' on page 39. Spanning 6 pages
- Lecture 09: Covering: The rank theorem as a generalization of a linear algebra fact and it's proof, basic definitions of submanifolds .
  Starting in 'Basic definitions' on page 39 and ending in 'Slice lemma[2]' on page 44. Spanning 5 pages

[2]Lee [2] calls it a theorem

[3] Lee [2] calls it a theorem

# Bibliography

[1] David Gale. “The Classification of 1-Manifolds: A Take-Home Exam”. In: (). URL: `https://www.math.uni-bonn.de/~lcote/1_man_classification.pdf`.

[2] John M. Lee. *Smooth Manifolds*. New York, NY: Springer New York, 2012. ISBN: 978-1-4419-9982-5. DOI: `10.1007/978-1-4419-9982-5`. URL: `https://doi.org/10.1007/978-1-4419-9982-5`.

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