Lecture notes on Markov Processes

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Based on the lectures of
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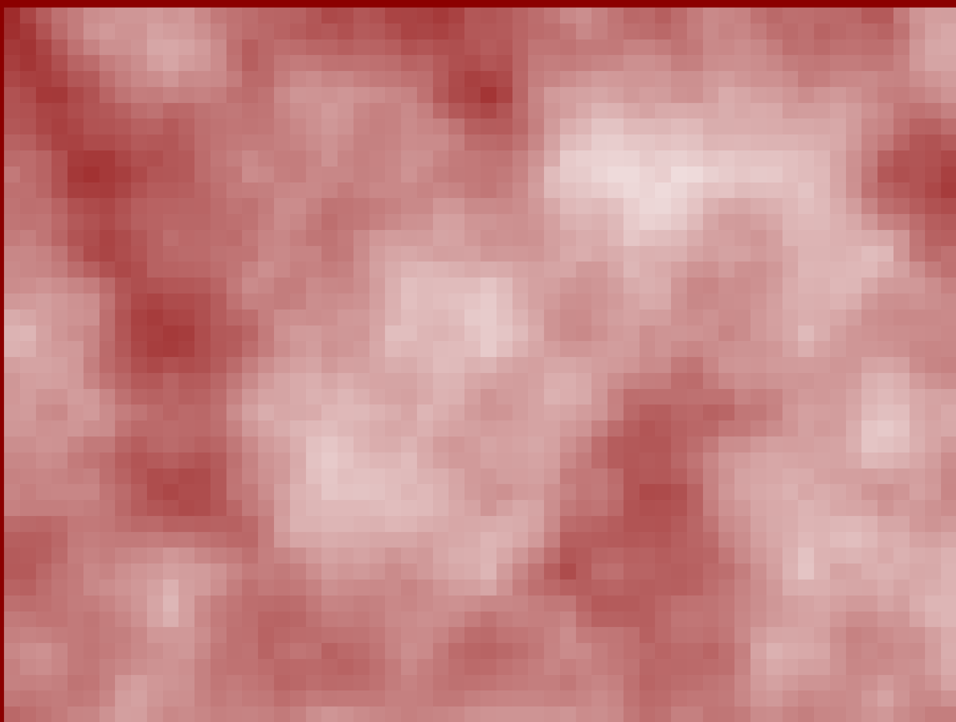

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Chapter 0: Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Ecampus: Ecampus link
- Basis: Basis link
- Website: None
- Time slot(s): **Tuesday 12-14** and **Thursday 12-14**
- Exams: Oral
- Exercises: To be handed in each **Friday** until **noon**

There are tutorials in the second week (14.10.24 and 16.10.24)!

Start of lecture 01 (08.10.2024)

Chapter 1: Continuous time Markov chains

Two motivating examples:

1.1 Markov chain, transition function, infinitesimal description

We have a **countable state space $\mathcal{S}$** and define

- $\Omega = \{$right-continuous functions $\omega : \mathbb{R}_+ \to \mathcal{S}$ with finitely many jumps in any finite time intervals$\}$
- σ-algebra $\mathcal{F} := \sigma(\{\omega \to \omega(t)$ which is measurable $\forall t \geq 0\})$ measurable w.r.t.?
- time shift operator: $(\theta_s \omega)(t) := \omega(s+t)$

Definition 1.1. *Denote $X(t,\omega) := \omega(t)$. Assume we have a collection of probability measures $\{\mathbb{P}^x, x \in \mathcal{S}\}$ on $(\Omega, \mathcal{F})$, a right-continuous filtration $\{\mathcal{F}_t, t \geq 0\}$ on $(\Omega, \mathcal{F})$. X is a* ***continuous time Markov chain (MC)*** *if*

(a) X is adapted to $\{\mathcal{F}_t, t \geq 0\}$

(b) ***Initial condition:*** $\mathbb{P}^x(X(0) = x) = 1$

(c) ***Markov property:*** *$\forall x \in \mathcal{S}, Y$ measurable function on $\Omega, s \geq 0$:*

$$\mathbb{E}^x\left(Y \circ \theta_s \mid \mathcal{F}_s\right) = \mathbb{E}^{X(s)}(Y), \mathbb{P}^x \ a.s.$$

Example. $Y = \max_{0 \leq t \leq 1} X(t) \to Y \circ \theta_s = \max_{s \leq t \leq s+1} X(t)$

Here: Time homogenous processes!

Definition 1.2. *A* ***transition function*** *is function $p_t(x,y), x, y \in \mathcal{S}, t \geq 0$ s.t.*

(a) ***Positivity:*** $p_t(x,y) \geq 0$

(b) ***Normalized:*** $\sum_{y \in \mathcal{S}} p_t(x,y) = 1$ *We will see while constructing, that this part is tricky, might just be ≤ 1*

(c) ***Continuity:*** $\lim_{t \downarrow 0} p_t(x,x) = p_0(x,x) = 1$

(d) ***Chapman-Kolmogorov equation:*** $\forall s,t \geq 0 \ \forall x,y \in \S$:

$$p_{s+t}(x,y) = \sum_{z \in \mathcal{S}} p_s(x,z) p_t(z,y)$$

Given a transition function, we can construct $\mathbb{P}^x$ as follows:
Finite dimensional distributions, $0 < t_1 < \cdots < t_n$:

$$\mathbb{P}^x(X(t_1) = x_1, X(t_2 = x_2), \ldots, X(t_n) = x_n) := p_{t_1}(x,x_1) \cdot p_{t_2 - t_1}(x_1,x_2) \cdot \cdots \cdot p_{t_n - t_{n-1}(x_{n-1},x_n)}$$

extend to full time $\mathbb{R}_+$ by the Kolmogorov(-Daniell) extension theorem, where the consistency,$\mathbb{P}^x(X(t) = y) = p_t(x, y)$, relations follow by the Chapman-Kolmogorov equation.

Example. $0 < t_2 < \cdots < t_n$

We are still in the same setting, therefore we use the same t_1 as previously

$$\sum_{x_1 \in \mathcal{S}} \mathbb{P}^x \left(X(t_1) = x_1, \ldots, X(t_n) = x_n\right) = \mathbb{P}^x \left(X(t_2) = x_2, \ldots, X(t_n) = x_n\right)$$

By modeling we often think at the basic biological/ physical properties of the system $\rightarrow$ typically we have *transition rates*, because for $x \neq y$: $\begin{cases} p_\epsilon(x, y) = O(\epsilon), & p_\epsilon(x, x) = 1 - O(\epsilon) \\ p_0(x, y) = 0, & p_0(x, x) = 1 \end{cases}$

Definition 1.3. *For a Markov chain X we define the **transition rates** from x to y ($x \neq y$) by:*

$$\tilde{q}(x, y) =:= \frac{d}{dt} p_t(x, y) \mid_{t=0}$$

Definition 1.4. *A **Q-matrix (or generator)** is a collection of numbers $\{q(x, y) : x, y \in \mathcal{S}\}$ s.t.:*

(a) $q(x, y) \geq 0 \forall x \neq y$

(b) $\sum_{y \in \mathcal{S}} q(x, y) = 0$, *Notation:* $c(x) := q(x, x) = \sum_{y \in \mathcal{S} \setminus \{x\}} q(x, y)$

$c(x)$ is the rate of leaving site x

> **Warning**
>
> It is not automatic that transition rates gives a Q-matrix For instance: $\tilde{q}(x, y) = \infty$

For **finite** state space,
Markov chain $\iff$ Transition function $\iff$ Q-matrix
Goal: Under which condition is the equivalence still true?

1.2 Examples

Example (Discrete to continuous MC). ***Given:*** *Markov chain Y in discrete time $t \in \mathbb{Z}_+$, i.e. transition matrix $P = (P(x, y))_{x, y \in \mathcal{S}}$*

$$\mathbb{P}(Y(n+1) = y \mid Y(n) = x) = P(x, y)$$

Consider a Poisson process(PP) with intensity 1, at each event time of the PP, there will be a jump of the continuous time MC X. The jumps follow the discrete time MC Y
$\implies p_t(x, y) = \sum_{n \geq 0} \frac{e^{-t} t^n}{n!} P^n(x, y)$
in this case $Q = P - 1$

Example (Finite $\mathcal{S}$). $p_t(x, y) = \left(e^{tQ}(x, y)\right) := \sum_{n \geq 0} \frac{t^n}{n!} Q^n(x, y)$

Here will be one of the main problems: If Q is finite / a normed operator, everything is well defined. Otherwise we have to use a different definition of the exponetial …

Example (Birth and death processes). $\mathcal{S} = \{0, 1, \ldots, \}, X(t) =$ *Population size at time t. Then*

- $q(k, k+1) = \rho_k, k \geq 0$
- $q(k+1, k) = \lambda_k, g \geq 1$
- $q(k, k) = -\rho_k - \lambda_k$ *with* $\lambda_0 = 0$
- *which implies* $q(k, l) = 0 \forall |k - l| \geq 2$

Depending on the choice of ρ_k, λ_k it is possible that the chain goes to ∞ in finite time.

Start of lecture 02 (10.10.2024)

1.3 From MC to Q-matrices

1.3.1 MC $\rightarrow$ Transition functions

Theorem 1.5. *Let X be a Markov chain. Then $p_t(x,y) := \mathbb{P}^x(X(t)=y)$.*

(a) $p_t(x,y)$ is a transition function

(b) $p_t(x,y)$ determines uniquely $\mathbb{P}^x$

Proof. **(a):** Positivity and normalization follows by the properties of $\mathbb{P}^x$.
$\tau := \inf\{t \in \mathbb{R}_+ \mid X(t) \neq X(0)\}$ is a.s. positive. $p_t(x,x) \geq \mathbb{P}^x(\tau > t)$, which implies
$1 = p_0(x,x) \geq \lim_{t\downarrow 0} p_t(x,x) \geq \lim_{t\downarrow 0} \mathbb{P}^x(\tau > t) = 1$.
We still need to verify the Chapman-Kolmogorov equation: Define
$Y = 1_{X(t)=y} \implies Y \circ \theta_s = 1_{X(t+s)=y}$

$$\begin{aligned} p_{s+t}(x,y) &= \mathbb{P}^x(X(s+t)=y) = \mathbb{E}^x\left(1_{X(s+t)=y}\right) \\ &= \mathbb{E}^x\left(\mathbb{E}^x(1_{X(s+t)=y} \mid \mathcal{F}_s)\right) \\ &\overset{\text{MP}}{=} \mathbb{E}^x\left(\mathbb{E}^{X(s)}(1_{X(t)=y})\right) \\ &= \sum_{z\in S} p_s(x,z) \underbrace{\mathbb{E}^z\left(1_{X(t)=y}\right)}_{=p_t(z,y)} \end{aligned}$$

(b): By the Markov property $\forall 0 < t_1 < \cdots < t_n, x_1, \ldots, x_n \in S$

$$\mathbb{P}^x\left(X(t_1)=x_1, \ldots X(t_n)=x_n\right) = p_{t_1}(x_1,x_1) \cdot p_{t_2-t_1}(x_1,x_2) \cdot \cdots \cdot p_{t_n-t_{n-1}}(x_{n-1},x_n)$$

using the Kolmogorov extension theorem $\mathbb{P}^x$ is uniquely determined by p_t. □

1.3.2 Transition functions to Q-matrices

Proposition 1.6. *Let $p_t(x,y)$ be a transition function.*

(a) $\forall t \geq 0, x \in S, p_t(x,x) > 0$

(b) If $p_t(x,x) = 1$ for some $t > 0$, then $p_s(x,x) = 1 \forall s \geq 0$

(c) $\forall x,y \in S, p_t(x,y)$ is uniformly continuous in t, due to

$$|p_{t+s}(x,y) - p_t(x,y)| \leq 1 - p_s(x,x)$$

Remark. *(a) differs from the discrete time markov chain $\to$ no issues of periodicity*

(b) The M.C. if it reaches site x, stays forever in x. x is an ***absorbing state***

Proof. **(a):** Since $\lim_{x\downarrow 0} p_t(x,x) = 1 \implies p_s(x,x) > 0$ for $s \in [0,t]$ for some small t. Therefore
$\forall s \in [0,t] : p_{t+s}(x,x) \overset{\text{C.Kolmogorov}}{\geq} p_t(x,x)p_s(x,x) > 0$, which we can iterate to get the claim.
(b): $p_{s+t}(x,x) = \sum_{z\in S} p_s(x,z)p_t(z,x) \leq p_s(x,x)p_t(x,x) + \underbrace{\sum_{z\neq x} p_s(x,z) \cdot 1}_{=1-p_s(x,x)}$ Then

still good, since $p_s(x,z)$ will be 0

$$p_{s+t}(x,x) \leq 1 - \underbrace{p_s(x,x)}_{\overset{(a)}{\geq} 0}(1 - p_t(x,x)).$$

If $p_{s+t}(x,x) = 1 \implies p_t(x,x) = 1$.
$\forall v \in [t+s, 2(t+s)] : p_v(x,x) \geq p_{\underbrace{v-(t+s)}_{\in[0,t+s]}}(x,x)p_{t+s}(x,x) = 1 \cdot 1 = 1$

(c):

$$\begin{aligned} p_{s+t}(x,y) - p_t(x,y) &= \sum_{z\in S} p_s(x,z)p_t(s,y) - p_t(x,y) \\ &= \underbrace{(p_s(x,x)-1)p_t(x,y)}_{\leq 0} + \underbrace{\sum_{z\neq x} p_s(x,z)p_t(z,y)}_{\geq 0} \end{aligned}$$

Since

$$|p_s(x,x)-1|p_t(x,y) \leq 1-p_s(x,x)$$

and

$$\sum_{z\neq x} p_s(x,z)\underbrace{p_t(z,y)}_{\leq 1} \leq 1-p_s(x,x)$$

$$\implies |p_{s+t}(x,y)-p_t(x,y)| \leq 1-p_s(x,y)$$

□

Theorem 1.7. *Let $p_t(x,y)$ be a transition function.*

(a) $\forall x \in S$ the right derivative

$$\tilde{c}(x) = -\tilde{q}(x,x) = -\frac{d}{dt}p_t(x,x)\mid_{t=0} \in [0,\infty]$$

exists and $p_t(x,x) \geq e^{-\tilde{c}(x)t}, \forall t>0$

(b) I tildec $<\infty \implies \forall y \neq x$, the right-derivative

$$\tilde{q}(x,y) \coloneqq \frac{d}{dt}p_t(x,y)\mid_{t=0} \in [0,\infty]$$

exists and $\sum_{y\in S}\tilde{q}(x,y) \leq 0$

*(c) If for some $x \in S, \tilde{c}(x)<\infty$ and $\sum_{y\in S}\tilde{q}(x,y)=0$, then $p_t(x,y)$ is C^1 in time for this x and $y \in S$. Moreover, it satisfies the **Kolmogorov- backwards equation**:*

$$\frac{d}{dt}p_t(x,y) = \sum_{z\in S}\tilde{q}(x,z)p_t(z,y) \tag{1.1}$$

Proof. **(a)** Let $f(t) \coloneqq -\ln(p_t(x,x)) \geq 0$. By $p_{s+t}(x,x) \geq p_s(x,x)p_t(x,x)$ it follows

$$f(t+s) \leq f(s)+f(t),$$

the function f is subadditive, which implies

In particular, the limit exists

$$\lim_{t\downarrow 0}\frac{f(t)}{t} = \sup_{t>0}\frac{f(t)}{t} \in [0,\infty]$$

$$\implies \lim_{t\downarrow 0}\frac{f(t)}{t} = \frac{d}{dt}f(t)\mid_{t=0} = \frac{-\frac{d}{dt}p_t(x,x)}{\underbrace{p_t(x,x)}_{\to 1}}\mid_{t=0} = \tilde{c}(x)$$

$$\implies \tilde{c}(x) \geq \frac{f(t)}{t} = -\frac{\ln p_t(x,x)}{t} \to p_t(x,x) \geq e^{-\tilde{c}(x)t}$$

(b) by (a): $1-p_t(x,x) \leq 1-e^{-\tilde{c}(x)t} \leq \tilde{c}(x)t$, which implies

$$\sum_{y\neq x} p_t(x,y) \leq \tilde{c}(x)t.$$

$$\implies \tilde{q}(x,y) \coloneqq \limsup_{t\downarrow 0}\frac{p_t(x,y)}{t} \in [0,\underbrace{\tilde{c}(x)}_{<\infty}]$$

Let $\delta>0, n\in\mathbb{N}$: Let $p_\delta = \{p_\delta(x,y)\}_{x,y\in S}$ be a discrete time chain.

$$p_{n\delta}(x,y) \geq \sum_{k=0}^{n-1}\underbrace{p_\delta(x,x)^k}_{\geq p_\delta(x,x)^n \geq e^{-\tilde{c}(x)n\delta}} p_\delta(x,y)p_{(n-k-1)\delta}(y,y)$$

$$\implies \frac{p_{n\delta}(x,y)}{n\delta} \geq \frac{p_{\delta(x,y)}}{\delta}e^{-\tilde{c}(x)n\delta}\cdot \inf_{0\leq s\leq n\delta}p_s(y,y)$$

Take a subsequence of $\delta \downarrow 0$ such that $n\delta \to t$ and $\lim_{\delta\downarrow 0} \frac{p_\delta(x,y)}{\delta} = \tilde{q}(x,y)$

$$\implies \liminf_{t\downarrow 0} \frac{p_t(x,y)}{t} \geq \tilde{q}(x,y)\cdot 1 \cdot 1$$

which implies that the limit exists.

$$\begin{aligned}
&\sum_{y\neq x} \frac{p_t(x,y)}{t} \leq \tilde{c}(x) \\
&\qquad \tilde{c}(x) \geq \liminf_{t\downarrow 0} \sum_{y\neq x} \frac{p_t(x,y)}{t} \overset{\text{Fatou}}{\geq} \sum_{y\neq x} \liminf_{t\downarrow 0} \frac{p_t(x,y)}{t} = \sum_{y\neq x} \tilde{q}(x,y) \\
&\qquad \implies \sum_y \tilde{q}(x,y) \leq 0
\end{aligned}$$

(c) By Chapman-Kolmogorov:

$$\begin{aligned}
&\frac{p_{t+\epsilon}(x,y) - p_t(x,y)}{\epsilon} - \sum_z \tilde{q}(x,z)p_t(z,y) \\
&= \sum_{z\in S} \underbrace{\left(\frac{p_\epsilon(x,z) - \overbrace{p_0(x,z)}^{\delta_{x,z}}}{\epsilon} - \tilde{q}(x,z) \right)}_{=(\star)} p_t(z,y)
\end{aligned}$$

For all $z : \lim_{\epsilon\to 0}(\star) = 0$
Take $T \subset S$, $|T| < \infty \implies \lim_{\epsilon\downarrow 0} \sum_{z\in T} (\star)\cdot p_t(z,y) = 0$. Let $x \in T$.

$$\begin{aligned}
&\sum_{z\notin T} \left| \frac{p_\epsilon(x,z)}{\epsilon} - \tilde{q}(x,z) \right| \underbrace{p_(z,y)}_{\leq 1} \\
&\leq \sum_{z\notin T} \frac{p_\epsilon(x,z)}{\epsilon} + \sum_{z\notin T} \tilde{q}(x,z) \overset{\sum q(x,\tilde{z})=0}{=} \frac{1}{\epsilon}\left(1 - \sum_{z\in T} p_\epsilon(x,z)\right) - \sum_{z\in T} \tilde{q}(x,z)
\end{aligned}$$

But $\sum_x \tilde{q}(x,z) = 0 \overset{\epsilon\downarrow 0}{\to} -2\sum_{z\in T} \tilde{q}(x,z) \overset{T\uparrow S}{\to} 0$ $\square$

Start of lecture 03 (15.10.2024)

Remark. *For M.C. also the* ***strong Markov property*** *holds:*
Let τ be a stopping time and $Y : \mathbb{R}_+ \times \Omega \mapsto Y_t(\omega)$[a] measurable, then

$$\mathbb{E}^x(Y \circ \theta_\tau \mid \mathcal{F}_t) = \mathbb{E}^x(Y_\tau), \mathbb{P} - \quad a.s.$$

on the set $\{\tau < \infty\}$.

[a] shorthand notation ...

$$\frac{d}{dt}p_t(x,y) = \lim_{\epsilon\downarrow 0} \frac{p_{t+\epsilon}(x,y) - p_t(x,y)}{\epsilon}$$

using $\sum_z p_t(x,z)p_\epsilon(z,y)$ instead of $\sum_z p_\epsilon(x,z)p_t(z,y)$ we get the **Kolmogorov Forward Equation**

$$\frac{d}{dt}p_t(x,y) = \sum_{z\in\mathcal{S}} p_t(x,z)q(z,y) \tag{1.2}$$

They are almost always equivalent. In computation one often uses the forward equation, in construction the backwards equation is preffered

1.4 From Q-matrix to the Markov Chains

Let $Q = (q(x,y))_{x,y\in\mathcal{S}}$ be a Q-matrix.

1.4.1 The backwards equation

Proposition 1.8. *Let $p_t(x,y)$ be a uniformly bounded function of x, y, t, then (a) is equivalent to (b):*

(a) $p_t(x,y)$ is C^1 in t, satisfies the KBE[a], with initial condition $p_0(x,y) = \delta_{x,y}$

(b) $p_t(x,y)$ is C^0 in t, and satisfies:

$$p_t(x,y) = \delta_{x,y} e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z\in\mathcal{S}\setminus\{x\}} q(x,z) p_s(z,y) \tag{1.3}$$

[a] Kolmogorov Backwards Equation

Proof. (a) $\implies$ (b): KBE $\iff$

$$\begin{aligned}
&\frac{d}{dt} p_t(x,y) = -c(x)p_t(x,y) + \sum_{z\in S\setminus\{x\}} q(x,z)p_t(z,y) \\
\implies &\frac{d}{dt}\left(e^{c(x)t}p_t(x,y)\right) = c(x)e^{c(x)t}p_t(x,y) + e^{c(x)t}\left(-c(x)p_t(x,y) + \sum_{z\neq x} q(x,z)p_t(z,y)\right) \\
&= e^{c(x)t}\sum_{z\neq x} q(x,z)p_t(z,y) \\
&\overset{\int_0^t}{=} e^{c(x)t}p_t(x,y) - p_0(x,y) = \int_0^t e^{c(x)s}\sum_{z\neq x} q(x,z)p_s(z,y)
\end{aligned}$$

(b) $\implies$ (a): The RHS of the equation is C^1 in t. The initial condition is also satisfied. Taking the derivate yields the $\frac{d}{dt}p_t(x,y) = -c(x)p_t(x,y) + \sum_{z\in S\setminus\{x\}} q(x,z)p_t(z,y)$ which is equivalent to the KBE. $\square$

We use proposition 1.8 to show existence of positive solution of the KBE. Idea: Use a **fixed point argument**:

- $p_t^{(0)}(x,y) = 0 \forall x, y \in \mathcal{S}, t \geq 0$
- For $n \geq 0 : p_t^{(n+1)}(x,y) = \delta_{x,y}e^{c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z\neq x} q(x,z)p_s^{(n)}(z,y)$

Lemma 1.9. $\forall n \geq 0$*:*

(a) $p_t^{(n)}(x,y) \geq 0 \forall x,y \in \mathcal{S}$

(b) $\sum_{y\in\mathcal{S}} p_t^{(n)}(x,y) \leq 1 \forall x \in \mathcal{S}, t \geq 0$

(c) $p_t^{(n+1)}(x,y) - p_t^{(n)}(x,y) \geq 0 \forall x,y \in \mathcal{S}, t \geq 0$

Proof. By induction. $n = 0$ is clearly satisfied for (a),(b),(c).
(a) is obvious, since RHS are all positive terms.

(b)

$$\begin{aligned}\sum_{y\in\mathcal{S}} p_t^{(n+1)}(x,y) &= e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \sum_{z\neq x} q(x,z) \underbrace{\sum_{y\in\mathcal{S}} p_s^{(n)}(z,y)}_{\leq 1} \\ &\leq e^{-c(x)t} + \int_0^t ds e^{-c(x)(t-s)} \underbrace{\sum_{z\neq x} q(x,z)}_{=c(x)} \\ &= e^{-c(x)t} + e^{-c(x)t} c(x) \int_0^t ds e^{c(x)s} = 1\end{aligned}$$

(c)

$$p_t^{(n+2)}(x,y) - p_t^{(n+1)}(x,y) = \int_0^t e^{-c(x)(t-s)} \sum_{z\neq x} q(x,z) \underbrace{\left(p_s^{(n+1)}(z,y) - p_s^{(n)}(z,y)\right)}_{\geq 0} \geq 0 \qquad \square$$

$\implies$ $\forall x,y,t$ we have $0 \leq p_t^0(x,y) \leq p_t^1(x,y) \leq \cdots \leq 1$.
$\implies$ there exists a limit, which we denote by $p_t^*(x,y)$

Definition 1.10. *The limit* $p_t^*(x,y) := \lim_{n\to\infty} p_t^n(x,y)$ *is called the* ***minimal solution of the KBE****.*

Question: Is p^* a transition function?

Theorem 1.11. p^* *satisfies:*

(a) $p_t^*(x,y) \geq 0$

(b) $\sum_{y\in S} p_t^*(x,y) \leq 1$

(c) The Chapman-Kolmogorov equation

(d) satisfies (1.3) and thus the KBE

Proof. (a) and (b) follow directly from lemma 1.9.
(d): Take the limits of $p_t^{(n)}(x,y)$ and use monotone convergence.
(c): Define $\Delta_t^{(n)} := p_t^{(n+1)}(x,y) - p_t^{(n)}(x,y) \geq 0$.
Notice $p_t^*(x,y) = \sum_{n\geq 0} \Delta_t^{(n)}(x,y)$ In lemma 1.12:
$Delta_{t+s}^{(n)}(x,y) = \sum_{z\in\mathcal{S}} \sum_{k=0}^n \Delta_s^{(k)}(x,z)\Delta_t^{(n-k)}(z,y)$ which then implies

$$\begin{aligned}p_{t+s}^*(x,y) &= \sum_{n\geq 0} \Delta_{t+s}^{(n)}(x,y) = \sum_{z\in\mathcal{S}} \sum_{n\geq 0} \sum_{k=0}^n \Delta_s^{(k)}(x,z)\Delta_t^{(n-k)}(z,y) \\ &= \sum_{z\in\mathcal{S}} \underbrace{\sum_{k\geq 0} \Delta_s^{(k)}(x,z)}_{p_s^*(x,z)} \underbrace{\sum_{n=k}^{\infty} \Delta_t^{(n-k)}(z,y)}_{p_t^*(z,y)} \\ &= \sum_{z\in\mathcal{S}} p_s^*(x,z) p_t^*(z,y)\end{aligned}$$

using monotone convergence. $\square$

Lemma 1.12. $\Delta_{t+s}^{(n)}(x,y) = \sum_{z\in\mathcal{S}} \sum_{k=0}^n \Delta_s^{(k)}(x,z)\Delta_t^{(n-k)}(z,y)$

Proof. Consider the Laplace transform of the equation w.r.t. s, t, since the functions are all positive!

$$\int_0^\infty ds \int_0^\infty dt e^{-\lambda s} e^{-\mu t \Delta_{s+t}^{(n)}} \stackrel{?}{=} \sum_{z\in\mathcal{S}} \sum_{k=0}^{n} \underbrace{\left[\int_0^\infty dse^{-\lambda s}\Delta_s^{(k)}(x,z)\right]}_{\Psi_{k,\lambda}(x,z)} \underbrace{\left[\int_0^t e^{-\mu t}\Delta_t^{(n-k)}(z,y)\right]}_{\Psi_{n-k,\mu}(z,y)}$$

Define $\int_0^t dse^{-\lambda s}\Delta_s^{(n)}(x,y) = \Psi_{n,\lambda}(x,y)$. Then for the RHS:

$$= \sum_{z\in\mathcal{S}} \sum_{k=0}^{n} \Psi_{k,\lambda}(x,z)\Psi_{n-k}\mu(z,y)$$

For the LHS:

$$\begin{aligned}
\int_0^\infty dsds \int_0^\infty dte^{-\lambda(s+t)}e^{-t(\mu-\lambda)}\Delta^{(n)}_{\underbrace{s+t}_{=:u}}(x,y) &= \int_0^\infty ds \int_s^\infty due^{-\lambda u}e^{-\mu(u-s)}\Delta_u^{(n)}(x,y) \\
&= \int_0^\infty du \int_u^\infty dse^{-(\lambda-\mu)s}e^{-\mu u}\Delta_u^{(n)}(x,y) = \frac{\Psi_{n,\mu}(x,y)-\Psi_{n,\lambda}(x,y)}{\lambda-\mu} \\
&\stackrel{?}{=} \sum_{k=0}^{n}\sum_{z\in\mathcal{S}} \Psi_{k,\lambda}(x,y)\Psi_{n-k,\mu}(z,y)
\end{aligned}$$

Another identity: $\Psi_{n+1,\lambda}(x,y) = \sum_{z\neq x} \frac{q(x,z)}{\lambda+c(x)}\Psi_{n,\lambda}(z,y)$
Define the matrix $A_\lambda(x,y) \coloneqq \frac{q(x,y)}{\lambda+c(x)}1_{z\neq x}$, then

$$\Psi_{n,\lambda} = (A_\lambda)^n\Psi_{0,\lambda}$$

Above the second equation is wrong, but the skip from the RHS to the 3rd line is correct

where $\Psi_{0,\lambda}(x,y) = \int_0^\infty dse^{-\lambda s}p_s^{(1)}(x,y)$ and $p_s^{(1)}(x,y) = \delta_{x,y}e^{-c(x)s}$. Then $\Psi_{0,\lambda}(x,y) = \delta_{x,y}\frac{1}{\lambda+c(x)}$.
$\implies$

$$\begin{aligned}
\frac{\Psi_{n,\mu}-\Psi_{n,\lambda}}{\lambda-\mu} &= \frac{(A_\mu)^n\Psi_{0,\mu}-(A_\lambda)^n\Psi_{0,\lambda}}{\lambda-\mu} \\
&= \frac{(A_\lambda)^n(\Psi_{0,\mu}-\Psi_{0,\lambda})}{\lambda-\mu}\sum_{k=0}^{n-1}(A_\lambda)^k\frac{A_\mu-A_\lambda}{\lambda-\mu}A^{n-k-1}\Psi_{0,\mu}
\end{aligned}$$

Reminder: For matrices $X^n - Y^n = (X-Y)X^{n-1} + Y(X-Y)X^{n-1} + \cdots + Y^{n-1}(X-Y)$, since they don't commute

$$\frac{\Psi_{0,\mu}-\Psi_0,\lambda}{\lambda-\mu}(x,y) = \frac{\delta_{x,y}}{\lambda-\mu}\left(\frac{1}{c(x)+\mu}-\frac{1}{c(x)+\lambda}\right) = \Psi_{0,\mu}(x,y)\cdot\Psi_{0,\lambda}(x,y).$$

Similarly

$$\frac{A_\mu-A_\lambda}{\lambda-\mu}(x,y) = \left(\Psi_{0,\lambda}A_\mu\right)(x,y)$$

Then

$$\begin{aligned}
\frac{(A_\lambda)^n(\Psi_{0,\mu}-\Psi_{0,\lambda})}{\lambda-\mu}\sum_{k=0}^{n-1}(A_\lambda)^k\frac{A_\mu-A_\lambda}{\lambda-\mu}A^{n-k-1}\Psi_{0,\mu} &= [(A_\lambda)^n\Psi_{0,\mu}](x,y) + \sum_{k=0}^{n-1}\sum_{z}\underbrace{\left[A_\lambda^k\Psi_{0,\lambda}\right]}_{=\Psi_{k,\lambda}(x,z)}(x,z)\underbrace{\left(A_\mu^{n-k}\Psi_{0,\mu}\right)(z,y)}_{=\Psi_{n-k,\mu}(z,y)} \\
&= \sum_{k=0}^{n}\sum_{\in\mathcal{S}}\Psi_{k,\lambda}(x,z)\Psi_{n-k,\mu(z,y)}
\end{aligned}$$

□

List of Lectures