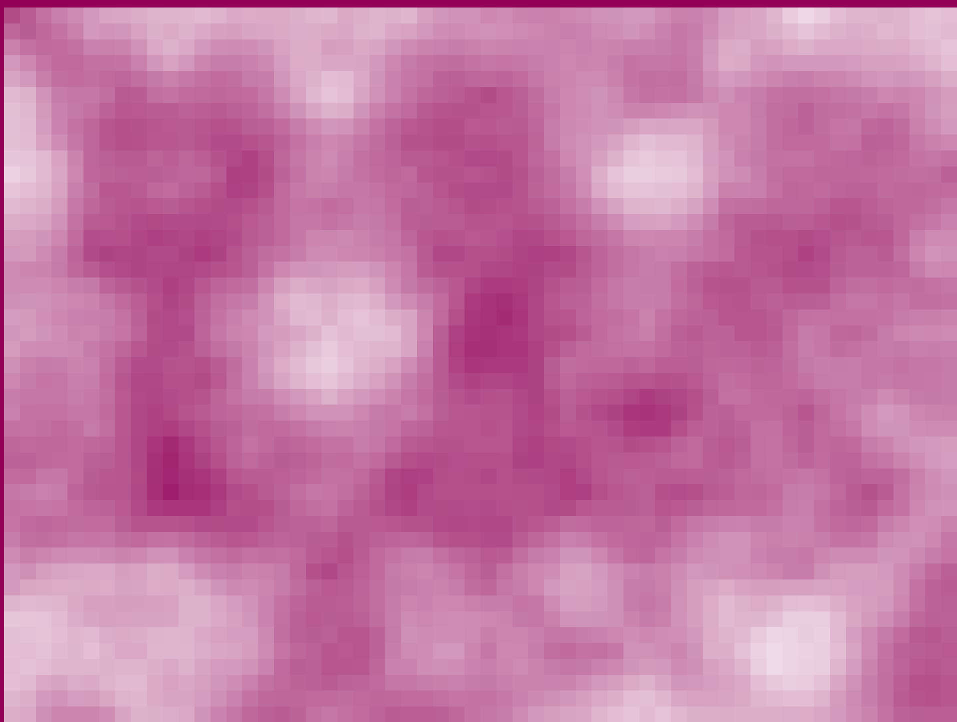
Lecture notes on Stochastic Analysis

Written by
Manuel Hinz
mh@mssh.dev or s6mlhinz@uni-bonn.de

Based on the lectures of
Prof. Dr. Patrik Ferrari
ferrari@uni-bonn.de



University of Bonn
Summer semester 2024
Last update: April 23, 2024

Contents

Chapter 0: Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Ecampus: Ecampus link
- Basis: Basis link
- Website: None
- Time slot(s): Tuesday 12-14 and Thursday 12-14
- Exams: Oral: 23-25 July, 30 Sept. maybe 27. Sept
- Deadlines: ?
- Exercises: One of Tue 16-18(for now this is preferred), Friday 10-12. Starting 16.04.

- First halve based on Eberle and / or Gubinelli (be careful with Notation of dimensions!) Start of lecture 01 (11.04.23)

Overview of the content

- Weak solutions of SDE
  - Martingale problem (characterization)
  - Time change (Dubin-Schwarz)
  - Change of measure (Girsonov)
- How to condition a diffusion to stay in a given domain forever (reflected BM, local time, ...)
- Lévy processes (a bit)
- Multiplicative stochastic heat-equation
  - relations with Kardar-Pavisi-Zhang class of growth models

Chapter 1:
Weak solutions of SDEs

SDE:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \ \textbf{Initial condition} \end{cases} \tag{1.1}$$

- $X_t \in \mathbb{R}^d, B:$ n-dim BM
- $b(t,x) : [b_k(t,x)]_{1\leq k\leq d}$: **drift vector**
- $\sigma(t,x) = [\sigma_{k,l}(t,x)]_{\substack{1\leq k\leq d \\ 1\leq l\leq n}}$: **dispersion matrix**
- $a(t,x) = \sigma(t,x)\cdot\sigma(t,x)^\intercal$: **diffusion matrix**

1.1 Strong solutions

Definition 1.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_t = \sigma(x_0, (B_s)_{0\leq s\leq t})$*

- *a d-dim process X_t is a* ***strong solution*** *of equation 1.1 if:*
  - $X_t = x_0$ *a.s.*
  - X_t *is adapted to* $\mathcal{F}_t \forall t \geq 0$
  - X *is a continuous semimartingale s.t.* $\forall t \geq 0$*:*

$$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2)ds < \infty \ a.s.$$

  - $X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$

In the last semester we proved:

Theorem 1.2. *Assume that b, σ are globally lipschitz with at most linear growth at ∞ (in space) $\implies \exists!$ strong solution of SDE.*

Foundations of Stochastic Analysis Thm 8.6

Added remark. *There exists $K > 0$ s.t. for all $x, y \in \mathbb{R}^d$: Globally Lipschitz:*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\|$$

Linear growth condition:

$$\|b(t,x)\| + \|\sigma(t,x)\| \leq K(1 + \|x\|)$$

Remark. *For strong solutions, $\mathcal{F}_t$ is given by the driving BM, wich is given to us.*
$\implies X_t(\omega) = \Phi(x_0(\omega), (B_s)_{0\leq s\leq t})$

1.2 Weak solutions

- For weak solutions we do not fix the driving brownian motion.

Definition 1.3. *A **weak solution** of equation 1.1 is a **pair** of adapted processes (X, B) to a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ s.t.*

- *B is a n-dim BM*
- *X is a d-dim continuous semimartingale with*
  1. $X_0 = x_0$ *a.s.*
  2. $\forall t \geq 0$
  $$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2) ds < \infty \ a.s.$$
  3. $X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$

Remark.
- *The filtration $(\mathcal{F}_t)_{t\geq 0}$ is not necessarily the one generated by B*
- *If X is adapted to the filtration generated by the BM $\implies$ we have strong solutions*
- *$\exists$ weak solution which are not strong solutions*
- *Warning: To give a weak solution we really need to provide $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, B)$*

Definition 1.4 (Uniqueness in law)**.** *An SDE 1.1 has **uniqueness in law** if given any two weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, B)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{B})$ satisfy:*

$$Law_{\mathbb{P}}(X) = Law_{\tilde{\mathbb{P}}}(\tilde{X})$$

They agree on any set in the sigma algebra

Definition 1.5 (Pathwise uniqueness)**.** *The SDE 1.1 has pathwise uniqueness if, whenever the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(B_t)_{t\geq 0}$ are fixed, then two solutions $X, \tilde{X}$ with $X_0 = \tilde{X}_0$ are indistinguishable.*

Example 1.6 (No strong solutions, no pathwise uniqueness, $\exists$ weak solution & and uniqueness in law by Tanaka)**.**

$$\begin{cases} dX_t = sgn(X_t) dB_t \\ X_0 = 0 \end{cases} \tag{1.2}$$

or more generally $X_0 = Y$, where

$$sgn(x) = \begin{cases} 1 & x > 0 \\ -1 x \leq 0 \end{cases}$$

Let W be a BM with $W_0 = Y$. Define

$$B_t := \int_0^t sgn(W_s) dW_s \ or \ dB_t = sgn(W_t) dW_t$$

$$\implies dW_t = sgn(W_t) dB_t$$
$$\implies W_t = y + \int_0^t sgn(W_s) dB_s$$

B_t is a local martingale with

$$\langle B \rangle_t = \int_0^t \underbrace{(sgn(W_s))^2}_{=1} \underbrace{d\langle W \rangle_s}_{=ds} = t$$

Also $B_0 = 0$, therefore B is a BM (see Lévy characterization) $\implies$ W solves the SDE. For $Y = 0$, W ***and*** *$-W$ solves the same SDE.*
$\implies$

- *exists weak solutions*
- *For $Y = 0$: no pathwise uniqueness*
- *Uniqueness in law (because the law is determined by X_t being a BM)*
- *No strong solution, because: $\mathcal{F}^B = \mathcal{F}^{|W|} \subsetneqq \mathcal{F}^W$*

Not a proof just yet, just the reason!

Example 1.7 (No solutions)**.**

$$\begin{cases} dX_t = -\frac{1}{2X_t} 1_{X_t \neq 0} dt + dB_t \\ X_0 = 0 \end{cases} \tag{1.3}$$

Assume there exists a solution. Use Itô formula for X_t^2, then:

$$\begin{aligned} X_t^2 &= 2\int_0^t X_s dX_s + \int_0^t 1 ds \\ &= -\int_0^t 1_{X_s \neq 0} ds + 2\int_0^t x_s dB_s + t \\ &= \int_0^t 1_{X_s = 0} ds + 2\int_0^t X_s dB_s \end{aligned}$$

We will prove $\int_0^t 1_{X_s=0} ds = 0 \implies X_t^2$ is a local martingale, $X_t^2 \geq 0$ (and therefore a supermartingale) and $X_0 = 0$ ($\implies \mathbb{E}(X_t^2) = 0$). If $X_t = 0 \implies \int_0^t 1_{X_s=0} ds = t \implies 0 = dB_t$ which are contradictions!

Remark. *If $X = \underbrace{M}_{\in \mathcal{M}_{loc}} + \underbrace{A}_{\in \mathcal{A}}$*

$$\implies \forall b \in \mathbb{R} \int_0^t 1_{X_s = b} d\langle M \rangle_s = 0$$

Sheet 0:

- Mail: Alexander.Becker@uni-bonn.de
- Exercise sheet handed in Fr 10 am via eCampus
- Groups of 3

Motivation:
in the last semester: Introduction to stochastic analysis

- Diffusion processes & SDEs

Here: Deepen the knowledge & have fun

- Learn SDE techniques and get other results
- Modify diffusion processes to behave in a certain way
- Ex. diffusion bridge (Brownian bridge)
- Condition a diffusion to stay in a domain
  - Ex. Condition BM to stay positive
  - Old SDE: $dB_t = dB_t$

- New SDE: $dX_t = \frac{1}{X_t}dx + dB_t \to P(X_t \in \cdot) = P(B_t \in \cdot | B > 0 \text{ forever})$
- $D \subset \mathbb{R}^d$ open domain, X diffusion process, with generator $L = \Sigma b\partial_i + \frac{1}{2}\sum a^{ij}\partial_i\partial_j$

$$Y :- (X | X \in D \text{ forever})$$

get drift term $\nabla \log \phi_0$, where ϕ_0 is the lowest eigenfuncion of $-L$ on D with dirichlet boundary.

Recap:
Brownian motion:

Added definition. $B_0 = 0$, *independent & $\mathcal{N}(0, t_i - t_{i-1})$ increments, $t \mapsto B_t(\omega)$ continuous.*

Regularity of path $t \mapsto B_t(\omega)$:

- nowhere differentiable
- $\alpha-$locally Hölder continuous $\iff \alpha < \frac{1}{2}$
- Quadratic variation $\langle B \rangle_t = t$
- Generator $\frac{\Delta}{2}$
- Recurrent $\iff ds2?$

Itô-Integral:

1. If X simple process $\implies$ RS-Integral
2. Itô isometry $\mathcal{E} \to \{L^2 - \text{local martingale}\}, \mathcal{E} \subset L^2(d[M])$ dense
3. general $X : \int XdM$ as L^2-limit

Added remark (Itô formula)**.**

- $$df(t, X_t) = \partial_t f(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}f(t, X_t)d\langle X \rangle_t$$
- *associative* $\int Xd(\int YdZ) = \int XYdZ$
- *If M local martingale $\implies \int XdM$ local martingale*

SDEs:
$DX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$

- ex./ uniqueness: b, σ locally Lipschitz $\implies$ strong ex.+pathwise uniqueness until explosion time
- globally Lipschitz (in space) and linear growth $(|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|))$, $e = \infty$.

Problem 00.1: SDE

Let B be a one-dimensional Brownian motion (starting from 0) and let $X_t = \sin(B_t)$.

1. Determine the SDE of X_t
2. Discuss the existence and/or uniqueness of strong solutions of the SDE
3. Determine whether the local martingale term is a martingale (Hint: for the integrability condition, think about the Itô isometry for instance)

Solution 00.1

1.:
Idea: Use Itô formula: $X_t = \sin(B_t) = f(B_t)$

$$dX_t = df(B_t) \overset{\text{Itô}}{=} \partial_x \underbrace{\cos(B_t)}_{\sqrt{1-X_t^2}} dB_t - \frac{1}{2}\underbrace{\sin(B_t)}_{X_t} dt$$

Careful: $\sqrt{1-X_t^2}$ is not inverse mapping, because it is always positive while $\cos(B_t)$ is not

2.:
Consider the SDE

$$dX_t = \begin{cases} -\frac{1}{2}X_t dt + \sqrt{1-X_t^2}dB_t \\ X_0 = x \in (-1,1) \end{cases}$$

Coefficients: $b : [-1,1] \to \mathbb{R}, b(x) = -\frac{1}{2}x$ and $\sigma : [-1,1] \to \mathbb{R}.\sigma(x) = \sqrt{1-x^2}$

$$[\sigma]_{1,2} = \sup \frac{|\sigma(x) - \sigma(y)|}{|x-y|^{1/2}} < \infty$$

Probably σ^2 Lipschitz $\implies$ σ Hölder $\frac{1}{2}$
3.:

Problem 00.2: Time change

Let B be a one-dimensional Brownian motion (starting from 0). Let $Y_t = \int_0^t s^2 dB_s$.

1. Determine the SDE of Y_t
2. Find A_t such that Y_{A_t} is a (stopped) Brownian motion

Solution 00.2

1.: By Definition

$$dY_t = t^2 dB_t$$

2.: We use the Dubin-Schwarz theorem:
$X \in \mathcal{M}^0_{\text{loc}}, \langle X \rangle_\infty = \infty, T_t \coloneqq \inf\{s \geq 0 | \langle X \rangle_s \geq t\} = X_t^{[-1]}$
$\implies B_t \coloneqq X_{T_t}$ 1 d BM w.r.t. $(F_{T_t})_{t \geq 0}, X_t = B_{\langle B \rangle_t}$
here: use $X_t = b(X_t)dt + \sigma(X_t)dB_t \implies d[X]_t = \sigma^2(X_t)dt$

Problem 00.3: SDE and PDE

Let f be a function supported on $[0,1]$, u the solution of

$$\frac{1}{2}u(t,x) = \frac{1}{2}x(1-x) + \frac{d^2}{dx^2}u(t,x), \qquad u(0,x) = f(x).$$

Consider also the one-dimensional SDE (also known as Wright-Fisher diffusion)

$$dX_t = \sqrt{X_t(1-X_t)dB_t}$$

with $X_0 = x \in (0,1)$.

1. For any fixed $t > 0$, define $M_s = u(t-s, X_s)$ for $s \in [0,t]$. Use Itô formula to show that M_s is a local martingale
2. Assume that f is bounded and there is a bounded solution of u. Show that $u(t,x) = \mathbb{E}_x(f(X_t))$.

Solution 00.3

1.:

$$
\begin{aligned}
dM_s &= du(t-s, X_s) \\
&= -\partial_s u(t-s, X_s)ds + \partial_x u(t-s, X_s) \underbrace{dX_s}_{b(X_s)ds+\sigma(X_s)dB_s \text{ by asso.}} + \frac{1}{2}\partial_x^2 u(t-s, X_s) \underbrace{d[X]_s}_{=\sigma^2(X_s)ds} \\
&= \underbrace{(-\partial_s u + b\partial_x u + \frac{1}{2}\sigma^2\partial_x^2 u)}_{=0}(t-s, X_s)ds + \partial_x u(t-s, X_s)\sigma(X_s)dB_s \\
&\implies dM_s = \partial_x u(t-s, X_s)\sigma(X_s)dB_s
\end{aligned}
$$

(i.e.: $M_t - M_0 = \int_0^t \dots dB_s$)
This is a purely stochastic integral against a (local) martingale $\implies$ martingale.
2.:

- M_s true martingale:
  1. $\forall t : \mathbb{E}(\sup_{[0,t]} |M|) < \infty$, for example: M bounded
  2. $\forall t : \mathbb{E}(\sup_{[0,t]} [M]_t) < \infty$

$M_s \coloneqq u(t-s, X_s), u$ bounded $\implies$ M bounded $\implies$ M true martingale
$w(s,x) \coloneqq u(t-s, x)$

$$
u(t,x) = w(0,x) = \mathbb{E}_x[w(0,X_0)] \overset{\text{martingale}}{=} \mathbb{E}_x[w(t,X_t)] = \mathbb{E}_x[(u(0,X_t))] \overset{\text{PDE}}{=} \mathbb{E}_x[f(X_t)]
$$

Feynman-Kac formula (later more general).
Also solvable by Kolmogorov Backward / forward equation

Start of lecture 02 (16.04.24)

Example 1.8 (Non-uniqueness of law, non pathwise uniqueness, with solutions)**.**

$$
\begin{cases} dX_t &= 1_{X_z} dB_t \\ x_0 &= 0 \end{cases}
$$

Then

$$
X_t = 0 \forall t \geq 0
$$

and

$$
X_t = B_t \forall t \geq 0
$$

both are solutions:

$$
X_t - B_t = -\int_0^t 1_{X_s=0} dB_s \implies \langle X - B \rangle_t = \int_0^t 1_{X_s=0} d\langle B \rangle_s = 0
$$

$X_t = \int_0^t 1_{X_s \neq 0} dB_s$
$= \int_0^t (1 - 1_{X_s=0} dB_s)$
$B_t - \int_0^t 1_{X_s=0} dB_s$

Let $\eta \sim Ber\left(\frac{1}{2}\right)$ *independent of* $(B_t)_{t\geq 0}$ *and define*

$$
\tilde{X}_t = \begin{cases} 0 & \eta = 0 \\ B_t & \eta = 1 \end{cases}
$$

$\implies$ $\tilde{X}_t$ *is adapted to* $\sigma(\eta(B_s)_{0\leq s\leq t})$*, but not to* $\sigma((B_s)_{0\leq s\leq t})$ *and therefore not a strong solution.*

Example 1.9 (No strong solution, weak solution, no pathwise uniqueness, no uniqueness in law)**.**

$$\begin{cases} dX_t &= 1_{X_t \neq 1} sgn(X_t) dB_t \\ X_0 &= 0 \end{cases}$$

Let Y_t be a solution of

$$\begin{cases} dY_t &= sgn(Y_t) dB_t \\ Y_0 &= 0 \end{cases}$$

$\implies X_t := Y_{t\wedge\tau}$, *where* $\tau = \inf\{s \geq 0 \mid Y_s = 1\}$ *is also a solution.*

Theorem 1.10 (Yamada-Watanabe)**.** *If both existence of weak solutions and pathwise uniqueness hold, uniqueness in law also holds.*
Moreover, $\forall$ choices of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(B_t)_{t\geq 0}$ then there exists a strong solution.

1.3 Lévy characterization

Example 1.11. *Consider the SDE*

$$\begin{cases} dX_t &= O_t dB_t \\ X_0 = x_0 \end{cases}$$

where both X_t and B_t are d-dimensional and O_t is an adapted process (matrix) s.t. $O_t^\intercal O_t = 1 \forall t \geq 0$ i.e. O_t is a rotation

$$\begin{aligned} \implies X_t^k &= X_0^k + \sum_{l=1}^d \int_0^t O_s^{k,l} dB_s^l \\ \implies \langle X^k, X^{\tilde{k}} \rangle_t &= \sum_{l,\tilde{l}}^d \int_0^t O_s^{k,l} O_s^{\tilde{k},\tilde{l}} d\langle B^l, B^{\tilde{l}} \rangle_s = \sum_{l=1}^d \int_0^t O_s^{k,l} O_s^{\tilde{k},l} ds \\ &= \int_0^t \underbrace{(O_s O_s^\intercal)^{k,\tilde{k}}}_{=1} ds = \delta_{k,\tilde{k}} t \overset{\text{Lévy}}{\implies} (X_t)_{t\geq 0} \text{ is also a } d\text{-dim BM starting from } x_0 \end{aligned}$$

Theorem 1.12 (Yamada,Watanabe)**.** *Let $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ and assume that there exist both a increasing function $\rho(u) \geq 0$ s.t. $|\sigma(x) - \sigma(y)| \leq \rho(|x-y|) \forall x, y \in \mathbb{R}$ s.t.*

$$\int_0^\infty \frac{1}{\rho(u)} du = \infty \implies 0 < u < C_1 \implies \rho(u) \leq C_2 u^{0.5}$$

and some increasing concave function $\gamma_1(u) \geq 0$ s.t.

$$|b(x) - b(y)| \leq \gamma_1(|x-y|) \forall x, y \in \mathbb{R}$$

and

$$\int_0^\infty \frac{1}{\gamma_1(u)} = \infty.$$

Then pathwise uniqueness holds.

Theorem 1.13 (Storokhod)**.** *Assume that σ, b are continuous bounded functions*
$\implies$ *there exist weak solutions to the SDE $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.*

1.4 Weak solutions and martingale problems

Let $(X_t)_{t\geq 0}$ be a weka solution of the SDE

$$\begin{cases} dX_t &= b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_0 &= 0 \end{cases}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
$\implies$ X_t is a semimartingale s.t. $X_t^k = X_0^k + \int_0^t b(s,X_s)ds + \sum_{l=1}^n \int_0^t \sigma_{k,l}(s,X_s)dB_s^l$ and

$$\langle X^i, X^j\rangle_t = \int_0^t \underbrace{\sum_{l=1}^n \sigma_{i,l}(s,X_s)\sigma_{j,l}^\intercal(s,X_s)}_{=a_{i,j}(s,X_s)} ds$$

1.4.1 Itô-Doeblin formula

Itô formula leads to

Proposition 1.14. *For $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, then*

$$f(t,X_t) = f(0,X_0) + \int_0^t (\sigma^\intercal \nabla f)(s,X_s)dB_s + \int_0^t \left[\left(\frac{\partial}{\partial s} + \mathcal{L}\right)\right](s,x_s)ds$$

where $(\mathcal{L}f)(t,x) = \frac{1}{2}\sum_{k,l=1}^n a_{k,l}(t,x)\frac{\partial^2}{\partial x_k \partial x_l} f(t,x) + \sum_{k=1}^n b_k(t,x)\frac{\partial}{\partial x_k} f(t,x)$.

*$\mathcal{L}$ is called the **generator***

Remark. *The Itô-Doeblin formula provides a connection between SDEs and PDEs as we have seen in the last part **Foundations of stochastic analysis**.*

Example 1.15. *Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}_d)$ be a solution of the PDE*

$$\frac{\partial}{\partial t} f(t,x) + (\mathcal{L}f)(t,x) = -g(t,x) \qquad t \geq 0, x \in U \subseteq \mathbb{R}^d$$
$$f(t,x) = \varphi(t,x) \qquad t \geq 0, x \in \partial U.$$

then $M_t \coloneqq f(t,X_t) + \int_0^t g(s,X_s)ds \in \mathcal{M}_{loc}$ by proposition 1.14 and if f,g are bounded $M_t \in \mathcal{M}$.

$$T \coloneqq \inf\{s \geq 0 | X_s \notin U\} \implies M_t^T \coloneqq M_{T\wedge t} \in \mathcal{M}.$$

Furthermore, if we assume $T < \infty$ a.s. :

$$\mathbb{E}[M_t] = \mathbb{E}[\varphi(T,X_T)] + \mathbb{E}[\int_0^T g(s,X_s)ds] = \varphi(0,X_0)$$

where $X_0 = x_0$.
There are two special cases:
$g = 0 \implies$ yields the exit distributions, while
$\varphi = 0, g = 1$ yields the mean exit times.

Example 1.16 (Feynman-Kac formula)**.** *Let $t \in \mathbb{R}_+$ be finite. Assume $f : \mathbb{R}^d \to \mathbb{R}$ and $K : [0,t] \times \mathbb{R}^d \to \mathbb{R}_+$ be continuous functions. Assume that u is a $C^{1,2}$-solution (bounded,for simplicity) of*

$$\begin{cases} \frac{\partial u}{\partial s}(s,x) = \frac{1}{2}\Delta u(s,x) - K(s,x)u(s,x) & s \in [0,t], x \in \mathbb{R}^d \\ u(0,x) = f(x) & x \in \mathbb{R}^d \end{cases}$$

Then u has the stochastic representation $u(t,x) = \mathbb{E}_x\left[f(X_t)\exp\left(-\int_0^t K(t-s,X_s)ds\right)\right]$, where X_t is BM starting from $X_0 = x$.

Sketch. 1. Define $r(s,x) \coloneqq u(t-s,x)$ for $s \in [0,t]$

2. Show: $M_s := \exp(-A_s) r(s,x)$ with $A_s = \int_0^s K(u, X_u) du$ is a local martingale.

□

Remark. *This is a reformulation of the formula from the last semester.*

1.4.2 Martingale problem

A solution of an SDE is generically defined up to some **explosion time ξ**, where it either diverges or it exists a given domain $U \subset \mathbb{R}^d$ (open).
$\implies$ For $k \in \mathbb{N}$ define $U_k := \{x \in U \mid |x| < k \wedge \operatorname{dist}(x, U^c) \geq \frac{1}{k}\}$ with $U = \bigcup_{k \geq 1} U_k$ and

$$T_k := \{t \geq 0 \mid x_t \notin U_k\}.$$

A solution of the SDE $b(t, X_t)dt + \sigma(t, X_t)dB_t$ is defined up to $\xi = \sup_{k \geq 1} T_k$.

Start of lecture 03 (18.04.24)

Added remark. *uniqueness of solution to the heat equation $\frac{1}{2}\Delta u - Ku$: not unique! Stochastic calculus: Paolo Baldi: 10.3-10.4*

Define $(\star)$:

$$dX_t b(t, X_t)dt + \sigma(t, X_t)dB_t \text{ with } X_0 = x_0$$

Theorem 1.17 (Martingale problem)**.** *If X is a solution of $(\star)$ up to time ζ, then $\forall f \in C^{1,2}(\mathbb{R}_+ \times U)$*

$$M_t = f(t, X_t) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right) f(s, x_s) ds, t < s$$

is a local martingale up to ζ and $M_t^{T_k}$ are localizing martingales.

Definition 1.18 (Martingale solutions)**.** *$(X_t)_{t \geq 0}$ is a **martingale solution** of $(\star)$ if $\forall f \in C^2(\mathbb{R}^d)$*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s) ds$$

is a continuous local martingale.

Theorem 1.19 (Equivalent definitions)**.** *The following are equivalent (~~for X_t being a solution of $(\star)$~~):*

(a) $\forall f \in C^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s) ds$$

is a local martingale

(b) The process in $\mathbb{R}^d$ given by

$$M_t := X_t - X_0 - \int_0^t b(s, X_s) ds$$

is a d-dimensional local martingale with $\langle M^i, M^j \rangle_t = \int_0^t a_{i,j}(s, X_s) ds = \langle X^i, X^j \rangle_t$

(c) $\forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$

$$\tilde{M}_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right)(s, X - s) ds$$

is a local martingale

ds

Proof. $\underline{c \implies a}$: by choosing f independent of t.
$\underline{a \implies b}$: 1.: Choosing $f(X) = X_i$ implies

We can maybe also proof this by calculating X^2?

$$M_t^f = X_t^i - X_0^i - \int_0^t b_i(s, X_s) ds \in \mathcal{M}_{\text{loc}}$$

2.: $f(X) = X_i X_j$:

$$\begin{aligned}(\mathcal{L}f)(x) &= \frac{1}{2}a_{ij} + \frac{1}{2}a_{ji} + b_i X_j b_j X_i \\ a = a^\intercal &\implies = a_{ij} b_i X_j b_j X_i \\ &\implies M_t^f X_t^i X_t^j - X_0^i X_0^j - \int_0^t \left[a_{ij}(s, X_s) + b_i(s, X_s) X_s^j + b_j(s, X_s) X_s^i\right] ds\end{aligned}$$

Here dX_s^i is the same as $b_i X_s^j$ is the same up to a local martingale term and $\langle X^i, X^j\rangle_t = \int_0^t a_{ij}(s, X_s) ds = \langle M^i, M^j\rangle_t$

$$\begin{aligned}X_t^i X_t^j - X_0^i X_0^j &\overset{\text{Integration by parts}}{=} \int_0^t X_s^i dX_s^j + \int_0^t X - s^j dX_s^i + \langle X^i, X^j\rangle_t \\ &= M_t^f + \int_0^t [a_{ij} + b_i X_s^j + b_j X_s^i] ds\end{aligned}$$

$\underline{b \implies c}$: By Proposition 1.14 If (use the next theorem) X was a weak solution $\implies \tilde{M}_t^f$ is a local martingale. □

Theorem 1.20. *Let $n = d$, assume $\sigma(t, x)$ is invertible $\forall t, x$ and $\sigma^{-1}(t, x)$ is uniformly bounded. T.f.a.e.:*

(a) $(X_t)_{t\geq 0}$ is a weak solution of the SDE $(\star)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; B)$

(b) $(X_t)_{t\geq 0}$ is a martingale solution of the SDE $(\star)$

This also works for $n \neq d$, but with a different proof

Proof. $a \implies b$: True
$b \implies a$: Goal construct a BM for the weak solution.
By proposition **??** $a \implies b$ $dM_t = dX_t - b(t, X_t)dt \in \mathcal{M}_{\text{loc}}$ and $d\langle M^i, M^j\rangle_t = a_{k,l}(t, X_t)dt$

$$\begin{aligned}\implies dX_t &= dM_t + b(t, X_t)dt \\ &= \sigma(t, X_t) d\tilde{B}_t + b(t, X - t)dt\end{aligned}$$

where $\tilde{B}_t \coloneqq \sigma(s, X_s)^{-1} dM_s$
To see: $\tilde{B}_t$ is a brownian motion.

$$\begin{aligned}\langle \tilde{B}^i, \tilde{B}^j\rangle_t &= \sum_{k,l} \int_0^t \sigma_{ij}^{-1} \sigma_{jl}^{-1} \underbrace{d\langle M^k, M^l\rangle_s}_{= \underbrace{a_{ij}}_{(\sigma^\intercal \sigma)_{kl}} ds} \\ &= \sum_{k,l,p} \int_0^t \sigma_{ik}^{-\intercal} \sigma_{kp} \sigma_{pl}^\intercal \sigma_{lj}^{-\intercal} ds \\ &= \delta_{ij} \int_0^t 1 ds = \delta_{ij} t\end{aligned}$$

Then by the Lévy characterization $\tilde{B}$ is a brownian motion. □

Added remark. *This is the first way to construct a weak solution: Solve a martingale problem! This is used a lot in practice.*

1.5 Weak solutions and time change

1.5.1 Time change

For $d = 1$:

Theorem 1.21. *[Dubins-Schwarz]*

- *Let $M \in \mathcal{M}^0_{loc}$ and $\langle M \rangle_\infty = \infty$ a.s.*
- *Let $T_t \coloneqq \inf\{s \geq |\langle M \rangle_s \geq t\}$*

This implies

1. *$t \mapsto M_{T_t}$ os a $(\mathcal{F}_{T_t})$ brownian motion*
2. *$M_t = B_{\langle M \rangle_t}$ for some standard brownian motion B*

$X_t = X_0 + \int_0^t b(s, X_s)ds + \underbrace{\int_0^t \sigma(s, X_s)dB_s}_{=M_t}$. If $\langle M \rangle_\infty = \infty$ a.s.:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \tilde{B}_{\int_0^t \sigma^2(s,X_s)ds}$$

1.5.2 Time change in a martingale problem

Consider $d = 1 = n$.

$$dY_t = \tilde{\sigma}(Y_t)dB_t \qquad (\star\star)$$

and $\tilde{\sigma}$ strictly positive positive continuous function.

$$\langle Y \rangle_t = \int_0^t \tilde{\sigma}(Y_s)^2 ds \eqqcolon A_t$$

By theorem 1.21 $\implies Y_t = W_{A_t}$ for some brownian motion W.
Assume $A_t\infty = \infty$ a.s.
$T_t \coloneqq \inf\{s \geq 0 \mid \langle Y \rangle_s \geq t\}$

$$\implies T_{A_t} = \inf\{s \geq 0 \mid \langle Y \rangle_s \geq \langle Y \rangle_t\} = t$$

$$1 = \frac{d}{dt}(T_{A_t}) = T'\underbrace{A_t}_{=u} \cdot A_t$$

$$\implies \underbrace{T'_u}_{=\frac{dT_u}{du}} = \frac{1}{A'_{T_u}} \implies T_u = \int_0^u \frac{1}{\tilde{\sigma}(Y_{T_s})^2}ds = \int_0^u \frac{1}{\tilde{\sigma}(W_s)^2}ds$$

$\implies$ to construct a solution of $(\star\star)$: Given $W \longrightarrow$ compute $T_u \longrightarrow$ determine
$A - t = T_t^{-1} \implies Y_t = W_{A_t}$

Start of lecture 04 (23-04-24)

Theorem 1.22. *Let $(X_u)_{u\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a weak solution of*

$$dX_u = b(X_u)du + \sigma(X_u)dB_u$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$,the drift and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are locally bounded, σ^{-1} exists for a.e. x and is locally bounded.
*Consider a **time change** $T_u \coloneqq \int_0^u \rho(X_s)ds$, where $\rho : \mathbb{R}^d \to \mathbb{R}_+$ s.t.*

$$T_u < \infty \forall u \geq 0 \text{ and } T_\infty = \infty \text{ a.s.}$$

$\implies$ *Then $Y_t := X_{A_t}$, where $A_t = T_t^{-1}$ is a weak solution of the SDE*

$$dY_t = \frac{b(Y_t)}{\rho(Y_t)}dt + \frac{\sigma(Y_t)}{\sqrt{\rho(Y_t)}}dB_t$$

Remark. *Special case: $d=1, b=0, \sigma=1$: Then X is a BM and $\rho = \frac{1}{\tilde{\sigma}^2(x)} \implies Y_t = X_{T_t^{-1}}$ solves*

$$dY_t = \tilde{\sigma}(Y_t)dB_t$$

Proof. By theorem 1.20, it is enough to see how the martingale problem after change of time becomes:

$$\mathcal{L} = \mathcal{L}_{X_t} \overset{\text{time change}}{\longrightarrow} \mathcal{L}_{Y_t} = \tilde{\mathcal{L}}$$

$Y_t = X_{A_t}; Y_0 = X_{A_0}$. For $f \in C^2 : M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$ is a local martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

The A_0 in the integral is probably 0, but it does not matter, we do a change of variables anyway.

$$\implies N_t^f := M_{A_t}^f = f(\underbrace{X_{A_t}}_{=Y_t}) - f(\underbrace{= X_{A_0}}_{Y_0}) - \int_{A_0}^{A_t} (\mathcal{L}f)(X_s)ds$$

is also a local martingale w.r.t. $(\mathcal{F}_{A_t})_{t\geq 0}$.
Change of variable (to get rid of the X_s in the integral):

$$\begin{aligned} \tau = T_s \leftrightarrow A_t = s &\implies X_s = X_{A_t} = Y_\tau \\ d\tau = \rho(X_s)ds & \end{aligned}$$

$$\implies N_t^f = f(Y_t) - f(Y_0) - \int_0^t (\mathcal{L}f)(Y_t)\frac{1}{\rho(Y_t)}d\tau$$

Since $\mathcal{L}f(x) = \sum_k b_k(x)\frac{\partial}{\partial x_k}f(x) + \sum_{k,l} a_{k,l}(x)\frac{\partial^2}{\partial x_k \partial x_l}f(x)$

$$\implies (\tilde{\mathcal{L}}f)(x) = \sum_k \frac{b_k(x)}{\rho(x)}\frac{\partial}{\partial x_k}f(x) + \sum_{k,l}\frac{\overbrace{a_{k,l}}^{(\sigma\sigma^\intercal)_{k,l}}}{\sqrt{\rho(x)\rho(x)}}(x)\frac{\partial^2}{\partial x_k \partial x_l}f(x)$$

$\implies$ It is a martingale problem for the SDE where the drift $\to \frac{\text{drift}}{\rho}$ and $\sigma \to \frac{\sigma}{\sqrt{\rho}}$

$\square$

1.5.3 Weak solutions in d=1

We will do both time and "space" changes.

- 1-d SDE:
$$\begin{cases} dX_t & = b(X_t)dt + \sigma(X_t)dB_t \\ X_0 = x_0 \in (\alpha, \beta) \end{cases} \tag{1.4}$$
- X_t a process in (α, β)
- Assume $b, \sigma : (\alpha, \beta) \to \mathbb{R}$ continuous, $\sigma(x) > 0 \forall x \in (\alpha, \beta)$
- Do a change of coordinates $Y_t := s(X_t)$ where $s : (\alpha, \beta) \to (s(\alpha), s(\beta))$, C^2 with $S'(x) > 0, x \in (\alpha, \beta)$.
- $s(x)$ is called the **scale function** and it is given by
$$s(x) = \int_{x_0}^x \exp\left(-\int_{x_0}^y dz \frac{2b(z)}{\sigma(z)^2}\right) dy$$
- s satisfies $\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0 \iff (\mathcal{L}s)(x) = 0$

Remark. *If $b(z) = 0 \implies s(x) = x - x_0 \implies s'(x) = 1$. If $s'(x) = 1$, we say that the process is in its "natural scale"*

By proposition 1.14: $\mathcal{L}s = 0, \dot{s} = 0$.
$\implies Y_t = s(X_t)$ is a local martingale satisfies $dY_t = s'(X_t)\sigma(X_t)dB_t$. the other terms cancel
$\iff Y_t$ is a solution of

$$\begin{cases} dY_t &= \tilde{\sigma}(Y_t)dB_t \\ Y_0 &= s(X_0) \end{cases} \tag{1.5}$$

where $\tilde{\sigma}(y) = s'(s^{-1}s(y))\sigma(s^{-1}(y))$.

Therefore we can write the original SDE in terms of a BM

Theorem 1.23. *The following are equivalent:*

1. *The process $(X_t)_{t<\xi}$, where ξ is the explosion time, on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; (B_t)_{t\geq 0})$ is a solution of (1.4) up to tje stopping time ξ*
2. *The process $Y_t = s(X_t)_{t<\xi}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; (B_t)_{t\geq 0})$ is a solution of (1.5) up to ξ*
3. *The process $(Y_t)_{t<\xi}$ has the representation $Y_t = \tilde{B}_{A_t}$, where $\tilde{B}$ is a BM starting at $\tilde{B}_0 = s(X_0)$ and $A_t = T_t^{-1}$ and $T_t = \int_0^t \frac{1}{\tilde{\sigma}^2(\tilde{B}_u)}du$*

s and A_t have the same definition as before

A degenerate case:

Let $\sigma(x) = |x|^\alpha$ for some $\alpha \in \left(0, \frac{1}{2}\right)$. $\implies$

$$\begin{cases} dY_t &= |Y_t|^\alpha dB_t \\ Y_0 &= y \end{cases} \tag{1.6}$$

$\implies T_t = \int_0^t \frac{1}{\sigma(\tilde{B}_u)^2}du$, $A_t = \int_0^t \sigma(Y_s)^2 ds$ and $Y_t = \tilde{B}_{A_t} \implies \tilde{B}_0 = y$.
$T_t < \infty$ a.s. ?

$$\begin{aligned} \mathbb{E}(T_t) &= \int_0^t \mathbb{E}\left(\frac{1}{\sigma(\tilde{B})^2}\right) du \\ &= \int_0^t \mathbb{E}\left(\frac{1}{|\tilde{B}|^{2\alpha}}\right) du \\ &= \int_0^t \int_{\mathbb{R}} dx \frac{1}{|x|^{2\alpha}} \frac{\exp(-\frac{(x-y)^2}{2u})}{\sqrt{2\pi u}} \overset{0<2\alpha<1}{<} \infty \end{aligned}$$

$\implies A_t = T_t^{-1}$, then $Y_t = \tilde{B}_{A_t}$ is a solution of (1.6), i.e. $\forall y \in \mathbb{R} \exists$ a non-trivial solution of (1.6).
For $Y_=0, Y_t = 0$ is also a solution $\implies$

- No uniqueness in law of (1.6)
- No pathwise uniqueness as well

Remark. *In general: uniqueness in law of 1-d SDEs is not to be expected if $\sigma(x) = 0$ somewhere (and σ continuous ...) (i.e. if σ is degenerate).*

By theorem 1.12 as soon as $\sigma(x) = |x|^\alpha$ for some $\alpha \geq \frac{1}{2}$, then one has pathwise uniqueness.

Hitting times and scale functions **Bessel process**:

$$\begin{cases} dR_t &= \frac{d-1}{2R_t}dt + dW_t \\ R_0 &= r_0 \in (0, \infty) \end{cases}$$

$\implies b(x) = \frac{d-1}{2x}, \sigma(x) = 1$.
The scale function satisfies $\mathcal{L}s(x) = \frac{1}{2}s''(x) + \frac{d-1}{2x}s'(x) = 0$

$$\implies s(x) = \begin{cases} \frac{x^{2-d}}{2-d} & d \neq 2 \\ \ln(x) & d = 2 \end{cases}$$

and therefore

$$(s_0, s_\infty) = \begin{cases} (0, \infty) & d < 2 \\ (-\infty, \infty) & d = 2 \\ (-\infty, 0 & d > 2) \end{cases}$$

Define the stopping time

$$T_a^R \coloneqq \inf\{t \geq 0 \mid R_t = a\}$$

Choose an $\alpha < r_0 < \beta$

$$\implies \mathbb{P}(T_\alpha^R < T_\beta^R) \overset{s' \geq 0}{=} \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

One computes WS exercises

$$\mathbb{P}(T_a^R < T_\beta^R) = \frac{s(\beta) - s(r_0)}{s(\beta) - s(\alpha)} = \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

This is generic, for Itô diffusions, provides that $\exists$ no killing in (α, β). unlike in 1.16