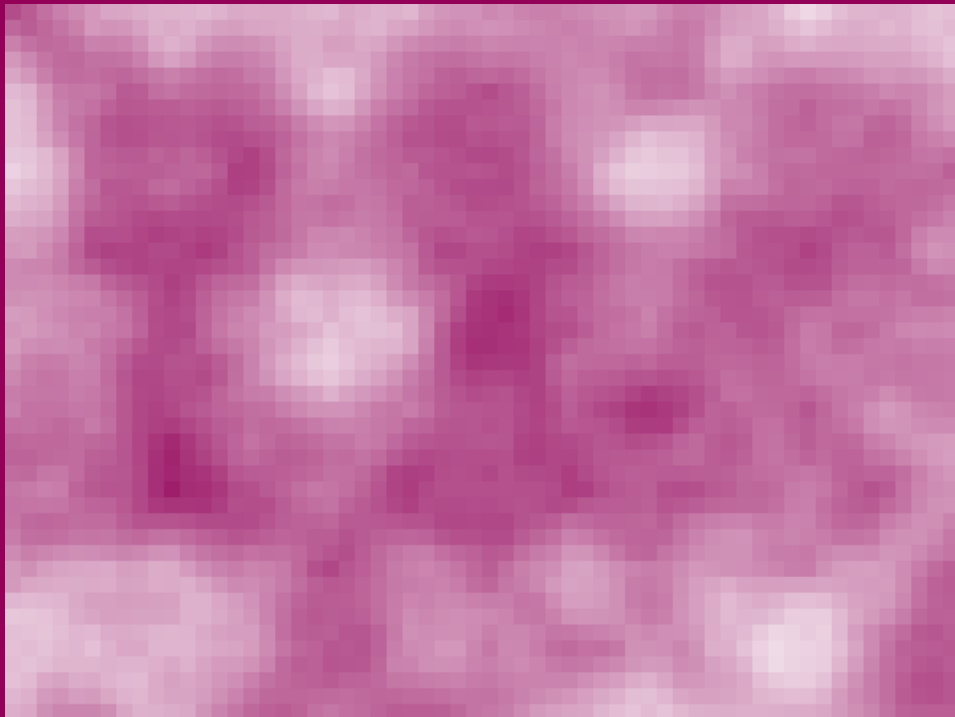# Lecture notes on Stochastic Analysis

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Based on the lectures of
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# Contents

# Chapter 0:
# Manuel's notes

> **Warning**
>
> These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
> Happy learning!

> **General Information**
>
> - Ecampus: Ecampus link
> - Basis: Basis link
> - Website: None
> - Time slot(s): Tuesday 12-14 and Thursday 12-14
> - Exams: Oral: 23-25 July, 30 Sept. maybe 27. Sept
> - Deadlines: ?
> - Exercises: One of Tue 16-18(for now this is preferred), Friday 10-12. Starting 16.04.

- First halve based on Eberle and / or Gubinelli ( be careful with Notation of dimensions!)

## Overview of the content

- Weak solutions of SDE
    · Martingale problem (characterization)
    · Time change (Dubin-Schwarz)
    · Change of measure (Girsonov)
- How to condition a diffusion to stay in a given domain forever (reflected BM, local time, ...)
- Lévy processes (a bit)
- Multiplicative stochastic heat-equation
    · relations with Kardar-Pavisi-Zhang class of growth models

# Chapter 1:
# Weak solutions of SDEs

<u>SDE:</u>

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \text{ \textbf{Initial condition}} \end{cases} \tag{1.1}$$

- $X_t \in \mathbb{R}^d, B : n\text{-dim BM}$

- $b(t, x) : [b_k(t, x)]_{1 \leq k \leq d}$: **drift vector**

- $\sigma(t, x) = [\sigma_{k,l}(t, x)]_{\substack{1 \leq k \leq d \\ 1 \leq l \leq n}}$: **dispersion matrix**

- $a(t, x) = \sigma(t, x) \cdot \sigma(t, x)^{\mathsf{T}}$: **diffusion matrix**

## 1.1  Strong solutions

**Definition 1.1.** *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_t = \sigma(x_0, (B_s)_{0 \leq s \leq t})$*

- *a d-dim process $X_t$ is a* ***strong solution*** *of equation 1.1 if:*

  · *$X_t = x_0$ a.s.*
  · *$X_t$ is adapted to $\mathcal{F}_t \forall t \geq 0$*
  · *$X$ is a continuous semimartingale s.t. $\forall t \geq 0$:*

$$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2) ds < \infty \text{ a.s.}$$

  · *$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$*

In the last semester we proved:

**Theorem 1.2.** *Assume that $b, \sigma$ are globally lipschitz with at most linear growth at $\infty$ (in space) $\implies \exists!$ strong solution of SDE.*  *Foundations of Stochastic Analysis Thm 8.6*

**Added remark.** *There exists $K > 0$ s.t. for all $x, y \in \mathbb{R}^d$: Globally Lipschitz:*

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|$$

*Linear growth condition:*
$$\|b(t, x)\| + \|\sigma(t, x)\| \leq K(1 + \|x\|)$$

**Remark.** *For strong solutions, $\mathcal{F}_t$ is given by the driving BM, wich is given to us.* *$\implies X_t(\omega) = \Phi(x_0(\omega), (B_s)_{0 \leq s \leq t})$*

## 1.2   Weak solutions

- For weak solutions we do not fix the driving brownian motion.

**Definition 1.3.** *A __weak solution__ of equation 1.1 is a __pair__ of adapted processes $(X, B)$ to a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ s.t.*

- *B is a n-dim BM*

- *X is a d-dim continuous semimartingale with*

  *1. $X_0 = x_0$ a.s.*
  *2. $\forall t \geq 0$*
  $$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2) ds < \infty \text{ a.s.}$$
  *3. $X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$*

**Remark.**    • *The filtration $(\mathcal{F}_t)_{t \geq 0}$ is not necessarily the one generated by B*

- *If X is adapted to the filtration generated by the BM $\implies$ we have strong solutions*

- *$\exists$ weak solution which are not strong solutions*

- *Warning: To give a weak solution we really need to provide $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X, B)$*

**Definition 1.4** (Uniqueness in law)**.** *An SDE 1.1 has __uniqueness in law__ if given any two weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, X, B)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{B})$ satisfy:*

$$Law_{\mathbb{P}}(X) = Law_{\tilde{\mathbb{P}}}(\tilde{X})$$

*They agree on any set in the sigma algebra*

**Definition 1.5** (Pathwise uniqueness)**.** *The SDE 1.1 has pathwise uniqueness if, whenever the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and $(B_t)_{t \geq 0}$ are fixed, then two solutions $X, \tilde{X}$ with $X_0 = \tilde{X}_0$ are indistinguishable.*

**Example 1.6** (No strong solutions, no pathwise uniqueness, $\exists$ weak solution & and uniqueness in law by Tanaka)**.**

$$\begin{cases} dX_t = sgn(X_t) dB_t \\ X_0 = 0 \end{cases} \tag{1.2}$$

*or more generally $X_0 = Y$, where*

$$sgn(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

*Let W be a BM with $W_0 = Y$. Define*

$$B_t := \int_0^t sgn(W_s) dW_s \text{ or } dB_t = sgn(W_t) dW_t$$

$$\implies dW_t = sgn(W_t) dB_t$$
$$\implies W_t = y + \int_0^t sgn(W_s) dB_s$$

*$B_t$ is a local martingale with*

$$\langle B \rangle_t = \int_0^t \underbrace{(sgn(W_s))^2}_{=1} \underbrace{d\langle W \rangle_s}_{=ds} = t$$

*Also $B_0 = 0$, therefore B is a BM (see Lévy characterization) $\implies$ W solves the SDE. For $Y = 0$, W **and** $-W$ solves the same SDE.*
$\implies$

- *exists weak solutions*

- *For $Y = 0$: no pathwise uniqueness*

- *Uniqueness in law (because the law is determined by $X_t$ being a BM)*

- *No strong solution, because: $\mathcal{F}^B = \mathcal{F}^{|W|} \subsetneq \mathcal{F}^W$*

*Not a proof just yet, just the reason!*

**Example 1.7** (No solutions)**.**

$$\begin{cases} dX_t = -\frac{1}{2X_t}1_{X_t \neq 0}dt + dB_t \\ X_0 = 0 \end{cases} \tag{1.3}$$

*Assume there exists a solution. Use Itô formula for $X_t^2$, then:*

$$X_t^2 = 2\int_0^t X_s dX_s + \int_0^t 1 ds$$

$$= -\int_0^t 1_{X_s \neq 0} ds + 2\int_0^t x_s dB_s + t$$

$$= \int_0^t 1_{X_s = 0} ds + 2\int_0^t X_s dB_s$$

*We will prove $\int_0^t 1_{X_s=0}ds = 0 \implies X_t^2$ is a local martingale, $X_t^2 \geq 0$ (and therefore a supermartingale) and $X_0 = 0$ ( $\implies \mathbb{E}(X_t^2) = 0$). If $X_t = 0 \implies \int_0^t 1_{X_s=0}ds = t \implies 0 = dB_t$ which are contradictions!*

**Remark.** *If $X = \underbrace{M}_{\in \mathcal{M}_{loc}} + \underbrace{A}_{\in \mathcal{A}}$*

$$\implies \forall b \in \mathbb{R} \int_0^t 1_{X_s=b}d\langle M\rangle_s = 0$$

# Sheet 0:

- Mail: Alexander.Becker@uni-bonn.de

- Exercise sheet handed in Fr 10 am via eCampus

- Groups of 3

Motivation:
in the last semester: Introduction to stochastic analysis

- Diffusion processes & SDEs

Here: Deepen the knowledge & have fun

- Learn SDE techniques and get other results

- Modify diffusion processes to behave in a certain way

- Ex. diffusion bridge (Brownian bridge)

- Condition a diffusion to stay in a domain

  - Ex. Condition BM to stay positive

  - Old SDE: $dB_t = dB_t$

- · New SDE: $dX_t = \frac{1}{X_t}dx + dB_t \to P(X_t \in \cdot) = P(B_t \in \cdot | B > 0 \text{ forever})$
- · $D \subset \mathbb{R}^d$ open domain, $X$ diffusion process, with generator $L = \Sigma b \partial_i + \frac{1}{2}\sum a^{ij}\partial_i\partial_j$

$$Y :- (X | X \in D \text{ forever})$$

get drift term $\nabla \log \phi_0$, where $\phi_0$ is the lowest eigenfuncion of $-L$ on $D$ with dirichlet boundary.

**Recap:**
Brownian motion:

**Added definition.** $B_0 = 0$, *independent* $\&$ $\mathcal{N}(0, t_i - t_{i-1})$ *increments,* $t \mapsto B_t(\omega)$ *continuous.*

Regularity of path $t \mapsto B_t(\omega)$:

- nowhere differentiable
- $\alpha-$locally Hölder continuous $\iff \alpha < \frac{1}{2}$
- Quadratic variation $\langle B \rangle_t = t$
- Generator $\frac{\Delta}{2}$
- Recurrent $\iff ds2$?

Itô-Integral:

1. If $X$ simple process $\implies$ RS-Integral
2. Itô isometry $\mathcal{E} \to \{L^2 - \text{local martingale}\}, \mathcal{E} \subset L^2(d[M])$ dense
3. general $X : \int X dM$ as $L^2$-limit

**Added remark** (Itô formula). •

$$df(t, X_t) = \partial_t f(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}f(t, X_t)d\langle X\rangle_t$$

- *associative* $\int X d(\int Y dZ) = \int XY dZ$
- *If $M$ local martingale* $\implies \int X dM$ *local martingale*

SDEs:
$DX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$

- ex./ uniqueness: $b, \sigma$ locally Lipschitz $\implies$ strong ex.+pathwise uniqueness until explosion time
- globally Lipschitz (in space) and linear growth $(|b(t, x)| + |\sigma(t, x)| \le C(1 + |x|))$, $e = \infty$.

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**Problem 00.1: SDE**

Let $B$ be a one-dimensional Brownian motion (starting from 0) and let $X_t = \sin(B_t)$.

1. Determine the SDE of $X_t$
2. Discuss the existence and/or uniqueness of strong solutions of the SDE
3. Determine whether the local martingale term is a martingale (Hint: for the integrability condition, think about the Itô isometry for instance)

### Solution 00.1

<u>1.:</u>
Idea: Use Itô formula: $X_t = \sin(B_t) = f(B_t)$

$$dX_t = df(B_t) \overset{\text{Itô}}{=} \partial_x \underbrace{\cos(B_t)}_{\sqrt{1-X_t^2}} dB_t - \frac{1}{2} \underbrace{\sin(B_t)}_{X_t} dt$$

Careful: $\sqrt{1-X_t^2}$ is not inverse mapping, because it is always positive while $\cos(B_t)$ is not

<u>2.:</u>
Consider the SDE

$$dX_t = \begin{cases} -\frac{1}{2}X_t dt + \sqrt{1-X_t^2} dB_t \\ X_0 = x \in (-1,1) \end{cases}$$

Coefficients: $b : [-1,1] \to \mathbb{R}, b(x) = -\frac{1}{2}x$ and $\sigma : [-1,1] \to \mathbb{R}. \sigma(x) = \sqrt{1-x^2}$

$$[\sigma]_{1,2} = \sup \frac{|\sigma(x) - \sigma(y)|}{|x-y|^{1/2}} < \infty$$

Probably $\sigma^2$ Lipschitz $\implies \sigma$ Hölder $\frac{1}{2}$
<u>3.:</u>

### Problem 00.2: Time change

Let $B$ be a one-dimensional Brownian motion (starting from 0). Let $Y_t = \int_0^t s^2 dB_s$.

1. Determine the SDE of $Y_t$

2. Find $A_t$ such that $Y_{A_t}$ is a (stopped) Brownian motion

### Solution 00.2

<u>1.:</u> By Definition
$$dY_t = t^2 dB_t$$

<u>2.:</u> We use the Dubin-Schwarz theorem:
$X_\in \mathcal{M}^0_{loc}, \langle X \rangle_\infty = \infty, T_t := \inf\{s \geq 0 | \langle X \rangle_s \geq t\} = X_t^{[-1]}$
$\implies B_t := X_{T_t}$ 1 d BM w.r.t. $(F_{T_t})_{t\geq 0}, X_t = B_{\langle B \rangle_t}$
here: use $X_t = b(X_t)dt + \sigma(X_t)dB_t \implies d[X]_t = \sigma^2(X_t)dt$

### Problem 00.3: SDE and PDE

Let $f$ be a function supported on $[0,1]$, $u$ the solution of

$$\frac{1}{2}u(t,x) = \frac{1}{2}x(1-x) + \frac{d^2}{dx^2}u(t,x), \qquad u(0,x) = f(x).$$

Consider also the one-dimensional SDE (also known as Wright-Fisher diffusion)

$$dX_t = \sqrt{X_t(1-X_t)dB_t}$$

with $X_0 = x \in (0,1)$.

1. For any fixed $t > 0$, define $M_s = u(t-s, X_s)$ for $s \in [0,t]$. Use Itô formula to show that $M_s$ is a local martingale

2. Assume that $f$ is bounded and there is a bounded solution of $u$. Show that $u(t,x) = \mathbb{E}_x(f(X_t))$.

> **Solution 00.3**
>
> <u>1.:</u>
>
> $dM_s = du(t - s, X_s)$
>
> $$= -\partial_s u(t - s, X_s)ds + \partial_x u(t - s, X_s)\underbrace{dX_s}_{b(X_s)ds+\sigma(X_s)dB_s \text{ by asso.}} + \frac{1}{2}\partial_x^2 u(t - s, X_s)\underbrace{d[X]_s}_{=\sigma^2(X_s)ds}$$
>
> $$= \underbrace{(-\partial_s u + b\partial_x u + \frac{1}{2}\sigma^2\partial_x^2 u)}_{=0}(t - s, X_s)ds + \partial_x u(t - s, X_s)\sigma(X_s)dB_s$$
>
> $$\implies dM_s = \partial_x u(t - s, X_s)\sigma(X_s)dB_s$$
>
> (i.e.: $M_t - M_0 = \int_0^t \dots dB_s$)
> This is a purely stochastic integral against a (local) martingale $\implies$ martingale.
> <u>2.:</u>
>
> - $M_s$ true martingale:
>
>   1. $\forall t : \mathbb{E}(\sup_{[0,t]} |M|) < \infty$, for example: $M$ bounded
>   2. $\forall t : \mathbb{E}(\sup_{[0,t]}[M]_t) < \infty$
>
> $M_s := u(t - s, X_s), u$ bounded $\implies$ $M$ bounded $\implies$ $M$ true martingale
> $w(s, x) := u(t - s, x)$
>
> $$u(t, x) = w(0, x) = \mathbb{E}_x[w(0, X_0)] \stackrel{\text{martingale}}{=} \mathbb{E}_x[w(t, X_t)] = \mathbb{E}_x[(u(0, X_t))] \stackrel{\text{PDE}}{=} \mathbb{E}_x[f(X_t)]$$
>
> Feynman-Kac formula (later more general).
> Also solvable by Kolmogorov Backward / forward equation

**Example 1.8** (Non-uniqueness of law, non pathwise uniqueness, with solutions)**.**

$$\begin{cases} dX_t &= 1_{X_z} dB_t \\ x_0 &= 0 \end{cases}$$

*Then*

$$X_t = 0 \forall t \geq 0$$

*and*

$$X_t = B_t \forall t \geq 0$$

*both are solutions:*

$$X_t - B_t = -\int_0^t 1_{X_s=0} dB_s \implies \langle X - B \rangle_t = \int_0^t 1_{X_s=0} d\langle B \rangle_s = 0$$

$X_t = \int_0^t 1_{X_s \neq 0} dB_s$
$= \int_0^t (1 - 1_{X_s=0} dB_s)$
$B_t - \int_0^t 1_{X_s=0} dB_s$

*Let $\eta \sim Ber\left(\frac{1}{2}\right)$ independent of $(B_t)_{t\geq 0}$ and define*

$$\tilde{X}_t = \begin{cases} 0 & \eta = 0 \\ B_t & \eta = 1 \end{cases}$$

$\implies \tilde{X}_t$ *is adapted to $\sigma(\eta(B_s)_{0\leq s\leq t})$, but not to $\sigma((B_s)_{0\leq s\leq t})$ and therefore not a* *strong solution.*

**Example 1.9** (No strong solution, weak solution, no pathwise uniqueness, no uniqueness in law)**.**

$$\begin{cases} dX_t & = 1_{X_t \neq 1} sgn(X_t) dB_t \\ X_0 & = 0 \end{cases}$$

*Let $Y_t$ be a solution of*

$$\begin{cases} dY_t & = sgn(Y_t) dB_t \\ Y_0 & = 0 \end{cases}$$

$\implies X_t := Y_{t \wedge \tau}$, where $\tau = \inf\{s \geq 0 \mid Y_s = 1\}$ is also a solution.

**Theorem 1.10** (Yamada-Watanabe)**.** *If both existence of weak solutions and pathwise uniqueness hold, uniqueness in law also holds.*
*Moreover, $\forall$ choices of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and $(B_t)_{t \geq 0}$ then there exists a strong solution.*

## 1.3 Lévy characterization

**Example 1.11.** *Consider the SDE*

$$\begin{cases} dX_t & = O_t dB_t \\ X_0 = x_0 \end{cases}$$

*where both $X_t$ and $B_t$ are d-dimensional and $O_t$ is an adapted process (matrix) s.t. $O_t^\intercal O_t = 1 \forall t \geq 0$   i.e. $O_t$ is a rotation*

$$\implies X_t^k = X_0^k + \sum_{l=1}^d \int_0^t O_s^{k,l} dB_s^l$$

$$\implies \langle X^k, X^{\tilde{k}} \rangle_t = \sum_{l,\tilde{l}}^d \int_0^t O_s^{k,l} O_s^{\tilde{k},\tilde{l}} d\langle B^l, B^{\tilde{l}} \rangle_s = \sum_{l=1}^d \int_0^t O_s^{k,l} O_s^{\tilde{k},l} ds$$

$$= \int_0^t \underbrace{(O_s O_s^\intercal)^{k,\tilde{k}}}_{=1} ds = \delta_{k,\tilde{k}} t \stackrel{L\acute{e}vy}{\implies} (X_t)_{t \geq 0} \text{ is also a d-dim BM starting from } x_0$$

**Theorem 1.12** (Yamada,Watanabe)**.** *Let $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ and assume that there exist both a increasing function $\rho(u) \geq 0$ s.t. $|\sigma(x) - \sigma(y)| \leq \rho(|x-y|) \forall x,y \in \mathbb{R}$ s.t.*

$$\int_0^\infty \frac{1}{\rho(u)} du = \infty \implies 0 < u < C_1 \implies \rho(u) \leq C_2 u^{0.5}$$

*and some increasing concave function $\gamma_1(u) \geq 0$ s.t.*

$$|b(x) - b(y)| \leq \gamma_1(|x-y|) \forall x,y \in \mathbb{R}$$

*and*

$$\int_0^\infty \frac{1}{\gamma_1(u)} = \infty.$$

*Then pathwise uniqueness holds.*

**Theorem 1.13** (Storokhod)**.** *Assume that $\sigma, b$ are continuous bounded functions*
$\implies$ *there exist weak solutions to the SDE $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.*

## 1.4   Weak solutions and martingale problems

Let $(X_t)_{t\geq 0}$ be a weka solution of the SDE

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 &= 0 \end{cases}$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

$\implies X_t$ is a semimartingale s.t. $X_t^k = X_0^k + \int_0^t b(s, X_s)ds + \sum_{l=1}^n \int_0^t \sigma_{k,l}(s, X_s)dB_s^l$ and

$$\langle X^i, X^j \rangle_t = \int_0^t \underbrace{\sum_{l=1}^n \sigma_{i,l}(s, X_s)\sigma_{j,l}^\intercal(s, X_s)}_{=a_{i,j}(s,X_s)} ds$$

### 1.4.1   Itô-Doeblin formula

Itô formula leads to

**Proposition 1.14.** *For $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, then*

$$f(t, X_t) = f(0, X_0) + \int_0^t (\sigma^\intercal \nabla f)(s, X_s)dB_s + \int_0^t \left[\left(\frac{\partial}{\partial s} + \mathcal{L}\right)\right](s, x_s)ds$$

*where $(\mathcal{L}f)(t, x) = \frac{1}{2}\sum_{k,l=1}^n a_{k,l}(t, x)\frac{\partial^2}{\partial x_k \partial x_l}f(t, x) + \sum_{k=1}^n b_k(t, x)\frac{\partial}{\partial x_k}f(t, x)$.*      $\mathcal{L}$ *is called the **generator***

**Remark.** *The Itô-Doeblin formula provides a connection between SDEs and PDEs as we have seen in the last part **Foundations of stochastic analysis**.*

**Example 1.15.** *Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}_d)$ be a solution of the PDE*

$$\frac{\partial}{\partial t}f(t, x) + (\mathcal{L}f)(t, x) = -g(t, x) \qquad\qquad t \geq 0, x \in U \subseteq \mathbb{R}^d$$

$$f(t, x) = \varphi(t, x) \qquad\qquad t \geq 0, x \in \partial U.$$

*then $M_t := f(t, X_t) + \int_0^t g(s, X_s)ds \in \mathcal{M}_{loc}$ by proposition 1.14 and if $f, g$ are bounded $M_t \in \mathcal{M}$.*

$$T := \inf\{s \geq 0 | X_s \notin U\} \implies M_t^T := M_{T\wedge t} \in \mathcal{M}.$$

*Furthermore, if we assume $T < \infty$ a.s. :*

$$\mathbb{E}[M_t] = \mathbb{E}[\varphi(T, X_T)] + \mathbb{E}[\int_0^T g(s, X_s)ds] = \varphi(0, X_0)$$

*where $X_0 = x_0$.*
*There are two special cases:*
*$g = 0 \implies$ yields the exit distributions, while*
*$\varphi = 0, g = 1$ yields the mean exit times.*

**Example 1.16** (Feynman-Kac formula)**.** *Let $t \in \mathbb{R}_+$ be finite. Assume $f : \mathbb{R}^d \to \mathbb{R}$ and $K : [0, t] \times \mathbb{R}^d \to \mathbb{R}_+$ be continuous functions. Assume that $u$ is a $C^{1,2}$-solution (bounded,for simplicity) of*

$$\begin{cases} \frac{\partial u}{\partial s}(s, x) = \frac{1}{2}\Delta u(s, x) - K(s, x)u(s, x) & s \in [0, t], x \in \mathbb{R}^d \\ u(0, x) = f(x) & x \in \mathbb{R}^d \end{cases}$$

*Then $u$ has the stochastic representation $u(t, x) = \mathbb{E}_x\left[f(X_t)\exp\left(-\int_0^t K(t - s, X_s)ds\right)\right]$, where $X_t$ is BM starting from $X_0 = x$.*

*Sketch.*     1. Define $r(s, x) := u(t - s, x)$ for $s \in [0, t]$

2. Show: $M_s := \exp(-A_s)r(s,x)$ with $A_s = \int_0^s K(u, X_u)du$ is a local martingale.

$\square$

**Remark.** *This is a reformulation of the formula from the last semester.*

### 1.4.2 Martingale problem

A solution of an SDE is generically defined up to some **explosion time** $\xi$, where it either diverges or it exists a given domain $U \subset \mathbb{R}^d$ (open).

$\implies$ For $k \in \mathbb{N}$ define $U_k := \{x \in U \mid |x| < k \wedge \operatorname{dist}(x, U^c) \geq \frac{1}{k}\}$ with $U = \bigcup_{k \geq 1} U_k$ and

$$T_k := \{t \geq 0 \mid x_t \notin U_k\}.$$

A solution of the SDE $b(t, X_t)dt + \sigma(t, X_t)dB_t$ is defined up to $\xi = \sup_{k \geq 1} T_k$.

Start of lecture 03 (18.04.24)

**Added remark.** *uniqueness of solution to the heat equation $\frac{1}{2}\Delta u - Ku$: not unique! Stochastic calculus: Paolo Baldi: 10.3-10.4*

Define $(\star)$:

$$dX_t b(t, X_t)dt + \sigma(t, X_t)dB_t \text{ with } X_0 = x_0$$

**Theorem 1.17** (Martingale problem). *If $X$ is a solution of $(\star)$ up to time $\zeta$, then $\forall f \in C^{1,2}(\mathbb{R}_+ \times U)$*

$$M_t = f(t, X_t) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right) f(s, x_s)ds, t < s$$

*is a local martingale up to $\zeta$ and $M_t^{T_k}$ are localizing martingales.*

**Definition 1.18** (Martingale solutions). *$(X_t)_{t \geq 0}$ is a **martingale solution** of $(\star)$ if $\forall f \in C^2(\mathbb{R}^d)$*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$$

*is a continuous local martingale.*

**Theorem 1.19** (Equivalent definitions). *The following are equivalent (for $X_t$ being a solution of $(\star)$):*

*(a)* $\forall f \in C^2(\mathbb{R}^d)$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)ds$$

*is a local martingale*

*(b) The process in $\mathbb{R}^d$ given by*

$$M_t := X_t - X_0 - \int_0^t b(s, X_s)ds$$

*is a $d$-dimensional local martingale with $\langle M^i, M^j \rangle_t = \int_0^t a_{i,j}(s, X_s)ds = \langle X^i, X^j \rangle_t$*

*(c)* $\forall f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$

$$\tilde{M}_t^f := f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \mathcal{L}\right)(s, X - s)ds$$

*is a local martingale*

*ds*

*Proof.* <u>$c \implies a$</u>: by choosing $f$ independent of $t$.
<u>$a \implies b$</u>: 1.: Choosing $f(X) = X_i$ implies

We can maybe also proof
this by calculating $X^2$?

$$M_t^f = X_t^i - X_0^i - \int_0^t b_i(s, X_s)ds \in \mathcal{M}_{\text{loc}}$$

2.: $f(X) = X_i X_j$:

$$(\mathcal{L}f)(x) = \frac{1}{2}a_{ij} + \frac{1}{2}a_{ji} + b_i X_j b_j X_i$$
$$a = a^\intercal \implies = a_{ij}b_i X_j b_j X_i$$
$$\implies M_t^f X_t^i X_t^j - X_0^i X_0^j - \int_0^t \left[ a_{ij}(s, X_s) + b_i(s, X_s)X_s^j + b_j(s, X_s)X_s^i \right] ds$$

Here $dX_s^i$ is the same as
$b_i X_s^j$ is the same up to a
local martingale term and
$\langle X^i, X^j \rangle_t =$
$\int_0^t a_{ij}(s, X_s)ds =$
$\langle M^i, M^j \rangle_t$

$$X_t^i X_t^j - X_0^i X_0^j \overset{\text{Integration by parts}}{=} \int_0^t X_s^i dX_s^j + \int_0^t X - s^j dX_s^i + \langle X^i, X^j \rangle_t$$
$$= M_t^f + \int_0^t [a_{ij} + b_i X_s^j + b_j X_s^i]ds$$

<u>$b \implies c$</u>: By Proposition 1.14 If (use the next theorem) $X$ was a weak solution $\implies \tilde{M}_t^f$ is a local martingale. $\qquad\square$

**Theorem 1.20.** *Let $n = d$, assume $\sigma(t, x)$ is invertible $\forall t, x$ and $\sigma^{-1}(t, x)$ is uniformly bounded.*

*(a) $(X_t)_{t \geq 0}$ is a weak solution of the SDE $(\star)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}; B)$*

*(b) $(X_t)_{t \geq 0}$ is a martingale solution of the SDE $(\star)$*

*This also works for $n \neq d$,
but with a different proof*

*Proof.* $a \implies b$: True
$b \implies a$: Goal construct a BM for the weak solution.
By proposition **??** $a \implies b$ $dM_t = dX_t - b(t, X_t)dt \in \mathcal{M}_{\text{loc}}$ and $d\langle M^i, M^j \rangle_t = a_{k,l}(t, X_t)dt$

$$\implies dX_t = dM_t + b(t, X_t)dt$$
$$= \sigma(t, X_t)d\tilde{B}_t + b(t, X - t)dt$$

where $\tilde{B}_t := \sigma(s, X_s)^{-1}dM_s$
To see: $\tilde{B}_t$ is a brownian motion.

$$\langle \tilde{B}^i, \tilde{B}^j \rangle_t = \sum_{k,l} \int_0^t \sigma_{ij}^{-1}\sigma_{jl}^{-1} \underbrace{d\langle M^k, M^l \rangle_s}_{\substack{= a_{ij} \\ (\sigma^\intercal \sigma)_{kl}}} ds$$
$$= \sum_{k,l,p} \int_0^t \sigma_{ik}^{-\intercal}\sigma_{kp}\sigma_{pl}^{\intercal}\sigma_{lj}^{-\intercal}ds$$
$$= \delta_{ij} \int_0^t 1ds = \delta_{ij}t$$

Then by the Lévy characterization $\tilde{B}$ is a brownian motion. $\qquad\square$

**Added remark.** *This is the first way to construct a weak solution: Solve a martingale problem!
This is used a lot in practice.*

## 1.5   Weak solutions and time change

### 1.5.1   Time change

For $d = 1$:

**Theorem 1.21.** *[Dubins-Schwarz]*

- *Let $M \in \mathcal{M}_{loc}^0$ and $\langle M \rangle_\infty = \infty$ a.s.*

- *Let $T_t := \inf\{s \geq |\langle M \rangle_s \geq t\}$*

*This implies*

1. *$t \mapsto M_{T_t}$ os a $(\mathcal{F}_{T_t})$ brownian motion*

2. *$M_t = B_{\langle M \rangle_t}$ for some standard brownian motion $B$*

$X_t = X_0 + \int_0^t b(s, X_s)ds + \underbrace{\int_0^t \sigma(s, X_s)dB_s}_{=M_t}$. If $\langle M \rangle_\infty = \infty$ a.s.:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \tilde{B}_{\int_0^t \sigma^2(s, X_s)ds}$$

### 1.5.2   Time change in a martingale problem

Consider $d = 1 = n$.

$$dY_t = \sigma(Y_t)dB_t \qquad (\star\star)$$

and $\sigma$ strictly positive positive continuous function.

$$\langle Y \rangle_t = \int_0^t \sigma(Y_s)^2 ds =: A_t$$

By theorem 1.21 $\implies Y_t = W_{A_t}$ for some brownian motion $W$.
Assume $A_t \infty = \infty$ a.s.
$T_t := \inf\{s \geq 0 \mid \langle Y \rangle_s \geq t\}$

$$\implies T_{A_t} = \inf\{s \geq 0 \mid \langle Y \rangle_s \geq \langle Y \rangle_t\} = t$$

$$1 = \frac{d}{dt}(T_{A_t}) = T'_{\underbrace{A_t}_{=u}} \cdot A_t$$

$$\implies \underbrace{T'_u}_{=\frac{dT_u}{du}} = \frac{1}{A'_{T_u}} \implies T_u = \int_0^u \frac{1}{\sigma(Y_{T_s})^2}ds = \int_0^u \frac{1}{\sigma(W_s)^2}ds$$

$\implies$ to construct a solution of $(\star\star)$: Given $W \longrightarrow$ compute $T_u \longrightarrow$ determine
$A - t = T_t^{-1} \implies Y_t = W_{A_t}$