1 Introduction

This document is a short script containing all statements and definitions of Algebra 1, as held by Prof. Franke in 2021. I will however change/add some formulations and definitions as I see fit. I am thankful for every correction, so if you find an error send me a mail (mh@mssh.dev). Most additions include the keyword "Bonus".

Unless otherwise specified, in this lecture all rings are supposed to be commutative rings with 1.

Kindest regards,

Manuel

2 Lecture 0N

Definition 1. *(Generated submodule)*

For a ring R, an R-module M and a subset $S \subseteq M$, the following subsets of M coincide:

- *The set of all sums $\sum_{s \in S} r_s \cdot s$, with all but finitely many $r_s \in R$ vanishing-*
- *The intersection of all (R)-submodules of M containing S.*
- *The smallest among all submodules of M containing S.*

This subset of M is called the submodule of M generated by S. If it coincides with M we say that M is generated by S. We call M finitely generated if it can be generated by a finite subset S.

Definition 2. *(Noetherian modules)*

For a ring R, an R-module M is noetherian if the following equivalent conditions hold:

- *All submodules of M are finitely generated.*
- *Every sequence $N_0 \subseteq N_1 \ldots$ of submodules of M terminates at some i with $N_j = N_i$ for $j \geq i$.*
- *Every set $\mathfrak{M} \neq \emptyset$ of submodules of M has a $\subseteq$-largest element.*

The case $M = R$ describes noetherian rings.

Proposition 3. *(Short exact sequences and finite generation)*

Let R be an arbitrary ring and

$$0 \to M' \xrightarrow{\iota} M \xrightarrow{p} M'' \to 0$$

a short exact seuqence of R-modules.

Let $(m'_i)_{i=1}^k$ be generators of M' and $(m''_i)_{i=1}^l$ of M'' as an R-module, and let $m_j \in M$ be chosen s.t. $p(m_j) = m''_j$. Then $(\iota(m'_1), \ldots, \iota(m'_k), m_1, \ldots, m_l)$ generate M as an R-module.

Corollary 4. *In particular, M is finitely generated if this holds for M' and M''.*

Fact

Let $M \xrightarrow{p} M'' \to 0$ be exact (in other words, p surjective). If M finietly generated (e.g. by $m_1, \ldots, m_k$), then M'' is finitely generated (namely by $p(m_1), \ldots, p(m_k)$).

Proposition 5. *If R is a Noetherian ring, then the polynomial rings $R[X_1, \ldots, X_n]$ in finitely many variables over R are also Noetherian.*

Remark 6. The polynomial ring in infinitely many variables $R[X_i | i \in \mathbb{N}]$ is not Noetherian unless $R = \{0\}$, as the ideal generated by the X_i is not finitely generated.

Remark 7. For a Noetherian ring R and an ideal $I \subseteq R$, R/I is Noetherian, for the preimage under $R \xrightarrow{\pi} R/I$ of an infinitely strictly ascending chain of ideals R/I would be a strictly ascending chain of ideals in R.

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Definition 8. *(Units of Rings)*

A ring R is called a domain if $0 \neq 1$ in R and $\mathrm{ab} = 0$ with $a, b \in R$ implies $a = 0$ or $b = 0$. A ring element x is called a unit if and only if there exists $i \in R$ with $\mathrm{ix} = 1$. This is obviously equivalent to $\mathrm{xR} = R$, The set of units in R is denoted $R^\times$ and called the group of units of R. It is a group under multiplication in R.

Remark 9. A ring R is a field if and only if the following two equivalent conditions hold:

- $R^\times = R \setminus \{0\}$.
- R contains precisely two ideals, namely $R \neq \{0\}$.

Every field is a domain.

Definition 10. *(proper ideals)*

For an ideal $I \subseteq R$ the following conditions are equivalent:

- $1 \notin I$.
- $I \neq R$.

Such an ideal is called proper.

Let alway R be an arbitrary Ring.

Definition 11. *(Prime ideals)*

For an ideal $\mathfrak{p} \subseteq R$, the following conditions are equivalent:

- *$R/\mathfrak{p}$ is a domain*
- *$\mathfrak{p}$ is a proper ideal satisfying the following condition: If $a, b \in R$ and $\mathrm{ab} \in \mathfrak{p}$ then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.*

Such an ideal is called a prime ideal.

Definition 12. *(Maximal ideals)*

For an ideal $\mathfrak{m} \subseteq R$ the following condition are equivalent:

- *$R/\mathfrak{m}$ is a field*
- *$\mathfrak{m}$ is a proper ideal, and if I is a proper ideal with $\mathfrak{m} \subseteq I$, then $\mathfrak{m} = I$.*

Such an ideal is called a maximal ideal.

Definition 13. *(Partial orders)*

by a partial order on a set X we understand a relation $\preceq$ on X s.t. the following conditions are satisfied for $x, y, z \in X$.

- *$x \preceq x$.*
- *$x \preceq y$ and $y \preceq x$ implies $x = y$.*
- *$x \preceq y$ and $y \preceq z$ implies $x \preceq z$.*

The order is linear or total if one of $x \preceq y$ or $y \preceq x$ always hold for $x, y \in X$.

Theorem 14. *(Zorn's Lemma, which depends on the Axiom of Choice)*

Let $(\mathfrak{M}, \preceq)$ be a partially order set with the conditions that every $\preceq$-linearly ordered subset $\mathfrak{L} \subseteq \mathfrak{M}$ is bounded in $\mathfrak{M}$ in the sense that there exists $b \in \mathfrak{M}$ (a bound of $\mathfrak{L}$ in $\mathfrak{M}$) with $I \preceq b$ for all $I \in \mathfrak{L}$. Then $\mathfrak{M}$ contains a $\preceq$-maximal x: $y \in \mathfrak{M}$ and $x \preceq y$ implies $x = y$. Note that every $b \in \mathfrak{M}$ is a bound of $\emptyset$. In the case of $\mathfrak{L} = \emptyset$ the boundedness condition just says $\mathfrak{M} \neq 0$. of course the maximal element is typically non-unique.

Corollary 15. *Every proper ideal $I \subseteq R$ is contained in a maximal ideal.*

Corollary 16. *If $R \neq \{0\}$, then R contains a maximal ideal. In particular, the set Spec R of prime ideals of R is $\neq \emptyset$.*

Definition 17. *(Multiplicatively equivalence)*

Let R be a ring. We say that $x, y \in R$ are multiplicatively equivalent and write $x \sim y$ if there is $\epsilon \in R^\times$ s.t. $x = \varepsilon y$.

From now on (until Lecture 1) let R be a domain.

Definition 18. *(Prime and irreducible elements)*

We say that $p \in R$ is a prime element if $p \neq 0$ and pR is a prime ideal. We say that p is irreducible if $p \notin \{0\} \cup R^\times$ and $p = \mathrm{ab}$ implies that one of a, b is $\in R^\times$ or equivalently that one of a or b is $\sim p$.

Definition 19. *((Unique)-decompositions)*

Every prime element is irreducible. But irreducible elements of domains are not necessarily prime.

By a decomposition of $r \in R \setminus \{0\}$ into prime elements (resp. irreducible elements) we understand a decomposition $r = \delta \prod_{i=1}^{k} p_i$ with $\delta \in R^\times$ and prime p_i. (resp. irreducible). The decomposition is called unique if for every similar decomposition $r = \varepsilon \prod_{i=1}^{l} q_i$, we have $k = l$ and there is $\pi \in S_k$ s.t. $p_i \sim q_{\pi(i)}$.

Proposition 20.

- *Every Noetherian domain has decomposition into irreducible elements.*
- *If R is a domain with decomposition into irreducible elements, then that decomposition is unique if and only if every irreducible element of R is prime or equivalently, if R has decomposition into prime elements (which then is unique).*

Definition 21. *A domain satisfying the second condition is called factorial or a unique factorization domain (UFD). For instance, every PID is a UDF (like $\mathbb{Z}$ or $\mathfrak{k}$ or $\mathfrak{k}[T]$ for any field $\mathfrak{k}$), and so are polynomial rings in any finite or infinite number of variables over a UFD.*

4 Lecture 1

Theorem 22. *(Hilbert's Basissatz)*

If R is a Noetherian ring, then the polynomial rings $R[X_1, \ldots, X_n]$ in finitely many variables is Noetherian.

Fact

- Every Noetherian module over an arbitrary ring is finitely generated.
- If R is a Noetherian ring, then an R-module is Noetherian if and only if it is finitely generated.

Fact

Every submodule of a Noetherian module is Noetherian.

Fact

Let $M \xrightarrow{p} M''$ be a surjective morphism if R-modules. If M is finitely generated (resp. Noetherian), then so is M''.

Fact

Let $M' \xrightarrow{f} M \xrightarrow{p} M'' \to 0$ be an exact sequence of R-modules. If M' and M'' are finitely generated (resp. Noetherian), then so is M.

Definition 23. *(Zeros of ideals)*

Let $\mathfrak{k}$ be a field, $R = \mathfrak{k}[X_1 \ldots, X_n]$ and I an ideal in R.

We say that $x \in \mathfrak{k}^n$ is a zero (Nullstelle) of I if $P(x) = 0$ for all $P \in I$. The notion of a zero in a field extension $\mathfrak{l}$ of $\mathfrak{k}$ is defined similarly. The set of zeros of I in $\mathfrak{k}$ will be denoted $V(I)$.

Remark 24. If I is the ideal generated by S, then x is a zero of I if and only if s(x) for all $s \in S$. Thus the zero set of ideals in R are precisely the solution set to systems of polynomial equations $P_s(x_1, x_2, \ldots, x_n) = 0$, $s \in S$. By the Hilbert Basissatz, for every such system there is a finite subset $\tilde{S}$ of S such that $P_s(x_1, \ldots, x_n) = 0$ for $s \in \tilde{S}$ has the same set of solutions.

Theorem 25. *(Hilbert's Nullstellensatz)*

Let $\mathfrak{k}$ be a field, $R = \mathfrak{k}[X_1 \ldots, X_n]$ and I an ideal in R.

If $\mathfrak{k}$ is algebraically closed and $I \subset R$ a proper ideal in R, then I has a zero in $\mathfrak{k}^n$.

Remark 26. If $n=1$ R is a principal ideal domain (PID) hence $I=\mathrm{PR}$ where $P\in R$. As I is proper, $P=0$ or P is non-constant. As $\mathfrak{k}$ is algebraically closed, P has a zero in $\mathfrak{k}$. Thus the theorem is trivial if $n=1$. Considering $I=P(X_1)R$ one also sees that the theorem fails for trivial reasons if $n>0$ and $\mathfrak{k}$ is not algebraically closed. We will however, encounter versions of the Nullstellensatz which hold for general fields.

Definition 27. *(Algebras, subalgebras and morphisms of algebras)*

Let R be a ring. An R-algebra (A,α) is a ring A with a ring homomorphism $R\xrightarrow{\alpha}A$. A subalgebra of A is a subring of A containing the image of α. A morphism of R-algebras $A\xrightarrow{f}\tilde{A}$ is a ring homomorphism with $\tilde{\alpha}=f\alpha$.

Remark 28. While this is sometimes the case, it is not assumed here that R is a subring of A. In fact, α is not even assumed to be injective. If (A,α) is an R-algebra, we have a ring homomorphism $R[X_1,\ldots,X_n]\xrightarrow{\alpha}A[X_1,\ldots,X_n]$ sending $P=\sum_{\beta\in\mathbb{N}^n}p_\beta X^\beta$ to $\alpha(P)=\sum_{\beta\in\mathbb{N}^n}\alpha(p_\beta)X^\beta$. We usually write $P(a_1,\ldots,a_n)$ for $(\alpha(P))(a_1,\ldots,a_n)$.

Definition 29. *(Generated algebras, algebras of finite type)*

The image of the ring homomorphism mentioned above is a subalgebra of A containing all a_i and is contained in every subalgebra of A containing all a_i. It thus also coincides with the intersection of all subalgebras containing all a_i. We call this R-subalgebra of A generated by the a_i. We say that A is an R-algebra of finite type if it can be generated in this sense by a finite subset of itself. For an arbitrary subset S of A, the subalgebra generated by S is the intersection of all subalgebras of A containing S, or equivalently, the union of the subalgebras generated in the above sense by the finite subsets of S, or equivalently, the image of the polynomial ring $R[X_s|s\in S]$ under the morphism of evaluation at S.

Theorem 30. *Hilbert Nullstellensatz (version II)*

Let L/Kbe an arbitrary field extension. Then L/K is a finite field extension if and only if L is a K-algebra of finite type.

5 Lecture 2

Let R be a ring and A an R-algebra. Then A is a module over itself and the ring homomorphism $R\to A$ allows us to derive an R-module structure on A from this A-module structure.

Definition 31. *We say A is finite over R, or that the R-algebra A is finite, or that A/R is finite if A is finitely generated as an R-module.*

Fact (Basic properties of finiteness)

A) Every ring is finite over itself.

B) A field extension is finite as a ring extension if and only if it is a finite filed extension.

C) Every finite algebra is of finite type.

D) If A/R and B/A are finite algebras, then so is B/R.

Definition 32. *(Integral elements, integral algebras)*

Let A be an R-algebra and $a \in A$.

A) There are $n \in \mathbb{N}$ $(r_i)_{i=0}^n$, $r_i \in R$, such that $a^n = \sum_{i=0}^{n-1} r_i a^i$.

B) There is a subalgebra $B \subseteq A$ finite over R and containing a.

We call $a \in A$ integral over R (ganz über R) if it satisfies these equivalent conditions. We say that A/R is integral or that A is integral over R, if every $a \in A$ is integral over R. The set of elements of A which are integral over R is called the integral closure of R in A.

Corollary 33.

A) Every finite R-algebra A is integral.

B) The integral closure of R in A is a R-subalgebra of A.

C) If A is an R-algebra, B an A-algebra and $b \in A$ integral over R, then it is integral over A.

D) If A is an integral R-Algebra and B any A-algebra and $b \in B$ is integral over A, then B is integral over R.

Definition 34. *(Determinants)*

For $A = (a_{ij}) \in \mathrm{Mat}(n, n, R)$ we define:

$$\det(A) = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Fact

The following rules for determinants hold:

A) det(AB)=det(A)det(B).

B) Development along a row or column.

C) Cramer's rule: $A \cdot \mathrm{Adj}(A) = \mathrm{Adj}(A) \cdot A = \det(A) \cdot \mathbf{1}_n$. In particular, A is invertible iff det(A) is a unit.

D) Caley Hamilton: If $P_A = \det(T \cdot \mathbf{1}_n - A)$, then $P_A(A) = 0$.

Fact

If A is an integral R-algebra of finite type, then it is a finite R-algebra.

Fact

If A is an R-algebra and B an A-algebra of finite type, then B is an R-algebra of finite type.

Fact (About integrality and fields)

Let B be a domain integral over its subring A. Then B is a field if and only if A is a field.

Theorem 35. *(Noether normalization theorem)*

Let K be a field and A a K-algebra of finite type. Then there are $a=(a_i)_{i=1}^n \in A$ which are algebraically independent over K in the sense that the ring homomorphism $K[X_1,\dots,X_n] \xrightarrow{\mathrm{ev}_a} A$, $\mathrm{ev}_a(P)=P(a_1,\dots,a_n)$, is injective and with the additional property that A is finite over the image of ev_a.

Definition 36. *An R-module is called finitely presented if it is isomorphic to the quotient module of R^k by a finitely generated submodule, for some $k\in\mathbb{N}$.*

6 Bonus: Definitions of Sheet 1

Definition 37. *Let $\mathfrak{k}$ be a field and consider the polynomial Ring $R=\mathfrak{k}[X_1,\dots,X_n]$ and an ideal $I\subset R$. The set of (simultaneous) zeros of I is denoted by $V(I)\subset \mathfrak{k}^n$.*

Definition 38. *Let M be an R-module. Define the annihilator of M by*

$$\mathrm{Ann}_R(M) := \{r\in R : r\cdot M = 0\}$$

Lemma 39. *Let M be an R-module.*

M is noetherian if and only if M is finitely generated and $R/\mathrm{Ann}_R(M)$ is a noetherian ring.

7 Bonus: Sheet 3

Definition 40. *Let X be a set. We call $\mathcal{F}\subset\mathfrak{P}(X)$ a filter if it satisfies:*

- *$X\in\mathcal{F}$;*
- *$\emptyset\notin\mathcal{F}$;*
- *If $A\in\mathcal{F}$ and $A\subset B$, then $B\in F$;*
- *If $A,B\in\mathcal{F}$, then $A\cap B\in\mathcal{F}$.*

Definition 41. *A filter is called an ultrafilter if it has the additional property:*

- *For any $A\subset X$ we have $A\in\mathcal{F}$ or $X\setminus A\in\mathcal{F}$.*

Proposition 42. *Every filter is an ultrafilter.*

Remark 43. The trivial filter on a set X is given by $\mathcal{F}=\{X\}$. Given an ultrafilter $\mathcal{F}$ on X and a map $f: X\to Y$. Define

$$f_*\mathcal{F} = \{A\subset Y : f^{-1}(A)\in\mathcal{F}\}.$$

This is a ultrafilter.

Definition 44. *Given a topologial space X and an ultrafilter $\mathcal{F}$ on X we say that $\mathcal{F}$ converges to $x\in X$ if*

$$\{U\subset X : U \text{ open}, x\in U\}\subset\mathcal{F}.$$

Definition 45. *For a field* $\mathfrak{k}$,$n \in \mathbb{N}$, $R = \mathfrak{k}[X_1, \ldots, X_n]$. *Given* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ *we put* $X^\alpha = \prod_{i=1}^n X_i^{\alpha_i}$. *We write* $\alpha | \beta$ *if* X^α *divides* X^β *in* R. *For* $P = \sum a_\alpha X^\alpha$ *let* $\mathrm{Le}(P) = \max_\prec \{\alpha \in \mathbb{N}^n : a_\alpha \neq 0\}$. *Where* $\prec$ *is a monomial order.*

Definition 46. *We call a linear order* $\preceq$ *a monomial order if it is translation invariant*

$$\alpha \preceq \beta \Rightarrow \alpha + \gamma \preceq \alpha + \gamma$$

and extends $|$

$$\alpha | \beta \Rightarrow \alpha \preceq \beta.$$

8 Lecture 3

Lemma 47. *Let* $S \subseteq \mathbb{N}^n$ *be a finite subset. Then there is* $k \in \mathbb{N}^n$ *s.t.* $k_1 = 1$ *and such that* $w_k(\alpha) \neq w_k(\beta)$ *if* α *and* β *are different elements of* S, *where* $w_k(\alpha) = \sum_{i=1}^n k_i \alpha_i$.

Definition 48. *For an ideal* I *in an arbitrary ring* R, *its radical* $\sqrt{I} := \bigcup_{n=0}^\infty \left\{ f \in R | f^n \in I \right\}$ *is an ideal in* R. *It is easy to see that* $\sqrt{\sqrt{I}} = \sqrt{I}$. *If* J *also is an ideal in* R, *let* $I \cdot J$ *be the ideal generated by* $\left\{ i \cdot j | i \in I, j \in J \right\}$. *Moreover, if* $(i_\lambda)_{\lambda \in \Lambda}$ *is a possibly infinite family of ideals in* R, *let* $\sum_{\lambda \in \Lambda} I_\lambda$ *be the set of* $\sum_{\lambda \in \Lambda} i_\lambda$ *with* $i_\lambda \in I_\lambda$ *and all but finitely many* i_λ *vanishing. It is easy to see that this coincides with the ideal generated by* $U_{\lambda \in \Lambda} I_\lambda$ *in* R. *Moreover,* $\bigcap_{\lambda \in \Lambda} I_\lambda$ *is also an ideal in* R.

Let $R = \mathfrak{k}[X_1, \ldots, X_n]$ where $\mathfrak{k}$ is an algebraically closed field.

Fact:

Let I,J and the $(i_\lambda)_{\lambda \in \Lambda}$ be ideals in R, where the index set Λ may be infinite.

A) We have $V(I) = V(\sqrt{I})$

B) We have $V(I) \subseteq V(J)$ when $J \subseteq I$ or more generally, $\sqrt{J} \subseteq \sqrt{I}$

C) We have $V(R) = \emptyset$ and $V(\{0\}) = \mathfrak{k}^n$.

D) We have $V(I \cap J) = V(I \cdot J) = V(I) \cup V(J)$

E) We have $V(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

Remark 49. There is no similar way to describe $V(\bigcap_{\lambda \in \Lambda} I_\lambda)$ in terms of the $V(I_\lambda)$ when Λ is infinite. For instance if $n = 1$ and $I_k = X_1^k R$ then $\bigcap_{k=0}^\infty I_k = \{0\}$ but $\bigcap_{k=0}^\infty V(I_k) = \{0\}$.

Corollary 50. *There is a topology on* $\mathfrak{k}^n$ *for which the set of closed sets coincides with the set* $\mathfrak{A}$ *of subsets of the form* $V(I)$ *for ideals* $I \subseteq R$.

Indeed, the previous fact shows that $\mathfrak{A}$ is closed under arbitrary intersections and finite unions and contains, in particular, $\emptyset$ and $\mathfrak{k}^n$.

Definition 51. *The topology on* $\mathfrak{k}^n$ *described in the previous corollary is called the* Zariski topology.

Example 52. If $n=1$ then R is a principal ideal domain (PID) hence every ideal in R is a principal ideal and the Zariski-closed subsets of $\mathfrak{k}$ are the subsets of the form $V(P)$ where $P \in R$. As $V(P)=\mathfrak{k}$ when $P=0$ and $V(P)$ is finite otherwise and $\{x_1,\ldots,x_n\}=V(\prod_{i=1}^{n}(T-x_i))$ the Zariski-closed subsets of $\mathfrak{k}$ are $\mathfrak{k}$ itself and the finite subsets. Because $\mathfrak{k}$ is infinite, this topology is not Hausdorff.

Definition 53. *For topological spaces X the separation properties T_{0-2} are defined as the condition that for arbitrary points $x \neq y$ of X, the following holds:*

- *T_0: There is an open subset U of X s.t. $U \cap \{x; y\}$ has precisely one element.*
- *T_1: There is an open subset U containing x but not y.*
- *T_2: There are disjoint open subsets U and V s.t. $x \in U$ and $y \in V$.*

The condition T_2 is the well-known Hausdorff condition.

Remark 54. Let $x \sim y$ denote the condition that the open subsets containing x are precisely the ones containing y. This is an equivalence relation, and T_0 holds if and only if $x \sim y$ implies $x=y$.

Fact:

A topological space is T_1 if and only if every point is closed.

Fact:

The Zariski topology on $\mathfrak{k}^n$ is T_1,but for $n \geq 1$ not Hausdorff. In fact, for $n \geq 1$ the intersection of two non-empty open sets of $\mathfrak{k}^n$ is always non-empty.

Definition 55. *A topological space X is called quasi-compact if every open covering of X has a finite subcovering. It is called compact if it is quasi-compact and Hausdorff.*

Remark 56. In the case of general topological spaces the equivalence between compactness and the existence of a converging subsequence of any sequence of points is lost. Compactness is, however, still equivalent to any ultrafilter having a unique limit. It is not necessary here to go into details as our topologies will not normally be Hausdorff.

Definition 57. *For a topological space X the following conditions are equivalent:*

A) Every open subset of X is quasi-compact.

B) Every descending sequence $A_0 \supseteq A_1 \supseteq \ldots$ of closed subsets of X stabilizes at some $i \in \mathbb{N}$ with $A_i = A_j$ for $j \geq i$.

C) Every set $\mathfrak{M} \neq \emptyset$ of closed subsets of X has a $\subseteq$-smallest element.

A topological space is called Noetherian if it satisfies these equivalent conditions.

Let $R=\mathfrak{k}[X_1,\ldots,X_n]$ where $\mathfrak{k}$ is an algebraically closed field. For $f \in R$, let $V(f)=V(\mathrm{fR})$ be the set of zeros of f on $\mathfrak{k}^n$.

Theorem 58. *(Hilbert's Nullstellensatz)*

Let I be an ideal of R. Then $V(I) \subseteq V(f)$ (in other words, f vanishes on the set of I) if and only if $f \in \sqrt{I}$.

Corollary 59. *There is a $\subseteq$-antimononic bijection between*

- *ideals $I \subseteq R$ satisfying $I = \sqrt{I}$.*
- *Zariski-closed subsets A of $\mathfrak{k}^n$.*

given by the mapping I to $A = V(I)$ and A to $I = \{f \in R | A \subseteq V(f)\}$.

Because it is antimononic, this bijection maps strictly decreasing chains of closed subsets of $\mathfrak{k}^n$ to strictly increasing chains of ideals in R. By the Basissatz R is noethrian- Thus

Corollary 60. *The topological space $\mathfrak{k}^n$ is Noetherian.*

9 Lecture 4

9.1 Irreducibility

Throughout this lecture, $\mathfrak{k}$ will be an algebraically closed field, and $\mathfrak{k}^n$ will be equipped with the Zariski topology.

Definition 61. *For a topological space X, we call $U \subset X$ dense if the intersection with an arbitrary non-empty open subset is non-empty.*

Definition 62. *A topological space is called irreducible if it is non-empty and satisfies the following equivalent conditions*

A) Every non-empty open subset U of X is dense.

B) The intersection of two non-empty open subsets U, V of X is non-empty.

C) If $X = A \cup B$, where A and B are closed subsets of X, then $A = X, B = X$.

D) Every open subset of X is connected.

Corollary 63. *Every irreducible topological space is connected.*

Example 64. We have seen in lecture 3 that the intersection of two non-empty open subsets of $\mathfrak{k}^n$ is non-empty. As a consequence of this, $\mathfrak{k}^n$ is irreducible.

Fact

Let X be an arbitrary topological space

A) A single point is always irreducible.

B) If X is Hausdorff then it is irreducible iff it has precisely one point.

C) A topological space is irreducible if and only if it cannot be written as a finite union of proper closed subsets (where the empty union is understood to be the empty set).

D) A topological space X is irreducible if and only any finite intersection of non-empty open subsets is non-empty (where the empty intersection in X is taken to be X).

Definition 65. *Let X be a topological space, $D \subset X$. The open subsets of the induced topology on D are just the open subsets of X intersected with D.*

Fact

If D is a dense subset of a topological space X, then X is irreducible if and only if D is irreducible with its induced topology.

Definition 66. *A subset Z of a topological space X is called irreducible if it is irreducible with its induced topology. We call Z an irreducible component of X if it is irreducible and if every irreducible subset Y with $Z \subseteq Y \subseteq X$ coincides with Z.*

Corollary 67.

- *A subset $Z \subseteq X$ is irreducible if and only if its closure $\bar{Z}$ in X is irreducible.*
- *Every irreducible component of X is a closed subset of X.*

Remark 68. Throughout this series of lectures it will be convenient to abbreviate "irreducible closed subset" to "irreducible subset".

Proposition 69. *Let X be a Noetherian topological space. Then X can be written as a finite union $X = \bigcup_{i=1}^{n} Z_i$ of irreducible closed subsets of X, and in addition one may also assume that $Z_i \not\subseteq Z_j$ when $i \neq j$. With this additional minimality condition on the decomposition, the number n is unique and the (Z_i) are unique up to permutation. In fact, $\{Z_1, \ldots, Z_n\}$ is the set of irreducible components of X.*

Remark 70. The existance proof of this is an example of Noetherian induction: If an assertion E about closed subsets of a Noetherian topological space X has the property that E holds for A if it holds for all proper subsets of A, then $E(A)$ holds for every closed subset $A \subseteq X$.

Let $R = \mathfrak{k}[X_1, \ldots, X_n]$.

Proposition 71. *Under the corespondence of ideals of R satisfying $I = \sqrt{I}$ and Zariski-closed subsets A of $\mathfrak{k}^n$, A is irreducible if and only if I is a prime ideal. Moreover, $\#A = 1$ if and only if I is a maximal ideal.*

9.2 Krull dimension

Definition 72. *Let Z be an irreducible subset of the topological space X. Let $\operatorname{codim}(Z, X)$ be the maximum of the length n of strictly increasing chains $Z \subseteq Z_0 \subseteq Z_1 \ldots \subseteq Z_n$ of irreducible closed subsets of X containing Z, or ∞ if such chains can be found for arbitrary n. Let $\dim X$ be the maximum of codimensions of irreducible subsets of X, or ∞ if irreducible subsets of arbitrary subsets may be found or $-\infty$ if $X = \emptyset$.*

Remark 73.

- In the situation of the definition, $\bar{Z}$ is irreducible by a previous fact. Hence $\operatorname{codim}(Z, X)$ is well-defnied, and one may assume without losing much generality that Z is closed.

- Because a point is always irreducible, every non-empty topological space has an irreducible subset, and for non-empty X $\dim X$ is ∞ or the maximum of codimensions in X of points of X.
- Even for Noetherian X, it may happen that $\operatorname{codim}(Z, X) = \infty$.
- Even if X is Noetherian and $\operatorname{codim}(Z, X)$ finite for all irreducible subsets Z of X, $\dim X$ may be infinite.

Fact

If $X = \{x\}$, then $\dim X = 0$.

Fact

For every $x \in \mathfrak{k}$,$\operatorname{codim}(\{x\}, \mathfrak{k}) = 1$. The only other irreducible closed subset of $\mathfrak{k}$ is $\mathfrak{k}$ itself, which has codimension zero.

Fact

Let $Y \subseteq X$ be irreducible and $U \subseteq X$ be an open subset such that $U \cap Y \neq \emptyset$. Then we have a bijection between

- the irreducible closed subsets A of X containing Y.
- the irreducible closed subsets B of U containing $Y \cap U$.

Sending A to $B = A \cap U$ and B to $A = \bar{B}$, its closure in X.

Corollary 74. *(Locality of Krull dimension)*

In the situation of the above fact, $\operatorname{codim}(Y, X) = \operatorname{codim}(Y \cap U, U)$.

Fact

Let $Z \subseteq Y \subseteq X$ be irreducible closed subsets of the topological space X. then

$$\operatorname{codim}(Z, Y) + \operatorname{codim}(Y, X) \leqslant \operatorname{codim}(Z, X). \tag{1}$$

Fact

In Y is an irreducible closed subset of the topological space X, then

$$\dim(Y) + \operatorname{codim}(Y, X) \leqslant \dim(X). \tag{2}$$

In generell, these inequalities may be strict.

Definition 75. *A topological space T is called catenary if equality holds in 1 whenever X is an irreducible closed subset of T.*

Theorem 76. *We have $\dim \mathfrak{k}^n = n$ and $\mathfrak{k}^n$ is catenary. Moreover, if X is an irreducible closed subset of $\mathfrak{k}^n$, then the equality holds in 2.*

Proposition 77. *Let $p \in R = \mathfrak{k}[X_1, \ldots, X_n]$ be a prime element. then the irreducible subset $X = V(p) \subseteq \mathfrak{k}^n$ has codimension one, and every codimension one subset of $\mathfrak{k}^n$ has this form.*

Lemma 78. *Every non-zero prime ideal $\mathfrak{p}$ of a factorial domain (UFD) R contains a prime element.*

10 Lecture 5

Throughout this lecture, $\mathfrak{k}$ will be an algebraically closed field, and $\mathfrak{k}^n$ will be equipped with the Zariski topology.

Definition 79. *Let X be a set and $\mathfrak{P}(X)$ the set of subsets of X. A Hull operator in X is a map $\mathfrak{P}(X) \xrightarrow{\mathcal{H}} \mathfrak{P}(X)$ with the following properties:*

1. *We have $A \subseteq \mathcal{H}(A)$ for all $A \in \mathfrak{P}(X)$.*
2. *If $A \subseteq B \subseteq X$, then $\mathcal{H}(A) \subseteq \mathcal{H}(B)$.*
3. *We have $\mathcal{H}(\mathcal{H}(X)) = \mathcal{H}(X)$.*

We call $\mathcal{H}$ matroidal if in addition the following conditions hold:

i. *If $m, n \in X$ and $A \subseteq X$ then $m \in \mathcal{H}(\{n\} \cup A) \setminus \mathcal{H}(A)$ if and only if $n \in \mathcal{H}(\{m\} \cup A) \setminus \mathcal{H}(A)$.*

ii. *We have $\mathcal{H}(A) = \bigcup_{F \subseteq A \text{ finite}} \mathcal{H}(F)$.*

In this case, a subset $S \subseteq X$ is called independent if $s \notin \mathcal{H}(S \setminus \{s\})$ for all $s \in S$, and generating if $X = \mathcal{H}(S)$. It is called a base if it is both generating and independet.

Remark 80. In most frequently studied case where there is a finite generating subset, condition ii) is automatic. This is not necessarily the canonical terminology.

Theorem 81. *If $\mathcal{H}$ is a matriodal hull operator on X, then a basis exists, every independent set is contained in a base, and two arbitrary bases have the same cardinality.*

Example 82. Let K be a field, V a K-vector space and $\mathcal{L}(T)$ the K-linear hull of T, for $T \subseteq V$. It is easy to see that $\mathcal{L}$ is a matroidal hull operator on V. The notion of base one obtains in one which, at least in the finite-dimensional case, is well-known from linear algebra.

Example 83. Let L/K be a field extension and let $\mathcal{H}(T)$ be the algebraic closure in L of the subfield of L generated by K and T (i.e. the intersection of all subfields of L containing $K \cup T$, or the field of quotients of the sub-K-algebra of L generated by T.). The related notion of independence is algebraic independence over K, and a base for $(L, \mathcal{H})$ is called an transcendence base. The transcendence degree $\operatorname{trdeg}(L/K)$ is definied as the cardinality of any transcendence base of L/K. It follows from the definition that L/K is algebraic if and only if $\operatorname{trdeg}(L/K) = 0$.

Remark 84. For the oral exam, it is not necessary to know about matroids or to show that $\mathcal{H}$ is a matroid.

Theorem 85. *(Eakin-Nagata)*

Let A be a subring of the Noetherian ring B. If the ring extension B/A is finite (i.e. B finitely generated as an A-module) then A is Noetherian.

The related result for being of finite type is

Proposition 86. *Let A be a subalgebra of the R-algebra B, where R is Noetherian. If B/R is of finite type and B/A finite, then A/R is also of finite type.*

Definition 87. *Let K be a field and $R = K[X_1, \ldots X_n]$. Let $K(X_1, \ldots X_n)$ denote the field of quotients of R.*

Lemma 88. *If $n>0$, then $K(X_1,\dots,X_n)/K$ is not of finite type.*

Theorem 89. *(Hilbert Nullstellensatz) If L/K is a field extension and L of finite type as a K-algebra, then this field extension is finite.*

Lemma 90. *There are infinitely many multiplicative equivalence classes of prime elements in $R{=}K[X_1,\dots,X_n]$.*

Lemma 91. *The ring extension $K(X_1,\dots,X_n)/K$ is not of finite type.*

From now until the end of this lecuture: "closed irreducible"→"irreducible".

Let $\mathfrak{K}(X)$ be the field of quotients $R/\mathfrak{p}$ for some $\mathfrak{p}$ prime ideal in R. $X=V(\mathfrak{p})$. As the elements of $\mathfrak{p}$ vanish on X, $X/\mathfrak{p}$ may be viewed as the ring of polynomials and $\mathfrak{K}$(X) as the field of rational functions on X.

Theorem 92. *If $X\subset\mathfrak{k}^n$ is irreducible, then* $\dim X=\operatorname{trdeg}\mathfrak{K}(X)/\mathfrak{k}$ *and* $\operatorname{codim}(X,\mathfrak{k}^n)=n-\operatorname{trdeg}\mathfrak{K}(X)/\mathfrak{k}$. *More generally, if $Y\subseteq\mathfrak{k}^n$ is irreducible and $X\subseteq Y$, then* $\operatorname{codim}(X,y)=\operatorname{trdeg}\mathfrak{K}(Y)/\mathfrak{k}-\operatorname{trdeg}\mathfrak{K}(X)/\mathfrak{k}$.

Remark 93. Thus, loosely speaking, the Krull dimension of X is equal to the maximal number of $\mathfrak{k}$-algebraically independent rational functions on X. This is yet another indication that the notion of dimension introduced in the previous lecture is the "correct" one.

Remark 94. The dimension theoretic theorem formulated in the previous lecture is easily derived from the theorem on this slide. The proof of this theorem is rather hard. It will occupy us for a few more lectures.

Lemma 95. *If K is an uncountable field, then the dimension of $K(T)$ as a K-vector space is uncountable.*

Theorem 96. *(Hilbert Nullstellensatz for uncountable fields)*

If K is an uncoutable field and L/K a field extension and L of finite type as a K-algebra, then this field extension is finite.

11 Bonus: Sheet 4

Definition 97. *The Zariski-Topology on* $\operatorname{Spec}R$ *is the topology, where subsets of the form*

$$V(I)=\{\mathfrak{p}\in\operatorname{Spec}R\colon I\subseteq\mathfrak{p}\}$$

are exactly the closed subsets.

Definition 98. *Let $R=\mathfrak{k}[X_1,\dots,X_n]$ and $g_1,\dots,g_k$ be non-zero elements of R. If those g_i generate an ideal I and each habe leading coefficients equal to 1 the following conditions are equivalent and define a Gröbner basis of the ideal I. Let* $\operatorname{Lc}(f)=a_{\operatorname{Le}(f)}$.

a) If $f\in I\setminus\{f\}$, then $\operatorname{Le}(g_i)|\operatorname{Le}(f)$ for some $1\leqslant i\leqslant k$.

b) If $f\in I$, then for any division with remainder of f by the g_i the remainder vanishes.

c) For arbitrary $1\leqslant i<j\leqslant k$, any division with remainder of $d_{i,j}$ by the (g_i) has remainder zero.

d) *For arbitrary* $1 \leqslant i < j \leqslant k$ *some division with remainder of the* $d_{i,j}$ *by the* (g_i) *has remainder zero.*

Where $d_{i,j} = X^{\beta_{ij}-\alpha_i} g_i - X^{\beta_{ij}-\alpha_j} g_j$, $\operatorname{lcm}(\alpha, \beta) = (\max(\alpha_i, \beta_i))_{i=0}^n$, $\operatorname{lcm}(X^\alpha, X^\beta) = X^{\operatorname{lcm}(\alpha,\beta)}$,$\alpha_i = \operatorname{Le}(g_i)$ *and* $\beta_{i,j} = \operatorname{lcm}(\alpha_i, \alpha_j)$.

12 Lecture 6

Throughout this lecture, $\mathfrak{k}$ will be an algebraically closed field, and $\mathfrak{k}^n$ will be equipped with the Zariski topology.

Definition 99. *Let* R *be a commutative ring.*

- *Let SpecR denote the set of prime ideals and* $\mathrm{mSpecR} \subseteq \mathrm{SpecR}$ *the set of maximal ideals of* R.
- *For an ideal* I *of* R, *let* $V(I) = \{\mathfrak{p} \in \mathrm{SpecR} | I \subseteq \mathfrak{p}\}$
- *We equip SpecR with the Zariski-Topology for which the closed subsets are the subsets of the form* $V(I)$, *where* I *runs over the set of ideals in* R.

Remark 100. When $R = \mathfrak{k}[X_1, \ldots, X_n]$, the notion $V(I)$ clashes with the previous notation $V(I)$ for the set of zeros in I. When several types of $V(I)$ will be in use, they will be distinguished using indices.

Remark 101. Let $(I_\lambda)_{\lambda \in \Lambda}$ and $(I_j)_{j=1}^n$ be ideals in R. where Λ may be infinite. It is not very hard to see that $V(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$ and $V(\bigcap_{j=1}^n I_j) = V(\prod_{j=1}^n I_j) = \bigcup_{j=1}^n V(I_j)$. Thus, the Zariski topology is indeed a topology.

Remark 102. Let $R = \mathfrak{k}[X_1, \ldots, X_n], \mathfrak{k} = \bar{\mathfrak{k}}$. We have a bijection (lecture 4: Irreducible subsets of $\mathfrak{k}^n$) between SpecR and the set of irreducible closed subsets of $\mathfrak{k}^n$ sending $p \in \operatorname{Spec} R$ to $V_{\mathfrak{k}^n}(\mathfrak{p})$ and identifyingthe one-point subsets with $\operatorname{mSpec} R$. The latter bijection defines a bijection $\mathfrak{k}^n \cong \operatorname{mSpec} R$ which is a homeomorphism if $\operatorname{mSpec} R$ is equipped with the induced topology from the Zariski topology on SpecR.

Definition 103. *A multiplicative subset of a Ring* R *is a subset* $S \subseteq R$ *s.t.* $\prod_{i=1}^n f_i \in S$ *when* $n \in \mathbb{N}$ *and all* $f_i \in S$. *The empty product is allowed, therefore* $1 \in S$.

Proposition 104. *In this situation, there is a ring homomorphism* $R \xrightarrow{i} R_S$ *s.t.* $i(S) \subseteq R_S^\times$ *and such that* i *has the universal property for such ring homomorphisms: If* $R \xrightarrow{j} T$ *is a ring homomorphism with* $j(S) \subseteq T^\times$, *then there is a unique ring homomorphism* $R_S \xrightarrow{\iota} T$ *with* $j = \iota i$.

Remark 105. The idea behind this is that R_S is obtained from R by inverting the elements of S. The construction is similar to the construction of the field of quotients. However, $i: r \mapsto [r, 1]$ is often not injective and $\operatorname{Ker}(i)$ is the ideal $\{r \in R | s \cdot r = 0 \text{ for some } s \in S\}$. In particular $(r = 1)$, R_S is the null ring iff $0 \in S$. It is easy to see that the universal property characterizes R_S uniquely up to unique isomorphism: If $R \xrightarrow{j} \mathrm{T}$ also has the universal property. then ι is an isomorphism.

The construction of $R_S = (R \times S)/\sim$ where $(r, s) \sim (\rho, \sigma)$ iff there is a $t \in S$ with $t\sigma r = \mathrm{t}s\rho$. Note that the factor t is not present in the normal textbook construction of the field of quotients but must not be ommitted if S contains zero divisors of R. Let $[r, s]$ denote the equivalence class of (r, s). This will normally be written $\frac{r}{s}$ when the proposition has been shown. The ring operations are $[r, s] + [\rho, \sigma] = [r\sigma + s\rho, s\sigma]$ and $[r, s][\rho, \sigma] = [r\rho, s\sigma]$. To verify the universal property one puts $\iota([r, s]) = \frac{j(r)}{j(s)}$.

Let R be a ring and S be a multiplicative subset of R. Let R_S be the localization, where we now write r/s for $[r,s]$. The ring homomorphism $R \xrightarrow{i} R_S$ is thus given by $i(r) = r/1$. The inverse image (preimage) under i of a subset $X \subseteq R_S$ will be denoted $X \sqcap R := i^{-1}(X)$.

Definition 106. *An ideal $I \subseteq R$ is called S-saturated if $s \in S, i \in R$ and $\mathrm{rs} \in I$ implies $r \in I$.*

Fact

A prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ is S-saturated iff $\mathfrak{p} \cap S = \emptyset$. Because the elements of S become units in R_S, $J \sqcap R$ is an S-saturated ideal in R when J is an ideal in R_S.

Proposition 107. *We have a bijection between the ideals $J \subseteq R_S$ and the S-saturated ideals $I \subseteq R$ sending J to $J \sqcap R$ and I to $J = \left\{ \frac{i}{s} | i \in I, s \in S \right\}$. Under this bijection, I is a prime if and only if J is.*

Corollary 108. *If R is Noetherian then so is R_S.*

Remark 109. The result about prime ideals gives us a bijection bweteen $\operatorname{Spec}(R_S)$ and $U = \operatorname{Spec} R \setminus \bigcup_{s \in S} V(s)$. One can show that the bijection is a homeomorphism if U is equipped with the topology induced from $\operatorname{Spec} R$. If $S = s^{\mathbb{N}} = \{s^n | n \in \mathbb{N}\}$ then $U = \operatorname{Spec} R \setminus V(s)$ is open, but not in general.

Fact

If I is S-saturated and $r \in R, s \in S$ then $\frac{r}{s} \in J$ if and only if $r \in I$.

Remark 110. Let R be a domain. If $S = R \setminus \{0\}$, then R_S is the field of quotients K of R. If $S \subseteq R \setminus \{0\}$, then

$$R_S \cong \left\{ \frac{a}{s} \in K | a \in R, s \in S \right\}.$$

In particular, the field of quotients of R and R_S are canonically isomorphic.

Remark 111. Let R be any ring, I and ideal in R. Even if I is not S-saturated, $J = I_S = \left\{ \frac{i}{s} | i \in I, s \in S \right\}$ is an deal in R_S, and $J \sqcap R$ equals $\{r \in R | r \cdot s \text{ for some } s \in S\}$, the S-saturation of I which is the smallest S-saturated ideal containing I. In this situation, if $\bar{\mathrm{S}}$ denotes the image of S in R/I, we have a canonical isomorphism $R_S/J \cong (R/I)_{\bar{S}}$.

Definition 112. *For a ring R and $\mathfrak{p} \in \operatorname{Spec} R$ let always $\mathfrak{k}(\mathfrak{p})$ denote the field of quotients of the domain $R/\mathfrak{p}$. This is called the residue field of $\mathfrak{p}$.*

Proposition 113. *Let $\mathfrak{k}$ be a field, A a $\mathfrak{k}$-algebra of finite type and $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} A$ with $\mathfrak{p} \subset \mathfrak{q}$. Then* $\operatorname{trdeg}(\mathfrak{k}(\mathfrak{p})/\mathfrak{k}) > \operatorname{trdeg}(\mathfrak{k}(\mathfrak{q})/\mathfrak{k})$.

Assume $\mathfrak{k}$ algebraically closed, X, Y, Z are irreducible, which means irreducible and closed, subsets of $\mathfrak{k}^n$.

Corollary 114. *We have* $\operatorname{codim}(X,Y) \leqslant \operatorname{trdeg}(\mathfrak{K}(Y)/\mathfrak{k}) - \operatorname{trdeg}(\mathfrak{K}(X)/\mathfrak{k})$.

Corollary 115. *For $X = \{z\}, Y = Z$ or $X = Z$ and $Y = \mathfrak{k}^n$, we have* $\dim Z \leqslant \operatorname{trdeg}(\mathfrak{K}(Z)/\mathfrak{k})$ *and* $\operatorname{codim}(Z, \mathfrak{k}^n) \leqslant n - \operatorname{trdeg}(\mathfrak{K}(Z)/\mathfrak{k})$. *In particular ($Z = \mathfrak{k}^n$),* $\dim \mathfrak{k}^n \leqslant n$ *and the opposite inequality was shown in lecture 4.*

Fact (About integrality and fields)

Let B be a domain integral over its subring A. Then B is a field if and only if A is a field.

Lemma 116. *There are $a_1,\dots,a_n \in A$ whose images in $A/\mathfrak{q}$ form a transcendence base for $\mathfrak{k}(\mathfrak{q})/\mathfrak{k}$.*

13 Bonus Sheet 5

Definition 117. *Let X be a topologial space and $\mathcal{B}$ be a set of open subsets of X.*

We call $\mathcal{B}$ a base for the topology of X if the following two equivalent conditions hold:

1. *If $x \in X$ and U is a neighbourhood of x, then there is $B \in \mathcal{B}$ s.t. $x \in B \subset U$.*
2. *Any open subset of X may be represented as a union of elements of $\mathcal{B}$.*

Lemma 118. *Let $\mathcal{B}$ be a base for the topology of X. X is quasi-compact if and only if any covering of X by the elements of $\mathcal{B}$ has a finite sub-covering.*

Lemma 119. *Let R be a ring and set $X = \operatorname{Spec} R$ equipped with the Zariski topology as defined on Sheet 4. Then $\{B_f : f \in R\}$ is a base of the topology, where $B_f := \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. X is quasi compact by the lemma above.*

14 Bonus: Nullstellensatz

14.1 Lecture 1

Theorem 120. *(Hilbert's Nullstellensatz)*

Let $\mathfrak{k}$ be a field, $R = \mathfrak{k}[X_1 \dots, X_n]$ and I an ideal in R.

If $\mathfrak{k}$ is algebraically closed and $I \subset R$ a proper ideal in R, then I has a zero in $\mathfrak{k}^n$.

Theorem 121. *Hilbert Nullstellensatz (version II)*

Let L/Kbe an arbitrary field extension. Then L/K is a finite field extension if and only if L is a K-algebra of finite type.

14.2 Lecture 3

Theorem 122. *(Hilbert's Nullstellensatz)*

Let I be an ideal of R. Then $V(I) \subseteq V(f)$ (in other words, f vanishes on the set of I) if and only if $f \in \sqrt{I}$.

14.3 Lecture 5

Theorem 123. *(Hilbert Nullstellensatz) If L/K is a field extension and L of finite type as a K-algebra, then this field extension is finite.*

Theorem 124. *(Hilbert Nullstellensatz for uncountable fields)*

If K is an uncoutable field and L/K a field extension and L of finite type as a K-algebra, then this field extension is finite.