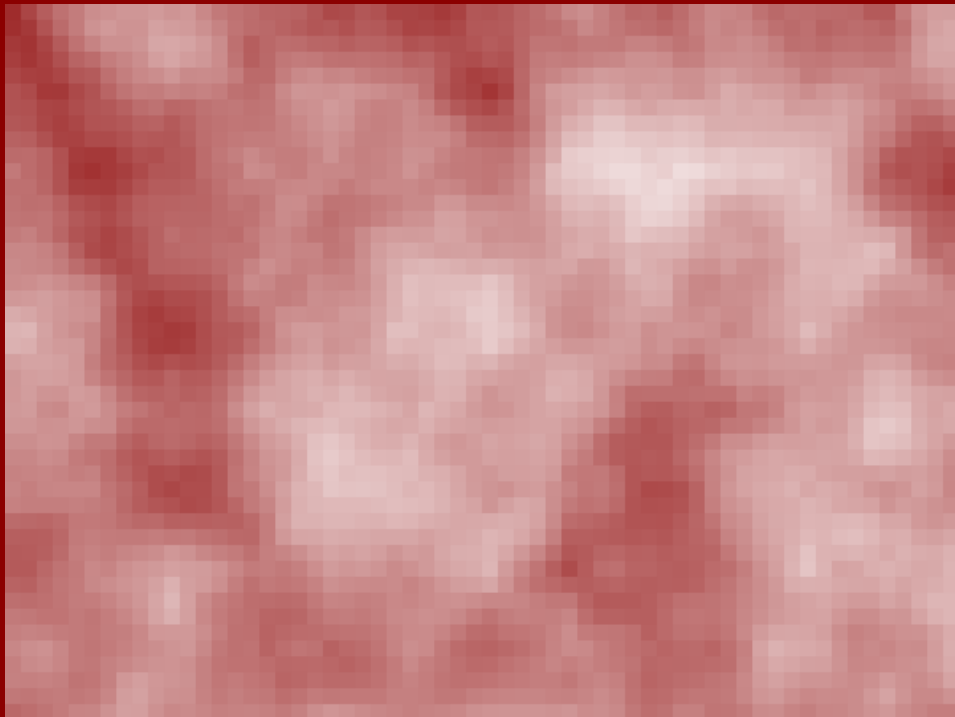

Lecture notes on Markov Processes

Written by
Manuel Hinz

mh@mssh.dev or s6mlhinz@uni-bonn.de

Based on the lectures of
Prof. Dr. Patrik Ferrari

ferrari@uni-bonn.de



Contents

Chapter 0	Manuel's notes	2
Chapter 1	Continuous time Markov chains	3
1.1	Markov chain, transition function, infinitesimal description	3
1.2	Examples	4
1.3	From MC to Q -matrices	4
1.3.1	MC \rightarrow Transition functions	4
1.3.2	Transition functions to Q -matrices	5
1.4	From Q -matrix to the Markov Chains	8
1.4.1	The backwards equation	8
1.4.2	The probabilistic construction	11
List of Lectures		14

Chapter 0:

Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Ecampus: [Ecampus link](#)
- Basis: [Basis link](#)
- Website: None
- Time slot(s): **Tuesday 12-14** and **Thursday 12-14**
- Exams: Oral
- Exercises: To be handed in each **Friday** until **noon**

There are tutorials in the second week (14.10.24 and 16.10.24)!

Start of lecture 01
(08.10.2024)

Chapter 1:

Continuous time Markov chains

Two motivating examples:

1.1 Markov chain, transition function, infinitesimal description

We have a countable state space \mathcal{S} and define

- $\Omega = \{\text{right-continuous functions } \omega : \mathbb{R}_+ \rightarrow \mathcal{S} \text{ with finitely many jumps in any finite time intervals}\}$
- $\sigma\text{-algebra } \mathcal{F} := \sigma(\{\omega \rightarrow \omega(t) \text{ which is measurable } \forall t \geq 0\})$ measurable w.r.t.?
- time shift operator: $(\theta_s \omega)(t) := \omega(s+t)$

Definition 1.1. Denote $X(t, \omega) := \omega(t)$. Assume we have a collection of probability measures $\{\mathbb{P}^x, x \in \mathcal{S}\}$ on (Ω, \mathcal{F}) , a right-continuous filtration $\{\mathcal{F}_t, t \geq 0\}$ on (Ω, \mathcal{F}) . X is a continuous time Markov chain (MC) if

- (a) X is adapted to $\{\mathcal{F}_t, t \geq 0\}$
- (b) Initial condition: $\mathbb{P}^x(X(0) = x) = 1$
- (c) Markov property: $\forall x \in \mathcal{S}, Y$ measurable function on $\Omega, s \geq 0$:

$$\mathbb{E}^x(Y \circ \theta_s \mid \mathcal{F}_s) = \mathbb{E}^{X(s)}(Y), \mathbb{P}^x \text{ a.s.}$$

Example. $Y = \max_{0 \leq t \leq 1} X(t) \rightarrow Y \circ \theta_s = \max_{s \leq t \leq s+1} X(t)$

Here: Time homogenous processes!

Definition 1.2. A transition function is function $p_t(x, y), x, y \in \mathcal{S}, t \geq 0$ s.t.

- (a) Positivity: $p_t(x, y) \geq 0$
- (b) Normalized: $\sum_{y \in \mathcal{S}} p_t(x, y) = 1$
- (c) Continuity: $\lim_{t \downarrow 0} p_t(x, x) = p_0(x, x) = 1$
- (d) Chapman-Kolmogorov equation: $\forall s, t \geq 0 \forall x, y \in \mathcal{S}$:

$$p_{s+t}(x, y) = \sum_{z \in \mathcal{S}} p_s(x, z) p_t(z, y)$$

We will see while constructing, that this part is tricky, might just be ≤ 1

Given a transition function, we can construct \mathbb{P}^x as follows:

Finite dimensional distributions, $0 < t_1 < \dots < t_n$:

$$\mathbb{P}^x(X(t_1) = x_1, X(t_2) = x_2, \dots, X(t_n) = x_n) := p_{t_1}(x, x_1) \cdot p_{t_2 - t_1}(x_1, x_2) \cdot \dots \cdot p_{t_n - t_{n-1}}(x_{n-1}, x_n)$$

extend to full time \mathbb{R}_+ by the Kolmogorov(-Daniell) extension theorem, where the consistency, $\mathbb{P}^x(X(t) = y) = p_t(x, y)$, relations follow by the Chapman-Kolmogorov equation.

Example. $0 < t_2 < \dots < t_n$

$$\sum_{x_1 \in \mathcal{S}} \mathbb{P}^x(X(t_1) = x_1, \dots, X(t_n) = x_n) = \mathbb{P}^x(X(t_2) = x_2, \dots, X(t_n) = x_n)$$

We are still in the same setting, therefore we use the same t_1 as previously

By modeling we often think at the basic biological/ physical properties of the system \rightarrow typically

we have *transition rates*, because for $x \neq y$:
$$\begin{cases} p_\epsilon(x, y) = O(\epsilon), & p_\epsilon(x, x) = 1 - O(\epsilon) \\ p_0(x, y) = 0, & p_0(x, x) = 1 \end{cases}$$

Definition 1.3. For a Markov chain X we define the **transition rates** from x to y ($x \neq y$) by:

$$\tilde{q}(x, y) := \frac{d}{dt} p_t(x, y) |_{t=0}$$

Definition 1.4. A ***Q-matrix (or generator)*** is a collection of numbers $\{q(x, y) : x, y \in \mathcal{S}\}$ s.t.:

- (a) $q(x, y) \geq 0 \forall x \neq y$
- (b) $\sum_{y \in \mathcal{S}} q(x, y) = 0$, Notation: $c(x) := q(x, x) = \sum_{y \in \mathcal{S} \setminus \{x\}} q(x, y)$

$c(x)$ is the rate of leaving site x

Warning

It is not automatic that transition rates gives a Q -matrix For instance: $\tilde{q}(x, y) = \infty$

For **finite** state space,

Markov chain \iff Transition function \iff Q -matrix

Goal: Under which condition is the equivalence still true?

1.2 Examples

Example (Discrete to continuous MC). **Given:** Markov chain Y in discrete time $t \in \mathbb{Z}_+$, i.e. transition matrix $P = (P(x, y))_{x, y \in \mathcal{S}}$

$$\mathbb{P}(Y(n+1) = y \mid Y(n) = x) = P(x, y)$$

Consider a Poisson process (PP) with intensity 1, at each event time of the PP, there will be a jump of the continuous time MC X . The jumps follow the discrete time MC Y

$$\implies p_t(x, y) = \sum_{n \geq 0} \frac{e^{-t} t^n}{n!} P^n(x, y)$$

in this case $Q = P - 1$

Example (Finite \mathcal{S}). $p_t(x, y) = (e^{tQ}(x, y)) := \sum_{n \geq 0} \frac{t^n}{n!} Q^n(x, y)$

Example (Birth and death processes). $\mathcal{S} = \{0, 1, \dots\}$, $X(t)$ = Population size at time t . Then

- $q(k, k+1) = \rho_k, k \geq 0$
- $q(k+1, k) = \lambda_k, k \geq 1$
- $q(k, k) = -\rho_k - \lambda_k$ with $\lambda_0 = 0$
- which implies $q(k, l) = 0 \forall |k-l| \geq 2$

Depending on the choice of ρ_k, λ_k it is possible that the chain goes to ∞ in finite time.

Here will be one of the main problems: If Q is finite / a normed operator, everything is well defined. Otherwise we have to use a different definition of the exponential ...

Start of lecture 02
(10.10.2024)

1.3 From MC to Q -matrices

1.3.1 MC \rightarrow Transition functions

Theorem 1.5. Let X be a Markov chain. Then $p_t(x, y) := \mathbb{P}^x(X(t) = y)$.

(a) $p_t(x, y)$ is a transition function

(b) $p_t(x, y)$ determines uniquely \mathbb{P}^x

Proof. **(a):** Positivity and normalization follows by the properties of \mathbb{P}^x .

$\tau := \inf\{t \in \mathbb{R}_+ \mid X(t) \neq X(0)\}$ is a.s. positive. $p_t(x, x) \geq \mathbb{P}^x(\tau > t)$, which implies

$$1 = p_0(x, x) \geq \lim_{t \downarrow 0} p_t(x, x) \geq \lim_{t \downarrow 0} \mathbb{P}^x(\tau > t) = 1.$$

We still need to verify the Chapman-Kolmogorov equation: Define

$$Y = 1_{X(t)=y} \implies Y \circ \theta_s = 1_{X(t+s)=y}$$

$$\begin{aligned} p_{s+t}(x, y) &= \mathbb{P}^x(X(s+t) = y) = \mathbb{E}^x(1_{X(s+t)=y}) \\ &= \mathbb{E}^x(\mathbb{E}^x(1_{X(s+t)=y} \mid \mathcal{F}_s)) \\ &\stackrel{\text{MP}}{=} \mathbb{E}^x\left(\mathbb{E}^{X(s)}(1_{X(t)=y})\right) \\ &= \sum_{z \in S} p_s(x, z) \underbrace{\mathbb{E}^z(1_{X(t)=y})}_{=p_t(z, y)} \end{aligned}$$

(b): By the Markov property $\forall 0 < t_1 < \dots < t_n, x_1, \dots, x_n \in S$

$$\mathbb{P}^x(X(t_1) = x_1, \dots, X(t_n) = x_n) = p_{t_1}(x_1, x_1) \cdot p_{t_2-t_1}(x_1, x_2) \cdot \dots \cdot p_{t_n-t_{n-1}}(x_{n-1}, x_n)$$

using the Kolmogorov extension theorem \mathbb{P}^x is uniquely determined by p_t . \square

1.3.2 Transition functions to Q -matrices

Proposition 1.6. Let $p_t(x, y)$ be a transition function.

(a) $\forall t \geq 0, x \in S, p_t(x, x) > 0$

(b) If $p_t(x, x) = 1$ for some $t > 0$, then $p_s(x, x) = 1 \forall s \geq 0$

(c) $\forall x, y \in S, p_t(x, y)$ is uniformly continuous in t , due to

$$|p_{t+s}(x, y) - p_t(x, y)| \leq 1 - p_s(x, x)$$

Remark. (a) differs from the discrete time markov chain \rightarrow no issues of periodicity

(b) The M.C. if it reaches site x , stays forever in x . x is an **absorbing state**

Proof. **(a):** Since $\lim_{x \downarrow 0} p_t(x, x) = 1 \implies p_s(x, x) > 0$ for $s \in [0, t]$ for some small t . Therefore

$\forall s \in [0, t] : p_{t+s}(x, x) \stackrel{\text{C.Kolmogorov}}{\geq} p_t(x, x)p_s(x, x) > 0$, which we can iterate to get the claim.

(b): $p_{s+t}(x, x) = \sum_{z \in S} p_s(x, z)p_t(z, x) \leq p_s(x, x)p_t(x, x) + \underbrace{\sum_{z \neq x} p_s(x, z) \cdot 1}_{=1-p_s(x, x)}$ Then

$$p_{s+t}(x, x) \leq 1 - \underbrace{p_s(x, x)}_{\stackrel{(a)}{\geq 0}}(1 - p_t(x, x)).$$

If $p_{s+t}(x, x) = 1 \implies p_t(x, x) = 1$.

$\forall v \in [t+s, 2(t+s)] : p_v(x, x) \geq \underbrace{p_v - (t+s)}_{\in [0, t+s]}(x, x)p_{t+s}(x, x) = 1 \cdot 1 = 1$

still good, since $p_s(x, z)$ will be 0

(c):

$$\begin{aligned} p_{s+t}(x, y) - p_t(x, y) &= \sum_{z \in S} p_s(x, z) p_t(z, y) - p_t(x, y) \\ &= \underbrace{(p_s(x, x) - 1) p_t(x, y)}_{\leq 0} + \underbrace{\sum_{z \neq x} p_s(x, z) p_t(z, y)}_{\geq 0} \end{aligned}$$

Since

$$|p_s(x, x) - 1| p_t(x, y) \leq 1 - p_s(x, x)$$

and

$$\begin{aligned} \sum_{z \neq x} p_s(x, z) \underbrace{p_t(z, y)}_{\leq 1} &\leq 1 - p_s(x, x) \\ \implies |p_{s+t}(x, y) - p_t(x, y)| &\leq 1 - p_s(x, y) \end{aligned}$$

□

Theorem 1.7. Let $p_t(x, y)$ be a transition function.

(a) $\forall x \in S$ the right derivative

$$\tilde{c}(x) = -\tilde{q}(x, x) = -\frac{d}{dt} p_t(x, x) \big|_{t=0} \in [0, \infty]$$

exists and $p_t(x, x) \geq e^{-\tilde{c}(x)t}, \forall t > 0$

(b) If $\tilde{c}(x) < \infty \implies \forall y \neq x$, the right-derivative

$$\tilde{q}(x, y) := \frac{d}{dt} p_t(x, y) \big|_{t=0} \in [0, \infty]$$

exists and $\sum_{y \in S} \tilde{q}(x, y) \leq 0$

(c) If for some $x \in S, \tilde{c}(x) < \infty$ and $\sum_{y \in S} \tilde{q}(x, y) = 0$, then $p_t(x, y)$ is C^1 in time for this x and $y \in S$. Moreover, it satisfies the **Kolmogorov- backwards equation**:

$$\frac{d}{dt} p_t(x, y) = \sum_{z \in S} \tilde{q}(x, z) p_t(z, y) \quad (1.1)$$

Proof. **(a)** Let $f(t) := -\ln(p_t(x, x)) \geq 0$. By $p_{s+t}(x, x) \geq p_s(x, x) p_t(x, x)$ it follows

$$f(t+s) \leq f(s) + f(t),$$

the function f is subadditive, which implies

$$\lim_{t \downarrow 0} \frac{f(t)}{t} = \sup_{t > 0} \frac{f(t)}{t} \in [0, \infty]$$

In particular, the limit exists

$$\begin{aligned} \implies \lim_{t \downarrow 0} \frac{f(t)}{t} &= \frac{d}{dt} f(t) \big|_{t=0} = \frac{-\frac{d}{dt} p_t(x, x)}{\underbrace{p_t(x, x)}_{\rightarrow 1}} \big|_{t=0} = \tilde{c}(x) \\ \implies \tilde{c}(x) &\geq \frac{f(t)}{t} = -\frac{\ln p_t(x, x)}{t} \rightarrow p_t(x, x) \geq e^{-\tilde{c}(x)t} \end{aligned}$$

(b) by (a): $1 - p_t(x, x) \leq 1 - e^{-\tilde{c}(x)t} \leq \tilde{c}(x)t$, which implies

$$\sum_{y \neq x} p_t(x, y) \leq \tilde{c}(x)t.$$

$$\implies \tilde{q}(x, y) := \limsup_{t \downarrow 0} \frac{p_t(x, y)}{t} \in [0, \underbrace{\tilde{c}(x)}_{< \infty}]$$

Let $\delta > 0, n \in \mathbb{N}$: Let $p_\delta = \{p_\delta(x, y)\}_{x, y \in S}$ be a discrete time chain.

$$\begin{aligned} p_{n\delta}(x, y) &\geq \sum_{k=0}^{n-1} \underbrace{p_\delta(x, x)^k}_{\geq p_\delta(x, x)^n \geq e^{-\tilde{c}(x)n\delta}} p_\delta(x, y) p_{(n-k-1)\delta}(y, y) \\ \implies \frac{p_{n\delta}(x, y)}{n\delta} &\geq \frac{p_\delta(x, y)}{\delta} e^{-\tilde{c}(x)n\delta} \cdot \inf_{0 \leq s \leq n\delta} p_s(y, y) \end{aligned}$$

Take a subsequence of $\delta \downarrow 0$ such that $n\delta \rightarrow t$ and $\lim_{\delta \downarrow 0} \frac{p_\delta(x, y)}{\delta} = \tilde{q}(x, y)$

$$\implies \liminf_{t \downarrow 0} \frac{p_t(x, y)}{t} \geq \tilde{q}(x, y) \cdot 1 \cdot 1$$

which implies that the limit exists.

$$\begin{aligned} \sum_{y \neq x} \frac{p_t(x, y)}{t} &\leq \tilde{c}(x) \\ \tilde{c}(x) &\geq \liminf_{t \downarrow 0} \sum_{y \neq x} \frac{p_t(x, y)}{t} \stackrel{\text{Fatou}}{\geq} \sum_{y \neq x} \liminf_{t \downarrow 0} \frac{p_t(x, y)}{t} = \sum_{y \neq x} \tilde{q}(x, y) \\ \implies \sum_y \tilde{q}(x, y) &\leq 0 \end{aligned}$$

(c) By Chapman-Kolmogorov:

$$\begin{aligned} &\frac{p_{t+\epsilon}(x, y) - p_t(x, y)}{\epsilon} - \sum_z \tilde{q}(x, z) p_t(z, y) \\ &= \sum_{z \in S} \underbrace{\left(\frac{p_\epsilon(x, z) - \overbrace{p_0(x, z)}^{\delta_{x, z}}}{\epsilon} - \tilde{q}(x, z) \right)}_{= (\star)} p_t(z, y) \end{aligned}$$

For all $z : \lim_{\epsilon \rightarrow 0} (\star) = 0$

Take $T \subset S, |T| < \infty \implies \lim_{\epsilon \downarrow 0} \sum_{z \in T} (\star) \cdot p_t(z, y) = 0$. Let $x \in T$.

$$\begin{aligned} &\sum_{z \notin T} \left| \frac{p_\epsilon(x, z)}{\epsilon} - \tilde{q}(x, z) \right| \underbrace{p_t(z, y)}_{\leq 1} \\ &\leq \sum_{z \notin T} \frac{p_\epsilon(x, z)}{\epsilon} + \sum_{z \notin T} \tilde{q}(x, z) \sum_{q(x, \tilde{z})=0} \frac{1}{\epsilon} \left(1 - \sum_{z \in T} p_\epsilon(x, z) \right) - \sum_{z \in T} \tilde{q}(x, z) \end{aligned}$$

But $\sum_x \tilde{q}(x, z) = 0 \xrightarrow{\epsilon \downarrow 0} -2 \sum_{z \in T} \tilde{q}(x, z) \xrightarrow{T \uparrow S} 0$ □

Start of lecture 03
(15.10.2024)

Remark. For M.C. also the **strong Markov property** holds:

Let τ be a stopping time and $Y : \mathbb{R}_+ \times \Omega \mapsto Y_t(\omega)$ ^a measurable, then

$$\mathbb{E}^x(Y \circ \theta_\tau \mid \mathcal{F}_t) = \mathbb{E}^x(Y_\tau), \mathbb{P} - a.s.$$

on the set $\{\tau < \infty\}$.

^ashorthand notation ...

$$\frac{d}{dt}p_t(x, y) = \lim_{\epsilon \downarrow 0} \frac{p_{t+\epsilon}(x, y) - p_t(x, y)}{\epsilon}$$

using $\sum_z p_t(x, z)p_\epsilon(z, y)$ instead of $\sum_z p_\epsilon(x, z)p_t(z, y)$ we get the **Kolmogorov Forward Equation**

$$\frac{d}{dt}p_t(x, y) = \sum_{z \in \mathcal{S}} p_t(x, z)q(z, y) \quad (1.2)$$

They are almost always equivalent. In computation one often uses the forward equation, in construction the backwards equation is preferred

1.4 From Q -matrix to the Markov Chains

Let $Q = (q(x, y))_{x, y \in \mathcal{S}}$ be a Q -matrix.

1.4.1 The backwards equation

Proposition 1.8. *Let $p_t(x, y)$ be a uniformly bounded function of x, y, t , then (a) is equivalent to (b):*

- (a) $p_t(x, y)$ is C^1 in t , satisfies the KBE^a, with initial condition $p_0(x, y) = \delta_{x, y}$
- (b) $p_t(x, y)$ is C^0 in t , and satisfies:

$$p_t(x, y) = \delta_{x, y}e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z \in \mathcal{S} \setminus \{x\}} q(x, z)p_s(z, y) \quad (1.3)$$

^aKolmogorov Backwards Equation

Proof. (a) \implies (b): KBE \iff

$$\begin{aligned} \frac{d}{dt}p_t(x, y) &= -c(x)p_t(x, y) + \sum_{z \in \mathcal{S} \setminus \{x\}} q(x, z)p_t(z, y) \\ \implies \frac{d}{dt} \left(e^{c(x)t} p_t(x, y) \right) &= c(x)e^{c(x)t} p_t(x, y) + e^{c(x)t} \left(-c(x)p_t(x, y) + \sum_{z \neq x} q(x, z)p_t(z, y) \right) \\ &= e^{c(x)t} \sum_{z \neq x} q(x, z)p_t(z, y) \\ &\stackrel{f_0}{=} e^{c(x)t} p_t(x, y) - p_0(x, y) = \int_0^t e^{c(x)s} \sum_{z \neq x} q(x, z)p_s(z, y) \end{aligned}$$

(b) \implies (a): The RHS of the equation is C^1 in t . The initial condition is also satisfied. Taking the derivate yields the $\frac{d}{dt}p_t(x, y) = -c(x)p_t(x, y) + \sum_{z \in \mathcal{S} \setminus \{x\}} q(x, z)p_t(z, y)$ which is equivalent to the KBE. \square

We use proposition 1.8 to show existence of positive solution of the KBE. Idea: Use a **fixed point argument**:

- $p_t^{(0)}(x, y) = 0 \forall x, y \in \mathcal{S}, t \geq 0$
- For $n \geq 0$: $p_t^{(n+1)}(x, y) = \delta_{x, y}e^{c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z \neq x} q(x, z)p_s^{(n)}(z, y)$

Lemma 1.9. $\forall n \geq 0$:

- (a) $p_t^{(n)}(x, y) \geq 0 \forall x, y \in \mathcal{S}$
- (b) $\sum_{y \in \mathcal{S}} p_t^{(n)}(x, y) \leq 1 \forall x \in \mathcal{S}, t \geq 0$
- (c) $p_t^{(n+1)}(x, y) - p_t^{(n)}(x, y) \geq 0 \forall x, y \in \mathcal{S}, t \geq 0$

Proof. By induction. $n = 0$ is clearly satisfied for (a),(b),(c).

(a) is obvious, since RHS are all positive terms.

(b)

$$\begin{aligned} \sum_{y \in \mathcal{S}} p_t^{(n+1)}(x, y) &= e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z \neq x} q(x, z) \underbrace{\sum_{y \in \mathcal{S}} p_s^{(n)}(z, y)}_{\leq 1} \\ &\leq e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \underbrace{\sum_{z \neq x} q(x, z)}_{=c(x)} \\ &= e^{-c(x)t} + e^{-c(x)t} c(x) \int_0^t ds e^{c(x)s} = 1 \end{aligned}$$

(c)

$$p_t^{(n+2)}(x, y) - p_t^{(n+1)}(x, y) = \int_0^t e^{-c(x)(t-s)} \sum_{z \neq x} q(x, z) \underbrace{\left(p_s^{(n+1)}(z, y) - p_s^{(n)}(z, y) \right)}_{\geq 0} \geq 0 \quad \square$$

$\Rightarrow \forall x, y, t$ we have $0 \leq p_t^0(x, y) \leq p_t^1(x, y) \leq \dots \leq 1$.

\Rightarrow there exists a limit, which we denote by $p_t^*(x, y)$

Definition 1.10. The limit $p_t^*(x, y) := \lim_{n \rightarrow \infty} p_t^n(x, y)$ is called the **minimal solution of the KBE**.

Question: Is p^* a transition function?

Theorem 1.11. p^* satisfies:

- (a) $p_t^*(x, y) \geq 0$
- (b) $\sum_{y \in \mathcal{S}} p_t^*(x, y) \leq 1$
- (c) The Chapman-Kolmogorov equation
- (d) satisfies (1.3) and thus the KBE

Proof. (a) and (b) follow directly from lemma 1.9.

(d): Take the limits of $p_t^{(n)}(x, y)$ and use monotone convergence.

(c): Define $\Delta_t^{(n)} := p_t^{(n+1)}(x, y) - p_t^{(n)}(x, y) \geq 0$.

Notice $p_t^*(x, y) = \sum_{n \geq 0} \Delta_t^{(n)}(x, y)$ In lemma 1.12:

$\Delta_{t+s}^{(n)}(x, y) = \sum_{z \in \mathcal{S}} \sum_{k=0}^n \Delta_s^{(k)}(x, z) \Delta_t^{(n-k)}(z, y)$ which then implies

$$\begin{aligned} p_{t+s}^*(x, y) &= \sum_{n \geq 0} \Delta_{t+s}^{(n)}(x, y) = \sum_{z \in \mathcal{S}} \sum_{n \geq 0} \sum_{k=0}^n \Delta_s^{(k)}(x, z) \Delta_t^{(n-k)}(z, y) \\ &= \sum_{z \in \mathcal{S}} \underbrace{\sum_{k \geq 0} \Delta_s^{(k)}(x, z)}_{p_s^*(x, z)} \underbrace{\sum_{n=k}^{\infty} \Delta_t^{(n-k)}(z, y)}_{p_t^*(z, y)} \\ &= \sum_{z \in \mathcal{S}} p_s^*(x, z) p_t^*(z, y) \end{aligned}$$

using monotone convergence. \square

Lemma 1.12. $\Delta_{t+s}^{(n)}(x, y) = \sum_{z \in \mathcal{S}} \sum_{k=0}^n \Delta_s^{(k)}(x, z) \Delta_t^{(n-k)}(z, y)$

Proof. Consider the Laplace transform of the equation w.r.t. s, t , since the functions are all positive!

$$\int_0^\infty ds \int_0^\infty dt e^{-\lambda s} e^{-\mu t} \Delta_{s+t}^{(n)} \stackrel{?}{=} \sum_{z \in S} \sum_{k=0}^n \underbrace{\left[\int_0^\infty ds e^{-\lambda s} \Delta_s^{(k)}(x, z) \right]}_{\Psi_{k,\lambda}(x,z)} \underbrace{\left[\int_0^t e^{-\mu t} \Delta_t^{(n-k)}(z, y) \right]}_{\Psi_{n-k,\mu}(z,y)}$$

Define $\int_0^t ds e^{-\lambda s} \Delta_s^{(n)}(x, y) = \Psi_{n,\lambda}(x, y)$. Then for the RHS:

$$= \sum_{z \in S} \sum_{k=0}^n \Psi_{k,\lambda}(x, z) \Psi_{n-k,\mu}(z, y)$$

For the LHS:

$$\begin{aligned} \int_0^\infty ds \int_0^\infty dt e^{-\lambda(s+t)} e^{-t(\mu-\lambda)} \Delta_{\underbrace{s+t}_{=:u}}^{(n)}(x, y) &= \int_0^\infty ds \int_s^\infty du e^{-\lambda u} e^{-\mu(u-s)} \Delta_u^{(n)}(x, y) \\ &= \int_0^\infty du \int_u^\infty ds e^{-(\lambda-\mu)s} e^{-\mu u} \Delta_u^{(n)}(x, y) = \frac{\Psi_{n,\mu}(x, y) - \Psi_{n,\lambda}(x, y)}{\lambda - \mu} \\ &\stackrel{?}{=} \sum_{k=0}^n \sum_{z \in S} \Psi_{k,\lambda}(x, y) \Psi_{n-k,\mu}(z, y) \end{aligned}$$

Another identity: $\Psi_{n+1,\lambda}(x, y) = \sum_{z \neq x} \frac{q(x,z)}{\lambda+c(x)} \Psi_{n,\lambda}(z, y)$

Define the matrix $A_\lambda(x, y) := \frac{q(x,y)}{\lambda+c(x)} 1_{z \neq x}$, then

$$\Psi_{n,\lambda} = (A_\lambda)^n \Psi_{0,\lambda}$$

where $\Psi_{0,\lambda}(x, y) = \int_0^\infty ds e^{-\lambda s} p_s^{(1)}(x, y)$ and $p_s^{(1)}(x, y) = \delta_{x,y} e^{-c(x)s}$. Then $\Psi_{0,\lambda}(x, y) = \delta_{x,y} \frac{1}{\lambda+c(x)}$.
 \Rightarrow

$$\begin{aligned} \frac{\Psi_{n,\mu} - \Psi_{n,\lambda}}{\lambda - \mu} &= \frac{(A_\mu)^n \Psi_{0,\mu} - (A_\lambda)^n \Psi_{0,\lambda}}{\lambda - \mu} \\ &= \frac{(A_\lambda)^n (\Psi_{0,\mu} - \Psi_{0,\lambda})}{\lambda - \mu} \sum_{k=0}^{n-1} (A_\lambda)^k \frac{A_\mu - A_\lambda}{\lambda - \mu} A^{n-k-1} \Psi_{0,\mu} \end{aligned}$$

$$\frac{\Psi_{0,\mu} - \Psi_{0,\lambda}}{\lambda - \mu}(x, y) = \frac{\delta_{x,y}}{\lambda - \mu} \left(\frac{1}{c(x) + \mu} - \frac{1}{c(x) + \lambda} \right) = \Psi_{0,\mu}(x, y) \cdot \Psi_{0,\lambda}(x, y).$$

Similarly

$$\frac{A_\mu - A_\lambda}{\lambda - \mu}(x, y) = (\Psi_{0,\lambda} A_\mu)(x, y)$$

Then

$$\begin{aligned} \frac{(A_\lambda)^n (\Psi_{0,\mu} - \Psi_{0,\lambda})}{\lambda - \mu} \sum_{k=0}^{n-1} (A_\lambda)^k \frac{A_\mu - A_\lambda}{\lambda - \mu} A^{n-k-1} \Psi_{0,\mu} &= [(A_\lambda)^n \Psi_{0,\mu}](x, y) + \sum_{k=0}^{n-1} \sum_z \underbrace{[A_\lambda^k \Psi_{0,\lambda}]}_{=\Psi_{k,\lambda}(x,z)} \underbrace{(A_\mu^{n-k} \Psi_{0,\mu})}_{=\Psi_{n-k,\mu}(z,y)}(z, y) \\ &= \sum_{k=0}^n \sum_{z \in S} \Psi_{k,\lambda}(x, z) \Psi_{n-k,\mu}(z, y) \quad \square \end{aligned}$$

Above the second equation is wrong, but the skip from the RHS to the 3rd line is correct

Reminder: For matrices $X^n - Y^n = (X - Y)X^{n-1} + Y(X - Y)X^{n-2} + \dots + Y^{n-1}(X - Y)$, since they don't commute

Start of lecture 04
(17.10.2024)

Theorem 1.13. (a) If $p_t(x, y)$ is a non-negative solution of the KBE 1.1 with $p_0(x, y) = \delta_{x,y} \Rightarrow p_t(x, y) \geq p_t^*(x, y)$

(b) If $\sum_{y \in \mathcal{S}} p_t^*(x, y) = 1, \forall x \in \mathcal{S} \implies p_t^*(x, y)$ is the unique transition function satisfying the KBE

Proof. **(a):** By induction: $p_t(x, y) \geq p_t^{(n)}(x, y) \forall n$.

For $n = 0$ $p_t(x, y) \geq 0 = p_t^{(0)}(x, y)$.

$$p_t^{(n+1)}(x, y) = \delta_{x,y} e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z \neq x} q(x, z) \underbrace{p_s^{(n)}(x, y)}_{\leq p_s(x, y)}$$

$$\begin{array}{c} p_t \text{ satisfies KBE} \\ \leq \\ p_t(x, y) \end{array}$$

(b):

$$\begin{aligned} &= \sum_{y \in \mathcal{S}} \underbrace{(p_t^*(x, y) - p_t^*(x, y))}_{\geq 0} \geq 0 \\ &= \underbrace{\sum_{y \in \mathcal{S}} p_t(x, y)}_{=1} - \underbrace{\sum_{y \in \mathcal{S}} p_t^*(x, y)}_{=1} \\ &\implies 0 \geq \sum_{y \in \mathcal{S}} (p_t^*(x, y) - p_t^*(x, y)) \geq 0 \\ &\implies p_t(x, y) = p_t^*(x, y) \quad \square \end{aligned}$$

Example 1.14. $\mathcal{S} = \{0, 1, \dots\}, q(i, j) = \begin{cases} \beta_i & j = i + 1 \\ -\beta_i & j = i \\ 0 & \text{otherwise} \end{cases}$ for some positive β_i . One shows

$$\sum_{j \in \mathcal{S}} p_t^*(i, j) = 1 \forall t, i \in \mathcal{S} \iff \sum_{k \geq 0} \frac{1}{\beta_k} < \infty.$$

What happens if $\sum_{k \geq 0} \frac{1}{\beta_k} = \infty$? Consider $N(t) = \#$ of jumps before time t , then $\mathbb{P}(n(t) = \infty) > 0$.

1.4.2 The probabilistic construction

Lemma 1.15. Let $X(t)$ be a continuous time M.C. and $\tau := \inf\{t : X(t) \neq X(0)\}$. Then there exists $c(x) = [0, \infty]$ s.t. $\mathbb{P}(\tau > t) = e^{-c(x)t}$

Proof. Use Markov property with $Y = 1_{X(r)=x, 0 \leq r \leq t-s}$:

$$\begin{aligned} \mathbb{P}^x(X(r) = x, s \leq r \leq t \mid \mathcal{F}_s) &= \mathbb{P}^{X(s)}(\tau > t - s) = (\star) \\ \mathbb{E}^x(1_{\tau > s} \cdot \mathbb{P}^x(X(r) = x, s \leq r \leq t \mid \mathcal{F}_s)) &= \mathbb{P}^x(\tau > t) \\ \mathbb{E}^x(1_{\tau > s} \cdot \mathbb{P}^x(X(r) = x, s \leq r \leq t \mid \mathcal{F}_s)) &\stackrel{(\star)}{=} \mathbb{E}^x\left(1_{\tau > s} \mathbb{P}^{X(s)}(\tau > t - s)\right) \\ &= \mathbb{P}^x(\tau > s) \mathbb{P}^x(\tau > t - s) \\ &\implies \mathbb{P}^x(\tau > t) = \mathbb{P}^x(\tau > s) \mathbb{P}^x(\tau > t - s) \end{aligned}$$

Either $\mathbb{P}^x(\tau > t) = 0 \forall t \implies c(x) = \infty$ (jump before t) or $\mathbb{P}^x(\tau > t) > 0 \forall t$. Define

$f(t) := -\ln \mathbb{P}^x(\tau > t)$ with $f(0) = 0 \implies f(t) \geq 0$. and

$f(t) = f(s) + f(t - s) \implies f(t) = k \cdot t$. \square

Idea: Divide the waiting times (which are exponentially distributed $\sim \exp(c(x)), \exp(c(y)), \dots$) and the **jumps**. The jumps are given by a discrete time M.C..

(1) Discrete time Markov chain $\{Z_n, n \geq 0\}$ with transition matrix:

$$P = (p(x, y))_{x, y \in \mathcal{S}}$$

If $c(x) = 0$:

$$p(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

If $c(x) > 0$:

$$p(x, y) \begin{cases} \frac{q(x, y)}{c(x)} & x \neq y \\ 0 & x = y \end{cases}.$$

(2) Initial distribution: $\pi = (\pi(x))_{x \in S}$ is a probability measure.

(3) Family of **independent** exponentially distributed waiting times independent $\tau_{x,i} \sim \exp(c(x))$, $x \in S$, $i \geq 1$.

If $c(x) = 0 \rightarrow \tau_{x,i} = \infty$.

The joint distribution of Z_n are given by

$$\mathbb{P}(Z_0 = x_0, \dots, Z_n = x_n) = \pi(x_0)p(x_0, x_1) \cdots p(x_{n-1}, x_n)$$

Remark. Careful, the waiting times are only independent given the positions of Z_n

$$\mathbb{P}(Z_0 = x_0, Z_1 = x_1, \dots, Z_n = x_n, \tau_0 > t_0, \tau_1 > t_1, \dots, \tau_n > t_n) = \pi(x_0)p(x_0, x_1) \cdots p(x_{n-1}, x_n)e^{-c(x_0)t_0} \cdots e^{-c(x_n)t_n}$$

where τ_k is the waiting time of the $(k+1)$ th jump.

This should be $k+1$, otherwise what does τ_{-1} mean?

$$\text{Denote by } N(t) := \begin{cases} \min(n \geq 0 \mid \tau_0 + \tau_1 + \dots + \tau_n > t) & \sum_{k \geq 0} \tau_k > t \\ \infty & \text{otherwise} \end{cases}$$

$$N(0) = 0.$$

Definition 1.16. We define the continuous time Markov chain $X(t)$ by $X(t) = Z_{N(t)}$ on $\{N(t) < \infty\}$.

Proposition 1.17. (a) $p_t^{(n)}(x, y) = \mathbb{P}(X(t) = y, N(t) < n \mid X(0) = x)$

$$(b) \quad p_t^*(x, y) = \mathbb{P}(X(t) = y, N(t) < \infty \mid X(0) = x)$$

$$(c) \quad \sum_{y \in S} p_t^*(x, y) = \mathbb{P}(N(t) < \infty \mid X(0) = x)$$

Proof. (b) and (c) follow from (a), therefore only (a) is proofed.

(a): By deriving the same iteration as $p_t^{(+1)}(x, y) = \dots$. For $c(x) > 0$ Condition on τ_0, Z_0, Z_1 :

$$\mathbb{P}(X(t) = y, N(t) < n+1 \mid \tau_0 = s, Z_1 = z, \underbrace{Z_0}_{=X(0)} = x) = \begin{cases} \delta_{x,y} & s > t \\ \mathbb{P}(X(t-s) = y, N(t-s) < n \mid X(0) = z) & s < t \end{cases}$$

$$\mathbb{P}(X(t) = y, N(t) < n+1 \mid X(0) = x)$$

$$= \int_0^\infty \sum_{z \in S} \mathbb{P}(X(t) = y, N(t) < n+1 \mid \tau_0 = s, Z_1 = z, X(0) = x) \cdot \underbrace{\mathbb{P}(\tau_0 \in ds, Z_1 = z \mid X(0) = x)}_{\text{density: } p(x, z)e^{-c(x)s}c(x)}$$

$$= \int_t^\infty ds \delta_{x,y} \cdot \underbrace{\sum_{z \in S} p(x, z) e^{-c(x)s} c(x)}_{=1} + \int_0^t ds \sum_{z \in S} \mathbb{P}(X(t-s) = y, N(t-s) < n \mid X(0) = z) p(x, z) c(x) e^{-c(x)s}$$

$$= \delta_{x,y} e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z \in S \setminus \{x\}} \mathbb{P}(X(t-s) = y, N(t-s) < n \mid X(0) = z) q(x, z)$$

$$\stackrel{s \mapsto t-s}{=} \delta_{x,y} e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{q(x, z)} \mathbb{P}(X(s) = y, N(s) < n \mid X(0) = z)$$

\Rightarrow Same I.C. and recursion as for $p_t^{(n)}(x, y)$. □

Theorem 1.18. The following are equivalent:

(a) The minimal solution $p_t^*(x, y)$ is stochastic

$$(b) \mathbb{P}(N(t) < \infty) = 1 \forall t$$

$$(c) \sum_{n \geq 0} \tau_n = \infty \text{ a.s.}$$

$$(d) \sum_{n \geq 0} \frac{1}{Z(n)} = \infty \text{ a.s.}$$

List of Lectures

- *Lecture 01: Introduction, elementary definitions of continuous time markov chains, examples*
- *Lecture 02: From a markov chain to the infinitesimal description*
- *Lecture 03:*
- *Lecture 04: Role of the $p_t^{(n)}$*