Lecture notes on Markov Processes

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Based on the lectures of
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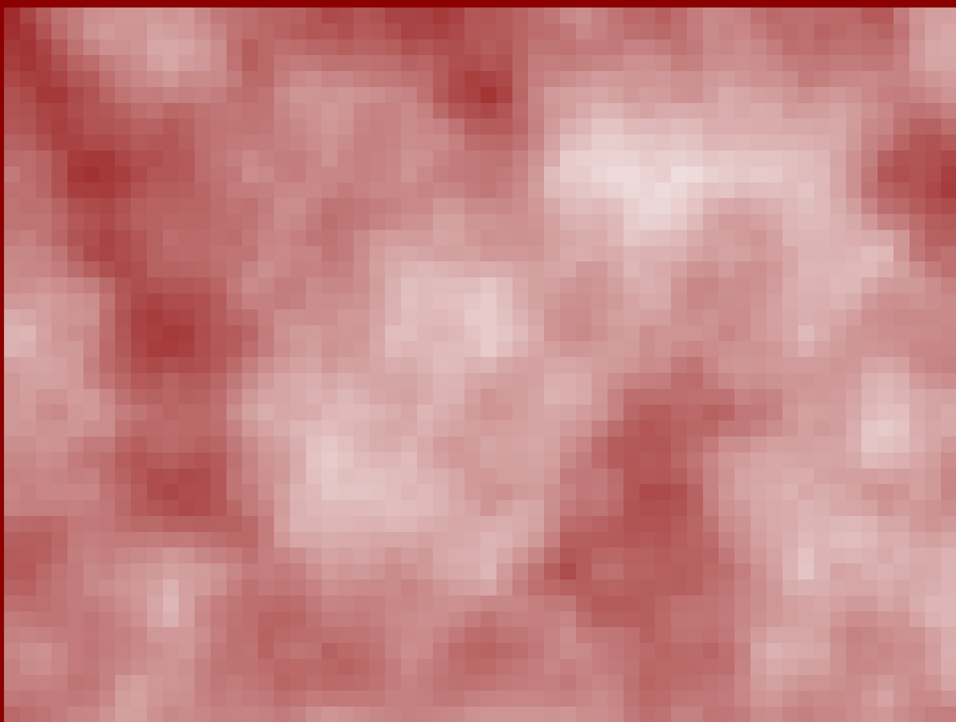

University of Bonn
Winter semester 2024
Last update: October 8, 2024

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Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Ecampus: Ecampus link
- Basis: Basis link
- Website: None
- Time slot(s): **Tuesday 12-14** and **Thursday 12-14**
- Exams: Oral
- Exercises: To be handed in each **Friday** until **noon**

There are tutorials in the second week (14.10.24 and 16.10.24)!

Start of lecture 01 (08.10.24)

Chapter 1: Continuous time Markov chains

Two motivating examples:

1.1 Markov chain, transition function, infinitesimal description

We have a **countable state space $\mathcal{S}$** and define

- $\Omega = \{$right-continuous functions $\omega : \mathbb{R}_+ \to \mathcal{S}$ with finitely many jumps in any finite time intervals$\}$
- σ-algebra $\mathcal{F} := \sigma(\{\omega \to \omega(t)$ which is measurable $\forall t \geq 0\})$ measurable w.r.t.?
- time shift operator: $(\theta_s \omega)(t) := \omega(s+t)$

Definition 1.1. *Denote $X(t,\omega) := \omega(t)$. Assume we have a collection of probability measures $\{\mathbb{P}^x, x \in \mathcal{S}\}$ on $(\Omega, \mathcal{F})$, a right-continuous filtration $\{\mathcal{F}_t, t \geq 0\}$ on $(\Omega, \mathcal{F})$. X is a **continuous time Markov chain (MC)** if*

(a) X is adapted to $\{\mathcal{F}_t, t \geq 0\}$

*(b) **Initial condition:** $\mathbb{P}^x(X(0) = x) = 1$*

*(c) **Markov property:** $\forall x \in \mathcal{S}, Y$ measurable function on $\Omega, s \geq 0$:*

$$\mathbb{E}^x\left(Y \circ \theta_s \mid \mathcal{F}_s\right) = \mathbb{E}^{X(s)}(Y), \mathbb{P}^x \ a.s.$$

Example. $Y = \max_{0 \leq t \leq 1} X(t) \to Y \circ \theta_s = \max_{s \leq t \leq s+1} X(t)$

Here: Time homogenous processes!

Definition 1.2. *A **transition function** is function $p_t(x,y), x, y \in \mathcal{S}, t \geq 0$ s.t.*

*(a) **Positivity:** $p_t(x,y) \geq 0$*

*(b) **Normalized:** $\sum_{y \in \mathcal{S}} p_t(x,y) = 1$*

We will see while constructing, that this part is tricky, might just be ≤ 1

*(c) **Continuity:** $\lim_{t \downarrow 0} p_t(x,x) = p_0(x,x) = 1$*

*(d) **Chapman-Kolmogorov equation:** $\forall s,t \geq 0 \ \forall x, y \in \S$:*

$$p_{s+t}(x,y) = \sum_{z \in \mathcal{S}} p_s(x,z) p_t(z,y)$$

Given a transition function, we can construct $\mathbb{P}^x$ as follows:
Finite dimensional distributions, $0 < t_1 < \cdots < t_n$:

$$\mathbb{P}^x(X(t_1) = x_1, X(t_2 = x_2), \ldots, X(t_n) = x_n) := p_{t_1}(x, x_1) \cdot p_{t_2 - t_1}(x_1, x_2) \cdot \cdots \cdot p_{t_n - t_{n-1}(x_{n-1}, x_n)}$$

extend to full time $\mathbb{R}_+$ by the Kolmogorov(-Daniell) extension theorem, where the consistency,$\mathbb{P}^x(X(t) = y) = p_t(x, y)$, relations follow by the Chapman-Kolmogorov equation.

Example. $0 < t_2 < \cdots < t_n$

We are still in the same setting, therefore we use the same t_1 as previously

$$\sum_{x_1 \in \mathcal{S}} \mathbb{P}^x \left(X(t_1) = x_1, \ldots, X(t_n) = x_n\right) = \mathbb{P}^x \left(X(t_2) = x_2, \ldots, X(t_n) = x_n\right)$$

By modeling we often think at the basic biological/ physical properties of the system $\to$ typically we have *transition rates*, because for $x \neq y$: $\begin{cases} p_\epsilon(x,y) = O(\epsilon), & p_\epsilon(x,x) = 1 - O(\epsilon) \\ p_0(x,y) = 0, & p_0(x,x) = 1 \end{cases}$

Definition 1.3. *For a Markov chain X we define the* ***transition rates*** *from x to y ($x \neq y$) by:*

$$\tilde{q}(x,y) =:= \frac{d}{dt} p_t(x,y) \mid_{t=0}$$

Definition 1.4. *A* ***Q-matrix (or generator)*** *is a collection of numbers $\{q(x,y) : x, y \in \mathcal{S}\}$ s.t.:*

(a) $q(x,y) \geq 0 \forall x \neq y$

(b) $\sum_{y \in \mathcal{S}} q(x,y) = 0$, *Notation:* $c(x) \coloneqq q(x,x) = \sum_{y \in \mathcal{S} \setminus \{x\}} q(x,y)$

$c(x)$ is the rate of leaving site x

Warning

It is not automatic that transition rates gives a Q-matrix For instance: $\tilde{q}(x,y) = \infty$

For **finite** state space,
Markov chain $\iff$ Transition function $\iff$ Q-matrix
Goal: Under which condition is the equivalence still true?

1.2 Examples

Example (Discrete to continuous MC). ***Given:*** *Markov chain Y in discrete time $t \in \mathbb{Z}_+$, i.e. transition matrix $P = (P(x,y))_{x,y \in \mathcal{S}}$*

$$\mathbb{P}(Y(n+1) = y \mid Y(n) = x) = P(x,y)$$

Consider a Poisson process(PP) with intensity 1, at each event time of the PP, there will be a jump of the continuous time MC X. The jumps follow the discrete time MC Y
$\implies p_t(x,y) = \sum_{n \geq 0} \frac{e^{-t} t^n}{n!} P^n(x,y)$
in this case $Q = P - 1$

Example (Finite $\mathcal{S}$). $p_t(x,y) = \left(e^{tQ}(x,y)\right) \coloneqq \sum_{n \geq 0} \frac{t^n}{n!} Q^n(x,y)$

Here will be one of the main problems: If Q is finite / a normed operator, everything is well defined. Otherwise we have to use a different definition of the exponetial ...

Example (Birth and death processes). $\mathcal{S} = \{0, 1, \ldots, \}, X(t) =$ *Population size at time t. Then*

- $q(k, k+1) = \rho_k, k \geq 0$
- $q(k+1, k) = \lambda_k, g \geq 1$
- $q(k,k) = -\rho_k - \lambda_k$ *with* $\lambda_0 = 0$
- *which implies* $q(k,l) = 0 \forall |k - l| \geq 2$

Depending on the choice of ρ_k, λ_k it is possible that the chain goes to ∞ in finite time.

List of Lectures