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# Lecture notes on Stochastic Analysis

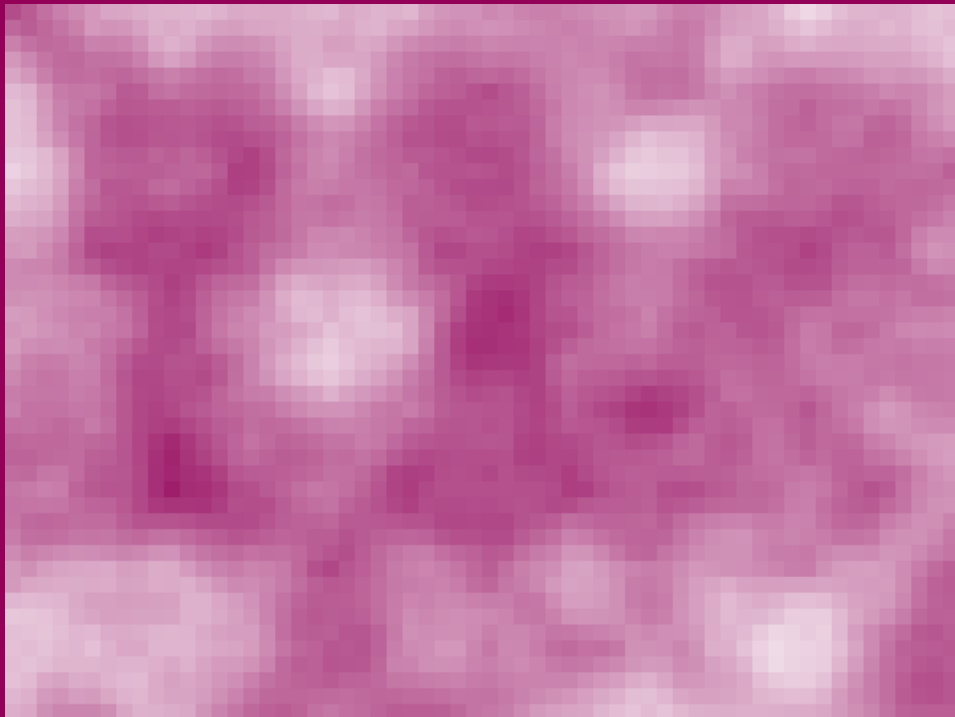
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# Contents

<b>Chapter 0</b>	<b>Manuel's notes</b>	<b>2</b>
<b>Chapter 1</b>	<b>Weak solutions of SDEs</b>	<b>3</b>
1.1	Strong solutions	3
1.2	Weak solutions	4
1.3	Lévy characterization	9
1.4	Weak solutions and martingale problems	10
1.4.1	Itô-Doeblin formula	10
1.4.2	Martingale problem	11
1.5	Weak solutions and time change	13
1.5.1	Time change	13
1.5.2	Time change in a martingale problem	13
1.5.3	Weak solutions in $d=1$	14
1.5.4	Uniqueness of martingale solution	16
<b>Chapter 2</b>	<b>SDE techniques</b>	<b>18</b>
2.1	Girsanov theorem	18
2.1.1	Drift transformation of SDE	19
2.2	Doob-h transform	20
2.3	Diffusion Bridges	21
<b>List of Lectures</b>		<b>22</b>

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# Chapter 0:

## Manuel's notes

### Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to [mh@mssh.dev](mailto:mh@mssh.dev) or [s6mlhinz@uni-bonn.de](mailto:s6mlhinz@uni-bonn.de).  
Happy learning!

### General Information

- Ecampus: [Ecampus link](#)
- Basis: [Basis link](#)
- Website: None
- Time slot(s): Tuesday 12-14 and Thursday 12-14
- Exams: Oral: 23-25 July, 30 Sept. maybe 27. Sept
- Deadlines: ?
- Exercises: One of Tue 16-18(for now this is preferred), Friday 10-12. Starting 16.04.

- First halve based on Eberle and / or Gubinelli ( be careful with Notation of dimensions!)

Start of lecture 01  
(11.04.23)

## Overview of the content

- Weak solutions of SDE
  - Martingale problem (characterization)
  - Time change (Dubin-Schwarz)
  - Change of measure (Girsonov)
- How to condition a diffusion to stay in a given domain forever (reflected BM, local time, ...)
- Lévy processes (a bit)
- Multiplicative stochastic heat-equation
  - relations with Kardar-Pavisi-Zhang class of growth models

# Chapter 1:

## Weak solutions of SDEs

SDE:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \text{ Initial condition} \end{cases} \quad (1.1)$$

- $X_t \in \mathbb{R}^d, B : n\text{-dim BM}$
- $b(t, x) : [b_k(t, x)]_{1 \leq k \leq d}$ : **drift vector**
- $\sigma(t, x) = [\sigma_{k,l}(t, x)]_{\substack{1 \leq k \leq d \\ 1 \leq l \leq n}}$ : **dispersion matrix**
- $a(t, x) = \sigma(t, x) \cdot \sigma(t, x)^\top$ : **diffusion matrix**

### 1.1 Strong solutions

**Definition 1.1.** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space with  $\mathcal{F}_t = \sigma(x_0, (B_s)_{0 \leq s \leq t})$

- a  $d$ -dim process  $X_t$  is a **strong solution** of equation 1.1 if:

- $X_t = x_0$  a.s.
- $X_t$  is adapted to  $\mathcal{F}_t \forall t \geq 0$
- $X$  is a continuous semimartingale s.t.  $\forall t \geq 0$ :

$$\int_0^t (\|b(s, X_s)\| + \|\sigma(s, X_s)\|^2) ds < \infty \text{ a.s.}$$

- $X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$

In the last semester we proved:

**Theorem 1.2.** Assume that  $b, \sigma$  are globally lipschitz with at most linear growth at  $\infty$  (in space)  
 $\implies \exists!$  strong solution of SDE.

*Foundations of Stochastic Analysis Thm 8.6*

**Added remark.** There exists  $K > 0$  s.t. for all  $x, y \in \mathbb{R}^d$ : Globally Lipschitz:

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|$$

Linear growth condition:

$$\|b(t, x)\| + \|\sigma(t, x)\| \leq K(1 + \|x\|)$$

**Remark.** For strong solutions,  $\mathcal{F}_t$  is given by the driving BM, wick is given to us.  
 $\implies X_t(\omega) = \Phi(x_0(\omega), (B_s)_{0 \leq s \leq t})$

## 1.2 Weak solutions

- For weak solutions we do not fix the driving brownian motion.

**Definition 1.3.** A **weak solution** of equation 1.1 is a **pair** of adapted processes  $(X, B)$  to a  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  s.t.

- $B$  is a  $n$ -dim BM
- $X$  is a  $d$ -dim continuous semimartingale with
  1.  $X_0 = x_0$  a.s.
  2.  $\forall t \geq 0$

$$\int_0^t (\|b(s, X_s)\| + \|\sigma(s, X_s)\|^2) ds < \infty \text{ a.s.}$$

$$3. X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$$

**Remark.** • The filtration  $(\mathcal{F}_t)_{t \geq 0}$  is not necessarily the one generated by  $B$

- If  $X$  is adapted to the filtration generated by the BM  $\implies$  we have strong solutions
- $\exists$  weak solution which are not strong solutions
- Warning: To give a weak solution we really need to provide  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X, B)$

**Definition 1.4** (Uniqueness in law). An SDE 1.1 has **uniqueness in law** if given any two weak solutions  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X, B)$  and  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{B})$  satisfy:

$$Law_{\mathbb{P}}(X) = Law_{\tilde{\mathbb{P}}}(\tilde{X})$$

They agree on any set in the sigma algebra

**Definition 1.5** (Pathwise uniqueness). The SDE 1.1 has pathwise uniqueness if, whenever the filtered space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and  $(B_t)_{t \geq 0}$  are fixed, then two solutions  $X, \tilde{X}$  with  $X_0 = \tilde{X}_0$  are indistinguishable.

**Example 1.6** (No strong solutions, no pathwise uniqueness,  $\exists$  weak solution & uniqueness in law by Tanaka).

$$\begin{cases} dX_t = \text{sgn}(X_t) dB_t \\ X_0 = 0 \end{cases} \quad (1.2)$$

or more generally  $X_0 = Y$ , where

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

Let  $W$  be a BM with  $W_0 = Y$ . Define

$$B_t := \int_0^t \text{sgn}(W_s) dW_s \text{ or } dB_t = \text{sgn}(W_t) dW_t$$

$$\implies dW_t = \text{sgn}(W_t) dB_t$$

$$\implies W_t = y + \int_0^t \text{sgn}(W_s) dB_s$$

$B_t$  is a local martingale with

$$\langle B \rangle_t = \int_0^t \underbrace{(\text{sgn}(W_s))^2}_{=1} \underbrace{d\langle W \rangle_s}_{=ds} = t$$

Also  $B_0 = 0$ , therefore  $B$  is a BM (see Lévy characterization)  $\implies W$  solves the SDE. For  $Y = 0$ ,  $W$  **and**  $-W$  solves the same SDE.

$\implies$

- exists weak solutions
- For  $Y = 0$ : no pathwise uniqueness
- Uniqueness in law (because the law is determined by  $X_t$  being a BM)
- No strong solution, because:  $\mathcal{F}^B = \mathcal{F}^{|W|} \subsetneq \mathcal{F}^W$

Not a proof just yet, just the reason!

**Example 1.7** (No solutions).

$$\begin{cases} dX_t = -\frac{1}{2X_t} 1_{X_t \neq 0} dt + dB_t \\ X_0 = 0 \end{cases} \quad (1.3)$$

Assume there exists a solution. Use Itô formula for  $X_t^2$ , then:

$$\begin{aligned} X_t^2 &= 2 \int_0^t X_s dX_s + \int_0^t 1 ds \\ &= - \int_0^t 1_{X_s \neq 0} ds + 2 \int_0^t x_s dB_s + t \\ &= \int_0^t 1_{X_s = 0} ds + 2 \int_0^t X_s dB_s \end{aligned}$$

We will prove  $\int_0^t 1_{X_s = 0} ds = 0 \implies X_t^2$  is a local martingale,  $X_t^2 \geq 0$  (and therefore a supermartingale) and  $X_0 = 0$  ( $\implies \mathbb{E}(X_t^2) = 0$ ). If  $X_t = 0 \implies \int_0^t 1_{X_s = 0} ds = t \implies 0 = dB_t$  which are contradictions!

**Remark.** If  $X = \underbrace{M}_{\in \mathcal{M}_{loc}} + \underbrace{A}_{\in \mathcal{A}}$

$$\implies \forall b \in \mathbb{R} \int_0^t 1_{X_s = b} d\langle M \rangle_s = 0$$

## Sheet 0:

- Mail: Alexander.Becker@uni-bonn.de
- Exercise sheet handed in Fr 10 am via eCampus
- Groups of 3

Motivation:

in the last semester: Introduction to stochastic analysis

- Diffusion processes & SDEs

Here: Deepen the knowledge & have fun

- Learn SDE techniques and get other results
- Modify diffusion processes to behave in a certain way
- Ex. diffusion bridge (Brownian bridge)
- Condition a diffusion to stay in a domain
  - Ex. Condition BM to stay positive
  - Old SDE:  $dB_t = dB_t$

- New SDE:  $dX_t = \frac{1}{X_t} dx + dB_t \rightarrow P(X_t \in \cdot) = P(B_t \in \cdot | B > 0 \text{ forever})$
- $D \subset \mathbb{R}^d$  open domain,  $X$  diffusion process, with generator  $L = \Sigma b \partial_i + \frac{1}{2} \sum a^{ij} \partial_i \partial_j$

$$Y := (X | X \in D \text{ forever})$$

get drift term  $\nabla \log \phi_0$ , where  $\phi_0$  is the lowest eigenfunction of  $-L$  on  $D$  with dirichlet boundary.

### Recap:

Brownian motion:

**Added definition.**  $B_0 = 0$ , independent  $\mathcal{E} \mathcal{N}(0, t_i - t_{i-1})$  increments,  $t \mapsto B_t(\omega)$  continuous.

Regularity of path  $t \mapsto B_t(\omega)$ :

- nowhere differentiable
- $\alpha$ -locally Hölder continuous  $\iff \alpha < \frac{1}{2}$
- Quadratic variation  $\langle B \rangle_t = t$
- Generator  $\frac{\Delta}{2}$
- Recurrent  $\iff ds^2?$

Itô-Integral:

1. If  $X$  simple process  $\implies$  RS-Integral
2. Itô isometry  $\mathcal{E} \rightarrow \{L^2 - \text{local martingale}\}, \mathcal{E} \subset L^2(d[M])$  dense
3. general  $X : \int X dM$  as  $L^2$ -limit

**Added remark** (Itô formula). •

$$df(t, X_t) = \partial_t f(t, X_t) dt + \partial_x f(t, X_t) dX_t + \frac{1}{2} f''(t, X_t) d\langle X \rangle_t$$

- associative  $\int X d(\int Y dZ) = \int XY dZ$
- If  $M$  local martingale  $\implies \int X dM$  local martingale

SDEs:

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t$$

- ex./ uniqueness:  $b, \sigma$  locally Lipschitz  $\implies$  strong ex.+pathwise uniqueness until explosion time
- globally Lipschitz (in space) and linear growth ( $|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|)$ ),  $e = \infty$ .

### Problem 00.1: SDE

Let  $B$  be a one-dimensional Brownian motion (starting from 0) and let  $X_t = \sin(B_t)$ .

1. Determine the SDE of  $X_t$
2. Discuss the existence and/or uniqueness of strong solutions of the SDE
3. Determine whether the local martingale term is a martingale (Hint: for the integrability condition, think about the Itô isometry for instance)

## Solution 00.1

1.:

Idea: Use Itô formula:  $X_t = \sin(B_t) = f(B_t)$

$$dX_t = df(B_t) \stackrel{\text{Itô}}{=} \underbrace{\partial_x \cos(B_t)}_{\sqrt{1-X_t^2}} dB_t - \frac{1}{2} \underbrace{\sin(B_t)}_{X_t} dt$$

Careful:  $\sqrt{1-X_t^2}$  is not inverse mapping, because it is always positive while  $\cos(B_t)$  is not

2.:

Consider the SDE

$$dX_t = \begin{cases} -\frac{1}{2}X_t dt + \sqrt{1-X_t^2} dB_t \\ X_0 = x \in (-1, 1) \end{cases}$$

Coefficients:  $b: [-1, 1] \rightarrow \mathbb{R}, b(x) = -\frac{1}{2}x$  and  $\sigma: [-1, 1] \rightarrow \mathbb{R}, \sigma(x) = \sqrt{1-x^2}$

$$[\sigma]_{1,2} = \sup \frac{|\sigma(x) - \sigma(y)|}{|x - y|^{1/2}} < \infty$$

Probably  $\sigma^2$  Lipschitz  $\implies \sigma$  Hölder  $\frac{1}{2}$

3.:

## Problem 00.2: Time change

Let  $B$  be a one-dimensional Brownian motion (starting from 0). Let  $Y_t = \int_0^t s^2 dB_s$ .

1. Determine the SDE of  $Y_t$
2. Find  $A_t$  such that  $Y_{A_t}$  is a (stopped) Brownian motion

## Solution 00.2

1.: By Definition

$$dY_t = t^2 dB_t$$

2.: We use the Dubin-Schwarz theorem:

$$X_{\infty} \in \mathcal{M}_{\text{loc}}^0, \langle X \rangle_{\infty} = \infty, T_t := \inf\{s \geq 0 | \langle X \rangle_s \geq t\} = X_t^{[-1]}$$

$$\implies B_t := X_{T_t} \text{ 1 d BM w.r.t. } (F_{T_t})_{t \geq 0}, X_t = B_{\langle B \rangle_t}$$

$$\text{here: use } X_t = b(X_t)dt + \sigma(X_t)dB_t \implies d[X]_t = \sigma^2(X_t)dt$$

## Problem 00.3: SDE and PDE

Let  $f$  be a function supported on  $[0, 1]$ ,  $u$  the solution of

$$\frac{1}{2}u(t, x) = \frac{1}{2}x(1-x) + \frac{d^2}{dx^2}u(t, x), \quad u(0, x) = f(x).$$

Consider also the one-dimensional SDE (also known as Wright-Fisher diffusion)

$$dX_t = \sqrt{X_t(1-X_t)}dB_t$$

with  $X_0 = x \in (0, 1)$ .

1. For any fixed  $t > 0$ , define  $M_s = u(t-s, X_s)$  for  $s \in [0, t]$ . Use Itô formula to show that  $M_s$  is a local martingale
2. Assume that  $f$  is bounded and there is a bounded solution of  $u$ . Show that  $u(t, x) = \mathbb{E}_x(f(X_t))$ .



## Solution 00.3

1.:

$$\begin{aligned}
dM_s &= du(t-s, X_s) \\
&= -\partial_s u(t-s, X_s)ds + \partial_x u(t-s, X_s) \underbrace{dX_s}_{b(X_s)ds + \sigma(X_s)dB_s \text{ by asso.}} + \frac{1}{2}\partial_x^2 u(t-s, X_s) \underbrace{d[X]_s}_{=\sigma^2(X_s)ds} \\
&= \underbrace{(-\partial_s u + b\partial_x u + \frac{1}{2}\sigma^2\partial_x^2 u)(t-s, X_s)}_{=0}ds + \partial_x u(t-s, X_s)\sigma(X_s)dB_s \\
&\implies dM_s = \partial_x u(t-s, X_s)\sigma(X_s)dB_s
\end{aligned}$$

(i.e.:  $M_t - M_0 = \int_0^t \dots dB_s$ )This is a purely stochastic integral against a (local) martingale  $\implies$  martingale.

2.:

- $M_s$  true martingale:

1.  $\forall t : \mathbb{E}(\sup_{[0,t]} |M|) < \infty$ , for example:  $M$  bounded
2.  $\forall t : \mathbb{E}(\sup_{[0,t]} [M]_t) < \infty$

 $M_s := u(t-s, X_s), u$  bounded  $\implies M$  bounded  $\implies M$  true martingale $w(s, x) := u(t-s, x)$ 

$$u(t, x) = w(0, x) = \mathbb{E}_x[w(0, X_0)] \stackrel{\text{martingale}}{=} \mathbb{E}_x[w(t, X_t)] = \mathbb{E}_x[(u(0, X_t))] \stackrel{\text{PDE}}{=} \mathbb{E}_x[f(X_t)]$$

Feynman-Kac formula (later more general).

Also solvable by Kolmogorov Backward / forward equation

Start of lecture 02  
(16.04.24)**Example 1.8** (Non-uniqueness of law, non pathwise uniqueness, with solutions).

$$\begin{cases} dX_t &= 1_{X_t \neq 0} dB_t \\ x_0 &= 0 \end{cases}$$

Then

$$X_t = 0 \forall t \geq 0$$

and

$$X_t = B_t \forall t \geq 0$$

both are solutions:

$$X_t - B_t = -\int_0^t 1_{X_s=0} dB_s \implies \langle X - B \rangle_t = \int_0^t 1_{X_s=0} d\langle B \rangle_s = 0$$

Let  $\eta \sim \text{Ber}(\frac{1}{2})$  independent of  $(B_t)_{t \geq 0}$  and define

$$\tilde{X}_t = \begin{cases} 0 & \eta = 0 \\ B_t & \eta = 1 \end{cases}$$

 $\implies \tilde{X}_t$  is adapted to  $\sigma(\eta(B_s)_{0 \leq s \leq t})$ , but not to  $\sigma((B_s)_{0 \leq s \leq t})$  and therefore not a **strong solution**.

$$\begin{aligned} X_t &= \int_0^t 1_{X_s \neq 0} dB_s \\ &= \int_0^t (1 - 1_{X_s=0}) dB_s \\ B_t - \int_0^t 1_{X_s=0} dB_s \end{aligned}$$

**Example 1.9** (No strong solution, weak solution, no pathwise uniqueness, no uniqueness in law).

$$\begin{cases} dX_t &= 1_{X_t \neq 1} \operatorname{sgn}(X_t) dB_t \\ X_0 &= 0 \end{cases}$$

Let  $Y_t$  be a solution of

$$\begin{cases} dY_t &= \operatorname{sgn}(Y_t) dB_t \\ Y_0 &= 0 \end{cases}$$

$\implies X_t := Y_{t \wedge \tau}$ , where  $\tau = \inf\{s \geq 0 \mid Y_s = 1\}$  is also a solution.

**Theorem 1.10** (Yamada-Watanabe). If both existence of weak solutions and pathwise uniqueness hold, uniqueness in law also holds.

Moreover,  $\forall$  choices of  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and  $(B_t)_{t \geq 0}$  then there exists a **strong solution**.

### 1.3 Lévy characterization

**Example 1.11.** Consider the SDE

$$\begin{cases} dX_t &= O_t dB_t \\ X_0 &= x_0 \end{cases}$$

where both  $X_t$  and  $B_t$  are  $d$ -dimensional and  $O_t$  is an adapted process (matrix) s.t.  $O_t^\top O_t = 1 \forall t \geq 0$  i.e.  $O_t$  is a rotation

$$\begin{aligned} \implies X_t^k &= X_0^k + \sum_{l=1}^d \int_0^t O_s^{k,l} dB_s^l \\ \implies \langle X^k, X^{\bar{k}} \rangle_t &= \sum_{l, \bar{l}} \int_0^t O_s^{k,l} O_s^{\bar{k}, \bar{l}} d\langle B^l, B^{\bar{l}} \rangle_s = \sum_{l=1}^d \int_0^t O_s^{k,l} O_s^{\bar{k}, l} ds \\ &= \int_0^t \underbrace{(O_s O_s^\top)^{k, \bar{k}}}_{=1} ds = \delta_{k, \bar{k}} t \xrightarrow{\text{Lévy}} (X_t)_{t \geq 0} \text{ is also a } d\text{-dim BM starting from } x_0 \end{aligned}$$

**Theorem 1.12** (Yamada, Watanabe). Let  $dX_t = b(X_t)dt + \sigma(X_t)dB_t$  and assume that there exist both a increasing function  $\rho(u) \geq 0$  s.t.  $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|) \forall x, y \in \mathbb{R}$  s.t.

$$\int_0^\infty \frac{1}{\rho(u)} du = \infty \implies 0 < u < C_1 \implies \rho(u) \leq C_2 u^{0.5}$$

and some increasing concave function  $\gamma_1(u) \geq 0$  s.t.

$$|b(x) - b(y)| \leq \gamma_1(|x - y|) \forall x, y \in \mathbb{R}$$

and

$$\int_0^\infty \frac{1}{\gamma_1(u)} du = \infty.$$

Then pathwise uniqueness holds.

**Theorem 1.13** (Storokhod). Assume that  $\sigma, b$  are continuous bounded functions

$\implies$  there exist weak solutions to the SDE  $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ .

## 1.4 Weak solutions and martingale problems

Let  $(X_t)_{t \geq 0}$  be a weak solution of the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = 0 \end{cases}$$

on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ .

$\implies X_t$  is a semimartingale s.t.  $X_t^k = X_0^k + \int_0^t b(s, X_s)ds + \sum_{l=1}^n \int_0^t \sigma_{k,l}(s, X_s)dB_s^l$  and

$$\langle X^i, X^j \rangle_t = \int_0^t \underbrace{\sum_{l=1}^n \sigma_{i,l}(s, X_s) \sigma_{j,l}^T(s, X_s)}_{=a_{i,j}(s, X_s)} ds$$

### 1.4.1 Itô-Doeblin formula

Itô formula leads to

**Proposition 1.14.** For  $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ , then

$$f(t, X_t) = f(0, X_0) + \int_0^t (\sigma^T \nabla f)(s, X_s) dB_s + \int_0^t \left[ \left( \frac{\partial}{\partial s} + \mathcal{L} \right) f \right](s, X_s) ds$$

where  $(\mathcal{L}f)(t, x) = \frac{1}{2} \sum_{k,l=1}^n a_{k,l}(t, x) \frac{\partial^2}{\partial x_k \partial x_l} f(t, x) + \sum_{k=1}^n b_k(t, x) \frac{\partial}{\partial x_k} f(t, x)$ .

$\mathcal{L}$  is called the **generator**

**Remark.** The Itô-Doeblin formula provides a connection between SDEs and PDEs as we have seen in the last part **Foundations of stochastic analysis**.

**Example 1.15.** Let  $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$  be a solution of the PDE

$$\begin{aligned} \frac{\partial}{\partial t} f(t, x) + (\mathcal{L}f)(t, x) &= -g(t, x) & t \geq 0, x \in U \subseteq \mathbb{R}^d \\ f(t, x) &= \varphi(t, x) & t \geq 0, x \in \partial U. \end{aligned}$$

then  $M_t := f(t, X_t) + \int_0^t g(s, X_s)ds \in \mathcal{M}_{loc}$  by proposition 1.14 and if  $f, g$  are bounded  $M_t \in \mathcal{M}$ .

$$T := \inf\{s \geq 0 | X_s \notin U\} \implies M_t^T := M_{t \wedge T} \in \mathcal{M}.$$

Furthermore, if we assume  $T < \infty$  a.s. :

$$\mathbb{E}[M_t] = \mathbb{E}[\varphi(T, X_T)] + \mathbb{E}\left[\int_0^T g(s, X_s)ds\right] = \varphi(0, X_0)$$

where  $X_0 = x_0$ .

There are two special cases:

$g = 0 \implies$  yields the exit distributions, while

$\varphi = 0, g = 1$  yields the mean exit times.

**Example 1.16** (Feynman-Kac formula). Let  $t \in \mathbb{R}_+$  be finite. Assume  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $K : [0, t] \times \mathbb{R}^d \rightarrow \mathbb{R}_+$  be continuous functions. Assume that  $u$  is a  $C^{1,2}$ -solution (bounded, for simplicity) of

$$\begin{cases} \frac{\partial u}{\partial s}(s, x) = \frac{1}{2} \Delta u(s, x) - K(s, x)u(s, x) & s \in [0, t], x \in \mathbb{R}^d \\ u(0, x) = f(x) & x \in \mathbb{R}^d \end{cases}$$

Then  $u$  has the stochastic representation  $u(t, x) = \mathbb{E}_x \left[ f(X_t) \exp \left( - \int_0^t K(t-s, X_s) ds \right) \right]$ , where  $X_t$  is BM starting from  $X_0 = x$ .

*Sketch.* 1. Define  $r(s, x) := u(t-s, x)$  for  $s \in [0, t]$

2. Show:  $M_s := \exp(-A_s)r(s, x)$  with  $A_s = \int_0^s K(u, X_u)du$  is a local martingale.  $\square$

**Remark.** This is a reformulation of the formula from the last semester.

### 1.4.2 Martingale problem

A solution of an SDE is generically defined up to some **explosion time**  $\xi$ , where it either diverges or it exists a given domain  $U \subset \mathbb{R}^d$  (open).

$\implies$  For  $k \in \mathbb{N}$  define  $U_k := \{x \in U \mid |x| < k \wedge \text{dist}(x, U^c) \geq \frac{1}{k}\}$  with  $U = \bigcup_{k \geq 1} U_k$  and

$$T_k := \{t \geq 0 \mid x_t \notin U_k\}.$$

A solution of the SDE  $b(t, X_t)dt + \sigma(t, X_t)dB_t$  is defined up to  $\xi = \sup_{k \geq 1} T_k$ .

Start of lecture 03  
(18.04.24)

**Added remark.** uniqueness of solution to the heat equation  $\frac{1}{2}\Delta u - Ku$ : not unique! Stochastic calculus: Paolo Baldi: 10.3-10.4

Define  $(\star)$ :

$$dX_t b(t, X_t)dt + \sigma(t, X_t)dB_t \text{ with } X_0 = x_0$$

**Theorem 1.17** (Martingale problem). If  $X$  is a solution of  $(\star)$  up to time  $\zeta$ , then  $\forall f \in C^{1,2}(\mathbb{R}_+ \times U)$

$$M_t = f(t, X_t) - \int_0^t \left( \frac{\partial}{\partial s} + \mathcal{L} \right) f(s, x_s) ds, t < s$$

is a local martingale up to  $\zeta$  and  $M_t^{T_k}$  are localizing martingales.

**Definition 1.18** (Martingale solutions).  $(X_t)_{t \geq 0}$  is a **martingale solution** of  $(\star)$  if  $\forall f \in C^2(\mathbb{R}^d)$

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s) ds$$

is a continuous local martingale.

**Theorem 1.19** (Equivalent definitions). The following are equivalent (for  $X_t$  being a solution of  $(\star)$ ):

(a)  $\forall f \in C^2(\mathbb{R}^d)$ ,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s) ds$$

is a local martingale

(b) The process in  $\mathbb{R}^d$  given by

$$M_t := X_t - X_0 - \int_0^t b(s, X_s) ds$$

is a  $d$ -dimensional local martingale with  $\langle M^i, M^j \rangle_t = \int_0^t a_{i,j}(s, X_s) ds = \langle X^i, X^j \rangle_t$

(c)  $\forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$

$$\tilde{M}_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left( \frac{\partial}{\partial s} + \mathcal{L} \right) f(s, X - s) ds$$

is a local martingale

ds

*Proof.*  $c \implies a$ : by choosing  $f$  independent of  $t$ .

$a \implies b$ : 1.: Choosing  $f(X) = X_i$  implies

$$M_t^f = X_t^i - X_0^i - \int_0^t b_i(s, X_s) ds \in \mathcal{M}_{\text{loc}}$$

2.:  $f(X) = X_i X_j$ :

$$\begin{aligned} (\mathcal{L}f)(x) &= \frac{1}{2}a_{ij} + \frac{1}{2}a_{ji} + b_i X_j b_j X_i \\ a &= a^\top \implies a_{ij} b_i X_j b_j X_i \\ \implies M_t^f X_t^i X_t^j - X_0^i X_0^j - \int_0^t [a_{ij}(s, X_s) + b_i(s, X_s) X_s^j + b_j(s, X_s) X_s^i] ds \end{aligned}$$

$$\begin{aligned} X_t^i X_t^j - X_0^i X_0^j &\stackrel{\text{Integration by parts}}{=} \int_0^t X_s^i dX_s^j + \int_0^t X_s^j dX_s^i + \langle X^i, X^j \rangle_t \\ &= M_t^f + \int_0^t [a_{ij} + b_i X_s^j + b_j X_s^i] ds \end{aligned}$$

We can maybe also proof this by calculating  $X^2$ ?

Here  $dX_s^i$  is the same as  $b_i X_s^j$  is the same up to a local martingale term and  $\langle X^i, X^j \rangle_t = \int_0^t a_{ij}(s, X_s) ds = \langle M^i, M^j \rangle_t$

$b \implies c$ : By Proposition 1.14 If (use the next theorem)  $X$  was a weak solution  $\implies \tilde{M}_t^f$  is a local martingale.  $\square$

**Theorem 1.20.** Let  $n = d$ , assume  $\sigma(t, x)$  is invertible  $\forall t, x$  and  $\sigma^{-1}(t, x)$  is uniformly bounded. T.f.a.e.:

- (a)  $(X_t)_{t \geq 0}$  is a weak solution of the SDE  $(\star)$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}; B)$
- (b)  $(X_t)_{t \geq 0}$  is a martingale solution of the SDE  $(\star)$

This also works for  $n \neq d$ , but with a different proof

*Proof.*  $a \implies b$ : True

$b \implies a$ : Goal construct a BM for the weak solution.

By proposition ??  $a \implies b$   $dM_t = dX_t - b(t, X_t)dt \in \mathcal{M}_{\text{loc}}$  and  $d\langle M^i, M^j \rangle_t = a_{k,l}(t, X_t)dt$

$$\begin{aligned} \implies dX_t &= dM_t + b(t, X_t)dt \\ &= \sigma(t, X_t) d\tilde{B}_t + b(t, X_t)dt \end{aligned}$$

where  $\tilde{B}_t := \sigma(s, X_s)^{-1} dM_s$

To see:  $\tilde{B}_t$  is a brownian motion.

$$\begin{aligned} \langle \tilde{B}^i, \tilde{B}^j \rangle_t &= \sum_{k,l} \int_0^t \sigma_{ij}^{-1} \sigma_{jl}^{-1} d\langle M^k, M^l \rangle_s \\ &\quad = \underbrace{a_{ij}}_{(\sigma^\top \sigma)_{kl}} ds \\ &= \sum_{k,l,p} \int_0^t \sigma_{ik}^{-1} \sigma_{kp} \sigma_{pl}^\top \sigma_{lj}^{-1} ds \\ &= \delta_{ij} \int_0^t 1 ds = \delta_{ij} t \end{aligned}$$

Then by the Lévy characterization  $\tilde{B}$  is a brownian motion.  $\square$

**Added remark.** This is the first way to construct a weak solution: Solve a martingale problem! This is used a lot in practice.

## 1.5 Weak solutions and time change

### 1.5.1 Time change

For  $d = 1$ :

**Theorem 1.21.** [Dubins-Schwarz]

- Let  $M \in \mathcal{M}_{loc}^0$  and  $\langle M \rangle_\infty = \infty$  a.s.
- Let  $T_t := \inf\{s \geq 0 : \langle M \rangle_s \geq t\}$

This implies

1.  $t \mapsto M_{T_t}$  is a  $(\mathcal{F}_{T_t})$  brownian motion
2.  $M_t = B_{\langle M \rangle_t}$  for some standard brownian motion  $B$

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \underbrace{\int_0^t \sigma(s, X_s) dB_s}_{=M_t}. \text{ If } \langle M \rangle_\infty = \infty \text{ a.s.:}$$

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \tilde{B}_{\int_0^t \sigma^2(s, X_s) ds}$$

### 1.5.2 Time change in a martingale problem

Consider  $d = 1 = n$ .

$$dY_t = \tilde{\sigma}(Y_t) dB_t \quad (\star\star)$$

and  $\tilde{\sigma}$  strictly positive continuous function.

$$\langle Y \rangle_t = \int_0^t \tilde{\sigma}(Y_s)^2 ds =: A_t$$

By theorem 1.21  $\implies Y_t = W_{A_t}$  for some brownian motion  $W$ .

Assume  $A_t \rightarrow \infty$  a.s.

$$T_t := \inf\{s \geq 0 : \langle Y \rangle_s \geq t\}$$

$$\implies T_{A_t} = \inf\{s \geq 0 : \langle Y \rangle_s \geq \langle Y \rangle_t\} = t$$

$$1 = \frac{d}{dt} (T_{A_t}) = \underbrace{T'_{A_t}}_{=u} \cdot A_t$$

$$\implies \underbrace{T'_u}_{=\frac{dT_u}{du}} = \frac{1}{A'_{T_u}} \implies T_u = \int_0^u \frac{1}{\tilde{\sigma}(Y_{T_s})^2} ds = \int_0^u \frac{1}{\tilde{\sigma}(W_s)^2} ds$$

$\implies$  to construct a solution of  $(\star\star)$ : Given  $W \rightarrow$  compute  $T_u \rightarrow$  determine  $A - t = T_t^{-1} \implies Y_t = W_{A_t}$

Start of lecture 04  
(23-04-24)

**Theorem 1.22.** Let  $(X_u)_{u \geq 0}$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a weak solution of

$$dX_u = b(X_u) du + \sigma(X_u) dB_u$$

where  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , the drift and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  are locally bounded,  $\sigma^{-1}$  exists for a.e.  $x$  and is locally bounded.

Consider a **time change**  $T_u := \int_0^u \rho(X_s) ds$ , where  $\rho : \mathbb{R}^d \rightarrow \mathbb{R}_+$  s.t.

$$T_u < \infty \forall u \geq 0 \text{ and } T_\infty = \infty \text{ a.s.}$$

$\implies$  Then  $Y_t := X_{A_t}$ , where  $A_t = T_t^{-1}$  is a weak solution of the SDE

$$dY_t = \frac{b(Y_t)}{\rho(Y_t)}dt + \frac{\sigma(Y_t)}{\sqrt{\rho(Y_t)}}dB_t$$

**Remark.** Special case:  $d = 1, b = 0, \sigma = 1$ : Then  $X$  is a BM and  $\rho = \frac{1}{\tilde{\sigma}^2(x)} \implies Y_t = X_{T_t^{-1}}$  solves

$$dY_t = \tilde{\sigma}(Y_t)dB_t$$

*Proof.* By theorem 1.20, it is enough to see how the martingale problem after change of time becomes:

$$\mathcal{L} = \mathcal{L}_{X_t} \xrightarrow{\text{time change}} \mathcal{L}_{Y_t} = \tilde{\mathcal{L}}$$

$Y_t = X_{A_t}; Y_0 = X_{A_0}$ . For  $f \in C^2$ :  $M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$  is a local martingale w.r.t.  $(\mathcal{F}_t)_{t \geq 0}$ .

$$\implies N_t^f := M_{A_t}^f = f(\underbrace{X_{A_t}}_{=Y_t}) - f(\underbrace{X_{A_0}}_{Y_0}) - \int_{A_0}^{A_t} (\mathcal{L}f)(X_s)ds$$

is also a local martingale w.r.t.  $(\mathcal{F}_{A_t})_{t \geq 0}$ .

Change of variable (to get rid of the  $X_s$  in the integral):

$$\begin{aligned} \tau = T_s &\leftrightarrow A_t = s \implies X_s = X_{A_t} = Y_\tau \\ d\tau &= \rho(X_s)ds \end{aligned}$$

$$\implies N_t^f = f(Y_t) - f(Y_0) - \int_0^t (\mathcal{L}f)(Y_t) \frac{1}{\rho(Y_t)} d\tau$$

Since  $\mathcal{L}f(x) = \sum_k b_k(x) \frac{\partial}{\partial x_k} f(x) + \sum_{k,l} a_{k,l}(x) \frac{\partial^2}{\partial x_k \partial x_l} f(x)$

$$\implies (\tilde{\mathcal{L}}f)(x) = \sum_k \frac{b_k(x)}{\rho(x)} \frac{\partial}{\partial x_k} f(x) + \sum_{k,l} \frac{\overbrace{a_{k,l}}^{(\sigma\sigma^\top)_{k,l}}}{\sqrt{\rho(x)\rho(x)}}(x) \frac{\partial^2}{\partial x_k \partial x_l} f(x)$$

$\implies$  It is a martingale problem for the SDE where the drift  $\rightarrow \frac{\text{drift}}{\rho}$  and  $\sigma \rightarrow \frac{\sigma}{\sqrt{\rho}}$

□

### 1.5.3 Weak solutions in d=1

We will do both time and “space” changes.

- 1-d SDE:

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dB_t \\ X_0 = x_0 \in (\alpha, \beta) \end{cases} \quad (1.4)$$

- $X_t$  a process in  $(\alpha, \beta)$
- Assume  $b, \sigma : (\alpha, \beta) \rightarrow \mathbb{R}$  continuous,  $\sigma(x) > 0 \forall x \in (\alpha, \beta)$
- Do a change of coordinates  $Y_t := s(X_t)$  where  $s : (\alpha, \beta) \rightarrow (s(\alpha), s(\beta))$ ,  $C^2$  with  $s'(x) > 0, x \in (\alpha, \beta)$ .
- $s(x)$  is called the **scale function** and it is given by

$$s(x) = \int_{x_0}^x \exp\left(-\int_{x_0}^y \frac{2b(z)}{\sigma(z)^2} dz\right) dy$$

- $s$  satisfies  $\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0 \iff (\mathcal{L}s)(x) = 0$

The  $A_0$  in the integral is probably 0, but it does not matter, we do a change of variables anyway.

**Remark.** If  $b(z) = 0 \implies s(x) = x - x_0 \implies s'(x) = 1$ . If  $s'(x) = 1$ , we say that the process is in its “natural scale”

By proposition 1.14:  $\mathcal{L}s = 0, \dot{s} = 0$ .

$\implies Y_t = s(X_t)$  is a local martingale satisfies  $dY_t = s'(X_t)\sigma(X_t)dB_t$ .

the other terms cancel

$\iff Y_t$  is a solution of

$$\begin{cases} dY_t &= \tilde{\sigma}(Y_t)dB_t \\ Y_0 &= s(X_0) \end{cases} \quad (1.5)$$

where  $\tilde{\sigma}(y) = s'(s^{-1}s(y))\sigma(s^{-1}(y))$ .

Therefore we can write the original SDE in terms of a BM

**Theorem 1.23.** The following are equivalent:

1. The process  $(X_t)_{t < \xi}$ , where  $\xi$  is the explosion time, on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}; (B_t)_{t \geq 0})$  is a solution of (1.4) up to the stopping time  $\xi$
2. The process  $Y_t = s(X_t)_{t < \xi}$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}; (B_t)_{t \geq 0})$  is a solution of (1.5) up to  $\xi$
3. The process  $(Y_t)_{t < \xi}$  has the representation  $Y_t = \tilde{B}_{A_t}$ , where  $\tilde{B}$  is a BM starting at  $\tilde{B}_0 = s(X_0)$  and  $A_t = T_t^{-1}$  and  $T_t = \int_0^t \frac{1}{\tilde{\sigma}^2(\tilde{B}_u)} du$

$s$  and  $A_t$  have the same definition as before

### A degenerate case:

Let  $\sigma(x) = |x|^\alpha$  for some  $\alpha \in (0, \frac{1}{2})$ .  $\implies$

$$\begin{cases} dY_t &= |Y_t|^\alpha dB_t \\ Y_0 &= y \end{cases} \quad (1.6)$$

$\implies T_t = \int_0^t \frac{1}{\sigma(\tilde{B}_u)^2} du$ ,  $A_t = \int_0^t \sigma(Y_s)^2 ds$  and  $Y_t = \tilde{B}_{A_t} \implies \tilde{B}_0 = y$ .  
 $T_t < \infty$  a.s. ?

$$\begin{aligned} \mathbb{E}(T_t) &= \int_0^t \mathbb{E} \left( \frac{1}{\sigma(\tilde{B})^2} \right) du \\ &= \int_0^t \mathbb{E} \left( \frac{1}{|\tilde{B}|^{2\alpha}} \right) du \\ &= \int_0^t \int_{\mathbb{R}} dx \frac{1}{|x|^{2\alpha}} \frac{\exp(-\frac{(x-y)^2}{2u})}{\sqrt{2\pi u}} \quad 0 < 2\alpha < 1 < \infty \end{aligned}$$

$\implies A_t = T_t^{-1}$ , then  $Y_t = \tilde{B}_{A_t}$  is a solution of (1.6), i.e.  $\forall y \in \mathbb{R} \exists$  a non-trivial solution of (1.6).

For  $Y=0, Y_t = 0$  is also a solution  $\implies$

- No uniqueness in law of (1.6)
- No pathwise uniqueness as well

**Remark.** In general: uniqueness in law of 1-d SDEs is not to be expected if  $\sigma(x) = 0$  somewhere (and  $\sigma$  continuous ...) (i.e. if  $\sigma$  is degenerate).

By theorem 1.12 as soon as  $\sigma(x) = |x|^\alpha$  for some  $\alpha \geq \frac{1}{2}$ , then one has pathwise uniqueness.

### Hitting times and scale functions Bessel process:

$$\begin{cases} dR_t &= \frac{d-1}{2R_t} dt + dW_t \\ R_0 &= r_0 \in (0, \infty) \end{cases}$$

$\implies b(x) = \frac{d-1}{2x}, \sigma(x) = 1$ .

The scale function satisfies  $\mathcal{L}s(x) = \frac{1}{2}s''(x) + \frac{d-1}{2x}s'(x) = 0$



$$\implies s(x) = \begin{cases} \frac{x^{2-d}}{2-d} & d \neq 2 \\ \ln(x) & d = 2 \end{cases}$$

and therefore

$$(s_0, s_\infty) = \begin{cases} (0, \infty) & d < 2 \\ (-\infty, \infty) & d = 2 \\ (-\infty, 0) & d > 2 \end{cases}$$

Define the stopping time

$$T_a^R := \inf\{t \geq 0 \mid R_t = a\}$$

Choose an  $\alpha < r_0 < \beta$

$$\implies \mathbb{P}(T_\alpha^R < T_\beta^R) \stackrel{s' \geq 0}{=} \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

One computes

$$\mathbb{P}(T_a^R < T_\beta^R) = \frac{s(\beta) - s(r_0)}{s(\beta) - s(\alpha)} = \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

This is generic, for Itô diffusions, provides that  $\exists$  no killing in  $(\alpha, \beta)$ .

WS exercises

unlike in 1.16  
Start of lecture 05  
(25.04.24)

### 1.5.4 Uniqueness of martingale solution

SDE  $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$  with generator

$$\mathcal{L} = \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l} (\sigma \sigma^\top)_{k,l} \frac{\partial^2}{\partial x_k \partial x_l}$$

**Definition 1.24.** Let  $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R}^d)$  with  $\sigma$ -algebra  $\mathcal{F}$ , canonical filtration  $(\mathcal{F}_t)_{t \geq 0}$ , canonical process  $Z_t(\omega) := \omega$ .

We say that  $\mathbb{P}$  on  $(\mathcal{C}, \mathcal{F})$  is a martingale solution for the generator  $\mathcal{L} \iff \forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R})$

$$M_t^f := f(t, Z_t) - f(0, Z_0) - \int_0^t \left( \frac{\partial}{\partial s} + \mathcal{L} \right) f(s, Z_s) ds \quad (1.7)$$

is a martingale w.r.t.  $\mathbb{P}$ .

**Definition 1.25.** A martingale problem (1.7) has a unique solution if for any two martingale solutions  $\mathbb{P} = \mathbb{Q}$  s.t.  $\text{Law}_{\mathbb{P}}(Z_0) = \text{Law}_{\mathbb{Q}}(Z_0)$

$$\implies \mathbb{P} = \mathbb{Q}$$

**Remark.** Uniqueness of martingale solutions corresponds to uniqueness in law of the weak solutions.

Backwards Kolmogorov Equation (BKE):

$$\frac{\partial}{\partial t} \varphi(t, x) = \mathcal{L} \varphi(t, x) \forall x \in \mathbb{R}^d, t \geq 0 \quad (1.8)$$

**Theorem 1.26.** Assume that  $\forall$  initial condition

$$\varphi(0, x) = \Psi(x), \Psi \in C_0^\infty(\mathbb{R}^d)$$

the (1.8) has a solution and  $\varphi$  bounded for all finite time intervals. We have uniqueness of martingale solutions and therefore uniqueness of weak solutions!

The Kolmogorov forward equation is (related to) the Fokker-Planck equation!

*Proof.* Prove that  $\forall 0 \leq t_1 < t_2 < \dots < t_n$ :

$$\text{Law}_{\mathbb{P}}(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}) = \text{Law}_{\mathbb{Q}}(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n})$$

### 1. One-time distribution:

$\forall 0 \leq s \leq r$ :

$$\left( \frac{\partial}{\partial s} + \mathcal{L} \right) \varphi(r-s, x) \stackrel{(1.8)}{=} 0$$

take  $t \in [0, r]$ :

$$\begin{aligned} M_t^r &:= \varphi(r-t, Z_t) - \varphi(r, Z_0) - \int_0^t \underbrace{(\partial_s + \mathcal{L})\varphi(r-s, Z_s)}_{=0} ds \\ &= \varphi(r-t, Z_t) - \varphi(r, Z_0) \text{ is a martingale} \end{aligned}$$

for any solution  $\mathbb{P}$ .

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{P}}(M_t^r - M_0^r \mid \mathcal{F}_t) = \mathbb{E}(\varphi(0, Z_r) - \varphi(r-t, Z_t) \mid \mathcal{F}_t) \\ \implies \forall 0 \leq t \leq r : \mathbb{E}_{\mathbb{P}}(\varphi(0, Z_r) \mid \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}(\varphi(r-t, Z_t) \mid \mathcal{F}_t) \\ &\stackrel{\text{a.s.}}{=} \varphi(r-t, Z_t) \\ \underbrace{\mathbb{E}(\varphi(0, Z_r))}_{\Psi(Z_r)} &\stackrel{t=0}{=} \mathbb{E}_{\mathbb{P}}(\varphi(r, Z_0)) \end{aligned}$$

$\forall$  other martingale solutions  $\mathbb{Q}$ :

$$\mathbb{E}_{\mathbb{Q}}(\Psi(Z_r)) = \mathbb{E}_{\mathbb{Q}}(\varphi(r, Z_0))$$

By assumption this implies  $\text{Law}_{\mathbb{P}}(Z_r) = \text{Law}_{\mathbb{Q}}(Z_r)$ .

### 2. Multi-time distributions:

For  $\Psi \in C_0^\infty$ , denote  $\varphi_\Psi$  the solution of (1.8) with initial condition  $\Psi$ :

$$\mathbb{E}_{\mathbb{P}}(\Psi(Z_r) \mid \mathcal{F}_t) = \varphi_\Psi(r-t, Z_t)$$

$0 \leq r_2 \leq r_1$  test for  $g \in C_0^\infty$ :

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(\Psi(Z_{r_1})g(Z_{r_2})) &= \mathbb{E}(\underbrace{\mathbb{E}(\varphi_\Psi(Z_{r_1}) \mid \mathcal{F}_{r_2})}_{\varphi_\Psi(r_1-r_2, Z_{r_2})} g(Z_{r_2})) \\ &= \mathbb{E}_{\mathbb{P}}(\varphi_\Psi(r_1-r_2, Z_{r_2})g(Z_{r_2})) \\ &\stackrel{1.}{=} \mathbb{E}_{\mathbb{Q}}(\varphi_\Psi(r_1-r_2, Z_{r_2})g(Z_{r_2})) \\ &= \mathbb{E}_{\mathbb{Q}}(\Psi(Z_{r_1})g(Z_{r_2})) \end{aligned}$$

Iterating yields the statement. □

This needs the boundedness of  $\varphi$ , otherwise it might only be a local martingale. There are softer conditions we can put on the coefficients to achieve the same result. This might not be needed, because  $\varphi$  is  $C^1$  in time anyways

# Chapter 2:

## SDE techniques

**Goal:** Study process by changing the measure.  
 E.g.:  $X_t = B_t$ .  $\underbrace{\text{Condition } X_t \geq 0 \forall t \geq 0}_{=: Y_t}$ .

### 2.1 Girsanov theorem

Given  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$  and two measures  $\mathbb{P}, \mathbb{Q}$   
 Assume  $\mathbb{P} \ll \mathbb{Q}$  and  $\mathbb{Q} \ll \mathbb{P}$  and let  $H = \frac{d\mathbb{Q}}{d\mathbb{P}}$

The notes of Eberle switches the roles of  $\mathbb{P}, \mathbb{Q}$ !

$$\implies Z_t := \mathbb{E}_{\mathbb{P}}(H \mid \mathcal{F}_t) = \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t}$$

is a martingale.

From last semester:  $\forall Y \in L^1(\mathbb{Q}) \cap L^1(\mathbb{P}), \mathcal{F}_t$  measurable:

$$\mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{F}_s) = \frac{bE_{\mathbb{P}}(Y \cdot Z_t \mid \mathcal{F}_s)}{Z_s} \forall s < t$$

If  $Z > 0$  is  $\mathcal{M}_{\text{loc}}$ ,  $\exists L \in \mathcal{M}_{\text{loc}}$  s.t.:

$$Z_t = e^{L_t - \frac{1}{2} \langle L \rangle_t} \rightarrow L_t = \ln(Z_0) + \int_0^t \frac{dZ_s}{Z_s}$$

There might be a problem, because  $\ln(Z_0)$  might not be integrable, and therefore not a local martingale!

**Theorem 2.1** (Girsanov). Assume  $Z > 0$  is a martingale. If  $M$  is a local martingale w.r.t.  $\mathbb{P}$ , then

$$\tilde{M}_t := M_t - \langle M, L \rangle_t$$

is a local martingale w.r.t.  $\mathbb{Q}$  and

$$\langle M \rangle_t = \langle \tilde{M} \rangle_t.$$

Moreover, if  $M$  is a BM w.r.t. to  $\mathbb{P}$ , then  $\tilde{M}$  is a BM w.r.t.  $\mathbb{Q}$ .

**Remark.** In applications, given  $\mathbb{P}, (Z_t)_{t \geq 0}$  a positive continuous martingale, define  $\mathbb{Q}$  on  $\mathcal{F}_{\infty} = \bigcup_{t \geq 0} \mathcal{F}_t$  s.t.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = Z_t$$

$Z$  is uniformly integrable  $\iff \mathbb{Q} \ll \mathbb{P}$ . **Problem:** In applications,  $Z$  is not necessarily uniformly integrable.

$\implies$  restrict to  $[0, T] \implies$  all fine.

$\mathbb{Q} \rightarrow \mathbb{Q}_T$  as in the last semester.

**Added example.** Let  $\gamma \in \mathbb{R}^d$ ,  $(B_t)_{t \geq 0}$  standard BM.

Let  $L_t := \gamma \cdot B_t$  and  $Z_t = \exp(L_t - \frac{1}{2}\langle L \rangle_t) = \exp(\gamma \cdot B_t - \frac{1}{2}|\gamma|^2 t)$

**Remark.**  $\lim_{M \rightarrow \infty} \sup_{t \geq 0} \mathbb{E}(|Z_t| 1_{|Z_t| > M}) \neq 0$ .

Define  $\mathbb{Q}$  on  $\mathcal{F}_\infty$  s.t.  $\tilde{B}_t^k = B_t^k - \langle L, B^k \rangle = B_t^k - \gamma^k t$  is a BM with drift. Show:  $\mathbb{Q} \ll \mathbb{P}$ .

Construct  $\mathbb{Q}$  via  $Z$

$$A = \left\{ \lim_{t \rightarrow \infty} \frac{\tilde{B}_t + \gamma t}{t} = \gamma \right\} \in \mathcal{F}_\infty$$

but:  $\tilde{B}$  is a BM w.r.t.  $\mathbb{Q} \implies \mathbb{Q}(A) = 1$ ,  $\tilde{B}$  is a BM with drift  $-\gamma$  w.r.t.  $\mathbb{P}(A) = 0$ .

Start of lecture 06  
(30.04.24)

### 2.1.1 Drift transformation of SDE

We once again start with

$$\star = \begin{cases} dX_t &= b(t, X_t) \\ X_0 &= x_0 \end{cases}$$

with drift  $b$  continuous.

**Goal:** Get a weak solution of  $\star$ .

Let  $(X_t)_{t \geq 0}$  be a BM in  $(\Omega, \mathcal{F}(\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ ,  $X_0 = x_0$ .

**Assume:**  $Z_t := \exp \left( \underbrace{\int_0^t b(s, X_s) dX_s}_{=L_t} - \frac{1}{2} \int_0^t |b(s, X_s)|^2 ds \right)$  is a martingale w.r.t.  $\mathbb{P}$ .

**Remark.** The assumption holds if

$$|b(t, x)| \leq C(1 + \|x\|) \text{ for some } C.$$

$$\implies \mathbb{E}_{\mathbb{P}}(Z_t) = 1 \forall t \geq 0$$

By Girsanov

$$\tilde{X}_t = X_t - \langle L, X \rangle_t$$

is a  $\mathbb{Q}$ -BM.

But

$$d\langle L, X \rangle_t = b \cdot dX_t \cdot dX_t = b \cdot dt$$

$$\implies \tilde{X}_t = X_t - \int_0^t b(s, X_s) ds$$

$$\text{w.r.t. } \mathbb{Q} : dX_t = b(t, X_t) dt + \underbrace{d\tilde{B}_t}_{d\tilde{X}_t}$$

Generalization: Start with

$$\star \star dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t$$

where  $X, B$  are  $d$ -dimensional.

**Proposition 2.2.** Assume  $(X, B)$  is a weak solution of  $\star \star$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ . If

$$Z_t = \exp \left( \int_0^t c(s, X_s) dB_s - \frac{1}{2} \int_0^t c(s, X_s)^2 ds \right)$$

is a martingale w.r.t.  $\mathbb{P}$ ,  $\mathbb{Q} \ll \mathbb{P}$  and  $\mathcal{F}_t$  s.t.  $Z_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}} \text{Vert}_{\mathcal{F}_t}$ .

In practice this is surprisingly useful in practice!

this is an implicit condition for  $c$

Then  $(X_t)_{t \geq 0}$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$  is a weak solution of

$$dX_t = [b(t, X_t) + \sigma(t, X_t)c(t, X_t)dt] + \sigma(t, X_t)d\tilde{B}_t$$

where  $\tilde{B}$  is a  $d$ -dim BM.

*Proof.*

$$\begin{aligned} d\langle L, B \rangle_t &\stackrel{L_t = \int_0^t c(s, X_s)dB_s}{=} c(t, X_t)dt \\ \implies \tilde{B}_t &:= B_t - \langle L, B \rangle_t \text{ is a } \mathbb{Q}\text{-BM} \\ \implies dB_t &= c(t, X_t)dt + d\tilde{B}_t \end{aligned}$$

From  $\star\star$ :  $dX_t = b(t, X_t)dt + \sigma(t, X_t)c(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t = \star\star\star$  □

Measure	SDE	Generators
$\mathbb{P}$	$dX_t b \cdot dt + \sigma \cdot dB_t$	$\mathcal{L} \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l} (\sigma \sigma^\top)_{k,l} \frac{\partial^2}{\partial x_k \partial x_l}$
$\mathbb{Q}$	$[b + \sigma \cdot c]dt + \sigma d\tilde{B}_t$	$\tilde{\mathcal{L}} = \mathcal{L} + \sum_k \sum_l \sigma_{k,l} c_l \frac{\partial}{\partial x_k} = \mathcal{L} + c \sigma^\top \nabla$

$t \leq T$  for  $\mathbb{Q}_t$ .

## 2.2 Doob-h transform

1. From  $B_t \rightarrow B_t$  conditioned on  $\{B_1 = 0\}$  (measure zero set)
2. From  $B_t \rightarrow B_t$  conditioned on  $\{B_t \geq 0 \forall t \geq 0\}$  (measure zero set)
3. From  $B_t \rightarrow B_t$  conditioned on  $\{B - t \geq 0 \forall t \in [0, 1], B_1 = 0\}$

Essentially three cases in 1d

Let  $(X_t, B_t)_{t \geq 0}$  be a weak solution of

$$(\star) = \begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \text{ fixed} \end{cases}$$

Assume there exists  $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$  s.t.  $h > 0$  satisfying

$$\left(\frac{\partial}{\partial t} + \mathcal{L}\right)h = 0 \forall t \in [0, T], x \in \mathbb{R}^d$$

where  $\mathcal{L}$  is the generator of the SDE  $(\star)$ .

By theorem 1.14  $Z_t := h(t, X_t) = h(0, X_0) + \int_0^t (\sigma \nabla h)(s, X_s)dB_s$ .

$Z_t$  is a positive local martingale. Assume  $Z_t$  is a martingale.

W.l.o.g.  $Z_0 = 1$  (if not  $h \mapsto \frac{h}{h(0, X_0)}$ ).

$\implies d\mathbb{Q}_T = Z_T \cdot d\mathbb{P}$ . Let  $L_t$  s.t.  $Z_t = \exp(L_t - \frac{1}{2}\langle L \rangle_t)$ .

Girsanov, since  $B_t$  is a  $\mathbb{P}$ -BM,

$\tilde{B}_t := B_t - \langle L, B \rangle_t$  is a  $\mathbb{Q}_T$ -BM, where

$$\langle L, B \rangle_t = \sigma^\top \cdot \nabla \ln h \cdot dB_t$$

because of

$$dL_t = \frac{dZ_t}{Z_t} = \frac{\sigma^\top \cdot \nabla h \cdot dB_t}{h(t, X_t)} = \sigma^\top \cdot \nabla \ln h \cdot dB_t$$

And therefore  $dB_t = d\tilde{B}_t + \sigma^\top \nabla \ln h dt$ .

From  $(\star) \implies dX_t = \underbrace{(b(t, X_t) + \sigma(t, X_t)\sigma^\top(t, X_t)\nabla h(t, X_t))}_{:= \tilde{b}}dt + \sigma(t, X_t)d\tilde{B}_t$

**Proposition 2.3.** Let  $X$  be a weak solution of  $dX_t = b(t, X_t) + \sigma(t, X_t)dB_t$  on  $[0, T]$  under  $\mathbb{P}$ . Then under  $\mathbb{Q}_T$ ,  $X$  is a weak solution of the SDE

$$\begin{cases} dX_t &= \tilde{b}(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t \\ X_0 &= x_0 \end{cases}$$

where  $\tilde{b} = b + \sigma\sigma^\top \nabla \ln h$ .

**Remark.** In applications, often  $h(x) \not\equiv 0$  for all  $x$ !

E.g.:  $h(t, x) = x \implies \left(\frac{\partial}{\partial t} + \mathcal{L}\right)h = 0$  ( $X_t = B_t + x_0$ ), but  $h(x) \leq 0$  for  $x \leq 0$ .

In this case first do the construction on  $[0, \tau]$ , where  $\tau := \inf\{t > 0 \mid B_t = 0\}$

$\implies$  ok for  $B_t^\tau \xrightarrow{\text{magically}}$  the new SDE has drift:

$$\frac{\partial}{\partial x} \ln(h(x)) = \frac{1}{h(x)} \frac{\partial h(x)}{\partial x} = \frac{1}{x}$$

We realize there is no problem in the new measure

**Example 2.4.** If  $X_t = B_t$  and  $h(t, \cdot) = e^{\gamma x - \frac{1}{2}|\gamma|^2 t}$

$$\begin{aligned} \implies dX_t &= \nabla \ln h(t, X_t)dt + dB_t \\ &= \nabla(\gamma \cdot x)dt + dB_t \\ &= \gamma dt + dB_t \end{aligned}$$

$\implies X_t$  is a BM with drift  $\gamma$  w.r.t  $\mathbb{Q}$ .

## 2.3 Diffusion Bridges

Consider a Markov process  $(X_t)_{t \geq 0}$  which is a diffusion starting from  $X_0 = x_0 \in \mathbb{R}^d$ .

We want to condition on the event  $\{X_T = y\}$  for some given  $T > 0, y \in \mathbb{R}^d$ .

**Goal:** Find,  $\forall y \in \mathbb{R}^d, \mathbb{Q}^y$  which is the conditional measure on  $\{X_T = y\}$ .

Assume:  $(X_t)_{t \geq 0}$  has transition density  $P$  s.t.

$$\forall 0 \leq s \leq t \leq T, \mathbb{P}(X_t \in dz \mid X_s = x) = p(s, x; t, z)dz.$$

$X_t$  solves

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \end{cases}$$

Define  $h^y(s, x) := \frac{p(s, x; T, y)}{p(0, x_0; T, y)}$  for  $s \in [0, T), x \in \mathbb{R}^d$ .

**Remark.**  $s < T$  since otherwise we get ingeneral singularities.

**Lemma 2.5.** Let  $Z_t^y := h^y(t, X_t)$  is a martingale.

*Proof.* Let  $0 \leq t < T$ :

$$\begin{aligned} \mathbb{E}(Z_t^y \mid \mathcal{F}_s) &= \mathbb{E}(h^y(t, X_t) \mid \mathcal{F}_s) \\ &\stackrel{\text{MP}}{=} \mathbb{E}(h^y(t, X_t \mid X_s)) \\ &= \int_{\mathbb{R}^d} h^y(t, x) p(s, X_s; t, x) dx \\ &= \int_{\mathbb{R}^d} \frac{p(t, x; T, y)}{p(0, x_0; T, y)} p(s, X_s; t, x) dx \\ &= \frac{1}{p(0, x_0; T, y) \int_{\mathbb{R}^d} p(s, X_s; t, x) p(t, x; T, y) dx} \\ &\stackrel{\text{Chapman-Kolmogorov}}{=} Z_s^y = \end{aligned}$$

$$\square \quad h^y(0, x_0) = 1$$

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# List of Lectures

- **Lecture 01:** Introduction, reminder of strong solutions, definition of weak solutions, uniqueness in law, pathwise uniqueness, and some examples
- **Lecture 02:** Further examples, Yamata-Watanabe theorems and Skorohod theorem (no proof), reminder of Lévy characterization, Ito-Doebelin formula
- **Lecture 03:** The martingale problem and one-to-one relation with weak solutions (special case of  $d = n$  proven); reminder of Dubins-Schwarz theorem
- **Lecture 04:** Transformation of SDE under time change, weak solutions for 1d SDEs, scale function and its relation to hitting times
- **Lecture 05:** Uniqueness of the solution of martingale problem, reminder of Girsanov theorem, changes of SDE under drift transformation
- **Lecture 06:** Drift transformation for SDE, Doob-h transform, start set-up for diffusion bridges