Lecture notes on Stochastic Analysis

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Based on the lectures of
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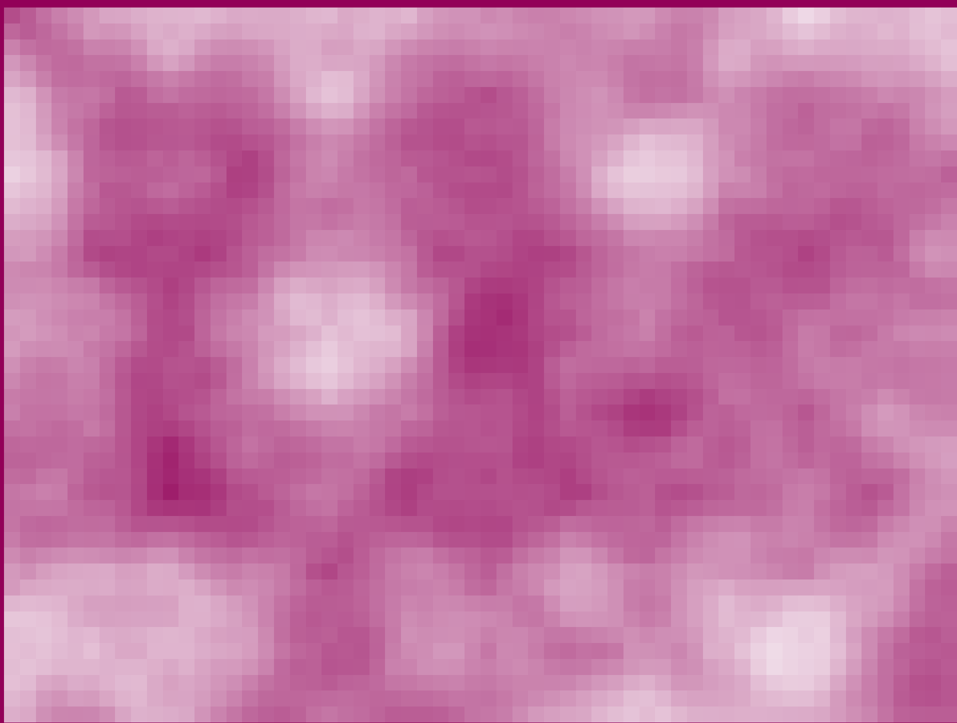

University of Bonn
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Contents

Chapter 0:
Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Ecampus: Ecampus link
- Basis: Basis link
- Website: None
- Time slot(s): Tuesday 12-14 and Thursday 12-14
- Exams: Oral: 23-25 July, 30 Sept. maybe 27. Sept
- Deadlines: ?
- Exercises: One of Tue 16-18(for now this is preferred), Friday 10-12. Starting 16.04.

- First halve based on Eberle and / or Gubinelli (be careful with Notation of dimensions!) Start of lecture 01 (11.04.23)

Overview of the content

- Weak solutions of SDE
  - Martingale problem (characterization)
  - Time change (Dubin-Schwarz)
  - Change of measure (Girsonov)
- How to condition a diffusion to stay in a given domain forever (reflected BM, local time, . . .)
- Lévy processes (a bit)
- Multiplicative stochastic heat-equation
  - relations with Kardar-Pavisi-Zhang class of growth models

Chapter 1: Weak solutions of SDEs

SDE:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \ \textbf{Initial condition} \end{cases} \tag{1.1}$$

- $X_t \in \mathbb{R}^d, B:$ n-dim BM
- $b(t, x) : [b_k(t, x)]_{1\leq k\leq d}$: **drift vector**
- $\sigma(t, x) = [\sigma_{k,l}(t, x)]_{\substack{1\leq k\leq d \\ 1\leq l\leq n}}$: **dispersion matrix**
- $a(t, x) = \sigma(t, x) \cdot \sigma(t, x)^\intercal$: **diffusion matrix**

1.1 Strong solutions

Definition 1.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_t = \sigma(x_0, (B_s)_{0\leq s\leq t})$*

- *a d-dim process X_t is a **strong solution** of equation 1.1 if:*
  - $X_t = x_0$ *a.s.*
  - X_t *is adapted to* $\mathcal{F}_t \forall t \geq 0$
  - X *is a continuous semimartingale s.t.* $\forall t \geq 0$:

$$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2)ds < \infty \ a.s.$$

  - $X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$

In the last semester we proved:

Theorem 1.2. *Assume that b, σ are globally lipschitz with at most linear growth at ∞ (in space) $\implies \exists!$ strong solution of SDE.*

Foundations of Stochastic Analysis Thm 8.6

Added remark. *There exists $K > 0$ s.t. for all $x, y \in \mathbb{R}^d$: Globally Lipschitz:*

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|$$

Linear growth condition:

$$\|b(t, x)\| + \|\sigma(t, x)\| \leq K(1 + \|x\|)$$

Remark. *For strong solutions, $\mathcal{F}_t$ is given by the driving BM, wich is given to us.*
$\implies X_t(\omega) = \Phi(x_0(\omega), (B_s)_{0\leq s\leq t})$

1.2 Weak solutions

- For weak solutions we do not fix the driving brownian motion.

Definition 1.3. *A **weak solution** of equation 1.1 is a **pair** of adapted processes (X, B) to a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ s.t.*

- *B is a n-dim BM*
- *X is a d-dim continuous semimartingale with*
  1. $X_0 = x_0$ *a.s.*
  2. $\forall t \geq 0$
  $$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2) ds < \infty \ a.s.$$
  3. $X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s) dB_s$

Remark.
- *The filtration $(\mathcal{F}_t)_{t\geq 0}$ is not necessarily the one generated by B*
- *If X is adapted to the filtration generated by the BM $\implies$ we have strong solutions*
- *$\exists$ weak solution which are not strong solutions*
- *Warning: To give a weak solution we really need to provide $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, B)$*

Definition 1.4 (Uniqueness in law)**.** *An SDE 1.1 has **uniqueness in law** if given any two weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, B)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{B})$ satisfy:*

$$Law_{\mathbb{P}}(X) = Law_{\tilde{\mathbb{P}}}(\tilde{X})$$

They agree on any set in the sigma algebra

Definition 1.5 (Pathwise uniqueness)**.** *The SDE 1.1 has pathwise uniqueness if, whenever the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(B_t)_{t\geq 0}$ are fixed, then two solutions $X, \tilde{X}$ with $X_0 = \tilde{X}_0$ are indistinguishable.*

Example 1.6 (No strong solutions, no pathwise uniqueness, $\exists$ weak solution & and uniqueness in law by Tanaka)**.**

$$\begin{cases} dX_t = sgn(X_t) dB_t \\ X_0 = 0 \end{cases} \tag{1.2}$$

or more generally $X_0 = Y$, where

$$sgn(x) = \begin{cases} 1 & x > 0 \\ -1 x \leq 0 \end{cases}$$

Let W be a BM with $W_0 = Y$. Define

$$B_t := \int_0^t sgn(W_s) dW_s \text{ or } dB_t = sgn(W_t) dW_t$$

$$\implies dW_t = sgn(W_t) dB_t$$
$$\implies W_t = y + \int_0^t sgn(W_s) dB_s$$

B_t is a local martingale with

$$\langle B \rangle_t = \int_0^t \underbrace{(sgn(W_s))^2}_{=1} \underbrace{d\langle W \rangle_s}_{=ds} = t$$

Also $B_0 = 0$, therefore B is a BM (see Lévy characterization) $\implies$ W solves the SDE. For $Y = 0$, W ***and*** *$-W$ solves the same SDE.*
$\implies$

- *exists weak solutions*
- *For $Y = 0$: no pathwise uniqueness*
- *Uniqueness in law (because the law is determined by X_t being a BM)*
- *No strong solution, because: $\mathcal{F}^B = \mathcal{F}^{|W|} \subsetneqq \mathcal{F}^W$*

Not a proof just yet, just the reason!

Example 1.7 (No solutions)**.**

$$\begin{cases} dX_t = -\frac{1}{2X_t} 1_{X_t \neq 0} dt + dB_t \\ X_0 = 0 \end{cases} \tag{1.3}$$

Assume there exists a solution. Use Itô formula for X_t^2, then:

$$\begin{aligned} X_t^2 &= 2\int_0^t X_s dX_s + \int_0^t 1 ds \\ &= -\int_0^t 1_{X_s \neq 0} ds + 2\int_0^t x_s dB_s + t \\ &= \int_0^t 1_{X_s = 0} ds + 2\int_0^t X_s dB_s \end{aligned}$$

We will prove $\int_0^t 1_{X_s=0} ds = 0 \implies X_t^2$ is a local martingale, $X_t^2 \geq 0$ (and therefore a supermartingale) and $X_0 = 0$ ($\implies \mathbb{E}(X_t^2) = 0$). If $X_t = 0 \implies \int_0^t 1_{X_s=0} ds = t \implies 0 = dB_t$ which are contradictions!

Remark. *If $X = \underbrace{M}_{\in \mathcal{M}_{loc}} + \underbrace{A}_{\in \mathcal{A}}$*

$$\implies \forall b \in \mathbb{R} \int_0^t 1_{X_s = b} d\langle M \rangle_s = 0$$

Sheet 0:

- Mail: Alexander.Becker@uni-bonn.de
- Exercise sheet handed in Fr 10 am via eCampus
- Groups of 3

Motivation:
in the last semester: Introduction to stochastic analysis

- Diffusion processes & SDEs

Here: Deepen the knowledge & have fun

- Learn SDE techniques and get other results
- Modify diffusion processes to behave in a certain way
- Ex. diffusion bridge (Brownian bridge)
- Condition a diffusion to stay in a domain
    - Ex. Condition BM to stay positive
    - Old SDE: $dB_t = dB_t$

- New SDE: $dX_t = \frac{1}{X_t}dx + dB_t \to P(X_t \in \cdot) = P(B_t \in \cdot | B > 0 \text{ forever})$
- $D \subset \mathbb{R}^d$ open domain, X diffusion process, with generator $L = \Sigma b\partial_i + \frac{1}{2}\sum a^{ij}\partial_i\partial_j$

$$Y :- (X | X \in D \text{ forever})$$

  get drift term $\nabla \log \phi_0$, where ϕ_0 is the lowest eigenfuncion of $-L$ on D with dirichlet boundary.

Recap:
Brownian motion:

Added definition. $B_0 = 0$, *independent & $\mathcal{N}(0, t_i - t_{i-1})$ increments, $t \mapsto B_t(\omega)$ continuous.*

Regularity of path $t \mapsto B_t(\omega)$:

- nowhere differentiable
- $\alpha-$locally Hölder continuous $\iff \alpha < \frac{1}{2}$
- Quadratic variation $\langle B \rangle_t = t$
- Generator $\frac{\Delta}{2}$
- Recurrent $\iff ds2?$

Itô-Integral:

1. If X simple process $\implies$ RS-Integral
2. Itô isometry $\mathcal{E} \to \{L^2 - \text{local martingale}\}, \mathcal{E} \subset L^2(d[M])$ dense
3. general $X : \int XdM$ as L^2-limit

Added remark (Itô formula)**.**

- $$df(t, X_t) = \partial_t f(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}f(t, X_t)d\langle X \rangle_t$$
- *associative* $\int Xd(\int YdZ) = \int XYdZ$
- *If M local martingale $\implies \int XdM$ local martingale*

SDEs:
$DX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$

- ex./ uniqueness: b, σ locally Lipschitz $\implies$ strong ex.+pathwise uniqueness until explosion time
- globally Lipschitz (in space) and linear growth $(|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|))$, $e = \infty$.

Problem 00.1: SDE

Let B be a one-dimensional Brownian motion (starting from 0) and let $X_t = \sin(B_t)$.

1. Determine the SDE of X_t
2. Discuss the existence and/or uniqueness of strong solutions of the SDE
3. Determine whether the local martingale term is a martingale (Hint: for the integrability condition, think about the Itô isometry for instance)

Solution 00.1

1.:
Idea: Use Itô formula: $X_t = \sin(B_t) = f(B_t)$

$$dX_t = df(B_t) \overset{\text{Itô}}{=} \partial_x \underbrace{\cos(B_t)}_{\sqrt{1-X_t^2}} dB_t - \frac{1}{2} \underbrace{\sin(B_t)}_{X_t} dt$$

Careful: $\sqrt{1-X_t^2}$ is not inverse mapping, because it is always positive while $\cos(B_t)$ is not

2.:
Consider the SDE

$$dX_t = \begin{cases} -\frac{1}{2} X_t dt + \sqrt{1-X_t^2} dB_t \\ X_0 = x \in (-1,1) \end{cases}$$

Coefficients: $b : [-1,1] \to \mathbb{R}, b(x) = -\frac{1}{2}x$ and $\sigma : [-1,1] \to \mathbb{R}.\sigma(x) = \sqrt{1-x^2}$

$$[\sigma]_{1,2} = \sup \frac{|\sigma(x) - \sigma(y)|}{|x-y|^{1/2}} < \infty$$

Probably σ^2 Lipschitz $\implies$ σ Hölder $\frac{1}{2}$
3.:

Problem 00.2: Time change

Let B be a one-dimensional Brownian motion (starting from 0). Let $Y_t = \int_0^t s^2 dB_s$.

1. Determine the SDE of Y_t
2. Find A_t such that Y_{A_t} is a (stopped) Brownian motion

Solution 00.2

1.: By Definition

$$dY_t = t^2 dB_t$$

2.: We use the Dubin-Schwarz theorem:
$X \in \mathcal{M}^0_{\text{loc}}, \langle X \rangle_\infty = \infty, T_t := \inf\{s \geq 0 | \langle X \rangle_s \geq t\} = X_t^{[-1]}$
$\implies B_t := X_{T_t}$ 1 d BM w.r.t. $(F_{T_t})_{t \geq 0}, X_t = B_{\langle B \rangle_t}$
here: use $X_t = b(X_t)dt + \sigma(X_t)dB_t \implies d[X]_t = \sigma^2(X_t)dt$

Problem 00.3: SDE and PDE

Let f be a function supported on $[0,1]$, u the solution of

$$\frac{1}{2}u(t,x) = \frac{1}{2}x(1-x) + \frac{d^2}{dx^2}u(t,x), \qquad u(0,x) = f(x).$$

Consider also the one-dimensional SDE (also known as Wright-Fisher diffusion)

$$dX_t = \sqrt{X_t(1-X_t)dB_t}$$

with $X_0 = x \in (0,1)$.

1. For any fixed $t > 0$, define $M_s = u(t-s, X_s)$ for $s \in [0,t]$. Use Itô formula to show that M_s is a local martingale
2. Assume that f is bounded and there is a bounded solution of u. Show that $u(t,x) = \mathbb{E}_x(f(X_t))$.

Solution 00.3

1.:

$$\begin{aligned}
dM_s &= du(t-s, X_s) \\
&= -\partial_s u(t-s,X_s)ds + \partial_x u(t-s,X_s) \underbrace{dX_s}_{b(X_s)ds+\sigma(X_s)dB_s \text{ by asso.}} + \frac{1}{2}\partial_x^2 u(t-s,X_s) \underbrace{d[X]_s}_{=\sigma^2(X_s)ds} \\
&= \underbrace{(-\partial_s u + b\partial_x u + \frac{1}{2}\sigma^2\partial_x^2 u)}_{=0}(t-s,X_s)ds + \partial_x u(t-s,X_s)\sigma(X_s)dB_s \\
&\implies dM_s = \partial_x u(t-s,X_s)\sigma(X_s)dB_s
\end{aligned}$$

(i.e.: $M_t - M_0 = \int_0^t \dots dB_s$)
This is a purely stochastic integral against a (local) martingale $\implies$ martingale.
2.:

- M_s true martingale:
  1. $\forall t : \mathbb{E}(\sup_{[0,t]} |M|) < \infty$, for example: M bounded
  2. $\forall t : \mathbb{E}(\sup_{[0,t]} [M]_t) < \infty$

$M_s \coloneqq u(t-s, X_s), u$ bounded $\implies$ M bounded $\implies$ M true martingale
$w(s,x) \coloneqq u(t-s,x)$

$$u(t,x) = w(0,x) = \mathbb{E}_x[w(0,X_0)] \overset{\text{martingale}}{=} \mathbb{E}_x[w(t,X_t)] = \mathbb{E}_x[(u(0,X_t))] \overset{\text{PDE}}{=} \mathbb{E}_x[f(X_t)]$$

Feynman-Kac formula (later more general).
Also solvable by Kolmogorov Backward / forward equation

Start of lecture 02 (16.04.24)

Example 1.8 (Non-uniqueness of law, non pathwise uniqueness, with solutions)**.**

$$\begin{cases} dX_t &= 1_{X_z} dB_t \\ x_0 &= 0 \end{cases}$$

Then

$$X_t = 0 \forall t \geq 0$$

and

$$X_t = B_t \forall t \geq 0$$

both are solutions:

$$X_t - B_t = -\int_0^t 1_{X_s=0} dB_s \implies \langle X-B\rangle_t = \int_0^t 1_{X_s=0} d\langle B\rangle_s = 0$$

$X_t = \int_0^t 1_{X_s\neq 0} dB_s$
$= \int_0^t (1 - 1_{X_s=0} dB_s)$
$B_t - \int_0^t 1_{X_s=0} dB_s$

Let $\eta \sim Ber\left(\frac{1}{2}\right)$ independent of $(B_t)_{t\geq 0}$ and define

$$\tilde{X}_t = \begin{cases} 0 & \eta = 0 \\ B_t & \eta = 1 \end{cases}$$

$\implies$ *$\tilde{X}_t$ is adapted to $\sigma(\eta(B_s)_{0\leq s\leq t})$, but not to $\sigma((B_s)_{0\leq s\leq t})$ and therefore not a strong solution.*

Example 1.9 (No strong solution, weak solution, no pathwise uniqueness, no uniqueness in law)**.**

$$\begin{cases} dX_t &= 1_{X_t \neq 1} sgn(X_t) dB_t \\ X_0 &= 0 \end{cases}$$

Let Y_t be a solution of

$$\begin{cases} dY_t &= sgn(Y_t) dB_t \\ Y_0 &= 0 \end{cases}$$

$\implies X_t := Y_{t\wedge\tau}$*, where* $\tau = \inf\{s \geq 0 \mid Y_s = 1\}$ *is also a solution.*

Theorem 1.10 (Yamada-Watanabe)**.** *If both existence of weak solutions and pathwise uniqueness hold, uniqueness in law also holds.*
Moreover, $\forall$ choices of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(B_t)_{t\geq 0}$ then there exists a strong solution.

1.3 Lévy characterization

Example 1.11. *Consider the SDE*

$$\begin{cases} dX_t &= O_t dB_t \\ X_0 = x_0 \end{cases}$$

where both X_t and B_t are d-dimensional and O_t is an adapted process (matrix) s.t. $O_t^\intercal O_t = 1 \forall t \geq 0$ *i.e. O_t is a rotation*

$$\implies X_t^k = X_0^k + \sum_{l=1}^{d} \int_0^t O_s^{k,l} dB_s^l$$

$$\implies \langle X^k, X^{\tilde{k}} \rangle_t = \sum_{l,\tilde{l}}^{d} \int_0^t O_s^{k,l} O_s^{\tilde{k},\tilde{l}} d\langle B^l, B^{\tilde{l}} \rangle_s = \sum_{l=1}^{d} \int_0^t O_s^{k,l} O_s^{\tilde{k},l} ds$$

$$= \int_0^t \underbrace{(O_s O_s^\intercal)^{k,\tilde{k}}}_{=1} ds = \delta_{k,\tilde{k}} t \overset{\text{Lévy}}{\implies} (X_t)_{t\geq 0} \text{ is also a d-dim BM starting from } x_0$$

Theorem 1.12 (Yamada,Watanabe)**.** *Let $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ and assume that there exist both a increasing function $\rho(u) \geq 0$ s.t.* $|\sigma(x) - \sigma(y)| \leq \rho(|x-y|) \forall x, y \in \mathbb{R}$ *s.t.*

$$\int_0^\infty \frac{1}{\rho(u)} du = \infty \implies 0 < u < C_1 \implies \rho(u) \leq C_2 u^{0.5}$$

and some increasing concave function $\gamma_1(u) \geq 0$ s.t.

$$|b(x) - b(y)| \leq \gamma_1(|x-y|) \forall x, y \in \mathbb{R}$$

and

$$\int_0^\infty \frac{1}{\gamma_1(u)} = \infty.$$

Then pathwise uniqueness holds.

Theorem 1.13 (Storokhod)**.** *Assume that σ, b are continuous bounded functions*
$\implies$ *there exist weak solutions to the SDE* $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

1.4 Weak solutions and martingale problems

Let $(X_t)_{t\geq 0}$ be a weka solution of the SDE

$$\begin{cases} dX_t &= b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_0 &= 0 \end{cases}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
$\implies$ X_t is a semimartingale s.t. $X_t^k = X_0^k + \int_0^t b(s,X_s)ds + \sum_{l=1}^n \int_0^t \sigma_{k,l}(s,X_s)dB_s^l$ and

$$\langle X^i, X^j \rangle_t = \int_0^t \underbrace{\sum_{l=1}^n \sigma_{i,l}(s,X_s)\sigma_{j,l}^\intercal(s,X_s)}_{=a_{i,j}(s,X_s)} ds$$

1.4.1 Itô-Doeblin formula

Itô formula leads to

Proposition 1.14. *For $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, then*

$$f(t,X_t) = f(0,X_0) + \int_0^t (\sigma^\intercal \nabla f)(s,X_s)dB_s + \int_0^t \left[\left(\frac{\partial}{\partial s} + \mathcal{L}\right)\right](s,x_s)ds$$

where $(\mathcal{L}f)(t,x) = \frac{1}{2}\sum_{k,l=1}^n a_{k,l}(t,x)\frac{\partial^2}{\partial x_k \partial x_l} f(t,x) + \sum_{k=1}^n b_k(t,x)\frac{\partial}{\partial x_k} f(t,x)$.

$\mathcal{L}$ is called the ***generator***

Remark. *The Itô-Doeblin formula provides a connection between SDEs and PDEs as we have seen in the last part* ***Foundations of stochastic analysis****.*

Example 1.15. *Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}_d)$ be a solution of the PDE*

$$\frac{\partial}{\partial t} f(t,x) + (\mathcal{L}f)(t,x) = -g(t,x) \qquad t \geq 0, x \in U \subseteq \mathbb{R}^d$$
$$f(t,x) = \varphi(t,x) \qquad t \geq 0, x \in \partial U.$$

then $M_t := f(t,X_t) + \int_0^t g(s,X_s)ds \in \mathcal{M}_{loc}$ by proposition 1.14 and if f,g are bounded $M_t \in \mathcal{M}$.

$$T := \inf\{s \geq 0 | X_s \notin U\} \implies M_t^T := M_{T\wedge t} \in \mathcal{M}.$$

Furthermore, if we assume $T < \infty$ a.s. :

$$\mathbb{E}[M_t] = \mathbb{E}[\varphi(T,X_T)] + \mathbb{E}[\int_0^T g(s,X_s)ds] = \varphi(0,X_0)$$

where $X_0 = x_0$.
There are two special cases:
$g = 0 \implies$ yields the exit distributions, while
$\varphi = 0, g = 1$ yields the mean exit times.

Example 1.16 (Feynman-Kac formula)**.** *Let $t \in \mathbb{R}_+$ be finite. Assume $f : \mathbb{R}^d \to \mathbb{R}$ and $K : [0,t] \times \mathbb{R}^d \to \mathbb{R}_+$ be continuous functions. Assume that u is a $C^{1,2}$-solution (bounded,for simplicity) of*

$$\begin{cases} \frac{\partial u}{\partial s}(s,x) = \frac{1}{2}\Delta u(s,x) - K(s,x)u(s,x) & s \in [0,t], x \in \mathbb{R}^d \\ u(0,x) = f(x) & x \in \mathbb{R}^d \end{cases}$$

Then u has the stochastic representation $u(t,x) = \mathbb{E}_x\left[f(X_t)\exp\left(-\int_0^t K(t-s,X_s)ds\right)\right]$, where X_t is BM starting from $X_0 = x$.

Sketch. 1. Define $r(s,x) := u(t-s,x)$ for $s \in [0,t]$

2. Show: $M_s := \exp(-A_s) r(s,x)$ with $A_s = \int_0^s K(u, X_u) du$ is a local martingale.

$\square$

Remark. *This is a reformulation of the formula from the last semester.*

1.4.2 Martingale problem

A solution of an SDE is generically defined up to some **explosion time** ξ, where it either diverges or it exists a given domain $U \subset \mathbb{R}^d$ (open).
$\implies$ For $k \in \mathbb{N}$ define $U_k := \{x \in U \mid |x| < k \wedge \operatorname{dist}(x, U^c) \geq \frac{1}{k}\}$ with $U = \bigcup_{k \geq 1} U_k$ and

$$T_k := \{t \geq 0 \mid x_t \notin U_k\}.$$

A solution of the SDE $b(t, X_t)dt + \sigma(t, X_t)dB_t$ is defined up to $\xi = \sup_{k \geq 1} T_k$.

Start of lecture 03 (18.04.24)

Added remark. *uniqueness of solution to the heat equation $\frac{1}{2}\Delta u - Ku$: not unique! Stochastic calculus: Paolo Baldi: 10.3-10.4*

Define $(\star)$:

$$dX_t b(t, X_t)dt + \sigma(t, X_t)dB_t \text{ with } X_0 = x_0$$

Theorem 1.17 (Martingale problem)**.** *If X is a solution of $(\star)$ up to time ζ, then $\forall f \in C^{1,2}(\mathbb{R}_+ \times U)$*

$$M_t = f(t, X_t) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L} \right) f(s, x_s) ds, t < s$$

is a local martingale up to ζ and $M_t^{T_k}$ are localizing martingales.

Definition 1.18 (Martingale solutions)**.** *$(X_t)_{t \geq 0}$ is a **martingale solution** of $(\star)$ if $\forall f \in C^2(\mathbb{R}^d)$*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L} f)(X_s) ds$$

is a continuous local martingale.

Theorem 1.19 (Equivalent definitions)**.** *The following are equivalent (~~for X_t being a solution of $(\star)$~~):*

(a) $\forall f \in C^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t (\mathcal{L} f)(X_s) ds$$

is a local martingale

(b) The process in $\mathbb{R}^d$ given by

$$M_t := X_t - X_0 - \int_0^t b(s, X_s) ds$$

is a d-dimensional local martingale with $\langle M^i, M^j \rangle_t = \int_0^t a_{i,j}(s, X_s) ds = \langle X^i, X^j \rangle_t$

(c) $\forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$

$$\tilde{M}_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L} \right) (s, X - s) ds$$

is a local martingale

ds

Proof. $\underline{c \implies a}$: by choosing f independent of t.
$\underline{a \implies b}$: 1.: Choosing $f(X) = X_i$ implies

We can maybe also proof this by calculating X^2?

$$M_t^f = X_t^i - X_0^i - \int_0^t b_i(s, X_s)ds \in \mathcal{M}_{\text{loc}}$$

2.: $f(X) = X_i X_j$:

$$\begin{aligned}(\mathcal{L}f)(x) &= \frac{1}{2}a_{ij} + \frac{1}{2}a_{ji} + b_i X_j b_j X_i \\ a = a^\intercal &\implies = a_{ij} b_i X_j b_j X_i \\ \implies M_t^f X_t^i X_t^j &- X_0^i X_0^j - \int_0^t \left[a_{ij}(s, X_s) + b_i(s, X_s)X_s^j + b_j(s, X_s)X_s^i\right] ds\end{aligned}$$

Here dX_s^i is the same as $b_i X_s^j$ is the same up to a local martingale term and $\langle X^i, X^j\rangle_t = \int_0^t a_{ij}(s, X_s)ds = \langle M^i, M^j\rangle_t$

$$\begin{aligned}X_t^i X_t^j - X_0^i X_0^j &\overset{\text{Integration by parts}}{=} \int_0^t X_s^i dX_s^j + \int_0^t X - s^j dX_s^i + \langle X^i, X^j\rangle_t \\ &= M_t^f + \int_0^t [a_{ij} + b_i X_s^j + b_j X_s^i]ds\end{aligned}$$

$\underline{b \implies c}$: By Proposition 1.14 If (use the next theorem) X was a weak solution $\implies \tilde{M}_t^f$ is a local martingale. $\square$

Theorem 1.20. *Let $n = d$, assume $\sigma(t,x)$ is invertible $\forall t, x$ and $\sigma^{-1}(t,x)$ is uniformly bounded. T.f.a.e.:*

(a) $(X_t)_{t\geq 0}$ is a weak solution of the SDE $(\star)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; B)$

(b) $(X_t)_{t\geq 0}$ is a martingale solution of the SDE $(\star)$

This also works for $n \neq d$, but with a different proof

Proof. $a \implies b$: True
$b \implies a$: Goal construct a BM for the weak solution.
By proposition **??** $a \implies b$ $dM_t = dX_t - b(t, X_t)dt \in \mathcal{M}_{\text{loc}}$ and $d\langle M^i, M^j\rangle_t = a_{k,l}(t, X_t)dt$

$$\begin{aligned}\implies dX_t &= dM_t + b(t, X_t)dt \\ &= \sigma(t, X_t)d\tilde{B}_t + b(t, X - t)dt\end{aligned}$$

where $\tilde{B}_t \coloneqq \sigma(s, X_s)^{-1}dM_s$
To see: $\tilde{B}_t$ is a brownian motion.

$$\begin{aligned}\langle \tilde{B}^i, \tilde{B}^j\rangle_t &= \sum_{k,l}\int_0^t \sigma_{ij}^{-1}\sigma_{jl}^{-1}\underbrace{d\langle M^k, M^l\rangle_s}_{= \underbrace{a_{ij}}_{(\sigma^\intercal\sigma)_{kl}} ds} \\ &= \sum_{k,l,p}\int_0^t \sigma_{ik}^{-\intercal}\sigma_{kp}\sigma_{pl}^\intercal\sigma_{lj}^{-\intercal}ds \\ &= \delta_{ij}\int_0^t 1ds = \delta_{ij}t\end{aligned}$$

Then by the Lévy characterization $\tilde{B}$ is a brownian motion. $\square$

Added remark. *This is the first way to construct a weak solution: Solve a martingale problem! This is used a lot in practice.*

1.5 Weak solutions and time change

1.5.1 Time change

For $d = 1$:

Theorem 1.21. *[Dubins-Schwarz]*

- *Let $M \in \mathcal{M}^0_{loc}$ and $\langle M \rangle_\infty = \infty$ a.s.*
- *Let $T_t \coloneqq \inf\{s \geq | \langle M \rangle_s \geq t\}$*

This implies

1. *$t \mapsto M_{T_t}$ os a $(\mathcal{F}_{T_t})$ brownian motion*
2. *$M_t = B_{\langle M \rangle_t}$ for some standard brownian motion B*

$X_t = X_0 + \int_0^t b(s, X_s) ds + \underbrace{\int_0^t \sigma(s, X_s) dB_s}_{= M_t}$. If $\langle M \rangle_\infty = \infty$ a.s.:

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \tilde{B}_{\int_0^t \sigma^2(s, X_s) ds}$$

1.5.2 Time change in a martingale problem

Consider $d = 1 = n$.

$$dY_t = \tilde{\sigma}(Y_t) dB_t \qquad (\star\star)$$

and $\tilde{\sigma}$ strictly positive positive continuous function.

$$\langle Y \rangle_t = \int_0^t \tilde{\sigma}(Y_s)^2 ds \eqqcolon A_t$$

By theorem 1.21 $\implies Y_t = W_{A_t}$ for some brownian motion W.
Assume $A_t \infty = \infty$ a.s.
$T_t \coloneqq \inf\{s \geq 0 \mid \langle Y \rangle_s \geq t\}$

$$\implies T_{A_t} = \inf\{s \geq 0 \mid \langle Y \rangle_s \geq \langle Y \rangle_t\} = t$$

$$1 = \frac{d}{dt}(T_{A_t}) = T'\underbrace{A_t}_{=u} \cdot A_t$$

$$\implies \underbrace{T'_u}_{= \frac{dT_u}{du}} = \frac{1}{A'_{T_u}} \implies T_u = \int_0^u \frac{1}{\tilde{\sigma}(Y_{T_s})^2} ds = \int_0^u \frac{1}{\tilde{\sigma}(W_s)^2} ds$$

$\implies$ to construct a solution of $(\star\star)$: Given $W \longrightarrow$ compute $T_u \longrightarrow$ determine $A - t = T_t^{-1} \implies Y_t = W_{A_t}$

Start of lecture 04 (23-04-24)

Theorem 1.22. *Let $(X_u)_{u \geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a weak solution of*

$$dX_u = b(X_u) du + \sigma(X_u) dB_u$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$,the drift and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ are locally bounded, σ^{-1} exists for a.e. x and is locally bounded.
Consider a time change $T_u \coloneqq \int_0^u \rho(X_s) ds$, where $\rho : \mathbb{R}^d \to \mathbb{R}_+$ s.t.

$$T_u < \infty \forall u \geq 0 \text{ and } T_\infty = \infty \text{ a.s.}$$

$\implies$ *Then* $Y_t := X_{A_t}$, *where* $A_t = T_t^{-1}$ *is a weak solution of the SDE*

$$dY_t = \frac{b(Y_t)}{\rho(Y_t)}dt + \frac{\sigma(Y_t)}{\sqrt{\rho(Y_t)}}dB_t$$

Remark. *Special case:* $d = 1, b = 0, \sigma = 1$*: Then* X *is a BM and* $\rho = \frac{1}{\tilde{\sigma}^2(x)} \implies Y_t = X_{T_t^{-1}}$ *solves*

$$dY_t = \tilde{\sigma}(Y_t)dB_t$$

Proof. By theorem 1.20, it is enough to see how the martingale problem after change of time becomes:

$$\mathcal{L} = \mathcal{L}_{X_t} \overset{\text{time change}}{\longrightarrow} \mathcal{L}_{Y_t} = \tilde{\mathcal{L}}$$

$Y_t = X_{A_t}; Y_0 = X_{A_0}$. For $f \in C^2 : M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$ is a local martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

The A_0 in the integral is probably 0, but it does not matter, we do a change of variables anyway.

$$\implies N_t^f := M_{A_t}^f = f(\underbrace{X_{A_t}}_{=Y_t}) - f(\underbrace{= X_{A_0}}_{Y_0}) - \int_{A_0}^{A_t} (\mathcal{L}f)(X_s)ds$$

is also a local martingale w.r.t. $(\mathcal{F}_{A_t})_{t\geq 0}$.
Change of variable (to get rid of the X_s in the integral):

$$\tau = T_s \leftrightarrow A_t = s \implies X_s = X_{A_t} = Y_\tau$$
$$d\tau = \rho(X_s)ds$$

$$\implies N_t^f = f(Y_t) - f(Y_0) - \int_0^t (\mathcal{L}f)(Y_t)\frac{1}{\rho(Y_t)}d\tau$$

Since $\mathcal{L}f(x) = \sum_k b_k(x)\frac{\partial}{\partial x_k}f(x) + \sum_{k,l} a_{k,l}(x)\frac{\partial^2}{\partial x_k \partial x_l}f(x)$

$$\implies (\tilde{\mathcal{L}}f)(x) = \sum_k \frac{b_k(x)}{\rho(x)}\frac{\partial}{\partial x_k}f(x) + \sum_{k,l}\frac{\overbrace{a_{k,l}}^{(\sigma\sigma^\intercal)_{k,l}}}{\sqrt{\rho(x)\rho(x)}}(x)\frac{\partial^2}{\partial x_k \partial x_l}f(x)$$

$\implies$ It is a martingale problem for the SDE where the drift $\rightarrow \frac{\text{drift}}{\rho}$ and $\sigma \rightarrow \frac{\sigma}{\sqrt{\rho}}$

□

1.5.3 Weak solutions in d=1

We will do both time and "space" changes.

- 1-d SDE:
$$\begin{cases} dX_t & = b(X_t)dt + \sigma(X_t)dB_t \\ X_0 = x_0 \in (\alpha, \beta) \end{cases} \tag{1.4}$$
- X_t a process in (α, β)
- Assume $b, \sigma : (\alpha, \beta) \rightarrow \mathbb{R}$ continuous, $\sigma(x) > 0 \forall x \in (\alpha, \beta)$
- Do a change of coordinates $Y_t := s(X_t)$ where $s : (\alpha, \beta) \rightarrow (s(\alpha), s(\beta))$, C^2 with $S'(x) > 0, x \in (\alpha, \beta)$.
- $s(x)$ is called the **scale function** and it is given by
$$s(x) = \int_{x_0}^x \exp\left(-\int_{x_0}^y dz\frac{2b(z)}{\sigma(z)^2}\right)dy$$
- s satisfies $\frac{1}{2}\sigma^2(x)s''(x) + b(x)s'(x) = 0 \iff (\mathcal{L}s)(x) = 0$

Remark. *If $b(z) = 0 \implies s(x) = x - x_0 \implies s'(x) = 1$. If $s'(x) = 1$, we say that the process is in its "natural scale"*

By proposition 1.14: $\mathcal{L}s = 0, \dot{s} = 0$.
$\implies Y_t = s(X_t)$ is a local martingale satisfies $dY_t = s'(X_t)\sigma(X_t)dB_t$. the other terms cancel
$\iff Y_t$ is a solution of

$$\begin{cases} dY_t &= \tilde{\sigma}(Y_t)dB_t \\ Y_0 &= s(X_0) \end{cases} \tag{1.5}$$

where $\tilde{\sigma}(y) = s'(s^{-1}s(y))\sigma(s^{-1}(y))$.

Therefore we can write the original SDE in terms of a BM

Theorem 1.23. *The following are equivalent:*

1. *The process $(X_t)_{t<\xi}$, where ξ is the explosion time, on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; (B_t)_{t\geq 0})$ is a solution of (1.4) up to tje stopping time ξ*
2. *The process $Y_t = s(X_t)_{t<\xi}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}; (B_t)_{t\geq 0})$ is a solution of (1.5) up to ξ*
3. *The process $(Y_t)_{t<\xi}$ has the representation $Y_t = \tilde{B}_{A_t}$, where $\tilde{B}$ is a BM starting at $\tilde{B}_0 = s(X_0)$ and $A_t = T_t^{-1}$ and $T_t = \int_0^t \frac{1}{\tilde{\sigma}^2(\tilde{B}_u)}du$*

s and A_t have the same definition as before

A degenerate case:

Let $\sigma(x) = |x|^\alpha$ for some $\alpha \in \left(0, \frac{1}{2}\right)$. $\implies$

$$\begin{cases} dY_t &= |Y_t|^\alpha dB_t \\ Y_0 &= y \end{cases} \tag{1.6}$$

$\implies T_t = \int_0^t \frac{1}{\sigma(\tilde{B}_u)^2}du$, $A_t = \int_0^t \sigma(Y_s)^2 ds$ and $Y_t = \tilde{B}_{A_t} \implies \tilde{B}_0 = y$.
$T_t < \infty$ a.s. ?

$$\begin{aligned} \mathbb{E}(T_t) &= \int_0^t \mathbb{E}\left(\frac{1}{\sigma(\tilde{B})^2}\right) du \\ &= \int_0^t \mathbb{E}\left(\frac{1}{|\tilde{B}|^{2\alpha}}\right) du \\ &= \int_0^t \int_{\mathbb{R}} dx \frac{1}{|x|^{2\alpha}} \frac{\exp(-\frac{(x-y)^2}{2u})}{\sqrt{2\pi u}} \overset{0<2\alpha<1}{<} \infty \end{aligned}$$

$\implies A_t = T_t^{-1}$, then $Y_t = \tilde{B}_{A_t}$ is a solution of (1.6), i.e. $\forall y \in \mathbb{R} \exists$ a non-trivial solution of (1.6).
For $Y_=0, Y_t = 0$ is also a solution $\implies$

- No uniqueness in law of (1.6)
- No pathwise uniqueness as well

Remark. *In general: uniqueness in law of 1-d SDEs is not to be expected if $\sigma(x) = 0$ somewhere (and σ continuous ...) (i.e. if σ is degenerate).*

By theorem 1.12 as soon as $\sigma(x) = |x|^\alpha$ for some $\alpha \geq \frac{1}{2}$, then one has pathwise uniqueness.

Hitting times and scale functions

Bessel process:

$$\begin{cases} dR_t &= \frac{d-1}{2R_t}dt + dW_t \\ R_0 &= r_0 \in (0, \infty) \end{cases}$$

$\implies b(x) = \frac{d-1}{2x}, \sigma(x) = 1$.
The scale function satisfies $\mathcal{L}s(x) = \frac{1}{2}s''(x) + \frac{d-1}{2x}s'(x) = 0$

$$\implies s(x) = \begin{cases} \frac{x^{2-d}}{2-d} & d \neq 2 \\ \ln(x) & d = 2 \end{cases}$$

and therefore

$$(s_0, s_\infty) = \begin{cases} (0, \infty) & d < 2 \\ (-\infty, \infty) & d = 2 \\ (-\infty, 0 & d > 2) \end{cases}$$

Define the stopping time

$$T_a^R := \inf\{t \geq 0 \mid R_t = a\}$$

Choose an $\alpha < r_0 < \beta$

$$\implies \mathbb{P}(T_\alpha^R < T_\beta^R) \overset{s' \geq 0}{=} \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

One computes

WS exercises

$$\mathbb{P}(T_a^R < T_\beta^R) = \frac{s(\beta) - s(r_0)}{s(\beta) - s(\alpha)} = \mathbb{P}(T_{s(\alpha)}^{s(R)} < T_{s(\beta)}^{s(R)})$$

This is generic, for Itô diffusions, provides that $\exists$ no killing in (α, β).

unlike in 1.16

Start of lecture 05 (25.04.24)

1.5.4 Uniqueness of martingale solution

SDE $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ with generator

$$\mathcal{L} = \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l} (\sigma\sigma^\intercal)_{k,l} \frac{\partial^2}{\partial x_k \partial x_l}$$

Definition 1.24. *Let $\mathcal{C} = C(\mathbb{R}_+, \mathbb{R}^d)$ with σ-algebra $\mathcal{F}$, canonical filtration $(\mathcal{F}_t)_{t\geq 0}$, canonical process $Z_t(\omega) := \omega$.*
*We say that $\mathbb{P}$ on $(\mathcal{C}, \mathcal{F})$ is a **martingale solution** for the generator $\mathcal{L} \iff \forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R})$*

$$M_t^f := f(t, Z_t) - f(0, Z_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right) f(s, Z_s) ds \tag{1.7}$$

is a martingale w.r.t. $\mathbb{P}$.

Definition 1.25. *A martingale problem (1.7) has a unique solution if for any two martingale solutions $\mathbb{P} = \mathbb{Q}$ s.t. $Law_{\mathbb{P}}(Z_0) = Law_{\mathbb{Q}}(Z_0)$*

$$\implies \mathbb{P} = \mathbb{Q}$$

Remark. *Uniqueness of martingale solutions corresponds to uniqueness in law of the weak solutions.*

Backwards Kolmogorov Equation (BKE):

$$\frac{\partial}{\partial t}\varphi(t, x) = \mathcal{L}\varphi(t, x) \forall x \in \mathbb{R}^d, t \geq 0 \tag{1.8}$$

Theorem 1.26. *Assume that $\forall$ initial condition*

$$\varphi(0, x) = \Psi(x), \Psi \in C_0^\infty(\mathbb{R}^d)$$

the (1.8) has a solution and φ bounded for all finite time intervals. We have uniqueness of martingale solutions and therefore uniqueness of weak solutions!

The Kolmogorov forward equation is (related to) the Folker-Plank equation!

Proof. Prove that $\forall 0 \le t_1 < t_2 < \cdots < t_n$:

$$\mathrm{Law}_{\mathbb{P}}(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}) = \mathrm{Law}_{\mathbb{Q}}(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n})$$

1. One-time distribution:
$\forall 0 \le s \le r$:

$$\left(\frac{\partial}{\partial s} + \mathcal{L}\right) \varphi(r-s, x) \overset{(1.8)}{=} 0$$

take $t \in [0, r]$:

$$\begin{aligned} M_t^r &:= \varphi(r-t, Z_t) - \varphi(r, Z_0) - \int_0^t \underbrace{(\partial_s + \mathcal{L})\varphi(r-s, Z_s)}_{=0} ds \\ &= \varphi(r-t, Z_t) - \varphi(r, Z_0) \text{ is a martingale} \end{aligned}$$

This needs the boundedness of φ, otherwise it might only be a local martingale. There are softer contidions we can put on the coefficients to achieve the same result. This might not be needed, because φ is C^1 in time anyways

for any solution $\mathbb{P}$.

$$\begin{aligned} 0 \quad &= \mathbb{E}_{\mathbb{P}}\left(M_r^r - M_t^r \mid \mathcal{F}_t\right) && = \mathbb{E}(\varphi(0, Z_r) - \varphi(r-t, Z_t) \mid \mathcal{F}_t) \\ \implies \quad \forall 0 \le t \le r : &\mathbb{E}_{\mathbb{P}}(\varphi(0, Z_r) \mid \mathcal{F}_t) && = \mathbb{E}_{\mathbb{P}}(\varphi(r-t, Z_t) \mid \mathcal{F}_t) \\ & && \overset{\text{a.s.}}{=} \varphi(r-t, Z_t) \\ &\mathbb{E}(\underbrace{\varphi(0, Z_r)}_{\Psi(Z_r)}) && \overset{t=0}{=} \mathbb{E}_b P(\varphi(r, Z_0)) \end{aligned}$$

$\forall$ other martingale solutions $\mathbb{Q}$:

$$\mathbb{E}_{\mathbb{Q}}(\Psi(Z_r)) = \mathbb{E}_{\mathbb{Q}}(\varphi(r, Z_0))$$

By assumption this implies $\mathrm{Law}_{\mathbb{P}}(Z_r) = \mathrm{Law}_{\mathbb{Q}}(Z_r)$.
2. Multi-time distributions:
For $\Psi \in C_0^\infty$, denote φ_Ψ the solution of (1.8) with initial condition Ψ:

$$\mathbb{E}_{\mathbb{P}}(\Psi(Z_r) \mid \mathcal{F}_t) = \varphi_\Psi(r-t, Z_t)$$

$0 \le r_2 \le r_1$ test for $g \in C_0^\infty$:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(\Psi(Z_{r_1}) g(X_{r_2})) &= \mathbb{E}(\underbrace{\mathbb{E}(\varphi_\Psi(Z_{r_1}) | \mathcal{F}_{r_2})}_{\varphi_\Psi(r_1 - r_2, Z_{r_2})} g(Z_{r_2})) \\ &= \mathbb{E}_{\mathbb{P}}(\varphi_\Psi(r_1 - r_2, Z_{r_2}) g(Z_{r_2})) \\ &\overset{1.}{=} \mathbb{E}_{\mathbb{Q}}(\varphi_\Psi(r_1 - r_2, Z_{r_2}) g(Z_{r_2})) \\ &= \mathbb{E}_{\mathbb{Q}}(\Psi(Z_{r_1}) g(Z_{r_2})) \end{aligned}$$

Iterating yields the statement.

□

Chapter 2: SDE techniques

Goal: Study process by changing the measure.
E.g.: $X_t = B_t$. $\underbrace{\text{Condition } X_t \geq 0 \forall t \geq 0}_{=:Y_t}$.

2.1 Girsanov theorem

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ and two measures $\mathbb{P}, \mathbb{Q}$
Assume $\mathbb{P} \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P}$ and let $H = \frac{d\mathbb{Q}}{d\mathbb{P}}$

The notes of Eberle switches the roles of $\mathbb{P}, \mathbb{Q}$!

$$\implies Z_t := \mathbb{E}_{\mathbb{P}}(H \mid \mathcal{F}_t) = \frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_t}$$

is a martingale.
From last semester: $\forall Y \in L^1(\mathbb{Q}) \cap L^1(\mathbb{P}), \mathcal{F}_t$ measurable:

$$\mathbb{E}_{\mathbb{Q}}(Y \mid \mathcal{F}_s) = \frac{bE_{\mathbb{P}}(Y \cdot Z_t \Big| \mathcal{F}_s)}{Z_s} \forall s < t$$

If $Z > 0$ is $\mathcal{M}_{\text{loc}}, \exists L \in \mathcal{M}_{\text{loc}}$ s.t.:

$$Z_t = e^{L_t - \frac{1}{2}\langle L \rangle_t} \to L_t = \ln(Z_0) + \int_0^t \frac{dZ_s}{Z_s}$$

There might be a problem, because $\ln(Z_0)$ might not be integrable, and therefore not a local martingale!

Theorem 2.1 (Girsanov). *Assume $Z > 0$ is a martingale. If M is a local martingale w.r.t. $\mathbb{P}$, then*

$$\tilde{M}_t := M_t - \langle M, L \rangle_t$$

is a local martingale w.r.t. $\mathbb{Q}$ and

$$\langle M \rangle_t = \langle \tilde{M} \rangle_t.$$

Moreover, if M is a BM w.r.t. to $\mathbb{P}$, then $\tilde{M}$ is a BM w.r.t. $\mathbb{Q}$.

Remark. *In applications, given $\mathbb{P}, (Z_t)_{t\geq 0}$ a positive continuous martingale, define $\mathbb{Q}$ on $\mathcal{F}_\infty = \bigcup_{t\geq 0} \mathcal{F}_t$ s.t.*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t$$

Z is uniformly integrable $\iff \mathbb{Q} \ll \mathbb{P}$. ***Problem:*** *In applications, Z is not necessarily uniformly integrable.*
$\implies$ restrict to $[0, T] \implies$ all fine.
$\mathbb{Q} \to \mathbb{Q}_T$ as in the last semester.

Added example. *Let $\gamma \in \mathbb{R}^d, (B_t)_{t\geq 0}$ standard BM.*
Let $L_t := \gamma \cdot B_t$ and $Z_t = \exp(L_t - \frac{1}{2}\langle L\rangle_t) = \exp(\gamma \cdot B_t - \frac{1}{2}|\gamma|^2 t)$

Remark. $\lim_{M\to\infty} \sup_{t\geq 0} \mathbb{E}(|Z_t| 1_{|Z_t|>M}) \neq 0$.
Define $\mathbb{Q}$ on $\mathcal{F}_\infty$ s.t. $\tilde{B}_t^k = B_t^k - \langle L, B^k\rangle = B_t^k - \gamma^k t$ is a BM with drift. Show: $Q \not\ll \mathbb{P}$. *Construct Q via Z*

$$A = \left\{ \lim_{t\to\infty} \frac{\tilde{B}_t + \gamma t}{t} = \gamma \right\} \in \mathcal{F}_\infty$$

but: $\tilde{B}$ is a BM w.r.t. $\mathbb{Q} \implies \mathbb{Q}(A) = 1$, $\tilde{B}$ is a BM with drift $-\gamma$ w.r.t $\mathbb{P}(A) = 0$.

Start of lecture 06 (30.04.24)

2.1.1 Drift transformation of SDE

We once again start with

$$\star = \begin{cases} dX_t &= b(t, X_t) \\ X_0 &= x_0 \end{cases}$$

with drift b continuous.
Goal: Get a weak solution of $\star$.
Let $(X_t)_{t\geq 0}$ be a BM in $(\Omega, \mathcal{F}(\mathcal{F}_t)_{t\geq 0}, \mathbb{P}), X_0 = x_0$.

Assume: $Z_t := \exp\left(\underbrace{\int_0^t b(s, X_s) dX_s}_{=L_t} - \frac{1}{2}\int_0^t |b(s, X_s)|^2 ds \right)$ is a martingale w.r.t. $\mathbb{P}$.

Remark. *Thr assumption holds if*

$$|b(t, x)| \leq C(1 + \|x\|) \text{ for some } C.$$

$\implies \mathbb{E}_{\mathbb{P}}(Z_t) = 1 \forall t \geq 0$
By Girsanov

$$\tilde{X}_t = X_t - \langle L, X\rangle_t$$

is a $\mathbb{Q}$-BM.
But

$$d\langle L, X\rangle_t = b \cdot dX_t \cdot dX_t = b \cdot dt$$
$$\implies \tilde{X}_t = X_t - \int_0^t b(s, X_s) ds$$
$$\text{w.r.t. } \mathbb{Q} : dX_t = b(t, X_t)dt + \underbrace{d\tilde{B}_t}_{d\tilde{X}_t}$$

Generalization: Start with

$$\star\star\, dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

where X, B are d-dimensional.

Proposition 2.2. *Assume (X, B) is a weak solution of $\star\star$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. If*

$$Z_t = \exp\left(\int_0^t c(s, X_s) dB_s - \frac{1}{2}\int_0^t c(s, X_s)^2 ds \right)$$

is a martingale w.r.t. $\mathbb{P}$, $\mathbb{Q} \ll \mathbb{P}$ and $\mathcal{F}_t$ s.t. $Z_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}} Vert_{\mathcal{F}_t}$.

In practice this is surprisingly useful in practice!

this is an implicit condition for c

Then $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ is a weak soltuion of

$$dX_t = [b(t, X_t) + \sigma(t, X_t)c(t, X_t)dt] + \sigma(t, X_t)d\tilde{B}_t$$

where $\tilde{B}$ is a d-dim BM.

Proof.

$$\begin{aligned} d\langle L, B\rangle_t &\overset{L_t = \int_0^t c(s,X_s)dB_s}{=} c(t, X_t)dt \\ &\implies \tilde{B}_t := B_t - \langle L, B\rangle_t \text{ is a } \mathbb{Q}\text{-BM} \\ &\implies dB_t = c(t, X_t)dt + d\tilde{B}_t \end{aligned}$$

From $\star\star$: $dX_t = b(t, X_t)dt + \sigma(t, X_t)c(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t = \star\star\star$ □

Measure	SDE	Generators
$\mathbb{P}$	$dX_t b \cdot dt + \sigma \cdot dB_t$	$\mathcal{L} \sum_k b_k \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k,l}(\sigma\sigma^\intercal)_{k,l}\frac{\partial^2}{\partial x_k \partial x_l}$
$\mathbb{Q}$	$[b + \sigma \cdot c]dt + \sigma d\tilde{B}_t$	$\tilde{\mathcal{L}} = \mathcal{L} + \sum_k \sum_l \sigma_{k,l}c_l \frac{\partial}{\partial x_k} = \mathcal{L} + c\sigma^\intercal \nabla$

$t \leq T$ for $\mathbb{Q}_t$.

2.2 Doob-h transform

Essentially three cases in 1d

1. From $B_t \to B_t$ conditioned on $\{B_1 = 0\}$ (measure zero set)
2. From $B_t \to B_t$ conditioned on $\{B_t \geq 0 \forall t \geq 0\}$ (measure zero set)
3. From $B_t \to B_t$ conditioned on $\{B_t \geq 0 \forall t \in [0, 1], B_1 = 0\}$

Let $(X_t, B_t)_{t\geq 0}$ be a weak solution of

$$(\star) = \begin{cases} dX_t &= b(t, X_t)dT + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \text{ fixed} \end{cases}$$

Assume there exists $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ s.t. $h > 0$ satisfying

$$(\frac{\partial}{\partial t} + \mathcal{L})h = 0 \forall t, \in [0, T], x \in \mathbb{R}^d$$

where $\mathcal{L}$ is the generator of the SDE $(\star)$.
By theorem 1.14 $Z_t := h(t, X_t) = h(0, X_0) + \int_0^t (\sigma\nabla h)(s, X_s)dB_s$.
Z_t is a positive local martingale. Assume Z_t is a martingale.
W.l.o.g. $Z_0 = 1$ (if not $h \mapsto \frac{h}{h(0,X_0)}$).
$\implies d\mathbb{Q}_T = Z_T \cdot d\mathbb{P}$. Let L_t s.t. $Z_t = \exp(L_t - \frac{1}{2}\langle L\rangle_t)$.
Girsanov, since B_t is a $\mathbb{P}$-BM,
$\tilde{B}_t := B_t - \langle L, B\rangle_t$ is a Q_T-BM, where

$$\langle L, B\rangle_t = \sigma^\intercal \cdot \nabla \ln h \cdot dB_t$$

because of

$$dL_t = \frac{dZ_t}{Z_t} = \frac{\sigma^\intercal \cdot \nabla h \cdot dB_t}{h(t, X_t)} = \sigma^\intercal \cdot \nabla \ln h \cdot dB_t$$

And therefore $dB_t = d\tilde{B}_t + \sigma^\intercal \nabla \ln h dt$.
From $(\star) \implies dX_t = \underbrace{(b(t, X_t) + \sigma(t, X_t)\sigma^\intercal(t, X_t)\nabla h(t, X_{)}}_{:=\tilde{b}})dt + \sigma(t, X - t)d\tilde{B}_t$

Proposition 2.3. *Let X be a weak solution of $dX_t = b(t, X_t) + \sigma(t, X_t)dB_t$ on $[0,T]$ under $\mathbb{P}$. Then under $\mathbb{Q}_T$, X is a weak solution of the SDE*

$$\begin{cases} dX_t &= \tilde{b}(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t \\ X_0 &= x_0 \end{cases}$$

where $\tilde{b} = b + \sigma\sigma^\intercal \nabla \ln h$.

Remark. *In applications, often $h(x) \not> 0$ for all x!*
E.g.: $h(t,x) = x \implies \left(\frac{\partial}{\partial t} + \mathcal{L}\right) h = 0$ $(X_t = B_t + x_0)$, but $h(x) \leq 0$ for $x \leq 0$.
In this case first do the construction on $[0, \tau]$, where $\tau \coloneqq \inf\{t > 0 \mid B_t = 0\}$
$\implies$ ok for $B_t^\tau \overset{\text{magically}}{\implies}$ the new SDE has drift:

$$\frac{\partial}{\partial x} \ln(h(x)) = \frac{1}{h(x)} \frac{\partial h(x)}{\partial x} = \frac{1}{x}$$

We realize there is no problem in the new measure

Added example. *If $X_t = B_t$ and $h(t,) = e^{\gamma x - \frac{1}{2}|\gamma|^2 t}$*

$$\begin{aligned} \implies dX_t &= \nabla \ln h(t, X_t)dt + dB_t \\ &= \nabla(\gamma \cdot x)dt + dB_t \\ &= \gamma d_t + dB_t \end{aligned}$$

$\implies$ X_t is a BM with drift γ w.r.t $\mathbb{Q}$.

2.3 Diffusion Bridges

Consider a Markov process $(X_t)_{t\geq 0}$ which is a diffusion starting from $X_0 = x_0 \in \mathbb{R}^d$.
We want to condition on the event $\{X_T = y\}$ for some given $T > 0, y \in \mathbb{R}^d$.
Goal: Find, $\forall y \in \mathbb{R}^d, \mathbb{Q}^y$ which is the conditional measure on $\{X_T = y\}$.
Assume: $(X_t)_{t\geq 0}$ has transition density P s.t.

This is not true in general, if we don't have unique solutions

$$\forall 0 \leq s \leq t \leq T, \mathbb{P}(X_t \in dz \mid X_s = x) = p(s, x; t, z)dz.$$

X_t solves

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= x_0 \end{cases}.$$

Define $h^y(s,x) \coloneqq \frac{p(s,x;T,y)}{p(0,x_0;T,y)}$ for $s \in [0,T), x \in \mathbb{R}^d$.

Remark. *$s < T$ since otherwise we get ingeneral singularities.*

Lemma 2.4. *Let $Z_t^y \coloneqq h^y(t, X_t)$ is a martingale.*

Proof. Let $0 \leq < t < T$:

$$\begin{aligned} \mathbb{E}(Z_t^y \mid \mathcal{F}_s) &= \mathbb{E}(h^y(t, X_t) \mid \mathcal{F}_s) \\ &\overset{\text{MP}}{=} \mathbb{E}(h^y(t, X_t \mid X_s)) \\ &= \int_{\mathbb{R}^d} h^y(t,x) p(s, X_s; t, x)dx \\ &= \int_{\mathbb{R}^d} \frac{p(t,x;T,y)}{p(0,x_0;T,y)} p(s, X_s; t, x)dx \\ &= \frac{1}{p(0,x_0;T,y) \int_{\mathbb{R}^d} p(s, X_s; t, x) p(t,x;T,y)dx} \\ &\overset{\text{Chapman-Kolmogorov}}{=} Z_s^y = \end{aligned}$$

□

$h^y(0, x_0) = 1$

Start of lecture 07 (02.05.24)

Added remark. ***Goal:*** *Find a family* $(\mathbb{Q}^y)_{y\in\mathbb{R}^d}$ *s.t. for* $A \in \mathcal{F}_T$*:*

$$\mathbb{P}(A) = \mathbb{E}(\mathbb{P}(A|X_T)) \overset{?}{=} \mathbb{E}(\mathbb{Q}^{X_T}(A))$$

and $\mathbb{Q}^y\left(\lim_{t\uparrow T} X_t = y\right) = 1$

Lemma 2.5. *We can take*

$$\mathbb{Q}^y(A) = \mathbb{E}(1_A \cdot h^y(s, X_s)) \forall A \in \mathcal{F}_s$$

for $s \in [0, T]$.

Proof. For all $A \in \mathcal{F}_s$, g bounded measurable:

$$\begin{aligned}
\mathbb{E}(1_A g(X_t)) &= \mathbb{E}(1_A \mathbb{E}(g(X_T|\mathcal{F}_s))) \\
&\overset{\text{MP}}{=} \mathbb{E}(1_A \mathbb{E}(g(X_T|X_s))) \\
&= \mathbb{E}\left(1_A \int_{\mathbb{R}^d} p(s, X_s; T, x) g(x) dx\right) \\
&= \int_{\mathbb{R}^d} dx g(x) \mathbb{E}(1_A p(s, X_s; T, x))
\end{aligned}$$

In particular,

$$\mathbb{P}(A) = \mathbb{E}(1_A) = \int_{\mathbb{R}^d} dx \mathbb{E}(1_A \cdot p(s, X_s; T, x))$$

But

$$\begin{aligned}
\int_{\mathbb{R}^d} dx p(0, X_0; T, x) Q^x(A) &= \int_{R^d} p(0, x_0; T, x) \underbrace{\mathbb{E}(1_A h^x(s, X_s))}_{=\int dz p(0,x_0;s,z) h^x(s,z) 1_A} \\
&= \int_{\mathbb{R}^d} \mathbb{E}(1_A p(s, X_s; T, x)) \\
&= \mathbb{P}(A).
\end{aligned}$$

□

Remark. $Z_t^y = h^y(t, X_t) = h(0, X_0) + \text{ martingale } + \underbrace{\int_0^t (\partial_t + \mathcal{L}) h^y(s, X_s) ds}_{=0}.$

One can verify: $(\partial_t + \mathcal{L}) h^y(t, X_t) = 0 \iff (\partial_t + \mathcal{L}) H^y(t, X_t) = 0$ *for* $H^y(t, x) = p(t, x; T, y)$.

$$\begin{aligned}
\frac{\partial}{\partial t} H^y(t, x) &= \lim_{\epsilon \downarrow 0} \frac{p(t, x; T, y) - p(t - \epsilon, x; T, y)}{\epsilon} \\
&= \lim_{\epsilon \downarrow 0} \frac{p(t, x; T, y) - \int \overbrace{p(t - \epsilon, x; t, z)}^{=e^{\epsilon L}(x,z) = \delta_{x-z} + \epsilon \mathcal{L}(x,z)} p(t, z; T, y) dz}{\epsilon} \\
&= -(\mathcal{L} H^y)(t, x)
\end{aligned}$$

$\implies$ *this is a doob-h transform with* $h = h^y$.

$$\frac{d\mathbb{Q}^y}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t^y.$$

Corollary 2.6. *Assume* $(t, x) \mapsto p(t, x; T, y)$ *is* $C^{1,2}([0, T] \times \mathbb{R}^d)$
$\implies$ *The Doob h-transform of the original process under* $\mathbb{Q}^y$ *satisfies the SDE:*

$$dX_t = \tilde{b}(t, X_t) dt + \sigma(t, X_t) d\tilde{B}_T$$

where $\tilde{B}_t$ *is a* $\mathbb{Q}^y$*-BM and* $\tilde{b}(t, X_t) = b(t, X_t) + (\sigma\sigma^\intercal \nabla \ln h^y)(t, X_t)$.

Remark. *Take $z \neq y$ and $\forall 0 < \epsilon < |z-y|$:*

$$\begin{aligned}\mathbb{Q}^y\left(\{|X_t - z| \leq \epsilon\}\right) &= \mathbb{E}\left(1_{|X_t-z|\leq\epsilon}\frac{p(t,X_t;T,y)}{p(0,x_0;T,y)}\right)\\ &= \int dx 1_{|x-z|\leq\epsilon}p(0,x_0,t,x)\frac{p(t,x;T,y)}{p(0,x_0;T,y)}\\ &\xrightarrow{t\to T} 0\end{aligned}$$

$\implies$ $\mathbb{P}\left(\lim_{t\uparrow T} X_t = y\right) = \mathbb{P}(X_t = y) = 0$, *but* $\mathbb{Q}^y\left(\lim_{t\uparrow T} X_t = y\right) = 1$
$\implies$ $\mathbb{Q}^y$ *is singular w.r.t.* $\mathbb{P}$.

Added example. $b = \gamma \in \mathbb{R}^d, \sigma = 1, X_0 = 0$ *and* $T = 1$. *Under* $\mathbb{P}$:

$$\begin{cases} dX_t = \gamma dt + dB_t \\ X_0 = 0\end{cases}$$

Under $\mathbb{Q}^y$*?*

$$h^y(t,x) = \frac{p(t,x;1,y)}{p(0,x_0,1,y)} = fct(\gamma,y)\cdot\exp\left(-\frac{(y-x)^2}{2(1-t)}\right)\exp\left((y-x)\gamma\right)$$

$\implies \ln(h^y(t,x)) = \ln f(\gamma,y) - \frac{(y-x)^2}{2(1-t)} + (y-x)\gamma \implies \nabla \ln h^y(t,x) = \frac{y-x}{1-t} - \gamma.$
Under $\mathbb{Q}^y$:

$$\begin{aligned} dX_t^k h^y &= \left(\gamma_k + \frac{y_k - X_t^k}{1-t}\right)dt + d\tilde{B}_t^k \\ &= \frac{Y_k - X_t^k}{1-t}dt + d\tilde{B}_t^k\end{aligned}$$

Indepdendent of γ!

for $k = 1, \ldots, d$

Lemma 2.7. *Let p be the transition density of X w.r.t. $\mathbb{P}$ and p^h the transition density of X w.r.t. $\mathbb{Q}^y$.*
Let $h^y(t,x) = \frac{p(t,x;T,y)}{p(0,x_0;T,y)}$.

$$\implies p^h(s,x;t,z) = \frac{1}{h^y(s,x)}p(s,x;t,z)h^y(t,z)$$

Useful in practice to sample!

Notice how the normalizations inside of h^y cancel!

Proof.

$$\begin{aligned} p^h(s,x;t,z)dz &= \mathbb{P}(X_t \in dz \mid X_s = x, X_T = y)\\ &\overset{0\leq s\leq t}{=} \frac{p(0,x_0;s,x)p(s,x;t,z)p(t,z;T,y)dz}{p(0,x_0;s,x)p(s,x;T,y)}\\ &= p(s,x;t,z)\frac{p(t,z;T,y)}{p(0,x_0;T,y)}\frac{1}{\frac{p(s,x;T,y)}{p(0,x_0;T,y)}}dz \end{aligned}$$

□

2.4 Brownian motion conditioned to stay positive forever

Goal: $(B_t)_{t\geq 0}$ with $B_0 = x_0 \geq 0$ and condition on

$$\{B_t \geq 0 \forall t \geq 0\}$$

$T_x = \inf\{t \geq 0 \mid B_t = x\}$ and take $0 < x_0 < R$.
$T_0 \wedge T_R = T_R$.
Let $\tilde{T}_R \coloneqq T_0 \wedge T_R$. Define the event $E_R = \{B_{\tilde{T}_R} = R\} = \{T_0 \wedge T_R = T_R\}$.

One verifies $\mathbb{P}(E_R) = \frac{x_0}{R} \in (0,1)$.
$\mathbb{P}(E_R) = \mathbb{E}(1_{B^{x_0}_{\tilde{T}_R}}) = \underbrace{\mathbb{E}(f(B^{x_0}_{\tilde{T}_R}))}_{u(x_0)}$, where

$$f(x) = \begin{cases} 1 & x = R \\ 0 & x = 0 \end{cases}.$$

u has a link to the PDE:

By Theorem 11.6 of the last semester

$$\begin{cases} \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x) = 0 & x \in (0,R) \\ u(x) = f(x) & x \in \{0,R\} \end{cases}$$

Define the conditional probability

later $R \to \infty$

$$Q^R(A) := \frac{\mathbb{P}(A \cap E_R)}{\mathbb{P}(E_R)}$$

Lemma 2.8. *Let $h(x) := \frac{\mathbb{P}_x(E_R)}{\mathbb{P}_{x_0}(E_R)} = \frac{x}{x_0}$, where $\mathbb{P}_x$ is the unconditional law for $B_0 = x$.*

$$Z_t = h(B_{t \cap \tilde{T}_R})$$

is a non-negative martingale.

Proof.

$$\begin{aligned}
\mathbb{Q}^R(A) &\overset{A \in \mathcal{F}_s}{=} \frac{\mathbb{E}(1_A \mathbb{E}(1_{E_R} \mid \mathcal{F}_s))}{\mathbb{P}(E_R)} = \frac{\mathbb{E}(1_A \mathbb{E}(1_{E_R} \cdot 1_{\tilde{T}_R > s} \mid \mathcal{F}_s)) + \mathbb{E}(1_A \mathbb{E}(1_{E_R} 1_{\tilde{T}_R \le s} \mid \mathcal{F}_s))}{\mathbb{P}(E_R)} \\
&= \frac{1}{\mathbb{P}(E_R)} \left[\mathbb{E}(1_A 1_{\tilde{T}_R > s} \underbrace{\mathbb{E}(1_{E_R} \mid \mathcal{F}_s)}_{\mathbb{P}_{B_s}(E_R)}) + \mathbb{E}(1_A \underbrace{1_{E_R}}_{1_{B_{\tilde{T}_R} = R}} 1_{\tilde{T}_R \le s}) \right] \\
&= \mathbb{E}(1_A 1_{\tilde{T}_R > s} h(B_s)) + \mathbb{E}(1_A 1_{\tilde{T}_R \le s} h(B_{T_R})) \\
&= \mathbb{E}(1_A h(B_{s \wedge \tilde{T}_R}))
\end{aligned}$$

$\implies \forall A \in \mathcal{F}_s : \mathbb{Q}^R(A) = \mathbb{E}_{\mathbb{P}}(1_A h(B_{s \wedge \tilde{T}_R}))$
$\implies h(B_{s \wedge \tilde{T}_R}) = \frac{d\mathbb{Q}^R}{d\mathbb{P}}|_{\mathcal{F}_s} \implies h \in (0, \frac{R}{x_0})$ is a martingale (See construction of Girsanov). □

Start of lecture 08 (07.05.24)
Start of lecture 09 (14.05.24)

Added example. *Discrete time M.C. with transition probability P aperiodic and irreducible: $\exists n_0 \forall n > n_0 (P)^n_{i,j} > 0 \implies \exists |\lambda_0| \le 1, |\lambda_1| > |\lambda_2| > |\lambda_3|$:*

$$(P)^n \underbrace{\varphi_0(i)}_{>0} = \lambda_0 \varphi_0(i)$$

$\lim_{n\to\infty} \underbrace{\overset{n}{P}}_{(1+L)} = (\varphi_0(1), \ldots, \varphi_0(d))$
where $(1+L) \to e^{tL}$, for which the real eigenvalues are positive.

2.5 Diffusion conditioned to stay in a domain

Domain $D \subset \mathbb{R}^d$: bounded, open, connected.
Diffusion with generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$$

$\iff$ SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \frac{1}{2}\sigma(t, X_t)dB_t \\ X_0 = x_0 \end{cases}$$

$a = \sigma\sigma^\intercal$.
Assume: b, σ are continuous, $\sigma \in C^1$ and let $\tau_D = \int\{t \geq 0 | X_t \notin D\}$
Key assumption in Pinsky's paper:

(a) $\mathbb{P}_x(\tau_D > t) \in \mathbb{C}^2(D)$

(b)

$$\begin{cases} \mathbb{P}_x(\tau_D > t) & = C_1\varphi_0(x)e^{-\lambda_0 t} + o\left(e^{-\lambda_0 t}\right) \\ \nabla\mathbb{P}_x(\tau_d > t) = C_1\nabla\varphi_0(x)e^{-\lambda_0 t} + o\left(e^{-\lambda_0 t}\right) \end{cases}$$

where

$$\begin{cases} -L\varphi_0(x) = \lambda_0\varphi_0(x) & x \in D \\ \varphi_0(x) = 0 & x \in \partial D \end{cases}$$

and λ_0 is the smallest eigenvalue of $-L$. Which means $\mathbb{P}(\tau_d > t) \overset{t\to\infty}{\to} 0$ in the non symmetric case take the real part first

Want: Condition X to stay in D forever.

(1) Condition X_t to $\{X_t \in D : 0 \leq t \leq T\}$: Forall $A \in \mathcal{F}_T$: define a measure

$$\mathbb{Q}^T(A) := \frac{\mathbb{P}_{x_0}(A \cap \{\tau_D > T\})}{\mathbb{P}_{x_0}(1_{\tau_D} > T)} = \frac{\mathbb{E}_{x_0}(1_A \cdot 1_{\tau > T})}{\mathbb{E}_{x_0}(1_{\tau > T})}$$

Lemma 2.9. *$\forall s < T, A \in \mathcal{F}_s$,*

$$\mathbb{Q}^T(A) = \mathbb{E}_{x_0}(1_A \cdot Z_s^T)$$

where $Z_s^T = \frac{g^{T-s}(X_{s\wedge\tau_D})}{g^T(x_0)}$ with $g^t(x) := \mathbb{P}_x(\tau_D > t)$.

Proof. Let $A \in \mathcal{F}_s, s < T$:

$$\begin{aligned} \mathbb{Q}^T(A) &= \frac{\mathbb{E}_{x_0}(1_A 1_{\tau_D > T})}{g^T(x_0)} = \frac{\mathbb{E}_{x_0}(1_A \mathbb{E}_{x_0}(1_{\tau_D > T} | \mathcal{F}_s))}{g^T(x_0)} \\ &\overset{\text{M.P.}}{=} \frac{\mathbb{E}_{x_0}(1_A \mathbb{E}_{x_0}(1_{\tau_D > T} | X_s))}{g^T(x_0)} \\ &= \frac{1}{g^T(x_0)}\left[\mathbb{E}_{x_0}(1_A \underbrace{\mathbb{E}_{x_0}(1_{\tau_D > T}1_{\tau_D > s} | X_s)}_{g^{T-s}(X_s)}) + \mathbb{E}_{x_0}(1_A \underbrace{\mathbb{E}_{x_0}(1_{\tau_D > T}1_{\tau_D < s} | X_s)}_{=0})\right] \\ &= \frac{1}{g^T(x_0)}\left(1_A g^{T-s}(X_{s\wedge\tau_D})\right) \end{aligned}$$

□

Lemma 2.10. *$Z_0^T = 1$ and $(Z_s^T)_{s\in[0,T]}$ is a martingale.*

Proof. By lemma 2.9 $\implies Z_s^T = \frac{d\mathbb{Q}^T}{d\mathbb{P}} |_{\mathcal{F}_s} \to$ is a martingale. □

Remark. *By construction, $\mathbb{Q}^T(\tau_D \leq T) = \frac{1}{g^T}(x_0)\mathbb{E}_{x_0}(1_{\tau_D \leq T}1_{\tau_D > t}) = 0$*

Assume that $g^t(x)$ is $C^{1,2}$ (1 in time, 2 in space) $\implies$ apply Itô and doob transform gives:

Proposition 2.11. *Let X be a weak solution of $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ under $\mathbb{P}$.*

$\implies$ *X is a weak solution of*

$$dX_t = \left(b(t,X_t) + \frac{a(t,X_t)\nabla g^{T-t}(X_t)}{g^{T-t}(X_t)}\right)dt + \sigma(t,X_t)d\tilde{B}_t, 0 \le t \le T$$

under $\mathbb{Q}^T$ provided $g^t(x) > 0 \forall x \in D, t \ge 0$.

What happens in the $T \to \infty$ limit?
By assumption (b)
$\lim_{T\to\infty} \frac{\nabla g^{T-t}(x)}{g^{T-t}(x)} = \frac{\nabla \varphi_0(x)}{\varphi_0(x)}$

Their operator is not the same as ours. The only difference is a drift term.

Remark. *If $g \in C^{1,2}$, it satisfies the parabolic PDE*

$$(\star) \begin{cases} \frac{\partial}{\partial t} g^t(x) = Lg^t(x) & x \in D \\ g^t(x) = 0 & x \in \partial D \end{cases}$$

Apply Theorem 11.5 from the WS with $A = L, u(t, x = g^t(x))$: $u(0,x) = \mathbb{E}_x(1_{\tau_D > 0}) = 0, u(t,x) = \mathbb{E}_x(1_{\tau_D > t}) = 0 \forall t, x \in \partial D$.

Pinsky proved that $\lim_{T\to\infty} Q^T = Q$ weak and that under Q the process satisfies

$$dX_t = \left[b(t,X_t) + a(t,X_t)\frac{\nabla \varphi_0(X_t)}{\varphi_0(X_t)}\right]dt + \sigma(t,X_t)dW_t.$$

Why is $\mathbb{P}_x(\tau_D > t) = C_1\varphi_0(x)e^{-\lambda_0 t} + o(e^{-\lambda_0 t})$

Added remark. *This is not the case for BM conditioned to stay in $D = (0,\infty)$. Reason: Spectrum of $-L$ is $\mathbb{R}_+$*

Since it satisfies the PDE $(\star)$, $g^t(x) = (e^{tL})g^0(x)$.
if the spectrum of $-L$ is discrete with eigenvalues $0 \le \lambda_0 < \lambda_1 \le \dots$, where there is a spectral gap between λ_0 and λ_1: $\exists$ eigenfunctions $\varphi_0(x), \varphi_1(x), \dots$, normalized as $\|\varphi_n\|_{L^2(D)} = 1 \implies +L\varphi_n(x) = -\lambda_n\varphi_n(x) \implies$ we can choose $\varphi_0, \varphi_1, \dots$ to be orthonormal: $(\varphi_i, \varphi_j)_{L^2(D)} = \delta_{ij}$.
$\implies 1 = \sum_{n\ge 0} \varphi_n\varphi_n^*$, since $\varphi_n\varphi_n^*$ is the projection onto the space generated by φ_m:

$$\varphi_n\varphi_n^* f = \varphi_n(\varphi_n, f)_{L^2(D)}$$

$$\begin{aligned} f(L)\varphi_n &= f(-\lambda_n)\varphi_n \\ g^t(x) &= (e^{tL})g^0(x) = e^{tL}1g^0(x) = \sum_{n\ge 0} e^{tL}\varphi_n(x)\varphi_n^* g^0 = \sum_{n\ge 0} e^{-\lambda nt}\varphi_n(x)(\varphi_m, g^0) \\ &= (\varphi_0, g^0)\varphi_0(x)e^{\lambda_0 t} + \underbrace{e^{-\lambda_0 t}\sum_{n\ge 0} e^{(\lambda_0-\lambda_n)t}\varphi_n(x)(\varphi_n, g^0)}_{o(e^{-\lambda_0 t})} \end{aligned}$$

Example 2.12. *One dimension.*
$L = \frac{1}{2}\frac{d^2}{dx^2}$ and $D = [0,R]$.
Solve $-L\varphi(n)(x) = \lambda_n\varphi_n(x)$ with $\varphi_n(x) = 0$ for $x \in \{0,R\}$.
Solutions $\varphi_n(x) = C_1 \sin\left(\frac{\pi x}{R}\cdot(n+1)\right)$ and $\lambda_n = \left(\frac{\pi(n+1)}{R}\right)^2 \implies \lambda_0 = \frac{\pi^2}{R^2}$.
For $x \in (0,R)$:

$$\mathbb{P}_x(\tau > t) \approx \tilde{C}_1\varphi_0(x)e^{-\frac{\pi^2 t}{R^2}}$$

$$\implies dX_t = \frac{\pi}{R}\frac{\cos(\frac{\pi X_t}{R})}{\sin(\frac{\pi X_t}{R})}d + dW_t$$

as $X_t \to 0$ (or $X_t \to \mathbb{R}$), drift $\approx \frac{1}{x_t}$ or $(\frac{-1}{R-X_t})$.
Also: formally $R \to \infty$ in the SDE, we get $dX_t = \frac{dt}{X_t} + dW_t$, which is the SDE we derived for BM conditioned to stay > 0 forever.

Start of lecture 10 (16.05.24)

Example 2.13 (Brownian Motion in a Weyl chamber)**.** *The **Weyl chamber** is defined as*

$$W^d = \{x \in \mathbb{R}^d \mid x_1 < x_2 < \cdots < x_d\}.$$

Given a d dim. Brownian motion $(B_t)_{t\geq 0}$ with $B_0 \in W^d$ what is the sde of this BM conditioned on staying in W^d forever.
What is the harmonic function vanishing at ∂W^d?

Lemma 2.14. *$h(x) := \prod_{1\leq k<l\leq d}(x_l - x_k)$ is harmonic and satisfies*

$$\frac{1}{2}\sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} h(x) = 0, x \in W^d$$

and $h(x) = 0$ for $x \in \partial W^d$, $h(x) > 0$ for $x \in W^d$.

Remark. $h(x) = \det(x_k^{l-1})$

Proof. Last two properties are clear. h is a polynomial, antisymmetric in each pair $x_l - x_k$ and has lowest possible power.
$\Delta h(x)$ is still antisymmetric, but with lower power $\implies \Delta h(x) = 0$. □

The BM conditioned to stay in the Weyl chamber will satisfy the SDE:

$$dX_t^k = \sum_{l\neq k} \frac{dt}{X_t^l - X_t^k} + dB_t^k, \qquad 1 \leq k \leq d$$

2.6 Stationary distribution for diffusions

let $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, where b, σ are time independent with generator

$$L = \sum_{k=1}^{d} b_k(x)\frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k,l=1}^{d} a_{k,l}(x)\frac{\partial^2}{\partial x_k \partial x_l}$$

Definition 2.15. *A probability measure μ **stationary (or invariant)** if for all $f \in C_0^\infty(\mathbb{R}^d)$*

or only test schwarzfunctions

$$\int_{\mathbb{R}^d} (Lf)(x)\mu(dx) = 0$$

Assume that $\mu \ll$ Lebesgue, i.e., $\mu(dx) = \rho(x)dx$ for some positive function $\rho(x) \in C^2$ with $\int \rho(x)dx = 1$.

Lemma 2.16. *μ is stationary with density ρ*

this also assumes b to be differentiable!

$$\iff L^*\rho(x) = 0 \text{ almost everywhere}$$

where L^* is the adjoint of L, given by

$$L^*\rho(x) = \frac{1}{2}\sum_{k,l=1}^{d} \frac{\partial^2}{\partial x_l \partial x_l}(a_{k,l(x)\rho(x)}) - \sum_{k=1}^{d} \frac{\partial}{\partial x_k}(b_k(x)\rho(x))$$

Proof.

$$\int_{\mathbb{R}^d}(Lf)(x)\rho(x)dx = 0$$
$$= \int_{\mathbb{R}^d} dx_1,\dots,dx_d\rho(x)\left(\frac{1}{2}\sum_{k,l=1}^{d}\frac{\partial^2}{\partial x_k\partial x_l}f(x)+\sum_{k=1}^{d}b_k(x)\frac{\partial}{\partial x_k}f(x)\right)$$

$$\int_{R^{d-1}}\prod_{l\neq k}dx_l \underbrace{\int_R dx_k\rho(x)b_k(x)\frac{\partial}{\partial x_k}f(x)}_{\overset{\text{I.b.p}}{=}-\int_R dx_k\frac{\partial}{\partial x_k}(\rho(x)b_k(x))\cdot f(x)}$$

□

Example 2.17 (1-dim. diffusion)**.** *Let $L = b(x)\frac{\partial}{\partial x}+\frac{1}{2}\sigma^2(x)\frac{\partial}{\partial x^2}$ for all $x \in\ R$.*

$$\implies L^*\rho(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\rho(x)\sigma^2(x)) - \frac{\partial}{\partial x}(\rho(x)b(x)) \overset{wanted}{=} 0$$

$\implies \exists c_1$ *s.t.* $\frac{1}{2}\frac{\partial}{\partial x}\underbrace{(\rho(x)\sigma^2(x))}_{=g(x)} - \underbrace{\rho(x)b(x)}_{=g(x)\frac{b(x)}{\sigma^2}} = c_1.$

*Let $s(x) = \int_{x_0}^x dye^{-\int_{x_0}^y dz\frac{2b(z)}{\sigma^2(z)}}$ be the **scale function**.*
The equation for c_1 is equivalent to

$$s'(x)\sigma^2(x)\rho(x) = c_2 + 2c_1s(x)$$

for some constant c_2,$s'(x)e^{-\int_{x_0}^y dz\frac{2b(z)}{\sigma^2(z)}} => 0$.
s satisfies $\frac{1}{2}\sigma^2(x)s''(x)+b(x)s'(x) = 0$, for the Bessel process $d = 2$, $s(x) = \ln(x)$
If $s(\mathbb{R}) = \mathbb{R}$, then $c_1 = 0$, which implies

$$\rho(x) = \frac{c_2}{\sigma^2(x)}\underbrace{e^{\int_{x_0}^x\frac{2b(y)}{\sigma^2(y)}dy}}_{\frac{1}{s'(x)}}$$

which satisfies positivity.

Lemma 2.18. *If $S(\mathbb{R}) = \mathbb{R}$, then there exists stationary measure with density $\rho(x)$ as described above.*

Counterexample:

- $\sigma = 1$, $x_0 = 0$, $b(z) = b > 0$.

$\implies s(x) = \frac{1-e^{-2bx}}{2b} \to s'(x) = e^{-2bx}$.

- $s(-\infty) = -\infty$
- $s(\infty) = \frac{1}{2b}$

$\implies$ Argument for $c_1 = 0$ does not work.

$$\implies \rho(x) = \tilde{c}_1 + \tilde{(c)}_2e^{2bx}$$

which can't be a density, because if at least $\tilde{c}_1$ or $\tilde{c}_2$ are $\neq 0 \implies$ it is not integrable.

Example 2.19. *Let $Lf(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}f(x)+b(x)\frac{\partial}{\partial x}f(x)$ where $b(x) = \frac{\partial}{\partial x}\ln h(x)$ for some $h(x) > 0$. Assume h is normalized as $\int_R(h(x))^2dx = 1$. $\implies$ **Claim:** The stationary density of the process with generator L is given by $\rho(x) = (h(x))^2$.*
Verify the claim:

Implicitly assumes $h \in L^2(\mathbb{R})$

$$\begin{aligned}L^*\rho(x) &\overset{?}{=} 0\\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x) - \frac{\partial}{\partial x}(b(x)\rho(x))\\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}(h(x))^2 - \frac{\partial}{\partial x}(\frac{h'(x)}{h(x)}h(x)^2)\\ &= \frac{\partial}{\partial x}(h(x)h'(x)) - \frac{\partial}{\partial x}(h'(x)h(x)) = 0\end{aligned}$$

Example **??** $\implies h(x) = c\sin(\frac{\pi x}{L})$

here $L \in \mathbb{R}$ is not the operator!

$$\begin{aligned}\int_0^L h(x)^2 dx = c^2\int_0^2 \sin^2(\frac{\pi x}{L})dx = 1\\ \implies c = \sqrt{\frac{2}{L}} \implies \rho(x) = \frac{2}{L}\sin^2(\frac{\pi x}{L}).\end{aligned}$$

2.7 Uniqueness in law and path integral formula

Consider the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + dB_t & \text{in } \mathbb{R}^d\\ x_0 = x_0\end{cases}$$

Assume

$$(\star)\forall T > 0 \int_0^T |b(s, X_s)|^2 ds < \infty \text{ a.s.}$$

.

Goal: Show uniqueness in law.
Consider any $(X, \mathcal{B}, \mathbb{P})$ weak solution satisfying $(\star)$ and define

$$\tau_n \coloneqq \inf\{t \geq 0 \mid \int_0^t |b(s, X_s)|^2 ds \geq n\}$$

which, by $(\star)$ goes to infinity.
For each n, define $\mathbb{Q}^n$:

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} = e^{\underbrace{-\int_0^{\tau_n} b(s, X_s)dB_s}_{L_{\tau_n}} - \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}.$$

By Girsanov: $\tilde{B}_t \coloneqq B_t - \langle B, L\rangle_t = B_t + \int_0^{t\wedge\tau_n} b(s, X_s)ds$ is BM w.r.t. $\mathbb{Q}^n$, which implies $X_t = \tilde{B}_t$ for all $t \leq \tau_n$ w.r.t. $\mathbb{Q}^n$.
Let events of (X, B) $\mathcal{F}_T$ measurable for some time T: are in A_T.

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(1_{(X,B)\in A_T}1_{T\leq\tau_n}\right) &= \mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,\mathcal{B})\in A_T}1_{T\leq\tau_n}e^{\int_0^{\tau_n} b(s,X_s)\underbrace{dB_s}_{=dX_s - b(s,X_s)ds} + \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)\\ &= \mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,B)\in A_T}1_{T\leq\tau_n}e^{\int_0^{\tau_n} b(s,X_s)dX_s - \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)\end{aligned}$$

B_s is adapted to $X_s \implies$ for some Φ: $B = \Phi(X) \implies$

$$\mathbb{E}_{\mathbb{Q}^n}\left(1_{(X,\Phi(X))\in A_T}1_{T\leq\tau_n}e^{\int_0^{\tau_n} b(s,X_s)dX_s - \frac{1}{2}\int_0^{\tau_n}|b(s,X_s)|^2 ds}\right)$$

But: Law of X w.r.t. $\mathbb{Q}^n$ is the law of BM. Take $n \to \infty$, then $\mathbb{Q}^n \to$ Wiener measure,
$T \to \infty \implies$ we look at every possible event, but the probability does only depend on X!

$$\mathbb{P}((X, B) \in \mathcal{B}(\mathcal{E}^d \times \mathcal{E}^d))$$

List of Lectures