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# Lecture notes on Stochastic Analysis

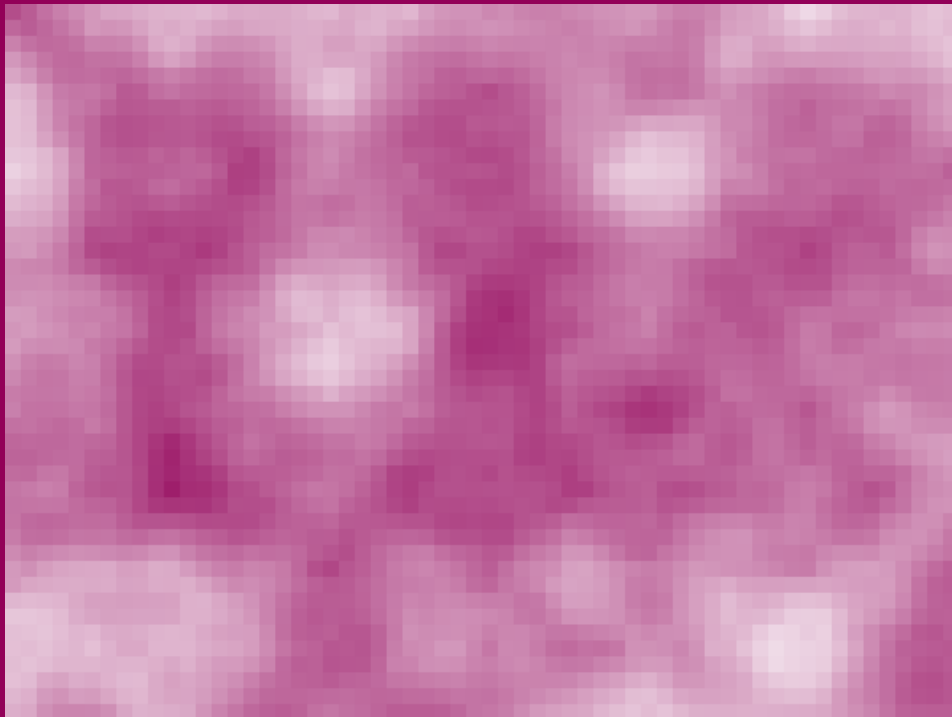
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# Chapter 0:

## Manuel's notes

### Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to [mh@mssh.dev](mailto:mh@mssh.dev) or [s6mlhinz@uni-bonn.de](mailto:s6mlhinz@uni-bonn.de).  
Happy learning!

### General Information

- Ecampus: [Ecampus link](#)
- Basis: [Basis link](#)
- Website: None
- Time slot(s): Tuesday 12-14 and Thursday 12-14
- Exams: Oral: 23-25 July, 30 Sept. maybe 27. Sept
- Deadlines: ?
- Exercises: One of Tue 16-18(for now this is preferred), Friday 10-12. Starting 16.04.

- First halve based on Eberle and / or Gubinelli ( be careful with Notation of dimensions!)

Start of lecture 01  
(11.04.23)

## Overview of the content

- Weak solutions of SDE
  - Martingale problem (characterization)
  - Time change (Dubin-Schwarz)
  - Change of measure (Girsanov)
- How to condition a diffusion to stay in a given domain forever (reflected BM, local time, ...)
- Lévy processes (a bit)
- Multiplicative stochastic heat-equation
  - relations with Kardar-Parisi-Zhang class of growth models

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# Chapter 1:

## Weak solutions of SDEs

SDE:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \text{ Initial condition} \end{cases} \quad (1.1)$$

- $X_t \in \mathbb{R}^d, B : n\text{-dim BM}$
- $b(t, x) : [b_k(t, x)]_{1 \leq k \leq d}$ : **drift vector**
- $\sigma(t, x) = [\sigma_{k,l}(t, x)]_{\substack{1 \leq k \leq d \\ 1 \leq l \leq n}}$ : **dispersion matrix**
- $a(t, x) = \sigma(t, x) \cdot \sigma(t, x)^\top$ : **diffusion matrix**

### 1.1 Strong solutions

**Definition 1.1.** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space with  $\mathcal{F}_t = \sigma(x_0, (B_s)_{0 \leq s \leq t})$

- a  $d$ -dim process  $X_t$  is a **strong solution** of equation 1.1 if:

- $X_t = x_0$  a.s.
- $X_t$  is adapted to  $\mathcal{F}_t \forall t \geq 0$
- $X$  is a continuous semimartingale s.t.  $\forall t \geq 0$ :

$$\int_0^t (\|b(s, X_s)\| + \|\sigma(s, X_s)\|^2) ds < \infty \text{ a.s.}$$

- $X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$

In the last semester we proved:

**Theorem 1.2.** Assume that  $b, \sigma$  are globally lipschitz with at most linear growth at  $\infty$  (in space)  
 $\implies \exists!$  strong solution of SDE.

*Foundations of Stochastic Analysis Thm 8.6*

**Added remark.** There exists  $K > 0$  s.t. for all  $x, y \in \mathbb{R}^d$ : Globally Lipschitz:

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|$$

Linear growth condition:

$$\|b(t, x)\| + \|\sigma(t, x)\| \leq K(1 + \|x\|)$$

**Remark.** For strong solutions,  $\mathcal{F}_t$  is given by the driving BM, which is given to us.

$$\implies X_t(\omega) = \Phi(x_0(\omega), (B_s)_{0 \leq s \leq t})$$

## 1.2 Weak solutions

- For weak solutions we do not fix the driving brownian motion.

**Definition 1.3.** A **weak solution** of equation 1.1 is a **pair** of adapted processes  $(X, B)$  to a  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  s.t.

- $B$  is a  $n$ -dim BM
- $X$  is a  $d$ -dim continuous semimartingale with
  1.  $X_0 = x_0$  a.s.
  2.  $\forall t \geq 0$

$$\int_0^t (\|b(s, X_s)\| + \|\sigma(s, X_s)\|^2) ds < \infty \text{ a.s.}$$

$$3. X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$$

**Remark.** • The filtration  $(\mathcal{F}_t)_{t \geq 0}$  is not necessarily the one generated by  $B$

- If  $X$  is adapted to the filtration generated by the BM  $\implies$  we have strong solutions
- $\exists$  weak solution which are not strong solutions
- Warning: To give a weak solution we really need to provide  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X, B)$

**Definition 1.4** (Uniqueness in law). An SDE 1.1 has **uniqueness in law** if given any two weak solutions  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X, B)$  and  $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{B})$  satisfy:

$$\text{Law}_{\mathbb{P}}(X) = \text{Law}_{\tilde{\mathbb{P}}}(\tilde{X})$$

They agree on any set in the sigma algebra

**Definition 1.5** (Pathwise uniqueness). The SDE 1.1 has pathwise uniqueness if, whenever the filtered space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and  $(B_t)_{t \geq 0}$  are fixed, then two solutions  $X, \tilde{X}$  with  $X_0 = \tilde{X}_0$  are indistinguishable.

**Example 1.6** (No strong solutions, no pathwise uniqueness,  $\exists$  weak solution & uniqueness in law by Tanaka).

$$\begin{cases} dX_t = \text{sgn}(X_t) dB_t \\ X_0 = 0 \end{cases} \quad (1.2)$$

or more generally  $X_0 = Y$ , where

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

Let  $W$  be a BM with  $W_0 = Y$ . Define

$$B_t := \int_0^t \text{sgn}(W_s) dW_s \text{ or } dB_t = \text{sgn}(W_t) dW_t$$

$$\implies dW_t = \text{sgn}(W_t) dB_t$$

$$\implies W_t = y + \int_0^t \text{sgn}(W_s) dB_s$$

$B_t$  is a local martingale with

$$\langle B \rangle_t = \int_0^t \underbrace{(\text{sgn}(W_s))^2}_{=1} \underbrace{d\langle W \rangle_s}_{=ds} = t$$

Also  $B_0 = 0$ , therefore  $B$  is a BM (see Lévy characterization)  $\implies W$  solves the SDE. For  $Y = 0$ ,  $W$  **and**  $-W$  solves the same SDE.

$\implies$

- exists weak solutions
- For  $Y = 0$ : no pathwise uniqueness
- Uniqueness in law (because the law is determined by  $X_t$  being a BM)
- No strong solution, because:  $\mathcal{F}^B = \mathcal{F}^{|W|} \subsetneq \mathcal{F}^W$

Not a proof just yet, just the reason!

**Example 1.7** (No solutions).

$$\begin{cases} dX_t = -\frac{1}{2X_t} 1_{X_t \neq 0} dt + dB_t \\ X_0 = 0 \end{cases} \quad (1.3)$$

Assume there exists a solution. Use Itô formula for  $X_t^2$ , then:

$$\begin{aligned} X_t^2 &= 2 \int_0^t X_s dX_s + \int_0^t 1 ds \\ &= - \int_0^t 1_{X_s \neq 0} ds + 2 \int_0^t x_s dB_s + t \\ &= \int_0^t 1_{X_s = 0} ds + 2 \int_0^t X_s dB_s \end{aligned}$$

We will prove  $\int_0^t 1_{X_s = 0} ds = 0 \implies X_t^2$  is a local martingale,  $X_t^2 \geq 0$  (and therefore a supermartingale) and  $X_0 = 0$  ( $\implies \mathbb{E}(X_t^2) = 0$ ). If  $X_t = 0 \implies \int_0^t 1_{X_s = 0} ds = t \implies 0 = dB_t$  which are contradictions!

**Remark.** If  $X = \underbrace{M}_{\in \mathcal{M}_{loc}} + \underbrace{A}_{\in \mathcal{A}}$

$$\implies \forall b \in \mathbb{R} \int_0^t 1_{X_s = b} d\langle M \rangle_s = 0$$

## Sheet 0:

- Mail: Alexander.Becker@uni-bonn.de
- Exercise sheet handed in Fr 10 am via eCampus
- Groups of 3

Motivation:

in the last semester: Introduction to stochastic analysis

- Diffusion processes & SDEs

Here: Deepen the knowledge & have fun

- Learn SDE techniques and get other results
- Modify diffusion processes to behave in a certain way
- Ex. diffusion bridge (Brownian bridge)
- Condition a diffusion to stay in a domain
  - Ex. Condition BM to stay positive
  - Old SDE:  $dB_t = dB_t$

- New SDE:  $dX_t = \frac{1}{X_t} dx + dB_t \rightarrow P(X_t \in \cdot) = P(B_t \in \cdot | B > 0 \text{ forever})$
- $D \subset \mathbb{R}^d$  open domain,  $X$  diffusion process, with generator  $L = \Sigma b \partial_i + \frac{1}{2} \sum a^{ij} \partial_i \partial_j$

$$Y := (X | X \in D \text{ forever})$$

get drift term  $\nabla \log \phi_0$ , where  $\phi_0$  is the lowest eigenfunction of  $-L$  on  $D$  with dirichlet boundary.

### Recap:

Brownian motion:

**Added definition.**  $B_0 = 0$ , independent  $\mathcal{E} \mathcal{N}(0, t_i - t_{i-1})$  increments,  $t \mapsto B_t(\omega)$  continuous.

Regularity of path  $t \mapsto B_t(\omega)$ :

- nowhere differentiable
- $\alpha$ -locally Hölder continuous  $\iff \alpha < \frac{1}{2}$
- Quadratic variation  $\langle B \rangle_t = t$
- Generator  $\frac{\Delta}{2}$
- Recurrent  $\iff ds^2?$

Itô-Integral:

1. If  $X$  simple process  $\implies$  RS-Integral
2. Itô isometry  $\mathcal{E} \rightarrow \{L^2 - \text{local martingale}\}, \mathcal{E} \subset L^2(d[M])$  dense
3. general  $X : \int X dM$  as  $L^2$ -limit

**Added remark** (Itô formula). •

$$df(t, X_t) = \partial_t f(t, X_t) dt + \partial_x f(t, X_t) dX_t + \frac{1}{2} f''(t, X_t) d\langle X \rangle_t$$

- associative  $\int X d(\int Y dZ) = \int XY dZ$
- If  $M$  local martingale  $\implies \int X dM$  local martingale

SDEs:

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t$$

- ex./ uniqueness:  $b, \sigma$  locally Lipschitz  $\implies$  strong ex.+pathwise uniqueness until explosion time
- globally Lipschitz (in space) and linear growth ( $|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|)$ ),  $e = \infty$ .

### Problem 00.1: SDE

Let  $B$  be a one-dimensional Brownian motion (starting from 0) and let  $X_t = \sin(B_t)$ .

1. Determine the SDE of  $X_t$
2. Discuss the existence and/or uniqueness of strong solutions of the SDE
3. Determine whether the local martingale term is a martingale (Hint: for the integrability condition, think about the Itô isometry for instance)



## Solution 00.1

1.:

Idea: Use Itô formula:  $X_t = \sin(B_t) = f(B_t)$

$$dX_t = df(B_t) \stackrel{\text{Itô}}{=} \underbrace{\partial_x \cos(B_t)}_{\sqrt{1-X_t^2}} dB_t - \frac{1}{2} \underbrace{\sin(B_t)}_{X_t} dt$$

Careful:  $\sqrt{1-X_t^2}$  is not inverse mapping, because it is always positive while  $\cos(B_t)$  is not

2.:

Consider the SDE

$$dX_t = \begin{cases} -\frac{1}{2}X_t dt + \sqrt{1-X_t^2} dB_t \\ X_0 = x \in (-1, 1) \end{cases}$$

Coefficients:  $b : [-1, 1] \rightarrow \mathbb{R}, b(x) = -\frac{1}{2}x$  and  $\sigma : [-1, 1] \rightarrow \mathbb{R}, \sigma(x) = \sqrt{1-x^2}$

$$[\sigma]_{1,2} = \sup \frac{|\sigma(x) - \sigma(y)|}{|x - y|^{1/2}} < \infty$$

Probably  $\sigma^2$  Lipschitz  $\implies \sigma$  Hölder  $\frac{1}{2}$

3.:

## Problem 00.2: Time change

Let  $B$  be a one-dimensional Brownian motion (starting from 0). Let  $Y_t = \int_0^t s^2 dB_s$ .

1. Determine the SDE of  $Y_t$
2. Find  $A_t$  such that  $Y_{A_t}$  is a (stopped) Brownian motion

## Solution 00.2

1.: By Definition

$$dY_t = t^2 dB_t$$

2.: We use the Dubin-Schwarz theorem:

$$X_{\infty} \in \mathcal{M}_{\text{loc}}^0, \langle X \rangle_{\infty} = \infty, T_t := \inf\{s \geq 0 | \langle X \rangle_s \geq t\} = X_t^{[-1]}$$

$$\implies B_t := X_{T_t} \text{ 1 d BM w.r.t. } (F_{T_t})_{t \geq 0}, X_t = B_{\langle B \rangle_t}$$

$$\text{here: use } X_t = b(X_t)dt + \sigma(X_t)dB_t \implies d[X]_t = \sigma^2(X_t)dt$$

## Problem 00.3: SDE and PDE

Let  $f$  be a function supported on  $[0, 1]$ ,  $u$  the solution of

$$\frac{1}{2}u(t, x) = \frac{1}{2}x(1-x) + \frac{d^2}{dx^2}u(t, x), \quad u(0, x) = f(x).$$

Consider also the one-dimensional SDE (also known as Wright-Fisher diffusion)

$$dX_t = \sqrt{X_t(1-X_t)}dB_t$$

with  $X_0 = x \in (0, 1)$ .

1. For any fixed  $t > 0$ , define  $M_s = u(t-s, X_s)$  for  $s \in [0, t]$ . Use Itô formula to show that  $M_s$  is a local martingale
2. Assume that  $f$  is bounded and there is a bounded solution of  $u$ . Show that  $u(t, x) = \mathbb{E}_x(f(X_t))$ .

## Solution 00.3

1.:

$$\begin{aligned}
 dM_s &= du(t-s, X_s) \\
 &= -\partial_s u(t-s, X_s)ds + \partial_x u(t-s, X_s) \underbrace{dX_s}_{b(X_s)ds + \sigma(X_s)dB_s \text{ by asso.}} + \frac{1}{2}\partial_x^2 u(t-s, X_s) \underbrace{d[X]_s}_{=\sigma^2(X_s)ds} \\
 &= \underbrace{(-\partial_s u + b\partial_x u + \frac{1}{2}\sigma^2\partial_x^2 u)(t-s, X_s)}_{=0}ds + \partial_x u(t-s, X_s)\sigma(X_s)dB_s \\
 &\implies dM_s = \partial_x u(t-s, X_s)\sigma(X_s)dB_s
 \end{aligned}$$

(i.e.:  $M_t - M_0 = \int_0^t \dots dB_s$ )

This is a purely stochastic integral against a (local) martingale  $\implies$  martingale.

2.:

- $M_s$  true martingale:

1.  $\forall t : \mathbb{E}(\sup_{[0,t]} |M|) < \infty$ , for example:  $M$  bounded
2.  $\forall t : \mathbb{E}(\sup_{[0,t]} [M]_t) < \infty$

$M_s := u(t-s, X_s), u$  bounded  $\implies M$  bounded  $\implies M$  true martingale

$w(s, x) := u(t-s, x)$

$$u(t, x) = w(0, x) = \mathbb{E}_x[w(0, X_0)] \stackrel{\text{martingale}}{=} \mathbb{E}_x[w(t, X_t)] = \mathbb{E}_x[(u(0, X_t))] \stackrel{\text{PDE}}{=} \mathbb{E}_x[f(X_t)]$$

Feynman-Kac formula (later more general).

Also solvable by Kolmogorov Backward / forward equation

Start of lecture 02  
(16.04.24)

**Example 1.8** (Non-uniqueness of law, non pathwise uniqueness, with solutions).

$$\begin{cases} dX_t &= 1_{X_t \neq 0} dB_t \\ x_0 &= 0 \end{cases}$$

Then

$$X_t = 0 \forall t \geq 0$$

and

$$X_t = B_t \forall t \geq 0$$

both are solutions:

$$X_t - B_t = -\int_0^t 1_{X_s=0} dB_s \implies \langle X - B \rangle_t = \int_0^t 1_{X_s=0} d\langle B \rangle_s = 0$$

Let  $\eta \sim \text{Ber}(\frac{1}{2})$  independent of  $(B_t)_{t \geq 0}$  and define

$$\tilde{X}_t = \begin{cases} 0 & \eta = 0 \\ B_t & \eta = 1 \end{cases}$$

$\implies \tilde{X}_t$  is adapted to  $\sigma(\eta(B_s)_{0 \leq s \leq t})$ , but not to  $\sigma((B_s)_{0 \leq s \leq t})$  and therefore not a **strong solution**.

$$\begin{aligned} X_t &= \int_0^t 1_{X_s \neq 0} dB_s \\ &= \int_0^t (1 - 1_{X_s=0}) dB_s \\ B_t - \int_0^t 1_{X_s=0} dB_s \end{aligned}$$

**Example 1.9** (No strong solution, weak solution, no pathwise uniqueness, no uniqueness in law).

$$\begin{cases} dX_t &= 1_{X_t \neq 1} \operatorname{sgn}(X_t) dB_t \\ X_0 &= 0 \end{cases}$$

Let  $Y_t$  be a solution of

$$\begin{cases} dY_t &= \operatorname{sgn}(Y_t) dB_t \\ Y_0 &= 0 \end{cases}$$

$\implies X_t := Y_{t \wedge \tau}$ , where  $\tau = \inf\{s \geq 0 \mid Y_s = 1\}$  is also a solution.

**Theorem 1.10** (Yamada-Watanabe). If both existence of weak solutions and pathwise uniqueness hold, uniqueness in law also holds.

Moreover,  $\forall$  choices of  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and  $(B_t)_{t \geq 0}$  then there exists a **strong solution**.

### 1.3 Lévy characterization

**Example 1.11.** Consider the SDE

$$\begin{cases} dX_t &= O_t dB_t \\ X_0 &= x_0 \end{cases}$$

where both  $X_t$  and  $B_t$  are  $d$ -dimensional and  $O_t$  is an adapted process (matrix) s.t.  $O_t^\top O_t = 1 \forall t \geq 0$  i.e.  $O_t$  is a rotation

$$\begin{aligned} \implies X_t^k &= X_0^k + \sum_{l=1}^d \int_0^t O_s^{k,l} dB_s^l \\ \implies \langle X^k, X^{\bar{k}} \rangle_t &= \sum_{l, \bar{l}} \int_0^t O_s^{k,l} O_s^{\bar{k}, \bar{l}} d\langle B^l, B^{\bar{l}} \rangle_s = \sum_{l=1}^d \int_0^t O_s^{k,l} O_s^{\bar{k}, l} ds \\ &= \int_0^t \underbrace{(O_s O_s^\top)^{k, \bar{k}}}_{=1} ds = \delta_{k, \bar{k}} t \xrightarrow{\text{Lévy}} (X_t)_{t \geq 0} \text{ is also a } d\text{-dim BM starting from } x_0 \end{aligned}$$

**Theorem 1.12** (Yamada, Watanabe). Let  $dX_t = b(X_t)dt + \sigma(X_t)dB_t$  and assume that there exist both a increasing function  $\rho(u) \geq 0$  s.t.  $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|) \forall x, y \in \mathbb{R}$  s.t.

$$\int_0^\infty \frac{1}{\rho(u)} du = \infty \implies 0 < u < C_1 \implies \rho(u) \leq C_2 u^{0.5}$$

and some increasing concave function  $\gamma_1(u) \geq 0$  s.t.

$$|b(x) - b(y)| \leq \gamma_1(|x - y|) \forall x, y \in \mathbb{R}$$

and

$$\int_0^\infty \frac{1}{\gamma_1(u)} du = \infty.$$

Then pathwise uniqueness holds.

**Theorem 1.13** (Storokhod). Assume that  $\sigma, b$  are continuous bounded functions

$\implies$  there exist weak solutions to the SDE  $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ .

## 1.4 Weak solutions and martingale problems

Let  $(X_t)_{t \geq 0}$  be a weak solution of the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = 0 \end{cases}$$

on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ .

$\implies X_t$  is a semimartingale s.t.  $X_t^k = X_0^k + \int_0^t b(s, X_s)ds + \sum_{l=1}^n \int_0^t \sigma_{k,l}(s, X_s)dB_s^l$  and

$$\langle X^i, X^j \rangle_t = \int_0^t \underbrace{\sum_{l=1}^n \sigma_{i,l}(s, X_s) \sigma_{j,l}^T(s, X_s)}_{=a_{i,j}(s, X_s)} ds$$

### 1.4.1 Itô-Doeblin formula

Itô formula leads to

**Proposition 1.14.** For  $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ , then

$$f(t, X_t) = f(0, X_0) + \int_0^t (\sigma^T \nabla f)(s, X_s) dB_s + \int_0^t \left[ \left( \frac{\partial}{\partial s} + \mathcal{L} \right) f(s, x_s) \right] ds$$

where  $(\mathcal{L}f)(t, x) = \frac{1}{2} \sum_{k,l=1}^n a_{k,l}(t, x) \frac{\partial^2}{\partial x_k \partial x_l} f(t, x) + \sum_{k=1}^n b_k(t, x) \frac{\partial}{\partial x_k} f(t, x)$ .

$\mathcal{L}$  is called the **generator**

**Remark.** The Itô-Doeblin formula provides a connection between SDEs and PDEs as we have seen in the last part **Foundations of stochastic analysis**.

**Example 1.15.** Let  $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$  be a solution of the PDE

$$\begin{aligned} \frac{\partial}{\partial t} f(t, x) + (\mathcal{L}f)(t, x) &= -g(t, x) & t \geq 0, x \in U \subseteq \mathbb{R}^d \\ f(t, x) &= \varphi(t, x) & t \geq 0, x \in \partial U. \end{aligned}$$

then  $M_t := f(t, X_t) + \int_0^t g(s, X_s)ds \in \mathcal{M}_{loc}$  by proposition 1.14 and if  $f, g$  are bounded  $M_t \in \mathcal{M}$ .

$$T := \inf\{s \geq 0 | X_s \notin U\} \implies M_t^T := M_{T \wedge t} \in \mathcal{M}.$$

Furthermore, if we assume  $T < \infty$  a.s. :

$$\mathbb{E}[M_t] = \mathbb{E}[\varphi(T, X_T)] + \mathbb{E}\left[\int_0^T g(s, X_s)ds\right] = \varphi(0, X_0)$$

where  $X_0 = x_0$ .

There are two special cases:

$g = 0 \implies$  yields the exit distributions, while

$\varphi = 0, g = 1$  yields the mean exit times.

**Example 1.16** (Feynman-Kac formula). Let  $t \in \mathbb{R}_+$  be finite. Assume  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $K : [0, t] \times \mathbb{R}^d \rightarrow \mathbb{R}_+$  be continuous functions. Assume that  $u$  is a  $C^{1,2}$ -solution (bounded, for simplicity) of

$$\begin{cases} \frac{\partial u}{\partial s}(s, x) = \frac{1}{2} \Delta u(s, x) - K(s, x)u(s, x) & s \in [0, t], x \in \mathbb{R}^d \\ u(0, x) = f(x) & x \in \mathbb{R}^d \end{cases}$$

Then  $u$  has the stochastic representation  $u(t, x) = \mathbb{E}_x \left[ f(X_t) \exp \left( - \int_0^t K(t-s, X_s) ds \right) \right]$ , where  $X_t$  is BM starting from  $X_0 = x$ .

*Sketch.* 1. Define  $r(s, x) := u(t-s, x)$  for  $s \in [0, t]$

2. Show:  $M_s := \exp(-A_s)r(s, x)$  with  $A_s = \int_0^s K(u, X_u)du$  is a local martingale.  $\square$

**Remark.** This is a reformulation of the formula from the last semester.

### 1.4.2 Martingale problem

A solution of an SDE is generically defined up to some **explosion time**  $\xi$ , where it either diverges or it exists a given domain  $U \subset \mathbb{R}^d$  (open).

$\implies$  For  $k \in \mathbb{N}$  define  $U_k := \{x \in U \mid |x| < k \wedge \text{dist}(x, U^c) \geq \frac{1}{k}\}$  with  $U = \bigcup_{k \geq 1} U_k$  and

$$T_k := \{t \geq 0 \mid x_t \notin U_k\}.$$

A solution of the SDE  $b(t, X_t)dt + \sigma(t, X_t)dB_t$  is defined up to  $\xi = \sup_{k \geq 1} T_k$ .

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**Added remark.** uniqueness of solution to the heat equation  $\frac{1}{2}\Delta u - Ku$ : not unique! Stochastic calculus: Paolo Baldi: 10.3-10.4

Define  $(\star)$ :

$$dX_t b(t, X_t)dt + \sigma(t, X_t)dB_t \text{ with } X_0 = x_0$$

**Theorem 1.17** (Martingale problem). If  $X$  is a solution of  $(\star)$  up to time  $\zeta$ , then  $\forall f \in C^{1,2}(\mathbb{R}_+ \times U)$

$$M_t = f(t, X_t) - \int_0^t \left( \frac{\partial}{\partial s} + \mathcal{L} \right) f(s, x_s) ds, t < s$$

is a local martingale up to  $\zeta$  and  $M_t^{T_k}$  are localizing martingales.

**Definition 1.18** (Martingale solutions).  $(X_t)_{t \geq 0}$  is a **martingale solution** of  $(\star)$  if  $\forall f \in C^2(\mathbb{R}^d)$

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s) ds$$

is a continuous local martingale.

**Theorem 1.19** (Equivalent definitions). The following are equivalent (for  $X_t$  being a solution of  $(\star)$ ):

(a)  $\forall f \in C^2(\mathbb{R}^d)$ ,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s) ds$$

is a local martingale

(b) The process in  $\mathbb{R}^d$  given by

$$M_t := X_t - X_0 - \int_0^t b(s, X_s) ds$$

is a  $d$ -dimensional local martingale with  $\langle M^i, M^j \rangle_t = \int_0^t a_{i,j}(s, X_s) ds = \langle X^i, X^j \rangle_t$

(c)  $\forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$

$$\tilde{M}_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left( \frac{\partial}{\partial s} + \mathcal{L} \right) f(s, X - s) ds$$

is a local martingale

ds

*Proof.*  $c \implies a$ : by choosing  $f$  independent of  $t$ .

$a \implies b$ : 1.: Choosing  $f(X) = X_i$  implies

$$M_t^f = X_t^i - X_0^i - \int_0^t b_i(s, X_s) ds \in \mathcal{M}_{\text{loc}}$$

2.:  $f(X) = X_i X_j$ :

$$\begin{aligned} (\mathcal{L}f)(x) &= \frac{1}{2}a_{ij} + \frac{1}{2}a_{ji} + b_i X_j b_j X_i \\ a &= a^\top \implies a_{ij} b_i X_j b_j X_i \\ \implies M_t^f X_t^i X_t^j - X_0^i X_0^j - \int_0^t [a_{ij}(s, X_s) + b_i(s, X_s) X_s^j + b_j(s, X_s) X_s^i] ds \end{aligned}$$

$$\begin{aligned} X_t^i X_t^j - X_0^i X_0^j &\stackrel{\text{Integration by parts}}{=} \int_0^t X_s^i dX_s^j + \int_0^t X_s^j dX_s^i + \langle X^i, X^j \rangle_t \\ &= M_t^f + \int_0^t [a_{ij} + b_i X_s^j + b_j X_s^i] ds \end{aligned}$$

We can maybe also proof this by calculating  $X^2$ ?

Here  $dX_s^i$  is the same as  $b_i X_s^j$  is the same up to a local martingale term and  $\langle X^i, X^j \rangle_t = \int_0^t a_{ij}(s, X_s) ds = \langle M^i, M^j \rangle_t$

$b \implies c$ : By Proposition 1.14 If (use the next theorem)  $X$  was a weak solution  $\implies \tilde{M}_t^f$  is a local martingale.  $\square$

**Theorem 1.20.** Let  $n = d$ , assume  $\sigma(t, x)$  is invertible  $\forall t, x$  and  $\sigma^{-1}(t, x)$  is uniformly bounded. T.f.a.e.:

- (a)  $(X_t)_{t \geq 0}$  is a weak solution of the SDE  $(\star)$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}; B)$
- (b)  $(X_t)_{t \geq 0}$  is a martingale solution of the SDE  $(\star)$

This also works for  $n \neq d$ , but with a different proof

*Proof.*  $a \implies b$ : True

$b \implies a$ : Goal construct a BM for the weak solution.

By proposition ??  $a \implies b$   $dM_t = dX_t - b(t, X_t)dt \in \mathcal{M}_{\text{loc}}$  and  $d\langle M^i, M^j \rangle_t = a_{k,l}(t, X_t)dt$

$$\begin{aligned} \implies dX_t &= dM_t + b(t, X_t)dt \\ &= \sigma(t, X_t) d\tilde{B}_t + b(t, X_t)dt \end{aligned}$$

where  $\tilde{B}_t := \sigma(s, X_s)^{-1} dM_s$

To see:  $\tilde{B}_t$  is a brownian motion.

$$\begin{aligned} \langle \tilde{B}^i, \tilde{B}^j \rangle_t &= \sum_{k,l} \int_0^t \sigma_{ij}^{-1} \sigma_{jl}^{-1} d\langle M^k, M^l \rangle_s \\ &\quad = \underbrace{a_{ij}}_{(\sigma^\top \sigma)_{kl}} ds \\ &= \sum_{k,l,p} \int_0^t \sigma_{ik}^{-1} \sigma_{kp} \sigma_{pl}^\top \sigma_{lj}^{-1} ds \\ &= \delta_{ij} \int_0^t 1 ds = \delta_{ij} t \end{aligned}$$

Then by the Lévy characterization  $\tilde{B}$  is a brownian motion.  $\square$

**Added remark.** This is the first way to construct a weak solution: Solve a martingale problem! This is used a lot in practice.

## 1.5 Weak solutions and time change

### 1.5.1 Time change

For  $d = 1$ :

**Theorem 1.21.** [Dubins-Schwarz]

- Let  $M \in \mathcal{M}_{loc}^0$  and  $\langle M \rangle_\infty = \infty$  a.s.
- Let  $T_t := \inf\{s \geq 0 : \langle M \rangle_s \geq t\}$

This implies

1.  $t \mapsto M_{T_t}$  is a  $(\mathcal{F}_{T_t})$  brownian motion
2.  $M_t = B_{\langle M \rangle_t}$  for some standard brownian motion  $B$

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \underbrace{\int_0^t \sigma(s, X_s) dB_s}_{=M_t}. \text{ If } \langle M \rangle_\infty = \infty \text{ a.s.:}$$

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \tilde{B}_{\int_0^t \sigma^2(s, X_s) ds}$$

### 1.5.2 Time change in a martingale problem

Consider  $d = 1 = n$ .

$$dY_t = \tilde{\sigma}(Y_t) dB_t \quad (\star\star)$$

and  $\tilde{\sigma}$  strictly positive continuous function.

$$\langle Y \rangle_t = \int_0^t \tilde{\sigma}(Y_s)^2 ds =: A_t$$

By theorem 1.21  $\implies Y_t = W_{A_t}$  for some brownian motion  $W$ .

Assume  $A_t \rightarrow \infty$  a.s.

$$T_t := \inf\{s \geq 0 : \langle Y \rangle_s \geq t\}$$

$$\implies T_{A_t} = \inf\{s \geq 0 : \langle Y \rangle_s \geq \langle Y \rangle_t\} = t$$

$$1 = \frac{d}{dt} (T_{A_t}) = \underbrace{T'_{A_t}}_{=u} \cdot A_t$$

$$\implies \underbrace{T'_u}_{=\frac{dT_u}{du}} = \frac{1}{A'_{T_u}} \implies T_u = \int_0^u \frac{1}{\tilde{\sigma}(Y_{T_s})^2} ds = \int_0^u \frac{1}{\tilde{\sigma}(W_s)^2} ds$$

$\implies$  to construct a solution of  $(\star\star)$ : Given  $W \rightarrow$  compute  $T_u \rightarrow$  determine  $A - t = T_t^{-1} \implies Y_t = W_{A_t}$

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**Theorem 1.22.** Let  $(X_u)_{u \geq 0}$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a weak solution of

$$dX_u = b(X_u) du + \sigma(X_u) dB_u$$

where  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , the drift and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  are locally bounded,  $\sigma^{-1}$  exists for a.e.  $x$  and is locally bounded.

Consider a **time change**  $T_u := \int_0^u \rho(X_s) ds$ , where  $\rho : \mathbb{R}^d \rightarrow \mathbb{R}_+$  s.t.

$$T_u < \infty \forall u \geq 0 \text{ and } T_\infty = \infty \text{ a.s.}$$

$\implies$  Then  $Y_t := X_{A_t}$ , where  $A_t = T_t^{-1}$  is a weak solution of the SDE

$$dY_t = \frac{b(Y_t)}{\rho(Y_t)}dt + \frac{\sigma(Y_t)}{\sqrt{\rho(Y_t)}}dB_t$$

**Remark.** Special case:  $d = 1, b = 0, \sigma = 1$ : Then  $X$  is a BM and  $\rho = \frac{1}{\tilde{\sigma}^2(x)} \implies Y_t = X_{T_t^{-1}}$  solves

$$dY_t = \tilde{\sigma}(Y_t)dB_t$$

*Proof.* By theorem 1.20, it is enough to see how the martingale problem after change of time becomes:

$$\mathcal{L} = \mathcal{L}_{X_t} \xrightarrow{\text{time change}} \mathcal{L}_{Y_t} = \tilde{\mathcal{L}}$$

$Y_t = X_{A_t}; Y_0 = X_{A_0}$ . For  $f \in C^2$ :  $M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$  is a local martingale w.r.t.  $(\mathcal{F}_t)_{t \geq 0}$ .

$$\implies N_t^f := M_{A_t}^f = f(\underbrace{X_{A_t}}_{=Y_t}) - f(\underbrace{X_{A_0}}_{Y_0}) - \int_{A_0}^{A_t} (\mathcal{L}f)(X_s)ds$$

is also a local martingale w.r.t.  $(\mathcal{F}_{A_t})_{t \geq 0}$ .

Change of variable (to get rid of the  $X_s$  in the integral):

$$\begin{aligned} \tau = T_s &\leftrightarrow A_t = s \implies X_s = X_{A_t} = Y_\tau \\ d\tau &= \rho(X_s)ds \end{aligned}$$

$$\implies N_t^f = f(Y_t) - f(Y_0) - \int_0^t (\mathcal{L}f)(Y_t) \frac{1}{\rho(Y_t)} d\tau$$

Since  $\mathcal{L}f(x) = \sum_k b_k(x) \frac{\partial}{\partial x_k} f(x) + \sum_{k,l} a_{k,l}(x) \frac{\partial^2}{\partial x_k \partial x_l} f(x)$

$$\implies (\tilde{\mathcal{L}}f)(x) = \sum_k \frac{b_k(x)}{\rho(x)} \frac{\partial}{\partial x_k} f(x) + \sum_{k,l} \frac{\overbrace{a_{k,l}}^{(\sigma\sigma^\top)_{k,l}}}{\sqrt{\rho(x)\rho(x)}}(x) \frac{\partial^2}{\partial x_k \partial x_l} f(x)$$

$\implies$  It is a martingale problem for the SDE where the drift  $\rightarrow \frac{\text{drift}}{\rho}$  and  $\sigma \rightarrow \frac{\sigma}{\sqrt{\rho}}$

□

### 1.5.3 Weak solutions in d=1

We will do both time and “space” changes.

- 1-d SDE:

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dB_t \\ X_0 = x_0 \in (\alpha, \beta) \end{cases} \quad (1.4)$$

- $X_t$  a process in  $(\alpha, \beta)$
- Assume  $b, \sigma : (\alpha, \beta) \rightarrow \mathbb{R}$  continuous,  $\sigma(x) > 0 \forall x \in (\alpha, \beta)$
- Do a change of coordinates  $Y_t := s(X_t)$  where  $s : (\alpha, \beta) \rightarrow (s(\alpha), s(\beta))$ ,  $C^2$  with  $s'(x) > 0, x \in (\alpha, \beta)$ .
- $s(x)$  is called the **scale function** and it is given by

$$s(x) = \int_{x_0}^x \exp\left(-\int_{x_0}^y \frac{2b(z)}{\sigma(z)^2} dz\right) dy$$

- $s$  satisfies  $\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0 \iff (\mathcal{L}s)(x) = 0$

The  $A_0$  in the integral is probably 0, but it does not matter, we do a change of variables anyway.



**Remark.** If  $b(z) = 0 \implies s(x) = x - x_0 \implies s'(x) = 1$ . If  $s'(x) = 1$ , we say that the process is in its “natural scale”

By proposition 1.14:  $\mathcal{L}s = 0, \dot{s} = 0$ .

$\implies Y_t = s(X_t)$  is a local martingale satisfies  $dY_t = s'(X_t)\sigma(X_t)dB_t$ .

the other terms cancel

$\iff Y_t$  is a solution of

$$\begin{cases} dY_t &= \tilde{\sigma}(Y_t)dB_t \\ Y_0 &= s(X_0) \end{cases} \quad (1.5)$$

where  $\tilde{\sigma}(y) = s'(s^{-1}s(y))\sigma(s^{-1}(y))$ .

Therefore we can write the original SDE in terms of a BM

**Theorem 1.23.** The following are equivalent:

1. The process  $(X_t)_{t < \xi}$ , where  $\xi$  is the explosion time, on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}; (B_t)_{t \geq 0})$  is a solution of (1.4) up to the stopping time  $\xi$
2. The process  $Y_t = s(X_t)_{t < \xi}$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}; (B_t)_{t \geq 0})$  is a solution of (1.5) up to  $\xi$
3. The process  $(Y_t)_{t < \xi}$  has the representation  $Y_t = \tilde{B}_{A_t}$ , where  $\tilde{B}$  is a BM starting at  $\tilde{B}_0 = s(X_0)$  and  $A_t = T_t^{-1}$  and  $T_t = \int_0^t \frac{1}{\tilde{\sigma}^2(\tilde{B}_u)} du$

$s$  and  $A_t$  have the same definition as before

### A degenerate case:

Let  $\sigma(x) = |x|^\alpha$  for some  $\alpha \in (0, \frac{1}{2})$ .  $\implies$

$$\begin{cases} dY_t &= |Y_t|^\alpha dB_t \\ Y_0 &= y \end{cases} \quad (1.6)$$

$\implies T_t = \int_0^t \frac{1}{\sigma(\tilde{B}_u)^2} du$ ,  $A_t = \int_0^t \sigma(Y_s)^2 ds$  and  $Y_t = \tilde{B}_{A_t} \implies \tilde{B}_0 = y$ .  
 $T_t < \infty$  a.s. ?

$$\begin{aligned} \mathbb{E}(T_t) &= \int_0^t \mathbb{E} \left( \frac{1}{\sigma(\tilde{B})^2} \right) du \\ &= \int_0^t \mathbb{E} \left( \frac{1}{|\tilde{B}|^{2\alpha}} \right) du \\ &= \int_0^t \int_{\mathbb{R}} dx \frac{1}{|x|^{2\alpha}} \frac{\exp(-\frac{(x-y)^2}{2u})}{\sqrt{2\pi u}} \quad 0 < 2\alpha < 1 < \infty \end{aligned}$$

$\implies A_t = T_t^{-1}$ , then  $Y_t = \tilde{B}_{A_t}$  is a solution of (1.6), i.e.  $\forall y \in \mathbb{R} \exists$  a non-trivial solution of (1.6).

For  $Y=0, Y_t=0$  is also a solution  $\implies$

- No uniqueness in law of (1.6)
- No pathwise uniqueness as well

**Remark.** In general: uniqueness in law of 1-d SDEs is not to be expected if  $\sigma(x) = 0$  somewhere (and  $\sigma$  continuous ...) (i.e. if  $\sigma$  is degenerate).

By theorem 1.12 as soon as  $\sigma(x) = |x|^\alpha$  for some  $\alpha \geq \frac{1}{2}$ , then one has pathwise uniqueness.

### Hitting times and scale functions Bessel process:

$$\begin{cases} dR_t &= \frac{d-1}{2R_t} dt + dW_t \\ R_0 &= r_0 \in (0, \infty) \end{cases}$$

$\implies b(x) = \frac{d-1}{2x}, \sigma(x) = 1$ .

The scale function satisfies  $\mathcal{L}s(x) = \frac{1}{2}s''(x) + \frac{d-1}{2x}s'(x) = 0$

$$\implies s(x) = \begin{cases} \frac{x^{2-d}}{2-d} & d \neq 2 \\ \ln(x) & d = 2 \end{cases}$$

and therefore

$$(s_0, s_\infty) = \begin{cases} (0, \infty) & d < 2 \\ (-\infty, \infty) & d = 2 \\ (-\infty, 0) & d > 2 \end{cases}$$

Define the stopping time

$$T_a^R := \inf\{t \geq 0 \mid R_t = a\}$$

Choose an  $\alpha < r_0 < \beta$

$$\implies \mathbb{P}(T_\alpha^R < T_\beta^R) \stackrel{s' \geq 0}{=} \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

One computes

$$\mathbb{P}(T_a^R < T_\beta^R) = \frac{s(\beta) - s(r_0)}{s(\beta) - s(\alpha)} = \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

This is generic, for Itô diffusions, provides that  $\exists$  no killing in  $(\alpha, \beta)$ .

WS exercises

unlike in 1.16  
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### 1.5.4 Uniqueness of martingale solution

SDE  $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$  with generator

$$\mathcal{L} = \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l} (\sigma \sigma^\top)_{k,l} \frac{\partial^2}{\partial x_k \partial x_l}$$

**Definition 1.24.** Let  $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R}^d)$  with  $\sigma$ -algebra  $\mathcal{F}$ , canonical filtration  $(\mathcal{F}_t)_{t \geq 0}$ , canonical process  $Z_t(\omega) := \omega$ .

We say that  $\mathbb{P}$  on  $(\mathcal{C}, \mathcal{F})$  is a martingale solution for the generator  $\mathcal{L} \iff \forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R})$

$$M_t^f := f(t, Z_t) - f(0, Z_0) - \int_0^t \left( \frac{\partial}{\partial s} + \mathcal{L} \right) f(s, Z_s) ds \quad (1.7)$$

is a martingale w.r.t.  $\mathbb{P}$ .

**Definition 1.25.** A martingale problem (1.7) has a unique solution if for any two martingale solutions  $\mathbb{P} = \mathbb{Q}$  s.t.  $\text{Law}_{\mathbb{P}}(Z_0) = \text{Law}_{\mathbb{Q}}(Z_0)$

$$\implies \mathbb{P} = \mathbb{Q}$$

**Remark.** Uniqueness of martingale solutions corresponds to uniqueness in law of the weak solutions.

Backwards Kolmogorov Equation (BKE):

$$\frac{\partial}{\partial t} \varphi(t, x) = \mathcal{L} \varphi(t, x) \forall x \in \mathbb{R}^d, t \geq 0 \quad (1.8)$$

**Theorem 1.26.** Assume that  $\forall$  initial condition

$$\varphi(0, x) = \Psi(x), \Psi \in C_0^\infty(\mathbb{R}^d)$$

the (1.8) has a solution and  $\varphi$  bounded for all finite time intervals. We have uniqueness of martingale solutions and therefore uniqueness of weak solutions!

The Kolmogorov forward equation is (related to) the Fokker-Planck equation!

*Proof.* Prove that  $\forall 0 \leq t_1 < t_2 < \dots < t_n$ :

$$\text{Law}_{\mathbb{P}}(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}) = \text{Law}_{\mathbb{Q}}(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n})$$

### 1. One-time distribution:

$\forall 0 \leq s \leq r$ :

$$\left( \frac{\partial}{\partial s} + \mathcal{L} \right) \varphi(r-s, x) \stackrel{(1.8)}{=} 0$$

take  $t \in [0, r]$ :

$$\begin{aligned} M_t^r &:= \varphi(r-t, Z_t) - \varphi(r, Z_0) - \int_0^t \underbrace{(\partial_s + \mathcal{L})\varphi(r-s, Z_s)}_{=0} ds \\ &= \varphi(r-t, Z_t) - \varphi(r, Z_0) \text{ is a martingale} \end{aligned}$$

for any solution  $\mathbb{P}$ .

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{P}}(M_t^r - M_0^r \mid \mathcal{F}_t) = \mathbb{E}(\varphi(0, Z_r) - \varphi(r-t, Z_t) \mid \mathcal{F}_t) \\ \implies \forall 0 \leq t \leq r : \mathbb{E}_{\mathbb{P}}(\varphi(0, Z_r) \mid \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}(\varphi(r-t, Z_t) \mid \mathcal{F}_t) \\ &\stackrel{\text{a.s.}}{=} \varphi(r-t, Z_t) \\ \underbrace{\mathbb{E}(\varphi(0, Z_r))}_{\Psi(Z_r)} &\stackrel{t=0}{=} \mathbb{E}_{\mathbb{P}}(\varphi(r, Z_0)) \end{aligned}$$

$\forall$  other martingale solutions  $\mathbb{Q}$ :

$$\mathbb{E}_{\mathbb{Q}}(\Psi(Z_r)) = \mathbb{E}_{\mathbb{Q}}(\varphi(r, Z_0))$$

By assumption this implies  $\text{Law}_{\mathbb{P}}(Z_r) = \text{Law}_{\mathbb{Q}}(Z_r)$ .

### 2. Multi-time distributions:

For  $\Psi \in C_0^\infty$ , denote  $\varphi_\Psi$  the solution of (1.8) with initial condition  $\Psi$ :

$$\mathbb{E}_{\mathbb{P}}(\Psi(Z_r) \mid \mathcal{F}_t) = \varphi_\Psi(r-t, Z_t)$$

$0 \leq r_2 \leq r_1$  test for  $g \in C_0^\infty$ :

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(\Psi(Z_{r_1})g(Z_{r_2})) &= \mathbb{E}(\underbrace{\mathbb{E}(\varphi_\Psi(Z_{r_1}) \mid \mathcal{F}_{r_2})}_{\varphi_\Psi(r_1-r_2, Z_{r_2})} g(Z_{r_2})) \\ &= \mathbb{E}_{\mathbb{P}}(\varphi_\Psi(r_1-r_2, Z_{r_2})g(Z_{r_2})) \\ &\stackrel{1.}{=} \mathbb{E}_{\mathbb{Q}}(\varphi_\Psi(r_1-r_2, Z_{r_2})g(Z_{r_2})) \\ &= \mathbb{E}_{\mathbb{Q}}(\Psi(Z_{r_1})g(Z_{r_2})) \end{aligned}$$

Iterating yields the statement. □

This needs the boundedness of  $\varphi$ , otherwise it might only be a local martingale. There are softer conditions we can put on the coefficients to achieve the same result. This might not be needed, because  $\varphi$  is  $C^1$  in time anyways

# Chapter 2:

## SDE techniques

**Goal:** Study process by changing the measure.  
 E.g.:  $X_t = B_t$ .  $\underbrace{\text{Condition } X_t \geq 0 \forall t \geq 0}_{=: Y_t}$ .

### 2.1 Girsanov theorem

Given  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$  and two measures  $\mathbb{P}, \mathbb{Q}$   
 Assume  $\mathbb{P} \ll \mathbb{Q}$  and  $\mathbb{Q} \ll \mathbb{P}$  and let  $H = \frac{d\mathbb{Q}}{d\mathbb{P}}$

The notes of Eberle switches the roles of  $\mathbb{P}, \mathbb{Q}$ !

$$\implies Z_t := \mathbb{E}_{\mathbb{P}}(H \mid \mathcal{F}_t) = \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t}$$

is a martingale.

From last semester:  $\forall Y \in L^1(\mathbb{Q}) \cap L^1(\mathbb{P}), \mathcal{F}_t$  measurable:

$$\mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{F}_s) = \frac{bE_{\mathbb{P}}(Y \cdot Z_t \mid \mathcal{F}_s)}{Z_s} \forall s < t$$

If  $Z > 0$  is  $\mathcal{M}_{\text{loc}}$ ,  $\exists L \in \mathcal{M}_{\text{loc}}$  s.t.:

$$Z_t = e^{L_t - \frac{1}{2} \langle L \rangle_t} \rightarrow L_t = \ln(Z_0) + \int_0^t \frac{dZ_s}{Z_s}$$

There might be a problem, because  $\ln(Z_0)$  might not be integrable, and therefore not a local martingale!

**Theorem 2.1** (Girsanov). Assume  $Z > 0$  is a martingale. If  $M$  is a local martingale w.r.t.  $\mathbb{P}$ , then

$$\tilde{M}_t := M_t - \langle M, L \rangle_t$$

is a local martingale w.r.t.  $\mathbb{Q}$  and

$$\langle M \rangle_t = \langle \tilde{M} \rangle_t.$$

Moreover, if  $M$  is a BM w.r.t. to  $\mathbb{P}$ , then  $\tilde{M}$  is a BM w.r.t.  $\mathbb{Q}$ .

**Remark.** In applications, given  $\mathbb{P}, (Z_t)_{t \geq 0}$  a positive continuous martingale, define  $\mathbb{Q}$  on  $\mathcal{F}_{\infty} = \bigcup_{t \geq 0} \mathcal{F}_t$  s.t.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = Z_t$$

$Z$  is uniformly integrable  $\iff \mathbb{Q} \ll \mathbb{P}$ . **Problem:** In applications,  $Z$  is not necessarily uniformly integrable.

$\implies$  restrict to  $[0, T] \implies$  all fine.

$\mathbb{Q} \rightarrow \mathbb{Q}_T$  as in the last semester.

**Example.** Let  $\gamma \in \mathbb{R}^d$ ,  $(B_t)_{t \geq 0}$  standard BM.

Let  $L_t := \gamma \cdot B_t$  and  $Z_t = \exp(L_t - \frac{1}{2} \langle L, L \rangle_t) = \exp(\gamma \cdot B_t - \frac{1}{2} |\gamma|^2 t)$

**Remark.**  $\lim_{M \rightarrow \infty} \sup_{t \geq 0} \mathbb{E}(|Z_t| 1_{|Z_t| > M}) \neq 0$ .

Define  $\mathbb{Q}$  on  $\mathcal{F}_\infty$  s.t.  $\tilde{B}_t^k = B_t^k - \langle L, B^k \rangle = B_t^k - \gamma^k t$  is a BM with drift. Show:  $\mathbb{Q} \ll \mathbb{P}$ .

Construct  $\mathbb{Q}$  via  $Z$

$$A = \left\{ \lim_{t \rightarrow \infty} \frac{\tilde{B}_t + \gamma t}{t} = \gamma \right\} \in \mathcal{F}_\infty$$

but:  $\tilde{B}$  is a BM w.r.t.  $\mathbb{Q} \implies \mathbb{Q}(A) = 1$ ,  $\tilde{B}$  is a BM with drift  $-\gamma$  w.r.t.  $\mathbb{P}(A) = 0$ .

Start of lecture 06  
(30.04.24)

### 2.1.1 Drift transformation of SDE

We once again start with

$$\star = \begin{cases} dX_t &= b(t, X_t) \\ X_0 &= x_0 \end{cases}$$

with drift  $b$  continuous.

**Goal:** Get a weak solution of  $\star$ .

Let  $(X_t)_{t \geq 0}$  be a BM in  $(\Omega, \mathcal{F}(\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ ,  $X_0 = x_0$ .

**Assume:**  $Z_t := \exp \left( \underbrace{\int_0^t b(s, X_s) dX_s}_{=L_t} - \frac{1}{2} \int_0^t |b(s, X_s)|^2 ds \right)$  is a martingale w.r.t.  $\mathbb{P}$ .

**Remark.** The assumption holds if

$$|b(t, x)| \leq C(1 + \|x\|) \text{ for some } C.$$

$$\implies \mathbb{E}_{\mathbb{P}}(Z_t) = 1 \forall t \geq 0$$

By Girsanov

$$\tilde{X}_t = X_t - \langle L, X \rangle_t$$

is a  $\mathbb{Q}$ -BM.

But

$$d\langle L, X \rangle_t = b \cdot dX_t \cdot dX_t = b \cdot dt$$

$$\implies \tilde{X}_t = X_t - \int_0^t b(s, X_s) ds$$

$$\text{w.r.t. } \mathbb{Q} : dX_t = b(t, X_t) dt + \underbrace{d\tilde{B}_t}_{d\tilde{X}_t}$$

Generalization: Start with

$$\star \star dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t$$

where  $X, B$  are  $d$ -dimensional.

**Proposition 2.2.** Assume  $(X, B)$  is a weak solution of  $\star \star$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ . If

$$Z_t = \exp \left( \int_0^t c(s, X_s) dB_s - \frac{1}{2} \int_0^t c(s, X_s)^2 ds \right)$$

is a martingale w.r.t.  $\mathbb{P}$ ,  $\mathbb{Q} \ll \mathbb{P}$  and  $\mathcal{F}_t$  s.t.  $Z_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}} \text{Vert}_{\mathcal{F}_t}$ .

In practice this is surprisingly useful in practice!

this is an implicit condition for  $c$

Then  $(X_t)_{t \geq 0}$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$  is a weak solution of

$$dX_t = [b(t, X_t) + \sigma(t, X_t)c(t, X_t)dt] + \sigma(t, X_t)d\tilde{B}_t$$

where  $\tilde{B}$  is a  $d$ -dim BM.

*Proof.*

$$\begin{aligned} d\langle L, B \rangle_t &\stackrel{L_t = \int_0^t c(s, X_s)dB_s}{=} c(t, X_t)dt \\ \implies \tilde{B}_t &:= B_t - \langle L, B \rangle_t \text{ is a } \mathbb{Q}\text{-BM} \\ \implies dB_t &= c(t, X_t)dt + d\tilde{B}_t \end{aligned}$$

From  $\star\star$ :  $dX_t = b(t, X_t)dt + \sigma(t, X_t)c(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t = \star\star\star$  □

Measure	SDE	Generators
$\mathbb{P}$	$dX_t b \cdot dt + \sigma \cdot dB_t$	$\mathcal{L} \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l} (\sigma \sigma^\top)_{k,l} \frac{\partial^2}{\partial x_k \partial x_l}$
$\mathbb{Q}$	$[b + \sigma \cdot c]dt + \sigma d\tilde{B}_t$	$\tilde{\mathcal{L}} = \mathcal{L} + \sum_k \sum_l \sigma_{k,l} c_l \frac{\partial}{\partial x_k} = \mathcal{L} + c \sigma^\top \nabla$

$t \leq T$  for  $\mathbb{Q}_t$ .

## 2.2 Doob-h transform

1. From  $B_t \rightarrow B_t$  conditioned on  $\{B_1 = 0\}$  (measure zero set)
2. From  $B_t \rightarrow B_t$  conditioned on  $\{B_t \geq 0 \forall t \geq 0\}$  (measure zero set)
3. From  $B_t \rightarrow B_t$  conditioned on  $\{B_t \geq 0 \forall t \in [0, 1], B_1 = 0\}$

Essentially three cases in 1d

Let  $(X_t, B_t)_{t \geq 0}$  be a weak solution of

$$(\star) = \begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \text{ fixed} \end{cases}$$

Assume there exists  $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$  s.t.  $h > 0$  satisfying

$$\left(\frac{\partial}{\partial t} + \mathcal{L}\right)h = 0 \forall t \in [0, T], x \in \mathbb{R}^d$$

where  $\mathcal{L}$  is the generator of the SDE  $(\star)$ .

By theorem 1.14  $Z_t := h(t, X_t) = h(0, X_0) + \int_0^t (\sigma \nabla h)(s, X_s)dB_s$ .

$Z_t$  is a positive local martingale. Assume  $Z_t$  is a martingale.

W.l.o.g.  $Z_0 = 1$  (if not  $h \mapsto \frac{h}{h(0, X_0)}$ ).

$\implies d\mathbb{Q}_T = Z_T \cdot d\mathbb{P}$ . Let  $L_t$  s.t.  $Z_t = \exp(L_t - \frac{1}{2}\langle L \rangle_t)$ .

Girsanov, since  $B_t$  is a  $\mathbb{P}$ -BM,

$\tilde{B}_t := B_t - \langle L, B \rangle_t$  is a  $\mathbb{Q}_T$ -BM, where

$$\langle L, B \rangle_t = \sigma^\top \cdot \nabla \ln h \cdot dB_t$$

because of

$$dL_t = \frac{dZ_t}{Z_t} = \frac{\sigma^\top \cdot \nabla h \cdot dB_t}{h(t, X_t)} = \sigma^\top \cdot \nabla \ln h \cdot dB_t$$

And therefore  $dB_t = d\tilde{B}_t + \sigma^\top \nabla \ln h dt$ .

From  $(\star) \implies dX_t = \underbrace{(b(t, X_t) + \sigma(t, X_t)\sigma^\top(t, X_t)\nabla h(t, X_t))}_{:= \tilde{b}}dt + \sigma(t, X_t)d\tilde{B}_t$

**Proposition 2.3.** Let  $X$  be a weak solution of  $dX_t = b(t, X_t) + \sigma(t, X_t)dB_t$  on  $[0, T]$  under  $\mathbb{P}$ . Then under  $\mathbb{Q}_T$ ,  $X$  is a weak solution of the SDE

$$\begin{cases} dX_t &= \tilde{b}(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t \\ X_0 &= x_0 \end{cases}$$

where  $\tilde{b} = b + \sigma\sigma^\top \nabla \ln h$ .

**Remark.** In applications, often  $h(x) \not\equiv 0$  for all  $x$ !

E.g.:  $h(t, x) = x \implies (\frac{\partial}{\partial t} + \mathcal{L})h = 0$  ( $X_t = B_t + x_0$ ), but  $h(x) \leq 0$  for  $x \leq 0$ .

In this case first do the construction on  $[0, \tau]$ , where  $\tau := \inf\{t > 0 \mid B_t = 0\}$

$\implies$  ok for  $B_t^\tau \xrightarrow{\text{magically}}$  the new SDE has drift:

$$\frac{\partial}{\partial x} \ln(h(x)) = \frac{1}{h(x)} \frac{\partial h(x)}{\partial x} = \frac{1}{x}$$

We realize there is no problem in the new measure

**Example.** If  $X_t = B_t$  and  $h(t, \cdot) = e^{\gamma x - \frac{1}{2}|\gamma|^2 t}$

$$\begin{aligned} \implies dX_t &= \nabla \ln h(t, X_t)dt + dB_t \\ &= \nabla(\gamma \cdot x)dt + dB_t \\ &= \gamma dt + dB_t \end{aligned}$$

$\implies X_t$  is a BM with drift  $\gamma$  w.r.t  $\mathbb{Q}$ .

## 2.3 Diffusion Bridges

Consider a Markov process  $(X_t)_{t \geq 0}$  which is a diffusion starting from  $X_0 = x_0 \in \mathbb{R}^d$ .

We want to condition on the event  $\{X_T = y\}$  for some given  $T > 0, y \in \mathbb{R}^d$ .

**Goal:** Find,  $\forall y \in \mathbb{R}^d, \mathbb{Q}^y$  which is the conditional measure on  $\{X_T = y\}$ .

Assume:  $(X_t)_{t \geq 0}$  has transition density  $P$  s.t.

$$\forall 0 \leq s \leq t \leq T, \mathbb{P}(X_t \in dz \mid X_s = x) = p(s, x; t, z)dz.$$

$X_t$  solves

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \end{cases}.$$

Define  $h^y(s, x) := \frac{p(s, x; T, y)}{p(0, x_0; T, y)}$  for  $s \in [0, T), x \in \mathbb{R}^d$ .

**Remark.**  $s < T$  since otherwise we get ingeneral singularities.

**Lemma 2.4.** Let  $Z_t^y := h^y(t, X_t)$  is a martingale.

*Proof.* Let  $0 \leq t < T$ :

$$\begin{aligned} \mathbb{E}(Z_t^y \mid \mathcal{F}_s) &= \mathbb{E}(h^y(t, X_t) \mid \mathcal{F}_s) \\ &\stackrel{\text{MP}}{=} \mathbb{E}(h^y(t, X_t) \mid X_s) \\ &= \int_{\mathbb{R}^d} h^y(t, x) p(s, X_s; t, x) dx \\ &= \int_{\mathbb{R}^d} \frac{p(t, x; T, y)}{p(0, x_0; T, y)} p(s, X_s; t, x) dx \\ &= \frac{1}{p(0, x_0; T, y) \int_{\mathbb{R}^d} p(s, X_s; t, x) p(t, x; T, y) dx} \\ &\stackrel{\text{Chapman-Kolmogorov}}{=} Z_s^y = \end{aligned}$$

$$\square \quad h^y(0, x_0) = 1$$

Start of lecture 07  
(02.05.24)

**Added remark.** *Goal:* Find a family  $(\mathbb{Q}^y)_{y \in \mathbb{R}^d}$  s.t. for  $A \in \mathcal{F}_T$ :

$$\mathbb{P}(A) = \mathbb{E}(\mathbb{P}(A|X_T)) \stackrel{?}{=} \mathbb{E}(\mathbb{Q}^{X_T}(A))$$

and  $\mathbb{Q}^y(\lim_{t \uparrow T} X_t = y) = 1$

**Lemma 2.5.** We can take

$$\mathbb{Q}^y(A) = \mathbb{E}(1_A \cdot h^y(s, X_s)) \forall A \in \mathcal{F}_s$$

for  $s \in [0, T]$ .

*Proof.* For all  $A \in \mathcal{F}_s$ ,  $g$  bounded measurable:

$$\begin{aligned} \mathbb{E}(1_A g(X_T)) &= \mathbb{E}(1_A \mathbb{E}(g(X_T) | \mathcal{F}_s)) \\ &\stackrel{\text{MP}}{=} \mathbb{E}(1_A \mathbb{E}(g(X_T) | X_s)) \\ &= \mathbb{E} \left( 1_A \int_{\mathbb{R}^d} p(s, X_s; T, x) g(x) dx \right) \\ &= \int_{\mathbb{R}^d} dx g(x) \mathbb{E}(1_A p(s, X_s; T, x)) \end{aligned}$$

In particular,

$$\mathbb{P}(A) = \mathbb{E}(1_A) = \int_{\mathbb{R}^d} dx \mathbb{E}(1_A \cdot p(s, X_s; T, x))$$

But

$$\begin{aligned} \int_{\mathbb{R}^d} dx p(0, X_0; T, x) \mathbb{Q}^x(A) &= \int_{\mathbb{R}^d} p(0, x_0; T, x) \underbrace{\mathbb{E}(1_A h^x(s, X_s))}_{= \int dz p(0, x_0; s, z) h^x(s, z) 1_A} \\ &= \int_{\mathbb{R}^d} \mathbb{E}(1_A p(s, X_s; T, x)) \\ &= \mathbb{P}(A). \end{aligned}$$

□

**Remark.**  $Z_t^y = h^y(t, X_t) = h(0, X_0) + \underbrace{\text{martingale}}_{=0} + \int_0^t (\partial_t + \mathcal{L})h^y(s, X_s) ds.$

One can verify:  $(\partial_t + \mathcal{L})h^y(t, X_t) = 0 \iff (\partial_t + \mathcal{L})H^y(t, X_t) = 0$  for  $H^y(t, x) = p(t, x; T, y).$

$$\begin{aligned} \frac{\partial}{\partial t} H^y(t, x) &= \lim_{\epsilon \downarrow 0} \frac{p(t, x; T, y) - p(t - \epsilon, x; T, y)}{\epsilon} \\ &= \lim_{\epsilon \downarrow 0} \frac{p(t, x; T, y) - \int \underbrace{p(t - \epsilon, x; t, z)}_{= e^{\epsilon \mathcal{L}}(x, z) = \delta_{x-z} + \epsilon \mathcal{L}(x, z)} p(t, z; T, y) dz}{\epsilon} \\ &= -(\mathcal{L}H^y)(t, x) \end{aligned}$$

$\implies$  this is a Doob-h transform with  $h = h^y$ .

$$\frac{d\mathbb{Q}^y}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t^y.$$

**Corollary 2.6.** Assume  $(t, x) \mapsto p(t, x; T, y)$  is  $C^{1,2}([0, T] \times \mathbb{R}^d)$

$\implies$  The Doob h-transform of the original process under  $\mathbb{Q}^y$  satisfies the SDE:

$$dX_t = \tilde{b}(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t$$

where  $\tilde{B}_t$  is a  $\mathbb{Q}^y$ -BM and  $\tilde{b}(t, X_t) = b(t, X_t) + (\sigma \sigma^\top \nabla \ln h^y)(t, X_t).$



**Remark.** Take  $z \neq y$  and  $\forall 0 < \epsilon < |z - y|$ :

$$\begin{aligned}\mathbb{Q}^y(\{|X_t - z| \leq \epsilon\}) &= \mathbb{E}\left(1_{|X_t - z| \leq \epsilon} \frac{p(t, X_t; T, y)}{p(0, x_0; T, y)}\right) \\ &= \int dx 1_{|x - z| \leq \epsilon} p(0, x_0, t, x) \frac{p(t, x; T, y)}{p(0, x_0; T, y)} \\ &\xrightarrow{t \rightarrow T} 0\end{aligned}$$

$\Rightarrow \mathbb{P}(\lim_{t \uparrow T} X_t = y) = \mathbb{P}(X_t = y) = 0$ , but  $\mathbb{Q}^y(\lim_{t \uparrow T} X_t = y) = 1$   
 $\Rightarrow \mathbb{Q}^y$  is singular w.r.t.  $\mathbb{P}$ .

**Example.**  $b = \gamma \in \mathbb{R}^d, \sigma = 1, X_0 = 0$  and  $T = 1$ . Under  $\mathbb{P}$ :

$$\begin{cases} dX_t = \gamma dt + dB_t \\ X_0 = 0 \end{cases}$$

Under  $\mathbb{Q}^y$ ?

$$h^y(t, x) = \frac{p(t, x; 1, y)}{p(0, x_0, 1, y)} = fct(\gamma, y) \cdot \exp\left(-\frac{(y-x)^2}{2(1-t)}\right) \exp((y-x)\gamma)$$

$$\Rightarrow \ln(h^y(t, x)) = \ln f(\gamma, y) - \frac{(y-x)^2}{2(1-t)} + (y-x)\gamma \Rightarrow \nabla \ln h^y(t, x) = \frac{y-x}{1-t} - \gamma.$$

Under  $\mathbb{Q}^y$ :

$$\begin{aligned}dX_t^k h^y &= \left(\gamma_k + \frac{y_k - X_t^k}{1-t}\right) dt + d\tilde{B}_t^k \\ &= \frac{Y_k - X_t^k}{1-t} dt + d\tilde{B}_t^k\end{aligned}$$

Independent of  $\gamma$ !

for  $k = 1, \dots, d$

**Lemma 2.7.** Let  $p$  be the transition density of  $X$  w.r.t.  $\mathbb{P}$  and  $p^h$  the transition density of  $X$  w.r.t.  $\mathbb{Q}^y$ .

Let  $h^y(t, x) = \frac{p(t, x; T, y)}{p(0, x_0; T, y)}$ .

Useful in practice to sample!

$$\Rightarrow p^h(s, x; t, z) = \frac{1}{h^y(s, x)} p(s, x; t, z) h^y(t, z)$$

Notice how the normalizations inside of  $h^y$  cancel!

**Added remark.** This is the first time my numbering is different from the handwritten notes, as they contain two environments numbered 2.5.

*Proof.*

$$\begin{aligned}p^h(s, x; t, z) dz &= \mathbb{P}(X_t \in dz \mid X_s = x, X_T = y) \\ &\stackrel{0 \leq s \leq t}{=} \frac{p(0, x_0; s, x) p(s, x; t, z) p(t, z; T, y) dz}{p(0, x_0; s, x) p(s, x; T, y)} \\ &= p(s, x; t, z) \frac{p(t, z; T, y)}{p(0, x_0; T, y)} \frac{1}{\frac{p(s, x; T, y)}{p(0, x_0; T, y)}} dz\end{aligned}$$

□

## 2.4 Brownian motion conditioned to stay positive forever

**Goal:**  $(B_t)_{t \geq 0}$  with  $B_0 = x_0 \geq 0$  and condition on

$$\{B_t \geq 0 \forall t \geq 0\}$$

$T_x = \inf\{t \geq 0 \mid B_t = x\}$  and take  $0 < x_0 < R$ .

$T_0 \wedge T_R = T_R$ .

Let  $\tilde{T}_R := T_0 \wedge T_R$ . Define the event  $E_R = \{B_{\tilde{T}_R} = R\} = \{T_0 \wedge T_R = T_R\}$ .

One verifies  $\mathbb{P}(E_R) = \frac{x_0}{R} \in (0, 1)$ .

$\mathbb{P}(E_R) = \mathbb{E}(1_{B_{\tilde{T}_R}^{x_0}}) = \underbrace{\mathbb{E}(f(B_{\tilde{T}_R}^{x_0}))}_{u(x_0)}$ , where

$$f(x) = \begin{cases} 1 & x = R \\ 0 & x = 0 \end{cases}.$$

$u$  has a link to the PDE:

$$\begin{cases} \frac{1}{2} \frac{\partial^2}{\partial x^2} u(x) = 0 & x \in (0, R) \\ u(x) = f(x) & x \in \{0, R\} \end{cases}$$

By Theorem 11.6 of the last semester

Define the conditional probability

later  $R \rightarrow \infty$

$$Q^R(A) := \frac{\mathbb{P}(A \cap E_R)}{\mathbb{P}(E_R)}$$

**Lemma 2.8.** Let  $h(x) := \frac{\mathbb{P}_x(E_R)}{\mathbb{P}_{x_0}(E_R)} = \frac{x}{x_0}$ , where  $\mathbb{P}_x$  is the unconditional law for  $B_0 = x$ .

$$Z_t = h(B_{t \wedge \tilde{T}_R})$$

is a non-negative martingale.

*Proof.*

$$\begin{aligned} \mathbb{Q}^R(A) &\stackrel{A \in \mathcal{F}_s}{=} \frac{\mathbb{E}(1_A \mathbb{E}(1_{E_R} \mid \mathcal{F}_s))}{\mathbb{P}(E_R)} = \frac{\mathbb{E}(1_A \mathbb{E}(1_{E_R} \cdot 1_{\tilde{T}_R > s} \mid \mathcal{F}_s)) + \mathbb{E}(1_A \mathbb{E}(1_{E_R} 1_{\tilde{T}_R \leq s} \mid \mathcal{F}_s))}{\mathbb{P}(E_R)} \\ &= \frac{1}{\mathbb{P}(E_R)} \left[ \mathbb{E}(1_A 1_{\tilde{T}_R > s} \underbrace{\mathbb{E}(1_{E_R} \mid \mathcal{F}_s)}_{\mathbb{P}_{B_s}(E_R)}) + \mathbb{E}(1_A \underbrace{1_{E_R}}_{1_{B_{\tilde{T}_R} = R}} 1_{\tilde{T}_R \leq s}) \right] \\ &= \mathbb{E}(1_A 1_{\tilde{T}_R > s} h(B_s)) + \mathbb{E}(1_A 1_{\tilde{T}_R \leq s} h(B_{\tilde{T}_R})) \\ &= \mathbb{E}(1_A h(B_{s \wedge \tilde{T}_R})) \end{aligned}$$

$$\implies \forall A \in \mathcal{F}_s : \mathbb{Q}^R(A) = \mathbb{E}_{\mathbb{P}}(1_A h(B_{s \wedge \tilde{T}_R}))$$

$$\implies h(B_{s \wedge \tilde{T}_R}) = \frac{d\mathbb{Q}^R}{d\mathbb{P}} \Big|_{\mathcal{F}_s} \implies h \in (0, \frac{R}{x_0}) \text{ is a martingale (See construction of Girsanov)}$$

□

Note that we can write  $Z_t^R := h(B_{s \wedge \tilde{T}_R}) = \exp(L_t - \frac{1}{2} \langle L \rangle_t)$  by choosing

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$$L_t = \int_0^{\tilde{T}_R} \frac{h'(B_s)}{h(B_s)} dB_s$$

since

$$dL_t = \frac{dZ_t^R}{Z_t^R} = \begin{cases} \frac{h'(B_t)dB_t}{h(B_t)} & t < \tilde{T}_R \\ 0 & t \geq \tilde{T}_R \end{cases}$$

**Proposition 2.9.** Under the measure  $\mathbb{Q}^R$ ,  $(B_t)_{t \geq 0}$  solves the SDE

$$dB_t = \frac{1_{t \leq \tilde{T}_R}}{B_t} dt + dW_t$$

for some  $(W_t)_{t \geq 0}$  a  $\mathbb{Q}^R$ -BM.

*Proof.* Apply Girsanov in which

$$d\langle B, L \rangle_t = dB_t \cdot dL_t \stackrel{h'(x)=\frac{1}{x_0}}{=} \frac{1}{B_t} dt \quad \square$$

Our goal was to condition the BM to stay positive forever. Sofar we have conditioned it to reach the level  $R$  before the level 0.

**Remark.**

$$\begin{aligned} \mathbb{Q}^R(T_0 < T_R) &= \mathbb{E}(1_{T_0 < T_R} h(B_{T_R \wedge T_0})) \\ &= \mathbb{E}(1_{T_0 < T_R} h(B_{T_0})) \\ &= \mathbb{E}(1_{T_0 < T_R} \underbrace{h(0)}_{=0}) = 0 \end{aligned}$$

$\implies$  under the new measure  $\mathbb{Q}^R$ , the BM indeed does not reach 0 before  $R$  as we wanted.

Finally we wan to take  $R \rightarrow \infty$ . It should

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**Example.** Discrete time M.C. with transition probability  $P$

aperiodic and irreducible:  $\exists n_0 \forall n > n_0 (P)_{i,j}^n > 0 \implies \exists |\lambda_0| \leq 1, |\lambda_1| > |\lambda_2| > |\lambda_3|$ :

$$(P)^n \underbrace{\varphi_0(i)}_{>0} = \lambda_0 \varphi_0(i)$$

$$\lim_{n \rightarrow \infty} \underbrace{P^n}_{(1+L)} = (\varphi_0(1), \dots, \varphi_0(d))$$

where  $(1+L) \rightarrow e^{tL}$ , for which the real eigenvalues are positive.

## 2.5 Diffusion conditioned to stay in a domain

Domain  $D \subset \mathbb{R}^d$ : bounded, open, connected.

Diffusion with generator

$$L = \frac{1}{2} \sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$$

$\iff$  SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \frac{1}{2}\sigma(t, X_t)dB_t \\ X_0 = x_0 \end{cases}$$

$a = \sigma \sigma^\top$ .

Assume:  $b, \sigma$  are continuous,  $\sigma \in C^1$  and let  $\tau_D = \int \{t \geq 0 | X_t \notin D\}$

**Key assumption** in Pinsky's paper:

$$(a) \quad \mathbb{P}_x(\tau_D > t) \in \mathcal{C}^2(D)$$

(b)

$$\begin{cases} \mathbb{P}_x(\tau_D > t) \\ \nabla \mathbb{P}_x(\tau_D > t) = C_1 \nabla \varphi_0(x) e^{-\lambda_0 t} + o(e^{-\lambda_0 t}) \end{cases} = C_1 \varphi_0(x) e^{-\lambda_0 t} + o(e^{-\lambda_0 t})$$

where

$$\begin{cases} -L\varphi_0(x) = \lambda_0 \varphi_0(x) & x \in D \\ \varphi_0(x) = 0 & x \in \partial D \end{cases}$$

and  $\lambda_0$  is the smallest eigenvalue of  $-L$ . Which means  $\mathbb{P}(\tau_d > t) \xrightarrow{t \rightarrow \infty} 0$

in the non symmetric case  
take the real part first

Want: Condition  $X$  to stay in  $D$  forever.

(1) Condition  $X_t$  to  $\{X_t \in D : 0 \leq t \leq T\}$ : For all  $A \in \mathcal{F}_T$ : define a measure

$$\mathbb{Q}^T(A) := \frac{\mathbb{P}_{x_0}(A \cap \{\tau_D > T\})}{\mathbb{P}_{x_0}(\tau_D > T)} = \frac{\mathbb{E}_{x_0}(1_A \cdot 1_{\tau_D > T})}{\mathbb{E}_{x_0}(1_{\tau_D > T})}$$

**Lemma 2.10.**  $\forall s < T, A \in \mathcal{F}_s$ ,

$$\mathbb{Q}^T(A) = \mathbb{E}_{x_0}(1_A \cdot Z_s^T)$$

where  $Z_s^T = \frac{g^{T-s}(X_{s \wedge \tau_D})}{g^T(x_0)}$  with  $g^t(x) := \mathbb{P}_x(\tau_D > t)$ .

*Proof.* Let  $A \in \mathcal{F}_s, s < T$ :

$$\begin{aligned} \mathbb{Q}^T(A) &= \frac{\mathbb{E}_{x_0}(1_A 1_{\tau_D > T})}{g^T(x_0)} = \frac{\mathbb{E}_{x_0}(1_A \mathbb{E}_{x_0}(1_{\tau_D > T} | \mathcal{F}_s))}{g^T(x_0)} \\ &\stackrel{\text{M.P.}}{=} \frac{\mathbb{E}_{x_0}(1_A \mathbb{E}_{x_0}(1_{\tau_D > T} | X_s))}{g^T(x_0)} \\ &= \frac{1}{g^T(x_0)} \left[ \mathbb{E}_{x_0}(1_A \underbrace{\mathbb{E}_{x_0}(1_{\tau_D > T} 1_{\tau_D > s} | X_s)}_{g^{T-s}(X_s)}) + \mathbb{E}_{x_0}(1_A \underbrace{\mathbb{E}_{x_0}(1_{\tau_D > T} 1_{\tau_D < s} | X_s)}_{=0}) \right] \\ &= \frac{1}{g^T(x_0)} (1_A g^{T-s}(X_{s \wedge \tau_D})) \end{aligned}$$

□

**Lemma 2.11.**  $Z_0^T = 1$  and  $(Z_s^T)_{s \in [0, T]}$  is a martingale.

*Proof.* By lemma 2.10  $\implies Z_s^T = \frac{d\mathbb{Q}^T}{d\mathbb{P}} |_{\mathcal{F}_s} \rightarrow$  is a martingale. □

**Remark.** By construction,  $\mathbb{Q}^T(\tau_D \leq T) = \frac{1}{g^T(x_0)} \mathbb{E}_{x_0}(1_{\tau_D \leq T} 1_{\tau_D > t}) = 0$

Assume that  $g^t(x)$  is  $C^{1,2}$  (1 in time, 2 in space)  $\implies$  apply Itô and doob transform gives:

**Proposition 2.12.** Let  $X$  be a weak solution of  $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$  under  $\mathbb{P}$ .  
 $\implies X$  is a weak solution of

$$dX_t = \left( b(t, X_t) + \frac{a(t, X_t) \nabla g^{T-t}(X_t)}{g^{T-t}(X_t)} \right) dt + \sigma(t, X_t) d\tilde{B}_t, 0 \leq t \leq T$$

under  $\mathbb{Q}^T$  provided  $g^t(x) > 0 \forall x \in D, t \geq 0$ .

What happens in the  $T \rightarrow \infty$  limit?

By assumption (b)

$$\lim_{T \rightarrow \infty} \frac{\nabla g^{T-t}(x)}{g^{T-t}(x)} = \frac{\nabla \varphi_0(x)}{\varphi_0(x)}$$

**Remark.** If  $g \in C^{1,2}$ , it satisfies the parabolic PDE

$$(\star) \begin{cases} \frac{\partial}{\partial t} g^t(x) = Lg^t(x) & x \in D \\ g^t(x) = 0 & x \in \partial D \end{cases}$$

Apply Theorem 11.5 from the WS with  $A = L, u(t, x = g^t(x))$ :  $u(0, x) = \mathbb{E}_x(1_{\tau_D > 0}) = 0, u(t, x) = \mathbb{E}_x(1_{\tau_D > t}) = 0 \forall t, x \in \partial D$ .

Pinsky proved that  $\lim_{T \rightarrow \infty} \mathbb{Q}^T = Q$  weak and that under  $Q$  the process satisfies

$$dX_t = \left[ b(t, X_t) + a(t, X_t) \frac{\nabla \varphi_0(X_t)}{\varphi_0(X_t)} \right] dt + \sigma(t, X_t) dW_t.$$

Why is  $\mathbb{P}_x(\tau_D > t) = C_1 \varphi_0(x) e^{-\lambda_0 t} + o(e^{-\lambda_0 t})$

Their operator is not the same as ours. The only difference is a drift term.

**Added remark.** This is not the case for BM conditioned to stay in  $D = (0, \infty)$ . Reason: Spectrum of  $-L$  is  $\mathbb{R}_+$

Since it satisfies the PDE  $(\star)$ ,  $g^t(x) = (e^{tL})g^0(x)$ .

if the spectrum of  $-L$  is discrete with eigenvalues  $0 \leq \lambda_0 < \lambda_1 \leq \dots$ , where there is a spectral gap between  $\lambda_0$  and  $\lambda_1$ :  $\exists$  eigenfunctions  $\varphi_0(x), \varphi_1(x), \dots$ , normalized as

$\|\varphi_n\|_{L^2(D)} = 1 \implies +L\varphi_n(x) = -\lambda_n\varphi_n(x) \implies$  we can choose  $\varphi_0, \varphi_1, \dots$  to be orthonormal:  
 $(\varphi_i, \varphi_j)_{L^2(D)} = \delta_{ij}$ .

$\implies 1 = \sum_{n \geq 0} \varphi_n \varphi_n^*$ , since  $\varphi_n \varphi_n^*$  is the projection onto the space generated by  $\varphi_n$ :

$$\varphi_n \varphi_n^* f = \varphi_n (\varphi_n, f)_{L^2(D)}$$

$$f(L)\varphi_n = f(-\lambda_n)\varphi_n$$

$$\begin{aligned} g^t(x) &= (e^{tL})g^0(x) = e^{tL}1g^0(x) = \sum_{n \geq 0} e^{tL}\varphi_n(x)\varphi_n^*g^0 = \sum_{n \geq 0} e^{-\lambda_n t}\varphi_n(x)(\varphi_n, g^0) \\ &= (\varphi_0, g^0)\varphi_0(x)e^{\lambda_0 t} + \underbrace{e^{-\lambda_0 t} \sum_{n \geq 0} e^{(\lambda_0 - \lambda_n)t}\varphi_n(x)(\varphi_n, g^0)}_{o(e^{-\lambda_0 t})} \end{aligned}$$

**Example 2.13.** One dimension.

$L = \frac{1}{2} \frac{d^2}{dx^2}$  and  $D = [0, R]$ .

Solve  $-L\varphi(n)(x) = \lambda_n\varphi_n(x)$  with  $\varphi_n(x) = 0$  for  $x \in \{0, R\}$ .

Solutions  $\varphi_n(x) = C_1 \sin\left(\frac{\pi x}{R} \cdot (n+1)\right)$  and  $\lambda_n = \left(\frac{\pi(n+1)}{R}\right)^2 \implies \lambda_0 = \frac{\pi^2}{R^2}$ .

For  $x \in (0, R)$ :

$$\mathbb{P}_x(\tau > t) \approx \tilde{C}_1 \varphi_0(x) e^{-\frac{\pi^2 t}{R^2}}$$

$$\implies dX_t = \frac{\pi \cos(\frac{\pi X_t}{R})}{R \sin(\frac{\pi X_t}{R})} dt + dW_t$$

as  $X_t \rightarrow 0$  (or  $X_t \rightarrow R$ ), drift  $\approx \frac{1}{x_t}$  or  $(\frac{-1}{R-X_t})$ .

Also: formally  $R \rightarrow \infty$  in the SDE, we get  $dX_t = \frac{dt}{X_t} + dW_t$ , which is the SDE we derived for BM conditioned to stay  $> 0$  forever.

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**Example 2.14** (Brownian Motion in a Weyl chamber). The **Weyl chamber** is defined as

$$W^d = \{x \in \mathbb{R}^d \mid x_1 < x_2 < \dots < x_d\}.$$

Given a  $d$  dim. Brownian motion  $(B_t)_{t \geq 0}$  with  $B_0 \in W^d$  what is the sde of this BM conditioned on staying in  $W^d$  forever.

What is the harmonic function vanishing at  $\partial W^d$ ?

**Lemma 2.15.**  $h(x) := \prod_{1 \leq k < l \leq d} (x_l - x_k)$  is harmonic and satisfies

$$\frac{1}{2} \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2} h(x) = 0, x \in W^d$$

and  $h(x) = 0$  for  $x \in \partial W^d$ ,  $h(x) > 0$  for  $x \in W^d$ .

**Remark.**  $h(x) = \det(x_k^{l-1})$

*Proof.* Last two properties are clear.  $h$  is a polynomial, antisymmetric in each pair  $x_l - x_k$  and has lowest possible power.

$\Delta h(x)$  is still antisymmetric, but with lower power  $\implies \Delta h(x) = 0$ . □

The BM conditioned to stay in the Weyl chamber will satisfy the SDE:

$$dX_t^k = \sum_{l \neq k} \frac{dt}{X_t^l - X_t^k} + dB_t^k, \quad 1 \leq k \leq d$$

## 2.6 Stationary distribution for diffusions

let  $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ , where  $b, \sigma$  are time independent with generator

$$L = \sum_{k=1}^d b_k(x) \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l=1}^d a_{k,l}(x) \frac{\partial^2}{\partial x_k \partial x_l}$$

**Definition 2.16.** A probability measure  $\mu$  stationary (or invariant) if for all  $f \in C_0^\infty(\mathbb{R}^d)$

or only test schwarzfunctions

$$\int_{\mathbb{R}^d} (Lf)(x) \mu(dx) = 0$$

Assume that  $\mu \ll \text{Lebesgue}$ , i.e.,  $\mu(dx) = \rho(x)dx$  for some positive function  $\rho(x) \in C^2$  with  $\int \rho(x)dx = 1$ .

**Lemma 2.17.**  $\mu$  is stationary with density  $\rho$

$$\iff L^* \rho(x) = 0 \text{ almost everywhere}$$

this also assumes  $b$  to be differentiable!

where  $L^*$  is the adjoint of  $L$ , given by

$$L^* \rho(x) = \frac{1}{2} \sum_{k,l=1}^d \frac{\partial^2}{\partial x_l \partial x_l} (a_{k,l}(x) \rho(x)) - \sum_{k=1}^d \frac{\partial}{\partial x_k} (b_k(x) \rho(x))$$

*Proof.*

$$\begin{aligned} \int_{\mathbb{R}^d} (Lf)(x) \rho(x) dx &= 0 \\ &= \int_{\mathbb{R}^d} dx_1, \dots, dx_d \rho(x) \left( \frac{1}{2} \sum_{k,l=1}^d \frac{\partial^2}{\partial x_k \partial x_l} f(x) + \sum_{k=1}^d b_k(x) \frac{\partial}{\partial x_k} f(x) \right) \\ &= \int_{\mathbb{R}^{d-1}} \prod_{l \neq k} dx_l \underbrace{\int_{\mathbb{R}} dx_k \rho(x) b_k(x) \frac{\partial}{\partial x_k} f(x)}_{\stackrel{\text{I.B.P.}}{=} - \int_{\mathbb{R}} dx_k \frac{\partial}{\partial x_k} (\rho(x) b_k(x)) \cdot f(x)} \end{aligned}$$

□

**Example 2.18** (1-dim. diffusion). Let  $L = b(x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2}$  for all  $x \in \mathbb{R}$ .

$$\implies L^* \rho(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\rho(x) \sigma^2(x)) - \frac{\partial}{\partial x} (\rho(x) b(x)) \stackrel{\text{wanted}}{=} 0$$

$$\implies \exists c_1 \text{ s.t. } \underbrace{\frac{1}{2} \frac{\partial}{\partial x} (\rho(x) \sigma^2(x))}_{=g(x)} - \underbrace{\rho(x) b(x)}_{=g(x) \frac{b(x)}{\sigma^2(x)}} = c_1.$$

Let  $s(x) = \int_{x_0}^x dy e^{-\int_{x_0}^y dz \frac{2b(z)}{\sigma^2(z)}}$  be the scale function.

The equation for  $c_1$  is equivalent to

$$s'(x) \sigma^2(x) \rho(x) = c_2 + 2c_1 s(x)$$

for some constant  $c_2, s'(x) e^{-\int_{x_0}^x dz \frac{2b(z)}{\sigma^2(z)}} = > 0$ .

$s$  satisfies  $\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0$ , for the Bessel process  $d = 2$ ,  $s(x) = \ln(x)$   
If  $s(\mathbb{R}) = \mathbb{R}$ , then  $c_1 = 0$ , which implies

$$\rho(x) = \frac{c_2}{\sigma^2(x)} \underbrace{e^{\int_{x_0}^x \frac{2b(y)}{\sigma^2(y)} dy}}_{\frac{1}{s'(x)}}$$

which satisfies positivity.

**Lemma 2.19.** If  $S(\mathbb{R}) = \mathbb{R}$ , then there exists stationary measure with density  $\rho(x)$  as described above.

### Counterexample:

- $\sigma = 1$ ,  $x_0 = 0$ ,  $b(z) = b > 0$ .

$$\implies s(x) = \frac{1-e^{-2bx}}{2b} \rightarrow s'(x) = e^{-2bx}.$$

- $s(-\infty) = -\infty$
- $s(\infty) = \frac{1}{2b}$

$\implies$  Argument for  $c_1 = 0$  does not work.

$$\implies \rho(x) = \tilde{c}_1 + \tilde{c}_2 e^{2bx}$$

which can't be a density, because if at least  $\tilde{c}_1$  or  $\tilde{c}_2$  are  $\neq 0 \implies$  it is not integrable.

**Example 2.20.** Let  $Lf(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}f(x) + b(x)\frac{\partial}{\partial x}f(x)$  where  $b(x) = \frac{\partial}{\partial x} \ln h(x)$  for some  $h(x) > 0$ . Assume  $h$  is normalized as  $\int_{\mathbb{R}} (h(x))^2 dx = 1$ .  $\implies$  **Claim:** The stationary density of the process with generator  $L$  is given by  $\rho(x) = (h(x))^2$ .  
Verify the claim:

Implicitly assumes  $h \in L^2(\mathbb{R})$

$$\begin{aligned} L^* \rho(x) &\stackrel{?}{=} 0 \\ &= \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho(x) - \frac{\partial}{\partial x} (b(x) \rho(x)) \\ &= \frac{1}{2} \frac{\partial^2}{\partial x^2} (h(x))^2 - \frac{\partial}{\partial x} \left( \frac{h'(x)}{h(x)} h(x)^2 \right) \\ &= \frac{\partial}{\partial x} (h(x) h'(x)) - \frac{\partial}{\partial x} (h'(x) h(x)) = 0 \end{aligned}$$

Example ??  $\implies h(x) = c \sin(\frac{\pi x}{L})$

here  $L \in \mathbb{R}$  is not the operator!

$$\begin{aligned} \int_0^L h(x)^2 dx &= c^2 \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx = 1 \\ \implies c &= \sqrt{\frac{2}{L}} \implies \rho(x) = \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right). \end{aligned}$$

## 2.7 Uniqueness in law and path integral formula

Consider the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + dB_t & \text{in } \mathbb{R}^d \\ x_0 = x_0 \end{cases}$$

Assume

$$(\star) \forall T > 0 \int_0^T |b(s, X_s)|^2 ds < \infty \text{ a.s.}$$

.

**Goal:** Show uniqueness in law.

Consider any  $(X, \mathcal{B}, \mathbb{P})$  weak solution satisfying  $(\star)$  and define

$$\tau_n := \inf\{t \geq 0 \mid \int_0^t |b(s, X_s)|^2 ds \geq n\}$$

which, by  $(\star)$  goes to infinity.

For each  $n$ , define  $\mathbb{Q}^n$  :

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = e^{\underbrace{-\int_0^{\tau_n} b(s, X_s) dB_s - \frac{1}{2} \int_0^{\tau_n} |b(s, X_s)|^2 ds}_{L_{\tau_n}}}.$$

By Girsanov:  $\tilde{B}_t := B_t - \langle B, L \rangle_t = B_t + \int_0^{t \wedge \tau_n} b(s, X_s) ds$  is BM w.r.t.  $\mathbb{Q}^n$ , which implies  $X_t = \tilde{B}_t$  for all  $t \leq \tau_n$  w.r.t.  $\mathbb{Q}^n$ .

Let events of  $(X, B)$   $\mathcal{F}_T$  measurable for some time  $T$ : are in  $A_T$ .

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(1_{(X, B) \in A_T} 1_{T \leq \tau_n}) &= \mathbb{E}_{\mathbb{Q}^n} \left( 1_{(X, B) \in A_T} 1_{T \leq \tau_n} e^{\underbrace{\int_0^{\tau_n} b(s, X_s) dB_s}_{=dX_s - b(s, X_s) ds} + \frac{1}{2} \int_0^{\tau_n} |b(s, X_s)|^2 ds} \right) \\ &= \mathbb{E}_{\mathbb{Q}^n} \left( 1_{(X, B) \in A_T} 1_{T \leq \tau_n} e^{\int_0^{\tau_n} b(s, X_s) dX_s - \frac{1}{2} \int_0^{\tau_n} |b(s, X_s)|^2 ds} \right) \end{aligned}$$

$B_s$  is adapted to  $X_s \implies$  for some  $\Phi$ :  $B = \Phi(X) \implies$

$$\mathbb{E}_{\mathbb{Q}^n} \left( 1_{(X, \Phi(X)) \in A_T} 1_{T \leq \tau_n} e^{\int_0^{\tau_n} b(s, X_s) dX_s - \frac{1}{2} \int_0^{\tau_n} |b(s, X_s)|^2 ds} \right)$$

But: Law of  $X$  w.r.t.  $\mathbb{Q}^n$  is the law of BM. Take  $n \rightarrow \infty$ , then  $\mathbb{Q}^n \rightarrow$  Wiener measure,  $T \rightarrow \infty \implies$  we look at every possible event, but the probability does only depend on  $X$ !

$$\mathbb{P}((X, B) \in \mathcal{B}(\mathcal{E}^d \times \mathcal{E}^d))$$

One case where the path integral formula is also numerically stable:

Assume that  $b(x) = -\nabla V(x)$  for some smooth  $V(x)$  (time independent). This is called drift of **gradient type**.

Apply Itô to  $V(x)$ :

$$V(\omega_T) = V(\omega_0) + \int_0^T \nabla V(\omega_s) d\omega_s + \frac{1}{2} \int_0^T \Delta V(\omega_s) ds$$

$$\implies \int_0^T b(s, \omega_s) d\omega_s = - \int_0^T \nabla V(\omega_s) d\omega_s$$

and therefore

$$\mathbb{P}(X \in \tilde{A}_T) = \int_{C^d} W(d\omega) \underbrace{e^{V(\omega_0) - V(\omega_T) + \frac{1}{2} \int_0^T [\Delta V(\omega_s) - (\nabla V(\omega_s))^2] ds}}_{=: \Psi(\omega)}$$

One application:

Let  $X_0 = x, f : \mathbb{R}^d \rightarrow \mathbb{R}$

$$\underbrace{bE_x(f(X_T))}_{(T(t)f)(x)} = \int_{\mathbb{R}^d} f(\omega_T) \Psi(\omega) W_x(d\omega)$$

where  $W_x$  is the wiener measure starting from  $x$  and  $T(t)$  is the semigroup of  $X$ .

$$|(T(t)f)(x)| \leq \|f e^{-V}\|_{\infty} e^{-\frac{1}{2} \int_0^t \inf_{x \in \mathbb{R}^d} (|\nabla V(x)|^2 - \Delta V(x)) ds}$$

If  $\inf_{x \in \mathbb{R}^d} (|\nabla V(x)|^2 - \Delta V(x)) \geq 2\lambda > 0 \implies$

$$\|e^{-V(x)(T(t)f)}\| \leq \|e^{-V} f\|_{\infty} e^{-\lambda t} \rightarrow \text{exponential decay}$$



# Chapter 3:

## Local times, Itô-Tanaka formula, reflected Brownian Motion

### 3.1 Extension of Itô formula to convex functions

If  $f \in C^2(\mathbb{R})$ ,  $f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_s$

**Proposition 3.1.** *Let  $f$  be a convex function on  $\mathbb{R}$ ,  $X_t$  a (continuous) semimartingale, then  $f(X_t)$  is a semimartingale:  $\exists A_t^f$  such that  $\forall t \geq 0$*

$$f(X_t) = f(X_0) + \int_0^t \underbrace{f'_-}_{\text{left derivative of } f}(X_s) dX_s + A_t^f$$

*We make a choice for the left derivative, the same works for the right one, but with different processes*

**Remark.** *This extends directly to the case that  $f = f_1 - f_2$  with  $f_1, f_2$  convex (But with bounded variation and not necessarily increasing  $A_t^f$ )*

*Sketch of Proof.* Consider a function  $\rho(x)$  s.t.

- $\rho(x) \geq 0$ ,  $\rho$  smooth
- $\rho(x) = 0$  if  $x \notin [0, 1]$
- $\int_0^1 \rho(x) dx = 1$

For any  $n \in \mathbb{N}$  set

$$f_n(x) := \int_{\mathbb{R}} n\rho(ny) f(x-y) dy$$

Verify:

- $f_n$  is  $C^2(\mathbb{R})$ ;  $f_n'' \geq 0$  (since  $f$  is convex)
- $f_n'(x) = \int_{\mathbb{R}} n\rho(ny) \cdot f'_-(x-y) dy$
- $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$
- $f_n'(x) \xrightarrow{n \rightarrow \infty} f'_-(x)$

Itô formula to  $f_n(X_t)$ :

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s) dX_s + \underbrace{\frac{1}{2} \int_0^t f_n''(X_s) d\langle X \rangle_s}_{A_t^{f_n} \in \mathcal{A}_+^0}$$

left derivative since we do convolution and not correlation with  $\rho$

Using stopping times: If  $X = \underbrace{M_+}_{\in \mathcal{M}_{\text{loc}}^0} + \underbrace{A}_{\in \mathcal{A}}$

$$\forall m \geq 1 : T_m = \inf\{t \geq 0 \mid |X_t| + \langle M \rangle_t + \int_0^t |dA_s| \geq m\}$$

$$f_n(X_{t \wedge T_m}) = f(X_0) + \underbrace{\int_0^{t \wedge T_m} f'_n(X_s) dX_s}_{\xrightarrow{n \rightarrow \infty, \mathbb{P}} \int_0^{t \wedge T_m} f'_-(X_s) dX_s} + \underbrace{\frac{1}{2} \int_0^{t \wedge T_m} f''_n(X_s) d\langle M \rangle_s}_{A_{t \wedge T_m}^{f_n}}$$

$\implies$  set  $A_t^{f,m} := f(X_{t \wedge T_m}) - f(X_0) - \int_0^{T_m} f'_-(X_s) dX_s$  and  $A_{t \wedge T_m}^{f_n} \xrightarrow{n \rightarrow \infty, \mathbb{P}} A_t^{f,m} = A_{t \wedge T_m}^f$ ; Take  $m \rightarrow \infty$  and get a process  $A_t^f$  since  $f''_n \geq 0 \implies A_t^f$  is increasing and in  $\mathcal{A}_0$ .  $\square$

Let  $f(x) = |x|$ . Then

$$f'_-(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

we therefore define

$$\text{sgn}(x) := \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

as the left derivative of  $|x|$ .

**Proposition 3.2** (Tanaka's formula). *Let  $X$  be a continuous semimartingale and  $a \in \mathbb{R}$ . Then there  $\exists$  increasing process  $(L_t^a(X))_{t \geq 0}$  s.t.:*

$$(a) \quad |X_t - a| = |X_0 - a| + \int_0^t \text{sgn}(X_s - a) dX_s + L_t^a(X)$$

$$(b) \quad (X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{X_s > a} dX_s + \frac{1}{2} L_t^a(X)$$

$$(c) \quad (X_t - a)^- = (X_0 - a)^- - \int_0^t 1_{X_s \leq a} dX_s + \frac{1}{2} L_t^a(X)$$

The process  $L_t^a(X)$  is called the **local time of  $X$  at level  $a$** . For any stopping time  $T$ :

$$L_t^a(X^T) = L_{t \wedge T}^a(X)$$

*Proof.* Taking  $f(x) = |x - a|$  in proposition 3.1 yields (a). Therefore

$$L_t^a(X) := |X_t - a| - |X_0 - a| - \int_0^t \text{sgn}(X_s - a) dX_s$$

Applying prop. 3.1 to  $f(x) = (x - a)^+$  and  $f(x) = (x - a)^-$

$$\implies \exists A_t^{a,(+)}, A_t^{a,(-)} \in \mathcal{A}_0^+$$

such that

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{X_s > a} dX_s + A_t^{a,(+)}$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t 1_{X_s \leq a} dX_s + A_t^{a,(-)}$$

$$X_t - a \stackrel{\text{Itô}}{=} X_0 - a + \int_0^t dX_s$$

$$X_t - a = (X_t - a)^+ - (X_t - a)^- = X_0 - a + \int_0^t dX_s + A_t^{a,(+)} - A_t^{a,(-)}$$

which holds if and only if  $A_t^{a,(+)} = A_t^{a,(-)}$ . Furthermore

$$|X_t - a| = |X_0 - a| + \int_0^t \text{sgn}(X_s - a) dX_s + \underbrace{A_t^{a,(+)} + A_t^{a,(-)}}_{=L_t^a(X)} \quad \square$$

To see:  $L_t^a(X) \rightarrow$  positive measure  $dL_t^a(X)$ , since  $L_t^a(X)$  is increasing.

$$\int_0^t \underbrace{f(x)}_{\geq 0} dL_s^a(X) \stackrel{?}{=} 0 \text{ if } f(x) > 0 \text{ for } x \neq a$$

**Proposition 3.3.** Let  $dL_t^a(X)$  be the measure associated with the increasing process  $L_t^a(X)$ . Then  $dL_t^a(X)$  is supported on  $\{s \geq 0 \mid X_s = a\}$ .

But this is not necessarily the full support!

*Proof.* Let  $Y_t = |X_t - a|$ .

$$\begin{aligned} \Rightarrow Y_t^2 &= (X_t - a)^2 \stackrel{\text{Itô to } f(X_t) = (X_t - a)^2}{=} (X_0 - a)^2 + \int_0^t (X_s - a) dX_s + 2\langle X \rangle_t \\ Y_t^2 &\stackrel{\text{Itô to } f(Y_t) = Y_t^2}{=} (X_0 - a)^2 + 2 \int_0^t \underbrace{Y_s dY_s}_{= |X_s - a| \cdot (\text{sgn}(X_s - a) dX_s + dL_s^a(X))} + \underbrace{\langle Y \rangle_t}_{\langle X \rangle_t \text{ by prop 3.1}} \\ &= (X_0 - a)^2 + 2 \int_0^t (X_s - a) dX_s + 2 \int_0^t |X_s - a| dL_s^a(X) + \langle X \rangle_t \\ &\Rightarrow \int_0^t |X_s - a| dL_s^a(X) = 0 \end{aligned}$$

and therefore the support is indeed contained in  $\{s \geq 0 \mid X_s = a\}$   $\square$

**Remark.** If  $f$  is convex  $\Rightarrow f'_-(x)$  is increasing and left continuous. Therefore there exists a unique measure  $f''(dx)$  on  $\mathbb{R}_+$  such that

$$f''([a, b]) = f'_-(b) - f'_-(a)$$

If  $f \in C^2(\mathbb{R})$ , then  $f''(dx) = f''(x)dx$ , i.e. it has the density given by the second derivative. For all convex functions  $f$  with  $f''(dx) = 0$  for all  $|x| \geq K$

$$\Rightarrow \exists \alpha, \beta \in \mathbb{R} : f(x) = \alpha + \beta x + \frac{1}{2} \int_{\mathbb{R}} |x - a| f''(da)$$

and  $f'_-(x) = \beta + \frac{1}{2} \int_{\mathbb{R}} \text{sgn}(x - a) f''(da)$  in a weak sense.

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**Theorem 3.4** (Itô-Tanaka-formula). If  $f$  is a difference of convex functions,  $X$  is a continuous semimartingale

$$\Rightarrow f(X_t) - f(X_0) + \int_0^t f'_-(X_s) dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a(X) f''(da)$$

, where  $f''(da)$  is the measure associated with  $f'_-$ .

*Sketch of proof.* (Full proof: Thm 9.6 LeGall book)

Once localized  $\rightarrow$  assume  $f''(da) = 0$  on  $[-\kappa, \kappa]^c$

$$\begin{aligned} \Rightarrow f(X_t) &= \alpha + \beta X_t + \int_{-\kappa}^{\kappa} |X_s - a| f''(da) \\ &\stackrel{\text{prop. 3.2}}{=} \alpha + \beta X_t + \frac{1}{2} \int |X_0 - a| f''(da) + \frac{1}{2} \int \int_0^t \text{sgn}(X_s - a) dX_s f''(da) + \frac{1}{2} \int L_t^a(X) f''(da) \\ &= \underbrace{\alpha + \beta X_0 + \frac{1}{2} \int |X_0 - a| f''(da)}_{=f(X_0)} + \underbrace{\beta \int_0^t dX_s + \frac{1}{2} \int \int_0^t \text{sgn}(X_s - a) f''(da) dX_s}_{= \int_0^t f'_-(X_s) dX_s} + \frac{1}{2} \int L_t^a(X) f''(da) \end{aligned}$$

$\square$

**Corollary 3.5** (Occupation time formula). Almost surely,  $\forall t \geq 0$ , non-negative measurable function  $\varphi$

$$\int_0^t \varphi(X_s) d\langle X \rangle_s = \int_{\mathbb{R}} \varphi(a) L_t^a(X) da$$

**Example.**  $X = BM$ ,  $\varphi(X) = 1_A(X)$ ,  $A$  Borel set in  $\mathbb{R}$

$$\implies \int_0^t 1_A(X_s) ds$$

is the time spend by  $(X_s, s \in [0, t])$  in  $A$  and the RHS is

$$\int_A L_t^a(X) da$$

$\implies L_t^a$  density of time spend by Brownian motion in  $a$ .

Think  $A = [a, a + \epsilon]$

*Proof.* (For  $\varphi$  with bounded support)

Find  $f \in C^2$  s.t.  $f''(x) = \varphi(x)$ .

$\implies$  Itô-Tanaka-formula:

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s) dX_s + \frac{1}{2} \int L_t^a(X) \varphi(a) da$$

By Itô-formula:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t \varphi(X_s) d\langle X \rangle_s$$

The claim follows from comparing the terms.  $\square$

Explicit representation of the local time:

**Lemma 3.6.** Let  $X$  be a continuous semimartingale

$\implies$  a.s.  $\forall a \in \mathbb{R}, t \geq 0$ :

$$L_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t 1_{a \leq X_s \leq a+\epsilon} d\langle X \rangle_s$$

*Proof.* Take  $\varphi(x) = \frac{1}{\epsilon} 1_{[a \leq X \leq a+\epsilon]}$  in corollary 3.5:

See Thm 9.4 in LeGall for more details

$$\implies \frac{1}{\epsilon} \int_0^t 1_{a \leq X_s \leq a+\epsilon} d\langle X \rangle_s = \frac{1}{\epsilon} \int_a^{a+\epsilon} L_t^{\tilde{a}}(X) d\tilde{a} \stackrel{\tilde{a} \rightarrow L_t^{\tilde{a}}}{=} L_t^a(X) \quad \square$$

Similarly:

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t 1_{[a-\epsilon \leq X_s < a]} d\langle X \rangle_s &= L_t^{a-}(X) \\ \implies \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{[a-\epsilon \leq X_s \leq a+\epsilon]} d\langle X \rangle_s &= \frac{1}{2} (L_t^{a-}(X) + L_t^a(X)) \end{aligned}$$

$$\begin{aligned} L_t^a(X) - L_t^{a-}(X) &\stackrel{\text{prop 3.2}}{=} |X_t - a| - |X_0 - a| - \int_0^t \text{sgn}(X_s - a) dX_s - (|X_t - a_-| - |X_0 - a_-| - \int_0^t \text{sgn}(X_s - a_-) dX_s) \\ &= 2 \int_0^t 1_{X_s = a} dX_s \stackrel{X_t = M_t + A_t}{=} 2 \int_0^t 1_{X_s = a} dA_s \end{aligned}$$

where the first two terms are equal, since  $X$  is a continuous semimartingale. This then implies, if  $X$  is a martingale (and therefore  $A_t = 0$ ) then

$$L_t^a(X) = L_t^{a-}(X) \implies L_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{[a-\epsilon \leq X_s \leq a+\epsilon]} d\langle X \rangle_s.$$

In particular, if  $X$  is a BM:

$$L_t^{a-}(X) \implies L_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^1 1_{[a-\epsilon \leq X_s \leq a+\epsilon]} ds$$

## 3.2 Brownian motion and local time

Let  $B$  be a standard BM, then by Tanaka:

$$|B_t| = \underbrace{|B_0|}_{=0} + \int_0^t \operatorname{sgn}(B_s) dB_s + \underbrace{L_t}_{:=L_t^0(B)}$$

We want to define the process  $R_t = |B_t|$  be the **BM reflected at 0**.  
We want to study  $R_t$  and  $S_t^B := \sup_{0 \leq s \leq t} B_s, L_t(B)$

### 3.2.1 No strong solution of the Tanaka SDE

$$(\star) = \begin{cases} dX_t &= \operatorname{sgn}(X_t) dB_t \\ X_0 &= 0 \end{cases}$$

We already know:  $B_t := \int_0^t \operatorname{sgn}(W_s) dW_s$ , where  $W$  is a BM  $\implies BM$  is a BM (using Levy characterization) and

$$\int_0^t \operatorname{sgn}(W_s) dB_s = \int_0^t \operatorname{sgn}(W_s)^2 dW_s = W_t \rightarrow \operatorname{sgn}(W_t) dB_t = dW_t$$

and therefore  $(W, B)$  is a weak solution of  $(\star)$ .

Assume  $(X, B)$  is a strong solution of  $(\star) \implies dX_t = \operatorname{sgn}(X_t) dB_t$  with  $X$  is a BM.

Itô-Tanaka-formula:

$$\begin{aligned} |X_t| &= \int_0^t \operatorname{sgn}(X_s) dX_s + L_t^0(X) \\ \implies \int_0^t \operatorname{sgn}(X_s) dX_s &= |X_t| - L_t^0(X) = |X_t| - \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{|X_s| \leq \epsilon} ds \in \mathcal{F}_t^{|X|} \end{aligned}$$

But also:

$$\int_0^t \operatorname{sgn}(X_s) dX_s = \int_0^t (\operatorname{sgn}(X_s))^2 dB_s = B_t$$

$$B_t = |X_t| - \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{|X_s| \leq \epsilon} ds$$

$\implies B_t$  is  $\mathcal{F}_t^{|X|}$ -measurable and therefore  $B$  is  $\mathcal{F}^{|X|}$  measurable.

If  $(X, B)$  is a strong solution, then  $X$  is measurable w.r.t.  $\mathcal{F}^B$ .  $\implies \mathcal{F}^X \subset \mathcal{F}^B \subseteq \mathcal{F}^{|X|}$  which is wrong, since we can not recall the value of  $X$  knowing only its absolute value.

### 3.2.2 Reflected SDE

**Definition 3.7.** The family  $(X, l, W)$  is a weak solution of the **reflected SDE** (One-dimensional):

$$dX_t = dW_t + dl_t$$

if:

- $W$  is a BM
- $X$  is a positive continuous process
- $l$  is a positive non-decreasing continuous process

s.t.

$$\int_0^\infty 1_{X_s} > 0 dl_s = 0$$

The solution is called **strong**, if  $(X, l)$  is adapted to  $W$ .

**Example.** •  $X_t := |B_t|$

•  $W_t := \int_0^t \text{sgn}(B_s) dB_s$

•  $l_t = L_t^0(B)$

This is a weak solution of the **1 dimensional reflected SDE**.

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(06.06.24)

**Lemma 3.8** (Skorokhod lemma for uniqueness of reflected SDE). *Let  $Y(t)$  be a continuous function with  $y(0) \geq 0$ . Then there exist a unique pair  $(z, a)$  of functions:*

- both continuous
- $a(0) = 0, a$  is non-decreasing s.t.

(a)  $z(t) = y(t) + a(t)$

(b)  $\int_0^\infty 1_{z(s) > 0} da(s) = 0$

Moreover, the function  $a$  has the representation

$$a(t) = \sup_{0 \leq s \leq t} [y(s)]_- = \sup_{0 \leq s \leq t} \max\{0, -y(s)\}$$

*Proof.* Postponed :) □

Consequence of Lemma 3.8:

$$l_t(\omega) := \sup_{0 \leq s \leq t} (-W_s(\omega))_+ = \sup_{0 \leq s \leq t} (-W_s(\omega)) =: S_t^{-W}(\omega)$$

given  $W(\omega), X_t(\omega) = W_t(\omega) + l_t(\omega) = W_t(\omega) + \sup_{0 \leq s \leq t} (-W_s(\omega))$ .

$\implies \exists!$  strong solution of the reflected SDE. Notation:

$$R_t := |B_t| = W_t + L_t = W_t + S_t^{-W}$$

We can start with  $W$  and explicitly construct a strong solution!

**Theorem 3.9.**

$$\begin{aligned} \text{Law}(|B|, \underbrace{L}_{=L^0(B)}) &\stackrel{(1)}{=} \text{Law}(L + W, L) \\ &\stackrel{(2)}{=} \text{Law}(W + S^{-W}, S^{-W}) \\ &\stackrel{(3)}{=} \text{Law}(S^{-\tilde{W}} - \tilde{W}, S^{\tilde{W}}) \end{aligned}$$

with  $\tilde{W}$  a generic BM.

*Proof.* (1) is from  $R_t = |B_t| = W_t + L_t^0(B)$  (as a process)

(2) by the skorokhod representation:  $L_t = S_t^{-W}$

(3)  $W \stackrel{d}{=} -\tilde{W}$  □

$\forall t \geq 0$ :

$$S_t := \sup_{0 \leq s \leq t} B_s \stackrel{(d)}{=} |B_t| \stackrel{(d)}{=} L_t^0(B)$$

By the reflection principle:

$$\mathbb{P}(S_t \in dx, B_t \in dy) \stackrel{2.2 \text{ LeGall}}{=} \frac{2(2x-y)}{\sqrt{2\pi t^3}} e^{-\frac{(2x-y)^2}{2t}} 1_{x>0, y<x} dx dy$$

and

$$\text{Law}(|B|) = \text{Law}(S^W - W)$$

Something is wrong here?  
Or at the very least confusing

Two points of view:

1.  $B \rightarrow \mathbb{R}_t := |B_t|$ ,  $W_t := \int_0^t \text{sgn}(B_s) dB_s$ ,  $L_t := L_t^0(B)$  satisfies

$$R_t = W_t + L_t$$

2. Given  $W_t$  ( $\forall \omega$ ): Define  $l_t := \Phi_t(W) = \sup_{0 \leq s \leq t} (-W_s)$ ,  $X_t = W_t + l_t$   
 $\text{Law}(R, L) = \text{Law}(X, l)$ .

An alternative definition of reflected Brownian motion is

$$X_t := \underbrace{W_t + \Phi_t(W)}_{\text{Skorokhod map}}$$

**Added remark.** *Careful:*

$$|W_t(\omega)| \neq W_t(\omega) + \Phi_t(W(\omega))$$

Or

$$z(t) \neq |y(t)|$$

### 3.3 Reflected BM on a curve

In section 3.2, we have reflected BM on  $g(t) = 0 \forall t \geq 0$ .

$$X_t = W_t + \sup_{0 \leq s \leq t} (-W_s)$$

Skorokhod lemma; from continuous function  $y(t)$ ,  $y(0) \geq 0$ :

$$z(t) := y(t) + \sup_{0 \leq s \leq t} (y(s) - 0)_-$$

Let  $D_g := \{(x, t) \mid x \geq g(t)\}$ .

Let  $(W_t)_{t \geq 0}$  with  $W_0 \geq g(0)$ , then there exists a unique, continuous non-decreasing process  $L_t$  with  $L_0 = 0$  s.t.

$$X_t := W_t + L_t \geq g(t) \forall t \geq 0$$

and  $L_t$  may increase only when  $X_t$  is at the boundary of  $D_g$ .

$$\int_0^\infty 1_{(g(s), \infty)}(X_s) dL_s = 0$$

$X_t$  has the formula:

$$L_t = \sup_{0 \leq s \leq t} (W_s - g(s))_-$$

To see this claim: Consider  $\tilde{W}_t = W_t - g(t)$

$$\implies \tilde{X}_t := X_t - g(t)$$

is constructed by the Skorokhod map and

$$\tilde{X}_t = \tilde{W}_t + L_t \geq 0$$

**Definition 3.10.** The **BM reflected by a continuous function  $g(t)$**  with  $X_0 \geq g(0)$  is given by the following construction

- Given  $W$  a BM with  $W_0 = X_0 \geq g(0)$ , define

$$X_t := W_t + \sup_{0 \leq s \leq t} (W_s - g(s))_-$$

- Alternatively with  $W_0 = 0$ :

$$X_t = X_0 + W_t + \sup_{0 \leq s \leq t} (W_s - g(s))_-$$

$D_g$  is the space above the curve  $(t, g(t))$ , not sure why the coordinates are flipped tho?

**Added remark.** Think about sticky BM, which is the limit of some random walk which stays on the boundary with some positive probability.

The exercise talks shows two reflected BMs, whose difference lies in the Weil chamber and solves the SDE for a process conditioned to be positive forever.

SDE:  $dX_t = dW_t + dL_t$  or  $X_t = X_0 + \underbrace{W_t}_{W_0=0} + L_t^g(X)$ , where  $L_t^g(X)$  is the local time of  $X$  spend at the function  $g$ .

**Example.** Let  $B = (B_1(t), B_2(t))$  a 2-dimensional BM with  $B_1(0) \leq B_2(0)$  s.t.  $B_1$  is not depending on  $B_2$ , but  $B_2$  is reflected upwards by  $(B_1(t), t \geq 0) = g \Rightarrow (B_1(t), \tilde{B}_2(t)) \in D = \{(x_1, x_2 \in \mathbb{R}^d \mid x_1 \leq x_2)\}$

Start with 2-dim BM then define the reflected BM...

**Proposition 3.11.** Denote by  $p_t(x, y)$  be the transition probe density of  $B$ :

$$p_t(B_1(t) \in dy_1, \tilde{B}_2(t) \in dy_2 \mid B_1(0) = 0, \tilde{B}_2(0) = x_2) = p_t((x_1, x_2), (y_1, y_2)) dy_1 dy_2$$

It satisfies the following equations:

$$\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) p_t((x_1, x_2), (y_1, y_2)) = \frac{\partial}{\partial t} p_t((x_1, x_2), (y_1, y_2))$$

for  $(x_1, x_2), (y_1, y_2) \in D \setminus \partial D, t > 0$ .

$$p_t((x_1, x_2), (y_1, y_2)) dy_1 dy_2 \xrightarrow{t \rightarrow 0} \delta_x(y)$$

as well as

$$\frac{\partial}{\partial x_2} p_t(x, y) \mid_{x=y} = 0$$

*Proof.* Let  $g$  be any smooth test function in  $D$ , set  $f(t, x) := \int dy g(y) p_t(x, y)$ . The undefined integrals are over  $D$ , for Itô we assume  $f \in C^2$ .

(1) SDE:

$$\begin{cases} dB_t^1 = dW_t^1 \\ d\tilde{B}_t^2 = dW_t^2 + dL_t^{B^1}(\tilde{B}_2) \end{cases}$$

By Itô: Let  $0 \leq t \leq T$ :

$$\begin{aligned} & f(T-t, B-t) \\ &= f(T, B_0) + \int dy g(y) [-\dot{P}_{T-S}(B_s, y) ds] + \int dy g(y) \int_0^t \partial_{x_1} p_{T-S}(B_s, y) dW_s^1 \\ &+ \int_0^t \partial_{x_2} p_{T-S}(B_s, y) (dW_s^1 + dL_s) \\ &+ \int dy g(y) \left[ \frac{1}{2} \int_0^t \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} p_{T-S}(B_s, y) ds \right) \right] \end{aligned}$$

(2) Martingale: Let  $t \leq T$

$$\begin{aligned} \mathbb{E}(g(B_t) \mid \mathcal{F}_t) &\stackrel{\text{MP}}{=} \mathbb{E}(g(B_t) \mid B_t) \\ &= \int dy g(y) p_{T-t}(B_t, y) = \underbrace{f(T-t, B_t)}_{\text{Martingale}} \end{aligned}$$

□

Important information:  
where is the boundary  
condition coming from?  
Not immediatly obvious!



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# Chapter 4:

## Levy processes

### 4.1 Basics

Start of lecture 14  
(11.06.24)

Poisson process (Picture) ... With  $t_{n+1} - t_n$  are iid.,  $\sim \exp(\lambda)$  (here it is the parameter, NOT the mean!) and  $N_t \sim \text{Poi}(\lambda, t)$ .

$$\mathbb{P}(N_{t+\epsilon} = y + 1 | N_t = y) = \lambda\epsilon + O(\epsilon^2)$$

BM	Poisson process
	Monotone increasing

BM:

A real valued process  $B = (B_t)_t$  satisfying:

- (a)  $t \mapsto B_t$  is continuous
- (b)  $\mathbb{P}(B_0 = 0) = 1$
- (c) For  $0 \leq s \leq t$ :  $B_t - B_s \stackrel{d}{=} B_{t-s}$
- (d) For  $0 \leq s \leq t$ :  $B_t - B_s$  is independent of  $\{B_u, u \leq s\}$
- (e)  $\forall t > 0 B_t \sim \mathcal{N}(0, t)$

Poisson process:

A process  $N = (N_t)$  with values in  $D$  is called a Poisson process with parameter  $\lambda > 0$ , if

- (a)  $t \mapsto N_t$  is  $\mathbb{P}$ . a.s. cadlag.
- (b)  $\mathbb{P}(N_0 = 0) = 1$
- (c) For  $0 \leq s \leq t$ :  $N_t - N_s \stackrel{d}{=} M_{t-s}$
- (d) For  $0 \leq s \leq t$ :  $N_t - N_s$  is independent of  $\{N_u, u \leq s\}$
- (e)  $\forall t > 0 N_t \sim \text{Poi}(\lambda t)$

Notice how b,c,d are the same!

Differences:

BM:

- continuous paths
- unbounded variation

PP:

- pure jump process

- bounded variation

Common features:

- both have càdlàg trajectories
- stationary, independent increments

**Definition 4.1** (Lévy Process). A process  $X = (X_t)_{t \geq 0}$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  is a **Lévy process**, if:

- (a) The trajectories are  $\mathbb{P}$ . a.s. càdlàg
- (b)  $\mathbb{P}(X_0 = 0) = 1$
- (c) For  $0 \leq s \leq t$ :  $X_t - X_s \stackrel{d}{=} X_{t-s}$
- (d) For  $0 \leq s \leq t$ :  $X_t - X_s$  is independent of  $\{X_u, u \leq s\}$ .

**Remark.** Lévy processes are Markov processes.

**Definition 4.2.** A real-valued random variable  $X$  has a **infinitely divisible distribution**, if for all  $n \geq 1$ : There exists r.v.  $X_1^n, \dots, X_n^n$  iid. s.t.

$$X \stackrel{d}{=} X_1^n + \dots + X_n^n.$$

Let  $X \sim \mu$ :  $\mu$  **is infinitely divisible** if for all  $n \geq 1$ :  $\exists$  prob. measure  $\mu_n$  s.t.  $\mu = (\mu_n)^{*n}$ , i.e. the  $n$ -fold convolution.

**Remark.** Setting  $\psi(u) = -\log(\mathbb{E}(e^{iuX}))$ , for  $u \in \mathbb{R}$ ,  $\implies \psi$  is called the **characteristic exponent of  $X$** . If  $X$  is infinitely divisible, then

$$\begin{aligned} \mathbb{E}(e^{iuX}) &= \mathbb{E}(e^{iu(X_1^n + \dots + X_n^n)}) \\ &= \mathbb{E}(e^{iuX})^n \\ \implies \psi(u) &= n\psi_n(u), \end{aligned}$$

where  $\psi_n$  is the characteristic exponent of  $X_1^n$ .

We can change the definition to  $\mathbb{E}(e^{iuX}) = e^{-\psi(u)}$  to avoid taking logarithms of complex numbers.

**Theorem 4.3** (Lévy-Khintchine formula). A probability law  $\mu$  of real valued real random variable is infinitely divisible with characteristic exponent

$$\psi(u\theta) = -\log \int_{\mathbb{R}} e^{i\theta x} \mu(dx)$$

if and only if  $\exists$  a triple  $(a, \sigma, \nu)$ ,  $a \in \mathbb{R}, \sigma > 0, \mu$  a measure supported on  $\mathbb{R} \setminus \{0\}$  satisfying  $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$  s.t.

$$\psi(\theta) = ia\theta + \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x 1_{|x|<1}) \nu(dx)$$

$\nu$  does not have to have finite measure!

**Added remark.**  $\nu$  is called the **Lévy measure**.

**Added remark.** What happens for  $x$  small?

$$(1 - e^{i\theta x} - i\theta x 1_{|x|<1}) \approx 1 - (1 + i\theta x - \frac{\theta^2 x^2}{2}) + i\theta x = \frac{\theta^2 x^2}{2}$$

**Added remark.** The 1 in the indicator function is arbitrary, tho other choices yield other formulas!

### 4.1.1 Relation of infinitely divisible processes with Levy processes

Let  $X$  be a Levy process,  $\forall t > 0$ ,  $X_t$  is infinitely divisible, since

$$(\star) \quad X_t = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \cdots + \left(X_t - X_{\frac{t(n-1)}{n}}\right)$$

For all  $\theta \in \mathbb{R}, \geq 0$ :  $\psi_t(\theta) := -\ln(\mathbb{E}(e^{i\theta X_t}))$

$$\stackrel{(\star)}{\implies} \quad \forall m, n \geq 1 : \frac{m}{n} \psi_1(\theta) = \frac{1}{m} \psi_m(\theta) = \frac{n}{m} \psi_{\frac{m}{n}}(\theta)$$

and therefore for  $t \geq 0$  rational:  $\psi_t(\theta) = t\psi_1(\theta)$ . It then follows that  $\forall t \in \mathbb{R}_+ : \psi_t(\theta) = t\psi_1(\theta)$ .

For Lévy process:

$$\mathbb{E}(e^{i\theta X_t}) = e^{-t\psi_1(\theta)}$$

**Definition 4.4.** We refer to  $\psi(\theta) := \psi_1(\theta)$  to be the characteristic exponent of the Lévy process.

**Theorem 4.5.** Let  $a \in \mathbb{R}, \sigma \geq 0, \nu$  a measure supported on  $\mathbb{R} \setminus \{0\}$  s.t.

$$\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$$

Define

$$\psi(\theta) := ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x) 1_{|x|<1} \nu(dx)$$

$\implies$  then there exists a Lévy process with characteristic exponent  $\psi(\theta)$ .

**Proposition 4.6.** Let  $F$  be a distribution function on  $\mathbb{R}_+$ . Then  $F$  is the distribution function of an infinitely divisible law

$\iff \forall \lambda > 0 : F^{\mathcal{L}}(\lambda) = \int_0^\infty e^{-\lambda x} F(dx) = e^{-c\lambda - \int_0^\infty (1 - e^{-\lambda x}) \mu(dx)}$ , where  $c \in \mathbb{R}$ ,  $\mu$  is a measure on  $(0, \infty)$ , s.t.  $\int_0^\infty (1 \wedge x) \mu(dx) < \infty$ .

*Proof.*  $\implies$  : Let  $X$  have distribution  $F : F(x) = \mathbb{P}(X \leq x)$ .  $X_1^n + \cdots + X_n^n$  with  $\mathbb{P}(X_1^n \leq x) =: F_n(x)$ .

$$F^{\mathcal{L}}(\lambda) = \int_0^\infty e^{-\lambda x} F(dx) = (F_n^{\mathcal{L}}(\lambda))^n,$$

since

$$\begin{aligned} F^{\mathcal{L}}(\lambda) &= \int_0^\infty e^{-\lambda x} \mathbb{P}(X_1 + \cdots + X_n \in dx) \\ &= \int_0^\infty e^{-\lambda x} \int \mathbb{P}(X_1 \in dz) \mathbb{P}(X_2 + \cdots + X_n + z \in dx) \\ &= \int_0^\infty \mathbb{P}(X_1 \in dz) \int_z^\infty \mathbb{P}(X_2 + \cdots + X_n + z \in dx) e^{-\lambda x} \\ &\stackrel{x=\tilde{x}+z}{=} \int_0^\infty \mathbb{P}(X_2 + \cdots + X_n \in d\tilde{x}) e^{-\lambda \tilde{x}} e^{-\lambda z} \\ &= \left( \int_0^\infty \mathbb{P}(X_1 \in dz) e^{-\lambda z} \right)^n = (F_n^{\mathcal{L}}(\lambda))^n \end{aligned}$$

Since  $F$  is a distribution on  $\mathbb{R}_+ \implies F_n^{\mathcal{L}} \in (0, 1]$ , which means

$$F_n^{\mathcal{L}}(\lambda) \xrightarrow{n \rightarrow \infty} 1$$

uniformly for  $\lambda$  in compact sets.

$$\implies \ln F^{\mathcal{L}}(\lambda) = n \log F_n^{\mathcal{L}}(\lambda) = n \log(1 - (1 - F_n^{\mathcal{L}}(\lambda))) \leq -n(1 - F_n^{\mathcal{L}}(\lambda))$$

1.:  $\forall \delta > 0, K < \infty, \exists n_0 < \infty$  s.t.  $n \geq n_0, \lambda \leq K \implies 1 - F_n^{\mathcal{L}}(\lambda) \leq \delta$ .

This last inequality is almost an equality for large  $n$ !

2.:  $\forall \delta > 0, C < \infty$  s.t.  $\forall 0 \leq x \leq \delta$ ,

$$-x(1 + cx) \leq \log(1 - x) \leq -x$$

Apply it to  $x = 1 - F_n^{\mathcal{L}}(\lambda)$

$$\implies -(1 - F_n^{\mathcal{L}}(\lambda))[1 + C(1 - F_n^{\mathcal{L}}(\lambda))] \leq \log F_n^{\mathcal{L}}(\lambda) - n(1 - F_n^{\mathcal{L}}(\lambda))$$

As  $n \rightarrow \infty$ , uniformly in bounded set (for  $\lambda$ ):

$$-n(1 - F_n^{\mathcal{L}}(\lambda)) \rightarrow \log F^{\mathcal{L}}(\lambda).$$

$$\begin{aligned} n(1 - F_n^{\mathcal{L}}(\lambda)) &= \int (1 - e^{-\lambda x}) F_n(dx) \\ &= \int \frac{1 - e^{-\lambda x}}{1 - e^{-x}} \underbrace{n(1 - e^{-x}) F_n(dx)}_{=: m_n(dx)} \end{aligned}$$

$m_n$  is a measure on  $(0, \infty)$  with total mass is  $n(1 - F_n^{\mathcal{L}}(1)) \rightarrow \log F^{\mathcal{L}}(1)$   
 $\implies \exists$  measure  $m$  on  $[0, \infty]$  s.t.  $m_n \rightarrow m$  and

$$n(1 - F_n^{\mathcal{L}}(\lambda)) \rightarrow m(\{0\})\lambda + \int_0^\infty \frac{1 - e^{-\lambda x}}{1 - e^{-x}} m(dx) + m(\{0\})$$

Let  $\lambda = 0$ :  $m(\{\infty\}) = -\log F^{\mathcal{L}}(0) = -\log 1 = 0$ . Then with  $c = m(\{0\})$

$$\mu(dx)(1 - e^{-x}) = m(dx)$$

$\Leftarrow \dots$

□

Question: What are  $a, \sigma, \nu$ ? Is there a relation between  $\nu$  and the jumps of the Lévy process?

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(13.06.24)

## 4.2 Examples

### 4.2.1 Poisson process

Let  $\lambda > 0$  and  $X \sim \text{Poi}(\lambda)$ ,

$$\begin{aligned} \mathbb{E}(e^{i\theta X}) &= \sum_{n \geq 0} e^{-\theta n} \frac{e^{-\lambda} \lambda^n}{n!} \\ &= e^{-\lambda} e^{\lambda e^{i\theta}} \\ &= e^{-\lambda(1 - e^{i\theta})} = \left( e^{-\frac{\lambda}{n}(1 - e^{i\theta})} \right)^n \end{aligned}$$

$\implies X$  is infinitely divisible.

To get the right characteristic exponent

$$-\log \mathbb{E}(e^{i\theta X}) = \lambda(1 - e^{i\theta})$$

which means  $\psi(\theta) = \lambda(1 - e^{i\theta})$ .

Take  $a = 0, \sigma = 0, \nu(dx) = \lambda \delta_1(x)$ , then  $\psi(\theta) = (1 - e^{i\theta})\lambda$ .

The Lévy process with characteristic exponent  $\psi(\theta)$  is called the Poisson process with parameter  $\lambda > 0$ .

**Added remark.** If you have a Lévy process on  $\mathbb{Z}_{\geq 0}$ ,  $(X_t)$  with  $X_0 = 0$  and with jumps  $+1 \implies$  This is a Poisson process.

### 4.2.2 Compound Poisson process

Let  $N$  be a Poisson random variable with parameter  $\lambda > 0$ ,  $\{\xi_k\}_{k \geq 1}$  iid r.v. (independent of  $N$ ) with distribution  $F$  on  $\mathbb{R} \setminus \{0\}$ .

Let  $X := \sum_{k=1}^N \xi_k$ :

$$\begin{aligned} \Rightarrow \mathbb{E}(e^{i\theta X}) &= \sum_{n \geq 0} \mathbb{E}\left(e^{i\theta(\xi_1 + \dots + \xi_n)}\right) \frac{e^{-\lambda \lambda^n}}{n!} \\ &= \sum_{n \geq 0} \frac{e^{-\lambda} \lambda^n}{n!} \left(\int e^{i\theta x} F(dx)\right)^n \\ &= e^{-\lambda} e^{\lambda \int e^{i\theta x} F(dx)} = e^{-\lambda \int F(dx)} e^{\lambda \int e^{i\theta x} F(dx)} \\ &= e^{-\lambda \int (1 - e^{i\theta x}) F(dx)} \\ &= e^{\underbrace{-\lambda \int (1 - e^{i\theta x}) F(dx)}_{=\psi(\theta)}} \end{aligned}$$

$\Rightarrow X$  is infinitely divisible with  $\psi(\theta)$  having  $\sigma = 0, \nu(dx) = \lambda F(dx), a = -\int |x| < 1 x F(dx)$   
How to get the process?

1. Choose Poisson process with parameter  $\lambda$ ,  $(N_t)_{t \geq 0}$
2.  $X_t = \sum_{k=1}^{N_t} \xi_k$  is called the **compound Poisson process**

Is  $(X_t)_{t \geq 0}$  a Lévy process?

Then

- $X_0 = 0$
- $\forall 0 \leq s < t: X_t = X_s + \sum_{k=N_s+1}^{N_t} \xi_k$

which implies  $X_t - X_s \stackrel{d}{=} \sum_{k=1}^{N_t - N_s} \xi_k \stackrel{d}{=} X_{t-s}$  and everything is independent, but  $N_t - N_s = N_{t-s}$ .  
Since  $N_t$  are càdlàc  $\Rightarrow X_t$  is càdlàg.  $(X_t)_{t \geq 0}$  is a Lévy process with

$$\psi(\theta) = \lambda \int_{\mathbb{R} \setminus \{0\}} (1 - e^{i\theta x}) F(dx).$$

Let  $\Delta X_t := X_t - X_{t-}$  and for  $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ ,

$$\tilde{N}_t(B) := \#\{s \in [0, t] : \Delta X_s \in B\}$$

the number of jumps of size  $B$  in the time span  $[0, t]$  and  $t_k$  are the jump times.

$(X_t)_t$  is a Lévy process  $\Rightarrow (\tilde{N}_t(B))$  has stationary independent increments.

$\tilde{N}_{t+s}(B) - \tilde{N}_t(B) \stackrel{d}{=} \tilde{N}_t(B)$  for all  $t, s \geq 0$  and  $B$ .

Therefore  $(\tilde{N}_t(B))_t$  is a Lévy process with jumps of size  $+1$ ,  $\tilde{N}_0(B) = 0$ , which implies  $\tilde{N}_t(B)$  is a poisson process. The parameter of  $\tilde{N}_t(B)$  is

$$\mathbb{E}(\tilde{N}_t(B)) = \underbrace{t \cdot \lambda}_{=\mathbb{E}(N_t)} \mathbb{P}(\xi \in B) = t \lambda \underbrace{\int_B F(dx)}_{=\nu(B)} = t\nu(B).$$

$\Rightarrow \nu(B)$  is the parameter of  $\tilde{N}_t(B)$ <sup>1</sup>.

**Added remark.** Suppose we had a Levy process with finite measure (remove  $ia\theta$  and  $i\theta x 1_{|x| < 1}$ ).  
Then we can normalize and define  $F \Rightarrow$  we get a compound poisson process!

Add a drift:

$$X_t = \sum_{k=1}^{N_t} \xi_k + ct, t \geq 0$$

Then Lévy process of  $X_t$  has characteristic exponent

$$\psi(\theta) = \lambda \int (1 - e^{i\theta x}) F(dx) - ic\theta$$

<sup>1</sup>As a process, not as a r.v.

The  $\xi$  give us the size of the jumps

We can think of a poisson process as the set of jumping points (times). This is sometimes called poisson jump process  
Notation  $0 =: \sum_{k=m}^{\infty} (\dots)$  for  $m > n$ .

### 4.2.3 Brownian motion with drift

Let  $\mu_{s,\gamma}(dx) := \frac{e^{-\frac{(x-\gamma)^2}{2s^2}}}{\sqrt{2\pi s^2}} dx$ ,  $s > 0, \gamma \in \mathbb{R}$ .

If  $X \sim \mu_{s,\gamma}$ , then

$$\mathbb{E}(e^{i\theta X}) = e^{-\frac{1}{2}s^2\theta^2 + i\theta\gamma} = \left( e^{-\frac{1}{2}\frac{s^2}{\sqrt{n}}^2\theta^2 + i\theta\frac{\gamma}{n}} \right)^n$$

$\implies X$  is infinitely divisible with characteristic exponent  $\psi(\theta)$  with  $\nu = 0, \sigma = s^2, a = -\gamma$ .

$$\implies \psi(\theta) = -i\theta\gamma + \frac{1}{2}s^2\theta^2.$$

and therefore

$$X_t := sB_t + \gamma t,$$

where  $B_t$  is a standard BM, is the Lévy process with characteristic exponent  $\psi(\theta)$ .

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### 4.2.4 The Gamma process

Given  $\alpha, \beta > 0$ , define the probability measure

$$\mu_{\alpha,\beta}(dx) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} dx 1_{\mathbb{R}_+ \setminus \{0\}}$$

where

$$\Gamma(z) := \int_0^\infty u^{z-1} e^{-u} du.$$

$\mu_{\alpha,\beta}$  is the Gamma( $\alpha, \beta$ ) distribution. For  $\beta = 1$  it is the exponential distribution.

Let  $X \sim X_{\alpha,\beta}$ .

$$\begin{aligned} \mathbb{E}(e^{i\theta X}) &= \int_{\mathbb{R}_+} e^{i\theta x} \mu_{\alpha,\beta}(dx) \\ &= \int_{\mathbb{R}_+} \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} (e^{-\alpha} \cdot e^{i\theta})^x dx \\ &\stackrel{(\alpha-i\theta)x=:u}{=} \frac{\alpha^\beta}{\Gamma(\beta)} \frac{1}{(\alpha-i\theta)^\beta} \underbrace{\int_0^{\infty(\alpha-i\theta)} u^{\beta-1} e^{-u} du}_{\text{By Cauchy Residue} = \int_0^\infty u^{\beta-1} e^{-u} du} \\ \implies \mathbb{E}(e^{i\theta X}) &= \frac{1}{(1 - \frac{i\theta}{\alpha})^\beta} = \left( \frac{1}{(1 - \frac{i\theta}{\alpha})^{\frac{\beta}{n}}} \right)^n \end{aligned}$$

$\implies X$  is infinitely divisible.

The triple  $(\alpha, \sigma, \nu)$  is given by:

- $\sigma = 0$
- $\nu(dx) = \beta x^{-1} e^{-\alpha x} dx$  on  $\mathbb{R}_+ \setminus \{0\}$
- $a = -\int_0^1 x\nu(dx) = -\frac{\beta}{\alpha}(1 - e^{-\alpha})$ . This is the correction term for the indicator function!

**Lemma 4.7.** For  $\alpha, \beta > 0, z \in \mathbb{C}$  with  $\text{Re}(z) \leq 0$ ,

$$\frac{1}{(1 - \frac{z}{\alpha})^\beta} = \exp \left( - \int_0^\infty (1 - e^{zx}) \beta x^{-1} e^{-\alpha x} dx \right)$$

Apply to  $z = i\theta$  to get the above!

*Proof.* Take the logarithm:

$$-\beta \log \left(1 - \frac{z}{\alpha}\right) \stackrel{?}{=} - \int_0^\infty (1 - e^{zx}) \beta x^{-1} e^{-\alpha x} dx$$

First look at  $z = 0: 0 = 0$ , which is of course true.

Now look at the derivative in  $z$  (after canceling the  $-\beta$  from both sides):

$$-\frac{1}{\alpha - z} = \frac{-\frac{1}{\alpha}}{1 - \frac{z}{\alpha}} \stackrel{?}{=} - \int_0^\infty \underbrace{x \cdot x^{-1}}_{=1} e^{-\alpha x} dx = - \int_0^\infty e^{-(\alpha - z)x} dx \stackrel{\text{Re}(\alpha - z) > 0}{=} - \frac{1}{\alpha - z}$$

□

Using  $z = i\theta$  in lemma 4.7:

$$\implies \Psi(\theta) = -\ln \left( \frac{1}{(1 - \frac{i\theta}{\alpha})^\beta} \right) = \int_0^\infty (1 - e^{i\theta x}) \beta x^{-1} e^{-\alpha x} dx \rightarrow \nu(dx)$$

**Added remark.** For the Gamma process is the Lévy measure  $\infty$ :  $\int_0^\infty \nu(dx) = \infty$ .

Properties:

- $0 \leq s \leq t$   $X_t \stackrel{d}{=} X_s + \tilde{X}_{t-s}$ , where  $\tilde{X}_{t-s}$  is an independent copy of  $X_{t-s}$
- $X_s < X_t$  a.s. for  $s < t$  (the Lévy measure is on  $\mathbb{R} > 0$ )
- it is not a compound Poisson process (since it does not have finite Lévy measure)

### 4.3 Some properties of the jumps

For compound poisson process:

$$N_t(B) := \# \text{ jumps of size } B \text{ in the time interval } [0, t]$$

$N_t(B) \sim \text{Poi}(t\nu(B))$  for  $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ .  $t \mapsto N_t(B)$  is a Poisson process with parameter / intensity / rate  $\lambda = \nu(B)$ .

**Added remark.** Suppose you want to simulate the poisson process. Sample exponential random variables, by sampling uniformly from  $(0,1)$  and then transform them by  $-\log$ .

Let  $T_0 = 0$ ,

$$T_{k+1} = \inf\{\xi_t > T_k \mid N_t(\mathbb{R}_{>0}) > N_{T_k}(\mathbb{R}_{>0})\}$$

This implies  $T_{k+1} - T_k \sim \exp(\nu(\mathbb{R}_{>0}))$ . The jump sizes of the compound Poisson process  $X_t$  are distributed as:

$$\mathcal{P}(X_{T_1} - X_{T_1^-} \in dy) = \frac{\nu(dy)}{\int_{\mathbb{R} \setminus \{0\}} \nu(dx)}$$

Argument for almost sure no step at time 0:  $\mathcal{P}(N_t(\mathbb{R}_{>0}) \geq 2) = O(t^2)$ .

$$\begin{aligned} \implies \mathbb{P}(N_t(\mathbb{R}_{>0}) = 1) &= e^{-t\nu(\mathbb{R}_+)} t\nu(\mathbb{R}_+) = t\nu(\mathbb{R}_+) + O(t^2) \\ \mathbb{P}(N_t(\mathbb{R}_{>0}) = 0) &= 1 - t\nu(\mathbb{R}_+) + O(t^2) \end{aligned}$$

$$\frac{\mathbb{P}(N_t(B) = 1 \cap N_t(\mathbb{R}_+) = 1)}{\mathbb{P}(N_t(\mathbb{R}_+) = 1)} \approx \frac{t\nu(B)}{t\nu(\mathbb{R}_+)}$$

Case  $\int_{\mathbb{R} \setminus \{0\}} \nu(dx) = \infty$ .

Problem: First jump  $\sim \exp(\nu(\mathbb{R} \setminus \{0\})) = 0$

a bit informal

$\Rightarrow$  The notion of first, second,  $\dots$ , jumps does not make sense:

$$\inf\{t > 0 \mid X_t - X_{t-} \neq 0\} = 0 \text{ a.s.}$$

Let  $A \subset \mathbb{R} \setminus \{0\}$  s.t.  $\nu(A) < \infty$ . This implies the first jump of size  $A$ ,  $T_1^A$  is well defined with  $T_1^A \sim \exp(\nu(A))$ :

$$\Rightarrow \mathbb{P}(X_{T_1^A} - X_{T_1^A-} \in dy) = \frac{\nu(dy)}{\nu(A)}$$

$\Rightarrow$  take  $A^0 = (1, \infty)$ ,  $A^1(\frac{1}{2}, 1]$ ,  $A^2 = (\frac{1}{4}, \frac{1}{2}]$

$\Rightarrow$  Construct Lévy process (compound poisson processes) by restricting the jumps to the ones of size  $A^n : (X_t^n)_{t \geq 0}$

$$X_t := \sum_{n \geq 0} X_t^n$$

where  $X_t^0, X_t^1, \dots$  are independent. The  $(X_t^n)$  are compound poisson processes with jumps in  $A^n$ :

$$T_k^{A^n} \sim \exp(\nu(A^n)) \text{ with jumps of size } \frac{\nu(dy)}{\nu(A^n)}$$

$\Rightarrow (X_t)_{t \geq 0}$  is a Lévy process with Lévy measure  $\nu$  ( $\sigma = 0$ ).

**Added remark.** We use that the sum of independent Lévy processes is still a Lévy process, since the exponential characteristic is the sum of the exponential characteristics

By theorem 4.3 we can decompose:

$$\Psi(\theta) = \underbrace{(ia\theta + \frac{1}{2}\sigma^2\theta^2)}_{=\Psi^{(1)}(\theta)} + \underbrace{\nu(\mathbb{R} \setminus \{(-1, 1)\}) \int_{|x| \geq 1} (1 - \exp(i\theta x)) \frac{\nu(dx)}{\nu(\mathbb{R} \setminus \{(-1, 1)\})}}_{=\Psi^{(2)}(\theta)} + \underbrace{\int_{0 < |x| < 1} (1 - \exp(i\theta x) + i\theta x) dx}_{=\Psi^{(3)}(\theta)}$$

**Theorem 4.8** (Lévy-Itô decomposition). Given  $a \in \mathbb{R}, \sigma > 0, \nu$  a measure on  $\mathbb{R} \setminus \{0\}$  s.t.  $\int_{\mathbb{R} \setminus \{0\}} (1 \wedge x^2) \nu(dx) < \infty$

$\Rightarrow \exists (\Omega, \mathcal{F}, \mathbb{P})$  on which 3 independent Lévy processes  $X^{(1)}, X^{(2)}, X^{(3)}$  with characteristic exponents  $\Psi^{(1)}(\theta), \Psi^{(2)}(\theta), \Psi^{(3)}(\theta)$  are defined. Then  $X_t := X^{(1)} + X^{(2)} + X^{(3)}$  is a Lévy process with characteristic exponent  $\Psi(\theta)$ .

- $X_t^{(1)}$  is a BM with drift:  $X_t^{(1)} = \sigma B_t - at, t \geq 0$ , where  $B_t$  is a standard BM
- $X_t^{(2)}$  is a compound Poisson process
- $X_t^{(3)}$  is constructed as above
- $\int_{0 < |x| < 1} (1 - \exp(i\theta x)) \nu(dx) = \sum_{n \geq 0} \lambda_n \int_{2^{-(n+1)} \leq |x| < 2^{-n}} F_n(dx) + i\theta \lambda_n \int_{2^{-(n+1)} \leq |x| < 2^{-n}} x F_n(dx)$ , where  $\lambda_n = \nu(2^{-(n+1)} \leq |x| < 2^{-n})$ ,  $F_n(dx) = \frac{1}{\lambda_n} \nu(dx)$ .

Next lecture has useful information about simulations!

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**Remark.** Càdlàg paths have at most countably many jumps.

Why do we have the condition  $\int (1 \wedge x^2) \nu(dx) < \infty$ ?

$$\mathbb{P}(|X_t| \geq M) \leq ? \text{ s.t. } \xrightarrow{M \rightarrow \infty} 0.$$

$$X_t^n = \sum_{k=1}^{\infty} \xi_k 1_{k \leq N_t(A^n)}$$



Take  $\nu$  supported on  $(0, 1]$  and  $A_n = (\frac{1}{2^n}, \frac{1}{2^{n-1}}]$ ,  $\nu(A_n) < \infty$

The computation uses independence, not true for higher moments!

$$\begin{aligned} \mathbb{P}(|X_t| \geq M) &\stackrel{\text{Chebyshev}}{\leq} \frac{\text{Var}(X_t)}{M^2} \\ \text{Var}(X_t) &= \sum_{i=1}^{\infty} \text{Var}(X_t^n) = \sum_{n=1}^{\infty} \underbrace{\sum_{m=0}^{\infty} \mathbb{P}(N_t(A_n) = m) \cdot m}_{\mathbb{E}(\# \text{ jumps of size in } A_n = t\nu(A_n))} \cdot \underbrace{\text{Var}(\xi_1^n)}_{\leq \mathbb{E}((\xi_1^n)^2) = \int_{A_n} \frac{x^2 \nu(dx)}{\nu(A_n)}} \\ &\leq t \sum_{n=1}^{\infty} \int_{A_n} x^2 \nu(dx) = t \int_{(0,1]} x^2 \nu(dx) < \infty \end{aligned}$$

$X_t$  is bounded a.s., for all given  $t$

no divergence in finite time

## 4.4 Some properties of Poisson processes

**Lemma 4.9** (Superposition). *Let  $(X_t^k)_{t \geq 0}, k = 1, \dots, N$  independent Poisson processes with intensity  $\lambda_k > 0$ .*

*Then  $(Y_t)_{t \geq 0} = X_t^1 + \dots + X_t^N$  is a Poisson process with intensity  $\lambda = \lambda_1 + \dots + \lambda_N$ .*

Also true for  $N = \infty$ , if  $\lambda < \infty$

*Proof.* Follows by computing the characteristic exponents / functions □

**Added remark.** Why is it called **Superposition**?

Instead of jumps look at the events of the jumps for each  $X_t^1$ . If you take the union (superimpose the pictures), you get the jump times of  $Y$ . Not related to quantum mechanics.

**Lemma 4.10** (Thinning property). *Let  $(X_t)_{t \geq 0}$  be a poisson process with intensity  $\lambda$  and a  $p \in [0, 1]$  fixed.*

*Construct two processes  $(X_t^1)_{t \geq 0}, (X_t^2)_{t \geq 0}$  as follows:*

- *Each jump of  $X_t$  is randomly and independently assigned to be a jump of  $X_t^1$  with probability  $p$  and if not assigned to  $X_t^1$  it is assigned to  $X_t^2$ .*
- *Then  $(X_t^1)$  is a Poisson process with intensity  $p\lambda$*
- *$(X_t^2)$  is a Poisson process with intensity  $(1-p)\lambda$*
- *and they are independent*

*Simplified proof.*

Just showing independence, other things are easy to check

$$\begin{aligned} \mathbb{P}(X_t^1 = k, X_t^2 = l) &= \mathbb{P}(X_t^1 = k, X_t^1 + X_t^2 = k + l) \\ &= \mathbb{P}(\underbrace{X_t^1 + X_t^2}_{X_t} = k + l) \mathbb{P}(X_t^1 = k \mid \underbrace{X_t^1 + X_t^2}_{X_t} = k + l) \\ &= \frac{e^{-\lambda t} (\lambda t)^{k+l}}{(k+l)!} \binom{l+k}{k} p^k (1-p)^l \\ &= \frac{e^{-\lambda p t} (\lambda p t)^k}{k!} \frac{e^{-\lambda (1-p) t} (\lambda (1-p) t)^l}{l!} \end{aligned} \quad \square$$

**Added remark.** Let  $X_t = X_t^1 - X_t^2$ ,  $X_t^1$  is a Poisson process with intensity  $p$  and  $X_t^2$  is a poisson process with intensity  $1-p$ . Therefore has intensity<sup>a</sup> 1

<sup>a</sup>intensity: expected number of jumps per unit time

This is not a Poisson process, since it might be negative!

How do we simulate this?

Simple solution: generate two Poisson process, by generating exponential variables

( $\sim \exp(p)$ ,  $\sim \exp(1-p)$ ). This yields to lists of jumping timings, which we can join (after sorting).

What do we do if we have not two, but many of them?

Draw many exponential variables  $\sim \exp(1)$ , then for each time choose a random poisson process, which you want to attribute the jump to.

For two we get the drift:  $\mathbb{E}(X_t) = t(p(1-p)) = (t(2p-1))$

two random walks  $X_t, \tilde{X}_t$  as above with  $X_0 < \tilde{X}_0$ , add constraint  $X_t < \tilde{X}_t$  meaning that jumps which would violate the condition are suppressed.

For all  $x \in Z$  associate a random poisson process  $N_t^x$  with intensity 1.

For intensity  $\frac{1}{2}$ : First with probability  $1/2$  we throw away the jump and then we flip a coin.

Again, we simulate one process with intensity as the sum of the intensities, then we split randomly!

When we draw poisson processes, they look different from reality, normally we would expect clumping!

**Added remark.** Some problems with infinitely many processes, jumps at 0?

**Added remark.** Lévy process with  $\Psi(\theta) = i\theta + \frac{1}{2}i\theta^2\sigma^2 + \int_{\mathbb{R}}(1 - e^{i\theta x} + i\theta x 1_{|x|<1})\nu(dx)$  Has generator

Not asked in the exam

$$(Lf)(x) = \frac{1}{2}\sigma^2 \frac{d^2}{dx^2}f(x) - a \frac{d}{dx}f(x) + \int_{\mathbb{R}} \left[ f(x+y) - f(x) - y \frac{d}{dx}f(x) 1_{|x|<1} \right] \nu(dy)$$

for  $f \in C_0^\infty(\mathbb{R})$ .

For poisson processes with intensity  $\lambda$ :

$$(Lf)(x) = \lambda(f(x+1) - f(x)).$$

A way of defining  $L$ :

$$(Lf)(x) := \lim_{t \downarrow 0} \frac{\mathbb{E}(f(X_t) \mid X_0 = 0) - f(x)}{t}$$

As  $t \downarrow 0$ :

$$\mathbb{P}(X_t = x+1 \mid X_0 = x) = \lambda t + O(t^2)$$

$$\mathbb{P}(X_t = x \mid X_0 = x) = 1 - \lambda t + O(t^2)$$

$$\mathbb{P}(X_t \notin x, x+1 \mid X_0 = x) = O(t^2)$$

$$\implies \frac{1}{t} \mathbb{E}(f(X_t) \mid X_0 = x) = (1-\lambda)f(x) + \lambda f(x+1) + (t) \xrightarrow{t \rightarrow 0} \lambda(f(x+1) - f(x))$$

Consider a  $d$ -dimensional Poisson process with intensity  $\lambda$ , i.e.

$X_t = (X_t^1, \dots, X_t^d)$  where the components are Poisson processes with intensity  $\lambda$  and independent.

$\implies$  Generator

$$Lf(x) = \sum_{k=1}^d L_k f(x), \quad L_k f(x_1, \dots, x_d) = \lambda(f(\dots, x_k+1, \dots) - f(\dots, x_k, \dots)) = \lambda(f(x+e_k) - f(x))$$

Let  $X_0 = (X_0^1, \dots, X_0^d) \in W_d = \{x \in \mathbb{Z}^d \mid x_1 < x_2 < \dots < x_d\}$

Want: Process  $X_t$  conditioned to stay forever in  $W_d$ .

**Lemma 4.11.** Let

Like for BM!

$$h(x) := \prod_{1 \leq k < l \leq d} (X_l - X_k),$$

$h$  satisfies

(a)  $h(x) = 0$  for  $x \in \partial W_d$  (two of the  $x_i$  are equal)

(b)  $h(x) > 0$  for  $x \in W_d$

(c)  $Lh(x) = 0$  for  $x \in W_d$

**Remark.** To prove use  $h(x) = \det(X_k^{j-1})_{1 \leq k, j \leq d}$  or  $e^{tL}h(x) = h(x)$ ,  $t > 0$ .

This conditioned process has generator:

$$L^h f(x) = \frac{1}{h(x)} (L(hf))(x) = Lf(x) + \sum_{k=1}^d \frac{\nabla_k h(x)}{h(x)}, \quad \nabla_k g(x) := g(x + e_k) - g(x)$$

Let  $x, y \in W_d$ :

$$P_t(x, y) = \frac{h(y)}{h(x)} \mathbb{P}(\tilde{X}(t) = y, T > t \mid \tilde{X}_0 = x)$$

where  $\tilde{X}(t)$  are independent components and  $T$  is the hitting time  $\inf\{s \geq 0 \mid \tilde{X}(s) \notin W_d\}$ .

**Lemma 4.12** (Karlin-McGregor formula).

$$\mathbb{P}(\tilde{X}(t) = y, T > t \mid \tilde{X}(0) = x) = \det(\varphi_t(x_i, y_j)_{1 \leq i, j \leq d})$$

where

$$\varphi_t(x, y) = \varphi_t(0, y - x) = \mathbb{P}(X_t^1 = y \mid X_0^1 = x) = \frac{e^{-\lambda t} (\lambda t)^{y-x}}{(y-x)!} 1_{y \geq x}.$$

*Notice, we only need to compute the interactions of two particles!*

Special case:  $\tilde{X}(0) = (0, 1, \dots, d-1)$  the lemma 4.12 implies

$$\mathbb{P}(X_t = x) = \text{const}(h(x))^2 \prod_{k=1}^d \frac{e^{-\lambda t} (\lambda t)^{x_k}}{x_k!}$$

Analogue of BM conditioned to stay in the continuous Weyl chamber,  $B(0) = 0$ , is

$$\mathbb{P}(B(t) \in dx) = \text{const}(h(x))^2 \prod_{k=1}^d \left( \frac{e^{-\frac{x_k^2}{2t}}}{\sqrt{2\pi t}} dx_k \right)$$

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# Chapter 5:

## The Kardar-Parisi-Zhang (KPZ) equation

### 5.1 Heuristic derivation

Start of lecture 18  
(25.06.24)

**Want:** describe the evolution of an interface

$$x \mapsto h(x, t) \in \mathbb{R} \quad x \in \mathbb{R}, t \geq 0$$

where the growth is given by some local process  
The analogue in 0-dimensions would be

$$h(t) \stackrel{t \in \mathbb{Z}}{=} \sum_{k=1}^t X_k,$$

where the  $X_k$  are iid r.v. with bounded variation. We know:

$$\frac{h(t)}{t} \xrightarrow{\text{a.s.}} \mu \in \mathbb{R} \text{ (law of large numbers)}$$

and

$$\frac{h(t) - \mu t}{\sigma \sqrt{t}} \xrightarrow{\mathbb{P}} \mathcal{N}(0, 1) \text{ (CLT)}$$

We call the first case **Macroscopic** and the second case **Fluctuations** or **mesoscopic**.  
From a macroscopic point of view:

$$\implies \delta_t h = v(\partial_x h)$$

for some function  $v$  (Hamitten-Jacobi equation).

**Question:** How to add noise?

Consider some discrete examples

**Ballistic deposition:**  $\forall x \in \mathbb{Z}$ : independent Poisson process (intensity 1),  $h(x, t) \in \mathbb{Z}, t \geq 0$

If at time  $t$ , there is a jump of the PP at site  $x$ , either

(a) add it on top of  $h(x, t)$ :  $h(x, t) \rightarrow h(x, t) + 1$

(b) stick to one of the neighboring sites if they are higher:  $h(x, t) \rightarrow \max\{h(x+1, t), h(x-1, t)\}$

We can use one Poisson process like previously for a finite window.

For the evolution:

$$h(x, t) = \max\{h(x, t_-), h(x-1, t_-), h(x+1, t_-)\}$$

**Question:** How fast do the fluctuations grow?

Simulations shows

$$\sqrt{\text{Var}_x(h(x, t))} \approx t^{\frac{1}{3}} \text{ for large } t$$

If there would not be any interaction between sites  $\xrightarrow{\text{CLT}} t^{\frac{1}{2}} \implies$  Stickiness reduces the fluctuations.

Influence of Stickiness is two-fold:

- (a) growth is depending on the slope (of the starting / boundary condition / the previous state?) → already modelled by  $v(\partial_x h)$
- (b) some regularization going on → simplest regularization in the continuum → Heat equation:

$$\partial_t h = \partial_x^2 h$$

$$\implies \partial_t h = v(\partial_x h) + \partial_x^2 h$$

**Question:** What kind of noise should we consider?

Simplest would be to consider **Gaussian noise**, without correlation in space and time (like in the discrete model).

**Informally:** the space-time white noise  $\xi(x, t)$  is the centered **Gaussian process** with variance  $\mathbb{E}(\xi(x, t)\xi(y, s)) = \delta(x - y)\delta(t - s)$

**Added remark (WARNING).**  $\xi$  is not a function, but a distribution!

**Definition 5.1.** The **space time white noise**  $\xi$  is the Gaussian process satisfying

$$\forall \varphi_1(x, t), \varphi_2(x, t) \in C_0^\infty$$

$$\implies \mathbb{E} \left( \int dx dt \varphi_1(x, t) \xi(x, t) \right) = 0 \text{ centered}$$

and

$$\mathbb{E} \left( \int \varphi_1(x, t) \xi(x, t) dx dt \cdot \int \varphi_2(y, s) \xi(y, s) dy ds \right) = \int \varphi_1(x, t) \varphi_2(x, t) dx dt$$

*This is a bit informally written, since  $\xi$  is a distribution and not absolutely continuous w.r.t. Lebesgue*

**Added remark.** Careful:  $\mathbb{E}(\xi(x, t)^2) = \infty$ . Pointwise this makes no sense!

$\implies$  We would get:

$$\partial_t h = \partial_x^2 h + v(\partial_x h) + \xi$$

Physicists argued: small  $\partial_x h$

- $v(\partial_x h) \approx v(0) + v'(0)\partial_x h + \frac{1}{2}v''(0)(\partial_x h)^2 + \dots$
- The first two terms can be removed wlog, since

$$\partial_t h = v(0) \rightarrow h_0 + v(0)t \implies h(x, t) + v(0)t =: \tilde{h}(x, t)$$

$$\implies \partial_t \tilde{h} = \partial_x^2 \tilde{h} + v'(0)\partial_x \tilde{h} + \dots$$

- Physics should not depend on the reference system we use!

Then **KPZ equation:**

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

Or by keeping track of physical constants:

$$\partial_t h = \nu \partial_x^2 h + \lambda (\partial_x h)^2 + \sqrt{D} \xi$$

**Problem:**  $|\partial_x h|$  is NOT SMALL!!!

Locally,  $h$  is like a Brownian motion →  $\partial_x h$  is a distribution!

$$\begin{aligned} \int \varphi(x) \partial_x h(x) dx &= - \int \varphi'(x) h(x) dx \\ \int \varphi(x) (\partial_x h(x))^2 dx &\stackrel{???}{=} - \int \varphi'(x) h(x) dx \end{aligned}$$

## 5.2 White noise

**Example 5.2.** Let  $S$  be a nice subset of  $\mathbb{R}^d$  and  $e_1, \dots$  an orthonormal basis of  $L^2(S)$ . Let  $\xi_1, \dots$ , iid  $\mathcal{N}(0, 1)$  r.v.

These might be eigenvectors of some operator

$$\implies \xi(x) = \sum_{n \geq 1} \xi_n e_n(x)$$

is a white noise on  $S$ .

*Proof.* Let  $\varphi_1, \varphi_2 : S \rightarrow \mathbb{R}$  be test functions.

$$\mathbb{E} \left( \int_S \varphi_1 \xi(x) dx \right) = \mathbb{E} \left( \int_S \varphi_1 \sum_{n \geq 1} \xi_n e_n(x) dx \right) = \sum_{n \geq 1} \mathbb{E} \left( \underbrace{\xi_n}_{\mathbb{E}(\xi)=0} \int_S \varphi_1 e_n(x) dx \right) = 0$$

and

$$\begin{aligned} \mathbb{E} \left( \int_S \varphi_1(x) \xi(x) dx \int_S \varphi_2(y) \xi(y) dy \right) &= \sum_{n, m \geq 1} \mathbb{E}(\xi_n \xi_m) \int_S \int_S \varphi_1(x) \varphi_2(y) e_n(x) e_m(y) dx dy \\ &= \int dx dy \varphi_1(x) \varphi_2(y) \underbrace{\sum_{n \geq 1} e_n(x) e_n(y)}_{=1_{x=y}} = \int dx \varphi_1(x) \varphi_2(x). \end{aligned}$$

□

**Added remark.** For orthonormal systems:

$$f(x) = \sum_{n \geq 1} e_n(x) (e_n, f) = \int dy f(y) \underbrace{\sum_{n \geq 1} e_n(x) e_n(y)}_{=\delta(x-y)}$$

**Example 5.3.** Let  $S = \mathbb{R}_+$  and  $\xi$  white noise on  $S$ .

$$\implies B(t) = \int_S 1_{[0, t)}(s) \xi(s) ds$$

is a Brownian motion.

Start of lecture 19  
(27.06.24)

**Lemma 5.4.** Let  $\xi$  be a white noise on  $\mathbb{R} \times \mathbb{R}_+$ . Then

$$\xi(x, t) \stackrel{d}{=} \epsilon^{\frac{z+1}{2}} \xi(\epsilon^1 x, \epsilon^z t)$$

where  $z$  is called the **dynamical exponent** (in time).

*Proof.*

$$\mathbb{E} \left( \left( \int \varphi_1(x, t) \xi(x, t) dx dt \int \varphi_2(y, s) \xi(y, s) dy ds \right) \right) \stackrel{\text{def}}{=} \int \varphi_1(x, t) \varphi_2(x, t) dx dt$$

But also

$$\begin{aligned} &\mathbb{E} \left( \int \phi_1(x, t) \epsilon^{\frac{z+1}{2}} \xi(\epsilon x, \epsilon^z t) dx dt \int \phi_2(y, s) \epsilon^{\frac{z+1}{2}} \xi(\epsilon y, \epsilon^z s) dy ds \right) \\ &= \epsilon^{z+1} \mathbb{E} \left( \varphi_1(\epsilon^{-1} x, \epsilon^{-z} t) \xi(x, t) dx dt \epsilon^{-(z+1)} \varphi_2(\epsilon^{-1} y, \epsilon^{-z} s) \xi(y, s) dy ds \epsilon^{-(z+1)} \right) \\ &= \epsilon^{-(z+1)} \int \varphi_1(\epsilon^{-1} x, \epsilon^{-z} t) \varphi_2(\epsilon^{-1} x, \epsilon^{-z} t) dx dt \\ &= \int \varphi_1(x, t) \varphi_2(x, t) dx dt \end{aligned}$$

using  $\tilde{x} = \epsilon x \mapsto x$  and  $t = \epsilon^z t \mapsto t$ .

□

### 5.3 The 1:2:3 scaling

Start with

$$\partial_t h = \nu \partial_x^2 h + \lambda (\partial_x h)^2 + \sqrt{D} \xi.$$

Let us see that wlog we can set  $\nu = \lambda = \frac{1}{2}, D = 1$ . Define

$$\tilde{h}(x, t) := \alpha h(\gamma x, \beta t)$$

which implies

$$\partial_t \tilde{h}(x, t) = \beta \alpha \dot{h}(\gamma x, \beta t)$$

using

- $\partial_t = \frac{1}{\alpha \beta} \partial_t \tilde{h}$
- $\partial_x h = \frac{1}{\alpha \gamma} \partial_x \tilde{h}$
- $\partial_x^2 h = \frac{1}{\alpha \gamma^2} \partial_x^2 \tilde{h}$

we get

$$\begin{aligned} \partial_t \tilde{h} &= \alpha \beta \partial_t h \\ &= \alpha \beta \left[ \nu \frac{1}{\alpha \gamma^2} \partial_x^2 \tilde{h} + \lambda \frac{1}{\alpha^2 \gamma^2} (\partial_x \tilde{h})^2 + \sqrt{D} \frac{1}{\sqrt{\beta \gamma}} \xi \right] \\ &= \underbrace{\frac{\beta \nu}{\gamma^2}}_{=\frac{1}{2}} \partial_x^2 \tilde{h} + \underbrace{\frac{\lambda \beta}{\alpha \gamma^2}}_{=\frac{1}{2}} (\partial_x \tilde{h})^2 + \underbrace{\sqrt{D} \alpha \sqrt{\frac{\beta}{\gamma}}}_{=1} \xi \end{aligned}$$

which implies  $\alpha = \frac{\lambda}{\nu}, \beta = \frac{2\nu^5}{D^2 \lambda^4}, \gamma = \frac{2\nu^3}{D \lambda^2}$ . The standard choice we will use below is

$$\partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} (\partial_x h)^2 + \xi$$

which is the **KPZ equation**.

Start with  $\partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} (\partial_x h)^2 + \xi$ . Define

$$h_\epsilon(x, t) := \underbrace{\epsilon^b}_{T^{-b/z} x} h(\underbrace{\epsilon^{-1} x}_{x T^{1/z}}, \underbrace{\epsilon^{-z} t}_{=t T})$$

$$h_\epsilon(x, 1) := \epsilon^b h(\epsilon^{-1} x, \epsilon^{-z} 1)$$

If  $b, z$  choosen s.t.  $\lim_{\epsilon \downarrow 0} h_\epsilon(x, 1)$  is well defined ( and non-trivial), then the Fluctuations of  $h$  at large time  $T$  is  $O(T^{b/z})$  and correlations are non-trivial over distances  $O(T^{\frac{1}{z}})$ .

Discrete model: Picture here ...

If  $(p - q)T^c \rightarrow 0$  the height converges to the height of KPZ eq.

Stationary measure for the increment are simple random walks.

Under scaling: One gets BM increments. If one starts with a 2-sided BM, more precisely

$$h(x, 0) = \sqrt{2}B(x) \implies h(x, t) - h(0, t) \stackrel{d}{=} \sqrt{2}B(x)$$

And therefore  $b = \frac{1}{2}$ .

- $\partial_t h = \epsilon^{z-b} \partial_t h_\epsilon$
- $\partial_x h = \epsilon^{1-b} \partial_x h_\epsilon$
- $\partial_x^2 h = \epsilon^{2-b} \partial_x^2 h_\epsilon$
- $\xi(x, \epsilon^{-1}, t \epsilon^{-z}) \stackrel{d}{=} \epsilon^{\frac{z+1}{2}} \xi(x, t)$

$h_\epsilon$  solves

$$\begin{aligned} \implies \partial_t h_\epsilon &= \frac{1}{2} \epsilon^{2-z} \partial_x h_\epsilon + \frac{1}{2} \epsilon^{2-z-b} (\partial_x h_\epsilon)^2 + \epsilon^{b-\frac{z+1}{2}+\frac{1}{2}} \xi \\ &\stackrel{b=\frac{1}{2}}{=} \frac{1}{2} \epsilon^{2-z} \partial_x^2 h_\epsilon + \frac{1}{2} \epsilon^{\frac{3}{2}-z} (\partial_x h_\epsilon)^2 + \epsilon^{1-\frac{z}{2}} \xi \end{aligned}$$

The nonlinear term  $\frac{1}{2} \epsilon^{\frac{3}{2}-z} (\partial_x h_\epsilon)^2$  is still relevant, but not  $\rightarrow \infty$ . This gives the idea  $z = \frac{3}{2}$ .

$$\partial_t h_\epsilon = \frac{1}{2} \epsilon^{\frac{1}{2}} \partial_x^2 h_\epsilon + \frac{1}{2} (\partial_x h_\epsilon)^2 + \epsilon^{\frac{1}{4}} \xi$$

$\implies$  at large time  $t$ , fluctuations of  $h(x, t) \approx t^{\frac{1}{3}}$ , correlations in space  $\approx t^{\frac{2}{3}}$ .

**Question: Are the  $\epsilon^{\frac{1}{2}}$  and  $\epsilon^{\frac{1}{4}}$  terms relevant as  $\epsilon \rightarrow 0$ ?**

1.:  $\partial_t h_\epsilon = \frac{1}{2} (\partial_x h_\epsilon)^2$  is not well-posed. with the term  $\sqrt{\epsilon} \partial_x^2 h_\epsilon$

$$\implies \partial_t h = \frac{1}{2} (\partial_x h)^2 + \frac{1}{2} \sqrt{\epsilon} \partial_x^2 h$$

which is the viscosity solution. Then taking take  $\epsilon \rightarrow 0$  gives

$$h(x, t) = \sup_y (h(y, 0) - \frac{(x-y)^2}{2t})$$

If you start with  $h_\epsilon(x, 0) = \sqrt{2}B(x) \implies h(x, t)$  has some explicit expression. This solution do not satisfy  $h(x, t) - h(x, 0) \stackrel{d}{=} \sqrt{2}B(x)$ . Therefore  $\epsilon^{\frac{1}{4}} \xi$  is relevant as  $\epsilon \rightarrow 0$ .

## 5.4 The linearized case: Edwards-Wilkinson equation

Let  $\lambda = 0 \implies$

$$\begin{cases} \partial_t h &= \frac{1}{2} \partial_x^2 h + \xi \\ h(x, 0) &= h_0(x) \end{cases} \quad (5.1)$$

It is linear!

**Remark.**

$$\begin{cases} \frac{\partial}{\partial t} h &= Ah + f \\ h(0) &= h_0 \end{cases} \implies$$

denote by  $\tilde{h}(t) := e^{At} h_0$  as the solution with  $f = 0$ . The variation of constant formula gives:

$$h(t) = e^{At} h_0 + \int_0^t e^{A(t-s)} f(s) ds$$

Without  $\Xi$ :  $h(x, t) = \int_{\mathbb{R}} P_t(x-y) h_0(y) dy$ ,  $P_t(x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$

$$\implies h(x, t) = \int_{\mathbb{R}} P_t(x-y) h_0(y) dy + \int_0^t \int_{\mathbb{R}} P_{t-s}(x-y) \xi(y, s) dy ds$$

Scaling of EW:

$$h_\epsilon(x, t) := \epsilon^{\frac{1}{2}} h(\epsilon^{-1} x, \epsilon^{-z} t) = \epsilon^{\frac{1}{2}} h(\epsilon^{-1} x, \epsilon^{-2} t)$$

yields ( with the same calculations as before, which leads to  $z = 2$ )

$$\partial_t h_\epsilon = \frac{1}{2} \epsilon^{2-z} \partial_x^2 h_\epsilon + \epsilon^{1-\frac{z}{2}} \xi = \frac{1}{2} \partial_x^2 h_\epsilon + \xi$$

Therefore, at large time  $t$ , fluctuations will be  $O(t^{\frac{1}{z}}) = O(t^{\frac{1}{2}})$ . The nonlinear term in the KPZ increases the size of the fluctuations!



## 5.5 The Cole-Hopf transformation

We have:  $\partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} (\partial_x h)^2 + \xi$

This sadly only works in 1 dimension

**Lemma 5.5.** Assume that  $h$  solves

$$\partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} (\partial_x h)^2 + V$$

with  $V$  some nice  $V$  in  $(x, t)$ .

Then  $z = e^h$  solves

$$\partial_t z = \frac{1}{2} \partial_x^2 z + zV$$

*Proof.*  $h = \ln(z)$ , and therefore

$$\frac{1}{2} \frac{(\partial_x z)^2}{z^2} + \frac{1}{2} \left( \frac{\partial_x^2 z}{z} - \frac{(\partial_x z)^2}{z^2} \right) + V = \partial_t h = \frac{1}{z} \partial_t z$$

□

Suppose that we regularize  $\xi$  in space:

$$\xi \rightarrow \xi_\epsilon := \int dy \rho_\epsilon(x-y) \xi(y, t)$$

where  $\rho_\epsilon = \frac{e^{-\frac{x^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}}$ .

Solve  $\partial_t z = \frac{1}{2} \partial_x^2 z + z \xi_\epsilon$ . Set  $h(x, t) := \ln(z(x, t))$ .

Exercise: In the above setting,  $h$  solves:

$$\partial_t h = \frac{1}{2} \partial_x^2 h + \left( \frac{1}{2} (\partial_x h)^2 - \frac{C}{\epsilon} \right) + \xi_\epsilon$$

for some explicit  $C$ .

If one solves  $\partial_t z = \frac{1}{2} \partial_x^2 z + z \xi$  and define  $h(x, t) := \ln(z(x, t))$ . This is called the **Cole-Hopf solution of the KPZ eq.**

Exam:

Start of lecture 20  
(02.07.24)

- Speak about topic we choose (freely): 15 min discussion
- He will not talk about us about the last part
- Talk the rest of the time about the rest of the lecture

If we are smooth enough, the  $C$  in the Cole-Hopf-equation will be 0.

In order to have a limit for  $\epsilon \rightarrow 0$ :

$$h(x, t) := \tilde{h}(x, t) - \frac{c}{\epsilon} t$$

$\xi_\epsilon$  is still centered Gaussian with covariance:

$$\begin{aligned} \mathbb{E}(\xi_\epsilon(x, t) \xi_\epsilon(y, s)) &= \mathbb{E} \left( \int_{\mathbb{R}} d\tilde{x} \rho_\epsilon(x - \tilde{x}) \xi(\tilde{x}, t) \int_{\mathbb{R}} d\tilde{y} \rho_\epsilon(y - \tilde{y}) \xi(\tilde{y}, s) \right) \\ &= \delta(t - s) \cdot \underbrace{\int_{\mathbb{R}} d\tilde{x} \rho_\epsilon(x - \tilde{x}) \rho_\epsilon(y - \tilde{x})}_{=: C_\epsilon(x-y)} \end{aligned}$$

In particular,  $C_\epsilon(0) = \frac{1}{\sqrt{2\pi\epsilon}} = \frac{c}{\epsilon}$ .

Let  $z = e^h$  be the solution of

$$(\star) \quad \partial_t z = \frac{1}{2} \partial_x^2 z + z \cdot \xi_\epsilon$$

- One checks  $z^\epsilon \xrightarrow{\epsilon \rightarrow 0} z$  uniformly on compact sets.
- $z(x, t) > 0$  (starting from  $z(x, 0) > 0$ ), then define  $h(x, t) = \ln(z(x, t))$
- Define  $h^\epsilon := \ln z^\epsilon(x, t)$ . it solves the following equation:

$$\partial_x h^\epsilon = \frac{1}{2} \partial_x^2 h^\epsilon + \frac{1}{2} (\partial_x h^\epsilon)^2 - C_\epsilon(0) + \xi_\epsilon$$

- In differential notation

$$dz^\epsilon = \frac{1}{2} \partial_x^2 z^\epsilon dt + z^\epsilon \underbrace{\xi_\epsilon dt}_{=d\xi_\epsilon}$$

Therefore

$$\begin{aligned} dh^\epsilon &= d \ln z^\epsilon \stackrel{\text{Itô}}{=} \frac{1}{z^\epsilon} dz^\epsilon - \frac{1}{2} \frac{1}{(z^\epsilon)^2} (dz^\epsilon)^2 \\ &= \frac{1}{2} \frac{1}{z^\epsilon} \partial_x^2 z^\epsilon dt + d\xi_\epsilon - \frac{1}{2} (d\xi_\epsilon)^2 \\ &= \frac{1}{2} \partial_x^2 h^\epsilon dt + \frac{1}{2} (\partial_x h^\epsilon)^2 dt + d\xi_\epsilon - \frac{1}{2} C_\epsilon(0) dt \end{aligned}$$

which implies

$$\partial_t h^\epsilon = \frac{1}{2} \partial_x^2 h^\epsilon + \frac{1}{\epsilon} ((\partial_x h^\epsilon)^2 - C_\epsilon(0)) + \xi_\epsilon$$

The Cole-Hopf solution of the KPZ equation is

$$h(x, t) = \ln(z(x, t))$$

where  $z$  solves

$$\begin{cases} \partial_t z &= \frac{1}{2} \partial_x^2 z + z \cdot \xi \\ z(x, 0) &= e^{h(x, 0)} \end{cases}$$

The difficulty in the multiplicative **stochastic heat equation (SHE)** is the term  $z\xi$ . Still it is much better than squaring a distribution.

Since  $h(x, t)$  will be locally BM,  $z(x, t) = e^{h(x, t)}$  is also locally gaussian. In the integral form:

$$z\xi \rightarrow \int_0^t z \underbrace{\xi ds}_{\approx dB_s}$$

### 5.5.1 Mild solutions

**Definition 5.6.** Let  $\begin{cases} \partial_t z &= \frac{1}{2} \partial_x^2 z + z \cdot \xi \\ z(x, 0) &= z_0(x) \end{cases}$  be  $(\star)$ .  $z$  is a **mild solution** of  $(\star)$ , if

$$z(x, t) = \int_{\mathbb{R}} P_t(x - y) z_0(y) dy + \int_0^t ds \int_{\mathbb{R}} dy P_{t-s}(x - y) z(y, s) \xi(y, s) \quad (A)$$

where  $z$  is predictable and

$$\int_0^t ds \int_{\mathbb{R}} (P_{t-s}(x - y))^2 \mathbb{E}(z(y, s))^2 < \infty \quad (B)$$

**Theorem 5.7.** Suppose that  $\sup_{x \in \mathbb{R}} \mathbb{E}(z_0^2(x)) < \infty$ . Then there exists a unique, progressively measurable  $z(x, t)$  satisfying (A) and (B).

*Proof.* Let  $z^0(x, t) = 0$ . Define iteratively

$$z^{n+1}(x, t) = \int_{\mathbb{R}} P_t(x - y) z_0(y) dy + \int_0^t ds \int_{\mathbb{R}} P_{t-s}(x - y) z^n(y, s) \xi(y, s). \text{ Let } \delta z^n(x, t) = z^{n+1}(x, t) - z^n(x, t).$$

$$\delta z^{n+1} = \int_0^t ds \int_{\mathbb{R}} dy P_{t-s}(x - y) \delta z^n(y, s) \xi(y, s)$$

$$\mathbb{E}(|\delta z^{n+1}|^2) = \int_0^t ds \int_{\mathbb{R}} dy (P_{t-s}(x - y))^2 \mathbb{E}(|\delta z^n(y, s)|^2).$$

Define  $f^n(t) := \sup_{\substack{x \in \mathbb{R} \\ 0 \leq s \leq t}} \mathbb{E}(|\delta z^n(x, s)|^2)$ . Assume that  $f^n(t) < \infty$

For  $n = 0$  it is true:  $\delta z^1(x, t) = z^1(x, t) = \int_{\mathbb{R}} P_t(x - y) z_0(y) dy$  and therefore

$$\mathbb{E} \left( \left| \int_{\mathbb{R}} P_t(x - y) z_0(y) dy \right|^2 \right) \stackrel{\text{Jensen}}{\leq} \mathbb{E} \left( \int_{\mathbb{R}} P_t(x - y) |z_0(y)|^2 dy \right) = \int_{\mathbb{R}} P_t(x - y) \mathbb{E}(|z_0(y)|^2) dy \leq \sup_{y \in \mathbb{R}} \mathbb{E}(|z_0(y)|^2)$$

Now we bound  $f^{n+1}$ :

$$\begin{aligned} f^{n+1}(t) &= \sup_{\substack{x \in \mathbb{R} \\ 0 \leq s \leq t}} \int_0^s d\tilde{s} \int_{\mathbb{R}} dy (P_{t-\tilde{s}}(x - y))^2 f^n(\tilde{s}) \\ &\leq \sup_{0 \leq s \leq t} \int_0^s d\tilde{s} f^n(\tilde{s}) \frac{C}{\sqrt{s - \tilde{s}}} \\ &\leq \int_0^t d\tilde{s} \frac{f^n(\tilde{s})}{\sqrt{t - \tilde{s}}} C_2 \leq \tilde{C} \int_0^{\tilde{s}} d\tilde{s} \int_0^{\tilde{s}} du \frac{f^{n-1}(u)}{\sqrt{t - \tilde{s}} \sqrt{\tilde{s} - u}} \\ &= C_3 \int_0^t du f^{n-1}(u) \underbrace{\int_u^t d\tilde{s} \frac{1}{\sqrt{\tilde{s} - u}} \sqrt{t - \tilde{s}}}_{\leq C_4} \\ &\implies f^{n+1}(t) \leq \hat{C} \int_0^t du f^{n-1}(u) \leq (\hat{C})^2 \int_0^t du \int_0^u du - f^{n-3}(u) \\ &\implies f^{2n}(t) \leq (\hat{C})^n \frac{t^n}{n!} \sup_{y \in \mathbb{R}} \mathbb{E}(|z_0(y)|^2) \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

where the convergence happens very fast! There there exists a solution  $z$  satisfying (A) and (B) □

For  $z_0 = e^{h_0(x)}$ ,  $h_0(x) = B(x)1_{x \geq 0}$ , we can not use 5.7! Our result is a bit to stiff.

**Lemma 5.8.** *Let  $z(x, t)$  satisfy (A) and (B) and*

$$\mathbb{E} \left( \left| \sup_{A \subseteq [-n, n]} \int_A z_0(x) dx \right| \right) \leq c e^{cn} \text{ for some } c \in (0, \infty)$$

*Then there exists  $\tilde{c} < \infty$ ,  $\mathbb{E}(z^2(x, t)) \leq \frac{\tilde{c}}{\sqrt{t}} e^{\tilde{c}(t+|x|)}$  for all  $x \in \mathbb{R}, t > 0$ .*

In the special case  $z_0(x) = \delta_0(x)$ , there exists  $c = c(T)$  s.t.  $\mathbb{E}(z^2(x, t)) \leq c(P_t(x))^2$  for all  $0 < t \leq T$ , where  $P_t = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$

We want to know: What is  $\mathbb{P}(z(x, t) \leq s)$  for a fixed  $x$ ?

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$$\begin{aligned} \mathbb{E} \left( \left| \int_0^t X_s \underbrace{\xi_s ds}_{dB_s} \right|^2 \right) &= \mathbb{E} \left( \int_0^t X_s^2 ds \right) \\ &= \int_0^t \mathbb{E}(X_s^2) ds \end{aligned}$$

$$\mathbb{E} \left( \left| \int_0^t ds \int dy p_{t-s}(x-y) \delta z^n(y, s) \xi(y, s) \right|^2 \right) \stackrel{?}{=} \int_0^t ds \int dy (P_{t-s}(x-y))^2 \mathbb{E}(|\delta z^n(y, s)|^2)$$

And a correction to the iteration from last time:

$$f^n(t) \sup_x \sup_{0 \leq s \leq t} \mathbb{E}(|\delta z^n(x, s)|^2)$$

instead consider

$$g_n(t) = \sup_x \mathbb{E}(|\delta z^n(x, s)|^2)$$

**End of correction**

**Question:**  $\mathbb{P}(h(0, t) \leq s) = ?$

## 5.6 Feynman-Kac representation

$$\begin{cases} u(x, 0) = f(x) \\ \partial_t u = \frac{1}{2} \partial_x^2 u + V(x, t) u \end{cases} \quad (5.2)$$

If  $V$  is nice enough (e.g. bounded)  $\implies$  (see lecture 20) a solution is

$$u(x, t) = \mathbb{E}_x \left( f(B_t) e^{\int_0^t V(t-s, B_s) ds} \right)$$

for some BM  $(B_t)_{t \geq 0}$  starting in  $x$  and  $\tilde{B}$  running backwards with  $B_t = x$ :

$$u(x, t) = \mathbb{E}_x \left( f(\tilde{B}_0) e^{\int_0^t V(s, \tilde{B}_s) ds} \right)$$

**Formally:**  $z(x, t) = \mathbb{E}_x(z(0, B_0)) \cdot e^{\int_0^t \xi(s, B_s) ds}$  with  $B$  BM from time  $t$  to time 0 with  $B_t = x$ .

Issue of this:

$$e^{\int_0^t \xi(s, B_s) ds} \sum_{n \geq 0} \frac{1}{N!} \int_0^t ds_1 \dots \int_0^t ds_n \xi(s_1, B_{s_1}) \dots \xi(s_n, B_{s_n})$$

We have to be careful about the diagonal, to avoid squaring the distribution  $\xi(s, B_s)$ .

$$\begin{aligned} e^{\int_0^t \xi(s, B_s) ds} &= \sum_{n \geq 0} \frac{1}{n!} \int_0^t ds_1 \dots \int_0^t ds_n \xi(s_1, B_{s_1}) \dots \xi(s_n, B_{s_n}) \\ &= \sum_{n \geq 0} \frac{1}{n!} \int_{0 < s_1 < \dots < s_n < t} ds_1 \dots ds_n \xi(s_1, B_{s_1}) \dots \xi(s_n, B_{s_n}) \end{aligned}$$

$$\implies z(x, t) = \sum_{n \geq 0} \int_{\mathbb{R}} dy \int_{0 < s_1 < \dots < s_n < t} ds_1 \dots ds_n \prod_{k=1}^N \int dx_k \xi(x_k, s_k) P_{x,t}(y, 0, x_1, s_1, \dots, x_n, s_n) z(y, 0)$$

### 5.6.1 A discrete example

Take for example  $\mathbb{Z}^2$  as time  $\times$  space. Starting in  $(x, t)$  look at a random walk with path  $\pi$  with nearest neighbor steps. To each  $(i, j) \in \mathbb{Z}^2$ -i iid random variable  $\omega(i, j)$

$$\implies Z(x, n) := \mathbb{E} \left( z(0, x_0) e^{-\beta \sum_{k=1}^n \omega(k, x_k)} \right)$$

We can imagine many other discretizations.

## 5.7 Moments of the multiplicative SHE

$$\mathbb{E}(e^{-\lambda z(x,t)}) = ?$$

if  $z(x, t) \geq 0$ , the laplace transform above is well defined.

$$= \mathbb{E} \left( \sum_{n \geq 0} \frac{(-\lambda)^n}{n!} z(x, t)^n \right) \stackrel{?}{=} \sum_{n \geq 0} \frac{(-\lambda)^n}{n!} \mathbb{E}(z(x, t)^n)$$

which is useful, since  $\mathbb{E}(z(x, t)^n)$  has explicit expressions:

**Strategy to compute these moments:** Start with noise  $\xi_\epsilon = \rho_\epsilon * \xi$ , with  $\rho_\epsilon(x, t) = \epsilon^{-1} \rho(\epsilon^{0.5} x, \epsilon^{0.5} t)$ , where  $\rho$  is even, smooth, compactly supported non-negative function with

$$\int \rho(x, t) dx dt = 1.$$

Then  $z^\epsilon(x, t) = \mathbb{E}_x \left( z(0, B_0) e^{\int_0^t x i_\epsilon(s, B_s) ds} e^{-C_\epsilon t} \right)$  and

$$z^\epsilon(x, t) \rightarrow z(x, t) \text{ nicely}$$

As  $\epsilon \rightarrow 0$

$$\mathbb{E}((z(x, t))^n) = \mathbb{E}_{(x, \dots, x)} \left( e^{-\frac{1}{2} \sum_{i < j} \int_0^t \text{sgn}(B_i - B_j) (dB_i - dB_j)} F(B(t)) \right)$$

for some  $n$ -dimensional BM,  $B_i(0) = x$  and

$$F(B) = e^{\frac{1}{2} \sum_{i < j} |B_i - B_j|} \prod_{i=1}^n z_0(B_i)$$

Using antisymmetry:

$$\begin{aligned} \sum_{i < j} \int_0^t \text{sgn}(B_i - B_j) (dB_i - dB_j) &= \sum_{i=1}^N \sum_{j \neq i} \int_0^t \text{sgn}(B_i - B_j) \underbrace{dB_i}_{=dB_i^i} \\ \mathbb{E}(e^{-\frac{1}{2} \sum_{i=1}^n (\sum_{j \neq i} \int_0^t \text{sgn}(B_i - B_j) dB_i)}) &= \mathbb{E} \left( e^{\frac{1}{8} \sum_{i=1}^n \int_0^t \sum_{j \neq i} \underbrace{(\text{sgn}(B_i - B_j))^2}_{=1}} ds \right) \end{aligned}$$

Trick:

$$\sum_{i=1}^n \left( \sum_{j \neq i} \text{sgn}(B_i - B_j) \right)^2 = \frac{1}{3} n(n^2 - 1)$$

$\implies \mathbb{E}(z^n(x, t)) \approx e^{\frac{n^3}{24}}$ . We then get

$$\sum_{n \geq 0} \frac{(-\lambda)^n}{n!} \underbrace{\mathbb{E}(z(x, t)^n)}_{\approx e^{\frac{1}{24} n^3}} \rightarrow \text{not ok}$$

$\implies$  The moments grow too fast and they do not determine the distribution of  $z(x, t)$ . there is some  $f$ :

$$e^{t^3} = \int dz e^{zt} f(z)$$

## 5.8 The q-TASEP

The solution of the multiplicative SHE with  $z_0(x) = \delta_0(x)$  is

$$z(x, t) = \mathbb{E}_x \left( \delta_0(B_0) e^{\int_0^t \xi(s, B_s) ds} \right)$$

Fix  $q \in (0, 1)$ .

**TASEP=Totally Asymmetric Exclusion Process**

- **Configuration space:** particle  $n$  at time  $t$  is at position  $x_n(t)$  with  $x_{n+1}(t) < x_n(t)$  for all  $t, n$ .
- **Dynamics:** each particle tries to jump to its right by one (+1) with **rate** 1. The jump takes place only if the arrival site is empty.

### 5.8.1 q-TASEP

here the jump rate of particle  $n$  is given by

$$1 - q^{X_{n-1}(t) - X_n(t) - 1}.$$

Consider the **initial condition**  $X_n(0) = -n, n \geq 1$ .

**Height function:**

$$\begin{cases} h(0, 0) = 0 & h(0, t) = 2(\# \text{ particles which jumped from to during } [0, t]) \\ h(x+1, t) - h(x, t) = 1 \end{cases}$$

$$= \begin{cases} h(0, 0) & = h(0, t) = 2(\# \text{ particles which jumped from to during } [0, t]) \\ \begin{cases} 1 & \nexists \text{ particle at time } t \\ -1 & \exists \text{ particle at time } t \end{cases} \end{cases}$$

One can compute: (using Macdonald polynomials)

$$\mathbb{E} \left( q^{\underbrace{l \cdot (x_n(t) + n)}_{\lambda_n(t=x_n(t)-n)}} \right)$$

$$x_n(0) = -n \implies x_n(t) + n \geq 0$$

$$0 \leq q^{l \cdot \lambda_n(t)} \leq n$$

**Introduce:**  $(x; q)_\infty := \prod_{k \geq 0} (1 - x^k)$

**Generating function**

$$\mathbb{E} \left( \frac{1}{(\zeta q^{\lambda_n(t)}; q)_\infty} \right)^{\text{Re} \zeta < 0} \mathbb{E} \left( \sum_{l \geq 0} \frac{q^{l \cdot \lambda_n(t) \zeta^l}}{(1-q) \cdots (1-q^l)} \right)^{|\zeta| \leq 1} \sum_{l \geq 0} \frac{\zeta^l}{(1-q) \cdots (1-q^l)} \mathbb{E}(q^{l \cdot \lambda_n(t)})$$

**magically** (algebra is the magician)

$$\sum_{l \geq 0} \frac{\zeta^l}{(1-q) \cdots (1-q^l)} \mathbb{E}(q^{l \cdot \lambda_n(t)}) = \det(1 + K_{\zeta, n})_{L^2(\mathbb{C})}$$

where  $K_\zeta$  is an operator and we use the Fredholm determinant

$$:= \sum_{m \geq 0} \frac{1}{m!} \int_{\mathbb{C}} dz_1 \cdots \int_{\mathbb{C}} dz_m \det[K_{\zeta, n}(z_i, z_j)]_{1 \leq i, j \leq m}$$

which is **purely analytic**, there is no randomness involved!

Next steps:

$q \rightarrow 1$  limit will lead to a semi-discrete polymov model  $\rightarrow$  solution  $z(x, t)$

**Added remark.** We can sometimes drop the condition  $|\zeta| < 1$  if the rhs is well defined, since we have to analytic functions which agree on an open set.

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# List of Lectures

- **Lecture 01:** Introduction, reminder of strong solutions, definition of weak solutions, uniqueness in law, pathwise uniqueness, and some examples
- **Lecture 02:** Further examples, Yamata-Watanabe theorems and Skorohod theorem (no proof), reminder of Lévy characterization, Ito-Doeblin formula
- **Lecture 03:** The martingale problem and one-to-one relation with weak solutions (special case of  $d = n$  proven); reminder of Dubins-Schwarz theorem
- **Lecture 04:** Transformation of SDE under time change, weak solutions for 1d SDEs, scale function and its relation to hitting times
- **Lecture 05:** Uniqueness of the solution of martingale problem, reminder of Girsanov theorem, changes of SDE under drift transformation
- **Lecture 06:** Drift transformation for SDE, Doob-h transform, start set-up for diffusion bridges
- **Lecture 07:** Diffusion bridges, set-up for Brownian motion conditioned to stay positive
- **Lecture 08:** Brownian motion conditioned to stay positive, Brownian excursion
- **Lecture 09:** Brownian motion conditioned to stay in a bounded domain
- **Lecture 10:** Brownian motion in the Weyl chamber, stationary measures for diffusions; Uniqueness in law via Girsanov
- **Lecture 11:** Path integral formula for drift of gradient type, extension of Ito to convex function, Tanaka formula and definition of local time
- **Lecture 12:** Ito-Tanaka formula, occupation time formula, explicit formula for the local time, non-existence of strong solutions for Tanaka SDE, definition of reflected SDE
- **Lecture 13:** Skorokhod lemma, uniqueness of (strong) solution of reflected SDE, definition of reflected Brownian motion through the Skorokhod map, reflected Brownian motion of a continuous curve and boundary
- **Lecture 14:** Definition of Lévy processes, relation with infinitely divisible distributions, Lévy-Khinchine formula
- **Lecture 15:** Poisson process, compound Poisson process, relation between Lévy measure and number of jumps
- **Lecture 16:** The Gamma process, construction of process with finite and infinite measure, Lévy-Itô decomposition
- **Lecture 17:** Superposition and thinning properties of Poisson process, Poisson processes conditioned of non-colliding
- **Lecture 18:** Introduction to KPZ, heuristic derivation, the white noise

- **Lecture 19:** Scaling of the white noise, 1:2:3 scaling of the KPZ equation
- **Lecture 20:** Cole-Hopf solution of KPZ, mild solution
- **Lecture 21:**