Lecture notes on Markov Processes

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Based on the lectures of
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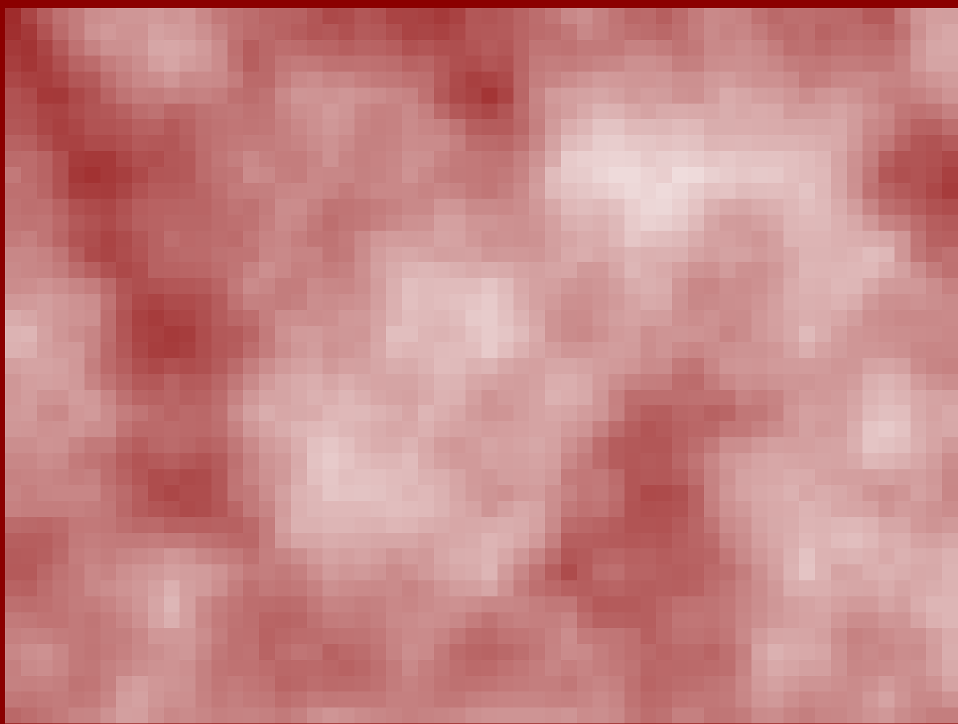

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Chapter 0:
Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Ecampus: Ecampus link
- Basis: Basis link
- Website: None
- Time slot(s): **Tuesday 12-14** and **Thursday 12-14**
- Exams: Oral
- Exercises: To be handed in each **Friday** until **noon**

There are tutorials in the second week (14.10.24 and 16.10.24)! Start of lecture 01 (08.10.2024)

Chapter 1: Continuous time Markov chains

Two motivating examples:

1.1 Markov chain, transition function, infinitesimal description

We have a **countable state space $\mathcal{S}$** and define

- $\Omega = \{$right-continuous functions $\omega : \mathbb{R}_+ \to \mathcal{S}$ with finitely many jumps in any finite time intervals$\}$
- σ-algebra $\mathcal{F} := \sigma(\{\omega \to \omega(t)$ which is measurable $\forall t \geq 0\})$ measurable w.r.t.?
- time shift operator: $(\theta_s \omega)(t) := \omega(s+t)$

Definition 1.1. *Denote $X(t,\omega) := \omega(t)$. Assume we have a collection of probability measures $\{\mathbb{P}^x, x \in \mathcal{S}\}$ on $(\Omega, \mathcal{F})$, a right-continuous filtration $\{\mathcal{F}_t, t \geq 0\}$ on $(\Omega, \mathcal{F})$. X is a* ***continuous time Markov chain (MC)*** *if*

(a) X is adapted to $\{\mathcal{F}_t, t \geq 0\}$

(b) ***Initial condition:*** $\mathbb{P}^x(X(0) = x) = 1$

(c) ***Markov property:*** *$\forall x \in \mathcal{S}, Y$ measurable function on $\Omega, s \geq 0$:*

$$\mathbb{E}^x\left(Y \circ \theta_s \mid \mathcal{F}_s\right) = \mathbb{E}^{X(s)}(Y), \mathbb{P}^x \text{ a.s.}$$

Example. $Y = \max_{0 \leq t \leq 1} X(t) \to Y \circ \theta_s = \max_{s \leq t \leq s+1} X(t)$

Here: Time homogenous processes!

Definition 1.2. *A* ***transition function*** *is function $p_t(x,y), x, y \in \mathcal{S}, t \geq 0$ s.t.*

(a) ***Positivity:*** $p_t(x,y) \geq 0$

(b) ***Normalized:*** $\sum_{y \in \mathcal{S}} p_t(x,y) = 1$ *We will see while constructing, that this part is tricky, might just be ≤ 1*

(c) ***Continuity:*** $\lim_{t \downarrow 0} p_t(x,x) = p_0(x,x) = 1$

(d) ***Chapman-Kolmogorov equation:*** *$\forall s, t \geq 0\ \forall x, y \in \S$:*

$$p_{s+t}(x,y) = \sum_{z \in \mathcal{S}} p_s(x,z) p_t(z,y)$$

Given a transition function, we can construct $\mathbb{P}^x$ as follows:
Finite dimensional distributions, $0 < t_1 < \cdots < t_n$:

$\mathbb{P}^x(X(t_1) = x_1, X(t_2 = x_2), \ldots, X(t_n) = x_n) := p_{t_1}(x, x_1) \cdot p_{t_2 - t_1}(x_1, x_2) \cdot \cdots \cdot p_{t_n - t_{n-1}(x_{n-1}, x_n)}$

extend to full time $\mathbb{R}_+$ by the Kolmogorov(-Daniell) extension theorem, where the consistency,$\mathbb{P}^x(X(t) = y) = p_t(x, y)$, relations follow by the Chapman-Kolmogorov equation.

Example. $0 < t_2 < \cdots < t_n$

$$\sum_{x_1 \in \mathcal{S}} \mathbb{P}^x\left(X(t_1) = x_1, \ldots, X(t_n) = x_n\right) = \mathbb{P}^x\left(X(t_2) = x_2, \ldots, X(t_n) = x_n\right)$$

We are still in the same setting, therefore we use the same t_1 as previously

By modeling we often think at the basic biological/ physical properties of the system → typically we have *transition rates*, because for $x \neq y$: $\begin{cases} p_\epsilon(x,y) = O(\epsilon), & p_\epsilon(x,x) = 1 - O(\epsilon) \\ p_0(x,y) = 0, & p_0(x,x) = 1 \end{cases}$

Definition 1.3. *For a Markov chain X we define the* ***transition rates*** *from x to y ($x \neq y$) by:*

$$\tilde{q}(x,y) =:= \frac{d}{dt} p_t(x,y) \mid_{t=0}$$

Definition 1.4. *A* ***Q-matrix (or generator)*** *is a collection of numbers $\{q(x,y) : x, y \in \mathcal{S}\}$ s.t.:*

(a) $q(x,y) \geq 0 \forall x \neq y$

(b) $\sum_{y \in \mathcal{S}} q(x,y) = 0$*, Notation:* $c(x) := q(x,x) = \sum_{y \in \mathcal{S} \setminus \{x\}} q(x,y)$

$c(x)$ is the rate of leaving site x

Warning

It is not automatic that transition rates gives a Q-matrix For instance: $\tilde{q}(x,y) = \infty$

For **finite** state space,
Markov chain $\iff$ Transition function $\iff$ Q-matrix
Goal: Under which condition is the equivalence still true?

1.2 Examples

Example (Discrete to continuous MC). ***Given:*** *Markov chain Y in discrete time $t \in \mathbb{Z}_+$, i.e. transition matrix $P = (P(x,y))_{x,y \in \mathcal{S}}$*

$$\mathbb{P}(Y(n+1) = y \mid Y(n) = x) = P(x,y)$$

Consider a Poisson process(PP) with intensity 1, at each event time of the PP, there will be a jump of the continuous time MC X. The jumps follow the discrete time MC Y
$\implies p_t(x,y) = \sum_{n \geq 0} \frac{e^{-t} t^n}{n!} P^n(x,y)$
in this case $Q = P - 1$

Example (Finite $\mathcal{S}$). $p_t(x,y) = \left(e^{tQ}(x,y)\right) := \sum_{n \geq 0} \frac{t^n}{n!} Q^n(x,y)$

Here will be one of the main problems: If Q is finite / a normed operator, everything is well defined. Otherwise we have to use a different definition of the exponetial ...

Example (Birth and death processes). $\mathcal{S} = \{0, 1, \ldots, \}, X(t) =$ *Population size at time t. Then*

- $q(k, k+1) = \rho_k, k \geq 0$
- $q(k+1, k) = \lambda_k, g \geq 1$
- $q(k,k) = -\rho_k - \lambda_k$ *with* $\lambda_0 = 0$
- *which implies* $q(k,l) = 0 \forall |k - l| \geq 2$

Depending on the choice of ρ_k, λ_k it is possible that the chain goes to ∞ in finite time.

Start of lecture 02 (10.10.2024)

1.3 From MC to Q-matrices

1.3.1 MC → Transition functions

Theorem 1.5. *Let X be a Markov chain. Then $p_t(x,y) \coloneqq \mathbb{P}^x(X(t)=y)$.*

(a) $p_t(x,y)$ is a transition function

(b) $p_t(x,y)$ determines uniquely $\mathbb{P}^x$

Proof. **(a):** Positivity and normalization follows by the properties of $\mathbb{P}^x$.
$\tau \coloneqq \inf\{t \in \mathbb{R}_+ \mid X(t) \neq X(0)\}$ is a.s. positive. $p_t(x,x) \geq \mathbb{P}^x(\tau > t)$, which implies
$1 = p_0(x,x) \geq \lim_{t\downarrow 0} p_t(x,x) \geq \lim_{t\downarrow 0} \mathbb{P}^x(\tau > t) = 1$.
We still need to verify the Chapman-Kolmogorov equation: Define
$Y = 1_{X(t)=y} \implies Y \circ \theta_s = 1_{X(t+s)=y}$

$$\begin{aligned} p_{s+t}(x,y) &= \mathbb{P}^x(X(s+t)=y) = \mathbb{E}^x\left(1_{X(s+t)=y}\right) \\ &= \mathbb{E}^x\left(\mathbb{E}^x(1_{X(s+t)=y} \mid \mathcal{F}_s)\right) \\ &\overset{\text{MP}}{=} \mathbb{E}^x\left(\mathbb{E}^{X(s)}(1_{X(t)=y})\right) \\ &= \sum_{z\in S} p_s(x,z) \underbrace{\mathbb{E}^z\left(1_{X(t)=y}\right)}_{=p_t(z,y)} \end{aligned}$$

(b): By the Markov property $\forall 0 < t_1 < \cdots < t_n, x_1, \ldots, x_n \in S$

$$\mathbb{P}^x\left(X(t_1)=x_1, \ldots X(t_n)=x_n\right) = p_{t_1}(x_1,x_1) \cdot p_{t_2-t_1}(x_1,x_2) \cdot \cdots \cdot p_{t_n-t_{n-1}}(x_{n-1},x_n)$$

using the Kolmogorov extension theorem $\mathbb{P}^x$ is uniquely determined by p_t. □

1.3.2 Transition functions to Q-matrices

Proposition 1.6. *Let $p_t(x,y)$ be a transition function.*

(a) $\forall t \geq 0, x \in S, p_t(x,x) > 0$

(b) If $p_t(x,x)=1$ for some $t>0$, then $p_s(x,x)=1 \forall s \geq 0$

(c) $\forall x,y \in S, p_t(x,y)$ is uniformly continuous in t, due to

$$|p_{t+s}(x,y) - p_t(x,y)| \leq 1 - p_s(x,x)$$

Remark. *(a) differs from the discrete time markov chain $\to$ no issues of periodicity*

*(b) The M.C. if it reaches site x, stays forever in x. x is an **absorbing state***

Proof. **(a):** Since $\lim_{x\downarrow 0} p_t(x,x) = 1 \implies p_s(x,x) > 0$ for $s \in [0,t]$ for some small t. Therefore
$\forall s \in [0,t] : p_{t+s}(x,x) \overset{\text{C.Kolmogorov}}{\geq} p_t(x,x)p_s(x,x) > 0$, which we can iterate to get the claim.
(b): $p_{s+t}(x,x) = \sum_{z\in S} p_s(x,z)p_t(z,x) \leq p_s(x,x)p_t(x,x) + \underbrace{\sum_{z\neq x} p_s(x,z) \cdot}_{=1-p_s(x,x)} 1$ Then

still good, since $p_s(x,z)$ will be 0

$$p_{s+t}(x,x) \leq 1 - \underbrace{p_s(x,x)}_{\overset{(a)}{\geq} 0}(1-p_t(x,x)).$$

If $p_{s+t}(x,x) = 1 \implies p_t(x,x) = 1$.
$\forall v \in [t+s, 2(t+s)] : p_v(x,x) \geq p_{\underbrace{v-(t+s)}_{\in[0,t+s]}}(x,x)p_{t+s}(x,x) = 1 \cdot 1 = 1$

(c):

$$\begin{aligned} p_{s+t}(x,y) - p_t(x,y) &= \sum_{z\in S} p_s(x,z)p_t(s,y) - p_t(x,y) \\ &= \underbrace{(p_s(x,x)-1)p_t(x,y)}_{\leq 0} + \underbrace{\sum_{z\neq x} p_s(x,z)p_t(z,y)}_{\geq 0} \end{aligned}$$

Since

$$|p_s(x,x)-1|p_t(x,y) \le 1-p_s(x,x)$$

and

$$\sum_{z\neq x} p_s(x,z)\underbrace{p_t(z,y)}_{\le 1} \le 1-p_s(x,x)$$

$$\implies |p_{s+t}(x,y)-p_t(x,y)| \le 1-p_s(x,y)$$

□

Theorem 1.7. *Let $p_t(x,y)$ be a transition function.*

(a) $\forall x\in S$ the right derivative

$$\tilde{c}(x) = -\tilde{q}(x,x) = -\frac{d}{dt}p_t(x,x)\mid_{t=0} \in [0,\infty]$$

exists and $p_t(x,x) \ge e^{-\tilde{c}(x)t}, \forall t>0$

(b) I tildec $<\infty \implies \forall y\neq x$, the right-derivative

$$\tilde{q}(x,y) := \frac{d}{dt}p_t(x,y)\mid_{t=0} \in [0,\infty]$$

exists and $\sum_{y\in S}\tilde{q}(x,y)\le 0$

*(c) If for some $x\in S, \tilde{c}(x)<\infty$ and $\sum_{y\in S}\tilde{q}(x,y)=0$, then $p_t(x,y)$ is C^1 in time for this x and $y\in S$. Moreover, it satisfies the **Kolmogorov- backwards equation**:*

$$\frac{d}{dt}p_t(x,y) = \sum_{z\in S}\tilde{q}(x,z)p_t(z,y) \tag{1.1}$$

Proof. **(a)** Let $f(t) := -\ln(p_t(x,x)) \ge 0$. By $p_{s+t}(x,x)\ge p_s(x,x)p_t(x,x)$ it follows

$$f(t+s)\le f(s)+f(t),$$

the function f is subadditive, which implies

In particular, the limit exists

$$\lim_{t\downarrow 0}\frac{f(t)}{t} = \sup_{t>0}\frac{f(t)}{t} \in [0,\infty]$$

$$\implies \lim_{t\downarrow 0}\frac{f(t)}{t} = \frac{d}{dt}f(t)\mid_{t=0} = \frac{-\frac{d}{dt}p_t(x,x)}{\underbrace{p_t(x,x)}_{\to 1}}\mid_{t=0} = \tilde{c}(x)$$

$$\implies \tilde{c}(x) \ge \frac{f(t)}{t} = -\frac{\ln p_t(x,x)}{t} \to p_t(x,x)\ge e^{-\tilde{c}(x)t}$$

(b) by (a): $1-p_t(x,x)\le 1-e^{-\tilde{c}(x)t}\le \tilde{c}(x)t$, which implies

$$\sum_{y\neq x}p_t(x,y)\le \tilde{c}(x)t.$$

$$\implies \tilde{q}(x,y) := \limsup_{t\downarrow 0}\frac{p_t(x,y)}{t} \in [0,\underbrace{\tilde{c}(x)}_{<\infty}]$$

Let $\delta>0, n\in\mathbb{N}$: Let $p_\delta = \{p_\delta(x,y)\}_{x,y\in S}$ be a discrete time chain.

$$p_{n\delta}(x,y) \ge \sum_{k=0}^{n-1}\underbrace{p_\delta(x,x)^k}_{\ge p_\delta(x,x)^n \ge e^{-\tilde{c}(x)n\delta}} p_\delta(x,y)p_{(n-k-1)\delta}(y,y)$$

$$\implies \frac{p_{n\delta}(x,y)}{n\delta} \ge \frac{p_{\delta(x,y)}}{\delta}e^{-\tilde{c}(x)n\delta}\cdot \inf_{0\le s\le n\delta}p_s(y,y)$$

Take a subsequence of $\delta \downarrow 0$ such that $n\delta \to t$ and $\lim_{\delta \downarrow 0} \frac{p_\delta(x,y)}{\delta} = \tilde{q}(x,y)$

$$\implies \liminf_{t \downarrow 0} \frac{p_t(x,y)}{t} \geq \tilde{q}(x,y) \cdot 1 \cdot 1$$

which implies that the limit exists.

$$\sum_{y \neq x} \frac{p_t(x,y)}{t} \leq \tilde{c}(x)$$

$$\tilde{c}(x) \geq \liminf_{t \downarrow 0} \sum_{y \neq x} \frac{p_t(x,y)}{t} \overset{\text{Fatou}}{\geq} \sum_{y \neq x} \liminf_{t \downarrow 0} \frac{p_t(x,y)}{t} = \sum_{y \neq x} \tilde{q}(x,y)$$

$$\implies \sum_{y} \tilde{q}(x,y) \leq 0$$

(c) By Chapman-Kolmogorov:

$$\frac{p_{t+\epsilon}(x,y) - p_t(x,y)}{\epsilon} - \sum_z \tilde{q}(x,z) p_t(z,y)$$

$$= \sum_{z \in S} \underbrace{\left(\frac{p_\epsilon(x,z) - \overbrace{p_0(x,z)}^{\delta_{x,z}}}{\epsilon} - \tilde{q}(x,z) \right)}_{=(\star)} p_t(z,y)$$

For all $z : \lim_{\epsilon \to 0} (\star) = 0$
Take $T \subset S$, $|T| < \infty \implies \lim_{\epsilon \downarrow 0} \sum_{z \in T} (\star) \cdot p_t(z,y) = 0$. Let $x \in T$.

$$\sum_{z \notin T} \left| \frac{p_\epsilon(x,z)}{\epsilon} - \tilde{q}(x,z) \right| \underbrace{p_(z,y)}_{\leq 1}$$

$$\leq \sum_{z \notin T} \frac{p_\epsilon(x,z)}{\epsilon} + \sum_{z \notin T} \tilde{q}(x,z) \overset{\sum q(x,\tilde{z})=0}{=} \frac{1}{\epsilon} \left(1 - \sum_{z \in T} p_\epsilon(x,z) \right) - \sum_{z \in T} \tilde{q}(x,z)$$

But $\sum_x \tilde{q}(x,z) = 0 \overset{\epsilon \downarrow 0}{\to} -2 \sum_{z \in T} \tilde{q}(x,z) \overset{T \uparrow S}{\to} 0$ $\square$

Start of lecture 03 (15.10.2024)

Remark. *For M.C. also the* ***strong Markov property*** *holds:*
Let τ be a stopping time and $Y : \mathbb{R}_+ \times \Omega \mapsto Y_t(\omega)$[a] measurable, then

$$\mathbb{E}^x(Y \circ \theta_\tau \mid \mathcal{F}_t) = \mathbb{E}^x(Y_\tau), \mathbb{P} - \quad a.s.$$

on the set $\{\tau < \infty\}$.

[a] shorthand notation ...

$$\frac{d}{dt} p_t(x,y) = \lim_{\epsilon \downarrow 0} \frac{p_{t+\epsilon}(x,y) - p_t(x,y)}{\epsilon}$$

using $\sum_z p_t(x,z) p_\epsilon(z,y)$ instead of $\sum_z p_\epsilon(x,z) p_t(z,y)$ we get the **Kolmogorov Forward Equation**

$$\frac{d}{dt} p_t(x,y) = \sum_{z \in \mathcal{S}} p_t(x,z) q(z,y) \tag{1.2}$$

They are almost always equivalent. In computation one often uses the forward equation, in construction the backwards equation is preffered

1.4 From Q-matrix to the Markov Chains

Let $Q = (q(x,y))_{x,y\in\mathcal{S}}$ be a Q-matrix.

1.4.1 The backwards equation

Proposition 1.8. *Let $p_t(x,y)$ be a uniformly bounded function of x, y, t, then (a) is equivalent to (b):*

(a) $p_t(x,y)$ is C^1 in t, satisfies the KBE[a], with initial condition $p_0(x,y) = \delta_{x,y}$

(b) $p_t(x,y)$ is C^0 in t, and satisfies:

$$p_t(x,y) = \delta_{x,y}e^{-c(x)t} + \int_0^t dse^{-c(x)(t-s)} \sum_{z\in\mathcal{S}\setminus\{x\}} q(x,z)p_s(z,y) \tag{1.3}$$

[a] Kolmogorov Backwards Equation

Proof. (a) $\implies$ (b): KBE $\iff$

$$\frac{d}{dt}p_t(x,y) = -c(x)p_t(x,y) + \sum_{z\in S\setminus\{x\}} q(x,z)p_t(z,y)$$

$$\begin{aligned}\implies \frac{d}{dt}\left(e^{c(x)t}p_t(x,y)\right) &= c(x)e^{c(x)t}p_t(x,y) + e^{c(x)t}\left(-c(x)p_t(x,y) + \sum_{z\neq x} q(x,z)p_t(z,y)\right)\\ &= e^{c(x)t}\sum_{z\neq x} q(x,z)p_t(z,y)\\ &\overset{\int_0^t}{=} e^{c(x)t}p_t(x,y) - p_0(x,y) = \int_0^t e^{c(x)s}\sum_{z\neq x} q(x,z)p_s(z,y)\end{aligned}$$

(b) $\implies$ (a): The RHS of the equation is C^1 in t. The initial condition is also satisfied. Taking the derivate yields the $\frac{d}{dt}p_t(x,y) = -c(x)p_t(x,y) + \sum_{z\in S\setminus\{x\}} q(x,z)p_t(z,y)$ which is equivalent to the KBE. $\square$

We use proposition 1.8 to show existence of positive solution of the KBE. Idea: Use a **fixed point argument**:

- $p_t^{(0)}(x,y) = 0 \forall x, y \in \mathcal{S}, t \geq 0$
- For $n \geq 0$: $p_t^{(n+1)}(x,y) = \delta_{x,y}e^{c(x)t} + \int_0^t dse^{-c(x)(t-s)}\sum_{z\neq x} q(x,z)p_s^{(n)}(z,y)$

Lemma 1.9. $\forall n \geq 0$*:*

(a) $p_t^{(n)}(x,y) \geq 0 \forall x, y \in \mathcal{S}$

(b) $\sum_{y\in\mathcal{S}} p_t^{(n)}(x,y) \leq 1 \forall x \in \mathcal{S}, t \geq 0$

(c) $p_t^{(n+1)}(x,y) - p_t^{(n)}(x,y) \geq 0 \forall x, y \in \mathcal{S}, t \geq 0$

Proof. By induction. $n = 0$ is clearly satisfied for (a),(b),(c).
(a) is obvious, since RHS are all positive terms.

(b)

$$\begin{aligned}\sum_{y\in\mathcal{S}} p_t^{(n+1)}(x,y) &= e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z\neq x} q(x,z) \underbrace{\sum_{y\in\mathcal{S}} p_s^{(n)}(z,y)}_{\leq 1} \\ &\leq e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \underbrace{\sum_{z\neq x} q(x,z)}_{=c(x)} \\ &= e^{-c(x)t} + e^{-c(x)t} c(x) \int_0^t ds e^{c(x)s} = 1\end{aligned}$$

(c)

$$p_t^{(n+2)}(x,y) - p_t^{(n+1)}(x,y) = \int_0^t e^{-c(x)(t-s)} \sum_{z\neq x} q(x,z) \underbrace{\left(p_s^{(n+1)}(z,y) - p_s^{(n)}(z,y)\right)}_{\geq 0} \geq 0 \qquad \square$$

$\implies$ $\forall x, y, t$ we have $0 \leq p_t^0(x,y) \leq p_t^1(x,y) \leq \cdots \leq 1$.
$\implies$ there exists a limit, which we denote by $p_t^*(x,y)$

Definition 1.10. *The limit $p_t^*(x,y) := \lim_{n\to\infty} p_t^n(x,y)$ is called the **minimal solution of the KBE**.*

Question: Is p^* a transition function?

Theorem 1.11. *p^* satisfies:*

(a) $p_t^*(x,y) \geq 0$

(b) $\sum_{y\in S} p_t^*(x,y) \leq 1$

(c) The Chapman-Kolmogorov equation

(d) satisfies (1.3) and thus the KBE

Proof. (a) and (b) follow directly from lemma 1.9.
(d): Take the limits of $p_t^{(n)}(x,y)$ and use monotone convergence.
(c): Define $\Delta_t^{(n)} := p_t^{(n+1)}(x,y) - p_t^{(n)}(x,y) \geq 0$.
Notice $p_t^*(x,y) = \sum_{n\geq 0} \Delta_t^{(n)}(x,y)$ In lemma 1.12:
$Delta_{t+s}^{(n)}(x,y) = \sum_{z\in\mathcal{S}} \sum_{k=0}^n \Delta_s^{(k)}(x,z)\Delta_t^{(n-k)}(z,y)$ which then implies

$$\begin{aligned}p_{t+s}^*(x,y) = \sum_{n\geq 0} \Delta_{t+s}^{(n)}(x,y) &= \sum_{z\in\mathcal{S}} \sum_{n\geq 0} \sum_{k=0}^n \Delta_s^{(k)}(x,z)\Delta_t^{(n-k)}(z,y) \\ &= \sum_{z\in\mathcal{S}} \underbrace{\sum_{k\geq 0} \Delta_s^{(k)}(x,z)}_{p_s^*(x,z)} \underbrace{\sum_{n=k}^{\infty} \Delta_t^{(n-k)}(z,y)}_{p_t^*(z,y)} \\ &= \sum_{z\in\mathcal{S}} p_s^*(x,z) p_t^*(z,y)\end{aligned}$$

using monotone convergence. $\square$

Lemma 1.12. $\Delta_{t+s}^{(n)}(x,y) = \sum_{z\in\mathcal{S}} \sum_{k=0}^n \Delta_s^{(k)}(x,z)\Delta_t^{(n-k)}(z,y)$

Proof. Consider the Laplace transform of the equation w.r.t. s, t, since the functions are all positive!

$$\int_0^\infty ds \int_0^\infty dt e^{-\lambda s} e^{-\mu t \Delta_{s+t}^{(n)}} \stackrel{?}{=} \sum_{z\in\mathcal{S}} \sum_{k=0}^{n} \underbrace{\left[\int_0^\infty ds e^{-\lambda s} \Delta_s^{(k)}(x,z)\right]}_{\Psi_{k,\lambda}(x,z)} \underbrace{\left[\int_0^t e^{-\mu t} \Delta_t^{(n-k)}(z,y)\right]}_{\Psi_{n-k,\mu}(z,y)}$$

Define $\int_0^t ds e^{-\lambda s} \Delta_s^{(n)}(x,y) = \Psi_{n,\lambda}(x,y)$. Then for the RHS:

$$= \sum_{z\in\mathcal{S}} \sum_{k=0}^{n} \Psi_{k,\lambda}(x,z) \Psi_{n-k}\mu(z,y)$$

For the LHS:

$$\begin{aligned}\int_0^\infty ds ds \int_0^\infty dt e^{-\lambda(s+t)} e^{-t(\mu-\lambda)} \Delta^{(n)}_{\underbrace{s+t}_{=:u}}(x,y) &= \int_0^\infty ds \int_s^\infty du e^{-\lambda u} e^{-\mu(u-s)} \Delta_u^{(n)}(x,y) \\ &= \int_0^\infty du \int_u^\infty ds e^{-(\lambda-\mu)s} e^{-\mu u} \Delta_u^{(n)}(x,y) = \frac{\Psi_{n,\mu}(x,y) - \Psi_{n,\lambda}(x,y)}{\lambda - \mu} \\ &\stackrel{?}{=} \sum_{k=0}^{n} \sum_{z\in\mathcal{S}} \Psi_{k,\lambda}(x,y) \Psi_{n-k,\mu}(z,y)\end{aligned}$$

Another identity: $\Psi_{n+1,\lambda}(x,y) = \sum_{z\neq x} \frac{q(x,z)}{\lambda + c(x)} \Psi_{n,\lambda}(z,y)$

Define the matrix $A_\lambda(x,y) := \frac{q(x,y)}{\lambda+c(x)} 1_{z\neq x}$, then

Above the second equation is wrong, but the skip from the RHS to the 3rd line is correct

$$\Psi_{n,\lambda} = (A_\lambda)^n \Psi_{0,\lambda}$$

where $\Psi_{0,\lambda}(x,y) = \int_0^\infty ds e^{-\lambda s} p_s^{(1)}(x,y)$ and $p_s^{(1)}(x,y) = \delta_{x,y} e^{-c(x)s}$. Then $\Psi_{0,\lambda}(x,y) = \delta_{x,y} \frac{1}{\lambda + c(x)}$.
$\implies$

Reminder: For matrices $X^n - Y^n = (X-Y)X^{n-1} + Y(X-Y)X^{n-1} + \cdots + Y^{n-1}(X-Y)$, since they don't commute

$$\begin{aligned}\frac{\Psi_{n,\mu} - \Psi_{n,\lambda}}{\lambda - \mu} &= \frac{(A_\mu)^n \Psi_{0,\mu} - (A_\lambda)^n \Psi_{0,\lambda}}{\lambda - \mu} \\ &= \frac{(A_\lambda)^n (\Psi_{0,\mu} - \Psi_{0,\lambda})}{\lambda - \mu} \sum_{k=0}^{n-1} (A_\lambda)^k \frac{A_\mu - A_\lambda}{\lambda - \mu} A^{n-k-1} \Psi_{0,\mu}\end{aligned}$$

$$\frac{\Psi_{0,\mu} - \Psi_0, \lambda}{\lambda - \mu}(x,y) = \frac{\delta_{x,y}}{\lambda - \mu}\left(\frac{1}{c(x)+\mu} - \frac{1}{c(x)+\lambda}\right) = \Psi_{0,\mu}(x,y) \cdot \Psi_{0,\lambda}(x,y).$$

Similarly

$$\frac{A_\mu - A_\lambda}{\lambda - \mu}(x,y) = \left(\Psi_{0,\lambda} A_\mu\right)(x,y)$$

Then

$$\begin{aligned}\frac{(A_\lambda)^n (\Psi_{0,\mu} - \Psi_{0,\lambda})}{\lambda - \mu} \sum_{k=0}^{n-1} (A_\lambda)^k \frac{A_\mu - A_\lambda}{\lambda - \mu} A^{n-k-1} \Psi_{0,\mu} &= [(A_\lambda)^n \Psi_{0,\mu}](x,y) + \sum_{k=0}^{n-1} \sum_z \underbrace{\left[A_\lambda^k \Psi_{0,\lambda}\right]}_{=\Psi_{k,\lambda}(x,z)}(x,z) \underbrace{\left(A_\mu^{n-k} \Psi_{0,\mu}\right)(z,y)}_{=\Psi_{n-k,\mu}(z,y)} \\ &= \sum_{k=0}^{n} \sum_{\in\mathcal{S}} \Psi_{k,\lambda}(x,z) \Psi_{n-k,\mu(z,y)}\end{aligned}$$

□

Start of lecture 04 (17.10.2024)

Theorem 1.13. *(a) If $p_t(x,y)$ is a non-negative solution of the KBE 1.1 with $p_0(x,y) = \delta_{x,y} \implies p_t(x,y) \geq p_t^*(x,y)$*

(b) If $\sum_{y\in\mathcal{S}} p_t^*(x,y) = 1, \forall x \in \mathcal{S} \implies p_t^*(x,y)$ is the unique transition function satisfying the KBE

Proof. **(a):** By induction: $p_t(x,y) \geq p_t^{(n)}(x,y)\forall n$.
For $n = 0$ $p_t(x,y) \geq 0 = p_t^{(0)}(x,y)$.

$$p_t^{(n+1)}(x,y) = \delta_{x,y}e^{-c(x)t} + \int_0^t dse^{-c(x)(t-s)}\sum_{z\neq x} q(x,z)\underbrace{p_s^{(n)}(x,y)}_{\leq p_s(x,y)}$$
$$\overset{p_t\text{satisfies KBE}}{\leq} \quad p_t(x,y)$$

(b):

$$\begin{aligned}
&= \sum_{y\in\mathcal{S}} \underbrace{(p_t^*(x,y) - p_t^*(x,y))}_{\geq 0} \geq 0 \\
&= \underbrace{\sum_{y\in\mathcal{S}} p_t(x,y)}_{=1} - \underbrace{\sum_{y\in\mathcal{S}} p_t^*(x,y)}_{=1} \\
&\implies 0 \geq \sum_{y\in\mathcal{S}} (p_t^*(x,y) - p_t^*(x,y)) \geq 0 \\
&\implies p_t(x,y) = p_t^*(x,y)
\end{aligned}$$

□

Example 1.14. $\mathcal{S} = \{0,1,\ldots,\}, q(i,j) = \begin{cases} \beta_i & j = i+1 \\ -\beta_i & j = i \\ 0 & \text{otherwise} \end{cases}$ *for some positive β_i. One shows $\sum_{j\in\mathcal{S}} p_t^*(i,j) = 1\forall t, i \in S \iff \sum_{k\geq 0}\frac{1}{\beta_k} < \infty$.*
What happens if $\sum_{k\geq 0}\frac{1}{\beta_i} = \infty$? Consider $N(t) = \#$ of jumps before time t,then $\mathbb{P}(n(t) = \infty) > 0$.

1.4.2 The probabilistic construction

Lemma 1.15. *Let $X(t)$ be a continuous time M.C. and $\tau := \inf\{t : X(t) \neq X(0)\}$. Then there exists $c(x) = [0,\infty]$ s.t. $\mathbb{P}(\tau > t) = e^{-c(x)t}$*

Proof. Use Markov property with $Y = 1_{X(r)=x, 0\leq r\leq t-s}$:

$$\begin{aligned}
\mathbb{P}^x\left(X(r) = x, s \leq r \leq t \mid \mathcal{F}_s\right) &= \mathbb{P}^{X(s)}(\tau > t-s) = (\star) \\
\mathbb{E}^x(1_{\tau>s}\cdot\mathbb{P}^x(X(r) = x, s\leq r\leq t \mid \mathcal{F}_s)) &= \mathbb{P}^x(\tau > t) \\
\mathbb{E}^x(1_{\tau>s}\cdot\mathbb{P}^x(X(r) = x, s\leq r\leq t \mid \mathcal{F}_s)) &\overset{(\star)}{=} \mathbb{E}^x\left(1_{\tau>s}\mathbb{P}^{X(s)}(\tau > t-s)\right) \\
&= \mathbb{P}^x(\tau > s)\mathbb{P}^x(\tau > t-s) \\
&\implies \mathbb{P}^x(\tau > t) = \mathbb{P}^x(\tau > s)\mathbb{P}(\tau > t-s)
\end{aligned}$$

Either $\mathbb{P}^x(\tau > t) = 0\forall t \implies c(x) = \infty$ (jump before t) or $\mathbb{P}^x(\tau > t) > 0\forall t$. Define $f(t) := -\ln\mathbb{P}^x(\tau > t)$ with $f(0) = 0 \implies f(t) \geq 0$. and
$f(t) = f(s) + f(t-s) \implies f(t) = k\cdot t$. □

Idea: Divide the waiting times (which are exponetially distributed $\sim \exp(c(x)), \exp(c(y)), \ldots$) and the **jumps**. The jumps are given by a discrete time M.C..
(1) Discrete time Markov chain $\{Z_n, n \geq 0\}$ with transition matrix:

$$P = (p(x,y))_{x,y\in\mathcal{S}}$$

If $c(x) = 0$:

$$p(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

If $c(x) > 0$:

$$p(x,y)\begin{cases}\frac{q(x,y)}{c(x)} & x \neq y\\ 0 & x = y\end{cases}.$$

(2) Initial distribution: $\pi = (\pi(x))_{x\in\mathcal{S}}$ is a probability measure.
(3) Family of **independent** exponetially distributed waiting timesindependent $\tau_{x,i} \sim \exp(c(x)), x \in \mathcal{S}, i \geq 1$.
If $c(x) = 0 \to \tau_{x,i} = \infty$.
The joint distribution of Z_n are given by

$$\mathbb{P}(Z_0 = x_0, \ldots, Z_n = x_n) = \pi(x_0)p(x_0,x_1)\cdot\cdots\cdot p(x_{n-1},x_n)$$

Remark. *Careful, the waiting times are only independent given the positions of Z_n*

$$\mathbb{P}(Z_0 = x_0, Z_1 = x_1, \ldots, Z_n = x_n, \tau_0 > t_0, \tau_1 > t_1, \ldots, \tau_n > t_n) = \pi(x_0)p(x_0,x_1)\cdot\cdots\cdot p(x_{n-1},x_n)e^{-c(x_0)t_0}\cdot\cdots\cdot e^(-c(x_n)t_n)$$

where tau_k is the waiting time of the $(k-1)$th jump.

This should be $k+1$, otherwise what does τ_{-1} mean?

Denote by $N(t) := \begin{cases}\min(n \geq 0) \mid \tau_0 + \tau_1 + \cdots + \tau_n > t & \sum_{k\geq 0}\tau_k > t\\ \infty & \textit{otherwise}\end{cases}$

$N(0) = 0$.

Definition 1.16. *We define the continuous time Markov chain $X(t)$ by $X(t) = Z_{N(t)}$ on $\{N(t) < \infty\}$.*

Proposition 1.17. *(a) $p_t^{(n)}(x,y) = \mathbb{P}(X(t) = y, N(t) < n \mid X(0) = x)$*

(b) $p_t^*(x,y) = \mathbb{P}(X(t) = y, N(t) < \infty \mid X(0) = x)$

(c) $\sum_{y\in\mathcal{S}} p_t^*(x,y) = \mathbb{P}(N(t) < \infty \mid X(0) = x)$

Proof. (b) and (c) follow from (a), therefore only (a) is proofed.
(a): By deriving the same iteration as $p_t^{(+1)}(x,y) = \ldots$. For $c(x) > 0$ Condition on τ_0, Z_0, Z_1 :

$$\mathbb{P}(X(t) = y, N(t) < n+1 \mid \tau_0 = s, Z_1 = z, \underbrace{Z_0}_{=X(0)} = x) = \begin{cases}\delta_{x,y} & s > t\\ \mathbb{P}(X(t-s) = y, N(t-s) < n \mid X(0) = z) & s < t\end{cases}$$

$$\begin{aligned}
&\mathbb{P}(X(t) = y, N(t) < n+1 \mid X(0) = x)\\
&= \int_0^\infty \sum_{z\in\mathcal{S}} \mathbb{P}(X(t) = y, N(t) < n+1 \mid \tau_0 = s, Z_1 = z, X(0) = x)\cdot \underbrace{\mathbb{P}(\tau_0 \in ds, Z_1 = z \mid X(0) = x)}_{\text{density: } p(x,z)e^{-c(x)s}c(x)}\\
&= \int_t^\infty ds\delta_{x,y}\cdot \underbrace{\sum_{z\in\mathcal{S}} p(x,z)}_{=1} e^{-c(x)s}c(x) + \int_0^t ds\sum_{z\in\mathcal{S}}\mathbb{P}(X(t-s) = y, N(t) < n \mid X(0) = z)p(x,z)c(x)e^{-c(x)s}\\
&= \delta_{x,y}e^{-c(x)t} + \int_0^t dse^{-c(x)\overbrace{s}^{t-s}} \sum_{z\in\mathcal{S}\setminus\{x\}} \mathbb{P}(X(\overbrace{t-s}^{s}) = y, N(t) < n \mid X(0) = z)q(x,z)\\
&\overset{s\mapsto t-s}{=} \delta_{x,y}e^{-c(x)t} + \int_0^t dse^{-c(x)(t-s)} \sum_{q(x,z)} \mathbb{P}(X(s) = y, N(s) < n \mid X(0) = z)
\end{aligned}$$

$\implies$ Same I.C. and recursion as for $p_t^{(n)}(x,y)$. ∎

Theorem 1.18. *The following are equivalent:*

(a) The minimal solution $p_t^*(x,y)$ is stochastic

(b) $\mathbb{P}(N(t) < \infty) = 1 \forall t$

(c) $\sum_{n \geq 0} \tau_n = \infty \ a.s.$

(d) $\sum_{n \geq 0} \frac{1}{Z(n)} = \infty \ a.s.$

List of Lectures