VELOCITY RESONANCE

The displacement of a driven oscillation is given by

$$x = A \sin(pt - \theta) \rightarrow (1)$$

Substitute for A

$$x = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} \sin(pt - \theta)$$

$\therefore$ The velocity of the driven oscillator at that instant is

$$v = \frac{dx}{dt} = pA\cos(pt - \theta) \rightarrow (2).$$

$$v = \frac{f_0}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}} p\cos(pt - \theta)$$

$$\left\langle v = v_0 \sin\left(\underbrace{pt - \theta}_{} + \pi/2\right)\right\rangle \rightarrow (3)$$

$\sin(\pi/2 + \theta) = \cos\theta$

Sin is +ve in 2nd quadrant

where $v_0 = \frac{f_0 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4k^2p^2}}$ & $\theta = \tan^{-1}\left(\frac{2kp}{\omega_0^2 - p^2}\right)$

v_0 is known as velocity Amplitude

Thus the velocity Amplitude v_0 varies with p.

When $p = 0$, $v = 0$. & maximum when $p = \omega_0$. Hence at the frequency $p = \omega_0$. (driving frequency is equal to the natural frequency). of the impressed force, the velocity has the maximum value & we call it as velocity Resonance.

At velocity Resonance $v_0 = \frac{f_0 \omega_0}{\sqrt{(\omega_0^2 - \omega_0^2)^2 + 4k^2\omega_0^2}}$

$$v_0 = \frac{f_0}{2k} = f_0 \tau$$

where $\tau = \frac{1}{2k}$

$\theta = \tan^{-1} \frac{2kp}{(\omega_0^2 - \omega_0^2)} = \pi/2$

$\angle \theta = \pi/2 \rangle$

$\theta = \tan^{-1} \frac{2kp}{(\omega_0^2 - p^2)}$ obtained by

~~f#o~~ derivation by dividing

$2kpA = f_0 \sin\theta$

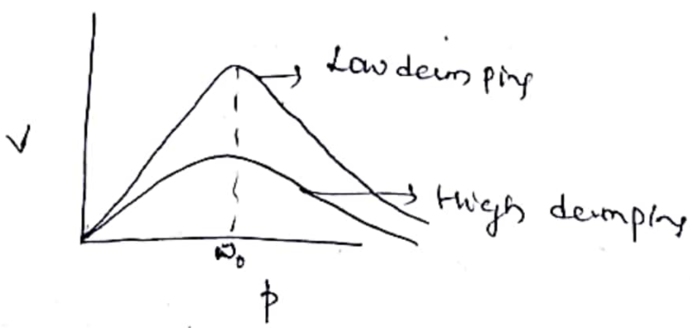
$(\omega_0^2 - p^2)A = f_0 \cos\theta$

consider eq^n (3) $V = v_0 \sin(pt \underbrace{- \theta + \pi/2})$

$\therefore$ velocity phase constant $= -\theta + \pi/2 = -\pi/2 + \pi/2 = 0$

Now, as we know the displacement at Resonance lags behind in phase by $\pi/2$ behind the driving force &, as we have just seen velocity then leads the displacement ~~in phase~~ in phase by $\pi/2$.

Therefore at Resonance, the velocity of driving oscillation is in phase with the driving force. this is the most favourable situation for transfer of energy from the applied force to the oscillator, because the rate of work done on the oscillator by the impressed force is FV which is always +ve for F & v in phase.



when $p \gtrless \omega_0$, the velocity amplitude is smaller than $p = \omega_0$.

Sharpness of Resonance

At Resonance the amplitude of the Oscillating system becomes maximum. It decreases from their maximum value with change in frequency of Impressed force. The Term Sharpness of resonance refers to rate of fall of amplitude with change in forced frequency on either side of Resonant frequency

The Rate at which the Amplitude changes corresponding to small change in frequency of Applied external force

change in Amp $= A_{max} - A$

$$= \frac{f_0}{2k\omega} - \frac{f_0}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}}$$

$\omega^2 - p^2 = 0$ beczq $\omega \simeq p$ small change in Applied freq

$$= \frac{f_0}{2k\omega} - \frac{f_0}{2kp}$$

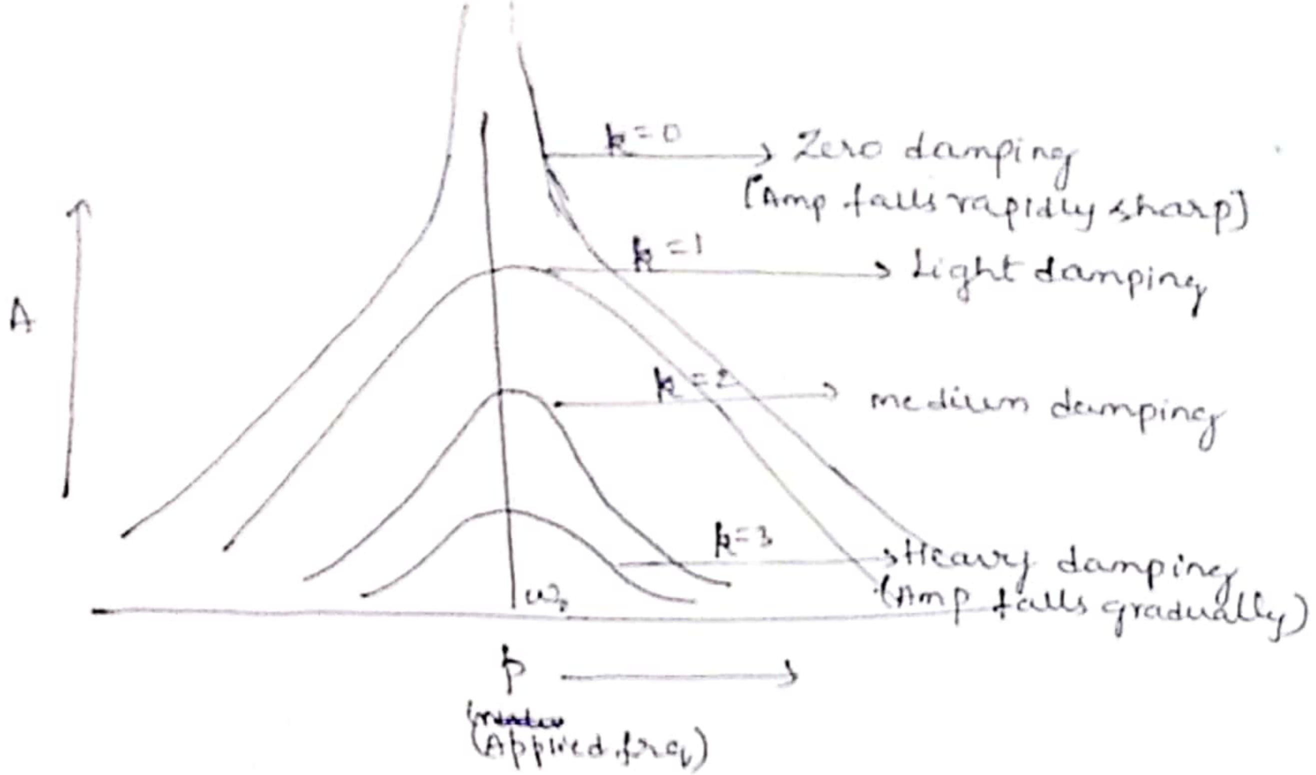
$$= \frac{f_0}{2k\omega p}(p-\omega)$$

$$\text{sharpness} = \frac{\text{change in Amp}}{\text{change in freq of Applied force}}$$

$$= \frac{\frac{f_0}{2k\omega p}\cancel{(p-\omega)}}{\cancel{(p-\omega)}}$$

$$\text{sharpness} = \frac{f_0}{2k\omega p}$$

Sharpness is inversly proportional to k



when the k is large curves are flat → Resonance is flat

if k is small → curves are sharp → peak

if k = 0, A is Infinity

when the damping is low, the amplitude falls very rapidly on either side of Resonant frequency of that the Resonance is sharp. On the other hand for high damping the ~~Resonance~~ Amplitude falls of very slowly on either side of resonance. Resonance is flat or sharp according to the damping for the oscillating system is large or small.

Example for flat resonance is Resonance of an air column of a aspirator bottle with tuning fork. Due to its large damping the air column respond with tuning fork over a wide range in the neighbourhood of resonance.

Thus in this case it is actually difficult to give an exact point of Resonance. & the resonce is said to be flat.

But in case of sonameter wire the damping is small responds only one particular freq i.e, its own natural frequency, hence resonance is sharp.

Quality factor

Q factor is a measure of sharpness of Resonance, which explains how fast energy decay in an oscillating system, Its a dimensionless parameter.

The mathematical representation is.

$$Q = 2\pi \frac{\text{Energy stored in oscillator}}{\text{Energy lost per cycle}} \Rightarrow 2\pi fr \times \frac{E_{stored}}{\text{power Loss}}$$

Energy stored = KE + PE

$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

$x = A\sin(pt-\theta)$

$\frac{dx}{dt} = Ap\cos(pt-\theta)$

$= \frac{1}{2}m\left[A^2p^2\cos^2(pt-\theta)\right] + \frac{1}{2}k\left[A^2\sin^2(pt-\theta)\right]$

$E_{stored} = \frac{1}{2}mA^2p^2\cos^2(pt-\theta) + \frac{1}{2}m\omega_0^2A^2\sin^2(pt-\theta) \longrightarrow (2)$

where $\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_0^2$

Energy loss per cycle = damping force x velocity

$= T.\gamma v \times v$

$= T.\gamma\left(\frac{dx}{dt}\right) \times \left(\frac{dx}{dt}\right)$

$= T\gamma\left[A^2p^2\cos^2(pt-\theta)\right]$

where $T = \frac{2\pi}{\omega} = \frac{2\pi}{p}$

$2k = \frac{r}{m}$ & $\tau = \frac{1}{2k}$ $\Rightarrow r = \frac{2km}{\cancel{m}}$ ... $= \frac{m}{\tau}$

(damping coeff)

Substitute $T = \frac{2\pi}{p}$ & $r = \frac{m}{\tau}$

$$E_{loss} = T \cdot r \cdot [A^2 p^2 \cos^2(pt - \theta)]$$

$$E_{loss} = \frac{2\pi}{p} \cdot \frac{m}{\tau} [A^2 p^2 \cos^2(pt-\theta)] \longrightarrow (3)$$

$$\varphi = 2\pi \cdot \frac{\frac{1}{2} m A^2 p^2 \cos^2(pt-\theta) + \frac{1}{2} m\omega_0^2 A^2 \sin^2(pt-\theta)}{\frac{2\pi}{p} \cdot \frac{m}{\tau} A^2 p^2 \cos^2(pt-\theta)}$$

[WKT $\cos^2\theta + \sin^2\theta = 1$, avg value of $\cos^2\theta$ or $\sin^2\theta = \frac{1}{2}$]

$$\therefore \varphi = 2\pi \cdot \frac{\frac{1}{2} m A^2 p^2 \overbrace{\cos^2(pt-\theta)}^{1/2} + \frac{1}{2} m\omega_0^2 A^2 \overbrace{\sin^2(pt-\theta)}^{1/2}}{\frac{2\pi}{p} \cdot \frac{m}{\tau} A^2 p^2 \overbrace{\cos^2(pt-\theta)}^{1/2}}$$

$$= \cancel{2\pi} \cdot \frac{\frac{1}{4} m A^2 p^2 + \frac{1}{4} m\omega_0^2 A^2}{\frac{1}{2} \cdot \frac{\cancel{2\pi}}{p} \cdot \frac{m}{\tau} A^2 p^2}$$

$$= \frac{\frac{1}{4} \cancel{m} A^2 (p^2 + \omega_0^2) \cdot 2p\tau}{\cancel{m} A^2 p^2}$$

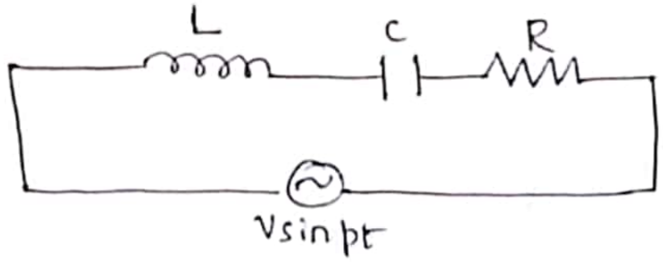
$$\varphi = \frac{1}{2} \frac{(p^2 + \omega_0^2)}{p} \cdot \tau$$

At Resonance $\omega_0 = p$. $\Rightarrow \varphi = \frac{1}{2}\left(\frac{2\omega_0^2}{\omega_0}\right)\tau$

$$\boxed{\varphi = \omega_0 \tau}$$

LCR Resonance

consider a series LCR ckt in which $V\sin pt$ is the applied AC voltage. the voltage drop across the Inductor L, Resistor R & capacitor C respectively.



$$V_L = L\frac{di}{dt}, \quad V_R = iR \quad \& \quad V_c = \frac{q}{C}.$$

Since the Applied voltage must be equal to the sum of the voltage drop across each of the elements.

$$V_L + V_R + V_c = V\sin pt. \longrightarrow (1)$$

or

$$L\frac{di}{dt} + Ri + \frac{1}{C}(q) = V\sin pt \qquad \text{But } i = \frac{dq}{dt}$$

$$\therefore \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V\sin pt \longrightarrow (1)$$

divide throughout by L we have

$$\frac{d^2q}{dt^2} + \left(\frac{R}{L}\right)\frac{dq}{dt} + \left(\frac{1}{LC}\right)q = \left(\frac{V}{L}\right)\sin pt \longrightarrow (2)$$

The above eqn resembles the eqn of motion of the vibrating sys given by,

$$\frac{d^2y}{dt^2} + \left(\frac{r}{m}\right)\frac{dy}{dt} + \left(\frac{k}{m}\right)y = \left(\frac{f}{m}\right)\sin pt \longrightarrow (3)$$

compare eqn (2) & (3)

$R \to r$, $L \to m$, $\frac{1}{C} \to k$, $q \to y$, $V \to f$, $R \to r$.

coz q & y are analogous quantity

Thus $\frac{1}{LC} \rightarrow \frac{k}{m} = \omega^2 \Rightarrow \omega = \sqrt{\frac{k}{m}}$

Resonance condn $\underset{\text{Applied freq}}{p} = \underset{\text{Natural freq}}{\omega} = \sqrt{\frac{k}{m}}$

$\left\langle \frac{1}{LC} = \omega^2 \right\rangle$

$\therefore$ In LCR ckt at Resonance $\omega^2 = \frac{1}{LC}$