Monotone decomposition of functions of bounded Variation

Manuel Hinz

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Introduction

In my seminar talk about functions of bounded variations I talked about the following theorems:

Theorem 1. *A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if there exist two monotone functions $f_1, f_2 : [a, b] \to \mathbb{R}$ s.t.*

$$f = f_1 - f_2.$$

If f is continuous we can require that both f_1 and f_2 are continuous.

Remark. *$f_1 \coloneqq V_a^x(f)$ and $f_2 = f_1 - f$ are used in the (constructive) proof of this theorem.*

Theorem 2. *$f \in \mathcal{L}^1[a, b]$ and $F(x) \coloneqq \int_a^x f(t)dt$ implies $F \in BV[a, b]$ and*

$$V_a^b F = \int_a^b |f(t)|dt.$$

Examples

sin

$$f_1(x) = V_a^x(\cos)$$

Therefore:

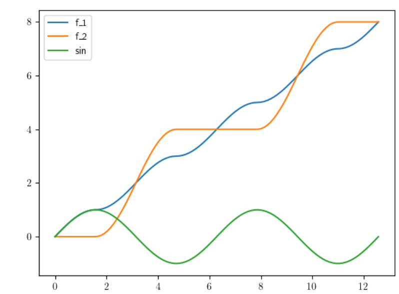

Figure 1: A monotone decomposition of sin

Polynomials

Let $P(x) := x^3 - 6x^2 + 4x + 12$. Therefore $P'(x) = 3 \cdot x^2 - 12 \cdot x + 4$:

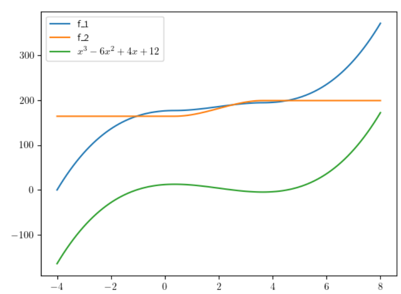

Figure 2: A monotone decomposition of P

Code

You can find the code used to generate the plots **here** . You can also produce your own plot using **generate_plot**. generate_plot needs both a function and the derivative of the function to work.