Lecture notes on Stochastic Analysis

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Based on the lectures of
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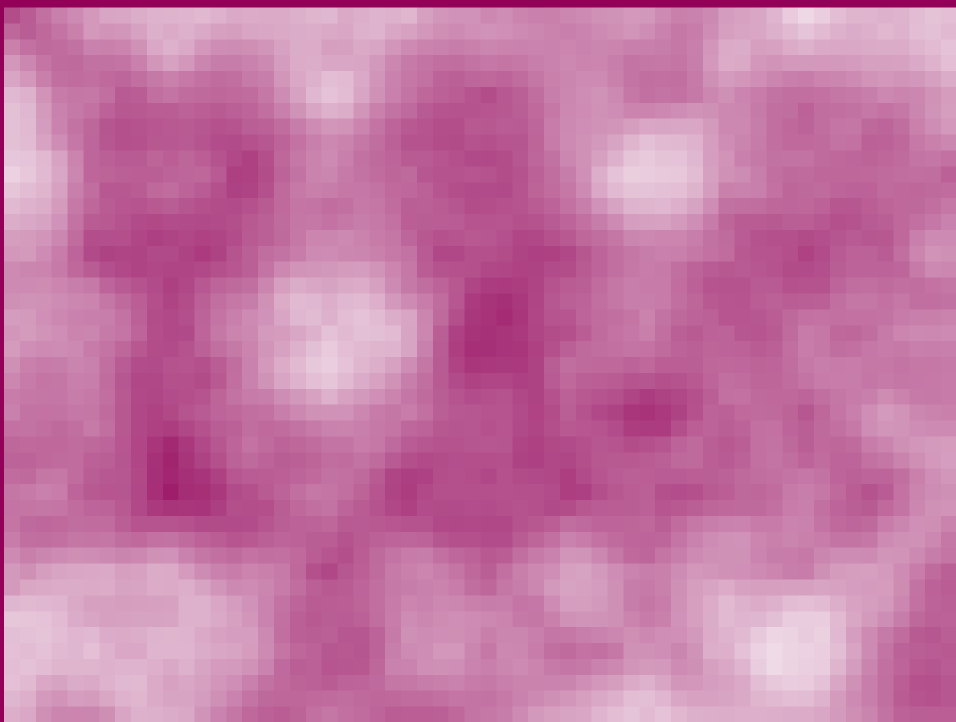

University of Bonn
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Manuel's notes

Warning

These are unofficial lecture notes written by a student. They are messy, will almost surely contain errors, typos and misunderstandings and may not be kept up to date! I do however try my best and use these notes to prepare for my exams. Feel free to email me any corrections to mh@mssh.dev or s6mlhinz@uni-bonn.de.
Happy learning!

General Information

- Ecampus: Ecampus link
- Basis: Basis link
- Website: None
- Time slot(s): Tuesday 12-14 and Thursday 12-14
- Exams: Oral: 23-25 July, 30 Sept. maybe 27. Sept
- Deadlines: ?
- Exercises: One of Tue 16-18(for now this is preferred), Friday 10-12. Starting 16.04.

- First halve based on Eberle and / or Gubinelli (be careful with Notation of dimensions!) Start of lecture 01 (11.04.23)

Overview of the content

- Weak solutions of SDE
  - Martingale problem (characterization)
  - Time change (Dubin-Schwarz)
  - Change of measure (Girsonov)
- How to condition a diffusion to stay in a given domain forever (reflected BM, local time, ...)
- Lévy processes (a bit)
- Multiplicative stochastic heat-equation
  - relations with Kardar-Pavisi-Zhang class of growth models

Chapter 1:
Weak solutions of SDEs

<u>SDE:</u>

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \\ X_0 = x_0 \ \textbf{Initial condition} \end{cases} \tag{1.1}$$

- $X_t \in \mathbb{R}^d, B:$ n-dim BM
- $b(t, x) : [b_k(t, x)]_{1 \leq k \leq d}$: **drift vector**
- $\sigma(t, x) = [\sigma_{k,l}(t, x)]_{\substack{1 \leq k \leq d \\ 1 \leq l \leq n}}$: **dispersion matrix**
- $a(t, x) = \sigma(t, x) \cdot \sigma(t, x)^\intercal$: **diffusion matrix**

1.1 Strong solutions

Definition 1.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ be a filtered probability space with $\mathcal{F}_t = \sigma(x_0, (B_s)_{0 \leq s \leq t})$*

- *a d-dim process X_t is a* ***strong solution*** *of equation 1.1 if:*
  - $X_t = x_0$ *a.s.*
  - X_t *is adapted to* $\mathcal{F}_t \forall t \geq 0$
  - X *is a continuous semimartingale s.t.* $\forall t \geq 0$*:*

$$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2)ds < \infty \ a.s.$$

  - $X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$

In the last semester we proved:

Theorem 1.2. *Assume that b, σ are globally lipschitz with at most linear growth at ∞ (in space) $\implies \exists!$ strong solution of SDE.*

Foundations of Stochastic Analysis Thm 8.6

Added remark. *There exists $K > 0$ s.t. for all $x, y \in \mathbb{R}^d$: Globally Lipschitz:*

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|$$

Linear growth condition:

$$\|b(t, x)\| + \|\sigma(t, x)\| \leq K(1 + \|x\|)$$

Remark. *For strong solutions, $\mathcal{F}_t$ is given by the driving BM, wich is given to us.*
$\implies X_t(\omega) = \Phi(x_0(\omega), (B_s)_{0 \leq s \leq t})$

1.2 Weak solutions

- For weak solutions we do not fix the driving brownian motion.

Definition 1.3. *A* ***weak solution*** *of equation 1.1 is a* ***pair*** *of adapted processes* (X, B) *to a* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *s.t.*

- *B is a n-dim BM*
- *X is a d-dim continuous semimartingale with*
    1. $X_0 = x_0$ *a.s.*
    2. $\forall t \geq 0$
    $$\int_0^t (\|b(s, X_t)\| + \|\sigma(t, X_t)|^2)ds < \infty \text{ a.s.}$$
    3. $X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$

Remark.
- *The filtration* $(\mathcal{F}_t)_{t\geq 0}$ *is not necessarily the one generated by B*
- *If X is adapted to the filtration generated by the BM* $\implies$ *we have strong solutions*
- $\exists$ *weak solution which are not strong solutions*
- *Warning: To give a weak solution we really need to provide* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, B)$

Definition 1.4 (Uniqueness in law)**.** *An SDE 1.1 has* ***uniqueness in law*** *if given any two weak solutions* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, X, B)$ *and* $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{X}, \tilde{B})$ *satisfy:*

$$Law_{\mathbb{P}}(X) = Law_{\tilde{\mathbb{P}}}(\tilde{X})$$

They agree on any set in the sigma algebra

Definition 1.5 (Pathwise uniqueness)**.** *The SDE 1.1 has pathwise uniqueness if, whenever the filtered space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *and* $(B_t)_{t\geq 0}$ *are fixed, then two solutions* $X, \tilde{X}$ *with* $X_0 = \tilde{X}_0$ *are indistinguishable.*

Example 1.6 (No strong solutions, no pathwise uniqueness, $\exists$ weak solution & and uniqueness in law by Tanaka)**.**

$$\begin{cases} dX_t = sgn(X_t)dB_t \\ X_0 = 0 \end{cases} \tag{1.2}$$

or more generally $X_0 = Y$, *where*

$$sgn(x) = \begin{cases} 1 & x > 0 \\ -1 x \leq 0 \end{cases}$$

Let W be a BM with $W_0 = Y$. *Define*

$$B_t := \int_0^t sgn(W_s)dW_s \text{ or } dB_t = sgn(W_t)dW_t$$

$$\implies dW_t = sgn(W_t)dB_t$$
$$\implies W_t = y + \int_0^t sgn(W_s)dB_s$$

B_t *is a local martingale with*

$$\langle B\rangle_t = \int_0^t \underbrace{(sgn(W_s))^2}_{=1} \underbrace{d\langle W\rangle_s}_{=ds} = t$$

Also $B_0 = 0$, therefore B is a BM (see Lévy characterization) $\implies W$ solves the SDE. For $Y = 0$, W ***and*** *$-W$ solves the same SDE.*
$\implies$

- *exists weak solutions*
- *For $Y = 0$: no pathwise uniqueness*
- *Uniqueness in law (because the law is determined by X_t being a BM)*
- *No strong solution, because: $\mathcal{F}^B = \mathcal{F}^{|W|} \subsetneq \mathcal{F}^W$*

Not a proof just yet, just the reason!

Example 1.7 (No solutions)**.**

$$\begin{cases} dX_t = -\frac{1}{2X_t} 1_{X_t \neq 0} dt + dB_t \\ X_0 = 0 \end{cases} \tag{1.3}$$

Assume there exists a solution. Use Itô formula for X_t^2, then:

$$\begin{aligned} X_t^2 &= 2\int_0^t X_s dX_s + \int_0^t 1 ds \\ &= -\int_0^t 1_{X_s \neq 0} ds + 2\int_0^t x_s dB_s + t \\ &= \int_0^t 1_{X_s = 0} ds + 2\int_0^t X_s dB_s \end{aligned}$$

We will prove $\int_0^t 1_{X_s=0} ds = 0 \implies X_t^2$ is a local martingale, $X_t^2 \geq 0$ (and therefore a supermartingale) and $X_0 = 0$ ($\implies \mathbb{E}(X_t^2) = 0$). If $X_t = 0 \implies \int_0^t 1_{X_s=0} ds = t \implies 0 = dB_t$ which are contradictions!

Remark. *If $X = \underbrace{M}_{\in \mathcal{M}_{loc}} + \underbrace{A}_{\in \mathcal{A}}$*

$$\implies \forall b \in \mathbb{R} \int_0^t 1_{X_s = b} d\langle M \rangle_s = 0$$