Lecture notes on Stochastic Analysis

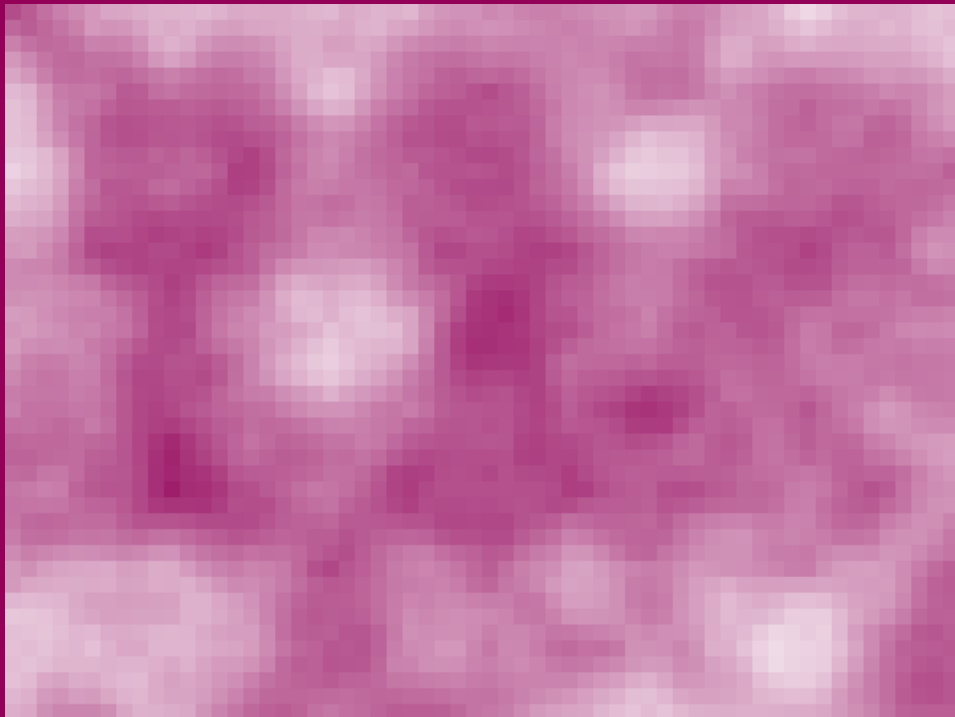Written by
Manuel Hinz
mh@mssh.dev or s6mlhinz@uni-bonn.de

Based on the lectures of
Prof. Dr. Patrik Ferrari
ferrari@uni-bonn.de

Contents

Chapter 0:
Manuel's notes

> **Warning**
>
> These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
> Happy learning!

> **General Information**
>
> - Ecampus: Ecampus link
> - Basis: Basis link
> - Website: None
> - Time slot(s): Tuesday 12-14 and Thursday 12-14
> - Exams: Oral: 23-25 July, 30 Sept. maybe 27. Sept
> - Deadlines: ?
> - Exercises: One of Tue 16-18(for now this is preferred), Friday 10-12. Starting 16.04.

- First halve based on Eberle and / or Gubinelli (be careful with Notation of dimensions!)

Overview of the content

- Weak solutions of SDE
  - · Martingale problem (characterization)
  - · Time change (Dubin-Schwarz)
  - · Change of measure (Girsonov)
- How to condition a diffusion to stay in a given domain forever (reflected BM, local time, . . .)
- Lévy processes (a bit)
- Multiplicative stochastic heat-equation
  - · relations with Kardar-Pavisi-Zhang class of growth models

Chapter 1:
Weak solutions of SDEs

SDE:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \textbf{ Initial condition} \end{cases} \tag{1.1}$$

- $X_t \in \mathbb{R}^d, B : n\text{-dim BM}$

- $b(t, x) : [b_k(t, x)]_{1 \leq k \leq d}$: **drift vector**

- $\sigma(t, x) = [\sigma_{k,l}(t, x)]_{\substack{1 \leq k \leq d \\ 1 \leq l \leq n}}$: **dispersion matrix**

- $a(t, x) = \sigma(t, x) \cdot \sigma(t, x)^\mathsf{T}$: **diffusion matrix**

1.1 Strong solutions

Definition 1.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_t = \sigma(x_0, (B_s)_{0 \leq s \leq t})$*

- *a d-dim process X_t is a **strong solution** of equation 1.1 if:*

  · *$X_t = x_0$ a.s.*

  · *X_t is adapted to $\mathcal{F}_t \forall t \geq 0$*

  · *X is a continuous semimartingale s.t. $\forall t \geq 0$:*

  $$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2)ds < \infty \text{ a.s.}$$

  · *$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$*

In the last semester we proved:

Theorem 1.2. *Assume that b, σ are globally lipschitz with at most linear growth at ∞ (in space) $\implies \exists!$ strong solution of SDE.*

Foundations of Stochastic Analysis Thm 8.6

Added remark. *There exists $K > 0$ s.t. for all $x, y \in \mathbb{R}^d$: Globally Lipschitz:*

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|$$

Linear growth condition:
$$\|b(t, x)\| + \|\sigma(t, x)\| \leq K(1 + \|x\|)$$

Remark. *For strong solutions, $\mathcal{F}_t$ is given by the driving BM, wich is given to us.*
$\implies X_t(\omega) = \Phi(x_0(\omega), (B_s)_{0 \leq s \leq t})$

1.2 Weak solutions

- For weak solutions we do not fix the driving brownian motion.

Definition 1.3. *A __weak solution__ of equation 1.1 is a **pair** of adapted processes (X, B) to a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ s.t.*

- *B is a n-dim BM*

- *X is a d-dim continuous semimartingale with*

    1. *$X_0 = x_0$ a.s.*
    2. *$\forall t \geq 0$*
    $$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2) ds < \infty \ a.s.$$

    3. *$X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$*

Remark. • *The filtration $(\mathcal{F}_t)_{t \geq 0}$ is not necessarily the one generated by B*

- *If X is adapted to the filtration generated by the BM $\implies$ we have strong solutions*

- *$\exists$ weak solution which are not strong solutions*

- *Warning: To give a weak solution we really need to provide $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X, B)$*

Definition 1.4 (Uniqueness in law)**.** *An SDE 1.1 has __uniqueness in law__ if given any two weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X, B)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{B})$ satisfy:*

They agree on any set in the sigma algebra

$$Law_{\mathbb{P}}(X) = Law_{\tilde{\mathbb{P}}}(\tilde{X})$$

Definition 1.5 (Pathwise uniqueness)**.** *The SDE 1.1 has pathwise uniqueness if, whenever the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and $(B_t)_{t \geq 0}$ are fixed, then two solutions $X, \tilde{X}$ with $X_0 = \tilde{X}_0$ are indistinguishable.*

Example 1.6 (No strong solutions, no pathwise uniqueness, $\exists$ weak solution & and uniqueness in law by Tanaka)**.**
$$\begin{cases} dX_t = sgn(X_t) dB_t \\ X_0 = 0 \end{cases} \tag{1.2}$$

or more generally $X_0 = Y$, where

$$sgn(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

Let W be a BM with $W_0 = Y$. Define

$$B_t := \int_0^t sgn(W_s) dW_s \ or \ dB_t = sgn(W_t) dW_t$$

$$\implies dW_t = sgn(W_t) dB_t$$
$$\implies W_t = y + \int_0^t sgn(W_s) dB_s$$

B_t is a local martingale with

$$\langle B \rangle_t = \int_0^t \underbrace{(sgn(W_s))^2}_{=1} \underbrace{d\langle W \rangle_s}_{=ds} = t$$

*Also $B_0 = 0$, therefore B is a BM (see Lévy characterization) $\implies$ W solves the SDE. For $Y = 0$, W **and** $-W$ solves the same SDE.*
$\implies$

- *exists weak solutions*

- *For $Y = 0$: no pathwise uniqueness*

- *Uniqueness in law (because the law is determined by X_t being a BM)*

- *No strong solution, because: $\mathcal{F}^B = \mathcal{F}^{|W|} \subsetneq \mathcal{F}^W$*

Not a proof just yet, just the reason!

Example 1.7 (No solutions).

$$\begin{cases} dX_t = -\frac{1}{2X_t}1_{X_t \neq 0}dt + dB_t \\ X_0 = 0 \end{cases} \tag{1.3}$$

Assume there exists a solution. Use Itô formula for X_t^2, then:

$$\begin{aligned} X_t^2 &= 2\int_0^t X_s dX_s + \int_0^t 1 ds \\ &= -\int_0^t 1_{X_s \neq 0} ds + 2\int_0^t x_s dB_s + t \\ &= \int_0^t 1_{X_s=0} ds + 2\int_0^t X_s dB_s \end{aligned}$$

We will prove $\int_0^t 1_{X_s=0}ds = 0 \implies X_t^2$ is a local martingale, $X_t^2 \geq 0$ (and therefore a supermartingale) and $X_0 = 0$ ($\implies \mathbb{E}(X_t^2) = 0$). If $X_t = 0 \implies \int_0^t 1_{X_s=0}ds = t \implies 0 = dB_t$ which are contradictions!

Remark. *If $X = \underbrace{M}_{\in \mathcal{M}_{loc}} + \underbrace{A}_{\in \mathcal{A}}$*

$$\implies \forall b \in \mathbb{R} \int_0^t 1_{X_s=b}d\langle M\rangle_s = 0$$

Sheet 0:

- Mail: Alexander.Becker@uni-bonn.de

- Exercise sheet handed in Fr 10 am via eCampus

- Groups of 3

Motivation:
in the last semester: Introduction to stochastic analysis

- Diffusion processes & SDEs

Here: Deepen the knowledge & have fun

- Learn SDE techniques and get other results

- Modify diffusion processes to behave in a certain way

- Ex. diffusion bridge (Brownian bridge)

- Condition a diffusion to stay in a domain

  · Ex. Condition BM to stay positive

  · Old SDE: $dB_t = dB_t$

- · New SDE: $dX_t = \frac{1}{X_t}dx + dB_t \to P(X_t \in \cdot) = P(B_t \in \cdot | B > 0 \text{ forever})$

- · $D \subset \mathbb{R}^d$ open domain, X diffusion process, with generator $L = \Sigma b\partial_i + \frac{1}{2}\sum a^{ij}\partial_i\partial_j$

$$Y := (X | X \in D \text{ forever})$$

get drift term $\nabla \log \phi_0$, where ϕ_0 is the lowest eigenfuncion of $-L$ on D with dirichlet boundary.

Recap:

Brownian motion:

Added definition. $B_0 = 0$, *independent & $\mathcal{N}(0, t_i - t_{i-1})$ increments, $t \mapsto B_t(\omega)$ continuous.*

Regularity of path $t \mapsto B_t(\omega)$:

- nowhere differentiable

- α−locally Hölder continuous $\iff \alpha < \frac{1}{2}$

- Quadratic variation $\langle B \rangle_t = t$

- Generator $\frac{\Delta}{2}$

- Recurrent $\iff ds2$?

Itô-Integral:

1. If X simple process $\implies$ RS-Integral

2. Itô isometry $\mathcal{E} \to \{L^2 - \text{local martingale}\}, \mathcal{E} \subset L^2(d[M])$ dense

3. general $X: \int X dM$ as L^2-limit

Added remark (Itô formula). •

$$df(t, X_t) = \partial_t f(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}f(t, X_t)d\langle X \rangle_t$$

- *associative $\int X d(\int Y dZ) = \int XY dZ$*

- *If M local martingale $\implies \int X dM$ local martingale*

SDEs:

$DX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$

- ex./ uniqueness: b, σ locally Lipschitz $\implies$ strong ex.+pathwise uniqueness until explosion time

- globally Lipschitz (in space) and linear growth ($|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|)$), $e = \infty$.

Problem 00.1: SDE

Let B be a one-dimensional Brownian motion (starting from 0) and let $X_t = \sin(B_t)$.

1. Determine the SDE of X_t

2. Discuss the existence and/or uniqueness of strong solutions of the SDE

3. Determine whether the local martingale term is a martingale (Hint: for the integrability condition, think about the Itô isometry for instance)

Solution 00.1

<u>1.:</u>
Idea: Use Itô formula: $X_t = \sin(B_t) = f(B_t)$

$$dX_t = df(B_t) \stackrel{\text{Itô}}{=} \partial_x \underbrace{\cos(B_t)}_{\sqrt{1-X_t^2}} dB_t - \frac{1}{2} \underbrace{\sin(B_t)}_{X_t} dt$$

Careful: $\sqrt{1-X_t^2}$ is not inverse mapping, because it is always positive while $\cos(B_t)$ is not

<u>2.:</u>
Consider the SDE
$$dX_t = \begin{cases} -\frac{1}{2}X_t dt + \sqrt{1-X_t^2} dB_t \\ X_0 = x \in (-1,1) \end{cases}$$

Coefficients: $b : [-1,1] \to \mathbb{R}, b(x) = -\frac{1}{2}x$ and $\sigma : [-1,1] \to \mathbb{R}.\sigma(x) = \sqrt{1-x^2}$

$$[\sigma]_{1,2} = \sup \frac{|\sigma(x) - \sigma(y)|}{|x-y|^{1/2}} < \infty$$

Probably σ^2 Lipschitz $\implies \sigma$ Hölder $\frac{1}{2}$
<u>3.:</u>

Problem 00.2: Time change

Let B be a one-dimensional Brownian motion (starting from 0). Let $Y_t = \int_0^t s^2 dB_s$.

1. Determine the SDE of Y_t

2. Find A_t such that Y_{A_t} is a (stopped) Brownian motion

Solution 00.2

<u>1.:</u> By Definition
$$dY_t = t^2 dB_t$$

<u>2.:</u> We use the Dubin-Schwarz theorem:
$X \in \mathcal{M}_{loc}^0, \langle X \rangle_\infty = \infty, T_t := \inf\{s \geq 0 | \langle X \rangle_s \geq t\} = X_t^{[-1]}$
$\implies B_t := X_{T_t}$ 1 d BM w.r.t. $(F_{T_t})_{t \geq 0}, X_t = B_{\langle B \rangle_t}$.
here: use $X_t = b(X_t)dt + \sigma(X_t)dB_t \implies d[X]_t = \sigma^2(X_t)dt$

Problem 00.3: SDE and PDE

Let f be a function supported on $[0,1]$, u the solution of

$$\frac{1}{2}u(t,x) = \frac{1}{2}x(1-x) + \frac{d^2}{dx^2}u(t,x), \qquad u(0,x) = f(x).$$

Consider also the one-dimensional SDE (also known as Wright-Fisher diffusion)

$$dX_t = \sqrt{X_t(1-X_t)dB_t}$$

with $X_0 = x \in (0,1)$.

1. For any fixed $t > 0$, define $M_s = u(t-s, X_s)$ for $s \in [0,t]$. Use Itô formula to show that M_s is a local martingale

2. Assume that f is bounded and there is a bounded solution of u. Show that $u(t,x) = \mathbb{E}_x(f(X_t))$.

Solution 00.3

<u>1.:</u>

$dM_s = du(t - s, X_s)$

$= -\partial_s u(t - s, X_s)ds + \partial_x u(t - s, X_s) \underbrace{dX_s}_{b(X_s)ds + \sigma(X_s)dB_s \text{ by asso.}} + \frac{1}{2}\partial_x^2 u(t - s, X_s) \underbrace{d[X]_s}_{=\sigma^2(X_s)ds}$

$= \underbrace{(-\partial_s u + b\partial_x u + \frac{1}{2}\sigma^2\partial_x^2 u)}_{=0}(t - s, X_s)ds + \partial_x u(t - s, X_s)\sigma(X_s)dB_s$

$\implies dM_s = \partial_x u(t - s, X_s)\sigma(X_s)dB_s$

(i.e.: $M_t - M_0 = \int_0^t \ldots dB_s$)
This is a purely stochastic integral against a (local) martingale $\implies$ martingale.
<u>2.:</u>

- M_s true martingale:

    1. $\forall t : \mathbb{E}(\sup_{[0,t]} |M|) < \infty$, for example: M bounded
    2. $\forall t : \mathbb{E}(\sup_{[0,t]}[M]_t) < \infty$

$M_s := u(t - s, X_s), u$ bounded $\implies$ M bounded $\implies$ M true martingale
$w(s, x) := u(t - s, x)$

$$u(t, x) = w(0, x) = \mathbb{E}_x[w(0, X_0)] \overset{\text{martingale}}{=} \mathbb{E}_x[w(t, X_t)] = \mathbb{E}_x[(u(0, X_t))] \overset{\text{PDE}}{=} \mathbb{E}_x[f(X_t)]$$

Feynman-Kac formula (later more general).
Also solvable by Kolmogorov Backward / forward equation

Example 1.8 (Non-uniqueness of law, non pathwise uniqueness, with solutions)**.**

$$\begin{cases} dX_t &= 1_{X_z}dB_t \\ x_0 &= 0 \end{cases}$$

Then
$$X_t = 0 \forall t \geq 0$$
and
$$X_t = B_t \forall t \geq 0$$

both are solutions:

$$X_t - B_t = -\int_0^t 1_{X_s=0}dB_s \implies \langle X - B\rangle_t = \int_0^t 1_{X_s=0}d\langle B\rangle_s = 0$$

Let $\eta \sim Ber\left(\frac{1}{2}\right)$ *independent of* $(B_t)_{t\geq 0}$ *and define*

$$\tilde{X}_t = \begin{cases} 0 & \eta = 0 \\ B_t & \eta = 1 \end{cases}$$

$\implies$ $\tilde{X}_t$ *is adapted to* $\sigma(\eta(B_s)_{0\leq s\leq t})$*, but not to* $\sigma((B_s)_{0\leq s\leq t})$ *and therefore not a strong solution.*

$X_t = \int_0^t 1_{X_s \neq 0}dB_s$
$= \int_0^t (1 - 1_{X_s=0}dB_s)$
$B_t - \int_0^t 1_{X_s=0}dB_s$

Example 1.9 (No strong solution, weak solution, no pathwise uniqueness, no uniqueness in law)**.**

$$\begin{cases} dX_t & = 1_{X_t \neq 1} sgn(X_t) dB_t \\ X_0 & = 0 \end{cases}$$

Let Y_t be a solution of

$$\begin{cases} dY_t & = sgn(Y_t) dB_t \\ Y_0 & = 0 \end{cases}$$

$\implies X_t := Y_{t \wedge \tau}$, *where* $\tau = \inf\{s \geq 0 \mid Y_s = 1\}$ *is also a solution.*

Theorem 1.10 (Yamada-Watanabe)**.** *If both existence of weak solutions and pathwise uniqueness hold, uniqueness in law also holds.*
Moreover, $\forall$ choices of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and $(B_t)_{t \geq 0}$ then there exists a strong solution.

1.3 Lévy characterization

Example 1.11. *Consider the SDE*

$$\begin{cases} dX_t & = O_t dB_t \\ X_0 = x_0 \end{cases}$$

where both X_t and B_t are d-dimensional and O_t is an adapted process (matrix) s.t. $O_t^{\mathsf{T}} O_t = 1 \forall t \geq 0$ i.e. O_t is a rotation

$$\implies X_t^k = X_0^k + \sum_{l=1}^{d} \int_0^t O_s^{k,l} dB_s^l$$

$$\implies \langle X^k, X^{\tilde{k}} \rangle_t = \sum_{l,\tilde{l}}^{d} \int_0^t O_s^{k,l} O_s^{\tilde{k},\tilde{l}} d\langle B^l, B^{\tilde{l}} \rangle_s = \sum_{l=1}^{d} \int_0^t O_s^{k,l} O_s^{\tilde{k},l} ds$$

$$= \int_0^t \underbrace{(O_s O_s^{\mathsf{T}})^{k,\tilde{k}}}_{=1} ds = \delta_{k,\tilde{k}} t \overset{L\acute{e}vy}{\implies} (X_t)_{t \geq 0} \text{ is also a d-dim BM starting from } x_0$$

Theorem 1.12 (Yamada,Watanabe)**.** *Let $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ and assume that there exist both a increasing function $\rho(u) \geq 0$ s.t. $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|) \forall x, y \in \mathbb{R}$ s.t.*

$$\int_0^\infty \frac{1}{\rho(u)} du = \infty \implies 0 < u < C_1 \implies \rho(u) \leq C_2 u^{0.5}$$

and some increasing concave function $\gamma_1(u) \geq 0$ s.t.

$$|b(x) - b(y)| \leq \gamma_1(|x - y|) \forall x, y \in \mathbb{R}$$

and

$$\int_0^\infty \frac{1}{\gamma_1(u)} = \infty.$$

Then pathwise uniqueness holds.

Theorem 1.13 (Storokhod)**.** *Assume that σ, b are continuous bounded functions*
$\implies$ *there exist weak solutions to the SDE $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.*

1.4 Weak solutions and martingale problems

Let $(X_t)_{t \geq 0}$ be a weka solution of the SDE

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= 0 \end{cases}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

$\implies X_t$ is a semimartingale s.t. $X_t^k = X_0^k + \int_0^t b(s, X_s)ds + \sum_{l=1}^n \int_0^t \sigma_{k,l}(s, X_s)dB_s^l$ and

$$\langle X^i, X^j \rangle_t = \int_0^t \underbrace{\sum_{l=1}^n \sigma_{i,l}(s, X_s)\sigma_{j,l}^\intercal(s, X_s)}_{=a_{i,j}(s, X_s)} ds$$

1.4.1 Itô-Doeblin formula

Itô formula leads to

Proposition 1.14. *For $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, then*

$$f(t, X_t) = f(0, X_0) + \int_0^t (\sigma^\intercal \nabla f)(s, X_s)dB_s + \int_0^t \left[\left(\frac{\partial}{\partial s} + \mathcal{L}\right)\right](s, x_s)ds$$

where $(\mathcal{L}f)(t, x) = \frac{1}{2}\sum_{k,l=1}^n a_{k,l}(t, x)\frac{\partial^2}{\partial x_k \partial x_l}f(t, x) + \sum_{k=1}^n b_k(t, x)\frac{\partial}{\partial x_k}f(t, x)$. $\mathcal{L}$ *is called the **generator***

Remark. *The Itô-Doeblin formula provides a connection between SDEs and PDEs as we have seen in the last part **Foundations of stochastic analysis**.*

Example 1.15. *Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}_d)$ be a solution of the PDE*

$$\frac{\partial}{\partial t}f(t, x) + (\mathcal{L}f)(t, x) = -g(t, x) \qquad\qquad t \geq 0, x \in U \subseteq \mathbb{R}^d$$

$$f(t, x) = \varphi(t, x) \qquad\qquad t \geq 0, x \in \partial U.$$

then $M_t := f(t, X_t) + \int_0^t g(s, X_s)ds \in \mathcal{M}_{loc}$ by proposition 1.14 and if f, g are bounded $M_t \in \mathcal{M}$.

$$T := \inf\{s \geq 0 | X_s \notin U\} \implies M_t^T := M_{T \wedge t} \in \mathcal{M}.$$

Furthermore, if we assume $T < \infty$ a.s. :

$$\mathbb{E}[M_t] = \mathbb{E}[\varphi(T, X_T)] + \mathbb{E}[\int_0^T g(s, X_s)ds] = \varphi(0, X_0)$$

where $X_0 = x_0$.
There are two special cases:
$g = 0 \implies$ yields the exit distributions, while
$\varphi = 0, g = 1$ yields the mean exit times.

Example 1.16 (Feynman-Kac formula)**.** *Let $t \in \mathbb{R}_+$ be finite. Assume $f : \mathbb{R}^d \to \mathbb{R}$ and $K : [0, t] \times \mathbb{R}^d \to \mathbb{R}_+$ be continuous functions. Assume that u is a $C^{1,2}$-solution (bounded, for simplicity) of*

$$\begin{cases} \frac{\partial u}{\partial s}(s, x) = \frac{1}{2}\Delta u(s, x) - K(s, x)u(s, x) & s \in [0, t], x \in \mathbb{R}^d \\ u(0, x) = f(x) & x \in \mathbb{R}^d \end{cases}$$

Then u has the stochastic representation $u(t, x) = \mathbb{E}_x \left[f(X_t)\exp\left(-\int_0^t K(t - s, X_s)ds\right)\right]$, where X_t is BM starting from $X_0 = x$.

Sketch.   1. Define $r(s, x) := u(t - s, x)$ for $s \in [0, t]$

2. Show: $M_s := \exp(-A_s)r(s,x)$ with $A_s = \int_0^s K(u, X_u)du$ is a local martingale.

□

Remark. *This is a reformulation of the formula from the last semester.*

1.4.2 Martingale problem

A solution of an SDE is generically defined up to some **explosion time ξ**, where it either diverges or it exists a given domain $U \subset \mathbb{R}^d$ (open).

$\implies$ For $k \in \mathbb{N}$ define $U_k := \{x \in U \mid |x| < k \wedge \text{dist}(x, U^c) \geq \frac{1}{k}\}$ with $U = \bigcup_{k \geq 1} U_k$ and

$$T_k := \{t \geq 0 \mid x_t \notin U_k\}.$$

A solution of the SDE $b(t, X_t)dt + \sigma(t, X_t)dB_t$ is defined up to $\xi = \sup_{k \geq 1} T_k$.

Start of lecture 03 (18.04.24)

Added remark. *uniqueness of solution to the heat equation $\frac{1}{2}\Delta u - Ku$: not unique! Stochastic calculus: Paolo Baldi: 10.3-10.4*

Define ($\star$):

$$dX_t b(t, X_t)dt + \sigma(t, X_t)dB_t \text{ with } X_0 = x_0$$

Theorem 1.17 (Martingale problem). *If X is a solution of ($\star$) up to time ζ, then $\forall f \in C^{1,2}(\mathbb{R}_+ \times U)$*

$$M_t = f(t, X_t) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right) f(s, x_s)ds, t < s$$

is a local martingale up to ζ and $M_t^{T_k}$ are localizing martingales.

Definition 1.18 (Martingale solutions). *$(X_t)_{t \geq 0}$ is a **martingale solution** of ($\star$) if $\forall f \in C^2(\mathbb{R}^d)$*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$$

is a continuous local martingale.

Theorem 1.19 (Equivalent definitions). *The following are equivalent (~~for X_t being a solution of~~ ~~($\star$)~~):*

(a) *$\forall f \in C^2(\mathbb{R}^d)$,*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$$

is a local martingale

(b) *The process in $\mathbb{R}^d$ given by*

$$M_t := X_t - X_0 - \int_0^t b(s, X_s)ds$$

is a d-dimensional local martingale with $\langle M^i, M^j \rangle_t = \int_0^t a_{i,j}(s, X_s)ds = \langle X^i, X^j \rangle_t$

(c) *$\forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$*

$$\tilde{M}_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right)(s, X - s)ds$$

is a local martingale

ds

Proof. <u>$c \implies a$</u>: by choosing f independent of t.
<u>$a \implies b$</u>: 1.: Choosing $f(X) = X_i$ implies

$$M_t^f = X_t^i - X_0^i - \int_0^t b_i(s, X_s) ds \in \mathcal{M}_{\text{loc}}$$

2.: $f(X) = X_i X_j$:

$$
\begin{aligned}
(\mathcal{L}f)(x) &= \frac{1}{2} a_{ij} + \frac{1}{2} a_{ji} + b_i X_j b_j X_i \\
a = a^\intercal &\implies = a_{ij} b_i X_j b_j X_i \\
\implies M_t^f X_t^i X_t^j &- X_0^i X_0^j - \int_0^t \left[a_{ij}(s, X_s) + b_i(s, X_s) X_s^j + b_j(s, X_s) X_s^i \right] ds
\end{aligned}
$$

$$
\begin{aligned}
X_t^i X_t^j - X_0^i X_0^j \overset{\text{Integration by parts}}{=} &\int_0^t X_s^i dX_s^j + \int_0^t X - s^j dX_s^i + \langle X^i, X^j \rangle_t \\
&= M_t^f + \int_0^t [a_{ij} + b_i X_s^j + b_j X_s^i] ds
\end{aligned}
$$

<u>$b \implies c$</u>: By Proposition 1.14 If (use the next theorem) X was a weak solution $\implies \tilde{M}_t^f$ is a local martingale. $\qquad \square$

> **Theorem 1.20.** *Let $n = d$, assume $\sigma(t, x)$ is invertible $\forall t, x$ and $\sigma^{-1}(t, x)$ is uniformly bounded. T.f.a.e.:*
>
> (a) *$(X_t)_{t \geq 0}$ is a weak solution of the SDE $(\star)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}; B)$*
>
> (b) *$(X_t)_{t \geq 0}$ is a martingale solution of the SDE $(\star)$*

Proof. $a \implies b$: True
$b \implies a$: Goal construct a BM for the weak solution.
By proposition **??** $a \implies b$ $dM_t = dX_t - b(t, X_t) dt \in \mathcal{M}_{\text{loc}}$ and $d\langle M^i, M^j \rangle_t = a_{k,l}(t, X_t) dt$

$$
\begin{aligned}
\implies dX_t &= dM_t + b(t, X_t) dt \\
&= \sigma(t, X_t) d\tilde{B}_t + b(t, X - t) dt
\end{aligned}
$$

where $\tilde{B}_t := \sigma(s, X_s)^{-1} dM_s$
To see: $\tilde{B}_t$ is a brownian motion.

$$
\begin{aligned}
\langle \tilde{B}^i, \tilde{B}^j \rangle_t &= \sum_{k,l} \int_0^t \sigma_{ij}^{-1} \sigma_{jl}^{-1} \underbrace{d\langle M^k, M^l \rangle_s}_{\substack{= a_{ij} \ ds \\ (\sigma^\intercal \sigma)_{kl}}} \\
&= \sum_{k,l,p} \int_0^t \sigma_{ik}^{-\intercal} \sigma_{kp} \sigma_{pl}^\intercal \sigma_{lj}^{-\intercal} ds \\
&= \delta_{ij} \int_0^t 1 ds = \delta_{ij} t
\end{aligned}
$$

Then by the Lévy characterization $\tilde{B}$ is a brownian motion. $\qquad \square$

> **Added remark.** *This is the first way to construct a weak solution: Solve a martingale problem! This is used a lot in practice.*

Margin notes:

We can maybe also proof this by calculating X^2?

Here dX_s^i is the same as $b_i X_s^j$ is the same up to a local martingale term and $\langle X^i, X^j \rangle_t = \int_0^t a_{ij}(s, X_s) ds = \langle M^i, M^j \rangle_t$

This also works for $n \neq d$, but with a different proof

1.5   Weak solutions and time change

1.5.1   Time change

For $d = 1$:

Theorem 1.21. *[Dubins-Schwarz]*

- *Let $M \in \mathcal{M}^0_{loc}$ and $\langle M \rangle_\infty = \infty$ a.s.*

- *Let $T_t := \inf\{s \geq |\langle M \rangle_s \geq t\}$*

This implies

1. *$t \mapsto M_{T_t}$ os a $(\mathcal{F}_{T_t})$ brownian motion*

2. *$M_t = B_{\langle M \rangle_t}$ for some standard brownian motion B*

$X_t = X_0 + \int_0^t b(s, X_s)ds + \underbrace{\int_0^t \sigma(s, X_s)dB_s}_{=M_t}$. If $\langle M \rangle_\infty = \infty$ a.s.:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \tilde{B}_{\int_0^t \sigma^2(s, X_s)ds}$$

1.5.2   Time change in a martingale problem

Consider $d = 1 = n$.

$$dY_t = \tilde{\sigma}(Y_t)dB_t \qquad\qquad (\star\star)$$

and $\tilde{\sigma}$ strictly positive positive continuous function.

$$\langle Y \rangle_t = \int_0^t \tilde{\sigma}(Y_s)^2 ds =: A_t$$

By theorem 1.21 $\implies Y_t = W_{A_t}$ for some brownian motion W.
Assume $A_t \infty = \infty$ a.s.
$T_t := \inf\{s \geq 0 \mid \langle Y \rangle_s \geq t\}$

$$\implies T_{A_t} = \inf\{s \geq 0 \mid \langle Y \rangle_s \geq \langle Y \rangle_t\} = t$$

$$1 = \frac{d}{dt}(T_{A_t}) = T'_{A_t} \cdot \underbrace{A_t}_{=u}$$

$$\implies \underbrace{T'_u}_{=\frac{dT_u}{du}} = \frac{1}{A'_{T_u}} \implies T_u = \int_0^u \frac{1}{\tilde{\sigma}(Y_{T_s})^2}ds = \int_0^u \frac{1}{\tilde{\sigma}(W_s)^2}ds$$

$\implies$ to construct a solution of $(\star\star)$: Given $W \longrightarrow$ compute $T_u \longrightarrow$ determine
$A - t = T_t^{-1} \implies Y_t = W_{A_t}$

Theorem 1.22. *Let $(X_u)_{u \geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a weak solution of*

$$dX_u = b(X_u)du + \sigma(X_u)dB_u$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, the drift and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are locally bounded, σ^{-1} exists for a.e. x and is locally bounded.
Consider a __time change__ $T_u := \int_0^u \rho(X_s)ds$, where $\rho : \mathbb{R}^d \to \mathbb{R}_+$ s.t.

$$T_u < \infty \forall u \geq 0 \text{ and } T_\infty = \infty \text{ a.s.}$$

$\implies$ *Then $Y_t := X_{A_t}$, where $A_t = T_t^{-1}$ is a weak solution of the SDE*

$$dY_t = \frac{b(Y_t)}{\rho(Y_t)}dt + \frac{\sigma(Y_t)}{\sqrt{\rho(Y_t)}}dB_t$$

Remark. *Special case: $d = 1, b = 0, \sigma = 1$: Then X is a BM and $\rho = \frac{1}{\tilde{\sigma}^2(x)}$ $\implies$ $Y_t = X_{T_t^{-1}}$ solves*

$$dY_t = \tilde{\sigma}(Y_t)dB_t$$

Proof. By theorem 1.20, it is enough to see how the martingale problem after change of time becomes:

$$\mathcal{L} = \mathcal{L}_{X_t} \xrightarrow{\text{time change}} \mathcal{L}_{Y_t} = \tilde{\mathcal{L}}$$

$Y_t = X_{A_t}; Y_0 = X_{A_0}$. For $f \in C^2 : M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$ is a local martingale w.r.t. $(\mathcal{F}_t)_{t \geq 0}$.

$$\implies N_t^f := M_{A_t}^f = f(\underbrace{X_{A_t}}_{=Y_t}) - f(\underbrace{= X_{A_0}}_{Y_0}) - \int_{A_0}^{A_t} (\mathcal{L}f)(X_s)ds$$

is also a local martingale w.r.t. $(\mathcal{F}_{A_t})_{t \geq 0}$.

The A_0 in the integral is probably 0, but it does not matter, we do a change of variables anyway.

Change of variable (to get rid of the X_s in the integral):

$$\tau = T_s \leftrightarrow A_t = s \implies X_s = X_{A_t} = Y_\tau$$
$$d\tau = \rho(X_s)ds$$

$$\implies N_t^f = f(Y_t) - f(Y_0) - \int_0^t (\mathcal{L}f)(Y_t)\frac{1}{\rho(Y_t)}d\tau$$

Since $\mathcal{L}f(x) = \sum_k b_k(x)\frac{\partial}{\partial x_k}f(x) + \sum_{k,l} a_{k,l}(x)\frac{\partial^2}{\partial x_k \partial x_l}f(x)$

$$\implies (\tilde{\mathcal{L}}f)(x) = \sum_k \frac{b_k(x)}{\rho(x)}\frac{\partial}{\partial x_k}f(x) + \sum_{k,l} \frac{\overbrace{a_{k,l}}^{(\sigma\sigma^\intercal)_{k,l}}}{\sqrt{\rho(x)\rho(x)}}(x)\frac{\partial^2}{\partial x_k \partial x_l}f(x)$$

$\implies$ It is a martingale problem for the SDE where the drift $\to \frac{\text{drift}}{\rho}$ and $\sigma \to \frac{\sigma}{\sqrt{\rho}}$

$\square$

1.5.3 Weak solutions in d=1

We will do both time and "space" changes.

- 1-d SDE:
$$\begin{cases} dX_t & = b(X_t)dt + \sigma(X_t)dB_t \\ X_0 = x_0 \in (\alpha, \beta) \end{cases} \tag{1.4}$$

- X_t a process in (α, β)

- Assume $b, \sigma : (\alpha, \beta) \to \mathbb{R}$ continuous, $\sigma(x) > 0 \forall x \in (\alpha, \beta)$

- Do a change of coordinates $Y_t := s(X_t)$ where $s : (\alpha, \beta) \to (s(\alpha), s(\beta))$, C^2 with $S'(x) > 0, x \in (\alpha, \beta)$.

- $s(x)$ is called the **scale function** and it is given by

$$s(x) = \int_{x_0}^x \exp\left(-\int_{x_0}^y dz \frac{2b(z)}{\sigma(z)^2}\right)dy$$

- s satisfies $\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0 \iff (\mathcal{L}s)(x) = 0$

Remark. *If $b(z) = 0 \implies s(x) = x - x_0 \implies s'(x) = 1$. If $s'(x) = 1$, we say that the process is in its "natural scale"*

By proposition 1.14: $\mathcal{L}s = 0, \dot{s} = 0$.
$\implies Y_t = s(X_t)$ is a local martingale satisfies $dY_t = s'(X_t)\sigma(X_t)dB_t$.

the other terms cancel

$\iff Y_t$ is a solution of

$$\begin{cases} dY_t & = \tilde{\sigma}(Y_t)dB_t \\ Y_0 & = s(X_0) \end{cases} \tag{1.5}$$

where $\tilde{\sigma}(y) = s'(s^{-1}s(y))\sigma(s^{-1}(y))$.

Therefore we can write the original SDE in terms of a BM

Theorem 1.23. *The following are equivalent:*

1. *The process $(X_t)_{t<\xi}$, where ξ is the explosion time, on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; (B_t)_{t\geq 0})$ is a solution of (1.4) up to tje stopping time ξ*

2. *The process $Y_t = s(X_t)_{t<\xi}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; (B_t)_{t\geq 0})$ is a solution of (1.5) up to ξ*

   s and A_t have the same definition as before

3. *The process $(Y_t)_{t<\xi}$ has the representation $Y_t = \tilde{B}_{A_t}$, where $\tilde{B}$ is a BM starting at $\tilde{B}_0 = s(X_0)$ and $A_t = T_t^{-1}$ and $T_t = \int_0^t \frac{1}{\tilde{\sigma}^2(\tilde{B}_u)}du$*

A degenerate case:
Let $\sigma(x) = |x|^\alpha$ for some $\alpha \in (0, \frac{1}{2})$. $\implies$

$$\begin{cases} dY_t & = |Y_t|^\alpha dB_t \\ Y_0 & = y \end{cases} \tag{1.6}$$

$\implies T_t = \int_0^t \frac{1}{\sigma(\tilde{B}_u)^2}du$, $A_t = \int_0^t \sigma(Y_s)^2 ds$ and $Y_t = \tilde{B}_{A_t} \implies \tilde{B}_0 = y$.
$T_t < \infty$ a.s. ?

$$\begin{aligned} \mathbb{E}(T_t) &= \int_0^t \mathbb{E}\left(\frac{1}{\sigma(\tilde{B})^2}\right) du \\ &= \int_0^t \mathbb{E}\left(\frac{1}{|\tilde{B}|^{2\alpha}}\right) du \\ &= \int_0^t \int_\mathbb{R} dx \frac{1}{|x|^{2\alpha}} \frac{\exp(-\frac{(x-y)^2}{2u})}{\sqrt{2\pi u}} \overset{0<2\alpha<1}{<} \infty \end{aligned}$$

$\implies A_t = T_t^{-1}$, then $Y_t = \tilde{B}_{A_t}$ is a solution of (1.6), i.e. $\forall y \in \mathbb{R}\exists$ a non-trivial solution of (1.6).
For $Y_{=0}, Y_t = 0$ is also a solution $\implies$

- No uniqueness in law of (1.6)

- No pathwise uniqueness as well

Remark. *In general: uniqueness in law of 1-d SDEs is not to be expected if $\sigma(x) = 0$ somewhere (and σ continuous . . .) (i.e. if σ is degenerate).*

By theorem 1.12 as soon as $\sigma(x) = |x|^\alpha$ for some $\alpha \geq \frac{1}{2}$, then one has pathwise uniqueness.
Hitting times and scale functions Bessel process:

$$\begin{cases} dR_t & = \frac{d-1}{2R_t}dt + dW_t \\ R_0 & = r_0 \in (0, \infty) \end{cases}$$

$\implies b(x) = \frac{d-1}{2x}, \sigma(x) = 1$.
The scale function satisfies $\mathcal{L}s(x) = \frac{1}{2}s''(x) + \frac{d-1}{2x}s'(x) = 0$

$$\implies s(x) = \begin{cases} \frac{x^{2-d}}{2-d} & d \neq 2 \\ \ln(x) & d = 2 \end{cases}$$

and therefore

$$(s_0, s_\infty) = \begin{cases} (0, \infty) & d < 2 \\ (-\infty, \infty) & d = 2 \\ (-\infty, 0 & d > 2) \end{cases}$$

Define the stopping time

$$T_a^R := \inf\{t \geq 0 \mid R_t = a\}$$

Choose an $\alpha < r_0 < \beta$

$$\implies \mathbb{P}(T_\alpha^R < T_\beta^R) \stackrel{s' \geq 0}{=} \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

One computes

$$\mathbb{P}(T_a^R < T_\beta^R) = \frac{s(\beta) - s(r_0)}{s(\beta) - s(\alpha)} = \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

This is generic, for Itô diffusions, provides that $\exists$ no killing in (α, β).

1.5.4 Uniqueness of martingale solution

SDE $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ with generator

$$\mathcal{L} = \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l} (\sigma \sigma^\intercal)_{k,l} \frac{\partial^2}{\partial x_k \partial x_l}$$

Definition 1.24. *Let $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R}^d)$ with σ-algebra $\mathcal{F}$, canonical filtration $(\mathcal{F}_t)_{t \geq 0}$, canonical process $Z_t(\omega) := \omega$.*
*We say that $\mathbb{P}$ on $(\mathcal{C}, \mathcal{F})$ is a **martingale solution** for the generator $\mathcal{L} \iff \forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R})$*

$$M_t^f := f(t, Z_t) - f(0, Z_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right) f(s, Z_s)ds \tag{1.7}$$

is a martingale w.r.t. $\mathbb{P}$.

Definition 1.25. *A martingale problem (1.7) has a unique solution if for any two martingale solutions $\mathbb{P} = \mathbb{Q}$ s.t. $Law_\mathbb{P}(Z_0) = Law_\mathbb{Q}(Z_0)$*

$$\implies \mathbb{P} = \mathbb{Q}$$

Remark. *Uniqueness of martingale solutions corresponds to uniqueness in law of the weak solutions.*

Backwards Kolmogorov Equation (BKE):

$$\frac{\partial}{\partial t}\varphi(t, x) = \mathcal{L}\varphi(t, x) \forall x \in \mathbb{R}^d, t \geq 0 \tag{1.8}$$

Theorem 1.26. *Assume that $\forall$ initial condition*

$$\varphi(0, x) = \Psi(x), \Psi \in C_0^\infty(\mathbb{R}^d)$$

the (1.8) has a solution and φ bounded for all finite time intervals. We have uniqueness of martingale solutions and therefore uniqueness of weak solutions!

Proof. Prove that $\forall 0 \leq t_1 < t_2 < \cdots < t_n$:

$$\text{Law}_{\mathbb{P}}(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n}) = \text{Law}_{\mathbb{Q}}(Z_{t_1}, Z_{t_2}, \ldots, Z_{t_n})$$

1. One-time distribution:
$\forall 0 \leq s \leq r$:

$$\left(\frac{\partial}{\partial s} + \mathcal{L}\right) \varphi(r - s, x) \overset{(1.8)}{=} 0$$

take $t \in [0, r]$:

$$M_t^r := \varphi(r - t, Z_t) - \varphi(r, Z_0) - \int_0^t \underbrace{(\partial_s + \mathcal{L})\varphi(r - s, Z_s)}_{=0}\, ds$$

$$= \varphi(r - t, Z_t) - \varphi(r, Z_0) \text{ is a martingale}$$

for any solution $\mathbb{P}$.

This needs the boundedness of φ, otherwise it might only be a local martingale. There are softer contidions we can put on the coefficients to achieve the same result. This might not be needed, because φ is C^1 in time anyways

$$
\begin{aligned}
0 &= \mathbb{E}_{\mathbb{P}}\left(M_r^r - M_t^r \mid \mathcal{F}_t\right) & &= \mathbb{E}(\varphi(0, Z_r) - \varphi(r - t, Z_t) \mid \mathcal{F}_t) \\
\implies & \forall 0 \leq t \leq r : \mathbb{E}_{\mathbb{P}}(\varphi(0, Z_r) \mid \mathcal{F}_t) & &= \mathbb{E}_{\mathbb{P}}(\varphi(r - t, Z_t) \mid \mathcal{F}_t) \\
& & &\overset{a.s.}{=} \varphi(r - t, Z_t) \\
& \mathbb{E}(\underbrace{\varphi(0, Z_r)}_{\Psi(Z_r)}) & &\overset{t=0}{=} \mathbb{E}_b P(\varphi(r, Z_0))
\end{aligned}
$$

$\forall$ other martingale solutions $\mathbb{Q}$:

$$\mathbb{E}_{\mathbb{Q}}(\Psi(Z_r)) = \mathbb{E}_{\mathbb{Q}}(\varphi(r, Z_0))$$

By assumption this implies $\text{Law}_{\mathbb{P}}(Z_r) = \text{Law}_{\mathbb{Q}}(Z_r)$.
2. Multi-time distributions:
For $\Psi \in C_0^\infty$, denote φ_Ψ the solution of (1.8) with initial condition Ψ:

$$\mathbb{E}_{\mathbb{P}}(\Psi(Z_r) \mid \mathcal{F}_t) = \varphi_\Psi(r - t, Z_t)$$

$0 \leq r_2 \leq r_1$ test for $g \in C_0^\infty$:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}(\Psi(Z_{r_1})g(X_{r_2})) &= \mathbb{E}(\underbrace{\mathbb{E}(\varphi_\Psi(Z_{r_1})|\mathcal{F}_{r_2})}_{\varphi_\Psi(r_1 - r_2, Z_{r_2})} g(Z_{r_2})) \\
&= \mathbb{E}_{\mathbb{P}}(\varphi_\Psi(r_1 - r_2, Z_{r_2})g(Z_{r_2})) \\
&\overset{1.}{=} \mathbb{E}_{\mathbb{Q}}(\varphi_\Psi(r_1 - r_2, Z_{r_2})g(Z_{r_2})) \\
&= \mathbb{E}_{\mathbb{Q}}(\Psi(Z_{r_1})g(Z_{r_2}))
\end{aligned}
$$

Iterating yields the statement.

$\square$

Chapter 2:
SDE techniques

Goal: Study process by changing the measure.

E.g.: $X_t = B_t$. $\underbrace{\text{Condition } X_t \geq 0 \forall t \geq 0}_{=:Y_t}$.

2.1 Girsanov theorem

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ and two measures $\mathbb{P}, \mathbb{Q}$

Assume $\mathbb{P} \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P}$ and let $H = \frac{d\mathbb{Q}}{d\mathbb{P}}$

The notes of Eberle switches the roles of $\mathbb{P}, \mathbb{Q}$!

$$\implies Z_t := \mathbb{E}_{\mathbb{P}}(H \mid \mathcal{F}_t) = \frac{d\mathbb{Q}}{d\mathbb{P}} \mid_{\mathcal{F}_t}$$

is a martingale.

From last semester: $\forall Y \in L^1(\mathbb{Q}) \cap L^1(\mathbb{P}), \mathcal{F}_t$ measurable:

$$\mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{F}_s) = \frac{bE_{\mathbb{P}}(Y \cdot Z_t \big| \mathcal{F}_s)}{Z_s} \forall s < t$$

If $Z > 0$ is $\mathcal{M}_{\text{loc}}, \exists L \in \mathcal{M}_{\text{loc}}$ s.t.:

$$Z_t = e^{L_t - \frac{1}{2}\langle L \rangle_t} \to L_t = \ln(Z_0) + \int_0^t \frac{dZ_s}{Z_s}$$

There might be a problem, because $\ln(Z_0)$ might not be integrable, and therefore not a local martingale!

Theorem 2.1 (Girsanov). *Assume $Z > 0$ is a martingale. If M is a local martingale w.r.t. $\mathbb{P}$, then*

$$\tilde{M}_t := M_t - \langle M, L \rangle_t$$

is a local martingale w.r.t. $\mathbb{Q}$ and

$$\langle M \rangle_t = \langle \tilde{M} \rangle_t.$$

Moreover, if M is a BM w.r.t. to $\mathbb{P}$, then $\tilde{M}$ is a BM w.r.t. $\mathbb{Q}$.

Remark. *In applications, given $\mathbb{P}, (Z_t)_{t \geq 0}$ a positive continuous martingale, define $\mathbb{Q}$ on $\mathcal{F}_\infty = \bigcup_{t \geq 0} \mathcal{F}_t$ s.t.*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t$$

Z is uniformly integrable $\iff \mathbb{Q} \ll \mathbb{P}$. **Problem:** *In applications, Z is not necessarily uniformly integrable.*

$\implies$ restrict to $[0, T] \implies$ all fine.

$\mathbb{Q} \to \mathbb{Q}_T$ as in the last semester.

Example. *Let $\gamma \in \mathbb{R}^d, (B_t)_{t\geq 0}$ standard BM.*
Let $L_t := \gamma \cdot B_t$ and $Z_t = \exp(L_t - \frac{1}{2}\langle L \rangle_t) = \exp(\gamma \cdot B_t - \frac{1}{2}|\gamma|^2 t)$

Remark. $\lim_{M\to\infty} \sup_{t\geq 0} \mathbb{E}(|Z_t| 1_{|Z_t|>M}) \neq 0.$
Define $\mathbb{Q}$ on $\mathcal{F}_\infty$ s.t. $\tilde{B}_t^k = B_t^k - \langle L, B^k \rangle = B_t^k - \gamma^k t$ is a BM with drift. Show: $Q \not\ll \mathbb{P}$. *Construct Q via Z*

$$A = \left\{ \lim_{t\to\infty} \frac{\tilde{B}_t + \gamma t}{t} = \gamma \right\} \in \mathcal{F}_\infty$$

but: $\tilde{B}$ is a BM w.r.t. $\mathbb{Q} \implies \mathbb{Q}(A) = 1$, $\tilde{B}$ is a BM with drift $-\gamma$ w.r.t $\mathbb{P}(A) = 0$.

2.1.1 Drift transformation of SDE

We once again start with

$$\star = \begin{cases} dX_t & = b(t, X_t) \\ X_0 & = x_0 \end{cases}$$

with drift b continuous.
Goal: Get a weak solution of $\star$.
Let $(X_t)_{t\geq 0}$ be a BM in $(\Omega, \mathcal{F}(\mathcal{F}_t)_{t\geq 0}, \mathbb{P}), X_0 = x_0$.

Assume: $Z_t := \exp\left(\underbrace{\int_0^t b(s, X_s)dX_s}_{=L_t} - \frac{1}{2}\int_0^t |b(s, X_s)|^2 ds \right)$ is a martingale w.r.t. $\mathbb{P}$.

Remark. *Thr assumption holds if*

$$|b(t, x)| \leq C(1 + \|x\|) \text{ for some } C.$$

$\implies \mathbb{E}_\mathbb{P}(Z_t) = 1 \forall t \geq 0$
By Girsanov

$$\tilde{X}_t = X_t - \langle L, X \rangle_t$$

is a $\mathbb{Q}$-BM.
But

$$d\langle L, X \rangle_t = b \cdot dX_t \cdot dX_t = b \cdot dt$$

$$\implies \tilde{X}_t = X_t - \int_0^t b(s, X_s)ds$$

$$\text{w.r.t. } \mathbb{Q}: dX_t = b(t, X_t)dt + \underbrace{d\tilde{B}_t}_{d\tilde{X}_t}$$

Generalization: Start with

$$\star\star \; dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

where X, B are d-dimensional.

Proposition 2.2. *Assume (X, B) is a weak solution of $\star\star$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. If* *In practice this is surprisingly useful in practice!*

$$Z_t = \exp\left(\int_0^t c(s, X_s)dB_s - \frac{1}{2}\int_0^t c(s, X_s)^2 ds \right)$$

is a martingale w.r.t. $\mathbb{P}$, $\mathbb{Q} \ll \mathbb{P}$ and $\mathcal{F}_t$ s.t. $Z_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}} Vert_{\mathcal{F}_t}$. *this is an implicit condition for c*

Then $(X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ is a weak soltuion of

$$dX_t = [b(t, X_t) + \sigma(t, X_t)c(t, X_t)dt] + \sigma(t, X_t)d\tilde{B}_t$$

where $\tilde{B}$ is a d-dim BM.

Proof.

$$d\langle L, B \rangle_t \overset{L_t = \int_0^t c(s, X_s)dB_s}{=} c(t, X_t)dt$$
$$\implies \tilde{B}_t := B_t - \langle L, B \rangle_t \text{ is a } \mathbb{Q}\text{-BM}$$
$$\implies dB_t = c(t, X_t)dt + d\tilde{B}_t$$

From $\star\star$: $dX_t = b(t, X_t)dt + \sigma(t, X_t)c(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t = \star\star\star$ $\qquad\qquad\square$

Measure	SDE	Generators
$\mathbb{P}$	$dX_t b \cdot dt + \sigma \cdot dB_t$	$\mathcal{L} \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l} (\sigma\sigma^\mathsf{T})_{k,l} \frac{\partial^2}{\partial x_k \partial x_l}$
$\mathbb{Q}$	$[b + \sigma \cdot c]dt + \sigma d\tilde{B}_t$	$\tilde{\mathcal{L}} = \mathcal{L} + \sum_k \sum_l \sigma_{k,l} c_l \frac{\partial}{\partial x_k} = \mathcal{L} + c\sigma^\mathsf{T} \nabla$

$t \leq T$ for $\mathbb{Q}_t$.

2.2 Doob-h transform

Essentially three cases in 1d

1. From $B_t \to B_t$ conditioned on $\{B_1 = 0\}$ (measure zero set)

2. From $B_t \to B_t$ conditioned on $\{B_t \geq 0 \forall t \geq 0\}$ (measure zero set)

3. From $B_t \to B_t$ conditioned on $\{B_t \geq 0 \forall t \in [0,1], B_1 = 0\}$

Let $(X_t, B_t)_{t \geq 0}$ be a weak solution of

$$(\star) = \begin{cases} dX_t & = b(t, X_t)dT + \sigma(t, X_t)dB_t \\ X_0 & = x_0 \text{ fixed} \end{cases}$$

Assume there exists $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ s.t. $h > 0$ satisfying

$$(\frac{\partial}{\partial t} + \mathcal{L})h = 0 \forall t, \in [0, T], x \in \mathbb{R}^d$$

where $\mathcal{L}$ is the generator of the SDE $(\star)$.
By theorem 1.14 $Z_t := h(t, X_t) = h(0, X_0) + \int_0^t (\sigma\nabla h)(s, X_s)dB_s$.
Z_t is a positive local martingale. Assume Z_t is a martingale.
W.l.o.g. $Z_0 = 1$ (if not $h \mapsto \frac{h}{h(0, X_0)}$).
$\implies d\mathbb{Q}_T = Z_T \cdot d\mathbb{P}$. Let L_t s.t. $Z_t = \exp(L_t - \frac{1}{2}\langle L \rangle_t)$.
Girsanov, since B_t is a $\mathbb{P}$-BM,
$\tilde{B}_t := B_t - \langle L, B \rangle_t$ is a Q_T-BM, where

$$\langle L, B \rangle_t = \sigma^\mathsf{T} \cdot \nabla \ln h \cdot dB_t$$

because of

$$dL_t = \frac{dZ_t}{Z_t} = \frac{\sigma^\mathsf{T} \cdot \nabla h \cdot dB_t}{h(t, X_t)} = \sigma^\mathsf{T} \cdot \nabla \ln h \cdot dB_t$$

And therefore $dB_t = d\tilde{B}_t + \sigma^\mathsf{T} \nabla \ln h dt$.
From $(\star) \implies dX_t = \underbrace{(b(t, X_t) + \sigma(t, X_t)\sigma^\mathsf{T}(t, X_t)\nabla h(t, X))}_{:=\tilde{b}}dt + \sigma(t, X-t)d\tilde{B}_t$

Proposition 2.3. *Let X be a weak solution of $dX_t = b(t, X_t) + \sigma(t, X_t)dB_t$ on $[0, T]$ under $\mathbb{P}$. Then under $\mathbb{Q}_T, X$ is a weak solution of the SDE*

$$\begin{cases} dX_t &= \tilde{b}(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t \\ X_0 &= x_0 \end{cases}$$

where $\tilde{b} = b + \sigma\sigma^{\mathsf{T}}\nabla \ln h$.

Remark. *In applications, often $h(x) \not> 0$ for all x!*
E.g.: $h(t, x) = x \implies \left(\frac{\partial}{\partial t} + \mathcal{L}\right) h = 0$ $(X_t = B_t + x_0)$, but $h(x) \le 0$ for $x \le 0$.
In this case first do the construction on $[0, \tau]$, where $\tau := \inf\{t > 0 \mid B_t = 0\}$
$\implies$ *ok for $B_t^\tau \stackrel{magically}{\implies}$ the new SDE has drift:*

$$\frac{\partial}{\partial x} \ln(h(x)) = \frac{1}{h(x)} \frac{\partial h(x)}{\partial x} = \frac{1}{x}$$

We realize there is no problem in the new measure

Example. *If $X_t = B_t$ and $h(t,) = e^{\gamma x - \frac{1}{2}|\gamma|^2 t}$*

$$\begin{aligned} \implies dX_t &= \nabla \ln h(t, X_t)dt + dB_t \\ &= \nabla(\gamma \cdot x)dt + dB_t \\ &= \gamma d_t + dB_t \end{aligned}$$

$\implies X_t$ *is a BM with drift γ w.r.t $\mathbb{Q}$.*

2.3 Diffusion Bridges

Consider a Markov process $(X_t)_{t \ge 0}$ which is a diffusion starting from $X_0 = x_0 \in \mathbb{R}^d$.
We want to condition on the event $\{X_T = y\}$ for some given $T > 0, y \in \mathbb{R}^d$.
Goal: Find, $\forall y \in \mathbb{R}^d, \mathbb{Q}^y$ which is the conditional measure on $\{X_T = y\}$.
Assume: $(X_t)_{t \ge 0}$ has transition density P s.t.

This is not true in general, if we don't have unique solutions

$$\forall 0 \le s \le t \le T, \mathbb{P}(X_t \in dz \mid X_s = x) = p(s, x; t, z)dz.$$

X_t solves

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \end{cases}.$$

Define $h^y(s, x) := \frac{p(s, x; T, y)}{p(0, x_0; T, y)}$ for $s \in [0, T), x \in \mathbb{R}^d$.

Remark. *$s < T$ since otherwise we get ingeneral singularities.*

Lemma 2.4. *Let $Z_t^y := h^y(t, X_t)$ is a martingale.*

Proof. Let $0 \le< t < T$:

$$\begin{aligned} \mathbb{E}(Z_t^y \mid \mathcal{F}_s) &= \mathbb{E}(h^y(t, X_t) \mid \mathcal{F}_s) \\ &\stackrel{\text{MP}}{=} \mathbb{E}(h^y(t, X_t \mid X_s)) \\ &= \int_{\mathbb{R}^d} h^y(t, x)p(s, X_s; t, x)dx \\ &= \int_{\mathbb{R}^d} \frac{p(t, x; T, y)}{p(0, x_0; T, y)}p(s, X_s; t, x)dx \\ &= \frac{1}{p(0, x_0; T, y) \int_{\mathbb{R}^d} p(s, X_s; t, x)p(t, x; T, y)dx} \\ &\stackrel{\text{Chapman-Kolmogorov}}{=} Z_s^y = \end{aligned}$$

$\square$ $h^y(0, x_0) = 1$

Added remark. <u>*Goal:*</u> *Find a family* $(\mathbb{Q}^y)_{y \in \mathbb{R}^d}$ *s.t. for* $A \in \mathcal{F}_T$:

$$\mathbb{P}(A) = \mathbb{E}(\mathbb{P}(A|X_T)) \stackrel{?}{=} \mathbb{E}(\mathbb{Q}^{X_T}(A))$$

and $\mathbb{Q}^y \left(\lim_{t \uparrow T} X_t = y \right) = 1$

Lemma 2.5. *We can take*
$$\mathbb{Q}^y(A) = \mathbb{E}(1_A \cdot h^y(s, X_s)) \forall A \in \mathcal{F}_s$$

for $s \in [0, T]$.

Proof. For all $A \in \mathcal{F}_s$, g bounded measurable:

$$\begin{aligned}
\mathbb{E}(1_A g(X_t)) &= \mathbb{E}(1_A \mathbb{E}(g(X_T|\mathcal{F}_s))) \\
&\stackrel{\text{MP}}{=} \mathbb{E}(1_A \mathbb{E}(g(X_T|X_s))) \\
&= \mathbb{E}\left(1_A \int_{\mathbb{R}^d} p(s, X_s; T, x) g(x) dx\right) \\
&= \int_{\mathbb{R}^d} dx g(x) \mathbb{E}(1_A p(s, X_s; T, x))
\end{aligned}$$

In particular,

$$\mathbb{P}(A) = \mathbb{E}(1_A) = \int_{\mathbb{R}^d} dx \mathbb{E}(1_A \cdot p(s, X_s; T, x))$$

But

$$\begin{aligned}
\int_{\mathbb{R}^d} dx p(0, X_0; T, x) Q^x(A) &= \int_{R^d} p(0, x_0; T, x) \underbrace{\mathbb{E}(1_A h^x(s, X_s))}_{= \int dz p(0, x_0; s, z) h^x(s, z) 1_A} \\
&= \int_{\mathbb{R}^d} \mathbb{E}(1_A p(s, X_s; T, x)) \\
&= \mathbb{P}(A). \qquad \qquad \square
\end{aligned}$$

Remark. $Z_t^y = h^y(t, X_t) = h(0, X_0) + \text{ martingale } + \underbrace{\int_0^t (\partial_t + \mathcal{L}) h^y(s, X_s) ds}_{=0}$.

One can verify: $(\partial_t + \mathcal{L}) h^y(t, X_t) = 0 \iff (\partial_t + \mathcal{L}) H^y(t, X_t) = 0$ *for* $H^y(t, x) = p(t, x; T, y)$.

$$\begin{aligned}
\frac{\partial}{\partial t} H^y(t, x) &= \lim_{\epsilon \downarrow 0} \frac{p(t, x; T, y) - p(t - \epsilon, x; T, y)}{\epsilon} \\
&= \lim_{\epsilon \downarrow 0} \frac{p(t, x; T, y) - \int \overbrace{p(t - \epsilon, x; t, z)}^{= e^{\epsilon L}(x, z) = \delta_{x-z} + \epsilon \mathcal{L}(x, z)} p(t, z; T, y) dz}{\epsilon} \\
&= -(\mathcal{L} H^y)(t, x)
\end{aligned}$$

$\implies$ *this is a doob-h transform with* $h = h^y$.

$$\frac{d\mathbb{Q}^y}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = Z_t^y.$$

Corollary 2.6. *Assume* $(t, x) \mapsto p(t, x; T, y)$ *is* $C^{1,2}([0, T] \times \mathbb{R}^d)$
$\implies$ *The Doob h-transform of the original process under* $\mathbb{Q}^y$ *satisfies the SDE:*
$$dX_t = \tilde{b}(t, X_t) dt + \sigma(t, X_t) d\tilde{B}_T$$

where $\tilde{B}_t$ *is a* $\mathbb{Q}^y$*-BM and* $\tilde{b}(t, X_t) = b(t, X_t) + (\sigma \sigma^\intercal \nabla \ln h^y)(t, X_t)$.

Remark. *Take $z \neq y$ and $\forall 0 < \epsilon < |z - y|$:*

$$\mathbb{Q}^y \left(\{|X_t - z| \leq \epsilon\} \right) = \mathbb{E} \left(1_{|X_t - z| \leq \epsilon} \frac{p(t, X_t; T, y)}{p(0, x_0; T, y)} \right)$$

$$= \int dx 1_{|x - z| \leq \epsilon} p(0, x_0, t, x) \frac{p(t, x; T, y)}{p(0, x_0; T, y)}$$

$$\xrightarrow{t \to T} 0$$

$\implies \mathbb{P}\left(\lim_{t \uparrow T} X_t = y \right) = \mathbb{P}(X_t = y) = 0,$ *but* $\mathbb{Q}^y \left(\lim_{t \uparrow T} X_t = y \right) = 1$
$\implies \mathbb{Q}^y$ *is singular w.r.t.* $\mathbb{P}$.

Example. *$b = \gamma \in \mathbb{R}^d, \sigma = 1, X_0 = 0$ and $T = 1$. Under $\mathbb{P}$:*

$$\begin{cases} dX_t = \gamma dt + dB_t \\ X_0 = 0 \end{cases}$$

Under $\mathbb{Q}^y$?

$$h^y(t, x) = \frac{p(t, x; 1, y)}{p(0, x_0, 1, y)} = fct(\gamma, y) \cdot \exp\left(-\frac{(y - x)^2}{2(1 - t)} \right) \exp\left((y - x)\gamma \right)$$

$\implies \ln(h^y(t, x)) = \ln f(\gamma, y) - \frac{(y-x)^2}{2(1-t)} + (y - x)\gamma \implies \nabla \ln h^y(t, x) = \frac{y-x}{1-t} - \gamma.$
Under $\mathbb{Q}^y$:

$$dX_t^k h^y = \left(\gamma_k + \frac{y_k - X_t^k}{1 - t} \right) dt + d\tilde{B}_t^k \qquad \text{\textit{Independent of γ!}}$$

$$= \frac{Y_k - X_t^k}{1 - t} dt + d\tilde{B}_t^k$$

for $k = 1, \ldots, d$

Lemma 2.7. *Let p be the transition density of X w.r.t. $\mathbb{P}$ and p^h the transition density of X w.r.t. $\mathbb{Q}^y$.* *Useful in practice to sample!*
Let $h^y(t, x) = \frac{p(t, x; T, y)}{p(0, x_0; T, y)}$.

$$\implies p^h(s, x; t, z) = \frac{1}{h^y(s, x)} p(s, x; t, z) h^y(t, z)$$

Notice how the normalizations inside of h^y cancel!

Added remark. *This is the first time my numbering is different from the handwritten notes, as they contain two environments numbered 2.5.*

Proof.

$$p^h(s, x; t, z)dz = \mathbb{P}(X_t \in dz \mid X_s = x, X_T = y)$$

$$\overset{0 \leq s \leq t}{=} \frac{p(0, x_0; s, x) p(s, x; t, z) p(t, z; T, y) dz}{p(0, x_0; s, x) p(s, x; T, y)}$$

$$= p(s, x; t, z) \frac{p(t, z; T, y)}{p(0, x_0; T, y)} \frac{1}{\frac{p(s, x; T, y)}{p(0, x_0; T, y)}} dz \qquad \square$$

2.4 Brownian motion conditioned to stay positive forever

<u>**Goal:**</u> $(B_t)_{t \geq 0}$ with $B_0 = x_0 \geq 0$ and condition on

$$\{B_t \geq 0 \forall t \geq 0\}$$

$T_x = \inf\{t \geq 0 \mid B_t = x\}$ and take $0 < x_0 < R$.
$T_0 \wedge T_R = T_R$.
Let $\tilde{T}_R := T_0 \wedge T_R$. Define the event $E_R = \{B_{\tilde{T}_R} = R\} = \{T_0 \wedge T_R = T_R\}$.
One verifies $\mathbb{P}(E_R) = \frac{x_0}{R} \in (0, 1)$.
$\mathbb{P}(E_R) = \mathbb{E}(1_{B_{\tilde{T}_R}^{x_0}}) = \underbrace{\mathbb{E}(f(B_{\tilde{T}_R}^{x_0}))}_{u(x_0)}$, where

$$f(x) = \begin{cases} 1 & x = R \\ 0 & x = 0 \end{cases}.$$

u has a link to the PDE:

By Theorem 11.6 of the last semester

$$\begin{cases} \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x) = 0 & x \in (0, R) \\ u(x) = f(x) & x \in \{0, R\} \end{cases}$$

Define the conditional probability

later $R \to \infty$

$$Q^R(A) := \frac{\mathbb{P}(A \cap E_R)}{\mathbb{P}(E_R)}$$

Lemma 2.8. *Let* $h(x) := \frac{\mathbb{P}_x(E_R)}{\mathbb{P}_{x_0}(E_R)} = \frac{x}{x_0}$, *where* $\mathbb{P}_x$ *is the unconditional law for* $B_0 = x$.

$$Z_t = h(B_{t \cap \tilde{T}_R})$$

is a non-negative martingale.

Proof.

$$Q^R(A) \overset{A \in \mathcal{F}_s}{=} \frac{\mathbb{E}(1_A\mathbb{E}(1_{E_R} \mid \mathcal{F}_s))}{\mathbb{P}(E_R)} = \frac{\mathbb{E}(1_A\mathbb{E}(1_{E_R} \cdot 1_{\tilde{T}_R > s} \mid \mathcal{F}_s)) + \mathbb{E}(1_A\mathbb{E}(1_{E_R}1_{\tilde{T}_R \leq s} \mid \mathcal{F}_s))}{\mathbb{P}(E_R)}$$

$$= \frac{1}{\mathbb{P}(E_R)}\left[\mathbb{E}(1_A 1_{\tilde{T}_R > s}\underbrace{\mathbb{E}(1_{E_R} \mid \mathcal{F}_s)}_{\mathbb{P}_{B_s}(E_R)})) + \mathbb{E}(1_A \underbrace{1_{E_R}}_{1_{B_{\tilde{T}_R} = R}} 1_{\tilde{T}_R \leq s})\right]$$

$$= \mathbb{E}(1_A 1_{\tilde{T}_R > s}h(B_s)) + \mathbb{E}(1_A 1_{\tilde{T}_R \leq s}h(B_{T_R}))$$

$$= \mathbb{E}(1_A h(B_{s \wedge \tilde{T}_R}))$$

$\implies \forall A \in \mathcal{F}_s : Q^R(A) = \mathbb{E}_{\mathbb{P}}(1_A h(B_{s \wedge \tilde{T}_R}))$
$\implies h(B_{s \wedge \tilde{T}_R}) = \frac{dQ^R}{d\mathbb{P}}|_{\mathcal{F}_s} \implies h \in (0, \frac{R}{x_0})$ is a martingale (See construction of Girsanov) $\qquad\square$

Note that we can write $Z_t^R =:= h(B_{s \wedge \tilde{T}_R}) = \exp(L_t - \frac{1}{2}\langle L \rangle_t)$ by choosing

Start of lecture 08 (07.05.24)

$$L_t = \int_0^{\tilde{T}_R} \frac{h'(B_s)}{h(B_s)} dB_s$$

since

$$dL_t = \frac{dZ_t^R}{Z_t^R} = \begin{cases} \frac{h'(B_t)dB_t}{h(B_t)} & t < \tilde{T}_R \\ 0 & t \geq \tilde{T}_R \end{cases}$$

Proposition 2.9. *Under the measure* Q^R, $(B_t)_{t \geq 0}$ *solves the SDE*

$$dB_t = \frac{1_{t\tilde{T}_R}}{B_t}dt + dW_t$$

for some $(W_t)_{t \geq 0}$ *a* Q^R-*BM.*

Proof. Apply Girsanov in which

$$d\langle B, L\rangle_t = dB_t \cdot dL_t \overset{h'(x)=\frac{1}{x_0}}{\underset{1_{t<\tilde{T}_R}}{}} \frac{1}{B_t} dt \qquad \square$$

Our goal was to condition the BM to stay positive forever. Sofar we have conditioned it to reach the level R before the level 0.

Remark.

$$\begin{aligned}
\mathbb{Q}^R(T_0 < T_R) &= \mathbb{E}(1_{T_0<T_R}h(B_{T_R \wedge T_0})) \\
&= \mathbb{E}(1_{T_0<T_R}h(B_{T_0})) \\
&= \mathbb{E}(1_{T_0<T_R}\underbrace{h(0)}_{=0}) = 0
\end{aligned}$$

$\implies$ *under the new measure $\mathbb{Q}^R$, the BM indeed does not reach 0 before R as we wanted.*

Finally we wan to take $R \to \infty$. It should

Example. *Discrete time M.C. with transition probability P aperiodic and irreducible: $\exists n_0 \forall n > n_0 (P)^n_{i,j} > 0 \implies \exists |\lambda_0| \le 1, |\lambda_1| > |\lambda_2| > |\lambda_3|$:*

$$(P)^n \underbrace{\varphi_0(i)}_{>0} = \lambda_0 \varphi_0(i)$$

$\lim_{n\to\infty} \underbrace{P}_{(1+L)}^n = (\varphi_0(1), \dots, \varphi_0(d))$

where $(1+L) \to e^{tL}$, for which the real eigenvalues are positive.

2.5 Diffusion conditioned to stay in a domain

Domain $D \subset \mathbb{R}^d$: bounded, open, connected.
Diffusion with generator

$$L = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$$

$\iff$ SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \frac{1}{2}\sigma(t, X_t)dB_t \\ X_0 = x_0 \end{cases}$$

$a = \sigma\sigma^\intercal$.
Assume: b, σ are continuous, $\sigma \in C^1$ and let $\tau_D = \int\{t \ge 0 | X_t \notin D\}$
Key assumption in Pinsky's paper:

(a) $\mathbb{P}_x(\tau_D > t) \in \mathbb{C}^2(D)$

(b)
$$\begin{cases} \mathbb{P}_x(\tau_D > t) & = C_1\varphi_0(x)e^{-\lambda_0 t} + o\left(e^{-\lambda_0 t}\right) \\ \nabla\mathbb{P}_x(\tau_d > t) = C_1\nabla\varphi_0(x)e^{-\lambda_0 t} + o\left(e^{-\lambda_0 t}\right) \end{cases}$$

where

$$\begin{cases} -L\varphi_0(x) = \lambda_0\varphi_0(x) & x \in D \\ \varphi_0(x) = 0 & x \in \partial D \end{cases}$$

and λ_0 is the smallest eigenvalue of $-L$. Which means $\mathbb{P}(\tau_d > t) \overset{t\to\infty}{\to} 0$

in the non symmetric case take the real part first

Want: Condition X to stay in D forever.

(1) Condition X_t to $\{X_t \in D : 0 \le t \le T\}$: Forall $A \in \mathcal{F}_T$: define a measure

$$\mathbb{Q}^T(A) := \frac{\mathbb{P}_{x_0}(A \cap \{\tau_D > T\})}{\mathbb{P}_{x_0}(1_{\tau_D} > T)} = \frac{\mathbb{E}_{x_0}(1_A \cdot 1_{\tau > T})}{\mathbb{E}_{x_0}(1_{\tau > T})}$$

Lemma 2.10. $\forall s < T, A \in \mathcal{F}_s$,
$$\mathbb{Q}^T(A) = \mathbb{E}_{x_0}(1_A \cdot Z_s^T)$$
where $Z_s^T = \frac{g^{T-s}(X_{s \wedge \tau_D})}{g^T(x_0)}$ with $g^t(x) := \mathbb{P}_x(\tau_D > t)$.

Proof. Let $A \in \mathcal{F}_s, s < T$:

$$\mathbb{Q}^T(A) = \frac{\mathbb{E}_{x_0}(1_A 1_{\tau_D > T})}{g^T(x_0)} = \frac{\mathbb{E}_{x_0}(1_A \mathbb{E}_{x_0}(1_{\tau_D > T} | \mathcal{F}_s))}{g^T(x_0)}$$

$$\overset{\text{M.P.}}{=} \frac{\mathbb{E}_{x_0}(1_A \mathbb{E}_{x_0}(1_{\tau_D > T} | X_s))}{g^T(x_0)}$$

$$= \frac{1}{g^T(x_0)} \left[\mathbb{E}_{x_0}(1_A \underbrace{\mathbb{E}_{x_0}(1_{\tau_D > T} 1_{\tau_D > s} | X_s)}_{g^{T-s}(X_s)}) + \mathbb{E}_{x_0}(1_A \underbrace{\mathbb{E}_{x_0}(1_{\tau_D > T} 1_{\tau_D < s} | X_s)}_{=0}) \right]$$

$$= \frac{1}{g^T(x_0)} \left(1_A g^{T-s}(X_{s \wedge \tau_D}) \right)$$

$\square$

Lemma 2.11. $Z_0^T = 1$ and $(Z_s^T)_{s \in [0,T]}$ *is a martingale.*

Proof. By lemma 2.10 $\implies Z_s^T = \frac{d\mathbb{Q}^T}{d\mathbb{P}} |_{\mathcal{F}_s} \to$ is a martingale. $\square$

Remark. *By construction,* $\mathbb{Q}^T(\tau_D \le T) = \frac{1}{g^T}(x_0)\mathbb{E}_{x_0}(1_{\tau_D \le T} 1_{\tau_D > t}) = 0$

Assume that $g^t(x)$ is $C^{1,2}$ (1 in time, 2 in space) $\implies$ apply Itô and doob transform gives:

Proposition 2.12. *Let X be a weak solution of $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ under $\mathbb{P}$.*
$\implies X$ *is a weak solution of*

$$dX_t = \left(b(t, X_t) + \frac{a(t, X_t)\nabla g^{T-t}(X_t)}{g^{T-t}(X_t)} \right) dt + \sigma(t, X_t)d\tilde{B}_t, 0 \le t \le T$$

under $\mathbb{Q}^T$ provided $g^t(x) > 0 \forall x \in D, t \ge 0$.

What happens in the $T \to \infty$ limit?
By assumption (b)
$\lim_{T \to \infty} \frac{\nabla g^{T-t}(x)}{g^{T-t}(x)} = \frac{\nabla \varphi_0(x)}{\varphi_0(x)}$

Their operator is not the same as ours. The only difference is a drift term.

Remark. *If $g \in C^{1,2}$, it satisfies the parabolic PDE*

$$(\star) \begin{cases} \frac{\partial}{\partial t} g^t(x) = L g^t(x) & x \in D \\ g^t(x) = 0 & x \in \partial D \end{cases}$$

Apply Theorem 11.5 from the WS with $A = L, u(t, x = g^t(x))$: $u(0, x) = \mathbb{E}_x(1_{\tau_D > 0}) = 0, u(t, x) = \mathbb{E}_x(1_{\tau_D > t}) = 0 \forall t, x \in \partial D$.

Pinsky proved that $\lim_{T \to \infty} Q^T = Q$ weak and that under Q the process satisfies

$$dX_t = \left[b(t, X_t) + a(t, X_t)\frac{\nabla \varphi_0(X_t)}{\varphi_0(X_t)} \right] dt + \sigma(t, X_t)dW_t.$$

Why is $\mathbb{P}_x(\tau_D > t) = C_1 \varphi_0(x)e^{-\lambda_0 t} + o(e^{-\lambda_0 t})$

Added remark. *This is not the case for BM conditioned to stay in $D = (0, \infty)$. Reason: Spectrum of $-L$ is $\mathbb{R}_+$*

Since it satisfies the PDE ($\star$), $g^t(x) = (e^{tL})g^0(x)$.

if the spectrum of $-L$ is discrete with eigenvalues $0 \leq \lambda_0 < \lambda_1 \leq \ldots$, where there is a spectral gap between λ_0 and λ_1: $\exists$ eigenfunctions $\varphi_0(x), \varphi_1(x), \ldots$, normalized as

$\|\varphi_n\|_{L^2(D)} = 1 \implies +L\varphi_n(x) = -\lambda_n\varphi_n(x) \implies$ we can choose $\varphi_0, \varphi_1, \ldots$ to be orthonormal:

$(\varphi_i, \varphi_j)_{L^2(D)} = \delta_{ij}$.

$\implies 1 = \sum_{n \geq 0} \varphi_n \varphi_n^*$, since $\varphi_n \varphi_n^*$ is the projection onto the space generated by φ_m:

$$\varphi_n \varphi_n^* f = \varphi_n(\varphi_n, f)_{L^2(D)}$$

$$f(L)\varphi_n = f(-\lambda_n)\varphi_n$$
$$g^t(x) = (e^{tL})g^0(x) = e^{tL} 1 g^0(x) = \sum_{n \geq 0} e^{tL}\varphi_n(x)\varphi_n^* g^0 = \sum_{n \geq 0} e^{-\lambda n t}\varphi_n(x)(\varphi_m, g^0)$$
$$= (\varphi_0, g^0)\varphi_0(x)e^{\lambda_0 t} + \underbrace{e^{-\lambda_0 t}\sum_{n \geq 0} e^{(\lambda_0 - \lambda_n)t}\varphi_n(x)(\varphi_n, g^0)}_{o(e^{-\lambda_0 t})}$$

Example 2.13. *One dimension.*
$L = \frac{1}{2}\frac{d^2}{dx^2}$ and $D = [0, R]$.
Solve $-L\varphi(n)(x) = \lambda_n\varphi_n(x)$ with $\varphi_n(x) = 0$ for $x \in \{0, R\}$.
Solutions $\varphi_n(x) = C_1 \sin\left(\frac{\pi x}{R} \cdot (n+1)\right)$ and $\lambda_n = \left(\frac{\pi(n+1)}{R}\right)^2 \implies \lambda_0 = \frac{\pi^2}{R^2}$.
For $x \in (0, R)$:

$$\mathbb{P}_x(\tau > t) \approx \tilde{C}_1\varphi_0(x)e^{-\frac{\pi^2 t}{R^2}}$$

$$\implies dX_t = \frac{\pi}{R}\frac{\cos(\frac{\pi X_t}{R})}{\sin(\frac{\pi X_t}{R})}d + dW_t$$

as $X_t \to 0$ (or $X_t \to \mathbb{R}$), drift $\approx \frac{1}{x_t}$ or $(\frac{-1}{R - X_t})$.
Also: formally $R \to \infty$ in the SDE, we get $dX_t = \frac{dt}{X_t} + dW_t$, which is the SDE we derived for BM conditioned to stay > 0 forever.

Example 2.14 (Brownian Motion in a Weyl chamber). *The **Weyl chamber** is defined as*

$$W^d = \{x \in \mathbb{R}^d \mid x_1 < x_2 < \cdots < x_d\}.$$

Given a d dim. Brownian motion $(B_t)_{t \geq 0}$ with $B_0 \in W^d$ what is the sde of this BM conditioned on staying in W^d forever.
What is the harmonic function vanishing at ∂W^d?

Lemma 2.15. *$h(x) := \prod_{1 \leq k < l \leq d}(x_l - x_k)$ is harmonic and satisfies*

$$\frac{1}{2}\sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}h(x) = 0, x \in W^d$$

and $h(x) = 0$ for $x \in \partial W^d$, $h(x) > 0$ for $x \in W^d$.

Remark. $h(x) = \det(x_k^{l-1})$

Proof. Last two properties are clear. h is a polynomial, antisymmetric in each pair $x_l - x_k$ and has lowest possible power.
$\Delta h(x)$ is still antisymmetric, but with lower power $\implies \Delta h(x) = 0$. $\qquad\square$

The BM conditioned to stay in the Weyl chamber will satisfy the SDE:

$$dX_t^k = \sum_{l \neq k} \frac{dt}{X_t^l - X_t^k} + dB_t^k, \qquad\qquad 1 \leq k \leq d$$

2.6 Stationary distribution for diffusions

let $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, where b, σ are time independent with generator

$$L = \sum_{k=1}^{d} b_k(x)\frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k,l=1}^{d} a_{k,l}(x)\frac{\partial^2}{\partial x_k \partial x_l}$$

Definition 2.16. *A probability measure μ* **stationary (or invariant)** *if for all $f \in C_0^\infty(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} (Lf)(x)\mu(dx) = 0$$

or only test schwarzfunctions

Assume that $\mu \ll$ Lebesgue, i.e., $\mu(dx) = \rho(x)dx$ for some positive function $\rho(x) \in C^2$ with $\int \rho(x)dx = 1$.

Lemma 2.17. *μ is stationary with density ρ*

$$\iff L^*\rho(x) = 0 \text{ almost everywhere}$$

where L^* is the adjoint of L, given by

this also assumes b to be differentiable!

$$L^*\rho(x) = \frac{1}{2}\sum_{k,l=1}^{d} \frac{\partial^2}{\partial x_l \partial x_l}(a_{k,l(x)\rho(x)}) - \sum_{k=1}^{d}\frac{\partial}{\partial x_k}(b_k(x)\rho(x))$$

Proof.

$$\int_{\mathbb{R}^d}(Lf)(x)\rho(x)dx = 0$$

$$= \int_{\mathbb{R}^d} dx_1, \ldots, dx_d \rho(x)\left(\frac{1}{2}\sum_{k,l=1}^{d}\frac{\partial^2}{\partial x_k \partial x_l}f(x) + \sum_{k=1}^{d}b_k(x)\frac{\partial}{\partial x_k}f(x)\right)$$

$$\int_{R^{d-1}}\prod_{l \neq k}dx_l \underbrace{\int_R dx_k \rho(x)b_k(x)\frac{\partial}{\partial x_k}f(x)}_{\text{I.b.p} = -\int_R dx_k \frac{\partial}{\partial x_k}(\rho(x)b_k(x))\cdot f(x)}$$

$\square$

Example 2.18 (1-dim. diffusion)**.** *Let $L = b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial}{\partial x^2}$ for all $x \in \ R$.*

$$\implies L^*\rho(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\rho(x)\sigma^2(x)) - \frac{\partial}{\partial x}(\rho(x)b(x)) \overset{wanted}{=} 0$$

$\implies \exists c_1 \ s.t. \ \frac{1}{2}\frac{\partial}{\partial x}\underbrace{(\rho(x)\sigma^2(x))}_{=g(x)} - \underbrace{\rho(x)b(x)}_{=g(x)\frac{b(x)}{\sigma^2}} = c_1.$

Let $s(x) = \int_{x_0}^{x} dy\, e^{-\int_{x_0}^{y} dz \frac{2b(z)}{\sigma^2(z)}}$ be the **scale function**.
The equation for c_1 is equivalent to

$$s'(x)\sigma^2(x)\rho(x) = c_2 + 2c_1 s(x)$$

for some constant $c_2, s'(x)e^{-\int_{x_0}^{y} dz \frac{2b(z)}{\sigma^2(z)}} => 0.$

s satisfies $\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0$, for the Bessel process $d = 2$, $s(x) = \ln(x)$
If $s(\mathbb{R}) = \mathbb{R}$, then $c_1 = 0$, which implies

$$\rho(x) = \frac{c_2}{\sigma^2(x)} \underbrace{e^{\int_{x_0}^x \frac{2b(y)}{\sigma^2(y)} dy}}_{\frac{1}{s'(x)}}$$

which satisfies positivity.

Lemma 2.19. If $S(\mathbb{R}) = \mathbb{R}$, then there exists stationary measure with density $\rho(x)$ as described above.

Counterexample:

- $\sigma = 1$, $x_0 = 0$, $b(z) = b > 0$.

$\implies s(x) = \frac{1 - e^{-2bx}}{2b} \to s'(x) = e^{-2bx}$.

- $s(-\infty) = -\infty$
- $s(\infty) = \frac{1}{2b}$

$\implies$ Argument for $c_1 = 0$ does not work.

$$\implies \rho(x) = \tilde{c}_1 + \tilde{(c)}_2 e^{2bx}$$

which can't be a density, because if at least $\tilde{c}_1$ or $\tilde{c}_2$ are $\neq 0 \implies$ it is not integrable.

Example 2.20. Let $Lf(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}f(x) + b(x)\frac{\partial}{\partial x}f(x)$ where $b(x) = \frac{\partial}{\partial x}\ln h(x)$ for some $h(x) > 0$.
Assume h is normalized as $\int_R (h(x))^2 dx = 1$. $\implies$ __Claim:__ The stationary density of the process with generator L is given by $\rho(x) = (h(x))^2$.
Verify the claim:

Implicitly assumes $h \in L^2(\mathbb{R})$

$$L^*\rho(x) \overset{?}{=} 0$$
$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x) - \frac{\partial}{\partial x}(b(x)\rho(x))$$
$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}(h(x))^2 - \frac{\partial}{\partial x}\left(\frac{h'(x)}{h(x)}h(x)^2\right)$$
$$= \frac{\partial}{\partial x}(h(x)h'(x)) - \frac{\partial}{\partial x}(h'(x)h(x)) = 0$$

Example ?? $\implies h(x) = c\sin(\frac{\pi x}{L})$

here $L \in \mathbb{R}$ is not the operator!

$$\int_0^L h(x)^2 dx = c^2 \int_0^2 \sin^2(\frac{\pi x}{L}) dx = 1$$
$$\implies c = \sqrt{\frac{2}{L}} \implies \rho(x) = \frac{2}{L}\sin^2(\frac{\pi x}{L}).$$

2.7 Uniqueness in law and path integral formula

Consider the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + dB_t & \text{in } \mathbb{R}^d \\ x_0 = x_0 \end{cases}$$

Assume

$$(\star) \forall T > 0 \int_0^T |b(s, X_s)|^2 ds < \infty \text{ a.s.}$$

.

Goal: Show uniqueness in law.

Consider any $(X, \mathcal{B}, \mathbb{P})$ weak solution satisfying $(\star)$ and define

$$\tau_n := \inf\{t \geq 0 \mid \int_0^t |b(s, X_s)|^2 ds \geq n\}$$

which, by $(\star)$ goes to infinity.

For each n, define $\mathbb{Q}^n$:

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = e^{\underbrace{-\int_0^{\tau_n} b(s,X_s)dB_s - \frac{1}{2}\int_0^{\tau_n} |b(s,X_s)|^2 ds}_{L_{\tau_n}}} .$$

By Girsanov: $\tilde{B}_t := B_t - \langle B, L \rangle_t = B_t + \int_0^{t \wedge \tau_n} b(s, X_s) ds$ is BM w.r.t. $\mathbb{Q}^n$, which implies $X_t = \tilde{B}_t$ for all $t \leq \tau_n$ w.r.t. $\mathbb{Q}^n$.

Let events of (X, B) $\mathcal{F}_T$ measurable for some time T: are in A_T.

$$\mathbb{E}_\mathbb{P}\left(1_{(X,B)\in A_T} 1_{T \leq \tau_n}\right) = \mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,\mathcal{B})\in A_T} 1_{T \leq \tau_n} e^{\int_0^{\tau_n} b(s,X_s)\underbrace{dB_s}_{=dX_s - b(s,X_s)ds} + \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)$$

$$= \mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,B)\in A_T} 1_{T \leq \tau_n} e^{\int_0^{\tau_n} b(s,X_s)dX_s - \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)$$

B_s is adapted to $X_s \implies$ for some Φ: $B = \Phi(X) \implies$

$$\mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,\Phi(X))\in A_T} 1_{T \leq \tau_n} e^{\int_0^{\tau_n} b(s,X_s)dX_s - \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)$$

But: Law of X w.r.t. $\mathbb{Q}^n$ is the law of BM. Take $n \to \infty$, then $\mathbb{Q}^n \to$ Wiener measure, $T \to \infty \implies$ we look at every possible event, but the probability does only depend on X!

$$\mathbb{P}((X, B) \in \mathcal{B}(\mathcal{E}^d \times \mathcal{E}^d))$$

One case where the path integral formula is also numerically stable:

Assume that $b(x) = -\nabla V(x)$ for some smooth $V(x)$ (time independent). This is called drift of **gradient type**.

Apply Itô to $V(x)$:

$$V(\omega_T) = V(\omega_0) + \int_0^T \nabla V(\omega_s) dw_s + \frac{1}{2}\int_0^T \Delta V(\omega_s) ds$$

$\implies \int_0^T b(s, \omega_s)d\omega_s = -\int_0^T \nabla V(\omega_s)d\omega_s$

and therefore

$$\mathbb{P}(X \in \tilde{A}_T) = \int_{C^d} W(d\omega) \underbrace{e^{V(\omega_0) - V(\omega_T) + \frac{1}{2}\int_0^T [\Delta V(\omega_s) - (\nabla V(\omega_s))^2]ds}}_{=:\Psi(\omega)}$$

One application:

Let $X_0 = x, f : \mathbb{R}^d \to \mathbb{R}$

$$\underbrace{bE_x(f(X_T))}_{(T(t)f)(x)} = \int_{\xi^d} f(\omega_T)\Psi(\omega)W_x(d\omega)$$

where W_x is the wiener measure starting from x and $T(t)$ is the semigroup of X.

$$|(T(t)f)(x)| \leq \|fe^{-V}\|_\infty e^{-\frac{1}{2}\int_0^t \inf_{x \in \mathbb{R}^d}(|\nabla V(x)|^2 - \Delta V(x))ds}$$

If $\inf_{x \in \mathbb{R}^d}(|\nabla V(x)|^2 - \Delta V(x)) \geq 2\lambda > 0 \implies$

$$\|e^{-V(x)(T(t)f)}\| \leq \|e^{-V}f\|_\infty e^{-\lambda t} \to \text{exponential decay}$$

Chapter 3:
Local times, Itô-Tanaka formula, reflected Brownian Motion

3.1 Extension of Itô formula to convex functions

If $f \in C^2(\mathbb{R})$, $f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle x \rangle_s$

Proposition 3.1. *Let f be a convex function on $\mathbb{R}$, X_t a (continuous) semimartingale, then $f(X_t)$ is a semimartingale: $\exists A_t^f$ such that $\forall t \geq 0$*

$$f(X_t) = f(X_0) + \int_0^t \underbrace{f'_-}_{\text{left derivative of } f}(X_s)dX_s + A_t^f$$

We make a choice for the left derivative, the same works for the right one, but with different processes

Remark. *This extends directly to the case that $f = f_1 - f_2$ with f_1, f_2 convex (But with bounded variation and not necessarily increasing A_t^f)*

Sketch of Proof. Consider a function $\rho(x)$ s.t.

- $\rho(x) \geq 0$, ρ smooth
- $\rho(x) = 0$ if $x \notin [0,1]$
- $\int_0^1 \rho(x)dx = 1$

For any $n \in \mathbb{N}$ set

$$f_n(x) := \int_{\mathbb{R}} n\rho(ny)f(x-y)dy$$

Verify:

- f_n is $C^2(\mathbb{R})$; $f_n'' \geq 0$ (since f is convex)
- $f_n'(x) = \int_{\mathbb{R}} n\rho(ny) \cdot f'_-(x-y)dy$
- $f_n(x) \overset{n\to\infty}{\to} f(x)$
- $f_n'(x) \overset{n\to\infty}{\to} f'_-(x)$

left derivative since we do convolution and not correlation with ρ

Itô formula to $f_n(X_t)$:

$$f_n(X_t) = f_n(X_0) + \int_0^t f_n'(X_s)dX_s + \underbrace{\frac{1}{2}\int_0^t f_n''(X_s)\langle X \rangle_s}_{A_t^{f_n} \in \mathcal{A}_+^0}$$

Using stopping times: If $X = \underbrace{M_+}_{\in \mathcal{M}_{\text{loc}}^0} + \underbrace{A}_{\in \mathcal{A}}$

$\forall m \geq 1 : T_m = \inf\{t \geq 0 \mid |X_t| + \langle M \rangle_t + \int_0^t |dA_s| \geq m\}$

$$f_n(X_{t \wedge T_m}) = f(X_0) + \underbrace{\int_0^{t \wedge T_m} f_n'(X_s)dX_s}_{n \to \infty, \mathbb{P} \int_0^{t \wedge T_m} f_-'(X_s)dX_s} + \underbrace{\frac{1}{2}\int_0^{t \wedge T_m} f_n''(X_s)d\langle M \rangle_s}_{A_{t \wedge T_m}^{f_n}}$$

$\implies$ set $A_t^{f,m} := f(X_{t \wedge T_m}) - f(X_0) - \int_0^{T_m} f_-'(X_s)dX_s$ and $A_{t \wedge T_m}^{f_n} \stackrel{n \to \infty, \mathbb{P}}{\to} A_t^{f,m} = A_{t \wedge T_m}^f$; Take
$m \to \infty$ and get a process A_t^f since $f_n'' \geq 0 \implies A_t^f$ is increasing and in $\mathcal{A}_0$. $\qquad \square$

Let $f(x) = |x|$. Then

$$f_-'(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

we therefore define

$$\mathrm{sgn}(x) := \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

as the left derivative of $|x|$.

> **Proposition 3.2** (Tanaka's formula). *Let X be a continuous semimartingale and $a \in \mathbb{R}$. Then there $\exists$ increasing process $(L_t^a(X))_{t \geq 0}$ s.t.:*
>
> *(a) $|X_t - a| = |X_0 - a| + \int_0^t sgn(X_s - a)dX_s + L_t^a(X)$*
>
> *(b) $(X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{X_s > a}dX_s + \frac{1}{2}L_t^a(X)$*
>
> *(c) $(X_t - a)^- = (X_0 - a)^- - \int_0^t 1_{X_s \leq a}dX_s + \frac{1}{2}L_t^a(X)$*
>
> *The process $L_t^a(X)$ is called the **local time of** X at **level** a. For any stopping time T:*
>
> $$L_t^a(X^T) = L_{t \wedge T}^a(X)$$

Proof. Taking $f(x) = |x - a|$ in proposition 3.1 yields (a). Therefore

$$L_t^a(X) := |X_t - a| - |X_0 - a| - \int_0^t \mathrm{sgn}(X_s - a)dX_s$$

Applying prop. 3.1 to $f(x) = (x-a)^+$ and $f(x) = (x-a)^-$

$$\implies \exists A_t^{a,(+)}, A_t^{a,(-)} \in \mathcal{A}_0^+$$

such that

$$(X_t - a)^+ = (X_0 - a)^+ \int_0^t 1_{X_s > a}dX_s A_t^{a,(+)}$$

$$(X_t - a)^- = (X_0 - a)^- \int_0^t 1_{X_s \leq a}dX_s A_t^{a,(-)}$$

$$X_t - a \stackrel{\text{Itô}}{=} X_0 - a + \int_0^t dX_s$$

$$X_t - a = (X_t - a)^+ - (X_t - a)^- = X_0 - a + \int_0^t dX_s + A_t^{a,(+)} - A_t^{a,(-)}$$

which holds if and only if $A_t^{a,(+)} = A_t^{a,(-)}$. Furthermore

$$|X_t - a| = |X_0 - a| + \int_0^1 \mathrm{sgn}(X_s - a)dX_s + \underbrace{A_t^{a,(+)} + A_t^{a,(-)}}_{=L_t^a(X)} \qquad \square$$

<u>To see:</u> $L_t^a(X) \to$ positive measure $dL_t^a(X)$, since $L_t^a(X)$ is increasing.

$$\int_0^t \underbrace{f(x)}_{\geq 0} dL_s^a(X) \stackrel{?}{=} 0 \text{ if } f(x) > 0 \text{ for } x \neq a$$

Proposition 3.3. *Let $dL_t^a(X)$ be the measure associated with the increasing process $L_t^a(X)$. Then $dL_t^a(X)$ is supported on $\{s \geq 0 \mid X_s = a\}$.*

But this is not necessarily the full support!

Proof. Let $Y_t = |X_t - a|$.

$$\implies Y_t^2 = (X_t - a)^2 \overset{\text{Itô to } f(X_t)=(X_t-a)^2}{=} (X_0 - a)^2 + \int_0^t (X_s - a)dX_s + 2\langle X \rangle_t$$

$$Y_t^2 \overset{\text{Itô to } f(Y_t)=Y_t^2}{=} (X_0 - a)^2 + 2\int_0^t \underbrace{Y_s dY_s}_{=|X_s-a|\cdot(\text{sgn}(X_s-a)dX_s+dL_s^a(X))} + \underbrace{\langle Y \rangle_t}_{\langle X \rangle_t \text{ by prop 3.1}}$$

$$= (X_0 - a)^2 + 2\int_0^t (X_s - a)dX_s + 2\int_0^t |X_s - a|dL_s^a(X) + \langle X \rangle_t$$

$$\implies \int_0^t |X_s - a|dL_s^a(X) = 0$$

and therefore the support is indeed contained in $\{s \geq 0 \mid X_s = a\}$ $\qquad\square$

Remark. *If f is convex $\implies f_-'(x)$ is increasing and left continuous. Therefore there exists a unique measure $f''(dx)$ on $\mathbb{R}_+$ such that*

$$f''([a,b]) = f_-'(b) - f_-'(a)$$

If $f \in C^2(\mathbb{R})$, then $f''(dx) = f''(x)dx$, i.e. it has the density given by the second derivative. For all convex functions f with $f''(dx) = 0$ for all $|x| \geq K$

$$\implies \exists \alpha, \beta \in \mathbb{R} : f(x) = \alpha + \beta x + \frac{1}{2}\int_{\mathbb{R}} |x - a|f''(da)$$

and $f_-'(x) = \beta + \frac{1}{2}\int_{\mathbb{R}} sgn(x - a)f''(da)$ in a weak sense.

Start of lecture 12 (04.06.24)

Theorem 3.4 (Itô-Tanaka-formula). *If f is a difference of convex functions, X is a continuous semimartingale*

$$\implies f(X_t)f(X_0) + \int_0^t f_-'(X_s)dX_s + \frac{1}{2}\int_R L_t^a(x)f''(da)$$

, where $f''(da)$ is the measure associated with f_-'.

Sketch of proof. (Full proof: Thm 9.6 LeGall book)
Once localized $\rightarrow$ assume $f''(da) = 0$ on $[-\kappa, \kappa]^c$

$$\implies f(X_t) = \alpha + \beta X_t + \int_{-\kappa}^{\kappa} |X_s - a|f''(da)$$

$$\overset{\text{prop. 3.2}}{=} \alpha + \beta X_t + \frac{1}{2}\int |X_0 - a|f''(da) + \frac{1}{2}\int\int_0^t \text{sgn}(X_s - a)dX_s f''(da) + \frac{1}{2}\int L_t^a(X)f''(da)$$

$$= \underbrace{\alpha + \beta X_0 + \frac{1}{2}\int |X_0 - a|f''(da)}_{=f(X_0)} + \underbrace{\beta\int_0^t dX_s + \frac{1}{2}\int_0^t \int \text{sgn}(X_s - a)f''(da)dX_s}_{=\int_0^1 f_-'(X_s dX_s)} + \frac{1}{2}\int L_t^a f''(da)$$

$\qquad\square$

Corollary 3.5 (Occupation time formula). *Almost surely, $\forall t \geq 0$, <u>non-negative</u> measurable function φ*

$$\int_0^t \varphi(X_s)d\langle X \rangle_s = \int_{\mathbb{R}} \varphi(a)L_t^a(X)da$$

Example. $X =$ BM, $\varphi(X) = 1_A(X)$, A Borel set in $\mathbb{R}$

$$\implies \int_0^t 1_A(X_s)ds$$

is the time spend by $(X_s, s \in [0,t])$ in A and the RHS is

Think $A = [a, a + \epsilon]$

$$\int_A L_t^a(X)da$$

$\implies L_t^a$ density of time spend by Brownian motion in a.

Proof. (For φ with bounded support)
Find $f \in C^2$ s.t. $f''(x) = \varphi(x)$.
$\implies$ Itô-Tanaka-formula:

Take f as the twice integrated φ on the support of φ until x

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}\int L_t^a(X)\varphi(a)da$$

By Itô-formula:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t \varphi(X_s)d\langle X\rangle_s$$

The claim follows from comparing the terms. $\qquad\qquad\square$

Explicit representation of the local time:

Lemma 3.6. *Let X be a continuous semimartingale*
$\implies$ *a.s. $\forall a \in \mathbb{R}, t \geq 0$:*
$$L_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\int_0^t 1_{a \leq X_s \leq a+\epsilon}d\langle X\rangle_s$$

Proof. Take $\varphi(x) = \frac{1}{\epsilon}1_{[a \leq X \leq a+\epsilon]}$ in corollary 3.5:

See Thm 9.4 in LeGall for more details

$$\implies \frac{1}{\epsilon}\int_0^t 1_{a \leq X_s \leq a+\epsilon}d\langle X\rangle_s = \frac{1}{\epsilon}\int_a^{a+\epsilon} L_t^{\tilde{a}}(X)d\tilde{a} \stackrel{\tilde{a} \to L_t^{\tilde{a}}}{=} L_t^a(X) \qquad\square$$

Similarly:

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\int_0^t 1_{[a-\epsilon \leq X_s < a]}d\langle X\rangle_s = L_t^{a-}(X)$$

$$\implies \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\int_0^t 1_{[a-\epsilon \leq X_s \leq a+\epsilon]}d\langle X\rangle_s = \frac{1}{2}(L_t^{a-}(X) + L_t^a(X))$$

$$L_t^a(X) - L_t^{a-}(X) \stackrel{\text{prop } 3.2}{=} |X_t - a| - |X_0 - a| - \int_0^t \text{sgn}(X_s - a)dX_s - (|X_t - a_-| - |X_0 - a_-| - \int_0^t \text{sgn}(X_s - a_-)dX_s)$$

$$= 2\int_0^t 1_{X_s = a}dX_s \stackrel{X_t = M_t + A_t}{=} 2\int_0^t 1_{X_s = a}dA_s$$

where the first two terms are equal, since X is a continuous semimartingale. This then implies, if X is a martingale (and therefore $A_t = 0$) then
$L_t^a(X) = L_t^{a-}(X) \implies L_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\int_0^1 1_{[a-\epsilon \leq X_s \leq a+\epsilon]}d\langle X\rangle_s$.
In particular, if X is a BM:

$$L_t^{a-}(X) \implies L_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\int_0^1 1_{[a-\epsilon \leq X_s \leq a+\epsilon]}ds$$

3.2 Brownian motion and local time

Let B be a standard BM, then by Tanaka:

$$|B_t| = |\underbrace{B_0}_{=0}| + \int_0^t \text{sgn}(B_s)dB_s + \underbrace{L_t}_{:=L_t^0(B)}$$

We want to define the process $R_t = |B_t|$ be the **BM reflected at** 0.
We want to study R_t and $S_t^B := \sup_{0 \leq s \leq t} B_s, L_t(B)$

3.2.1 No strong solution of the Tanaka SDE

$$(\star) = \begin{cases} dX_t & = \text{sgn}(X_t)dB_t \\ X_0 & = 0 \end{cases}$$

We already know: $B_t := \int_0^t \text{sgn}(W_s)dW_s$, where W is a BM $\implies BM$ is a BM (using Levy characterization) and

$$\int_0^t \text{sgn}(W_s)dB_s = \int_0^t \text{sgn}(W_s)^2 dW_s = W_t \to \text{sgn}(W_t)dB_t = dW_t$$

and therefore (W, B) is a weak solution of $(\star)$.
Assume (X, B) is a strong solution of $(\star) \implies dX_t = \text{sgn}(X_t)dB_t$ with X is a BM.
Itô-Tanaka-formula:

$$|X_t| = \int_0^t \text{sgn}(X_s)dX_s + L_t^0(X)$$

$$\implies \int_0^t \text{sgn}(X_s)dX_s = |X_t| - L_t^0(X) = |X_t| - \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{|X_s| \leq \epsilon}ds \in \mathcal{F}_t^{|X|}$$

But also:

$$\int_0^t \text{sgn}(X_s)dX_s = \int_0^t (\text{sgn}(X_s))^2 dB_s = B_t$$

$$B_t = |X_t| - \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{|X_s| \leq \epsilon}ds$$

$\implies B_t$ is $\mathcal{F}_t^{|X|}$-measurable and therefore B is $\mathcal{F}^{|X|}$ measurable.
If (X, B) is a strong solution, then X is measurable w.r.t. $\mathcal{F}^B$. $\implies \mathcal{F}^X \subset \mathcal{F}^B \subseteq \mathcal{F}^{|X|}$ which is wrong, since we can not recall the value of X knowing only its absolute value.

3.2.2 Reflected SDE

Definition 3.7. *The family (X, l, W) is a weak solution of the **reflected SDE** (One-dimensional):*

$$dX_t = dW_t + dl_t$$

if:

- *W is a BM*

- *X is a positive continuous process*

- *l is a positive non-decreasing continuous process*

s.t.

$$\int_0^\infty 1_{X_s} > 0 dl_s = 0$$

*The solution is called **strong**, if (X, l) is adapted to W.*

Example.   • $X_t := |B_t|$

- $W_t := \int_0^t sgn(B_s)dB_s$

- $l_t = L_t^0(B)$

*This is a weak solution of the **1 dimensional reflected SDE**.*

Lemma 3.8 (Skorokhod lemma for uniqueness of reflected SDE). *Let $Y(t)$ be a continuous function with $y(0) \geq 0$. Then there exist a unique pair (z, a) of functions:*

- *both continuous*

- *$a(0) = 0, a$ is non-decreasing s.t.*

  (a) $z(t) = y(t) + a(t)$

  (b) $\int_0^\infty 1_{z(s)>0}da(s) = 0$

Moreover, the function a has the representation

$$a(t) = \sup_{0 \leq s \leq t} [y(s)]_- = \sup_{0 \leq s \leq t} \max\{0, -y(s)\}$$

Proof. Postponed :) □

Consequence of Lemma 3.8:

$$l_t(\omega) := \sup_{0 \leq s \leq t} (-W_s(\omega))_+ = \sup_{0 \leq s \leq t} (-W_s(\omega)) =: S_t^{-W}(\omega)$$

given $W(\omega), X_t(\omega) = W_t(\omega) + l_t(\omega) = W_t(\omega) + \sup_{0 \leq s \leq t}(-W_s(\omega))$.
$\implies \exists!$ strong solution of the reflected SDE. Notation:

$$R_t := |B_t| = W_t + L_t = W_t + S_t^{-W}$$

Theorem 3.9.

$$Law(|B|, \underbrace{L}_{=L^0(B)}) \overset{(1)}{=} Law(L + W, L)$$

$$\overset{(2)}{=} Law(W + S^{-W}, S^{-W})$$

$$\overset{(3)}{=} Law(S^{-\tilde{W}} - \tilde{W}, S^{\tilde{W}})$$

with $\tilde{W}$ a generic BM.

Proof. (1) is from $R_t = |B_t| = W_t + L_t^0(B)$ (as a process)
(2) by the skorokhod representation: $L_t = S_t^{-W}$
(3) $W \overset{d}{=} -\tilde{W}$ □

$\forall t \geq 0$:

$$S_t := \sup_{0 \leq s \leq t} B_s \overset{(d)}{=} |B_t| \overset{(d)}{=} L_t^0(B)$$

By the reflection principle:

$$\mathbb{P}(S_t \in dx, B_t \in dy) \overset{2.2 \text{ LeGall}}{=} \frac{2(2x-y)}{\sqrt{2\pi t^3}} e^{-\frac{(2x-y)^2}{2t}} 1_{x>0, y<x} dxdy$$

and

$$Law(|B|) = Law(S^W - W)$$

Two points of view:

1. $B \to \mathbb{R}_t := |B_t|, W_t := \int_0^t \mathrm{sgn}(B_s) dB_s, L_t := L_t^0(B)$ satisfies

$$R_t = W_t + L_t$$

2. Given $W_t (\forall \omega)$: Define $l_t := \Phi_t(W) = \sup_{0 \le s \le t}(-W_s), X_t = W_t + l_t$

$\mathrm{Law}(R, L) = \mathrm{Law}(X, l)$.

An alternative definition of reflected Brownian motion is

$$X_t := \underbrace{W_t + \Phi_t(W)}_{\textbf{Skorokhod map}}$$

Added remark. *Careful:*

$$|W_t(\omega)| \ne W_t(\omega) + \Phi_t(W(\omega))$$

Or

$$z(t) \ne |y(t)|$$

3.3 Reflected BM on a curve

In section 3.2, we have reflected BM on $g(t) = 0 \forall t \ge 0$.

$$X_t = W_t + \sup_{0 \le s \le t}(-W_s)$$

Skorokhod lemma; from continuous function $y(t), y(0) \ge 0$:

$$z(t) := y(t) + \sup_{0 \le s \le t}(y(s) - 0)_-$$

Let $D_g := \{(x, t) \mid x \ge g(t)\}$.

Let $(W_t)_{t \ge 0}$ with $W_0 \ge g(0)$, then there exists a unique, continuous non-decreasing process L_t with $L_0 = 0$ s.t.

$$X_t := W_t + L_t \ge g(t) \forall t \ge 0$$

and L_t may increase only when X_t is at the boundary of D_g.

$$\int_0^\infty 1_{(g(s), \infty)}(X_s) dL_s = 0$$

X_t has the formula:

$$L_t = \sup_{0 \le s \le t}(W_s - g(s))_-$$

To see this claim: Consider $\tilde{W}_t = W_t - g(t)$

$$\implies \tilde{X}_t := X_t - g(t)$$

is constructed by the Skorokhod map and

$$\tilde{X}_t = \tilde{W}_t + L_t \ge 0$$

D_g is the space above the curve $(t, g(t))$, not sure why the coordinates are fliped tho?

Definition 3.10. *The **BM reflected by a continuous function** $g(t)$ with $X_0 \ge g(0)$ is given by the following construction*

- *Given W a BM with $W_0 = X_0 \ge g(0)$, define*

$$X_t := W_t + \sup_{0 \le s \le t}(W_s - g(s))_-$$

- *Alternatively with $W_0 = 0$:*

$$X_t = X_0 + W_t + \sup_{0 \le s \le t}(W_s - g(s))_-$$

Added remark. *Think about sticky BM, which is the limit of some random walk which stays on the boundary with some positive probability.*
The exercise talks shows two reflected BMs, whose difference lies in the Weil chamber and solves the SDE for a process conditioned to be positive forever.

SDE: $dX_t = dW_t + dL_t$ or $X_t = X_0 + \underbrace{W_t}_{W_0=0} + L_t^g(X)$, where $L_t^g(X)$ is the local time of X spend at

the function g.

Example. *Let $B = (B_1(t), B_2(t))$ a 2-dimensional BM with $B_1(0) \leq B_2(0)$ s.t. B_1 is not depending*   *Start with 2-dim BM then*
on B_2, but B_2 is reflected upwards by $(B_1(t), t \geq 0) = g$ *define the reflected BM...*
$\implies (B_1(t), \tilde{B}_2(t)) \in D = \{(x_1, x_2 \in \mathbb{R}^d \mid x_1 \leq x_2)\}$

Proposition 3.11. *Denote by $p_t(x, y)$ be the transition probe density of B:*

$$p_t(B_1(t) \in dy_1, \tilde{B}_2(t) \in dy_2 \mid B_1(0) = 0, \tilde{B}_2(0) = x_2) = p_t((x_1, x_2), (y_1, y_2))dy_1 dy_2$$

It satisfies the following equations:

$$\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) p_t((x_1, x_2), (y_1, y_2)) = \frac{\partial}{\partial} p_t((x_1, x_2), (y_1, y_2))$$

for $(x_1, x_2), (y_1, y_2) \in D \setminus \partial D, t > 0$.

$$p_t((x_1, x_2), (y_1, y_2))dy_1 dy_2 \stackrel{t \to 0}{\to} \delta_x(y)$$

as well as

$$\frac{\partial}{\partial x_2} p_t(x, y) \mid_{x=y} = 0$$

Proof. Let g be any smooth test function in D, set $f(t, x) := \int dy g(y) p_t(x, y)$. The undefined   Important information:
integrals are over D, for Itô we assume $f \in C^2$.   where is the boundary
(1) SDE: condition coming from?
    Not immediatly obvious!

$$\begin{cases} dB_t^1 = dW_t^1 \\ d\tilde{B}_t^2 = dW_t^2 + dL_t^{B^1}(\tilde{B}_2) \end{cases}$$

By Itô: Let $0 \leq t \leq T$:

$$f(T - t, B - t)$$
$$= f(T, B_0) + \int dy g(y)[-\dot{P}_{T-S}(B_s, y)ds] + \int dy g(y) \int_0^t \partial x_1 p_{T-S}(B_s, y)dW_s^1$$
$$+ \int_0^t \partial x_2 p_{T-S}(B_s, y)(dW_s^1 + dL_s)$$
$$+ \int dy g(y) \left[\frac{1}{2}\int_0^t \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} p_{T-S}(B_s, y)ds\right)\right]$$

(2) Martingale: Let $t \leq T$

$$\mathbb{E}(g(B_t) \mid \mathcal{F}_t) \stackrel{\text{MP}}{=} \mathbb{E}(g(B_t) \mid B_t)$$
$$= \int dy g(y) p_{T-t}(B_t, y) = \underbrace{f(T - t, B_t)}_{\text{Martingale}} \qquad \square$$

Chapter 4:
Levy processes

4.1 Basics

Poisson process (Picture) ... With $t_{n+1} - t_n$ are iid., $\sim \exp(\lambda)$ (here it is the parameter, NOT the mean!) and $N_t \sim \text{Poi}(\lambda, t)$.

$$\mathbb{P}(N_{t+\epsilon} = y + 1 | N_t = y) = \lambda\epsilon + O(\epsilon^2)$$

BM	Poisson process
	Monotone increasing

BM:
A real valued process $B = (B_t)_t$ satisfying:

(a) $t \mapsto B_t$ is continuous

(b) $\mathbb{P}(B_0 = 0) = 1$

(c) For $0 \leq s \leq t : B_t - B_s \overset{d}{=} B_{t-s}$

(d) For $0 \leq s \leq t : B_t - B_s$ is independent of $\{B_u, u \leq s\}$

(e) $\forall t > 0 B_t \sim \mathcal{N}(0, t)$

Poisson process:
A process $N = (N_t)$ with values in D is called a Poisson process with parameter $\lambda > 0$, if

(a) $t \mapsto N_t$ is $\mathbb{P}$. a.s. **cadlag**.

(b) $\mathbb{P}(N_0 = 0) = 1$

(c) For $0 \leq s \leq t : N_t - N_s \overset{d}{=} M_{t-s}$

(d) For $0 \leq s \leq t : N_t - N_s$ is independent of $\{N_u, u \leq s\}$

(e) $\forall t > 0 N_t \sim \text{Poi}(\lambda t)$

Notice how b,c,d are the same!
Differences:
BM:

- continuous paths

- unbounded variation

PP:

- pure jump process

- bounded variation

Common features:

- both have càdlàg trajectories

- stationary, independent increments

Definition 4.1 (Lévy Process). *A process $X = (X_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a **Lévy process**, if:*

(a) *The trajectories are $\mathbb{P}$. a.s. càdlàg*

(b) $\mathbb{P}(X_0 = 0) = 1$

(c) *For $0 \leq s \leq t : X_t - X_s \overset{d}{=} X_{t-s}$*

(d) *For $0 \leq s \leq t : X_t - X_s$ is independent of $\{X_u, u \leq s\}$.*

Remark. *Lévy processes are Markov processes.*

Definition 4.2. *A real-valued random variable X has a **infinitely divisible distribution**, if for all $n \geq 1$: There exists r.v. $X_1^n, \ldots, X_n^n$ iid. s.t.*

$$X \overset{d}{=} X_1^n + \ldots X_n^n.$$

*Let $X \sim \mu : \mu$ **is infinitely divisible** if for all $n \geq 1$: $\exists$ prob. measure μ_n s.t. $\mu = (\mu_n)^{*n}$, i.e. the n-fold convolution.*

Remark. *Setting $\psi(u) = -\log(\mathbb{E}(e^{iuX}))$, for $u \in \mathbb{R}$, $\implies$ ψ is called the **charateristic exponent of X**. If X is infinitely divisible, then*

$$\begin{aligned} \mathbb{E}(e^{iuX}) &= \mathbb{E}(e^{iu(X_1^n + \cdots + X_n^b)}) \\ &= \mathbb{E}(e^{iuX})^n \\ &\implies \psi(u) = n\psi_n(u), \end{aligned}$$

where ψ_n is the characteristic exponent of X_1^n.

We can change the definition to $\mathbb{E}(e^{iuX}) = e^{-\psi(u)}$ to avoid taking logarithms of complex numbers.

Theorem 4.3 (Lévy-Khintchine formula). *A probability law μ of real valued real random variable is infinitely divisible with characteristic exponent*

$$\psi(u\theta) = -\log \int_{\mathbb{R}} e^{i\theta x} \mu(dx)$$

if and only if $\exists$ a triple (a, σ, ν), $a \in \mathbb{R}, \sigma > 0$,$\mu$ a measure supported on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\nu d(x) < \infty$ s.t.

$$\psi(\theta) = ia\theta + \frac{1}{2}\theta^2\sigma^2 + \int_R (1 - e^{i\theta x} + i\theta x 1_{|x|<1})\nu(dx)$$

ν does not have to have finite measure!

Added remark. *ν is called the **Lévy measure**.*

Added remark. *What happens for x small?*

$$(1 - e^{i\theta x} - i\theta x 1_{|x|<1}) \approx 1 - (1 + i\theta x - \frac{\theta^2 x^2}{2}) + i\theta x = \frac{\theta^2 x^2}{2}$$

Added remark. *The 1 in the indicator function is arbitrary, tho other choices yield other formulas!*

4.1.1   Relation of infinitely divisible processes with Levy processes

Let X be a Levy process, $\forall t > 0$, X_t is infinitely divisible, since

$$(\star)\ X_t = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \cdots + \left(X_t - X_{\frac{t(n-1)}{n}}\right)$$

For all $\theta \in \mathbb{R}, \geq 0$: $\psi_t(\theta) := -\ln(\mathbb{E}(e^{i\theta X_t}))$

$$\stackrel{(\star)}{\Longrightarrow}\ \forall m, n \geq 1 : \frac{m}{m}\psi_1(\theta) = \frac{1}{m}\psi_m(\theta) = \frac{n}{m}\psi_{\frac{m}{n}}(\theta)$$

and therefore for $t \geq 0$ rational: $\psi_t(\theta) = t\psi_1(\theta)$. It then follows that $\forall t \in \mathbb{R}_+ : \psi_t(\theta) = t\psi_1(\theta)$. For Lévy process:

$$\mathbb{E}(e^{i\theta X_t}) = e^{-t\psi_1(\theta)}$$

Definition 4.4. *We refer to $\psi(\theta) := \psi_1(\theta)$ to be the characteristic exponent of the Lévy process.*

Theorem 4.5. *Let $a \in \mathbb{R}, \sigma \geq 0, \nu$ a measure supported on $\mathbb{R} \setminus \{0\}$ s.t.*

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty$$

Define

$$\psi(\theta) := ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x)1_{|x|<1}\nu(dx)$$

$\implies$ *then there exists a Lévy process with characteristic exponent $\psi(\theta)$.*

Proposition 4.6. *Let F be a distribution function on $\mathbb{R}_+$. Then F is the distribution function of an infinitely divisible law*
$\iff \forall \lambda > 0 : F^{\mathcal{L}}(\lambda) = \int_0^\infty e^{-\lambda x}F(dx) = e^{-c\lambda - \int_0^\infty (1-e^{-\lambda x})\mu(dx)}$, *where $c \in \mathbb{R}$, μ is a measure on $(0,\infty)$, s.t. $\int_0^\infty (1 \wedge x)\mu(dx) < \infty$.*

Proof. $\implies$: Let X have distribution $F : F(x) = \mathbb{P}(X \leq x)$. $X_1^n + \cdots + X_n^n$ with $\mathbb{P}(X_1^n \leq x) =: F_n(x)$.

$$F^{\mathcal{L}}(\lambda) = \int_0^\infty e^{-\lambda x}F(dx) = (F_n^{\mathcal{L}}(\lambda))^n,$$

since

$$
\begin{aligned}
F^{\mathcal{L}}(\lambda) &= \int_0^\infty e^{-\lambda x}\mathbb{P}(X_1 + \cdots + X_n \in dx) \\
&= \int_0^\infty e^{-\lambda x} \int \mathbb{P}(X_1 \in dz)\mathbb{P}(X_2 + \cdots + X_n + z \in dx) \\
&= \int_0^\infty \mathbb{P}(X_1 \in dz) \int_z^\infty \mathbb{P}(X_2 + \cdots + X_n + z \in dx)e^{-\lambda x} \\
&\stackrel{x = \tilde{x} + z}{=} \int_0^\infty \mathbb{P}(X_2 + \cdots + X_n \in d\tilde{x})e^{-\lambda\tilde{x}}e^{-\lambda z} \\
&= \left(\int_0^\infty \mathbb{P}(X_1 \in dz)e^{-\lambda z}\right)^n = \left(F_n^{\mathcal{L}}(\lambda)\right)^n
\end{aligned}
$$

Since F is a distribution on $\mathbb{R}_+ \implies F_n^{\mathcal{L}} \in (0,1]$, which means

$$F_n^{\mathcal{L}}(\lambda) \stackrel{n\to\infty}{\to} 1$$

uniformly for λ in compact sets.

$$\implies \ln F^{\mathcal{L}}(\lambda) = n\log F_n^{\mathcal{L}}(\lambda) = n\log(1 - (1 - F_n^{\mathcal{L}}(\lambda))) \leq -n(1 - F_n^{\mathcal{L}}(\lambda))$$

This last inequality is almost an equality for large n!

1.: $\forall \delta > 0, K < \infty, \exists n_0 < \infty$ s.t. $n \geq n_0, \lambda \leq k \implies 1 - F_n^{\mathcal{L}}(\lambda) \leq \delta$.

2.: $\forall \delta > 0, C < \infty$ s.t. $\forall 0 \leq x \leq \delta$,

$$-x(1+cx) \leq \log(1-x) \leq -x$$

Apply it to $x = 1 - F_n^{\mathcal{L}}(\lambda)$

$$\implies -(1 - F_n^{\mathcal{L}}(\lambda))[1 + C(1 - F_n^{\mathcal{L}}(\lambda))] \leq \log F_n^{\mathcal{L}}(\lambda) - n(1 - F_n^{\mathcal{L}}(\lambda))$$

As $n \to \infty$, uniformly in bounded set (for λ):

$$-n(1 - F_n^{\mathcal{L}}(\lambda)) \to \log F^{\mathcal{L}}(\lambda).$$

$$\begin{aligned}
n(1 - F_n^{\mathcal{L}}(\lambda)) &= \int (1 - e^{-\lambda x}) F_n(dx) \\
&= \int \frac{1 - e^{-\lambda x}}{1 - e^{-x}} \underbrace{n(1 - e^{-x}) F_n(dx)}_{-:m_n(dx)}
\end{aligned}$$

m_n is a measure on $(0, \infty)$ with total mass is $n(1 - F_n^{\mathcal{L}}(1)) \to \log F^{\mathcal{L}}(1)$
$\implies \exists$ measure m on $[0, \infty]$ s.t. $m_n \to m$ and

$$n(1 - F_n^{\mathcal{L}}(\lambda)) \to m(\{0\})\lambda + \int_0^\infty \frac{1 - e^{-\lambda x}}{1 - e^{-x}} m(dx) + m(\{0\})$$

Let $\lambda = 0$: $m(\{\infty\}) = -\log F^{\mathcal{L}}(0) = -\log 1 = 0$. Then with $c = m(\{0\})$

$$\mu(dx)(1 - e^{-x}) = m(dx)$$

$\Longleftarrow$ ... $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Question: What are a, σ, ν? Is there a relation between ν and the jumps of the Lévy process?

4.2   Examples

4.2.1   Poisson process

Let $\lambda > 0$ and $X \sim \text{Poi}(\lambda)$,

$$\begin{aligned}
\mathbb{E}(e^{i\theta X}) &= \sum_{n \geq 0} e^{-\theta n} \frac{e^{-\lambda} \lambda^n}{n!} \\
&= e^{-\lambda} e^{\lambda e^{i\theta}} \\
&= e^{-\lambda(1 - e^{i\theta})} = \left(e^{-\frac{\lambda}{n}(1 - e^{i\theta})}\right)^n
\end{aligned}$$

$\implies X$ is infinitely divisible.
To get the right characteristic exponent

$$-\log \mathbb{E}(e^{i\theta X}) = e^{-\lambda} e^{\lambda e^{i\theta}}$$

which means $\psi(\theta) = \lambda(1 - e^{i\theta})$.
Take $a = 0, \sigma = 0, \nu(dx) = \lambda \delta_1(x)$, then $\psi(\theta) = (1 - e^{i\theta}\lambda)$.
The Lévy process with characteristic exponent $\psi(\theta)$ is called the Poisson process with parameter $\lambda > 0$.

Added remark. *If you have a Lévy process on $\mathbb{Z}_{\geq 0}$, (X_t) with $X_0 = 0$ and with jumps $+1 \implies$ This is a Poisson process.*

4.2.2 Compound Poisson process

Let N be a Poisson random variable with parameter $\lambda > 0$, $\{\xi_k\}_{k \geq 1}$ iid r.v. (independent of N) with distribution F on $\mathbb{R} \setminus \{0\}$.

Let $X =:= \sum_{k=1}^{N} \xi_k$:

The ξ give us the size of the jumps

$$\implies \mathbb{E}\left(e^{i\theta X}\right) = \sum_{n \geq 0} \mathbb{E}\left(e^{i\theta(\xi_1 + \cdots + \xi_n)}\right) \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \sum_{n \geq 0} \frac{e^{-\lambda}\lambda^n}{n!}\left(\int e^{i\theta X} F(dx)\right)^n$$

$$= e^{-\lambda}e^{\lambda \int e^{i\theta X} F(dx)} = e^{-\lambda \int F(dx)}e^{\lambda \int e^{i\theta X} F(dx)}$$

$$= e^{\underbrace{-\lambda \int (1 - e^{i\theta X})F(dx)}_{=\psi(\theta)}}$$

$\implies$ X is infinitely divisible with $\psi(\theta)$ having $\sigma = 0, \nu(dx) = \lambda F(dx), a = -\int_{|x| < 1} xF(dx)$

How to get the process?

We can think of a poisson process sa the set of jumping points (times). This is sometimes called poisson jump process

1. Choose Poisson process with parameter λ, $(N_t)_{t \geq 0}$

2. $X_t = \sum_{k=1}^{N_t} \xi_k$ is called the **compound Poisson process**

Is $(X_t)_{t \geq 0}$ a Lévy process?

Notation $0 =: \sum_{k=m}^n (\ldots)$ for $m > n$.

Then

- $X_0 = 0$

- $\forall 0 \leq s < t$: $X_t = X_s + \sum_{k=N_s+1}^{N_t} \xi_k$

which implies $X_t - X_s \overset{d}{=} \sum_{k=1}^{N_t - N_s} \xi_k \overset{d}{=} X_{t-s}$ and everything is independent, but $N_t - N_s = N_{t-s}$.

Since N_t are càdlàc $\implies$ X_t is càdlàg. $(X_t)_{t \geq 0}$ is a Lévy process with

$$\psi(\theta) = \lambda \int_{\mathbb{R} \setminus \{0\}} (1 - e^{i\theta x} F(dx)).$$

Let $\Delta X_t := X_t - X_{t^-}$ and for $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$,

$$\tilde{N}_t(B) := \#\{s \in [0,t] : \Delta X_s \in B\}$$

the number of jumps of size B in the time span $[0,t]$ and t_k are the jump times.

$(X_t)_t$ is a Lévy process $\implies$ $(\tilde{N}_t(B))$ has stationary independent increments.

$\tilde{N}_{t+s}(B) - \tilde{N}_{(t}(B) \overset{d}{=} \tilde{N}_t(B)$ for all $t, s \geq 0$ and B.

Therefore $(\tilde{N}_t(B))_t$ is a Lévy process with jumps of size $+1$, $\tilde{N}_0(B) = 0$, which implies $\tilde{N}_t(B)$ is a poisson process. The parameter of $\tilde{N}_t(B)$ is

$$\mathbb{E}(\tilde{N}_t(B)) = \underbrace{t \cdot \lambda}_{=\mathbb{E}(N_t)} \mathbb{P}(\xi \in B) = t\lambda \underbrace{\int_B F(dx)}_{=\nu(dx)} = t\nu(B).$$

$\implies$ $\nu(B)$ is the parameter of $\tilde{N}_t(B)$[1].

Added remark. *Suppose we had a Levy process with finite measure (remove $ia\theta$ and $i\theta x 1_{|x|<1}$). Then we can normalize and define $F \implies$ we get a compound poisson process!*

Add a drift:

$$X_t = \sum_{k=1}^{N_t} \xi_k + ct, t \geq 0$$

Then Lévy process of X_t has characteristic exponent

$$\psi(\theta) = \lambda \int (1 - {}^{i\theta X})F(dx)\underline{-ic\theta}$$

---

[1]As a process, not as a r.v.

4.2.3 Brownian motion with drift

Let $\mu_{s,\gamma}(dx) := \frac{e^{-\frac{(x-\gamma)^2}{2s^2}}dx}{\sqrt{2\pi s^2}}$, $s > 0, \gamma \in \mathbb{R}$.

If $X \sim \mu_{s,\gamma}$, then

$$\mathbb{E}(e^{i\theta X}) = e^{-\frac{1}{2}s^2\theta^2 + i\theta\gamma} = \left(e^{-\frac{1}{2}\frac{s}{\sqrt{n}}^2\theta^2 + i\theta\frac{\gamma}{n}}\right)^n$$

$\implies X$ is infinitely divisible with characteristic exponent $\psi(\theta)$ with $\nu = 0, \sigma = s^2, a = -\gamma$.

$$\implies \psi(\theta) = -i\theta\gamma + \frac{1}{2}s^2\theta^2.$$

and therefore

$$X_t := sB_t + \gamma t,$$

where B_t is a standard BM, is the Lévy process with characteristic exponent $\psi(\theta)$.

4.2.4 The Gamma process

Given $\alpha, \beta > 0$, define the probability measure

$$\mu_{\alpha,\beta}(dx) = \frac{\alpha^\beta}{\Gamma(\beta)}x^{\beta-1}e^{-\alpha x}dx 1_{R_+\backslash\{0\}}$$

where

$$\Gamma(z) := \int_0^\infty u^{z-1}e^{-u}du.$$

$\mu_{\alpha,\beta}$ is the Gamma(α,β) distribution. For $\beta = 1$ it is the exponential distribution.

Let $X \sim X_{\alpha,\beta}$.

$$\mathbb{E}(e^{i\theta X}) = \int_{\mathbb{R}_+} e^{i\theta x}\mu_{\alpha,\beta}(dx)$$

$$= \int_{\mathbb{R}_+} \frac{\alpha^\beta}{\Gamma(\beta)}x^{\beta-1}\left(e^{-\alpha}\cdot e^{i\theta}\right)^x dx$$

$$\stackrel{(\alpha-i\theta)x=:u}{=} \frac{\alpha^\beta}{\Gamma(\beta)}\frac{1}{(\alpha-i\theta)^\beta} \underbrace{\int_0^{\infty(\alpha-i\theta)} u^{\beta-1}e^{-u}du}_{\text{By Cauchy Residue}=\int_0^\infty u^{\beta-1}e^{-u}du}$$

$$\implies \mathbb{E}(e^{i\theta X}) = \frac{1}{\left(1-\frac{i\theta}{\alpha}\right)^\beta} = \left(\frac{1}{\left(1-\frac{i\theta}{\alpha}\right)^{\frac{\beta}{n}}}\right)^n$$

$\implies X$ is infinitely divisible.

The triple (α, σ, ν) is given by:

- $\sigma = 0$

- $\nu(dx) = \beta x^{-1}e^{-\alpha x}dx$ on $\mathbb{R}_+ \backslash \{0\}$

- $a = -\int_0^1 x\nu(dx) = -\frac{\beta}{\alpha}(1-e^{-\alpha})$. This is the correction term for the indicator function!

Lemma 4.7. *For $\alpha, \beta > 0, z \in \mathbb{C}$ with $Re(z) \leq 0$,*

$$\frac{1}{\left(1-\frac{z}{\alpha}\right)^\beta} = \exp\left(-\int_0^\infty (1-e^{zx})\beta x^{-1}e^{-\alpha x}dx\right)$$

Apply to $z = i\theta$ to get the above!

Proof. Take the logarithm:

$$-\beta \log\left(1 - \frac{z}{\alpha}\right) \stackrel{?}{=} -\int_0^\infty (1 - e^{zx})\beta x^{-1} e^{-\alpha x} dx$$

First look at $z = 0$: $0 = 0$, which is of course true.
Now look at the derivative in z (after canceling the $-\beta$ from both sides):

$$-\frac{1}{\alpha - z} = \frac{-\frac{1}{\alpha}}{1 - \frac{z}{\alpha}} \stackrel{?}{=} -\int_0^\infty \underbrace{x \cdot x^{-1}}_{=1} e^{-\alpha x} dx = -\int_0^\infty e^{-(\alpha - z)x} dx \stackrel{\mathrm{Re}(\alpha - z) > 0}{=} -\frac{1}{\alpha - z}$$

$\square$

Using $z = i\theta$ in lemma 4.7:

$$\implies \Psi(\theta) = -\ln\left(\frac{1}{(1 - \frac{i\theta}{\alpha})^\beta}\right) = \int_0^\infty (1 - e^{i\theta x})\beta x^{-1} e^{-\alpha x} dx \to \nu(dx)$$

Added remark. *For the Gamma process is the Lévy measure ∞: $\int_0^\infty \nu(dx) = \infty$.*

Properties:

- $0 \le s \le t$ $X_t \stackrel{d}{=} X_t = X_s + \tilde{X}_{t-s}$, where $\tilde{X}_{t-s}$ is an independent copy of X_{t-s}

- $X_s < X_t$ a.s. for $s < t$ (the Lévy measure is on $\mathbb{R} > 0$)

- it is <u>not</u> a compound Poisson process (since it does not have finite Lévy measure)

4.3 Some properties of the jumps

For compound poisson process:

$$N_t(B) := \# \text{ jumps of size } B \text{ in the time interval } [0, t]$$

$N_t(B) \sim \mathrm{Poi}(t\nu(B))$ for $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$. $t \mapsto N_t(B)$ is a Poisson process with parameter / intensity / rate $\lambda = \nu(B)$.

Added remark. *Suppose you want ot simulate the poisson process. Sample exponential random variables, by sampling uniformly form (0,1) and then transform them by $-\log$.*

Let $T_0 = 0$,
$$T_{k+1} = \inf\{\xi_t > T_k \mid N_t(\mathbb{R}_{>0}) > N_{T_k}(\mathbb{R}_{>0})\}$$

This implies $T_{k+1} - T_k \sim \exp(\nu(\mathcal{R}_{>\prime}))$. The jump sizes of the compound Poisson process X_t are distributed as:

$$\mathcal{P}\left(X_{T_1} - X_{T_1^-} \in dy\right) = \frac{\nu(dy)}{\int_{\mathbb{R} \setminus \{0\}} \nu(dx)}$$

Argument for almost sure no step at time 0: $\mathcal{P}(N_t(\mathbb{R}_{>0}) \ge 2) = O(t^2)$.

$$\implies \mathbb{P}(N_t(\mathbb{R}_{>0}) = 1) = e^{-t\nu(\mathbb{R}_+)} t\nu(\mathbb{R}_+) = t\nu(\mathbb{R}_+) + O(t^2)$$
$$\mathbb{P}(N_t(\mathbb{R}_{>0}) = 0) = 1 - t\nu(\mathbb{R}_+) + O(t^2)$$

$$\frac{\mathbb{P}(N_t(B) = 1 \cap N_t(\mathbb{R}_+) = 1)}{\mathbb{P}(N_t(R_+) = 1)} \approx \frac{t\nu(B)}{t\nu(\mathbb{R}_+)}$$

Case $\int_{\mathbb{R} \setminus \{0\}} \nu(dx) = \infty$.
Problem: First jump $\sim \exp(\nu(\mathbb{R} \setminus \{0\})) = 0$ a bit informal

$\Longrightarrow$ The notion of first, second,..., jumps does not make sense:
$\inf\{t > 0 \mid X_t - X_{t^-} \neq 0\} = 0$ a.s. .
Let $A \subset \mathbb{R} \setminus \{0\}$ s.t. $\nu(A) < \infty$. This implies the first jump of size A, T_1^A is well defined with
$T_1^A \sim \exp(\nu(A))$:

$$\Longrightarrow \mathbb{P}(X_{T_1^A} - X_{T_1^A-} \in dy) = \frac{\nu(dy)}{\nu(A)}$$

$\Longrightarrow$ take $A^0 = (1, \infty), A^1(\frac{1}{2}, 1], A^2 = (\frac{1}{4}, \frac{1}{2}]$
$\Longrightarrow$ Construct Lévy process (compound poisson processes) by restricting the jumps to the ones
of size $A^n : (X_t^n)_{t \geq 0}$

$$X_t := \sum_{n \geq 0} X_t^n$$

where $X_t^0, X_t^1, \ldots$ are independent. The (X_t^n) are compound poisson processes with jumps in A^n:
$T_k^{A_n} \sim \exp(\nu(A^n))$ with jumps of size $\frac{\nu(dy)}{\nu(A^n)}$
$\Longrightarrow (X_t)_{t \geq 0}$ is a Lévy process with Levy measure ν ($\sigma = 0$).

Added remark. *We use that the sum of independent Lévy processes is still a Lévy process, since the exponential characteristic is the sum of the exponential characteristics*

By theorem 4.3 we can decompose:

$$\Psi(\theta) = \underbrace{(ia\theta + \frac{1}{2}\sigma^2\theta^2)}_{=\Psi^{(1)}(\theta)} + \underbrace{\nu(\mathbb{R} \setminus \{(-1,1)\}) \int_{|x| \geq 1} (1 - \exp(i\theta x)) \frac{\nu(dx)}{\nu(\mathbb{R} \setminus \{(-1,1)\})}}_{=\Psi^{(2)}(\theta)} + \underbrace{\int_{0 < |x| <} (1 - \exp(i\theta x) + i\theta x) dx}_{=\Psi^{(3)}(\theta)}$$

Theorem 4.8 (Lévy-Itô decomposition). *Given $a \in \mathbb{R} \sigma > 0, \nu$ a measure on $\mathbb{R} \setminus \{0\}$ s.t. $\int_{\mathbb{R} \setminus \{0\}} (1 \wedge x^2)\nu(dx) < \infty$*
$\Longrightarrow \exists (\Omega, \mathcal{F}, \mathbb{P})$ *on which 3 independent Lévy processes $X^{(1)}, X^{(2)}, X^{(3)}$ with characteristic exponents $\Psi^{(1)}(\theta), \Psi^{(2)}(\theta), \Psi^{(3)}(\theta)$ are defined. Then $X_t := X^{(1)} + X^{(2)} + X^{(3)}$ is a Lévy process with characteristic exponent $\Psi(\theta)$.*

- *$X_t^{(1)}$ is a BM with drift: $X_t^{(1)} = \sigma B_t - at, t \geq 0$, where B_t is a standard BM*

- *$X_t^{(2)}$ is a compound Poisson process*

- *$X_t^{(3)}$ is constructed as above*

- *$\int_{0 < |x| < 1}(1 - \exp(i\theta x))\nu(dx) = \sum_{n \geq 0} \lambda_n \int_{2^{-(n+1)} \leq |x| < 2^{-n}} F_n(dx) + i\theta\lambda_n \int_{2^{-(n+1)} \leq |x| < 2^{-n}} x F_n(dx)$, where $\lambda_N = \nu(2^{-(n+1)} \leq |x| < 2^{-n})$, $F_n(dx) = \frac{1}{\lambda_n}\nu(dx)$.*

Next lecture has useful information about simulations!

Remark. *Càdlàg paths have at most countably many jumps.*

Why do we have the condition $\int(1 \wedge x^2)\nu(dx) < \infty$?
$\mathbb{P}(|X_t| \geq M) \leq ?$ s.t. $\overset{M \to \infty}{\to} 0$.

$$X_t^n = \sum_{k=1}^{\infty} \xi_k 1_{k \leq N_t(A^n)}$$

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Take ν supported on $(0,1]$ and $A_n = \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$, $\nu(A_n) < \infty$

$$\mathbb{P}(|X_t| \geq M) \stackrel{\text{Chebyshev}}{\leq} \frac{\text{Var}(X_t)}{M^2}$$

$$\text{Var}(X_t) = \sum_{i=1}^{\infty} \text{Var}(X_t^n) = \sum_{n=1}^{\infty} \underbrace{\sum_{m=0}^{\infty} \mathbb{P}(N_t(A_n) = m) \cdot m}_{\mathbb{E}(\# \text{ jumps of size in } A_n) = t\nu(A_n)} \cdot \underbrace{\text{Var}(\xi_1^n)}_{\leq \mathbb{E}((\xi_1^n)^2) = \int_{A_n} \frac{x^2 \nu(dx)}{\nu(A_n)}}$$

$$\leq t \sum_{n=1}^{\infty} \int_{A_n} x^2 \nu(dx) = t \int_{(0,1]} x^2 \nu(dx) < \infty$$

X_t is bounded a.s., for all given t

The computation uses independence, not true for higher moments!

no divergence in finite time

4.4 Some properties of Poisson processes

Lemma 4.9 (Superposition)**.** *Let $(X_t^k)_{t \geq 0}, k = 1, \ldots, N$ independent Poisson processes with intensity $\lambda_k > 0$.*
Then $(Y_t)_{t \geq 0} = X_t^1 + \cdots + X_t^N$ is a Poisson process with intensity $\lambda = \lambda_1 + \cdots + \lambda_n$.

Proof. Follows by computing the characteristic exponents / functions □

Also true for $N = \infty$, if $\lambda < \infty$

Added remark. *Why is it called **Superposition**?*
Instead of jumps look at the events of the jumps for each X_t^1. If you take the union (superimpose the pictures), you get the jump times of Y. Not related to quantum mechanics.

Lemma 4.10 (Thining property)**.** *Let $(X_t)_{t \geq 0}$ be a poisson process with intensity λ and a $p \in [0,1]$ fixed.*
Construct two processes $(X_t^1)_{t \geq 0}, (X_t^2)_{t \geq 0}$ as follows:

- *Each jump of X_t is randomly and independently assigned to be a jump of X_t^1 with probability p and if not assigned to X_t^1 it is assigned to X_t^2.*

- *Then (X_t^1) is a Poisson process with intensity $p\lambda$*

- *(X_t^2) is a Poisson process with intensity $(1-p)\lambda$*

- *and they are independent*

Simplified proof.

$$\mathbb{P}(X_t^1 = k, X_t^2 = l) = \mathbb{P}(X_t^1 = k, X_t^1 + X_t^2 = k + l)$$

$$= \mathbb{P}(\underbrace{X_t^1 + X_t^2}_{X_t} = k + l) \mathbb{P}(X_t^1 = k \mid \underbrace{X_t^1 + X_t^2}_{X_t} = k + l)$$

$$= \frac{e^{-\lambda t}(\lambda t)^{k+l}}{(k+l)!} \binom{l+k}{k} p^k (1-p)^l$$

$$= \frac{e^{-\lambda p t}(\lambda p t)^k}{k!} \frac{e^{-\lambda(1-p)t}(\lambda(1-p)l)^l}{l!} \qquad \square$$

Just showing independence, other things are easy to check

Added remark. *Let $X_t = X_t^1 - X_t^2$, X_t^1 is a Poisson process with intensity p and X_t^2 is a poisson process with intensity $1 - p$. Therefore has intensity[a] 1*

This is not a Poisson process, since it might be negative!

---

[a]intensity: expected number of jumps per unit time

How do we simulate this?
Simple solution: generate two Poisson process, by generating exponential variables
$(\sim \exp(p), \sim \exp(1-p))$. This yields to lists of jumping timings, which we can join (after sorting).
What do we do if we have not two, but many of them?

Draw many exponential variables $\sim \exp(1)$, then for each time choose a random poisson process, which you want to attribute the jump to.

For two we get the drift: $\mathbb{E}(X_t) = t(p(1-p)) = (t(2p-1))$

two random walks $X_t, \tilde{X}_t$ as above with $X_0 < \tilde{X}_0$, add constraint $X_t < \tilde{X}_t$ meaning that jumps which would violate the condition are suppressed.

For all $x \in Z$ associate a random poisson process N_t^x with intensity 1.

For intensity $\frac{1}{2}$: First with probability $1/2$ we throw away the jump and then we flip a coin. Again, we simulate one process with intensity as the sum of the intensities, then we split randomly!

Added remark. *Some problems with infinitely many processes, jumps at 0?*

Added remark. *Lévy process with $\Psi(\theta) = ia\theta + \frac{1}{2}i\theta^2\sigma^2 + \int_R(1 - e^{i\theta x} + i\theta x 1_{|x|<1})\nu(dx)$ Has generator*

Not asked in the exam

$$(Lf)(x) = \frac{1}{2}\sigma^2\frac{d^2}{dx^2}f(x) - a\frac{d}{dx}f(x) + \int_{\mathbb{R}}\left[f(x+y) - f(x) - y\frac{d}{dx}f(x)1_{|x|<1}\right]\nu(dy)$$

for $f \in C_0^\infty(\mathbb{R})$.

For poisson processes with intensity λ:

$$(Lf)(x) = \lambda(f(x+1) - f(x)).$$

A way of defining L:

$$(Lf)(x) := \lim_{t\downarrow 0}\frac{\mathbb{E}(f(X_t) \mid X_0 = 0) - f(x)}{t}$$

As $t \downarrow 0$:

$$\mathbb{P}(X_t = x + 1 \mid X_0 = x) = \lambda t + O(t^2)$$
$$\mathbb{P}(X_t = x + 1 \mid X_0 = x) = 1 - \lambda t + O(t^2)$$
$$\mathbb{P}(X_t \notin x, x+1 \mid X_0 = x) = O(t^2)$$
$$\implies \frac{1}{t}\mathbb{E}(f(X_t) \mid X_0 = x) = (1 - \lambda)f(x) + \lambda f(x+1) + (t) \stackrel{t\to 0}{=} \lambda(f(x+1) - f(x))$$

Consider a d-dimensional Poisson process with intensity λ, i.e.
$X_t = (X_t^1, \ldots, X_t^d)$ where the components are Poisson processes with intensity λ and independent.

$\implies$ Generator

$$Lf(x) = \sum k = 1^d L_k f(x), \ L_k f(x_1, \ldots, x_d) = \lambda(f(\ldots, x_k+1, \ldots) - f(\ldots, x_k, \ldots)) = \lambda(f(x+e_k) - f(x))$$

Let $X_0 = (X_0^1, \ldots, X_0^t) \in W_d = \{x \in \mathbb{Z}^d \mid x_1 < x_2 < \cdots < x_d\}$

Want: Process X_t conditioned to stay forever in W_d.

Lemma 4.11. *Let*

Like for BM!

$$h(x) := \prod_{1 \le k < kl \le d}(X_l - X_k),$$

h satisfies

(a) *$h(x) = 0$ for $x \in \partial W_d$ (two of the x_i are equal)*

(b) *$h(x) > 0$ for $x \in W_d$*

(c) *$Lh(x) = 0$ for $x \in W_d$*

Remark. *To prove use $h(x) = \det(X_k^{j-1})_{1 \le k, j \le d}$ or $e^{tL}h(x) = h(x), \ t > 0$.*

This conditioned process has generator:

$$L^h f(x) = \frac{1}{h(x)}(L(hf))(x) = Lf(x) + \sum_{k=1}^{d} \frac{\nabla_k h(x)}{h(x)}, \ \nabla_k g(x) := g(x + e_k) - g(x)$$

Let $x, y \in W_d$:

$$P_t(x, y) = \frac{h(y)}{h(x)} \mathbb{P}(\tilde{X}(t) = y, T > t \mid \tilde{X}_0 = x)$$

where $\tilde{X}(t)$ are independent components and T is the hitting time $\inf\{s \geq 0 \mid \tilde{X}(s) \notin W_d\}$.

Lemma 4.12 (Karlin-McGregor formula).

$$\mathbb{P}(\tilde{X}(t) = y, T > t \mid \tilde{X}(0) = x) = \det(\varphi_t(x_i, y_j)_{1 \leq i,j \leq d})$$

where

$$\varphi_t(x, y) = \varphi_t(0, y - x) = \mathbb{P}(X_t^1 = y \mid X_0^1 = x) = \frac{e^{-\lambda t}(\lambda t)^{y-x}}{(y-x)!} 1_{y \geq x}.$$

Notice, we only need to compute the interactions of two particles!

Special case: $\tilde{X}(0) = (0, 1, \ldots, d-1)$ the lemma 4.12 implies

$$\mathbb{P}(X_t = x) = \text{const}(h(x))^2 \prod_{k=1}^{d} \frac{e^{-\lambda t}(\lambda t)^{x_k}}{x_k!}$$

Analogue of BM conditioned to stay in the continuous Weyl chamber, $B(0) = 0$, is

$$\mathbb{P}(B(t) \in dx) = \text{const}(h(x))^2 \prod_{k=1}^{d} \left(\frac{e^{-\frac{x_k^2}{2t}}}{\sqrt{2\pi t}} dx_k \right)$$

Chapter 5:
The Kardar-Parisi-Zhang (KPZ) equation

5.1   Heuristic derivation

Want: describe the evolution of an interface

$$x \mapsto h(x,t) \in \mathbb{R} \; x \in \mathbb{R}, t \geq 0$$

where the growth is given by some local process
The analogue in 0-dimensions would be

$$h(t) \overset{t \in \mathbb{Z}}{=} \sum_{k=1}^{t} X_k,$$

where the X_k are iid r.v. with bounded variation. We know:

$$\frac{h(t)}{t} \overset{\text{a.s.}}{\to} \mu \in \mathbb{R} \text{(law of large numbers)}$$

and

$$\frac{h(t) - \mu t}{\sigma} t^{\frac{1}{2}} \overset{\mathbb{P}}{\to} \mathcal{N}(0,1) \text{ (CLT)}$$

We call the first case **Macroscopic** and the second case **Fluctuations** or **mesoscopic**.
From a macroscopic point of view:

$$\implies \delta_t h = v(\partial_x h)$$

for some function v (Hamitten-Jacobi equation).
Question: How to add noise?
Consider some discrete examples
Ballistic deposition: $\forall x \in \mathbb{Z}$: independent Poisson process (intensity 1), $h(x,t) \in \mathbb{Z}, t \geq 0$
If at time t, there is a jump of the PP at site x, either

   (a) add it on top of $h(x,t)$: $h(x,t) \to h(x,t) + 1$

   (b) stick to one of the neighboring sites if they are higher: $h(x,t) \to \max\{h(x+1,t), h(x-1,t)\}$

We can use one Poisson process like previously for a finite window.
For the evolution:

$$h(x,t) = \max\{h(x,t_-), h(x-1,t_-), h(x+1,t_-)\}$$

Question: How fast do the fluctuations grow?
Simulations shows

$$\sqrt{\text{Var}_x(h(x,t))} \approx t^{\frac{1}{3}} \text{ for large } t$$

If there would not be any interaction between sites $\overset{\text{CLT}}{\to} t^{\frac{1}{2}} \implies$ Stickiness reduces the fluctuations.
Influence of Stickiness is two-fold:

(a) growth is depending on the slope (of the starting / boundary condition / the previous state?) → already modelled by $v(\partial_x h)$

(b) some regularization going on → simplest regularization in the continuum → Heat equation:

$$\partial_t h = \partial_x^2 h$$

$\implies \partial_t h = v(\partial_x h) + \partial_x^2 h$

Question: What kind of noise should we consider?
Simplest would be to consider **Gaussian noise**, without correlation in space and time (like in the discrete model).
Informally: the space-time white noise $\xi(x,t)$ is the centered **Gaussian process** with variance
$\mathbb{E}(\xi(x,t)\xi(y,t)) = \delta(x-y)\delta(t-s)$

Added remark (WARNING). *ξ is not a function, but a distribution!*

Definition 5.1. *The __space time white noise__ ξ is the Gaussian process satisfying*

$$\forall \varphi_1(x,t), \varphi_2(x,t) \in C_0^\infty$$

$$\implies \mathbb{E}\left(\int dxdt\,\varphi_1(x,t)\xi(x,t)\right) = 0 \ centered$$

and

$$\mathbb{E}\left(\int \varphi_1(x,t)\xi(x,t)dxdt \cdot \int \varphi_2(y,s)\xi(y,s)dyds\right) = \int \varphi_1(x,t)\varphi_2(x,t)dxdt$$

This is a bit informally wirtten, since ξ is a distribution and not absolutely continuous w.r.t. Lebesgue

Added remark. *Careful: $\mathbb{E}(\xi(x,t)^2) = \infty$. Pointwise this makes no sense!*

$\implies$ We would get:

$$\partial_t h = \partial_x^2 h + v(\partial_x h) + \xi$$

Physicists argued: small $\partial_x h$

- $v(\partial_x h) \approx v(0) + v'(0)\partial_x h + \frac{1}{2}v''(0)(\partial_x h)^2 + \dots$

- The first two terms can be removed wlog, since

$$\partial_t h = v(0) \to h_0 + v(0)t \implies h(x,t) + v(0)t =: \tilde{h}(x,t)$$

$\implies \partial_t \tilde{h} = \partial_x^2 \tilde{h} + v'(0)\partial_x \tilde{h} + \dots$

- Physics should not depend on the reference system we use!

Then **KPZ equation:**

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

Or by keeping track of physical constants:

$$\partial_t h = \nu \partial_x^2 h + \lambda(\partial_x h)^2 + \sqrt{D}\xi$$

Problem: $|\partial_x h|$ is NOT SMALL!!!
Locally, h is like a Brownian motion → $\partial_x h$ is a distribution!

$$\int \varphi(x)\partial_x h(x)dx = -\int \varphi'h(x)dx$$

$$\int \varphi(x)(\partial_x h(x))^2 dx \overset{????}{=} -\int \varphi'h(x)dx$$

5.2   White noise

Example 5.2. *Let S be a nice subset of $\mathbb{R}^d$ and $e_1, \ldots$ an orthonormal basis of $L^2(S)$. Let $\xi_1, \ldots,$*   *These might be*
iid $\mathcal{N}(0,1)$ r.v.   *eigenvectors of some*

$$\implies \xi(x) = \sum_{n \geq 1} \xi_n e_n(x)$$

operator

is a white noise on S.

Proof. Let $\varphi_1, \varphi_2 : S \to \mathbb{R}$ be test functions.

$$\mathbb{E}\left(\int_S \varphi_1 \xi(x) dx\right) = \mathbb{E}\left(\int_S \varphi_1 \sum_{n \geq 1} \xi_n e_n(x) dx\right) = \sum_{n \geq 1} \mathbb{E}(\underbrace{\xi_n}_{\mathbb{E}(\xi)=0} \int_S \varphi_1 e_n(x) dx = 0)$$

and

$$\mathbb{E}\left(\int_S \varphi_1(x)\xi(x)dx \int_S \varphi_2(y)\xi(y)dy\right) = \sum_{n,m \geq 1} \mathbb{E}(\xi_n \xi_m) \int_S \int_S \varphi_1(x)\varphi_2(y)e_n(x)e_m(y)dxdy$$

$$= \int dxdy \varphi_1(x)\varphi_2(y) \underbrace{\sum_{n \geq 1} e_n(x)e_m(x)}_{=1_{x=y}} = \int dx \varphi_1(x)\varphi_2(y).$$

$\square$

Added remark. *For orthonormal systems:*

$$f(x) = \sum_{n \geq 1} e_n(x)(e_n, f) = \int dy f(y) \underbrace{\sum_{n \geq 1} e_n(x)e_n(y)}_{=\delta(x-y)}$$

Example 5.3. *Let $S = \mathbb{R}_+$ and ξ white noise on S.*

$$\implies B(t) = \int_S 1_{[0,t)}(s)\xi(s)ds$$

is a Brownian motion.

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Lemma 5.4. *Let ξ be a white noise on $\mathbb{R} \times \mathbb{R}_+$. Then*

$$\xi(x,t) \overset{d}{=} \epsilon^{\frac{z+1}{2}} \xi(\epsilon^1 x, \epsilon^z t)$$

*where z is called the **dynamical exponent** (in time).*

Proof.

$$\mathbb{E}\left(\left(\int \varphi_1(x,t)\xi(x,t)dxdt \int \varphi_2(y,s)\xi(y,s)dyds\right)\right) \overset{\text{def}}{=} \int \varphi_1(x,t)\varphi_2(x,t)dxdt$$

But also

$$\mathbb{E}\left(\int \phi_1(x,t)\epsilon^{\frac{z+1}{2}}\xi(\epsilon x, \epsilon^z t)dxdt \int \phi_2(y,s)\epsilon^{\frac{z+1}{2}}\xi(\epsilon y, \epsilon^z s)dyds\right)$$

$$= \epsilon^{z+1}\mathbb{E}\left(\varphi_1(\epsilon^{-1}x, \epsilon^{-z}t)\xi(x,t)dxdt\epsilon^{-(z+1)}\varphi_2(\epsilon^{-1}y, \epsilon^{-z}s)\xi(y,s)dyds\epsilon^{-(z+1)}\right)$$

$$= \epsilon^{-(z+1)}\int \varphi_1(\epsilon^{-1}x, \epsilon^{-z}t)\varphi_2(\epsilon^{-1}x, \epsilon^{-z}t)dxdt$$

$$= \int \varphi_1(x,t)\varphi_2(x,t)dxdt$$

using $\tilde{x} = \epsilon x \mapsto x$ and $t = \epsilon^z t \mapsto t$. $\square$

5.3   The 1:2:3 scaling

Start with

$$\partial_t h = \nu \partial_x^2 h + \lambda (\partial_x h)^2 + \sqrt{D}\xi.$$

Let us see that wlog we can set $\nu = \lambda = \frac{1}{2}, D = 1$. Define

$$\tilde{h}(x,t) := \alpha h(\gamma x, \beta t)$$

which implies

$$\partial_t \tilde{h}(x,t) = \beta \alpha \dot{h}(\gamma x, \beta t)$$

using

- $\partial_t = \frac{1}{\alpha\beta}\partial_t\tilde{h}$

- $\partial_x h = \frac{1}{\alpha\gamma}\partial_x\tilde{h}$

- $\partial_x^2 h = \frac{1}{\alpha\gamma^2}\partial_x^2\tilde{h}$

we get

$$\partial_t \tilde{h} = \alpha\beta\partial_t h$$
$$= \alpha\beta\left[\nu\frac{1}{\alpha\gamma^2}\partial_x^2\tilde{h} + \lambda\frac{1}{\alpha^2\gamma^2}(\partial_x\tilde{h})^2 + \sqrt{D}\frac{1}{\sqrt{\beta\gamma}}\xi\right]$$
$$= \underbrace{\frac{\beta\nu}{\gamma^2}}_{=\frac{1}{2}}\partial_x^2\tilde{h} + \underbrace{\frac{\lambda\beta}{\alpha\gamma^2}}_{=\frac{1}{2}}(\partial_x\tilde{h})^2 + \underbrace{\sqrt{D}\alpha\sqrt{\frac{\beta}{\gamma}}}_{=1}\xi$$

which implies $\alpha = \frac{\lambda}{\nu}, \beta = \frac{2\nu^5}{D^2\lambda^4}, \gamma = \frac{2\nu^3}{D\lambda^2}$. The standard choice we will use below is

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi$$

which is the **KPZ equation**.
Start with $\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi$. Define

$$h_\epsilon(x,t) := \underbrace{\epsilon^b}_{T^{-b/z}x}\, h(\underbrace{\epsilon^{-1}x}_{xT^{1/z}}, \underbrace{\epsilon^{-z}t}_{=:tT})$$

$$h_\epsilon(x,1) := \epsilon^b h(\epsilon^{-1}x, \epsilon^{-z}1)$$

If b, z choosen s.t. $\lim_{\epsilon\downarrow 0} h_\epsilon(x,1)$ is well defined (and non-trivial), then the Fluctuations of h at large time T is $O(T^{b/z})$ and correlations are non-trivial over distances $O(T^{\frac{1}{z}})$.
Discrete model: Picture here ...
If $(p-q)T^c \to 0$ the height converges to the height of KPZ eq.
Stationary measure for the increment are simple random walks.
Under scaling: One gets BM increments. If one starts with a 2-sided BM, more precisely

$$h(x,0) = \sqrt{2}B(x) \implies h(x,t) - h(0,t) \overset{d}{=} \sqrt{2}B(x)$$

And therefore $b = \frac{1}{2}$.

- $\partial_t h = \epsilon^{z-b}\partial_t h_\epsilon$

- $\partial_x h = \epsilon^{1-b}\partial_t h_\epsilon$

- $\partial_x^2 h = \epsilon^{2-b}\partial_x^2 h_\epsilon$

- $xi(x, \epsilon^{-1}, t\epsilon^{-z}) \overset{d}{=} \epsilon^{\frac{z+1}{2}}\xi(x,t)$

h_ϵ solves

$$\implies \partial_t h_\epsilon = \frac{1}{2}\epsilon^{2-z}\partial_x h_\epsilon + \frac{1}{2}\epsilon^{2-z-b}(\partial_x h_\epsilon)^2 + \epsilon^{b-\frac{z+1}{2}+\frac{1}{2}}\xi$$

$$\overset{b=\frac{1}{2}}{=} \frac{1}{2}\epsilon^{2-z}\partial_x^2 h_\epsilon + \frac{1}{2}\epsilon^{\frac{3}{2}-z}(\partial_x h_\epsilon)^2 + \epsilon^{1-\frac{z}{2}}\xi$$

The nonlinear term $\frac{1}{2}\epsilon^{\frac{3}{2}-z}(\partial_x h_\epsilon)^2$ is still relevant, but not $\to \infty$. This gives the idea $z = \frac{3}{2}$.

$$\partial_t h_\epsilon = \frac{1}{2}\epsilon^{\frac{1}{2}}\partial_x^2 h_\epsilon + \frac{1}{2}(\partial_x h_\epsilon)^2 + \epsilon^{\frac{1}{4}}\xi$$

$\implies$ at large time t, fluctuations of $h(x,t) \approx t^{\frac{1}{3}}$, correlations in space $\approx t^{\frac{2}{3}}$.

Question: Are the $\epsilon^{\frac{1}{2}}$ and $\epsilon^{\frac{1}{4}}$ terms relevant as $\epsilon \to 0$?

1.: $\partial_t h_\epsilon = \frac{1}{2}(\partial_x h_\epsilon)^2$ is not well-posed. with the term $\sqrt{\epsilon}\partial_x^2 h_\epsilon$

$$\implies \partial_t h = \frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\sqrt{\epsilon}\partial_x^2 h$$

which is the viscocity solution. Then taking take $\epsilon \to 0$ gives

$$h(x,t) = \sup_y(h(y,0) - \frac{(x-y)^2}{2t})$$

If you start with $h_\epsilon(x,0) = \sqrt{2}B(x) \implies h(x,t)$ has some explicit expression. This solution do not satisfy $h(x,t) - h(x,0) \overset{d}{=} \sqrt{2}B(x)$. Therefore $\epsilon^{\frac{1}{4}}\xi$ is relevant as $\epsilon \to 0$.

5.4 The linearized case: Edwards-Wilkinsen equation

Let $\lambda = 0 \implies$

$$\begin{cases} \partial_t h &= \frac{1}{2}\partial_x^2 h + \xi \\ h(x,0) &= h_0(x) \end{cases} \tag{5.1}$$

It is linear!

Remark.

$$\begin{cases} \frac{\partial}{\partial t}h &= Ah + f \\ h(0) &= h_0 \end{cases} \implies$$

denote by $\tilde{h}(t) := e^{At}h_0$ as the solution with $f = 0$. The variation of constant formula gives:

$$h(t) = e^{At}h_0 + \int_0^t e^{A(t-s)}f(s)ds$$

Without Ξ: $h(x,t) = \int_\mathbb{R} P_t(x-y)h_0(y)dy, P_t(x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$

$$\implies h(x,t) = \int_\mathbb{R} P_t(x-y)h_0(y)dy + \int_0^t \int_\mathbb{R} P_{t-s}(x-y)\xi(y,s)dyds$$

Scaling of EW:

$$h_\epsilon(x,t) := \epsilon^{\frac{1}{2}}h(\epsilon^{-1}x, \epsilon^{-z}t) = \epsilon^{\frac{1}{2}}h(\epsilon^{-1}x, \epsilon^{-2}t)$$

yields (with the same calculations as before, which leads to $z = 2$)

$$\partial_t h_\epsilon = \frac{1}{2}\epsilon^{2-z}\partial_x^2 h_\epsilon + \epsilon^{1-\frac{z}{2}}\xi = \frac{1}{2}\partial_x^2 h_\epsilon + \xi$$

Therefore, at large time t, fluctuations will be $O(t^{\frac{b}{z}}) = O(t^{\frac{1}{4}})$. The nonlinear term in the KPZ increases the size of the fluctuations!

5.5 The Cole-Hopf transformation

We have: $\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi$

This sadly only works in 1 dimension

Lemma 5.5. *Assume that h solves*

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + V$$

with V some nice V in (x,t).
Then $z = e^h$ solves

$$\partial_t z = \frac{1}{2}\partial_x^2 z + zV$$

Proof. $h = \ln(z)$, and therefore

$$\frac{1}{2}\frac{(\partial_x z)^2}{z^2} + \frac{1}{2}\left(\frac{\partial_x^2 z}{z} - \frac{(\partial_x z)^2}{z^2}\right) + V = \partial_t h = \frac{1}{z}\partial_t z$$

$\square$

Suppose that we regularize ξ in space:

$$\xi \to \xi_\epsilon := \int dy \rho_\epsilon(x-y)\xi(y,t)$$

where $\rho_\epsilon = \frac{e^{\frac{-x^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}}$.
Solve $\partial_t z = \frac{1}{2}\partial_x^2 z + z\xi_\epsilon$. Set $h(x,t) := \ln(z(x,t))$.
Exercise: In the above setting, h solves:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \left(\frac{1}{2}(\partial_x h)^2 - \frac{C}{\epsilon}\right) + \xi_\epsilon$$

for some explicit C.
If one solves $\partial_t z = \frac{1}{2}\partial_x^2 z + z\xi$ and define $h(x,t) := \ln(z(x,t))$. This is called the
Cole-Hopf solution of the KPZ eq..
Exam:

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- Speak about topic we choose (freely): 15 min discussion

- He will not talk about us about the last part

- Talk the rest of the time about the rest of the lecture

If we are smooth enough, the C in the Cole-Hopf-equation will be 0.
In order to have a limit for $\epsilon \to 0$:

$$h(x,t) := \tilde{h}(x,t) - \frac{c}{\epsilon}t$$

ξ_ϵ is still centered Gaussian with covariance:

$$\mathbb{E}(\xi_\epsilon(x,t)\xi_\epsilon(y,s)) = \mathbb{E}\left(\int_{\mathbb{R}} d\tilde{x}\rho_\epsilon(x-\tilde{x})\xi(\tilde{x},t)\int_{\mathbb{R}} d\tilde{y}\rho_\epsilon(y-\tilde{y})\xi(\tilde{y},s)\right)$$

$$= \delta(t-s)\cdot\underbrace{\int_R d\tilde{x}\rho_\epsilon(x-\tilde{x})\rho_\epsilon(y-\tilde{x})}_{=:C_\epsilon(x-y)}$$

In particular, $C_\epsilon(0) = \frac{1}{\sqrt{2\pi\epsilon}} = \frac{c}{\epsilon}$.
Let $z =^\epsilon$ be the solution of

$$(\star)\ \partial_t z = \frac{1}{2}\partial_x^2 z + z \cdot \xi_\epsilon$$

- One checks $z^\epsilon \overset{\epsilon \to 0}{\to} z$ uniformly on compact sets.

- $z(x,t) > 0$ (starting from $z(x,0) > 0$), then define $h(x,t) = \ln(z(x,t))$

- Define $h^\epsilon :- \ln z^\epsilon(x,t)$. it solves the following equation:

$$\partial_x h^\epsilon = \frac{1}{2}\partial_x^2 h^\epsilon + \frac{1}{2}(\partial_x h^\epsilon)^2 - C_\epsilon(0) + \xi_\epsilon$$

- In differential notation

$$dz^\epsilon = \frac{1}{2}\partial_x^2 z^\epsilon dt + z^\epsilon \underbrace{\xi_\epsilon dt}_{=d\xi_\epsilon}$$

Therefore

$$dh^\epsilon = d\ln z^\epsilon \overset{\text{Itô}}{=} \frac{1}{z^\epsilon}dz^\epsilon - \frac{1}{2}\frac{1}{(z^\epsilon)^2}(dz^\epsilon)^2$$

$$= \frac{1}{2}\frac{1}{z^\epsilon}\partial_x^2 z^\epsilon dt + d\xi_\epsilon - \frac{1}{2}(d\xi_\epsilon)^2$$

$$= \frac{1}{2}\partial_x^2 h^\epsilon dt + \frac{1}{2}(\partial_x h^\epsilon)^2 dt + d\xi_\epsilon - \frac{1}{2}C_\epsilon(0)dt$$

which implies

$$\partial_t h^\epsilon = \frac{1}{2}\partial_x^2 h^\epsilon + \frac{1}{\epsilon}\left((\partial_x h^\epsilon)^2 - C_\epsilon(0)\right) + \xi_\epsilon$$

The Cole-Hopf solution of the KPZ equation is

$$h(x,t) = \ln(z(x,t))$$

where z solves

$$\begin{cases} \partial_t z & = \frac{1}{2}\partial_x^2 z + z \cdot \xi \\ z(x,0) & = e^{h(x,0)} \end{cases}$$

The difficulty in the multiplicative **stochastic heat equation (SHE)** is the term $z\xi$. Still it is much better than squaring a distribution.

Since $h(x,t)$ will be locally BM, $z(x,t) = e^{h(x,t)}$ is also locally gaussian. In the integral form:

$$z\xi \to \int_0^t z \underbrace{\xi ds}_{\approx dB_s}$$

5.5.1   Mild solutions

Definition 5.6. *Let* $\begin{cases} \partial_t z & = \frac{1}{2}\partial_x^2 z + z \cdot \xi \\ z(x,0) & = z_0(x) \end{cases}$ *be* $(\star)$. z *is a* **mild solution** *of* $(\star)$, *if*

$$z(x,t) = \int_R P_t(x-y)z_0(y)dy + \int_0^t ds \int_{\mathbb{R}} dy P_{t-s}(x-y)z(y,s)\xi(y,s) \ (A)$$

where z is predictable and

$$\int_0^t ds \int_{\mathbb{R}} (P_{t-s}(x-y))^2 \mathbb{E}(z(y,s))^2 < \infty \ (B)$$

Theorem 5.7. *Suppose that* $\sup_{x \in \mathbb{R}} \mathbb{E}(z_0^2(x)) < \infty$. *Then there exists a unique, progressively measurable $z(x,t)$ satisfying (A) and (B).*

Proof. Let $z^0(x,t) = 0$. Define iteratively
$z^{n+1}(x,t) = \int_{\mathbb{R}} P_t(x-y)z_0(y)dy + \int_0^t ds \int_{\mathbb{R}} P_{t-s}(x-y)z^n(y,s)\xi(y,s)$. Let
$\delta z^n(x,t) = z^{n+1}(x,t) - z^n(x,t)$.

$$\delta z^{n+1} = \int_0^t ds \int_{\mathbb{R}} dy P_{t-s}(x-y)\delta z^m(y,s)\xi(y,s)$$

$$\mathbb{E}\left(|\delta z^{n+1}|^2\right) = \int_0^t ds \int_{\mathbb{R}} dy (P_{t-s}(x-y))^2 \mathbb{E}(|\delta z^n(y,s)|^2).$$

Define $f^n(t) := \sup_{\substack{x \in \mathbb{R} \\ 0 \le s \le t}} \mathbb{E}(|\delta z^n(x,s)|^2)$. Assume that $f^n(t) < \infty$

For $n = 0$ it is true: $\delta z^1(x,t) = z^1(x,t) = \int_{\mathbb{R}} P_t(x-y)z_0(y)dy$ and therefore

$$\mathbb{E}\left(\left|\int_{\mathbb{R}} P_t(x-y)z_0(y)\right|^2\right) \overset{\text{Jensen}}{\le} \mathbb{E}\left(\int_{\mathbb{R}} P_t(x-y)|z_0(y)|^2 dy\right) = \int_{\mathbb{R}} P_t(x-y)\mathbb{E}(|z_0(y)|^2)dy \le \sup_{y \in \mathbb{R}} \mathbb{E}(|z_0(y|^2)$$

Now we bound f^{n+1}:

$$f^{n+1}(t) = \sup_{\substack{x \in \mathbb{R} \\ 0 \le s \le t}} \int_0^s d\tilde{s} \int_{\mathbb{R}} dy (P_{t-\tilde{s}}(x-y))^2 f^n(\tilde{s})$$

$$\le \sup_{0 \le s \le t} \int_0^s d\tilde{s} f^n(\tilde{s}) \frac{C}{\sqrt{s-\tilde{s}}}$$

$$\le \int_0^t d\tilde{s} \frac{f^n(\tilde{s})}{\sqrt{t-\tilde{s}}} C_2 \le \tilde{C} \int d\tilde{s} \int_0^{\tilde{s}} du \frac{f^{n-1}(u)}{\sqrt{t-\tilde{s}}\sqrt{\tilde{s}-u}}$$

$$= C_3 \int_0^t du f^{n-1}(u) \underbrace{\int_u^t d\tilde{s} \frac{1}{\sqrt{\tilde{s}-u}} \sqrt{t-\tilde{s}}}_{\le C_4}$$

$$\implies f^{n+1}(t) \le \hat{C} \int_0^t du f^{n-1}(u) \le (\hat{C})^2 \int_0^t du \int_0^u du - f^{n-3}(u)$$

$$\implies f^{2n}(t) \le (\hat{C})^n \frac{t^n}{n!} \sup_{y \in \mathbb{R}} \mathbb{E}(|z_0(y)|^2) \overset{n \to \infty}{\to} 0$$

where the convergence happens very fast! There there exists a solution z satisfying (A) and
(B) $\qquad\square$

For $z_0 = e^{h_0(x)}, h_0(x) = B(x)1_{x \ge 0}$, we can not use 5.7! Our result is a bit to stiff.

Lemma 5.8. *Let $z(x,t)$ satisfy (A) and (B) and*

$$\mathbb{E}\left(\left|\sup_{A \subseteq [-n,n]} \int_A z_0(x)dx\right|\right) \le ce^{cn} \text{ for some } c \in (0, \infty)$$

Then there exists $\tilde{c} < \infty$, $\mathbb{E}(z^2(x,t)) \le \frac{\tilde{c}}{\sqrt{t}} e^{\tilde{c}(t+|x|)}$ for all $x \in \mathbb{R}, t > 0$.

In the special case $z_0(x) = \delta_0(x)$, there exists $c = c(T)$ s.t. $\mathbb{E}(z^2(x,t)) \le c(P_t(x))^2$ for all
$0 < t \le T$, where $P_t = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$
We want to know: What is $\mathbb{P}(z(x,t) \le s)$ for a fixed x?

List of Lectures