Foundations of Stochastic Analysis

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Start of lecture 1 (10.10.2023)

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- Information on ecampus
- Exercise groups
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    * Name(s) of exercise partner students
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- Exercise sheets on ecampus, by Friday
- Oral exam **first** oder third week after lecture period
- Most content done by christmas! -¿ learn for the exam in the break

Revision: Stochastic processes

- First part: mostly repetition in continuous time.
- Important: conditional expectation, short review on friday

1 Introduction

1.1 Motivation

Brownian motion: Limit of random walk:
Let $X_1, X_2, \dots$ iid. random variable with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$ and

$$S_{t,n} = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor tn \rfloor} X_k$$

By CLT $S_{t,n} \overset{\to\infty}{\to}$ Gaussian random variable. Brownian motion: $B_t = \lim_{n\to\infty} S_{t,n}$
example[Population growth] Let S_t be the size of a population at time t, let R be the average growth rate.

$$\implies \frac{dS_t}{dt} = RS_t \implies S_t = S_0 e^{Rt}$$

There are fluctuations

$$\implies \frac{dS_t}{dt} = (R + \underbrace{N_t}_{\text{noise term}})S_t$$

example
example[Langvin equation] Describes the evolution of a small particle in a fluid:

$$m\frac{dV_t}{dt} = -\eta V_t + \underbrace{N_t}_{\text{noise term}}$$

Where m is the mass of the particle, V_t is the speed at time t and η is the viscosity coefficient. example
example[Dirichlet problems] Let f be a harmonic function on a domain $D \subset \mathbb{R}^d$, i.e. $\Delta f = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} f = 0$.
If f is kown on ∂D:

$$\implies x \in D \setminus \partial D : f(x) = \mathbb{E}(f(B_\tau^x))$$

where B_t^x is Brownian motion starting at x, $\tau = \inf\{t > 0 | B_t^x \in \partial D\}$.
example
What should N_t be ?
1. Trial: Take N_t to be a fully random function in continuous time, i.e., analog of a sequence of random variables.
$\implies$ Ask for

1. N_t is independent of N_s for $t \neq t$
2. N_t has a distribution independent of t
3. $\mathbb{E}(N_t) = 0$

Problem:
N_t is not measurable, expect when $N_t = 0$ (why?).
Let ν be the distribution of N_t.
$\implies \exists a \in \mathbb{R} : p = \mathbb{P}(N_t \leq a) \in (0,1)$. Let $E = \{t \geq 0 | N_t \leq a\}$. Therefore E is not Lebesgue-measurable: If E where measurable $\implies \forall c < d : \text{leb}(E \cap (c,d)) = p \cdot (d-c)$ But if measurable: $\forall \alpha < 1 \exists (c,d)$ s.t. $\text{leb}(E \cap (c,d)) > \alpha \cdot (d-c) \implies$ contradiction.
In the examples, we were interested not directly to N_t, but to integrals:

$$\frac{dS_t}{dt} = (R + N_t)S_t \to \int_0^u \frac{dS_t}{dt} = S_u - S_0 = \int_0^u RS_t dt + \int_0^u N_t S_t dt$$

2. Trial: Let $B_t \coloneqq \int_0^t N_s ds$.
$\implies$ Ask for :

(BM1) For $0 = t_0 < t_1 <, \dots, t_n$, the random variables

$$B_{t_{j+1}} - B_{t_j} j = 0, \dots, n-1$$

are **independent**

(BM2) B_t has stationary increments:

$$B_{t_1+s} - B_{u_1+s}, \dots, B_{t_n+s} - B_{u_n+s}$$

is independent of $s \geq 0$ for all $u_i < t_i$ for $i = 1, \dots, n$

(BM3) $\mathbb{E}(B_t) = 0$.

Add: **normalization**:

(BM4) $\mathbb{E}(B_1^2) = 1$

Add: **continuity**

(BM5) $t \mapsto B_t$ is (almost surely) continuous.

We will see that a process with (BM1) - (BM5) exists (Wiener process / Brownian Motion).

Lemma 1.1. *Property (BM5) implies* $\forall \epsilon > 0$*:*

$$\lim_{n\to\infty} \mathbb{P}\left(\left|B_{t+\frac{1}{n}} - B_t\right| > \epsilon\right) = 0$$

Proof. Implicit assumption: $B_0 = 0$ (otherwise we have to carry the B_0 term) Let $H_t \coloneqq \sup_{1\leq k\leq n}\left|B_{\frac{k}{n}} - \frac{k-1}{n}\right|$

$$\implies \forall \epsilon > 0 : \mathbb{P}(H_n) > \epsilon \overset{\to\infty}{\to} 0$$

But $\mathbb{P}(H_n > \epsilon) = 1 - \mathbb{P}(H_n \leq \epsilon) = 1 - \mathbb{P}(\left|B_{\frac{1}{n}-B_0}\right|, \dots, \left|B_{\frac{n}{n}-B_{\frac{n-1}{n}}}\right| \leq \epsilon)$

$$\overset{\text{BM1}}{=} 1 - \prod_{k=0}^{n} \mathbb{P}(\left|B_{\frac{k}{n}} - B_{\frac{k-1}{n}}\right| \leq \epsilon) \overset{\text{BM2}}{=} 1 - \mathbb{P}(\left|B_{\frac{1}{n}} - B_{\frac{0}{n}}\right| \leq \epsilon)^n$$

$$= 1 - (\underbrace{1 - \mathbb{P}(\left|B_{\frac{1}{n}}\right|) > 0}_{\leq \exp(-n\mathbb{P}(\left|B_{\frac{1}{n}}\right|>\epsilon))}) \overset{1-x\leq e^{-x}}{=} 1 - \exp(-n\mathbb{P}(\left|B_{\frac{1}{n}}\right| > \epsilon))$$

$$\implies n \to \infty, n\mathbb{P}(\left|B_{\frac{1}{n}-B_0}\right| > \epsilon) \overset{n\to\infty}{\to} 0$$

$\square$

Lemma 1.2. $\forall s < t : B_t - B_s \sim \mathcal{N}(0, t-s)$, *i.e.:*

$$\mathbb{P}(\mathbb{B}_{\approx} - \mathbb{B}_{\sim}) \leq \frown = \frac{1}{\sqrt[2]{2\pi/(t-s)}} \int_{-\infty}^{\infty} dy e^{-\frac{y}{2/(t-s)}}$$

Proof. Take $s = 0$, w.l.o.g..
Let $B_t = \sum_{k=0}^{n} X_{n,k}, X_{n,k} = B_{\frac{kt}{n}} - B_{\frac{(k-1)t}{n}}$ are iid. By (BM3) $\implies \mathbb{E}(X_{n,k}) = 0$ and $\mathbb{E}(B_{\frac{t}{n}}) = 0$.
Assume: $\mathbb{E}(B_1^2) = 1 \implies \operatorname{Var}(B_1^2) \underbrace{=}_{t=1} = \sum_{k=1}^{n} \operatorname{Var}(X_{n,k}) \implies \operatorname{Var}(X_{n,k}) = \frac{t}{n} \implies \operatorname{Var}(B_t) = t.$

CLT finishes the prove.

$\square$

Let ($\tilde{\text{BM}}$2): For $s, t \geq 0$:

$$\mathbb{P}(\mathbb{B}_{\approx+\sim} - \mathbb{B}_{\sim} \in \mathbb{A}) = \frac{1}{\sqrt{2\pi t}} \int_A dx e^{-\frac{x^2}{2t}}$$

Definition 1.3. *A one-dimensional Brownian Motion is a real valued process $B_t, t \geq 0$, s.t.: (BM1), ($\tilde{\text{BM}}$2),(BM5).*

Remark. *The standard brownian motion also has $B_0 = 0$.*

Next: Construct a process satisfying this definition and study its properties.

———————————————End of lecture 1 (10.10.2023)———————————————

———————————————Start of lecture 2 (13.10.2023)———————————————

Definition. *A one-dimensional* ***standard*** *Brownian motion is a real-valued stoch. process:*

1. $B_0 = 0$
2. $\forall n \geq 1, 0 = t_0 < t_1 < \cdots < t_n B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent r.v. wth* $B_{t_k} - B_{t_{k-1}} \sim \mathcal{N}(0, t_k - t_{k-1})$
3. *for almost all $\omega \in \Omega$*

$$t \mapsto B_t(\omega)$$

are continuous.

1.2 A review of conditional expectation

1.2.1 Definitions

Definition 1.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob. space and $\mathcal{G} \subset \mathcal{F}$ is sub-σ-algebra. Let $X \in L^1(\mathbb{P})$. The* ***conditional expectation of** X **given** $\mathcal{G}$*, *$\mathbb{E}(X|\mathcal{G})$ is any random variable Y s.t.*

1. *Y is $\mathcal{G}$-measurable*
2. $\forall A \in \mathcal{G}$, $\int_A Y d\mathbb{P} = \int_A X d\mathbb{P}$

Remark. *If $Y, \tilde{Y}$ satisfy 1.4 $\implies Y = \tilde{Y}$ a.s..*

In words: In $\mathcal{G}$ we have partial information, for $A \subset \mathcal{G}$ we know wether it occurs or not.

$$\mathbb{E}[X|\mathcal{G}]$$

is the best guess of X given the information in $\mathcal{G}$.

1.2.2 Examples

Example 1. *Let X be $\mathcal{G}$ measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.*

Example 2. *Let X be independent of $\mathcal{G} \implies \mathbb{E}|\mathcal{G} = \mathbb{E}(X)$ a.s..*

Remark. *$\forall B \in \mathcal{B}(\mathbb{R}), A \in \mathcal{G}, \mathbb{P}(X \in B \cap A) = \mathbb{P}(X \in B)\mathbb{P}(A)$.*
$Y := \mathbb{E}(X) \implies Y$ $\mathcal{G}$ measurable. Let $A \in \mathcal{G}$:

$$\int_A \mathbb{E}(X) d\mathbb{P} = \mathbb{E}(X)\mathbb{P}(A) = \mathbb{E}(X)\mathbb{E}(1_A)$$

$$\overset{X, 1_A \text{ in dependent}}{=} \mathbb{E}(X 1_A) = \int_A X d\mathbb{P}$$

Example 3. *Assume* $\Omega = \bigcup_{k\geq 1} \Omega_k$ *disjoint union with* $\mathbb{P}(\Omega_k) > 0$. *Let* $\mathcal{G} = \sigma(\Omega_1, \ldots, \Omega_n)$

$$\implies \forall k \geq 1 Y := \mathbb{E}(X|\mathcal{G}) = \frac{\mathbb{E}(X 1_{\Omega_k})}{\mathbb{P}(\Omega_k)} \text{ on } \Omega_k$$

If $\omega \in \Omega_k$ *what is the best guess of* $X(\omega)$. Y *is constant on* $\Omega_k \implies Y$ *is measurable. Since* $\mathcal{G}$ *is generated by the* Ω_k *(smallest elements), we have to verify the second property only for* $A = \Omega_k \forall k$.

Remark. *If* $\mathcal{G} = \{\emptyset, \Omega\} \implies$

$$\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X) \text{ a.s.}$$

Proposition 1.5. $X \in L^1(\mathbb{P})$

1. $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$
2. *If* $\mathcal{G}_1 \subset \mathcal{G}_2$ *are two sub-*σ*-algebras,*

$$\implies \mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) = \mathbb{E}(\mathbb{X}|\mathcal{G}_{\nVdash}) \text{ a.s.}$$

$$\implies \mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(\mathbb{X}|\mathcal{G}_{\nVdash}) \text{ a.s.}$$

3. *If* X *is* $\mathcal{G}$*-measurable,* Y *a random variable,* $\mathbb{E}(|Y|) < \infty, \mathbb{E}(|XY|) < \infty$

$$\implies \mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G}) \text{ a.s.}$$

1.2.3 Geometric interpretation

Proposition 1.6. *Let* X *be a r.v. with* $\mathbb{E}(|X|^2) < \infty \implies \mathbb{E}(X|\mathcal{G})$ *is the random variable* Y *which is* $\mathcal{G}$*-measurable and minimizes*

$$\mathbb{E}((X-Y)^2)$$

<u>**Notation**</u>

$$L^2(\mathcal{G}) = \{Y \mathcal{G}\text{-measurable} | \mathbb{E}(|Y|^2) < \infty\}$$

Proof. $L^2(\mathcal{F})$ is Hilbert space and $L^2(\mathcal{G})$ is a closed subspace.
If $Z \in L^2(\mathcal{G}) \implies \mathbb{E}(Z \cdot \mathbb{E}(X|\mathcal{G})) = \mathbb{E}(\mathbb{E}(ZX|\mathcal{G})) = \mathbb{E}(ZX)$

$$\implies \mathbb{E}(Z(X - \mathbb{E}(X|\mathcal{G}))) = 0$$

If $Y \in L^2(\mathcal{G})$ and $Y = \mathbb{E}(X|\mathcal{G}) + Z$

$$\implies \mathbb{E}(|X-Y|^2) = \mathbb{E}((X - \mathbb{E}(X|\mathcal{G}) - Z)^2)$$

$$= \mathbb{E}((X - \mathbb{E}(X|\mathcal{G}))^2) + \mathbb{E}(Z^2) - \underbrace{2\mathbb{E}(\mathbb{Z}(\mathbb{X} - \mathbb{E}(\mathbb{X}|\mathcal{G})))}_{=0}$$

$\implies$ Minimize $\mathbb{E}((X-Y)^2) \iff \mathbb{E}(Z^2) = 0 \implies Z = 0$

□

1.2.4 Random walk

Let $X_1, \ldots, X_n$ iid random variables with $\mathbb{E}(X_i) = 0$; let $S_n := X_1 + \cdots + X_n$. Let $\mathcal{G}_n = \sigma(X_1, \ldots, X_n), n \geq 1$

$$\implies \mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots$$

$$\implies \forall m \leq n : \exists \mathcal{E}(S_n|\mathcal{G}_n) = S_m \text{a.s.}$$

<u>**Ideed:**</u> $Y = S_n$ is $\mathcal{G}_m$-measurable.
Let $A \in \mathcal{G}_m \implies \int_A S_m d\mathcal{P} = \int_A S_n d\mathcal{P}$
But $\int_A S_n d\mathbb{P} = \int_A S_m d\mathbb{P} + \sum_{k=1}^n \underbrace{\int_A X_k d\mathbb{P}}_{\mathbb{E}(X_k 1_A) = \mathbb{E}(X_k)\nVdash_{\mathbb{A}} = 0}$
and $X_k, 1_A$ are independent by assumption.

1.2.5 Conditional densities

Definition 1.7. *Let X, Y be r. v. with $\mathbb{E}(|X|) < \infty$.*
$\implies$ *We define $\mathbb{E}(X|Y) := \mathbb{E}(X|\sigma(Y))$*

- Consider the case where (X, Y) have a density with respect to Lebesgue, i.e.

$$\forall B \in \mathcal{B}(\mathbb{R}^2)\mathbb{P}((X,Y) \in B) = \int_B f(x,y)dxdy$$

Remark. $\mathbb{P}(A|\mathcal{G}) := \mathbb{E}(1_A|\mathcal{G})$.

Assume $\int f(x,y)dx > 0 \forall y$

Proposition 1.8. *Let X be a r.v. and g a function s.t. $\mathbb{E}(|g(X)|) < \infty$.*

$$\implies \mathbb{E}(g(X)|Y) = h(Y)$$

where

$$h(Y) = \frac{\int g(x)f(x,y)dx}{\int f(x,y)dx}$$

Proof. How to derive?

$$\mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x \cap Y = y)}{\mathbb{P}(Y = y)} = \frac{f(x,y)}{\int f(x,y)dx}$$

$$\implies \mathbb{E}(g(X)|Y = y) = \frac{\int g(x)f(x,y)dx}{\int f(x,y)dx} = h(y)$$

$h(Y)$ is $\mathcal{G}$-measurable.
For $A \in \sigma(Y) \implies \exists \tilde{A} \in \mathcal{B}(\mathbb{R})$ s.t. $A = \{\omega : Y(\omega) = \tilde{A}\}$

$$\implies \int_A h(y)dy = \int_{\tilde{A}} dy \underbrace{\int dxh(yf(x,y))}_{=\int dxg(x)f(x,y)}$$

$$\implies \int dxg(x) \int_{\tilde{A}} dyf(x,y) = \mathbb{E}(g(X)1_{\tilde{A}}(Y)) = \mathbb{E}(g(X)1_A)$$

$\square$

End of lecture 2 (13.10.2023)