Lecture notes on Stochastic Analysis

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Based on the lectures of
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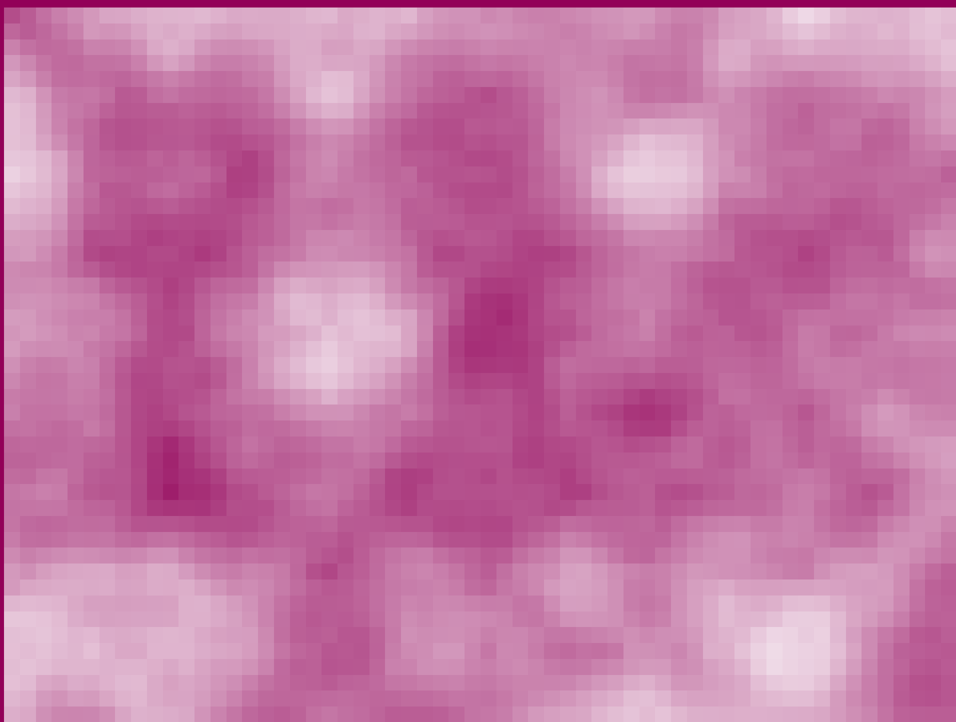

University of Bonn
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Contents

Chapter 0:
Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Ecampus: Ecampus link
- Basis: Basis link
- Website: None
- Time slot(s): Tuesday 12-14 and Thursday 12-14
- Exams: Oral: 23-25 July, 30 Sept. maybe 27. Sept
- Deadlines: ?
- Exercises: One of Tue 16-18(for now this is preferred), Friday 10-12. Starting 16.04.

- First halve based on Eberle and / or Gubinelli (be careful with Notation of dimensions!) Start of lecture 01 (11.04.23)

Overview of the content

- Weak solutions of SDE
  - Martingale problem (characterization)
  - Time change (Dubin-Schwarz)
  - Change of measure (Girsonov)
- How to condition a diffusion to stay in a given domain forever (reflected BM, local time, . . .)
- Lévy processes (a bit)
- Multiplicative stochastic heat-equation
  - relations with Kardar-Pavisi-Zhang class of growth models

Chapter 1:
Weak solutions of SDEs

SDE:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \ \textbf{Initial condition} \end{cases} \tag{1.1}$$

- $X_t \in \mathbb{R}^d, B:$ n-dim BM
- $b(t,x) : [b_k(t,x)]_{1\leq k\leq d}$: **drift vector**
- $\sigma(t,x) = [\sigma_{k,l}(t,x)]_{\substack{1\leq k\leq d \\ 1\leq l\leq n}}$: **dispersion matrix**
- $a(t,x) = \sigma(t,x)\cdot\sigma(t,x)^\intercal$: **diffusion matrix**

1.1 Strong solutions

Definition 1.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_t = \sigma(x_0, (B_s)_{0\leq s\leq t})$*

- *a d-dim process X_t is a **strong solution** of equation 1.1 if:*
  - $X_t = x_0$ *a.s.*
  - X_t *is adapted to* $\mathcal{F}_t \forall t \geq 0$
  - X *is a continuous semimartingale s.t.* $\forall t \geq 0$*:*

    $$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2)ds < \infty \text{ a.s.}$$

  - $X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$

In the last semester we proved:

Theorem 1.2. *Assume that b, σ are globally lipschitz with at most linear growth at ∞ (in space) $\implies \exists!$ strong solution of SDE.*

Foundations of Stochastic Analysis Thm 8.6

Added remark. *There exists $K > 0$ s.t. for all $x, y \in \mathbb{R}^d$: Globally Lipschitz:*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\|$$

Linear growth condition:

$$\|b(t,x)\| + \|\sigma(t,x)\| \leq K(1 + \|x\|)$$

Remark. *For strong solutions, $\mathcal{F}_t$ is given by the driving BM, wich is given to us.*
$\implies X_t(\omega) = \Phi(x_0(\omega), (B_s)_{0\leq s\leq t})$

1.2 Weak solutions

- For weak solutions we do not fix the driving brownian motion.

Definition 1.3. *A* ***weak solution*** *of equation 1.1 is a* ***pair*** *of adapted processes* (X, B) *to a* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *s.t.*

- *B is a n-dim BM*
- *X is a d-dim continuous semimartingale with*
  1. $X_0 = x_0$ *a.s.*
  2. $\forall t \geq 0$
  $$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2) ds < \infty \text{ a.s.}$$
  3. $X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$

Remark.
- *The filtration $(\mathcal{F}_t)_{t\geq 0}$ is not necessarily the one generated by B*
- *If X is adapted to the filtration generated by the BM $\implies$ we have strong solutions*
- *$\exists$ weak solution which are not strong solutions*
- *Warning: To give a weak solution we really need to provide $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, B)$*

Definition 1.4 (Uniqueness in law)**.** *An SDE 1.1 has* ***uniqueness in law*** *if given any two weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, B)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{B})$ satisfy:*

$$Law_{\mathbb{P}}(X) = Law_{\tilde{\mathbb{P}}}(\tilde{X})$$

They agree on any set in the sigma algebra

Definition 1.5 (Pathwise uniqueness)**.** *The SDE 1.1 has pathwise uniqueness if, whenever the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(B_t)_{t\geq 0}$ are fixed, then two solutions $X, \tilde{X}$ with $X_0 = \tilde{X}_0$ are indistinguishable.*

Example 1.6 (No strong solutions, no pathwise uniqueness, $\exists$ weak solution & and uniqueness in law by Tanaka)**.**

$$\begin{cases} dX_t = sgn(X_t) dB_t \\ X_0 = 0 \end{cases} \tag{1.2}$$

or more generally $X_0 = Y$, where

$$sgn(x) = \begin{cases} 1 & x > 0 \\ -1 x \leq 0 \end{cases}$$

Let W be a BM with $W_0 = Y$. Define

$$B_t := \int_0^t sgn(W_s) dW_s \text{ or } dB_t = sgn(W_t) dW_t$$

$$\implies dW_t = sgn(W_t) dB_t$$
$$\implies W_t = y + \int_0^t sgn(W_s) dB_s$$

B_t is a local martingale with

$$\langle B \rangle_t = \int_0^t \underbrace{(sgn(W_s))^2}_{=1} \underbrace{d\langle W \rangle_s}_{=ds} = t$$

*Also $B_0 = 0$, therefore B is a BM (see Lévy characterization) $\implies$ W solves the SDE. For $Y = 0$, W **and** $-W$ solves the same SDE.*
$\implies$

- *exists weak solutions*
- *For $Y = 0$: no pathwise uniqueness*
- *Uniqueness in law (because the law is determined by X_t being a BM)*
- *No strong solution, because: $\mathcal{F}^B = \mathcal{F}^{|W|} \subsetneq \mathcal{F}^W$*

Not a proof just yet, just the reason!

Example 1.7 (No solutions)**.**

$$\begin{cases} dX_t = -\frac{1}{2X_t} 1_{X_t \neq 0} dt + dB_t \\ X_0 = 0 \end{cases} \tag{1.3}$$

Assume there exists a solution. Use Itô formula for X_t^2, then:

$$\begin{aligned} X_t^2 &= 2\int_0^t X_s dX_s + \int_0^t 1 ds \\ &= -\int_0^t 1_{X_s \neq 0} ds + 2\int_0^t x_s dB_s + t \\ &= \int_0^t 1_{X_s = 0} ds + 2\int_0^t X_s dB_s \end{aligned}$$

We will prove $\int_0^t 1_{X_s=0} ds = 0 \implies X_t^2$ is a local martingale, $X_t^2 \geq 0$ (and therefore a supermartingale) and $X_0 = 0$ ($\implies \mathbb{E}(X_t^2) = 0$). If $X_t = 0 \implies \int_0^t 1_{X_s=0} ds = t \implies 0 = dB_t$ which are contradictions!

Remark. *If $X = \underbrace{M}_{\in \mathcal{M}_{loc}} + \underbrace{A}_{\in \mathcal{A}}$*

$$\implies \forall b \in \mathbb{R} \int_0^t 1_{X_s = b} d\langle M \rangle_s = 0$$

Sheet 0:

- Mail: Alexander.Becker@uni-bonn.de
- Exercise sheet handed in Fr 10 am via eCampus
- Groups of 3

Motivation:
in the last semester: Introduction to stochastic analysis

- Diffusion processes & SDEs

Here: Deepen the knowledge & have fun

- Learn SDE techniques and get other results
- Modify diffusion processes to behave in a certain way
- Ex. diffusion bridge (Brownian bridge)
- Condition a diffusion to stay in a domain
    - Ex. Condition BM to stay positive
    - Old SDE: $dB_t = dB_t$

- New SDE: $dX_t = \frac{1}{X_t}dx + dB_t \to P(X_t \in \cdot) = P(B_t \in \cdot | B > 0 \text{ forever})$
- $D \subset \mathbb{R}^d$ open domain, X diffusion process, with generator $L = \Sigma b\partial_i + \frac{1}{2}\sum a^{ij}\partial_i\partial_j$

$$Y :- (X|X \in D \text{ forever})$$

get drift term $\nabla \log \phi_0$, where ϕ_0 is the lowest eigenfuncion of $-L$ on D with dirichlet boundary.

Recap:
Brownian motion:

Added definition. $B_0 = 0$, *independent & $\mathcal{N}(0, t_i - t_{i-1})$ increments, $t \mapsto B_t(\omega)$ continuous.*

Regularity of path $t \mapsto B_t(\omega)$:

- nowhere differentiable
- $\alpha-$locally Hölder continuous $\iff \alpha < \frac{1}{2}$
- Quadratic variation $\langle B \rangle_t = t$
- Generator $\frac{\Delta}{2}$
- Recurrent $\iff ds2?$

Itô-Integral:

1. If X simple process $\implies$ RS-Integral
2. Itô isometry $\mathcal{E} \to \{L^2 - \text{local martingale}\}, \mathcal{E} \subset L^2(d[M])$ dense
3. general $X : \int XdM$ as L^2-limit

Added remark (Itô formula)**.**

- $$df(t, X_t) = \partial_t f(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}f(t, X_t)d\langle X \rangle_t$$
- *associative* $\int Xd(\int YdZ) = \int XYdZ$
- *If M local martingale $\implies \int XdM$ local martingale*

SDEs:
$DX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$

- ex./ uniqueness: b, σ locally Lipschitz $\implies$ strong ex.+pathwise uniqueness until explosion time
- globally Lipschitz (in space) and linear growth $(|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|))$, $e = \infty$.

Problem 00.1: SDE

Let B be a one-dimensional Brownian motion (starting from 0) and let $X_t = \sin(B_t)$.

1. Determine the SDE of X_t
2. Discuss the existence and/or uniqueness of strong solutions of the SDE
3. Determine whether the local martingale term is a martingale (Hint: for the integrability condition, think about the Itô isometry for instance)

Solution 00.1

1.:
Idea: Use Itô formula: $X_t = \sin(B_t) = f(B_t)$

$$dX_t = df(B_t) \overset{\text{Itô}}{=} \partial_x \underbrace{\cos(B_t)}_{\sqrt{1-X_t^2}} dB_t - \frac{1}{2}\underbrace{\sin(B_t)}_{X_t} dt$$

Careful: $\sqrt{1-X_t^2}$ is not inverse mapping, because it is always positive while $\cos(B_t)$ is not

2.:
Consider the SDE

$$dX_t = \begin{cases} -\frac{1}{2}X_t dt + \sqrt{1-X_t^2} dB_t \\ X_0 = x \in (-1,1) \end{cases}$$

Coefficients: $b : [-1,1] \to \mathbb{R}, b(x) = -\frac{1}{2}x$ and $\sigma : [-1,1] \to \mathbb{R}.\sigma(x) = \sqrt{1-x^2}$

$$[\sigma]_{1,2} = \sup \frac{|\sigma(x)-\sigma(y)|}{|x-y|^{1/2}} < \infty$$

Probably σ^2 Lipschitz $\implies$ σ Hölder $\frac{1}{2}$
3.:

Problem 00.2: Time change

Let B be a one-dimensional Brownian motion (starting from 0). Let $Y_t = \int_0^t s^2 dB_s$.

1. Determine the SDE of Y_t
2. Find A_t such that Y_{A_t} is a (stopped) Brownian motion

Solution 00.2

1.: By Definition

$$dY_t = t^2 dB_t$$

2.: We use the Dubin-Schwarz theorem:
$X \in \mathcal{M}^0_{\text{loc}}, \langle X \rangle_\infty = \infty, T_t \coloneqq \inf\{s \geq 0 | \langle X \rangle_s \geq t\} = X_t^{[-1]}$
$\implies B_t \coloneqq X_{T_t}$ 1 d BM w.r.t. $(F_{T_t})_{t\geq 0}, X_t = B_{\langle B \rangle_t}$
here: use $X_t = b(X_t)dt + \sigma(X_t)dB_t \implies d[X]_t = \sigma^2(X_t)dt$

Problem 00.3: SDE and PDE

Let f be a function supported on $[0,1]$, u the solution of

$$\frac{1}{2}u(t,x) = \frac{1}{2}x(1-x) + \frac{d^2}{dx^2}u(t,x), \qquad u(0,x) = f(x).$$

Consider also the one-dimensional SDE (also known as Wright-Fisher diffusion)

$$dX_t = \sqrt{X_t(1-X_t)dB_t}$$

with $X_0 = x \in (0,1)$.

1. For any fixed $t > 0$, define $M_s = u(t-s, X_s)$ for $s \in [0,t]$. Use Itô formula to show that M_s is a local martingale
2. Assume that f is bounded and there is a bounded solution of u. Show that $u(t,x) = \mathbb{E}_x(f(X_t))$.

Solution 00.3

1.:

$$\begin{aligned}
dM_s &= du(t-s,X_s)\\
&= -\partial_s u(t-s,X_s)ds + \partial_x u(t-s,X_s) \underbrace{dX_s}_{b(X_s)ds+\sigma(X_s)dB_s \text{ by asso.}} + \frac{1}{2}\partial_x^2 u(t-s,X_s) \underbrace{d[X]_s}_{=\sigma^2(X_s)ds}\\
&= \underbrace{(-\partial_s u + b\partial_x u + \frac{1}{2}\sigma^2\partial_x^2 u)}_{=0}(t-s,X_s)ds + \partial_x u(t-s,X_s)\sigma(X_s)dB_s\\
&\implies dM_s = \partial_x u(t-s,X_s)\sigma(X_s)dB_s
\end{aligned}$$

(i.e.: $M_t - M_0 = \int_0^t \dots dB_s$)
This is a purely stochastic integral against a (local) martingale $\implies$ martingale.
2.:

- M_s true martingale:
  1. $\forall t : \mathbb{E}(\sup_{[0,t]} |M|) < \infty$, for example: M bounded
  2. $\forall t : \mathbb{E}(\sup_{[0,t]}[M]_t) < \infty$

$M_s \coloneqq u(t-s,X_s), u$ bounded $\implies$ M bounded $\implies$ M true martingale
$w(s,x) \coloneqq u(t-s,x)$

$$u(t,x) = w(0,x) = \mathbb{E}_x[w(0,X_0)] \overset{\text{martingale}}{=} \mathbb{E}_x[w(t,X_t)] = \mathbb{E}_x[(u(0,X_t))] \overset{\text{PDE}}{=} \mathbb{E}_x[f(X_t)]$$

Feynman-Kac formula (later more general).
Also solvable by Kolmogorov Backward / forward equation

Start of lecture 02 (16.04.24)

Example 1.8 (Non-uniqueness of law, non pathwise uniqueness, with solutions)**.**

$$\begin{cases} dX_t &= 1_{X_z} dB_t \\ x_0 &= 0 \end{cases}$$

Then

$$X_t = 0 \forall t \geq 0$$

and

$$X_t = B_t \forall t \geq 0$$

both are solutions:

$$X_t - B_t = -\int_0^t 1_{X_s=0} dB_s \implies \langle X - B\rangle_t = \int_0^t 1_{X_s=0} d\langle B\rangle_s = 0$$

$X_t = \int_0^t 1_{X_s \neq 0} dB_s$
$= \int_0^t (1 - 1_{X_s=0} dB_s)$
$B_t - \int_0^t 1_{X_s=0} dB_s$

Let $\eta \sim Ber\left(\frac{1}{2}\right)$ *independent of* $(B_t)_{t\geq 0}$ *and define*

$$\tilde{X}_t = \begin{cases} 0 & \eta = 0 \\ B_t & \eta = 1 \end{cases}$$

$\implies$ $\tilde{X}_t$ *is adapted to* $\sigma(\eta(B_s)_{0\leq s\leq t})$*, but not to* $\sigma((B_s)_{0\leq s\leq t})$ *and therefore not a strong solution.*

Example 1.9 (No strong solution, weak solution, no pathwise uniqueness, no uniqueness in law)**.**

$$\begin{cases} dX_t &= 1_{X_t \neq 1} sgn(X_t) dB_t \\ X_0 &= 0 \end{cases}$$

Let Y_t be a solution of

$$\begin{cases} dY_t &= sgn(Y_t) dB_t \\ Y_0 &= 0 \end{cases}$$

$\implies X_t := Y_{t\wedge\tau}$, *where* $\tau = \inf\{s \geq 0 \mid Y_s = 1\}$ *is also a solution.*

Theorem 1.10 (Yamada-Watanabe)**.** *If both existence of weak solutions and pathwise uniqueness hold, uniqueness in law also holds.*
Moreover, $\forall$ choices of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(B_t)_{t\geq 0}$ then there exists a strong solution.

1.3 Lévy characterization

Example 1.11. *Consider the SDE*

$$\begin{cases} dX_t &= O_t dB_t \\ X_0 = x_0 \end{cases}$$

where both X_t and B_t are d-dimensional and O_t is an adapted process (matrix) s.t. $O_t^\intercal O_t = 1 \forall t \geq 0$ *i.e. O_t is a rotation*

$$\implies X_t^k = X_0^k + \sum_{l=1}^d \int_0^t O_s^{k,l} dB_s^l$$

$$\implies \langle X^k, X^{\tilde{k}} \rangle_t = \sum_{l,\tilde{l}}^d \int_0^t O_s^{k,l} O_s^{\tilde{k},\tilde{l}} d\langle B^l, B^{\tilde{l}} \rangle_s = \sum_{l=1}^d \int_0^t O_s^{k,l} O_s^{\tilde{k},l} ds$$

$$= \int_0^t \underbrace{(O_s O_s^\intercal)^{k,\tilde{k}}}_{=1} ds = \delta_{k,\tilde{k}} t \overset{\text{Lévy}}{\implies} (X_t)_{t\geq 0} \text{ is also a d-dim BM starting from } x_0$$

Theorem 1.12 (Yamada,Watanabe)**.** *Let $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ and assume that there exist both a increasing function $\rho(u) \geq 0$ s.t. $|\sigma(x) - \sigma(y)| \leq \rho(|x-y|) \forall x, y \in \mathbb{R}$ s.t.*

$$\int_0^\infty \frac{1}{\rho(u)} du = \infty \implies 0 < u < C_1 \implies \rho(u) \leq C_2 u^{0.5}$$

and some increasing concave function $\gamma_1(u) \geq 0$ s.t.

$$|b(x) - b(y)| \leq \gamma_1(|x-y|) \forall x, y \in \mathbb{R}$$

and

$$\int_0^\infty \frac{1}{\gamma_1(u)} = \infty.$$

Then pathwise uniqueness holds.

Theorem 1.13 (Storokhod)**.** *Assume that σ, b are continuous bounded functions*
$\implies$ *there exist weak solutions to the SDE $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.*

1.4 Weak solutions and martingale problems

Let $(X_t)_{t\geq 0}$ be a weka solution of the SDE

$$\begin{cases} dX_t &= b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_0 &= 0 \end{cases}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
$\implies$ X_t is a semimartingale s.t. $X_t^k = X_0^k + \int_0^t b(s,X_s)ds + \sum_{l=1}^n \int_0^t \sigma_{k,l}(s,X_s)dB_s^l$ and

$$\langle X^i, X^j\rangle_t = \int_0^t \underbrace{\sum_{l=1}^n \sigma_{i,l}(s,X_s)\sigma_{j,l}^{\intercal}(s,X_s)}_{=a_{i,j}(s,X_s)} ds$$

1.4.1 Itô-Doeblin formula

Itô formula leads to

Proposition 1.14. *For $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, then*

$$f(t,X_t) = f(0,X_0) + \int_0^t (\sigma^{\intercal}\nabla f)(s,X_s)dB_s + \int_0^t \left[\left(\frac{\partial}{\partial s} + \mathcal{L}\right)\right](s,x_s)ds$$

where $(\mathcal{L}f)(t,x) = \frac{1}{2}\sum_{k,l=1}^n a_{k,l}(t,x)\frac{\partial^2}{\partial x_k \partial x_l}f(t,x) + \sum_{k=1}^n b_k(t,x)\frac{\partial}{\partial x_k}f(t,x)$.

$\mathcal{L}$ is called the ***generator***

Remark. *The Itô-Doeblin formula provides a connection between SDEs and PDEs as we have seen in the last part* ***Foundations of stochastic analysis****.*

Example 1.15. *Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}_d)$ be a solution of the PDE*

$$\begin{aligned} \frac{\partial}{\partial t}f(t,x) + (\mathcal{L}f)(t,x) &= -g(t,x) && t\geq 0, x \in U \subseteq \mathbb{R}^d \\ f(t,x) &= \varphi(t,x) && t \geq 0, x \in \partial U. \end{aligned}$$

then $M_t := f(t,X_t) + \int_0^t g(s,X_s)ds \in \mathcal{M}_{loc}$ by proposition 1.14 and if f,g are bounded $M_t \in \mathcal{M}$.

$$T := \inf\{s \geq 0 | X_s \notin U\} \implies M_t^T := M_{T\wedge t} \in \mathcal{M}.$$

Furthermore, if we assume $T < \infty$ a.s. :

$$\mathbb{E}[M_t] = \mathbb{E}[\varphi(T,X_T)] + \mathbb{E}[\int_0^T g(s,X_s)ds] = \varphi(0,X_0)$$

where $X_0 = x_0$.
There are two special cases:
$g = 0 \implies$ yields the exit distributions, while
$\varphi = 0, g = 1$ yields the mean exit times.

Example 1.16 (Feynman-Kac formula)**.** *Let $t \in \mathbb{R}_+$ be finite. Assume $f : \mathbb{R}^d \to \mathbb{R}$ and $K : [0,t]\times\mathbb{R}^d \to \mathbb{R}_+$ be continuous functions. Assume that u is a $C^{1,2}$-solution (bounded,for simplicity) of*

$$\begin{cases} \frac{\partial u}{\partial s}(s,x) = \frac{1}{2}\Delta u(s,x) - K(s,x)u(s,x) & s \in [0,t], x \in \mathbb{R}^d \\ u(0,x) = f(x) & x \in \mathbb{R}^d \end{cases}$$

Then u has the stochastic representation $u(t,x) = \mathbb{E}_x\left[f(X_t)\exp\left(-\int_0^t K(t-s,X_s)ds\right)\right]$, where X_t is BM starting from $X_0 = x$.

Sketch. 1. Define $r(s,x) := u(t-s,x)$ for $s \in [0,t]$

2. Show: $M_s \coloneqq \exp(-A_s) r(s,x)$ with $A_s = \int_0^s K(u, X_u)du$ is a local martingale.

$\square$

Remark. *This is a reformulation of the formula from the last semester.*

1.4.2 Martingale problem

A solution of an SDE is generically defined up to some **explosion time ξ**, where it either diverges or it exists a given domain $U \subset \mathbb{R}^d$ (open).
$\implies$ For $k \in \mathbb{N}$ define $U_k \coloneqq \{x \in U \mid |x| < k \wedge \mathrm{dist}(x, U^c) \geq \frac{1}{k}\}$ with $U = \bigcup_{k\geq 1} U_k$ and

$$T_k \coloneqq \{t \geq 0 \mid x_t \notin U_k\}.$$

A solution of the SDE $b(t, X_t)dt + \sigma(t, X_t)dB_t$ is defined up to $\xi = \sup_{k\geq 1} T_k$.

Start of lecture 03 (18.04.24)

Added remark. *uniqueness of solution to the heat equation $\frac{1}{2}\Delta u - Ku$: not unique! Stochastic calculus: Paolo Baldi: 10.3-10.4*

Define $(\star)$:

$$dX_t b(t, X_t)dt + \sigma(t, X_t)dB_t \text{ with } X_0 = x_0$$

Theorem 1.17 (Martingale problem)**.** *If X is a solution of $(\star)$ up to time ζ, then $\forall f \in C^{1,2}(\mathbb{R}_+ \times U)$*

$$M_t = f(t, X_t) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right) f(s, x_s)ds, t < s$$

is a local martingale up to ζ and $M_t^{T_k}$ are localizing martingales.

Definition 1.18 (Martingale solutions)**.** *$(X_t)_{t\geq 0}$ is a **martingale solution** of $(\star)$ if $\forall f \in C^2(\mathbb{R}^d)$*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$$

is a continuous local martingale.

Theorem 1.19 (Equivalent definitions)**.** *The following are equivalent (~~for X_t being a solution of $(\star)$~~):*

(a) $\forall f \in C^2(\mathbb{R}^d)$,

$$M_t^f \coloneqq f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$$

is a local martingale

(b) The process in $\mathbb{R}^d$ given by

$$M_t \coloneqq X_t - X_0 - \int_0^t b(s, X_s)ds$$

is a d-dimensional local martingale with $\langle M^i, M^j\rangle_t = \int_0^t a_{i,j}(s, X_s)ds = \langle X^i, X^j\rangle_t$

(c) $\forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$

$$\tilde{M}_t^f \coloneqq f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right)(s, X - s)ds$$

is a local martingale

ds

Proof. $\underline{c \implies a}$: by choosing f independent of t.
$\underline{a \implies b}$: 1.: Choosing $f(X) = X_i$ implies

We can maybe also proof this by calculating X^2?

$$M_t^f = X_t^i - X_0^i - \int_0^t b_i(s, X_s)ds \in \mathcal{M}_{\text{loc}}$$

2.: $f(X) = X_i X_j$:

$$\begin{aligned}(\mathcal{L}f)(x) &= \frac{1}{2}a_{ij} + \frac{1}{2}a_{ji} + b_i X_j b_j X_i \\ a = a^\intercal &\implies = a_{ij} b_i X_j b_j X_i \\ &\implies M_t^f X_t^i X_t^j - X_0^i X_0^j - \int_0^t \left[a_{ij}(s, X_s) + b_i(s, X_s)X_s^j + b_j(s, X_s)X_s^i\right] ds\end{aligned}$$

Here dX_s^i is the same as $b_i X_s^j$ is the same up to a local martingale term and $\langle X^i, X^j\rangle_t = \int_0^t a_{ij}(s, X_s)ds = \langle M^i, M^j\rangle_t$

$$\begin{aligned}X_t^i X_t^j - X_0^i X_0^j &\overset{\text{Integration by parts}}{=} \int_0^t X_s^i dX_s^j + \int_0^t X - s^j dX_s^i + \langle X^i, X^j\rangle_t \\ &= M_t^f + \int_0^t [a_{ij} + b_i X_s^j + b_j X_s^i] ds\end{aligned}$$

$\underline{b \implies c}$: By Proposition 1.14 If (use the next theorem) X was a weak solution $\implies \tilde{M}_t^f$ is a local martingale. □

Theorem 1.20. *Let $n = d$, assume $\sigma(t, x)$ is invertible $\forall t, x$ and $\sigma^{-1}(t, x)$ is uniformly bounded. T.f.a.e.:*

(a) $(X_t)_{t\geq 0}$ is a weak solution of the SDE $(\star)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; B)$

(b) $(X_t)_{t\geq 0}$ is a martingale solution of the SDE $(\star)$

This also works for $n \neq d$, but with a different proof

Proof. $a \implies b$: True
$b \implies a$: Goal construct a BM for the weak solution.
By proposition **??** $a \implies b$ $dM_t = dX_t - b(t, X_t)dt \in \mathcal{M}_{\text{loc}}$ and $d\langle M^i, M^j\rangle_t = a_{k,l}(t, X_t)dt$

$$\begin{aligned}\implies dX_t &= dM_t + b(t, X_t)dt \\ &= \sigma(t, X_t)d\tilde{B}_t + b(t, X - t)dt\end{aligned}$$

where $\tilde{B}_t \coloneqq \sigma(s, X_s)^{-1} dM_s$
To see: $\tilde{B}_t$ is a brownian motion.

$$\begin{aligned}\langle \tilde{B}^i, \tilde{B}^j\rangle_t &= \sum_{k,l} \int_0^t \sigma_{ij}^{-1} \sigma_{jl}^{-1} \underbrace{d\langle M^k, M^l\rangle_s}_{= \underbrace{a_{ij}}_{(\sigma^\intercal \sigma)_{kl}} ds} \\ &= \sum_{k,l,p} \int_0^t \sigma_{ik}^{-\intercal} \sigma_{kp} \sigma_{pl}^\intercal \sigma_{lj}^{-\intercal} ds \\ &= \delta_{ij} \int_0^t 1 ds = \delta_{ij} t\end{aligned}$$

Then by the Lévy characterization $\tilde{B}$ is a brownian motion. □

Added remark. *This is the first way to construct a weak solution: Solve a martingale problem! This is used a lot in practice.*

1.5 Weak solutions and time change

1.5.1 Time change

For $d = 1$:

Theorem 1.21. *[Dubins-Schwarz]*

- *Let* $M \in \mathcal{M}^0_{loc}$ *and* $\langle M \rangle_\infty = \infty$ *a.s.*
- *Let* $T_t \coloneqq \inf\{s \geq |\langle M \rangle_s \geq t\}$

This implies

1. $t \mapsto M_{T_t}$ *os a* $(\mathcal{F}_{T_t})$ *brownian motion*
2. $M_t = B_{\langle M \rangle_t}$ *for some standard brownian motion* B

$X_t = X_0 + \int_0^t b(s, X_s)ds + \underbrace{\int_0^t \sigma(s, X_s)dB_s}_{=M_t}$. If $\langle M \rangle_\infty = \infty$ a.s.:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \tilde{B}_{\int_0^t \sigma^2(s,X_s)ds}$$

1.5.2 Time change in a martingale problem

Consider $d = 1 = n$.

$$dY_t = \tilde{\sigma}(Y_t)dB_t \qquad (\star\star)$$

and $\tilde{\sigma}$ strictly positive positive continuous function.

$$\langle Y \rangle_t = \int_0^t \tilde{\sigma}(Y_s)^2 ds =: A_t$$

By theorem 1.21 $\implies Y_t = W_{A_t}$ for some brownian motion W.
Assume $A_t \infty = \infty$ a.s.
$T_t \coloneqq \inf\{s \geq 0 \mid \langle Y \rangle_s \geq t\}$

$$\implies T_{A_t} = \inf\{s \geq 0 \mid \langle Y \rangle_s \geq \langle Y \rangle_t\} = t$$

$$1 = \frac{d}{dt}(T_{A_t}) = T'\underbrace{A_t}_{=u} \cdot A_t$$

$$\implies \underbrace{T'_u}_{=\frac{dT_u}{du}} = \frac{1}{A'_{T_u}} \implies T_u = \int_0^u \frac{1}{\tilde{\sigma}(Y_{T_s})^2} ds = \int_0^u \frac{1}{\tilde{\sigma}(W_s)^2} ds$$

$\implies$ to construct a solution of $(\star\star)$: Given $W \longrightarrow$ compute $T_u \longrightarrow$ determine $A - t = T_t^{-1} \implies Y_t = W_{A_t}$

Start of lecture 04 (23-04-24)

Theorem 1.22. *Let* $(X_u)_{u \geq 0}$ *on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ *be a weak solution of*

$$dX_u = b(X_u)du + \sigma(X_u)dB_u$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$*,the drift and* $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ *are locally bounded,* σ^{-1} *exists for a.e.* x *and is locally bounded.*
Consider a ***time change*** $T_u \coloneqq \int_0^u \rho(X_s)ds$*, where* $\rho : \mathbb{R}^d \to \mathbb{R}_+$ *s.t.*

$$T_u < \infty \forall u \geq 0 \text{ and } T_\infty = \infty \text{ a.s.}$$

$\implies$ *Then $Y_t := X_{A_t}$, where $A_t = T_t^{-1}$ is a weak solution of the SDE*

$$dY_t = \frac{b(Y_t)}{\rho(Y_t)}dt + \frac{\sigma(Y_t)}{\sqrt{\rho(Y_t)}}dB_t$$

Remark. *Special case: $d = 1, b = 0, \sigma = 1$: Then X is a BM and $\rho = \frac{1}{\tilde{\sigma}^2(x)}$ $\implies$ $Y_t = X_{T_t^{-1}}$ solves*

$$dY_t = \tilde{\sigma}(Y_t)dB_t$$

Proof. By theorem 1.20, it is enough to see how the martingale problem after change of time becomes:

$$\mathcal{L} = \mathcal{L}_{X_t} \overset{\text{time change}}{\longrightarrow} \mathcal{L}_{Y_t} = \tilde{\mathcal{L}}$$

$Y_t = X_{A_t}; Y_0 = X_{A_0}$. For $f \in C^2 : M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$ is a local martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

The A_0 in the integral is probably 0, but it does not matter, we do a change of variables anyway.

$$\implies N_t^f := M_{A_t}^f = f(\underbrace{X_{A_t}}_{=Y_t}) - f(\underbrace{= X_{A_0}}_{Y_0}) - \int_{A_0}^{A_t} (\mathcal{L}f)(X_s)ds$$

is also a local martingale w.r.t. $(\mathcal{F}_{A_t})_{t\geq 0}$.
Change of variable (to get rid of the X_s in the integral):

$$\begin{aligned} &\tau = T_s \leftrightarrow A_t = s \implies X_s = X_{A_t} = Y_\tau \\ &d\tau = \rho(X_s)ds \end{aligned}$$

$$\implies N_t^f = f(Y_t) - f(Y_0) - \int_0^t (\mathcal{L}f)(Y_t)\frac{1}{\rho(Y_t)}d\tau$$

Since $\mathcal{L}f(x) = \sum_k b_k(x)\frac{\partial}{\partial x_k}f(x) + \sum_{k,l} a_{k,l}(x)\frac{\partial^2}{\partial x_k \partial x_l}f(x)$

$$\implies (\tilde{\mathcal{L}}f)(x) = \sum_k \frac{b_k(x)}{\rho(x)}\frac{\partial}{\partial x_k}f(x) + \sum_{k,l}\frac{\overbrace{a_{k,l}}^{(\sigma\sigma^\intercal)_{k,l}}}{\sqrt{\rho(x)\rho(x)}}(x)\frac{\partial^2}{\partial x_k \partial x_l}f(x)$$

$\implies$ It is a martingale problem for the SDE where the drift $\to \frac{\text{drift}}{\rho}$ and $\sigma \to \frac{\sigma}{\sqrt{\rho}}$

$\square$

1.5.3 Weak solutions in d=1

We will do both time and "space" changes.

- 1-d SDE:
$$\begin{cases} dX_t & = b(X_t)dt + \sigma(X_t)dB_t \\ X_0 = x_0 \in (\alpha, \beta) \end{cases} \tag{1.4}$$
- X_t a process in (α, β)
- Assume $b, \sigma : (\alpha, \beta) \to \mathbb{R}$ continuous, $\sigma(x) > 0 \forall x \in (\alpha, \beta)$
- Do a change of coordinates $Y_t := s(X_t)$ where $s : (\alpha, \beta) \to (s(\alpha), s(\beta))$, C^2 with $S'(x) > 0, x \in (\alpha, \beta)$.
- $s(x)$ is called the **scale function** and it is given by
$$s(x) = \int_{x_0}^x \exp\left(-\int_{x_0}^y dz \frac{2b(z)}{\sigma(z)^2}\right)dy$$
- s satisfies $\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0 \iff (\mathcal{L}s)(x) = 0$

Remark. *If $b(z)=0 \implies s(x)=x-x_0 \implies s'(x)=1$. If $s'(x)=1$, we say that the process is in its "natural scale"*

By proposition 1.14: $\mathcal{L}s=0, \dot{s}=0$.
$\implies Y_t=s(X_t)$ is a local martingale satisfies $dY_t=s'(X_t)\sigma(X_t)dB_t$. the other terms cancel
$\iff Y_t$ is a solution of

$$\begin{cases} dY_t &= \tilde{\sigma}(Y_t)dB_t \\ Y_0 &= s(X_0) \end{cases} \tag{1.5}$$

where $\tilde{\sigma}(y)=s'(s^{-1}s(y))\sigma(s^{-1}(y))$.

Therefore we can write the original SDE in terms of a BM

Theorem 1.23. *The following are equivalent:*

1. *The process $(X_t)_{t<\xi}$, where ξ is the explosion time, on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P};(B_t)_{t\geq 0})$ is a solution of (1.4) up to tje stopping time ξ*
2. *The process $Y_t=s(X_t)_{t<\xi}$ on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P};(B_t)_{t\geq 0})$ is a solution of (1.5) up to ξ*
3. *The process $(Y_t)_{t<\xi}$ has the representation $Y_t=\tilde{B}_{A_t}$, where $\tilde{B}$ is a BM starting at $\tilde{B}_0=s(X_0)$ and $A_t=T_t^{-1}$ and $T_t=\int_0^t \frac{1}{\tilde{\sigma}^2(\tilde{B}_u)}du$*

s and A_t have the same definition as before

A degenerate case:

Let $\sigma(x)=|x|^\alpha$ for some $\alpha\in\left(0,\frac{1}{2}\right)$. $\implies$

$$\begin{cases} dY_t &= |Y_t|^\alpha dB_t \\ Y_0 &= y \end{cases} \tag{1.6}$$

$\implies T_t=\int_0^t \frac{1}{\sigma(\tilde{B}_u)^2}du$, $A_t=\int_0^t \sigma(Y_s)^2 ds$ and $Y_t=\tilde{B}_{A_t} \implies \tilde{B}_0=y$.
$T_t<\infty$ a.s. ?

$$\begin{aligned} \mathbb{E}(T_t) &= \int_0^t \mathbb{E}\left(\frac{1}{\sigma(\tilde{B})^2}\right)du \\ &= \int_0^t \mathbb{E}\left(\frac{1}{|\tilde{B}|^{2\alpha}}\right)du \\ &= \int_0^t \int_{\mathbb{R}} dx \frac{1}{|x|^{2\alpha}} \frac{\exp(-\frac{(x-y)^2}{2u})}{\sqrt{2\pi u}} \overset{0<2\alpha<1}{<} \infty \end{aligned}$$

$\implies A_t=T_t^{-1}$, then $Y_t=\tilde{B}_{A_t}$ is a solution of (1.6), i.e. $\forall y\in\mathbb{R}\exists$ a non-trivial solution of (1.6).
For $Y_=0, Y_t=0$ is also a solution $\implies$

- No uniqueness in law of (1.6)
- No pathwise uniqueness as well

Remark. *In general: uniqueness in law of 1-d SDEs is not to be expected if $\sigma(x)=0$ somewhere (and σ continuous ...) (i.e. if σ is degenerate).*

By theorem 1.12 as soon as $\sigma(x)=|x|^\alpha$ for some $\alpha\geq\frac{1}{2}$, then one has pathwise uniqueness.

Hitting times and scale functions

Bessel process:

$$\begin{cases} dR_t &= \frac{d-1}{2R_t}dt+dW_t \\ R_0 &= r_0\in(0,\infty) \end{cases}$$

$\implies b(x)=\frac{d-1}{2x}, \sigma(x)=1$.
The scale function satisfies $\mathcal{L}s(x)=\frac{1}{2}s''(x)+\frac{d-1}{2x}s'(x)=0$

$$\implies s(x) = \begin{cases} \frac{x^{2-d}}{2-d} & d \neq 2 \\ \ln(x) & d = 2 \end{cases}$$

and therefore

$$(s_0, s_\infty) = \begin{cases} (0, \infty) & d < 2 \\ (-\infty, \infty) & d = 2 \\ (-\infty, 0 & d > 2) \end{cases}$$

Define the stopping time

$$T_a^R := \inf\{t \geq 0 \mid R_t = a\}$$

Choose an $\alpha < r_0 < \beta$

$$\implies \mathbb{P}(T_\alpha^R < T_\beta^R) \overset{s' \geq 0}{=} \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

One computes

WS exercises

$$\mathbb{P}(T_a^R < T_\beta^R) = \frac{s(\beta) - s(r_0)}{s(\beta) - s(\alpha)} = \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

This is generic, for Itô diffusions, provides that $\exists$ no killing in (α, β).

unlike in 1.16

Start of lecture 05 (25.04.24)

1.5.4 Uniqueness of martingale solution

SDE $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ with generator

$$\mathcal{L} = \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l} (\sigma\sigma^\intercal)_{k,l} \frac{\partial^2}{\partial x_k \partial x_l}$$

Definition 1.24. *Let $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R}^d)$ with σ-algebra $\mathcal{F}$, canonical filtration $(\mathcal{F}_t)_{t\geq 0}$, canonical process $Z_t(\omega) := \omega$.*
We say that $\mathbb{P}$ on $(\mathcal{C}, \mathcal{F})$ is a ***martingale solution*** *for the generator $\mathcal{L}$ $\iff$ $\forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R})$*

$$M_t^f := f(t, Z_t) - f(0, Z_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right) f(s, Z_s) ds \tag{1.7}$$

is a martingale w.r.t. $\mathbb{P}$.

Definition 1.25. *A martingale problem (1.7) has a unique solution if for any two martingale solutions $\mathbb{P} = \mathbb{Q}$ s.t. $Law_{\mathbb{P}}(Z_0) = Law_{\mathbb{Q}}(Z_0)$*

$$\implies \mathbb{P} = \mathbb{Q}$$

Remark. *Uniqueness of martingale solutions corresponds to uniqueness in law of the weak solutions.*

Backwards Kolmogorov Equation (BKE):

$$\frac{\partial}{\partial t}\varphi(t, x) = \mathcal{L}\varphi(t, x) \forall x \in \mathbb{R}^d, t \geq 0 \tag{1.8}$$

Theorem 1.26. *Assume that $\forall$ initial condition*

$$\varphi(0, x) = \Psi(x), \Psi \in C_0^\infty(\mathbb{R}^d)$$

the (1.8) has a solution and φ bounded for all finite time intervals. We have uniqueness of martingale solutions and therefore uniqueness of weak solutions!

The Kolmogorov forward equation is (related to) the Folker-Plank equation!

Proof. Prove that $\forall 0 \le t_1 < t_2 < \cdots < t_n$:

$$\mathrm{Law}_{\mathbb{P}}(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}) = \mathrm{Law}_{\mathbb{Q}}(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n})$$

1. One-time distribution:
$\forall 0 \le s \le r$:

$$\left(\frac{\partial}{\partial s} + \mathcal{L}\right) \varphi(r-s, x) \overset{(1.8)}{=} 0$$

take $t \in [0, r]$:

$$\begin{aligned} M_t^r &:= \varphi(r-t, Z_t) - \varphi(r, Z_0) - \int_0^t \underbrace{(\partial_s + \mathcal{L})\varphi(r-s, Z_s)}_{=0} ds \\ &= \varphi(r-t, Z_t) - \varphi(r, Z_0) \text{ is a martingale} \end{aligned}$$

This needs the boundedness of φ, otherwise it might only be a local martingale. There are softer contidions we can put on the coefficients to achieve the same result. This might not be needed, because φ is C^1 in time anyways

for any solution $\mathbb{P}$.

$$\begin{aligned} 0 \qquad &= \mathbb{E}_{\mathbb{P}}\left(M_r^r - M_t^r \mid \mathcal{F}_t\right) &&= \mathbb{E}(\varphi(0, Z_r) - \varphi(r-t, Z_t) \mid \mathcal{F}_t) \\ \implies \quad &\forall 0 \le t \le r : \mathbb{E}_{\mathbb{P}}(\varphi(0, Z_r) \mid \mathcal{F}_t) &&= \mathbb{E}_{\mathbb{P}}(\varphi(r-t, Z_t) \mid \mathcal{F}_t) \\ & &&\overset{\text{a.s.}}{=} \varphi(r-t, Z_t) \\ &\mathbb{E}(\underbrace{\varphi(0, Z_r)}_{\Psi(Z_r)}) &&\overset{t=0}{=} \mathbb{E}_b P(\varphi(r, Z_0)) \end{aligned}$$

$\forall$ other martingale solutions $\mathbb{Q}$:

$$\mathbb{E}_{\mathbb{Q}}(\Psi(Z_r)) = \mathbb{E}_{\mathbb{Q}}(\varphi(r, Z_0))$$

By assumption this implies $\mathrm{Law}_{\mathbb{P}}(Z_r) = \mathrm{Law}_{\mathbb{Q}}(Z_r)$.
2. Multi-time distributions:
For $\Psi \in C_0^\infty$, denote φ_Ψ the solution of (1.8) with initial condition Ψ:

$$\mathbb{E}_{\mathbb{P}}(\Psi(Z_r) \mid \mathcal{F}_t) = \varphi_\Psi(r-t, Z_t)$$

$0 \le r_2 \le r_1$ test for $g \in C_0^\infty$:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(\Psi(Z_{r_1}) g(X_{r_2})) &= \mathbb{E}(\underbrace{\mathbb{E}(\varphi_\Psi(Z_{r_1}) | \mathcal{F}_{r_2})}_{\varphi_\Psi(r_1 - r_2, Z_{r_2})} g(Z_{r_2})) \\ &= \mathbb{E}_{\mathbb{P}}(\varphi_\Psi(r_1 - r_2, Z_{r_2}) g(Z_{r_2})) \\ &\overset{1.}{=} \mathbb{E}_{\mathbb{Q}}(\varphi_\Psi(r_1 - r_2, Z_{r_2}) g(Z_{r_2})) \\ &= \mathbb{E}_{\mathbb{Q}}(\Psi(Z_{r_1}) g(Z_{r_2})) \end{aligned}$$

Iterating yields the statement.

□

Chapter 2: SDE techniques

Goal: Study process by changing the measure.
E.g.: $X_t = B_t$. $\underbrace{\text{Condition } X_t \geq 0 \forall t \geq 0}_{=:Y_t}$.

2.1 Girsanov theorem

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ and two measures $\mathbb{P}, \mathbb{Q}$
Assume $\mathbb{P} \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P}$ and let $H = \frac{d\mathbb{Q}}{d\mathbb{P}}$

The notes of Eberle switches the roles of $\mathbb{P}, \mathbb{Q}$!

$$\implies Z_t := \mathbb{E}_{\mathbb{P}}(H \mid \mathcal{F}_t) = \frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_t}$$

is a martingale.
From last semester: $\forall Y \in L^1(\mathbb{Q}) \cap L^1(\mathbb{P}), \mathcal{F}_t$ measurable:

$$\mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{F}_s) = \frac{bE_{\mathbb{P}}(Y \cdot Z_t \Big| \mathcal{F}_s)}{Z_s} \forall s < t$$

If $Z > 0$ is $\mathcal{M}_{\text{loc}}, \exists L \in \mathcal{M}_{\text{loc}}$ s.t.:

$$Z_t = e^{L_t - \frac{1}{2}\langle L \rangle_t} \to L_t = \ln(Z_0) + \int_0^t \frac{dZ_s}{Z_s}$$

There might be a problem, because $\ln(Z_0)$ might not be integrable, and therefore not a local martingale!

Theorem 2.1 (Girsanov)**.** *Assume $Z > 0$ is a martingale. If M is a local martingale w.r.t. $\mathbb{P}$, then*

$$\tilde{M}_t := M_t - \langle M, L \rangle_t$$

is a local martingale w.r.t. $\mathbb{Q}$ and

$$\langle M \rangle_t = \langle \tilde{M} \rangle_t.$$

Moreover, if M is a BM w.r.t. to $\mathbb{P}$, then $\tilde{M}$ is a BM w.r.t. $\mathbb{Q}$.

Remark. *In applications, given $\mathbb{P}, (Z_t)_{t\geq 0}$ a positive continuous martingale, define $\mathbb{Q}$ on $\mathcal{F}_\infty = \bigcup_{t\geq 0} \mathcal{F}_t$ s.t.*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t$$

Z is uniformly integrable $\iff \mathbb{Q} \ll \mathbb{P}$. ***Problem:*** *In applications, Z is not necessarily uniformly integrable.*
$\implies$ restrict to $[0, T] \implies$ all fine.
$\mathbb{Q} \to \mathbb{Q}_T$ as in the last semester.

Example. *Let $\gamma \in \mathbb{R}^d, (B_t)_{t\geq 0}$ standard BM.*
Let $L_t \coloneqq \gamma \cdot B_t$ and $Z_t = \exp(L_t - \frac{1}{2}\langle L\rangle_t) = \exp(\gamma \cdot B_t - \frac{1}{2}|\gamma|^2 t)$

Remark. $\lim_{M\to\infty} \sup_{t\geq 0} \mathbb{E}(|Z_t| 1_{|Z_t|>M}) \neq 0$.
Define $\mathbb{Q}$ on $\mathcal{F}_\infty$ s.t. $\tilde{B}_t^k = B_t^k - \langle L, B^k\rangle = B_t^k - \gamma^k t$ is a BM with drift. Show: $Q \not\ll \mathbb{P}$. *Construct Q via Z*

$$A = \left\{ \lim_{t\to\infty} \frac{\tilde{B}_t + \gamma t}{t} = \gamma \right\} \in \mathcal{F}_\infty$$

but: $\tilde{B}$ is a BM w.r.t. $\mathbb{Q} \implies \mathbb{Q}(A) = 1$, $\tilde{B}$ is a BM with drift $-\gamma$ w.r.t $\mathbb{P}(A) = 0$.

Start of lecture 06 (30.04.24)

2.1.1 Drift transformation of SDE

We once again start with

$$\star = \begin{cases} dX_t &= b(t, X_t) \\ X_0 &= x_0 \end{cases}$$

with drift b continuous.
Goal: Get a weak solution of $\star$.
Let $(X_t)_{t\geq 0}$ be a BM in $(\Omega, \mathcal{F}(\mathcal{F}_t)_{t\geq 0}, \mathbb{P}), X_0 = x_0$.
Assume: $Z_t \coloneqq \exp\left(\underbrace{\int_0^t b(s, X_s) dX_s}_{=L_t} - \frac{1}{2}\int_0^t |b(s, X_s)|^2 ds \right)$ is a martingale w.r.t. $\mathbb{P}$.

Remark. *Thr assumption holds if*

$$|b(t,x)| \leq C(1 + \|x\|) \text{ for some } C.$$

$\implies \mathbb{E}_{\mathbb{P}}(Z_t) = 1 \forall t \geq 0$
By Girsanov

$$\tilde{X}_t = X_t - \langle L, X\rangle_t$$

is a $\mathbb{Q}$-BM.
But

$$d\langle L, X\rangle_t = b \cdot dX_t \cdot dX_t = b \cdot dt$$
$$\implies \tilde{X}_t = X_t - \int_0^t b(s, X_s) ds$$
$$\text{w.r.t. } \mathbb{Q} : dX_t = b(t, X_t)dt + \underbrace{d\tilde{B}_t}_{d\tilde{X}_t}$$

Generalization: Start with

$$\star\star\, dX_t = b(t, X_t)dt + \sigma(t, X_t) dB_t$$

where X, B are d-dimensional.

Proposition 2.2. *Assume (X, B) is a weak solution of $\star\star$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. If* *In practice this is surprisingly useful in practice!*

$$Z_t = \exp\left(\int_0^t c(s, X_s) dB_s - \frac{1}{2}\int_0^t c(s, X_s)^2 ds \right)$$

is a martingale w.r.t. $\mathbb{P}$, $\mathbb{Q} \ll \mathbb{P}$ and $\mathcal{F}_t$ s.t. $Z_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}} Vert_{\mathcal{F}_t}$. *this is an implicit condition for c*

Then $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ is a weak soltuion of

$$dX_t = [b(t, X_t) + \sigma(t, X_t)c(t, X_t)dt] + \sigma(t, X_t)d\tilde{B}_t$$

where $\tilde{B}$ is a d-dim BM.

Proof.

$$\begin{aligned} d\langle L, B\rangle_t &\overset{L_t = \int_0^t c(s, X_s)dB_s}{=} c(t, X_t)dt \\ &\implies \tilde{B}_t := B_t - \langle L, B\rangle_t \text{ is a } \mathbb{Q}\text{-BM} \\ &\implies dB_t = c(t, X_t)dt + d\tilde{B}_t \end{aligned}$$

From $\star\star$: $dX_t = b(t, X_t)dt + \sigma(t, X_t)c(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t = \star\star\star$ □

Measure	SDE	Generators
$\mathbb{P}$	$dX_t b \cdot dt + \sigma \cdot dB_t$	$\mathcal{L} \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k,l}(\sigma\sigma^\intercal)_{k,l}\frac{\partial^2}{\partial x_k \partial x_l}$
$\mathbb{Q}$	$[b + \sigma \cdot c]dt + \sigma d\tilde{B}_t$	$\tilde{\mathcal{L}} = \mathcal{L} + \sum_k \sum_l \sigma_{k,l} c_l \frac{\partial}{\partial x_k} = \mathcal{L} + c\sigma^\intercal \nabla$

$t \leq T$ for $\mathbb{Q}_t$.

2.2 Doob-h transform

Essentially three cases in 1d

1. From $B_t \to B_t$ conditioned on $\{B_1 = 0\}$ (measure zero set)
2. From $B_t \to B_t$ conditioned on $\{B_t \geq 0 \forall t \geq 0\}$ (measure zero set)
3. From $B_t \to B_t$ conditioned on $\{B_t \geq 0 \forall t \in [0, 1], B_1 = 0\}$

Let $(X_t, B_t)_{t\geq 0}$ be a weak solution of

$$(\star) = \begin{cases} dX_t &= b(t, X_t)dT + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \text{ fixed} \end{cases}$$

Assume there exists $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ s.t. $h > 0$ satisfying

$$(\frac{\partial}{\partial t} + \mathcal{L})h = 0 \forall t, \in [0, T], x \in \mathbb{R}^d$$

where $\mathcal{L}$ is the generator of the SDE $(\star)$.
By theorem 1.14 $Z_t := h(t, X_t) = h(0, X_0) + \int_0^t (\sigma \nabla h)(s, X_s)dB_s$.
Z_t is a positive local martingale. Assume Z_t is a martingale.
W.l.o.g. $Z_0 = 1$ (if not $h \mapsto \frac{h}{h(0, X_0)}$).
$\implies d\mathbb{Q}_T = Z_T \cdot d\mathbb{P}$. Let L_t s.t. $Z_t = \exp(L_t - \frac{1}{2}\langle L\rangle_t)$.
Girsanov, since B_t is a $\mathbb{P}$-BM,
$\tilde{B}_t := B_t - \langle L, B\rangle_t$ is a Q_T-BM, where

$$\langle L, B\rangle_t = \sigma^\intercal \cdot \nabla \ln h \cdot dB_t$$

because of

$$dL_t = \frac{dZ_t}{Z_t} = \frac{\sigma^\intercal \cdot \nabla h \cdot dB_t}{h(t, X_t)} = \sigma^\intercal \cdot \nabla \ln h \cdot dB_t$$

And therefore $dB_t = d\tilde{B}_t + \sigma^\intercal \nabla \ln h dt$.
From $(\star) \implies dX_t = (\underbrace{b(t, X_t) + \sigma(t, X_t)\sigma^\intercal(t, X_t)\nabla h(t, X_)}_{:=\tilde{b}})dt + \sigma(t, X - t)d\tilde{B}_t$

Proposition 2.3. *Let X be a weak solution of $dX_t = b(t,X_t) + \sigma(t,X_t)dB_t$ on $[0,T]$ under $\mathbb{P}$. Then under $\mathbb{Q}_T, X$ is a weak solution of the SDE*

$$\begin{cases} dX_t &= \tilde{b}(t,X_t)dt + \sigma(t,X_t)d\tilde{B}_t \\ X_0 &= x_0 \end{cases}$$

where $\tilde{b} = b + \sigma\sigma^\intercal \nabla \ln h$.

Remark. *In applications, often $h(x) \not> 0$ for all x!*
E.g.: $h(t,x) = x \implies \left(\frac{\partial}{\partial t} + \mathcal{L}\right) h = 0$ $(X_t = B_t + x_0)$, but $h(x) \leq 0$ for $x \leq 0$.
In this case first do the construction on $[0,\tau]$, where $\tau \coloneqq \inf\{t > 0 \mid B_t = 0\}$
$\implies$ *ok for $B_t^\tau \overset{magically}{\implies}$ the new SDE has drift:*

We realize there is no problem in the new measure

$$\frac{\partial}{\partial x}\ln(h(x)) = \frac{1}{h(x)}\frac{\partial h(x)}{\partial x} = \frac{1}{x}$$

Example. *If $X_t = B_t$ and $h(t,) = e^{\gamma x - \frac{1}{2}|\gamma|^2 t}$*

$$\begin{aligned} \implies dX_t &= \nabla \ln h(t,X_t)dt + dB_t \\ &= \nabla(\gamma \cdot x)dt + dB_t \\ &= \gamma d_t + dB_t \end{aligned}$$

$\implies$ *X_t is a BM with drift γ w.r.t $\mathbb{Q}$.*

2.3 Diffusion Bridges

Consider a Markov process $(X_t)_{t\geq 0}$ which is a diffusion starting from $X_0 = x_0 \in \mathbb{R}^d$.
We want to condition on the event $\{X_T = y\}$ for some given $T > 0, y \in \mathbb{R}^d$.
Goal: Find, $\forall y \in \mathbb{R}^d, \mathbb{Q}^y$ which is the conditional measure on $\{X_T = y\}$.
Assume: $(X_t)_{t\geq 0}$ has transition density P s.t.

This is not true in general, if we don't have unique solutions

$$\forall 0 \leq s \leq t \leq T, \mathbb{P}(X_t \in dz \mid X_s = x) = p(s,x;t,z)dz.$$

X_t solves

$$\begin{cases} dX_t &= b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_0 &= x_0 \end{cases}.$$

Define $h^y(s,x) \coloneqq \frac{p(s,x;T,y)}{p(0,x_0;T,y)}$ for $s \in [0,T), x \in \mathbb{R}^d$.

Remark. *$s < T$ since otherwise we get ingeneral singularities.*

Lemma 2.4. *Let $Z_t^y \coloneqq h^y(t,X_t)$ is a martingale.*

Proof. Let $0 \leq < t < T$:

$$\begin{aligned} \mathbb{E}(Z_t^y \mid \mathcal{F}_s) &= \mathbb{E}(h^y(t,X_t) \mid \mathcal{F}_s) \\ &\overset{\text{MP}}{=} \mathbb{E}(h^y(t,X_t \mid X_s)) \\ &= \int_{\mathbb{R}^d} h^y(t,x)p(s,X_s;t,x)dx \\ &= \int_{\mathbb{R}^d} \frac{p(t,x;T,y)}{p(0,x_0;T,y)}p(s,X_s;t,x)dx \\ &= \frac{1}{p(0,x_0;T,y)\int_{\mathbb{R}^d} p(s,X_s;t,x)p(t,x;T,y)dx} \\ &\overset{\text{Chapman-Kolmogorov}}{=} Z_s^y = \end{aligned}$$

□ $h^y(0,x_0) = 1$

Start of lecture 07 (02.05.24)

Added remark. ***Goal:*** *Find a family* $(\mathbb{Q}^y)_{y\in\mathbb{R}^d}$ *s.t. for* $A \in \mathcal{F}_T$*:*

$$\mathbb{P}(A) = \mathbb{E}(\mathbb{P}(A|X_T)) \overset{?}{=} \mathbb{E}(\mathbb{Q}^{X_T}(A))$$

and $\mathbb{Q}^y\left(\lim_{t\uparrow T} X_t = y\right) = 1$

Lemma 2.5. *We can take*

$$\mathbb{Q}^y(A) = \mathbb{E}(1_A \cdot h^y(s, X_s)) \forall A \in \mathcal{F}_s$$

for $s \in [0, T]$.

Proof. For all $A \in \mathcal{F}_s$, g bounded measurable:

$$\begin{aligned}
\mathbb{E}(1_A g(X_t)) &= \mathbb{E}(1_A \mathbb{E}(g(X_T|\mathcal{F}_s))) \\
&\overset{\text{MP}}{=} \mathbb{E}(1_A \mathbb{E}(g(X_T|X_s))) \\
&= \mathbb{E}\left(1_A \int_{\mathbb{R}^d} p(s, X_s; T, x) g(x) dx\right) \\
&= \int_{\mathbb{R}^d} dx g(x) \mathbb{E}(1_A p(s, X_s; T, x))
\end{aligned}$$

In particular,

$$\mathbb{P}(A) = \mathbb{E}(1_A) = \int_{\mathbb{R}^d} dx \mathbb{E}(1_A \cdot p(s, X_s; T, x))$$

But

$$\begin{aligned}
\int_{\mathbb{R}^d} dx p(0, X_0; T, x) Q^x(A) &= \int_{R^d} p(0, x_0; T, x) \underbrace{\mathbb{E}(1_A h^x(s, X_s))}_{=\int dz p(0, x_0; s, z) h^x(s, z) 1_A} \\
&= \int_{\mathbb{R}^d} \mathbb{E}(1_A p(s, X_s; T, x)) \\
&= \mathbb{P}(A).
\end{aligned}$$

□

Remark. $Z_t^y = h^y(t, X_t) = h(0, X_0) +$ *martingale* $+ \underbrace{\int_0^t (\partial_t + \mathcal{L}) h^y(s, X_s) ds}_{=0}$.

One can verify: $(\partial_t + \mathcal{L}) h^y(t, X_t) = 0 \iff (\partial_t + \mathcal{L}) H^y(t, X_t) = 0$ *for* $H^y(t, x) = p(t, x; T, y)$.

$$\begin{aligned}
\frac{\partial}{\partial t} H^y(t, x) &= \lim_{\epsilon \downarrow 0} \frac{p(t, x; T, y) - p(t - \epsilon, x; T, y)}{\epsilon} \\
&= \lim_{\epsilon \downarrow 0} \frac{p(t, x; T, y) - \int \overbrace{p(t - \epsilon, x; t, z)}^{= e^{\epsilon L}(x, z) = \delta_{x-z} + \epsilon \mathcal{L}(x, z)} p(t, z; T, y) dz}{\epsilon} \\
&= -(\mathcal{L} H^y)(t, x)
\end{aligned}$$

$\implies$ *this is a doob-h transform with* $h = h^y$.

$$\frac{d\mathbb{Q}^y}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t^y.$$

Corollary 2.6. *Assume* $(t, x) \mapsto p(t, x; T, y)$ *is* $C^{1,2}([0, T] \times \mathbb{R}^d)$
$\implies$ *The Doob h-transform of the original process under* $\mathbb{Q}^y$ *satisfies the SDE:*

$$dX_t = \tilde{b}(t, X_t) dt + \sigma(t, X_t) d\tilde{B}_T$$

where $\tilde{B}_t$ *is a* $\mathbb{Q}^y$*-BM and* $\tilde{b}(t, X_t) = b(t, X_t) + (\sigma\sigma^\intercal \nabla \ln h^y)(t, X_t)$.

Remark. *Take $z \neq y$ and $\forall 0 < \epsilon < |z - y|$:*

$$\begin{aligned}\mathbb{Q}^y\left(\{|X_t - z| \leq \epsilon\}\right) &= \mathbb{E}\left(1_{|X_t - z| \leq \epsilon}\frac{p(t, X_t; T, y)}{p(0, x_0; T, y)}\right)\\ &= \int dx 1_{|x-z|\leq\epsilon} p(0, x_0, t, x)\frac{p(t, x; T, y)}{p(0, x_0; T, y)}\\ &\xrightarrow{t\to T} 0\end{aligned}$$

$\implies \mathbb{P}\left(\lim_{t\uparrow T} X_t = y\right) = \mathbb{P}(X_t = y) = 0,$ *but* $\mathbb{Q}^y\left(\lim_{t\uparrow T} X_t = y\right) = 1$
$\implies \mathbb{Q}^y$ *is singular w.r.t.* $\mathbb{P}$.

Example. $b = \gamma \in \mathbb{R}^d, \sigma = 1, X_0 = 0$ *and* $T = 1$. *Under* $\mathbb{P}$ *:*

$$\begin{cases} dX_t = \gamma dt + dB_t \\ X_0 = 0 \end{cases}$$

Under $\mathbb{Q}^y$*?*

$$h^y(t, x) = \frac{p(t, x; 1, y)}{p(0, x_0, 1, y)} = fct(\gamma, y) \cdot \exp\left(-\frac{(y-x)^2}{2(1-t)}\right)\exp\left((y - x)\gamma\right)$$

$\implies \ln(h^y(t, x)) = \ln f(\gamma, y) - \frac{(y-x)^2}{2(1-t)} + (y - x)\gamma \implies \nabla \ln h^y(t, x) = \frac{y-x}{1-t} - \gamma.$
Under $\mathbb{Q}^y$ *:*

$$\begin{aligned} dX^k_t h^y &= \left(\gamma_k + \frac{y_k - X^k_t}{1-t}\right)dt + d\tilde{B}^k_t\\ &= \frac{Y_k - X^k_t}{1-t}dt + d\tilde{B}^k_t\end{aligned}$$

Indepdendent of γ*!*

for $k = 1, \ldots, d$

Lemma 2.7. *Let* p *be the transition density of* X *w.r.t.* $\mathbb{P}$ *and* p^h *the transition density of* X *w.r.t.* $\mathbb{Q}^y$.
Let $h^y(t, x) = \frac{p(t,x;T,y)}{p(0,x_0;T,y)}$.

$$\implies p^h(s, x; t, z) = \frac{1}{h^y(s, x)}p(s, x; t, z)h^y(t, z)$$

Useful in practice to sample!

Notice how the normalizations inside of h^y *cancel!*

Added remark. *This is the first time my numbering is different from the handwritten notes, as they contain two environments numbered 2.5.*

Proof.

$$\begin{aligned} p^h(s, x; t, z)dz &= \mathbb{P}(X_t \in dz \mid X_s = x, X_T = y)\\ &\stackrel{0\leq s\leq t}{=} \frac{p(0, x_0; s, x)p(s, x; t, z)p(t, z; T, y)dz}{p(0, x_0; s, x)p(s, x; T, y)}\\ &= p(s, x; t, z)\frac{p(t, z; T, y)}{p(0, x_0; T, y)}\frac{1}{\frac{p(s,x;T,y)}{p(0,x_0;T,y)}}dz \end{aligned}$$

□

2.4 Brownian motion conditioned to stay positive forever

Goal: $(B_t)_{t\geq0}$ with $B_0 = x_0 \geq 0$ and condition on

$$\{B_t \geq 0 \forall t \geq 0\}$$

$T_x = \inf\{t \geq 0 \mid B_t = x\}$ and take $0 < x_0 < R$.
$T_0 \wedge T_R = T_R$.
Let $\tilde{T}_R \coloneqq T_0 \wedge T_R$. Define the event $E_R = \{B_{\tilde{T}_R} = R\} = \{T_0 \wedge T_R = T_R\}$.
One verifies $\mathbb{P}(E_R) = \frac{x_0}{R} \in (0,1)$.
$\mathbb{P}(E_R) = \mathbb{E}(1_{B^{x_0}_{\tilde{T}_R}}) = \underbrace{\mathbb{E}(f(B^{x_0}_{\tilde{T}_R}))}_{u(x_0)}$, where

$$f(x) = \begin{cases} 1 & x = R \\ 0 & x = 0 \end{cases}.$$

u has a link to the PDE:

By Theorem 11.6 of the last semester

$$\begin{cases} \frac{1}{2}\frac{\partial^2}{\partial x^2} u(x) = 0 & x \in (0,R) \\ u(x) = f(x) & x \in \{0, R\} \end{cases}$$

Define the conditional probability

later $R \to \infty$

$$Q^R(A) \coloneqq \frac{\mathbb{P}(A \cap E_R)}{\mathbb{P}(E_R)}$$

Lemma 2.8. *Let $h(x) \coloneqq \frac{\mathbb{P}_x(E_R)}{\mathbb{P}_{x_0}(E_R)} = \frac{x}{x_0}$, where $\mathbb{P}_x$ is the unconditional law for $B_0 = x$.*

$$Z_t = h(B_{t \cap \tilde{T}_R})$$

is a non-negative martingale.

Proof.

$$\mathbb{Q}^R(A) \overset{A \in \mathcal{F}_s}{=} \frac{\mathbb{E}(1_A \mathbb{E}(1_{E_R} \mid \mathcal{F}_s))}{\mathbb{P}(E_R)} = \frac{\mathbb{E}(1_A \mathbb{E}(1_{E_R} \cdot 1_{\tilde{T}_R > s} \mid \mathcal{F}_s)) + \mathbb{E}(1_A \mathbb{E}(1_{E_R} 1_{\tilde{T}_R \leq s} \mid \mathcal{F}_s))}{\mathbb{P}(E_R)}$$

$$\begin{aligned}
&= \frac{1}{\mathbb{P}(E_R)}\left[\mathbb{E}(1_A 1_{\tilde{T}_R > s} \underbrace{\mathbb{E}(1_{E_R} \mid \mathcal{F}_s)}_{\mathbb{P}_{B_s}(E_R)}) + \mathbb{E}(1_A \underbrace{1_{E_R}}_{1_{B_{\tilde{T}_R} = R}} 1_{\tilde{T}_R \leq s})\right] \\
&= \mathbb{E}(1_A 1_{\tilde{T}_R > s} h(B_s)) + \mathbb{E}(1_A 1_{\tilde{T}_R \leq s} h(B_{T_R})) \\
&= \mathbb{E}(1_A h(B_{s \wedge \tilde{T}_R}))
\end{aligned}$$

$\implies \forall A \in \mathcal{F}_s : \mathbb{Q}^R(A) = \mathbb{E}_{\mathbb{P}}(1_A h(B_{s \wedge \tilde{T}_R}))$
$\implies h(B_{s \wedge \tilde{T}_R}) = \frac{d\mathbb{Q}^R}{d\mathbb{P}}|_{\mathcal{F}_s} \implies h \in (0, \frac{R}{x_0})$ is a martingale (See construction of Girsanov) □

Note that we can write $Z^R_t =\coloneqq h(B_{s \wedge \tilde{T}_R}) = \exp(L_t - \frac{1}{2}\langle L \rangle_t)$ by choosing

Start of lecture 08 (07.05.24)

$$L_t = \int_0^{\tilde{T}_R} \frac{h'(B_s)}{h(B_s)} dB_s$$

since

$$dL_t = \frac{dZ^R_t}{Z^R_t} = \begin{cases} \frac{h'(B_t) dB_t}{h(B_t)} & t < \tilde{T}_R \\ 0 & t \geq \tilde{T}_R \end{cases}$$

Proposition 2.9. *Under the measure $\mathbb{Q}^R$, $(B_t)_{t \geq 0}$ solves the SDE*

$$dB_t = \frac{1_{t \tilde{T}_R}}{B_t} dt + dW_t$$

for some $(W_t)_{t \geq 0}$ a $\mathbb{Q}^R$-BM.

Proof. Apply Girsanov in which

$$d\langle B, L\rangle_t = dB_t \cdot dL_t \overset{h'(x)=\frac{1}{x_0}}{1_{t<\tilde{T}_R}} \frac{1}{B_t} dt \qquad \square$$

Our goal was to condition the BM to stay positive forever. Sofar we have conditioned it to reach the level R before the level 0.

Remark.

$$\begin{aligned}\mathbb{Q}^R(T_0 < T_R) &= \mathbb{E}(1_{T_0<T_R} h(B_{T_R \wedge T_0})) \\ &= \mathbb{E}(1_{T_0<T_R} h(B_{T_0})) \\ &= \mathbb{E}(1_{T_0<T_R} \underbrace{h(0)}_{=0}) = 0\end{aligned}$$

$\implies$ *under the new measure $\mathbb{Q}^R$, the BM indeed does not reach 0 before R as we wanted.*

Finally we wan to take $R \to \infty$. It should

Start of lecture 09 (14.05.24)

Example. *Discrete time M.C. with transition probability P aperiodic and irreducible: $\exists n_0 \forall n > n_0 (P)^n_{i,j} > 0 \implies \exists |\lambda_0| \leq 1, |\lambda_1| > |\lambda_2| > |\lambda_3|$:*

$$(P)^n \underbrace{\varphi_0(i)}_{>0} = \lambda_0 \varphi_0(i)$$

$\lim_{n\to\infty} \underbrace{\overset{n}{P}}_{(1+L)} = (\varphi_0(1), \dots, \varphi_0(d))$
where $(1+L) \to e^{tL}$, for which the real eigenvalues are positive.

2.5 Diffusion conditioned to stay in a domain

Domain $D \subset \mathbb{R}^d$: bounded, open, connected.
Diffusion with generator

$$L = \frac{1}{2} \sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$$

$\iff$ SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \frac{1}{2}\sigma(t, X_t)dB_t \\ X_0 = x_0 \end{cases}$$

$a = \sigma\sigma^\intercal$.
Assume: b, σ are continuous, $\sigma \in C^1$ and let $\tau_D = \int\{t \geq 0 | X_t \notin D\}$
Key assumption in Pinsky's paper:

(a) $\mathbb{P}_x(\tau_D > t) \in \mathbb{C}^2(D)$

(b)
$$\begin{cases} \mathbb{P}_x(\tau_D > t) & = C_1 \varphi_0(x) e^{-\lambda_0 t} + o\left(e^{-\lambda_0 t}\right) \\ \nabla \mathbb{P}_x(\tau_d > t) = C_1 \nabla\varphi_0(x) e^{-\lambda_0 t} + o\left(e^{-\lambda_0 t}\right) \end{cases}$$

where

$$\begin{cases} -L\varphi_0(x) = \lambda_0 \varphi_0(x) & x \in D \\ \varphi_0(x) = 0 & x \in \partial D \end{cases}$$

and λ_0 is the smallest eigenvalue of $-L$. Which means $\mathbb{P}(\tau_d > t) \overset{t\to\infty}{\to} 0$ — in the non symmetric case take the real part first

Want: Condition X to stay in D forever.

(1) Condition X_t to $\{X_t \in D : 0 \leq t \leq T\}$: Forall $A \in \mathcal{F}_T$: define a measure

$$\mathbb{Q}^T(A) := \frac{\mathbb{P}_{x_0}(A \cap \{\tau_D > T\})}{\mathbb{P}_{x_0}(1_{\tau_D} > T)} = \frac{\mathbb{E}_{x_0}(1_A \cdot 1_{\tau > T})}{\mathbb{E}_{x_0}(1_{\tau > T})}$$

Lemma 2.10. *$\forall s < T, A \in \mathcal{F}_s$,*

$$\mathbb{Q}^T(A) = \mathbb{E}_{x_0}(1_A \cdot Z_s^T)$$

where $Z_s^T = \frac{g^{T-s}(X_{s \wedge \tau_D})}{g^T(x_0)}$ with $g^t(x) := \mathbb{P}_x(\tau_D > t)$.

Proof. Let $A \in \mathcal{F}_s, s < T$:

$$\begin{aligned}
\mathbb{Q}^T(A) &= \frac{\mathbb{E}_{x_0}(1_A 1_{\tau_D > T})}{g^T(x_0)} = \frac{\mathbb{E}_{x_0}(1_A \mathbb{E}_{x_0}(1_{\tau_D > T} | \mathcal{F}_s))}{g^T(x_0)} \\
&\overset{\text{M.P.}}{=} \frac{\mathbb{E}_{x_0}(1_A \mathbb{E}_{x_0}(1_{\tau_D > T} | X_s))}{g^T(x_0)} \\
&= \frac{1}{g^T(x_0)} \left[\mathbb{E}_{x_0}(1_A \underbrace{\mathbb{E}_{x_0}(1_{\tau_D > T} 1_{\tau_D > s} | X_s)}_{g^{T-s}(X_s)}) + \mathbb{E}_{x_0}(1_A \underbrace{\mathbb{E}_{x_0}(1_{\tau_D > T} 1_{\tau_D < s} | X_s)}_{=0}) \right] \\
&= \frac{1}{g^T(x_0)} \left(1_A g^{T-s}(X_{s \wedge \tau_D})\right)
\end{aligned}$$

□

Lemma 2.11. *$Z_0^T = 1$ and $(Z_s^T)_{s \in [0,T]}$ is a martingale.*

Proof. By lemma 2.10 $\implies Z_s^T = \frac{d\mathbb{Q}^T}{d\mathbb{P}} |_{\mathcal{F}_s} \rightarrow$ is a martingale. □

Remark. *By construction, $\mathbb{Q}^T(\tau_D \leq T) = \frac{1}{g^T}(x_0) \mathbb{E}_{x_0}(1_{\tau_D \leq T} 1_{\tau_D > t}) = 0$*

Assume that $g^t(x)$ is $C^{1,2}$ (1 in time, 2 in space) $\implies$ apply Itô and doob transform gives:

Proposition 2.12. *Let X be a weak solution of $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ under $\mathbb{P}$.*
$\implies$ X is a weak solution of

$$dX_t = \left(b(t, X_t) + \frac{a(t, X_t) \nabla g^{T-t}(X_t)}{g^{T-t}(X_t)} \right) dt + \sigma(t, X_t) d\tilde{B}_t, 0 \leq t \leq T$$

under $\mathbb{Q}^T$ provided $g^t(x) > 0 \forall x \in D, t \geq 0$.

What happens in the $T \to \infty$ limit?
By assumption (b)
$\lim_{T \to \infty} \frac{\nabla g^{T-t}(x)}{g^{T-t}(x)} = \frac{\nabla \varphi_0(x)}{\varphi_0(x)}$

Their operator is not the same as ours. The only difference is a drift term.

Remark. *If $g \in C^{1,2}$, it satisfies the parabolic PDE*

$$(\star) \begin{cases} \frac{\partial}{\partial t} g^t(x) = L g^t(x) & x \in D \\ g^t(x) = 0 & x \in \partial D \end{cases}$$

Apply Theorem 11.5 from the WS with $A = L, u(t, x = g^t(x))$: $u(0, x) = \mathbb{E}_x(1_{\tau_D > 0}) = 0, u(t, x) = \mathbb{E}_x(1_{\tau_D > t}) = 0 \forall t, x \in \partial D$.

Pinsky proved that $\lim_{T \to \infty} Q^T = Q$ weak and that under Q the process satisfies

$$dX_t = \left[b(t, X_t) + a(t, X_t) \frac{\nabla \varphi_0(X_t)}{\varphi_0(X_t)} \right] dt + \sigma(t, X_t) dW_t.$$

Why is $\mathbb{P}_x(\tau_D > t) = C_1 \varphi_0(x) e^{-\lambda_0 t} + o(e^{-\lambda_0 t})$

Added remark. *This is not the case for BM conditioned to stay in $D = (0, \infty)$. Reason: Spectrum of $-L$ is $\mathbb{R}_+$*

Since it satisfies the PDE $(\star)$, $g^t(x) = (e^{tL})g^0(x)$.
if the spectrum of $-L$ is discrete with eigenvalues $0 \le \lambda_0 < \lambda_1 \le \dots$, where there is a spectral gap between λ_0 and λ_1: $\exists$ eigenfunctions $\varphi_0(x), \varphi_1(x), \dots$, normalized as $\|\varphi_n\|_{L^2(D)} = 1 \implies +L\varphi_n(x) = -\lambda_n\varphi_n(x) \implies$ we can choose $\varphi_0, \varphi_1, \dots$ to be orthonormal: $(\varphi_i, \varphi_j)_{L^2(D)} = \delta_{ij}$.
$\implies 1 = \sum_{n\ge 0} \varphi_n\varphi_n^*$, since $\varphi_n\varphi_n^*$ is the projection onto the space generated by φ_m:

$$\varphi_n\varphi_n^* f = \varphi_n(\varphi_n, f)_{L^2(D)}$$

$$\begin{aligned} f(L)\varphi_n &= f(-\lambda_n)\varphi_n \\ g^t(x) &= (e^{tL})g^0(x) = e^{tL}1g^0(x) = \sum_{n\ge 0} e^{tL}\varphi_n(x)\varphi_n^* g^0 = \sum_{n\ge 0} e^{-\lambda n t}\varphi_n(x)(\varphi_m, g^0) \\ &= (\varphi_0, g^0)\varphi_0(x)e^{\lambda_0 t} + \underbrace{e^{-\lambda_0 t}\sum_{n\ge 0} e^{(\lambda_0-\lambda_n)t}\varphi_n(x)(\varphi_n, g^0)}_{o(e^{-\lambda_0 t})} \end{aligned}$$

Example 2.13. *One dimension.*
$L = \frac{1}{2}\frac{d^2}{dx^2}$ *and* $D = [0, R]$.
Solve $-L\varphi(n)(x) = \lambda_n\varphi_n(x)$ *with* $\varphi_n(x) = 0$ *for* $x \in \{0, R\}$.
Solutions $\varphi_n(x) = C_1 \sin\left(\frac{\pi x}{R} \cdot (n+1)\right)$ *and* $\lambda_n = \left(\frac{\pi(n+1)}{R}\right)^2 \implies \lambda_0 = \frac{\pi^2}{R^2}$.
For $x \in (0, R)$:

$$\mathbb{P}_x(\tau > t) \approx \tilde{C}_1\varphi_0(x)e^{-\frac{\pi^2 t}{R^2}}$$

$$\implies dX_t = \frac{\pi}{R}\frac{\cos(\frac{\pi X_t}{R})}{\sin(\frac{\pi X_t}{R})}d + dW_t$$

as $X_t \to 0$ *(or* $X_t \to \mathbb{R}$*), drift* $\approx \frac{1}{x_t}$ *or* $(\frac{-1}{R-X_t})$.
Also: formally $R \to \infty$ *in the SDE, we get* $dX_t = \frac{dt}{X_t} + dW_t$*, which is the SDE we derived for BM conditioned to stay* > 0 *forever.*

Start of lecture 10 (16.05.24)

Example 2.14 (Brownian Motion in a Weyl chamber)**.** *The* ***Weyl chamber*** *is defined as*

$$W^d = \{x \in \mathbb{R}^d \mid x_1 < x_2 < \dots < x_d\}.$$

Given a d dim. Brownian motion $(B_t)_{t\ge 0}$ *with* $B_0 \in W^d$ *what is the sde of this BM conditioned on staying in* W^d *forever.*
What is the harmonic function vanishing at ∂W^d*?*

Lemma 2.15. $h(x) := \prod_{1\le k<l\le d}(x_l - x_k)$ *is harmonic and satisfies*

$$\frac{1}{2}\sum_{k=1}^{d}\frac{\partial^2}{\partial x_k^2}h(x) = 0, x \in W^d$$

and $h(x) = 0$ *for* $x \in \partial W^d$, $h(x) > 0$ *for* $x \in W^d$.

Remark. $h(x) = \det(x_k^{l-1})$

Proof. Last two properties are clear. h is a polynomial, antisymmetric in each pair $x_l - x_k$ and has lowest possible power.
$\Delta h(x)$ is still antisymmetric, but with lower power $\implies \Delta h(x) = 0$. ∎

The BM conditioned to stay in the Weyl chamber will satisfy the SDE:

$$dX_t^k = \sum_{l\neq k} \frac{dt}{X_t^l - X_t^k} + dB_t^k, \qquad 1 \leq k \leq d$$

2.6 Stationary distribution for diffusions

let $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, where b, σ are time independent with generator

$$L = \sum_{k=1}^{d} b_k(x)\frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k,l=1}^{d} a_{k,l}(x)\frac{\partial^2}{\partial x_k \partial x_l}$$

Definition 2.16. *A probability measure μ **stationary (or invariant)** if for all $f \in C_0^\infty(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} (Lf)(x)\mu(dx) = 0$$

or only test schwarzfunctions

Assume that $\mu \ll$ Lebesgue, i.e., $\mu(dx) = \rho(x)dx$ for some positive function $\rho(x) \in C^2$ with $\int \rho(x)dx = 1$.

Lemma 2.17. *μ is stationary with density ρ*

$$\iff L^*\rho(x) = 0 \text{ almost everywhere}$$

where L^* is the adjoint of L, given by

$$L^*\rho(x) = \frac{1}{2}\sum_{k,l=1}^{d} \frac{\partial^2}{\partial x_l \partial x_l}(a_{k,l(x)\rho(x)}) - \sum_{k=1}^{d}\frac{\partial}{\partial x_k}(b_k(x)\rho(x))$$

this also assumes b to be differentiable!

Proof.

$$\begin{aligned}&\int_{\mathbb{R}^d}(Lf)(x)\rho(x)dx = 0\\ &= \int_{\mathbb{R}^d} dx_1, \dots, dx_d \rho(x)\left(\frac{1}{2}\sum_{k,l=1}^{d}\frac{\partial^2}{\partial x_k \partial x_l}f(x) + \sum_{k=1}^{d} b_k(x)\frac{\partial}{\partial x_k}f(x)\right)\end{aligned}$$

$$\int_{R^{d-1}}\prod_{l\neq k} dx_l \underbrace{\int_R dx_k \rho(x) b_k(x)\frac{\partial}{\partial x_k}f(x)}_{\overset{\text{I.b.p}}{=} -\int_R dx_k \frac{\partial}{\partial x_k}(\rho(x)b_k(x))\cdot f(x)}$$

□

Example 2.18 (1-dim. diffusion)**.** *Let $L = b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial}{\partial x^2}$ for all $x \in R$.*

$$\implies L^*\rho(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\rho(x)\sigma^2(x)) - \frac{\partial}{\partial x}(\rho(x)b(x)) \overset{\text{wanted}}{=} 0$$

$\implies \exists c_1$ *s.t.* $\frac{1}{2}\frac{\partial}{\partial x}\underbrace{(\rho(x)\sigma^2(x))}_{=g(x)} - \underbrace{\rho(x)b(x)}_{=g(x)\frac{b(x)}{\sigma^2}} = c_1$.

*Let $s(x) = \int_{x_0}^{x} dy e^{-\int_{x_0}^{y} dz \frac{2b(z)}{\sigma^2(z)}}$ be the **scale function**.*
The equation for c_1 is equivalent to

$$s'(x)\sigma^2(x)\rho(x) = c_2 + 2c_1 s(x)$$

for some constant c_2,$s'(x)e^{-\int_{x_0}^{y} dz \frac{2b(z)}{\sigma^2(z)}} => 0$.

s satisfies $\frac{1}{2}\sigma^2(x)s''(x)+b(x)s'(x)=0$, for the Bessel process $d=2$, $s(x)=\ln(x)$
If $s(\mathbb{R})=\mathbb{R}$, then $c_1=0$, which implies

$$\rho(x)=\frac{c_2}{\sigma^2(x)}\underbrace{e^{\int_{x_0}^{x}\frac{2b(y)}{\sigma^2(y)}dy}}_{\frac{1}{s'(x)}}$$

which satisfies positivity.

Lemma 2.19. *If $S(\mathbb{R})=\mathbb{R}$, then there exists stationary measure with density $\rho(x)$ as described above.*

Counterexample:

- $\sigma=1$, $x_0=0$, $b(z)=b>0$.

$\implies s(x)=\frac{1-e^{-2bx}}{2b}\to s'(x)=e^{-2bx}$.

- $s(-\infty)=-\infty$
- $s(\infty)=\frac{1}{2b}$

$\implies$ Argument for $c_1=0$ does not work.

$$\implies \rho(x)=\tilde{c}_1+\tilde{(c)}_2e^{2bx}$$

which can't be a density, because if at least $\tilde{c}_1$ or $\tilde{c}_2$ are $\neq 0 \implies$ it is not integrable.

Example 2.20. *Let $Lf(x)=\frac{1}{2}\frac{\partial^2}{\partial x^2}f(x)+b(x)\frac{\partial}{\partial x}f(x)$ where $b(x)=\frac{\partial}{\partial x}\ln h(x)$ for some $h(x)>0$. Assume h is normalized as $\int_R(h(x))^2dx=1$. $\implies$ **Claim:** The stationary density of the process with generator L is given by $\rho(x)=(h(x))^2$.* *Implicitly assumes $h\in L^2(\mathbb{R})$*
Verify the claim:

$$\begin{aligned}L^*\rho(x)&\stackrel{?}{=}0\\
&=\frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x)-\frac{\partial}{\partial x}(b(x)\rho(x))\\
&=\frac{1}{2}\frac{\partial^2}{\partial x^2}(h(x))^2-\frac{\partial}{\partial x}(\frac{h'(x)}{h(x)}h(x)^2)\\
&=\frac{\partial}{\partial x}(h(x)h'(x))-\frac{\partial}{\partial x}(h'(x)h(x))=0\end{aligned}$$

*Example **??** $\implies h(x)=c\sin(\frac{\pi x}{L})$* *here $L\in\mathbb{R}$ is not the operator!*

$$\int_0^L h(x)^2dx=c^2\int_0^2\sin^2(\frac{\pi x}{L})dx=1$$
$$\implies c=\sqrt{\frac{2}{L}}\implies \rho(x)=\frac{2}{L}\sin^2(\frac{\pi x}{L}).$$

2.7 Uniqueness in law and path integral formula

Consider the SDE

$$\begin{cases}dX_t=b(t,X_t)dt+dB_t & \text{in } \mathbb{R}^d\\ x_0=x_0\end{cases}$$

Assume

$$(\star)\forall T>0\int_0^T|b(s,X_s)|^2ds<\infty \text{ a.s.}$$

.

Goal: Show uniqueness in law.
Consider any $(X,\mathcal{B},\mathbb{P})$ weak solution satisfying $(\star)$ and define

$$\tau_n := \inf\{t \geq 0 \mid \int_0^t |b(s,X_s)|^2 ds \geq n\}$$

which, by $(\star)$ goes to infinity.
For each n, define $\mathbb{Q}^n$:

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = e^{\underbrace{-\int_0^{\tau_n} b(s,X_s)dB_s}_{L_{\tau_n}} - \frac{1}{2}\int_0^{\tau_n} |b(s,X_s)|^2 ds}.$$

By Girsanov: $\tilde{B}_t := B_t - \langle B, L\rangle_t = B_t + \int_0^{t\wedge\tau_n} b(s,X_s)ds$ is BM w.r.t. $\mathbb{Q}^n$, which implies $X_t = \tilde{B}_t$ for all $t \leq \tau_n$ w.r.t. $\mathbb{Q}^n$.
Let events of (X,B) $\mathcal{F}_T$ measurable for some time T: are in A_T.

$$\begin{aligned}\mathbb{E}_\mathbb{P}\left(1_{(X,B)\in A_T}1_{T\leq\tau_n}\right) &= \mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,\mathcal{B})\in A_T}1_{T\leq\tau_n}e^{\int_0^{\tau_n} b(s,X_s)\underbrace{dB_s}_{=dX_s - b(s,X_s)ds} + \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)\\ &= \mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,B)\in A_T}1_{T\leq\tau_n}e^{\int_0^{\tau_n} b(s,X_s)dX_s - \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)\end{aligned}$$

B_s is adapted to $X_s \implies$ for some Φ: $B = \Phi(X) \implies$

$$\mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,\Phi(X))\in A_T}1_{T\leq\tau_n}e^{\int_0^{\tau_n} b(s,X_s)dX_s - \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)$$

But: Law of X w.r.t. $\mathbb{Q}^n$ is the law of BM. Take $n \to \infty$, then $\mathbb{Q}^n \to$ Wiener measure, $T \to \infty \implies$ we look at every possible event, but the probability does only depend on X!

$$\mathbb{P}((X,B) \in \mathcal{B}(\mathcal{E}^d \times \mathcal{E}^d))$$

One case where the path integral formula is also numerically stable: Start of lecture 11 (28.05.24)
Assume that $b(x) = -\nabla V(x)$ for some smooth $V(x)$ (time independent). This is called drift of **gradient type**.
Apply Itô to $V(x)$:

$$V(\omega_T) = V(\omega_0) + \int_0^T \nabla V(\omega_s)dw_s + \frac{1}{2}\int_0^T \Delta V(\omega_s)ds$$

$\implies \int_0^T b(s,\omega_s)d\omega_s = -\int_0^T \nabla V(\omega_s)d\omega_s$
and therefore

$$\mathbb{P}(X \in \tilde{A}_T) = \int_{C^d} W(d\omega)\underbrace{e^{V(\omega_0)-V(\omega_T)+\frac{1}{2}\int_0^T\left[\Delta V(\omega_s) - (\nabla V(\omega_s))^2\right]ds}}_{=:\Psi(\omega)}$$

One application:
Let $X_0 = x, f : \mathbb{R}^d \to \mathbb{R}$

$$\underbrace{bE_x(f(X_T))}_{(T(t)f)(x)} = \int_{\xi^d} f(\omega_T)\Psi(\omega)W_x(d\omega)$$

where W_x is the wiener measure starting from x and $T(t)$ is the semigroup of X.

$$|(T(t)f)(x)| \leq \|fe^{-V}\|_\infty e^{-\frac{1}{2}\int_0^t \inf_{x\in\mathbb{R}^d}(|\nabla V(x)|^2 - \Delta V(x))ds}$$

If $\inf_{x\in\mathbb{R}^d}(|\nabla V(x)|^2 - \Delta V(x)) \geq 2\lambda > 0 \implies$

$$\|e^{-V(x)(T(t)f)}\| \leq \|e^{-V}f\|_\infty e^{-\lambda t} \to \text{exponential decay}$$

Chapter 3: Local times, Itô-Tanaka formula, reflected Brownian Motion

3.1 Extension of Itô formula to convex functions

If $f \in C^2(\mathbb{R})$, $f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle x\rangle_s$

Proposition 3.1. *Let f be a convex function on $\mathbb{R}$, X_t a (continuous) semimartingale, then $f(X_t)$ is a semimartingale: $\exists A_t^f$ such that $\forall t \geq 0$*

$$f(X_t) = f(X_0) + \int_0^t \underbrace{f'_-}_{\text{left derivative of } f}(X_s)dX_s + A_t^f$$

We make a choice for the left derivative, the same works for the right one, but with different processes

Remark. *This extends directly to the case that $f = f_1 - f_2$ with f_1, f_2 convex (But with bounded variation and not necessarily increasing A_t^f)*

Sketch of Proof. Consider a function $\rho(x)$ s.t.

- $\rho(x) \geq 0$, ρ smooth
- $\rho(x) = 0$ if $x \notin [0, 1]$
- $\int_0^1 \rho(x)dx = 1$

For any $n \in \mathbb{N}$ set

$$f_n(x) := \int_{\mathbb{R}} n\rho(ny)f(x-y)dy$$

Verify:

- f_n is $C^2(\mathbb{R})$; $f_n'' \geq 0$ (since f is convex)
- $f_n'(x) = \int_{\mathbb{R}} n\rho(ny) \cdot f'_-(x-y)dy$
- $f_n(x) \overset{n\to\infty}{\to} f(x)$
- $f_n'(x) \overset{n\to\infty}{\to} f'_-(x)$

left derivative since we do convolution and not correlation with ρ

Itô formula to $f_n(X_t)$:

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)dX_s + \underbrace{\frac{1}{2}\int_0^t f_n''(X_s)\langle X\rangle_s}_{A_t^{f_n} \in \mathcal{A}_+^0}$$

Using stopping times: If $X = \underbrace{M_+}_{\in \mathcal{M}_{\text{loc}}^0} + \underbrace{A}_{\in \mathcal{A}}$

$\forall m \geq 1 : T_m = \inf\{t \geq 0 \mid |X_t| + \langle M\rangle_t + \int_0^t |dA_s| \geq m\}$

$$f_n(X_{t\wedge T_m}) = f(X_0) + \underbrace{\int_0^{t\wedge T_m} f_n'(X_s)dX_s}_{\overset{n\to\infty,\mathbb{P}}{\to} \int_0^{t\wedge T_m} f_-'(X_s)dX_s} + \underbrace{\frac{1}{2}\int_0^{t\wedge T_m} f_n''(X_s)d\langle M\rangle_s}_{A^{f_n}_{t\wedge T_m}}$$

$\implies$ set $A_t^{f,m} := f(X_{t\wedge T_m}) - f(X_0) - \int_0^{T_m} f_-'(X_s)dX_s$ and $A^{f_n}_{t\wedge T_m} \overset{n\to\infty,\mathbb{P}}{\to} A_t^{f,m} = A^f_{t\wedge T_m}$; Take $m\to\infty$ and get a process A_t^f since $f_n'' \geq 0 \implies A_t^f$ is increasing and in $\mathcal{A}_0$. ☐

Let $f(x) = |x|$. Then

$$f_-'(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

we therefore define

$$\operatorname{sgn}(x) := \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

as the left derivative of $|x|$.

Proposition 3.2 (Tanaka's formula). *Let X be a continuous semimartingale and $a \in \mathbb{R}$. Then there $\exists$ increasing process $(L_t^a(X))_{t\geq 0}$ s.t.:*

(a) $|X_t - a| = |X_0 - a| + \int_0^t sgn(X_s - a)dX_s + L_t^a(X)$

(b) $(X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{X_s > a}dX_s + \frac{1}{2}L_t^a(X)$

(c) $(X_t - a)^- = (X_0 - a)^- - \int_0^t 1_{X_s \leq a}dX_s + \frac{1}{2}L_t^a(X)$

The process $L_t^a(X)$ is called the ***local time of X*** *at* ***level a****. For any stopping time T:*

$$L_t^a(X^T) = L^a_{t\wedge T}(X)$$

Proof. Taking $f(x) = |x - a|$ in proposition 3.1 yields (a). Therefore

$$L_t^a(X) := |X_t - a| - |X_0 - a| - \int_0^t \operatorname{sgn}(X_s - a)dX_s$$

Applying prop. 3.1 to $f(x) = (x - a)^+$ and $f(x) = (x - a)^-$

$$\implies \exists A_t^{a,(+)}, A_t^{a,(-)} \in \mathcal{A}_0^+$$

such that

$$(X_t - a)^+ = (X_0 - a)^+ \int_0^t 1_{X_s > a}dX_s A_t^{a,(+)}$$
$$(X_t - a)^- = (X_0 - a)^- \int_0^t 1_{X_s \leq a}dX_s A_t^{a,(-)}$$

$$X_t - a \overset{\text{Itô}}{=} X_0 - a + \int_0^t dX_s$$
$$X_t - a = (X_t - a)^+ - (X_t - a)^- = X_0 - a + \int_0^t dX_s + A_t^{a,(+)} - A_t^{a,(-)}$$

which holds if and only if $A_t^{a,(+)} = A_t^{a,(-)}$. Furthermore

$$|X_t - a| = |X_0 - a| + \int_0^1 \operatorname{sgn}(X_s - a)dX_s + \underbrace{A_t^{a,(+)} + A_t^{a,(-)}}_{=L_t^a(X)}$$ ☐

<u>To see:</u> $L_t^a(X) \to$ positive measure $dL_t^a(X)$, since $L_t^a(X)$ is increasing.

$$\int_0^t \underbrace{f(x)}_{\geq 0} dL_s^a(X) \overset{?}{=} 0 \text{ if } f(x) > 0 \text{ for } x \neq a$$

Proposition 3.3. *Let $dL_t^a(X)$ be the measure associated with the increasing process $L_t^a(X)$. Then $dL_t^a(X)$ is supported on $\{s \geq 0 \mid X_s = a\}$.*

But this is not necessarily the full support!

Proof. Let $Y_t = |X_t - a|$.

$$\implies Y_t^2 = (X_t - a)^2 \overset{\text{Itô to } f(X_t)=(X_t-a)^2}{=} (X_0 - a)^2 + \int_0^t (X_s - a)dX_s + 2\langle X\rangle_t$$

$$Y_t^2 \overset{\text{Itô to } f(Y_t)=Y_t^2}{=} (X_0-a)^2 + 2\int_0^t \underbrace{Y_s dY_s}_{=|X_s-a|\cdot(\text{sgn}(X_s-a)dX_s + dL_s^a(X))} + \underbrace{\langle Y\rangle_t}_{\langle X\rangle_t \text{ by prop 3.1}}$$

$$= (X_0-a)^2 + 2\int_0^t (X_s-a)dX_s + 2\int_0^t |X_s-a| dL_s^a(X) + \langle X\rangle_t$$

$$\implies \int_0^t |X_s - a| dL_s^a(X) = 0$$

and therefore the support is indeed contained in $\{s \geq 0 \mid X_s = a\}$ □

Remark. *If f is convex $\implies$ $f'_-(x)$ is increasing and left continuous. Therefore there exists a unique measure $f''(dx)$ on $\mathbb{R}_+$ such that*

$$f''([a,b]) = f'_-(b) - f'_-(a)$$

If $f \in C^2(\mathbb{R})$, then $f''(dx) = f''(x)dx$, i.e. it has the density given by the second derivative.
For all convex functions f with $f''(dx) = 0$ for all $|x| \geq K$

$$\implies \exists \alpha, \beta \in \mathbb{R} : f(x) = \alpha + \beta x + \frac{1}{2}\int_{\mathbb{R}} |x-a| f''(da)$$

and $f'_-(x) = \beta + \frac{1}{2}\int_{\mathbb{R}} sgn(x-a) f''(da)$ in a weak sense.

Start of lecture 12 (04.06.24)

Theorem 3.4 (Itô-Tanaka-formula)**.** *If f is a difference of convex functions, X is a continuous semimartingale*

$$\implies f(X_t)f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}\int_R L_t^a(x) f''(da)$$

, where $f''(da)$ is the measure associated with f'_-.

Sketch of proof. (Full proof: Thm 9.6 LeGall book)
Once localized → assume $f''(da) = 0$ on $[-\kappa, \kappa]^c$

$$\implies f(X_t) = \alpha + \beta X_t + \int_{-\kappa}^{\kappa} |X_s - a| f''(da)$$

$$\overset{\text{prop. 3.2}}{=} \alpha + \beta X_t + \frac{1}{2}\int |X_0 - a| f''(da) + \frac{1}{2}\int\int_0^t \text{sgn}(X_s - a) dX_s f''(da) + \frac{1}{2}\int L_t^a(X) f''(da)$$

$$= \underbrace{\alpha + \beta X_0 + \frac{1}{2}\int |X_0 - a| f''(da)}_{=f(X_0)} + \underbrace{\beta\int_0^t dX_s + \frac{1}{2}\int_0^t \int \text{sgn}(X_s - a) f''(da) dX_s}_{=\int_0^1 f'_-(X_s dX_s)} + \frac{1}{2}\int L_t^a f''(da)$$

□

Corollary 3.5 (Occupation time formula)**.** *Almost surely, $\forall t \geq 0$, non-negative measurable function φ*

$$\int_0^t \varphi(X_s) d\langle X\rangle_s = \int_{\mathbb{R}} \varphi(a) L_t^a(X) da$$

Example. *X =BM, $\varphi(X) = 1_A(X)$, A Borel set in $\mathbb{R}$*

$$\implies \int_0^t 1_A(X_s)ds$$

is the time spend by $(X_s, s \in [0,t])$ in A and the RHS is

$$\int_A L_t^a(X)da$$

$\implies$ *L_t^a density of time spend by Brownian motion in a.*

Think $A = [a, a+\epsilon]$

Proof. (For φ with bounded support)
Find $f \in C^2$ s.t. $f''(x) = \varphi(x)$.
$\implies$ Itô-Tanaka-formula:

Take f as the twice integrated φ on the support of φ until x

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}\int L_t^a(X)\varphi(a)da$$

By Itô-formula:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t \varphi(X_s)d\langle X\rangle_s$$

The claim follows from comparing the terms. □

Explicit representation of the local time:

Lemma 3.6. *Let X be a continuous semimartingale*
$\implies$ *a.s. $\forall a \in \mathbb{R}, t \geq 0$:*

$$L_t^a(X) = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t 1_{a\leq X_s\leq a+\epsilon}d\langle X\rangle_s$$

Proof. Take $\varphi(x) = \frac{1}{\epsilon}1_{[a\leq X\leq a+\epsilon]}$ in corollary 3.5:

See Thm 9.4 in LeGall for more details

$$\implies \frac{1}{\epsilon}\int_0^t 1_{a\leq X_s\leq a+\epsilon}d\langle X\rangle_s = \frac{1}{\epsilon}\int_a^{a+\epsilon} L_t^{\tilde{a}}(X)d\tilde{a} \overset{\tilde{a}\to L_t^{\tilde{a}}}{=} L_t^a(X)$$ □

Similarly:

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t 1_{[a-\epsilon\leq X_s< a]}d\langle X\rangle_s = L_t^{a-}(X)$$

$$\implies \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t 1_{[a-\epsilon\leq X_s\leq a+\epsilon]}d\langle X\rangle_s = \frac{1}{2}(L_t^{a-}(X) + L_t^a(X))$$

$$\begin{aligned}L_t^a(X) - L_t^{a-}(X) &\overset{\text{prop 3.2}}{=} |X_t - a| - |X_0 - a| - \int_0^t \operatorname{sgn}(X_s - a)dX_s - (|X_t - a_-| - |X_0 - a_-| - \int_0^t \operatorname{sgn}(X_s - a_-)dX_s)\\ &= 2\int_0^t 1_{X_s=a}dX_s \overset{X_t=M_t+A_t}{=} 2\int_0^t 1_{X_s=a}dA_s\end{aligned}$$

where the first two terms are equal, since X is a continuous semimartingale. This then implies, if X is a martingale (and therefore $A_t = 0$) then
$L_t^a(X) = L_t^{a-}(X) \implies L_t^a(X) = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^1 1_{[a-\epsilon\leq X_s\leq a+\epsilon]}d\langle X\rangle_s$.
In particular, if X is a BM:

$$L_t^{a-}(X) \implies L_t^a(X) = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^1 1_{[a-\epsilon\leq X_s\leq a+\epsilon]}ds$$

3.2 Brownian motion and local time

Let B be a standard BM, then by Tanaka:

$$|B_t| = |\underbrace{B_0}_{=0}| + \int_0^t \operatorname{sgn}(B_s)dB_s + \underbrace{L_t}_{:=L_t^0(B)}$$

We want to define the process $R_t = |B_t|$ be the **BM reflected at 0**.
We want to study R_t and $S_t^B := \sup_{0\leq s\leq t} B_s, L_t(B)$

3.2.1 No strong solution of the Tanaka SDE

$$(\star) = \begin{cases} dX_t &= \operatorname{sgn}(X_t)dB_t \\ X_0 &= 0 \end{cases}$$

We already know: $B_t := \int_0^t \operatorname{sgn}(W_s)dW_s$, where W is a BM $\implies$ BM is a BM (using Levy characterization) and

$$\int_0^t \operatorname{sgn}(W_s)dB_s = \int_0^t \operatorname{sgn}(W_s)^2 dW_s = W_t \to \operatorname{sgn}(W_t)dB_t = dW_t$$

and therefore (W, B) is a weak solution of $(\star)$.
Assume (X, B) is a strong solution of $(\star)$ $\implies$ $dX_t = \operatorname{sgn}(X_t)dB_t$ with X is a BM.
Itô-Tanaka-formula:

$$|X_t| = \int_0^t \operatorname{sgn}(X_s)dX_s + L_t^0(X)$$

$$\implies \int_0^t \operatorname{sgn}(X_s)dX_s = |X_t| - L_t^0(X) = |X_t| - \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t 1_{|X_s|\leq\epsilon}ds \in \mathcal{F}_t^{|X|}$$

But also:

$$\int_0^t \operatorname{sgn}(X_s)dX_s = \int_0^t (\operatorname{sgn}(X_s))^2 dB_s = B_t$$

$$B_t = |X_t| - \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t 1_{|X_s|\leq\epsilon}ds$$

$\implies$ B_t is $\mathcal{F}_t^{|X|}$-measurable and therefore B is $\mathcal{F}^{|X|}$ measurable.
If (X, B) is a strong solution, then X is measurable w.r.t. $\mathcal{F}^B$. $\implies$ $\mathcal{F}^X \subset \mathcal{F}^B \subseteq \mathcal{F}^{|X|}$ which is wrong, since we can not recall the value of X knowing only its absolute value.

3.2.2 Reflected SDE

Definition 3.7. *The family (X, l, W) is a weak solution of the **reflected SDE** (One-dimensional):*

$$dX_t = dW_t + dl_t$$

if:

- *W is a BM*
- *X is a positive continuous process*
- *l is a positive non-decreasing continuous process*

s.t.

$$\int_0^\infty 1_{X_s} > 0 dl_s = 0$$

*The solution is called **strong**, if (X, l) is adapted to W.*

Example.
- $X_t := |B_t|$
- $W_t := \int_0^t sgn(B_s)dB_s$
- $l_t = L_t^0(B)$

This is a weak solution of the ***1 dimensional reflected SDE****.*

Start of lecture 13 (06.06.24)

Lemma 3.8 (Skorokhod lemma for uniqueness of reflected SDE)**.** *Let $Y(t)$ be a continuous function with $y(0) \geq 0$. Then there exist a unique pair (z, a) of functions:*

- *both continuous*
- *$a(0) = 0, a$ is non-decreasing s.t.*
  (a) $z(t) = y(t) + a(t)$
  (b) $\int_0^\infty 1_{z(s)>0} da(s) = 0$

Moreover, the function a has the representation

$$a(t) = \sup_{0 \leq s \leq t} [y(s)]_- = \sup_{0 \leq s \leq t} \max\{0, -y(s)\}$$

Proof. Postponed :) □

Consequence of Lemma 3.8:

$$l_t(\omega) := \sup_{0 \leq s \leq t} (-W_s(\omega))_+ = \sup_{0 \leq s \leq t} (-W_s(\omega)) =: S_t^{-W}(\omega)$$

given $W(\omega), X_t(\omega) = W_t(\omega) + l_t(\omega) = W_t(\omega) + \sup_{0 \leq s \leq t}(-W_s(\omega))$.
$\implies \exists!$ strong solution of the reflected SDE. Notation:

We can start with W and explicitly construct a strong solution!

$$R_t := |B_t| = W_t + L_t = W_t + S_t^{-W}$$

Theorem 3.9.

$$\begin{aligned} Law(|B|, \underbrace{L}_{=L^0(B)}) &\overset{(1)}{=} Law(L + W, L) \\ &\overset{(2)}{=} Law(W + S^{-W}, S^{-W}) \\ &\overset{(3)}{=} Law(S^{-\tilde{W}} - \tilde{W}, S^{\tilde{W}}) \end{aligned}$$

with $\tilde{W}$ a generic BM.

Proof. (1) is from $R_t = |B_t| = W_t + L_t^0(B)$ (as a process)
(2) by the skorokhod representation: $L_t = S_t^{-W}$
(3) $W \overset{d}{=} -\tilde{W}$ □

$\forall t \geq 0$:

Something is wrong here? Or at the very least confusing

$$S_t := \sup_{0 \leq s \leq t} B_s \overset{(d)}{=} |B_t| \overset{(d)}{=} L_t^0(B)$$

By the reflection principle:

$$\mathbb{P}(S_t \in dx, B_t \in dy) \overset{2.2 \text{ LeGall}}{=} \frac{2(2x - y)}{\sqrt{2\pi t^3}} e^{-\frac{(2x-y)^2}{2t}} 1_{x>0, y<x} dx dy$$

and

$$\text{Law}(|B|) = \text{Law}(S^W - W)$$

Two points of view:
1. $B \to \mathbb{R}_t := |B_t|, W_t := \int_0^t \operatorname{sgn}(B_s) dB_s, L_t := L_t^0(B)$ satisfies

$$R_t = W_t + L_t$$

2. Given W_t $(\forall \omega)$: Define $l_t := \Phi_t(W) = \sup_{0 \leq s \leq t}(-W_s), X_t = W_t + l_t$
$\text{Law}(R, L) = \text{Law}(X, l)$.
An alternative definition of reflected Brownian motion is

$$X_t := \underbrace{W_t + \Phi_t(W)}_{\textbf{Skorokhod map}}$$

Added remark. *Careful:*

$$|W_t(\omega)| \neq W_t(\omega) + \Phi_t(W(\omega))$$

Or

$$z(t) \neq |y(t)|$$

3.3 Reflected BM on a curve

In section 3.2, we have reflected BM on $g(t) = 0 \forall t \geq 0$.

$$X_t = W_t + \sup_{0 \leq s \leq t}(-W_s)$$

Skorokhod lemma; from continuous function $y(t), y(0) \geq 0$:

$$z(t) := y(t) + \sup_{0 \leq s \leq t}(y(s) - 0)_-$$

Let $D_g := \{(x, t) \mid x \geq g(t)\}$.
Let $(W_t)_{t \geq 0}$ with $W_0 \geq g(0)$, then there exists a unique, continuous non-decreasing process L_t with $L_0 = 0$ s.t.

$$X_t := W_t + L_t \geq g(t) \forall t \geq 0$$

and L_t may increase only when X_t is at the boundary of D_g.

D_g is the space above the curve $(t, g(t))$, not sure why the coordinates are fliped tho?

$$\int_0^\infty 1_{(g(s),\infty)}(X_s) dL_s = 0$$

X_t has the formula:

$$L_t = \sup_{0 \leq s \leq t}(W_s - g(s))_-$$

To see this claim: Consider $\tilde{W}_t = W_t - g(t)$

$$\implies \tilde{X}_t := X_t - g(t)$$

is constructed by the Skorokhod map and

$$\tilde{X}_t = \tilde{W}_t + L_t \geq 0$$

Definition 3.10. *The* ***BM reflected by a continuous function*** $g(t)$ *with* $X_0 \geq g(0)$ *is given by the following construction*

- *Given* W *a BM with* $W_0 = X_0 \geq g(0)$*, define*

$$X_t := W_t + \sup_{0 \leq s \leq t}(W_s - g(s))_-$$

- *Alternatively with* $W_0 = 0$ *:*

$$X_t = X_0 + W_t + \sup_{0 \leq s \leq t}(W_s - g(s))_-$$

Added remark. *Think about sticky BM, which is the limit of some random walk which stays on the boundary with some positive probability.*
The exercise talks shows two reflected BMs, whose difference lies in the Weil chamber and solves the SDE for a process conditioned to be positive forever.

SDE: $dX_t = dW_t + dL_t$ or $X_t = X_0 + \underbrace{W_t}_{W_0=0} + L_t^g(X)$, where $L_t^g(X)$ is the local time of X spend at the function g.

Example. *Let $B = (B_1(t), B_2(t))$ a 2-dimensional BM with $B_1(0) \leq B_2(0)$ s.t. B_1 is not depending on B_2, but B_2 is reflected upwards by $(B_1(t), t \geq 0) = g$*
$\implies (B_1(t), \tilde{B}_2(t)) \in D = \{(x_1, x_2 \in \mathbb{R}^d \mid x_1 \leq x_2)\}$

Start with 2-dim BM then define the reflected BM...

Proposition 3.11. *Denote by $p_t(x,y)$ be the transition probe density of B:*

$$p_t(B_1(t) \in dy_1, \tilde{B}_2(t) \in dy_2 \mid B_1(0) = 0, \tilde{B}_2(0) = x_2) = p_t((x_1,x_2),(y_1,y_2))dy_1dy_2$$

It satisfies the following equations:

$$\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) p_t((x_1,x_2),(y_1,y_2)) = \frac{\partial}{\partial} p_t((x_1,x_2),(y_1,y_2))$$

for $(x_1,x_2),(y_1,y_2) \in D \setminus \partial D, t > 0$.

$$p_t((x_1,x_2),(y_1,y_2))dy_1dy_2 \overset{t\to 0}{\to} \delta_x(y)$$

as well as

$$\frac{\partial}{\partial x_2} p_t(x,y) \mid_{x=y} = 0$$

Proof. Let g be any smooth test function in D, set $f(t,x) := \int dyg(y)p_t(x,y)$. The undefined integrals are over D, for Itô we assume $f \in C^2$.
(1) SDE:

$$\begin{cases} dB_t^1 = dW_t^1 \\ d\tilde{B}_t^2 = dW_t^2 + dL_t^{B^1}(\tilde{B}_2) \end{cases}$$

Important information: where is the boundary condition coming from? Not immediatly obvious!

By Itô: Let $0 \leq t \leq T$:

$$\begin{aligned}
&f(T-t, B-t) \\
&= f(T, B_0) + \int dyg(y)[-\dot{P}_{T-S}(B_s,y)ds] + \int dyg(y) \int_0^t \partial x_1 p_{T-S}(B_s,y)dW_s^1 \\
&+ \int_0^t \partial x_2 p_{T-S}(B_s,y)(dW_s^1 + dL_s) \\
&+ \int dyg(y)\left[\frac{1}{2}\int_0^t \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} p_{T-S}(B_s,y)ds\right)\right]
\end{aligned}$$

(2) Martingale: Let $t \leq T$

$$\begin{aligned}
\mathbb{E}(g(B_t) \mid \mathcal{F}_t) &\overset{\text{MP}}{=} \mathbb{E}(g(B_t) \mid B_t) \\
&= \int dyg(y)p_{T-t}(B_t,y) = \underbrace{f(T-t,B_t)}_{\text{Martingale}}
\end{aligned}$$

□

Chapter 4: Levy processes

4.1 Basics

Start of lecture 14 (11.06.24)

Poisson process (Picture) ... With $t_{n+1} - t_n$ are iid., $\sim \exp(\lambda)$ (here it is the parameter, NOT the mean!) and $N_t \sim \text{Poi}(\lambda, t)$.

$$\mathbb{P}(N_{t+\epsilon} = y + 1 | N_t = y) = \lambda\epsilon + O(\epsilon^2)$$

BM	Poisson process
	Monotone increasing

BM:
A real valued process $B = (B_t)_t$ satisfying:

(a) $t \mapsto B_t$ is continuous

(b) $\mathbb{P}(B_0 = 0) = 1$

(c) For $0 \le s \le t : B_t - B_s \overset{d}{=} B_{t-s}$

(d) For $0 \le s \le t : B_t - B_s$ is independent of $\{B_u, u \le s\}$

(e) $\forall t > 0 B_t \sim \mathcal{N}(0, t)$

Poisson process:
A process $N = (N_t)$ with values in D is called a Poisson process with parameter $\lambda > 0$, if

(a) $t \mapsto N_t$ is $\mathbb{P}$. a.s. **cadlag**.

(b) $\mathbb{P}(N_0 = 0) = 1$

(c) For $0 \le s \le t : N_t - N_s \overset{d}{=} M_{t-s}$

(d) For $0 \le s \le t : N_t - N_s$ is independent of $\{N_u, u \le s\}$

(e) $\forall t > 0 N_t \sim \text{Poi}(\lambda t)$

Notice how b,c,d are the same!
Differences:
BM:

- continuous paths
- unbounded variation

PP:

- pure jump process

- bounded variation

Common features:

- both have càdlàg trajectories
- stationary, independent increments

Definition 4.1 (Lévy Process)**.** *A process $X = (X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a* ***Lévy process****, if:*

(a) The trajectories are $\mathbb{P}$. a.s. càdlàg

(b) $\mathbb{P}(X_0 = 0) = 1$

(c) For $0 \leq s \leq t : X_t - X_s \overset{d}{=} X_{t-s}$

(d) For $0 \leq s \leq t : X_t - X_s$ is independent of $\{X_u, u \leq s\}$.

Remark. *Lévy processes are Markov processes.*

Definition 4.2. *A real-valued random variable X has a* ***infinitely divisible distribution****, if for all $n \geq 1$: There exists r.v. $X_1^n, \ldots, X_n^n$ iid. s.t.*

$$X \overset{d}{=} X_1^n + \ldots X_n^n.$$

Let $X \sim \mu : \mu$ ***is infinitely divisible*** *if for all $n \geq 1$: $\exists$ prob. measure μ_n s.t. $\mu = (\mu_n)^{*n}$, i.e. the n-fold convolution.*

Remark. *Setting $\psi(u) = -\log(\mathbb{E}(e^{iuX}))$, for $u \in \mathbb{R}$, $\implies$ ψ is called the* ***charateristic exponent of X****. If X is infinitely divisible, then*

$$\begin{aligned}\mathbb{E}(e^{iuX}) &= \mathbb{E}(e^{iu(X_1^n + \cdots + X_n^b)}) \\ &= \mathbb{E}(e^{iuX})^n \\ &\implies \psi(u) = n\psi_n(u),\end{aligned}$$

where ψ_n is the characteristic exponent of X_1^n.

We can change the definition to $\mathbb{E}(e^{iuX}) = e^{-\psi(u)}$ to avoid taking logarithms of complex numbers.

Theorem 4.3 (Lévy-Khintchine formula)**.** *A probability law μ of real valued real random variable is infinitely divisible with characteristic exponent*

$$\psi(u\theta) = -\log \int_{\mathbb{R}} e^{i\theta x} \mu(dx)$$

if and only if $\exists$ a triple (a, σ, ν), $a \in \mathbb{R}, \sigma > 0, \mu$ a measure supported on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2) \nu d(x) < \infty$ s.t.

$$\psi(\theta) = ia\theta + \frac{1}{2}\theta^2\sigma^2 + \int_R (1 - e^{i\theta x} + i\theta x 1_{|x|<1})\nu(dx)$$

ν does not have to have finite measure!

Added remark. *ν is called the* ***Lévy measure****.*

Added remark. *What happens for x small?*

$$(1 - e^{i\theta x} - i\theta x 1_{|x|<1}) \approx 1 - (1 + i\theta x - \frac{\theta^2 x^2}{2}) + i\theta x = \frac{\theta^2 x^2}{2}$$

Added remark. *The 1 in the indicator function is arbitrary, tho other choices yield other formulas!*

4.1.1 Relation of infinitely divisible processes with Levy processes

Let X be a Levy process, $\forall t > 0$, X_t is infinitely divisible, since

$$(\star)\ X_t = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \cdots + \left(X_t - X_{\frac{t(n-1)}{n}}\right)$$

For all $\theta \in \mathbb{R}, \geq 0$: $\psi_t(\theta) := -\ln(\mathbb{E}(e^{i\theta X_t}))$

$$\overset{(\star)}{\Longrightarrow}\ \forall m, n \geq 1 : \frac{m}{m}\psi_1(\theta) = \frac{1}{m}\psi_m(\theta) = \frac{n}{m}\psi_{\frac{m}{n}}(\theta)$$

and therefore for $t \geq 0$ rational: $\psi_t(\theta) = t\psi_1(\theta)$. It then follows that $\forall t \in \mathbb{R}_+ : \psi_t(\theta) = t\psi_1(\theta)$.
For Lévy process:

$$\mathbb{E}(e^{i\theta X_t}) = e^{-t\psi_1(\theta)}$$

Definition 4.4. *We refer to $\psi(\theta) := \psi_1(\theta)$ to be the characteristic exponent of the Lévy process.*

Theorem 4.5. *Let $a \in \mathbb{R}, \sigma \geq 0, \nu$ a measure supported on $\mathbb{R} \setminus \{0\}$ s.t.*

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty$$

Define

$$\psi(\theta) := ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x)1_{|x|<1}\nu(dx)$$

$\Longrightarrow$ *then there exists a Lévy process with characteristic exponent $\psi(\theta)$.*

Proposition 4.6. *Let F be a distribution function on $\mathbb{R}_+$. Then F is the distribution function of an infinitely divisible law*
$\iff$ $\forall \lambda > 0 : F^{\mathcal{L}}(\lambda) = \int_0^\infty e^{-\lambda x}F(dx) = e^{-c\lambda - \int_0^\infty (1-e^{-\lambda x})\mu(dx)}$, *where $c \in \mathbb{R}$, μ is a measure on $(0, \infty)$, s.t. $\int_0^\infty (1 \wedge x)\mu(dx) < \infty$.*

Proof. $\Longrightarrow$: Let X have distribution $F : F(x) = \mathbb{P}(X \leq x)$. $X_1^n + \cdots + X_n^n$ with $\mathbb{P}(X_1^n \leq x) =: F_n(x)$.

$$F^{\mathcal{L}}(\lambda) = \int_0^\infty e^{-\lambda x}F(dx) = (F_n^{\mathcal{L}}(\lambda))^n,$$

since

$$\begin{aligned}
F^{\mathcal{L}}(\lambda) &= \int_0^\infty e^{-\lambda x}\mathbb{P}(X_1 + \cdots + X_n \in dx) \\
&= \int_0^\infty e^{-\lambda x} \int \mathbb{P}(X_1 \in dz)\mathbb{P}(X_2 + \cdots + X_n + z \in dx) \\
&= \int_0^\infty \mathbb{P}(X_1 \in dz) \int_z^\infty \mathbb{P}(X_2 + \cdots + X_n + z \in dx)e^{-\lambda x} \\
&\overset{x=\tilde{x}+z}{=} \int_0^\infty \mathbb{P}(X_2 + \cdots + X_n \in d\tilde{x})e^{-\lambda\tilde{x}}e^{-\lambda z} \\
&= \left(\int_0^\infty \mathbb{P}(X_1 \in dz)e^{-\lambda z}\right)^n = \left(F_n^{\mathcal{L}}(\lambda)\right)^n
\end{aligned}$$

Since F is a distribution on $\mathbb{R}_+ \implies F_n^{\mathcal{L}} \in (0, 1]$, which means

$$F_n^{\mathcal{L}}(\lambda) \overset{n\to\infty}{\to} 1$$

uniformly for λ in compact sets.

$$\Longrightarrow\ \ln F^{\mathcal{L}}(\lambda) = n \log F_n^{\mathcal{L}}(\lambda) = n \log(1 - (1 - F_n^{\mathcal{L}}(\lambda))) \leq -n(1 - F_n^{\mathcal{L}}(\lambda))$$

This last inequality is almost an equality for large n!

1.: $\forall \delta > 0, K < \infty, \exists n_0 < \infty$ s.t. $n \geq n_0$, $\lambda \leq k \implies 1 - F_n^{\mathcal{L}}(\lambda) \leq \delta$.

2.: $\forall \delta > 0, C < \infty$ s.t. $\forall 0 \leq x \leq \delta$,

$$-x(1+cx) \leq \log(1-x) \leq -x$$

Apply it to $x = 1 - F_n^{\mathcal{L}}(\lambda)$

$$\implies -(1-F_n^{\mathcal{L}}(\lambda))[1+C(1-F_n^{\mathcal{L}}(\lambda))] \leq \log F_n^{\mathcal{L}}(\lambda) - n(1-F_n^{\mathcal{L}}(\lambda))$$

As $n \to \infty$, uniformly in bounded set (for λ):

$$-n(1-F_n^{\mathcal{L}}(\lambda)) \to \log F^{\mathcal{L}}(\lambda).$$

$$\begin{aligned} n(1-F_n^{\mathcal{L}}(\lambda)) &= \int (1-e^{-\lambda x}) F_n(dx) \\ &= \int \frac{1-e^{-\lambda x}}{1-e^{-x}} \underbrace{n(1-e^{-x})F_n(dx)}_{=:m_n(dx)} \end{aligned}$$

m_n is a measure on $(0,\infty)$ with total mass is $n(1-F_n^{\mathcal{L}}(1)) \to \log F^{\mathcal{L}}(1)$
$\implies \exists$ measure m on $[0,\infty]$ s.t. $m_n \to m$ and

$$n(1-F_n^{\mathcal{L}}(\lambda)) \to m(\{0\})\lambda + \int_0^\infty \frac{1-e^{-\lambda x}}{1-e^{-x}} m(dx) + m(\{0\})$$

Let $\lambda = 0$: $m(\{\infty\}) = -\log F^{\mathcal{L}}(0) = -\log 1 = 0$. Then with $c = m(\{0\})$

$$\mu(dx)(1-e^{-x}) = m(dx)$$

$\impliedby$ ... $\square$

Question: What are a, σ, ν? Is there a relation between ν and the jumps of the Lévy process? Start of lecture 15 (13.06.24)

4.2 Examples

4.2.1 Poisson process

Let $\lambda > 0$ and $X \sim \text{Poi}(\lambda)$,

$$\begin{aligned} \mathbb{E}(e^{i\theta X}) &= \sum_{n \geq 0} e^{-\theta n} \frac{e^{-\lambda}\lambda^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^{i\theta}} \\ &= e^{-\lambda(1-e^{i\theta})} = \left(e^{-\frac{\lambda}{n}(1-e^{i\theta})}\right)^n \end{aligned}$$

$\implies$ X is infinitely divisible.
To get the right characteristic exponent

$$-\log \mathbb{E}(e^{i\theta X}) = e^{-\lambda} e^{\lambda e^{i\theta}}$$

which means $\psi(\theta) = \lambda(1-e^{i\theta})$.
Take $a = 0, \sigma = 0, \nu(dx) = \lambda \delta_1(x)$, then $\psi(\theta) = (1 - e^{i\theta}\lambda)$.
The Lévy process with characteristic exponent $\psi(\theta)$ is called the Poisson process with parameter $\lambda > 0$.

Added remark. *If you have a Lévy process on $\mathbb{Z}_{\geq 0}$, (X_t) with $X_0 = 0$ and with jumps $+1$ $\implies$ This is a Poisson process.*

4.2.2 Compound Poisson process

Let N be a Poisson random variable with parameter $\lambda > 0$, $\{\xi_k\}_{k\geq 1}$ iid r.v. (independent of N) with distribution F on $\mathbb{R}\setminus\{0\}$.

The ξ give us the size of the jumps

Let $X =:= \sum_{k=1}^{N}\xi_k$:

$$\implies \mathbb{E}\left(e^{i\theta X}\right) = \sum_{n\geq 0}\mathbb{E}\left(e^{i\theta(\xi_1+\cdots+\xi_n)}\right)\frac{e^{-\lambda\lambda^n}}{n!}$$
$$= \sum_{n\geq 0}\frac{e^{-\lambda}\lambda^n}{n!}\left(\int e^{i\theta X}F(dx)\right)^n$$
$$= e^{-\lambda}e^{\lambda\int e^{i\theta X}F(dx)} = e^{-\lambda\int F(dx)}e^{\lambda\int e^{i\theta X}F(dx)}$$
$$= e^{-\lambda\underbrace{\int(1-e^{i\theta X})F(dx)}_{=\psi(\theta)}}$$

$\implies$ X is infinitely divisible with $\psi(\theta)$ having $\sigma = 0, \nu(dx) = \lambda F(dx), a = -\int |x| < 1xF(dx)$

How to get the process?

We can think of a poisson process sa the set of jumping points (times). This is sometimes called poisson jump process

1. Choose Poisson process with parameter λ, $(N_t)_{t\geq 0}$
2. $X_t = \sum_{k=1}^{N_t}\xi_k$ is called the **compound Poisson process**

Notation $0 =: \sum_{k=m}^{n}(\dots)$ for $m > n$.

Is $(X_t)_{t\geq 0}$ a Lévy process?

Then

- $X_0 = 0$
- $\forall 0 \leq s < t$: $X_t = X_s + \sum_{k=N_s+1}^{N_t}\xi_k$

which implies $X_t - X_s \overset{d}{=} \sum_{k=1}^{N_t-N_s}\xi_k \overset{d}{=} X_{t-s}$ and everything is independent, but $N_t - N_s = N_{t-s}$.
Since N_t are càdlàc $\implies$ X_t is càdlàg. $(X_t)_{t\geq 0}$ is a Lévy process with

$$\psi(\theta) = \lambda\int_{\mathbb{R}\setminus\{0\}}(1-e^{i\theta x}F(dx)).$$

Let $\Delta X_t := X_t - X_{t^-}$ and for $B \in \mathcal{B}(\mathbb{R}\setminus\{0\})$,

$$\tilde{N}_t(B) := \#\{s\in[0,t] : \Delta X_s \in B\}$$

the number of jumps of size B in the time span $[0,t]$ and t_k are the jump times.
$(X_t)_t$ is a Lévy process $\implies$ $(\tilde{N}_t(B))$ has stationary independent increments.
$\tilde{N}_{t+s}(B) - \tilde{N}_(B) \overset{d}{=} \tilde{N}_t(B)$ for all $t,s\geq 0$ and B.
Therefore $(\tilde{N}_t(B))_t$ is a Lévy process with jumps of size $+1$, $\tilde{N}_0(B) = 0$, which implies $\tilde{N}_t(B)$ is a poisson process. The parameter of $\tilde{N}_t(B)$ is

$$\mathbb{E}(\tilde{N}_t(B)) = \underbrace{t\cdot\lambda}_{=\mathbb{E}(N_t)}\mathbb{P}(\xi\in B) = t\underbrace{\lambda\int_B F(dx)}_{=\nu(dx)} = t\nu(B).$$

$\implies$ $\nu(B)$ is the parameter of $\tilde{N}_t(B)$[1].

Added remark. *Suppose we had a Levy process with finite measure (remove $ia\theta$ and $i\theta x 1_{|x|<1}$). Then we can normalize and define F $\implies$ we get a compound poisson process!*

Add a drift:

$$X_t = \sum_{k=1}^{N_t}\xi_k + ct, t\geq 0$$

Then Lévy process of X_t has characteristic exponent

$$\psi(\theta) = \lambda\int(1-^{i\theta X})F(dx)\underline{-ic\theta}$$

[1]As a process, not as a r.v.

4.2.3 Brownian motion with drift

Let $\mu_{s,\gamma}(dx) := \frac{e^{-\frac{(x-\gamma)^2}{2s^2}}dx}{\sqrt{2\pi s^2}}$, $s > 0, \gamma \in \mathbb{R}$.
If $X \sim \mu_{s,\gamma}$, then

$$\mathbb{E}(e^{i\theta X}) = e^{-\frac{1}{2}s^2\theta^2 + i\theta\gamma} = \left(e^{-\frac{1}{2}\frac{s}{\sqrt{n}}^2\theta^2 + i\theta\frac{\gamma}{n}}\right)^n$$

$\implies$ X is infinitely divisible with characteristic exponent $\psi(\theta)$ with $\nu = 0, \sigma = s^2, a = -\gamma$.

$$\implies \psi(\theta) = -i\theta\gamma + \frac{1}{2}s^2\theta^2.$$

and therefore

$$X_t := sB_t + \gamma t,$$

where B_t is a standard BM, is the Lévy process with characteristic exponent $\psi(\theta)$.

Start of lecture 16 (18.06.24)

4.2.4 The Gamma process

Given $\alpha, \beta > 0$, define the probability measure

$$\mu_{\alpha,\beta}(dx) = \frac{\alpha^\beta}{\Gamma(\beta)}x^{\beta-1}e^{-\alpha x}dx1_{R_+\setminus\{0\}}$$

where

$$\Gamma(z) := \int_0^\infty u^{z-1}e^{-u}du.$$

$\mu_{\alpha,\beta}$ is the Gamma(α,β) distribution. For $\beta = 1$ it is the exponential distribution.
Let $X \sim X_{\alpha,\beta}$.

$$\begin{aligned}
\mathbb{E}(e^{i\theta X}) &= \int_{\mathbb{R}_+} e^{i\theta x}\mu_{\alpha,\beta}(dx) \\
&= \int_{\mathbb{R}_+} \frac{\alpha^\beta}{\Gamma(\beta)}x^{\beta-1}\left(e^{-\alpha}\cdot e^{i\theta}\right)^x dx \\
&\overset{(\alpha-i\theta)x=:u}{=} \frac{\alpha^\beta}{\Gamma(\beta)}\frac{1}{(\alpha-i\theta)^\beta} \underbrace{\int_0^{\infty(\alpha-i\theta)} u^{\beta-1}e^{-u}du}_{\text{By Cauchy Residue}=\int_0^\infty u^{\beta-1}e^{-u}du}
\end{aligned}$$

$$\implies \mathbb{E}(e^{i\theta X}) = \frac{1}{(1-\frac{i\theta}{\alpha})^\beta} = \left(\frac{1}{(1-\frac{i\theta}{\alpha})^{\frac{\beta}{n}}}^n\right)$$

$\implies$ X is infinitely divisible.
The triple (α, σ, ν) is given by:

- $\sigma = 0$
- $\nu(dx) = \beta x^{-1}e^{-\alpha x}dx$ on $\mathbb{R}_+ \setminus \{0\}$
- $a = -\int_0^1 x\nu(dx) = -\frac{\beta}{\alpha}(1-e^{-\alpha})$. This is the correction term for the indicator function!

Lemma 4.7. *For $\alpha, \beta > 0, z \in \mathbb{C}$ with $Re(z) \leq 0$,*

$$\frac{1}{(1-\frac{z}{\alpha})^\beta} = \exp\left(-\int_0^\infty (1-e^{zx})\beta x^{-1}e^{-\alpha x}dx\right)$$

Apply to $z = i\theta$ to get the above!

Proof. Take the logarithm:

$$-\beta \log\left(1-\frac{z}{\alpha}\right) \stackrel{?}{=} -\int_0^\infty (1-e^{zx})\beta x^{-1}e^{-\alpha x}dx$$

First look at $z = 0$:$0 = 0$, which is of course true.
Now look at the derivative in z (after canceling the $-\beta$ from both sides):

$$-\frac{1}{\alpha - z} = \frac{-\frac{1}{\alpha}}{1-\frac{z}{\alpha}} \stackrel{?}{=} -\int_0^\infty \underbrace{x\cdot x^{-1}}_{=1} e^{-\alpha x}dx = -\int_0^\infty e^{-(\alpha-z)x}dx \stackrel{\mathrm{Re}(\alpha-z)>0}{=} -\frac{1}{\alpha-z}$$

□

Using $z = i\theta$ in lemma 4.7:

$$\implies \Psi(\theta) = -\ln\left(\frac{1}{(1-\frac{i\theta}{\alpha})^\beta}\right) = \int_0^\infty (1-e^{i\theta x})\beta x^{-1}e^{-\alpha x}dx \to \nu(dx)$$

Added remark. *For the Gamma process is the Lévy measure ∞: $\int_0^\infty \nu(dx) = \infty$.*

Properties:

- $0 \le s \le t$ $X_t \stackrel{d}{=} X_t = X_s + \tilde{X}_{t-s}$, where $\tilde{X}_{t-s}$ is an independent copy of X_{t-s}
- $X_s < X_t$ a.s. for $s < t$ (the Lévy measure is on $\mathbb{R} > 0$)
- it is not a compound Poisson process (since it does not have finite Lévy measure)

4.3 Some properties of the jumps

For compound poisson process:

$$N_t(B) := \#\text{ jumps of size } B \text{ in the time interval } [0,t]$$

$N_t(B) \sim \mathrm{Poi}(t\nu(B))$ for $B \in \mathcal{B}(\mathbb{R}\setminus\{0\})$. $t \mapsto N_t(B)$ is a Poisson process with parameter / intensity / rate $\lambda = \nu(B)$.

Added remark. *Suppose you want ot simulate the poisson process. Sample exponential random variables, by sampling uniformly form (0,1) and then transform them by $-\log$.*

Let $T_0 = 0$,

$$T_{k+1} = \inf\{\xi_t > T_k \mid N_t(\mathbb{R}_{>0}) > N_{T_k}(\mathbb{R}_{>0})\}$$

This implies $T_{k+1} - T_k \sim \exp(\nu(\mathcal{R}_{>\prime}))$. The jump sizes of the compound Poisson process X_t are distributed as:

$$\mathcal{P}\left(X_{T_1} - X_{T_1^-} \in dy\right) = \frac{\nu(dy)}{\int_{\mathbb{R}\setminus\{0\}}\nu(dx)}$$

Argument for almost sure no step at time 0: $\mathcal{P}(N_t(\mathbb{R}_{>0}) \ge 2) = O(t^2)$.

$$\begin{aligned} \implies \mathbb{P}(N_t(\mathbb{R}_{>0}) = 1) &= e^{-t\nu(\mathbb{R}_+)}t\nu(\mathbb{R}_+) = t\nu(\mathbb{R}_+) + O(t^2) \\ \mathbb{P}(N_t(\mathbb{R}_{>0}) = 0) &= 1 - t\nu(\mathbb{R}_+) + O(t^2) \end{aligned}$$

$$\frac{\mathbb{P}(N_t(B) = 1 \cap N_t(\mathbb{R}_+) = 1)}{\mathbb{P}(N_t(R_+) = 1)} \approx \frac{t\nu(B)}{t\nu(\mathbb{R}_+)}$$

Case $\int_{\mathbb{R}\setminus\{0\}}\nu(dx) = \infty$.
Problem: First jump $\sim \exp(\nu(\mathbb{R}\setminus\{0\})) = 0$ a bit informal

$\implies$ The notion of first, second,..., jumps does not make sense:
$\inf\{t > 0 \mid X_t - X_{t^-} \neq 0\} = 0$ a.s. .
Let $A \subset \mathbb{R} \setminus \{0\}$ s.t. $\nu(A) < \infty$. This implies the first jump of size A, T_1^A is well defined with $T_1^A \sim \exp(\nu(A))$:

$$\implies \mathbb{P}(X_{T_1^A} - X_{T_1^A-} \in dy) = \frac{\nu(dy)}{\nu(A)}$$

$\implies$ take $A^0 = (1, \infty), A^1(\frac{1}{2}, 1], A^2 = (\frac{1}{4}, \frac{1}{2}]$
$\implies$ Construct Lévy process (compound poisson processes) by restricting the jumps to the ones of size $A^n : (X_t^n)_{t\geq 0}$

$$X_t \coloneqq \sum_{n\geq 0} X_t^n$$

where $X_t^0, X_t^1, \ldots$ are independent. The (X_t^n) are compound poisson processes with jumps in A^n: $T_k^{A_n} \sim \exp(\nu(A^n))$ with jumps of size $\frac{\nu(dy)}{\nu(A^n)}$
$\implies$ $(X_t)_{t\geq 0}$ is a Lévy process with Levy measure ν $(\sigma = 0)$.

Added remark. *We use that the sum of independent Lévy processes is still a Lévy process, since the exponential characteristic is the sum of the exponential characteristics*

By theorem 4.3 we can decompose:

$$\Psi(\theta) = \underbrace{(ia\theta + \frac{1}{2}\sigma^2\theta^2)}_{=\Psi^{(1)}(\theta)} + \underbrace{\nu(\mathbb{R} \setminus \{(-1,1)\}) \int_{|x|\geq 1} (1 - \exp(i\theta x)) \frac{\nu(dx)}{\nu(\mathbb{R} \setminus \{(-1,1)\})}}_{=\Psi^{(2)}(\theta)} + \underbrace{\int_{0<|x|<} (1 - \exp(i\theta x) + i\theta x) dx}_{=\Psi^{(3)}(\theta)}$$

Theorem 4.8 (Lévy-Itô decomposition)**.** *Given $a \in \mathbb{R}\sigma > 0, \nu$ a measure on $\mathbb{R} \setminus \{0\}$ s.t. $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge x^2)\nu(dx) < \infty$*
$\implies$ *$\exists(\Omega, \mathcal{F}, \mathbb{P})$ on which 3 independent Lévy processes $X^{(1)}, X^{(2)}, X^{(3)}$ with characteristic exponents $\Psi^{(1)}(\theta), \Psi^{(2)}(\theta), \Psi^{(3)}(\theta)$ are defined. Then $X_t \coloneqq X^{(1)} + X^{(2)} + X^{(3)}$ is a Lévy process with characteristic exponent $\Psi(\theta)$.*

- *$X_t^{(1)}$ is a BM with drift: $X_t^{(1)} = \sigma B_t - at, t \geq 0$, where B_t is a standard BM*
- *$X_t^{(2)}$ is a compound Poisson process*
- *$X_t^{(3)}$ is constructed as above*
- *$\int_{0<|x|<1} (1 - \exp(i\theta x))\nu(dx) = \sum_{n\geq 0} \lambda_n \int_{2^{-(n+1)} \leq |x| < 2^{-n}} F_n(dx) + i\theta\lambda_n \int_{2^{-(n+1)} \leq |x| < 2^{-n}} x F_n(dx)$, where $\lambda_N = \nu(2^{-(n+1)} \leq |x| < 2^{-n})$, $F_n(dx) = \frac{1}{\lambda_n}\nu(dx)$.*

Next lecture has useful information about simulations!

Start of lecture 17 (20.06.24)

Remark. *Càdlàg paths have at most countably many jumps.*

Why do we have the condition $\int (1 \wedge x^2)\nu(dx) < \infty$?
$\mathbb{P}(|X_t| \geq M) \leq ?$ s.t. $\overset{M\to\infty}{\to} 0$.

$$X_t^n = \sum_{k=1}^{\infty} \xi_k 1_{k \leq N_t(A^n)}$$

Take ν supported on $(0,1]$ and $A_n = \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$, $\nu(A_n) < \infty$

The computation uses independence, not true for higher moments!

$$\mathbb{P}(|X_t| \geq M) \overset{\text{Chebyshev}}{\leq} \frac{\operatorname{Var}(X_t)}{M^2}$$

$$\begin{aligned}\operatorname{Var}(X_t) &= \sum_{i=1}^{\infty} \operatorname{Var}(X_t^n) = \sum_{n=1}^{\infty} \underbrace{\sum_{m=0}^{\infty} \mathbb{P}(N_t(A_n) = m) \cdot m}_{\mathbb{E}(\#\text{ jumps of size in } A_n) = t\nu(A_n)} \cdot \underbrace{\operatorname{Var}(\xi_1^n)}_{\leq \mathbb{E}((\xi_1^n)^2) = \int_{A_n} \frac{x^2\nu(dx)}{\nu(A_n)}} \\ &\leq t \sum_{n=1}^{\infty} \int_{A_n} x^2 \nu(dx) = t \int_{(0,1]} x^2 \nu(dx) < \infty\end{aligned}$$

X_t is bounded a.s., for all given t

no divergence in finite time

4.4 Some properties of Poisson processes

Lemma 4.9 (Superposition)**.** *Let $(X_t^k)_{t\geq 0}, k = 1, \ldots, N$ independent Poisson processes with intensity $\lambda_k > 0$.*
Then $(Y_t)_{t\geq 0} = X_t^1 + \cdots + X_t^N$ is a Poisson process with intensity $\lambda = \lambda_1 + \cdots + \lambda_n$.

Also true for $N = \infty$, if $\lambda < \infty$

Proof. Follows by computing the characteristic exponents / functions □

Added remark. *Why is it called **Superposition**?*
Instead of jumps look at the events of the jumps for each X_t^1. If you take the union (superimpose the pictures), you get the jump times of Y. Not related to quantum mechanics.

Lemma 4.10 (Thining property)**.** *Let $(X_t)_{t\geq 0}$ be a poisson process with intensity λ and a $p \in [0,1]$ fixed.*
Construct two processes $(X_t^1)_{t\geq 0}, (X_t^2)_{t\geq 0}$ as follows:

- *Each jump of X_t is randomly and independently assigned to be a jump of X_t^1 with probability p and if not assigned to X_t^1 it is assigned to X_t^2.*
- *Then (X_t^1) is a Poisson process with intensity $p\lambda$*
- *(X_t^2) is a Poisson process with intensity $(1-p)\lambda$*
- *and they are independent*

Simplified proof.

Just showing independence, other things are easy to check

$$\begin{aligned}\mathbb{P}(X_t^1 = k, X_t^2 = l) &= \mathbb{P}(X_t^1 = k, X_t^1 + X_t^2 = k + l) \\ &= \mathbb{P}(\underbrace{X_t^1 + X_t^2}_{X_t} = k + l)\mathbb{P}(X_t^1 = k \mid \underbrace{X_t^1 + X_t^2}_{X_t} = k + l) \\ &= \frac{e^{-\lambda t}(\lambda t)^{k+l}}{(k+l)!} \binom{l+k}{k} p^k (1-p)^l \\ &= \frac{e^{-\lambda p t}(\lambda p t)^k}{k!} \frac{e^{-\lambda(1-p)t}(\lambda(1-p)l)^l}{l!}\end{aligned}$$ □

Added remark. *Let $X_t = X_t^1 - X_t^2$, X_t^1 is a Poisson process with intensity p and X_t^2 is a poisson process with intensity $1-p$. Therefore has intensity*[a] *1*

[a]intensity: expected number of jumps per unit time

This is not a Poisson process, since it might be negative!

How do we simulate this?
Simple solution: generate two Poisson process, by generating exponential variables $(\sim \exp(p), \sim \exp(1-p))$. This yields to lists of jumping timings, which we can join (after sorting).
What do we do if we have not two, but many of them?

Draw many exponential variables $\sim \exp(1)$, then for each time choose a random poisson process, which you want to attribute the jump to.
For two we get the drift: $\mathbb{E}(X_t) = t(p(1-p)) = (t(2p-1))$
two random walks $X_t, \tilde{X}_t$ as above with $X_0 < \tilde{X}_0$, add constraint $X_t < \tilde{X}_t$ meaning that jumps which would violate the condition are suppressed.
For all $x \in Z$ associate a random poisson process N_t^x with intensity 1.
For intensity $\frac{1}{2}$: First with probability $1/2$ we throw away the jump and then we flip a coin.
Again, we simulate one process with intensity as the sum of the intensities, then we split randomly!

When we draw poisson processes, they look different from reality, normally we would expect clumping!

Added remark. *Some problems with infinitely many processes, jumps at 0?*

Added remark. *Lévy process with $\Psi(\theta) = ia\theta + \frac{1}{2}i\theta^2\sigma^2 + \int_R (1 - e^{i\theta x} + i\theta x 1_{|x|<1})\nu(dx)$ Has generator*

Not asked in the exam

$$(Lf)(x) = \frac{1}{2}\sigma^2 \frac{d^2}{dx^2} f(x) - a\frac{d}{dx} f(x) + \int_{\mathbb{R}} \left[f(x+y) - f(x) - y\frac{d}{dx} f(x) 1_{|x|<1} \right] \nu(dy)$$

for $f \in C_0^\infty(\mathbb{R})$.

For poisson processes with intensity λ:

$$(Lf)(x) = \lambda(f(x+1) - f(x)).$$

A way of defining L:

$$(Lf)(x) := \lim_{t \downarrow 0} \frac{\mathbb{E}(f(X_t) \mid X_0 = 0) - f(x)}{t}$$

As $t \downarrow 0$:

$$\mathbb{P}(X_t = x+1 \mid X_0 = x) = \lambda t + O(t^2)$$
$$\mathbb{P}(X_t = x+1 \mid X_0 = x) = 1 - \lambda t + O(t^2)$$
$$\mathbb{P}(X_t \notin x, x+1 \mid X_0 = x) = O(t^2)$$

$$\implies \frac{1}{t}\mathbb{E}(f(X_t) \mid X_0 = x) = (1-\lambda) f(x) + \lambda f(x+1) + (t) \overset{t \to 0}{=} \lambda(f(x+1) - f(x))$$

Consider a d-dimensional Poisson process with intensity λ, i.e.
$X_t = (X_t^1, \dots, X_t^d)$ where the components are Poisson processes with intensity λ and independent.
$\implies$ Generator

$$Lf(x) = \sum k = 1^d L_k f(x),\ L_k f(x_1, \dots, x_d) = \lambda(f(\dots, x_k+1, \dots) - f(\dots, x_k, \dots)) = \lambda(f(x+e_k) - f(x))$$

Let $X_0 = (X_0^1, \dots, X_0^t) \in W_d = \{x \in \mathbb{Z}^d \mid x_1 < x_2 < \dots < x_d\}$
Want: Process X_t conditioned to stay forever in W_d.

Lemma 4.11. *Let*

Like for BM!

$$h(x) := \prod_{1 \le k < kl \le d} (X_l - X_k),$$

h satisfies

(a) $h(x) = 0$ for $x \in \partial W_d$ (two of the x_i are equal)

(b) $h(x) > 0$ for $x \in W_d$

(c) $Lh(x) = 0$ for $x \in W_d$

Remark. *To prove use $h(x) = \det(X_k^{j-1})_{1 \le k,j \le d}$ or $e^{tL} h(x) = h(x),\ t > 0$.*

This conditioned process has generator:

$$L^h f(x) = \frac{1}{h(x)}(L(hf))(x) = Lf(x) + \sum_{k=1}^{d} \frac{\nabla_k h(x)}{h(x)},\ \nabla_k g(x) \coloneqq g(x+e_k) - g(x)$$

Let $x, y \in W_d$:

$$P_t(x,y) = \frac{h(y)}{h(x)} \mathbb{P}(\tilde{X}(t) = y, T > t \mid \tilde{X}_0 = x)$$

where $\tilde{X}(t)$ are independent components and T is the hitting time $\inf\{s \geq 0 \mid \tilde{X}(s) \notin W_d\}$.

Lemma 4.12 (Karlin-McGregor formula)**.**

$$\mathbb{P}(\tilde{X}(t) = y, T > t \mid \tilde{X}(0) = x) = \det(\varphi_t(x_i, y_j)_{1 \leq i,j \leq d})$$

where

$$\varphi_t(x,y) = \varphi_t(0, y-x) = \mathbb{P}(X_t^1 = y \mid X_0^1 = x) = \frac{e^{-\lambda t}(\lambda t)^{y-x}}{(y-x)!} 1_{y \geq x}.$$

Notice, we only need to compute the interactions of two particles!

Special case: $\tilde{X}(0) = (0, 1, \ldots, d-1)$ the lemma 4.12 implies

$$\mathbb{P}(X_t = x) = \text{const}(h(x))^2 \prod_{k=1}^{d} \frac{e^{-\lambda t}(\lambda t)^{x_k}}{x_k!}$$

Analogue of BM conditioned to stay in the continuous Weyl chamber, $B(0) = 0$, is

$$\mathbb{P}(B(t) \in dx) = \text{const}(h(x))^2 \prod_{k=1}^{d} \left(\frac{e^{-\frac{x_k^2}{2t}}}{\sqrt{2\pi t}} dx_k \right)$$

List of Lectures