Lecture notes on PDE and Modelling

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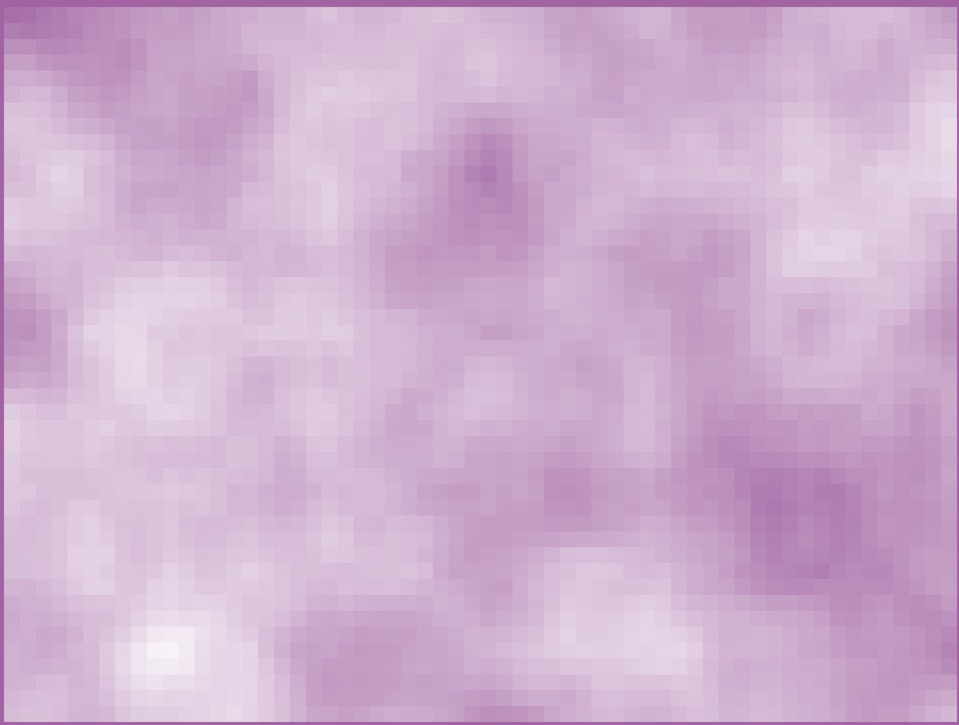

University of Bonn
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Chapter 0:
Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Basis: Basis
- Website: https://ins.uni-bonn.de/teachings/ss-2025-467-v5e1-advanced-topics/
- Time slot(s): **Wednesdays: 10-12** Zeichensaal and **Fridays: 08-10** Zeichensaal
- Exams: Oral or written depending on what the class wants
- Deadlines: Fridays at ?

0.1 Topics

Mostly about Continuum mechanics: fluids and solids

Start of lecture 01 (09.4.2025)

- He will follow the **rational mechanics** community
  - **Balance laws** (universal): Conservation of (mass, energy, momentum)
  - **Constitutive equations** (describe specific material): For example the relation between deformation and force needed to achieve the deformation
- Prequel to rational mechanics: **Kinematics**: Description of admissible states
- Sequel: **Second law of thermodynamics**: Entropy is increasing in a closed system[1]

These lead to formulation of equations:

1. Fluids (Euler equations, Navier-Stokes equations)
   - Scaling laws
   - Evolution equations, questions of singularities
   - Turbulence
2. Solids / Flosticity / Variational methods (minimization of functions)

[1]This is an inequality! This is a constraint / restriction

0.2 Organization

- 15 minute break after 45 min
- There are lecture notes
- Exercises: 50 percent, teams of 2, friday is the deadline, tutorials in the third week
- Books:
    - Morta, Gurtin: An introduction to continuous mechanics

Chapter 1:
General principles

1.1 Kinematics

1.1.1 Deformations

Let $\Omega \subset \mathbb{R}^n$ be a domain .

This somehow creates playing field
Open, connected set

Definition 1.1. *A C^k $(k \geq 1)$ deformation of a domain $\Omega \subset \mathbb{R}^n$ is a map $\varphi : \Omega \to \mathbb{R}^n$ with:*

1. *$\varphi \in C^k(\Omega, \mathbb{R}^n)$*
2. *φ has a continuous extension to $\overline{\Omega}$ and the extension is invertible*
3. *φ is **orientation preserving**: $\det D\varphi > 0$ in Ω.*

*Ω is often called the **reference configuration** of the body and $\varphi(\Omega)$ is often called the **deformed configuration** (position in physical space).*

Some linear algebra

Simplest example: $\varphi(x) = Ax + b$ with $b \in \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$ linear with $\det A > 0$. We often identify linear maps with the corresponding matrix with respect to the standard basis of $\mathbb{R}^n$.

$$Ae_j = \sum_{i=1}^n A_{ij} e_i$$

We also usually use the standard scalar product on $\mathbb{R}^n$

$$x \cdot y = \sum_{i=1}^n x_i y_i$$

Transpose is defined by

$$A^\intercal x \cdot y = x \cdot Ay$$

for all $x, y \in \mathbb{R}^n$.
A is symmetric if $A^\intercal = A$. A symmetric matrix A is positive semi-definite if

$$Ax \cdot x \geq 0$$

and positive definite if

$$Ax \cdot x > 0.$$

Definition 1.2.

$$O(n) = \{A : A^\intercal A = Id\}$$

$$SO(n) = \{A : A^\intercal A = Id, \det A = 1\}$$

Some basic facts:

- A isometry: $|Ax|^2 = |x|^2 \forall x \in \mathbb{R}^n$ which holds $\iff A \in O(n)$
- $A \in O(n) \implies \det A = \pm 1$

Norms on matrices:

- **Euclidean norm (Hilbert Schmidt norm)**:
  $|A|^2 = \operatorname{tr}(A^\intercal A) = \sum_{i,j=1}^n A_{ij}^2, A \cdot B = \operatorname{tr} A^\intercal B$
- **Operator norm** $\|A\| = \operatorname{supp}_{x \neq 0} \frac{|Ax|}{|x|} = \sup_{|x|=1} |Ax| \implies \|AB\| \leq \|A\| \|B\|$

Lemma 1.3 (Polar decomposition)**.** *Let* $A : \mathbb{R}^n \to \mathbb{R}^n$ *linear,* $\det A > 0$*.*

1. *There exists a unique pair* (R, U) *with* $R \in SO(n), U$ *symmetric s.t.*
$$A = RU$$
2. *there exists a unique pair* (R', V) *with* $R' \in SO(n), U$ *symmetric s.t.*
$$A = VR'$$

If $\det A = 0$ *this is still possible, but we lose uniqueness.*

Theorem 1.4. *Let* W *be symmetric and positive semi-definite. Then there exists a unique symmetric and positive semi-definite matrix* U*, such that*
$$W = U^2.$$

Notation: $U = \sqrt{W}, U = W^{\frac{1}{2}}$.

Proof. Just existence(not uniqueness).
Let $D = \operatorname{diag}(d_i), d_i \geq 0$, then $D^{\frac{1}{2}} = \operatorname{diag}(\sqrt{d_i})$. For W symmetric positive definite, we get
$$W = Q^\intercal D Q, Q \in \mathrm{SO}(n)$$
Then $U = Q^\intercal D^{\frac{1}{2}} Q$ is the square root:
$$U^2 = Q^\intercal D \underbrace{QQ^\intercal}_{=\mathrm{Id}}$$
□

Proof of 1.3. For (ii) apply (i) to $A^\intercal$.
For (i): Assume we already knew that there are (R, u) s.t.
$$A = RU.$$
Then
$$A^\intercal = U^\intercal R^\intercal = U R^\intercal$$
$$A^\intercal A = U \underbrace{R^\intercal R}_{=\mathrm{Id}} U = U^2$$
$$(A^\intercal A)^\intercal = A^\intercal A$$
$$(A^\intercal Ax \cdot x = (Ax \cdot Ax) \geq 0)$$
$$\implies U = (A^\intercal A)^{\frac{1}{2}}$$
If such a decomposition exists, U is unique by the formula.
For existence define $U = (A^\intercal A)^{\frac{1}{2}}$ and $R := AU^{-1}$. We have to check that $R \in \mathrm{SO}(n)$:
$$R^\intercal R = (U^{-1})^\intercal A^\intercal A U^{-1} = \mathrm{Id}$$
$$\det R = \det A \det(U^{-1}) > 0 \implies R \in \mathrm{SO}(n).$$
□

Lemma 1.5. *$A \in \mathbb{R}^{n\times n}$. Then there exists $R, Q \in SO(n)$ and $\lambda_1 \in \mathbb{R}, \lambda_2, \dots, \lambda_n \geq 0$ s.t. $|\lambda_1| \leq \lambda_2 \leq \dots \leq \lambda_n$:*

$$A = R \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q.$$

The $\lambda_1, \dots, \lambda_n$ are uniquely determined by A. The λ_i are called the singular values of A.

Proof. If $\det A > 0$ Polar $\overset{\Longrightarrow}{\text{decomp.}}$ $A = RU = \underbrace{R'Q^\intercal}_{R\in \mathrm{SO}(n)} DQ$.

If $\det A < 0$ consider $P = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \implies \det(AP) > 0$.

The $|A^\intercal A|$ are the eigenvalues of $(A^\intercal A)^{\frac{1}{2}}$.

□

We know $|A| = (\sum_{i=1}^n \lambda_i^2)^{\frac{1}{2}}, \|A\| = \lambda_n$ and

$$\det A = \prod_{i=1}^n \lambda_i$$

Rigid motion

Definition 1.6. *A deformation $\varphi : \Omega \to \mathbb{R}^n$ is called a **rigid deformation** if $D\varphi(x) \in SO(n) \forall x \in \Omega$.*

Theorem 1.7 (Liouville)**.** *Suppose $\varphi : \Omega \to \mathbb{R}^n$ is C^1, Ω is a domain and $\det D\varphi$. Then the following three statements are equivalent:*

1. *$D\varphi(x) \in SO(n) \forall x \in \Omega$ (Rigid motion)*
2. *φ is an **affine** rigid motion:*

$$\varphi(x) = Ax + b, A \in SO(n), b \in \mathbb{R}^n$$

3. *$|\varphi(x) - \varphi(y)| = |x.y| \forall x, y \in \mathbb{R}^n$*

Proof. Ideas (rest is homework):

- (ii) $\implies$ (iii) is clear (since we assume the determinant to be > 0)
- (iii) $\implies$ (ii): nice ex. (true in hilbert spaces). $\varphi((1-\lambda)x + \lambda y) = (1-\lambda)\varphi(x) + \lambda\varphi(y)$ By using that spheres are mapped to spheres (and two of those intersect in a single point)
- (ii) $\implies$ (i): trivial
- (i) $\implies$ (ii): goes via local version of (iii). Claim: for all $x_0 \in \Omega \exists r > 0, B_r(x_0) \subset \Omega$ and $|\varphi(x) - \varphi(y)| \leq |x-y| \forall x, y \in B_(0)$
- $\geq$Use inverse function theorem

□

This can be generalized to Sobolev spaces.

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