* If $y=(\sin^{-1}x)^2$ show that, (16)

$(1-x^2)y_{n+2} - x(2n+1)y_{n+1} - n^2y_n = 0$.

$\Rightarrow$ Consider. $(y = (\sin^{-1}x)^2)$

$\Rightarrow y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = 2(\sin^{-1}x)$

$(1-x^2)\cdot y_1^2 = 4(\sin^{-1}x)^2$ $\quad [y=(\sin^{-1}x)^2]$

$(1-x^2)\cdot y_1^2 = 4y$.

Differentiating both sides w.r.t x, we get.

$(1-x^2)\cdot 2y_1y_2 - 2xy_1^2 = 4y_1$ $\quad \div$ by $2y_1$

$(1-x^2)\,y_2 - xy_1 - 2 = 0$

Differentiating each term n times,

By Leibnitz's theorem, we get

$^nC_0(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - xy_{n+1} - {}^nC_1 y_n(1) = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}(2)\cdot y_n - xy_{n+1} - ny_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+n-n)y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$.

or $\quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1}$

* If $y = [\log(x+\sqrt{1+x^2})]^2$ show that

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} - n^2y_n = 0$.

→ We have,

$$y = \left[\log\left(x+\sqrt{1+x^2}\right)\right]^2$$

$$y_1 = 2\cdot\log\left(x+\sqrt{1+x^2}\right)\cdot\frac{1}{\left(x+\sqrt{1+x^2}\right)}\cdot\left[1+\frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\right]$$

$$y_1 = 2\log\left(x+\sqrt{1+x^2}\right)\cdot\left[\frac{1}{\cancel{x+\sqrt{1+x^2}}}\right]\left[\frac{\cancel{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}}\right]$$

$$y_1 = 2\log\left(x+\sqrt{1+x^2}\right)\cdot\frac{1}{\sqrt{1+x^2}}$$

$$\sqrt{1+x^2}\cdot y_1 = 2\cdot\log\left(x+\sqrt{1+x^2}\right).$$

S.B.S $(1+x^2)y_1^2 = 4\left[\log\left(x+\sqrt{1+x^2}\right)\right]^2$

$(1+x^2)\,y_1^2 = 4y$ since $\left[y = \left[\log\left(x+\sqrt{1+x^2}\right)\right]^2\right]$.

Diff Both sides w.r.t. x, we get

$$(1+x^2)\cdot 2y_1y_2 + 2xy_1^2 = 4y_1$$

dividing throughout by $2y_1$, we get

$$(1+x^2)\cdot y_2 + xy_1 - 2 = 0$$

diff each term n times, by using Leibnitz's th

we have,

$$(1+x^2)y_{n+2} + n\cdot 2xy_{n+1} + \frac{n(n-1)}{2}\cdot 2\cdot y_n + xy_{n+1} + n\cdot 1\cdot y_n = 0$$

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n)y_n = 0$$

⇒ $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

or $(1+x^2)y_{n+2} + (2n+1)xy_{n+1}$

* If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ show that

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + 2n^2 y_n = 0$$

Sol^n Consider, $\cos^{-1}\left[\frac{y}{b}\right] = \log\left[\frac{x}{n}\right]^n$

$$\frac{y}{b} = \cos\left[\log\left(\frac{x}{n}\right)^n\right]$$

$$y = b \cdot \cos\left[n \log\left(\frac{x}{n}\right)\right] \quad -① $$

$$\Rightarrow y_1 = -b\sin\left[n\log\left(\frac{x}{n}\right)\right] \cdot n \cdot \left(\frac{n}{x}\right) \cdot \frac{1}{n}$$

$$x y_1 = -n \cdot b\sin\left[n\log\left(\frac{x}{n}\right)\right]$$

diff both sides w.r.t x, we get

$$x y_2 + y_1 = -n\,b \cdot \cos\left[n \cdot \log\left(\frac{x}{n}\right)\right] \cdot n \cdot \left(\frac{n}{x}\right) \cdot \frac{1}{n}$$

from(1)

$$x^2 y_2 + x y_1 = -n^2 y$$

differentiating each term n times, by Leibnitz's thm we have,

$$x^2 y_{n+2} + {}^nC_1 \cdot 2x y_{n+1} + {}^nC_2 \cdot 2 \cdot y_n + x y_{n+1} + {}^nC_1\, y_n = -n^2 y_n$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x y_{n+1} + [n(n-1) + n + n^2]\,y_n = 0$$

$$x^2 y_{n+2} + (2n+1)\,x y_{n+1} + 2n^2 y_n = 0$$

* If $y^{1/m} + y^{-1/m} = 2x$,
$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Solⁿ: Let $y^{1/m} = z \Rightarrow y^{-1/m} = \frac{1}{z}$.

$$\Rightarrow y^{1/m} + y^{-1/m} = 2x$$
$$\Rightarrow z + \frac{1}{z} = 2x$$
$$z^2 + 1 = 2xz$$
$$z^2 - 2xz + 1 = 0$$
$$z = x \pm \sqrt{x^2-1}.$$

taking the positive sign we have.

$$y^{1/m} = x + \sqrt{x^2-1} \Rightarrow y = (x+\sqrt{x^2-1})^m.$$

diff B.S w.r.t 'x', we get

$$y_1 = m(x+\sqrt{x^2-1})^{m-1}\left(1 + \frac{2x}{2\sqrt{x^2-1}}\right)$$
$$y_1 = m(x+\sqrt{x^2-1})^{m-1}\left[\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right].$$
$$\sqrt{x^2-1}\cdot y_1 = m(x+\sqrt{x^2-1})^m \quad [\text{Since } y = (x+\sqrt{x^2-1})^m]$$
$$\sqrt{x^2-1}\cdot y_1 = my$$

S.B.S $(x^2-1)y_1^2 = m^2y^2.$

Diff B.S w.r.t 'x'

$$(x^2-1)\cdot 2y_1y_2 + 2xy_1^2 - m^2 2yy_1 = 0.$$

÷ by $2y_1$ we get.

$$(x^2-1)y_2 + xy_1 - m^2y = 0$$

Diff each term n times by using Leibnitz's thm,

$$(x^2-1)y_{n+2} + {}^nC_1\, 2xy_{n+1} + {}^nC_2\, 2y_n + xy_{n+1} + {}^nC_1\, y_n - m^2y_n = 0.$$
$$(x^2-1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_2 + xy_{n+1} + ny_n - m^2y_n = 0$$
$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-n+n-m^2)y_n = 0$$
$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)\cdot y_n = 0$$

or $(x^2-1)y_{n+2} + (2n+1)xy_{n+1}$

* If $y = a\cdot\cos(\log x) + b\sin(\log x)$ S.T. (20)

$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$.

Sol^n $y = a\cos(\log x) + b\sin(\log x)$. —— (1)

diff $y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cdot\cos(\log x)\cdot\frac{1}{x}$

$xy_1 = -a\sin(\log x) + b\cdot\cos(\log x)$

diff $xy_2 + y_1 = -a\cdot\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$.

$x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$ (by (1))

$x^2y_2 + xy_1 = -y$

diff each term n times by Leibnitz thm, we get

$x^2y_{n+2} + nC_1\, 2x y_{n+1} + nC_2\, 2y_n + xy_{n+1} + nC_1 y_n + y_n = 0$

$x^2y_{n+2} + 2n\,x y_{n+1} + n(n-1)y_n + xy_{n+1} + n\,y_n + y_n = 0$

$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2-n+n+1)y_n = 0$.

$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$

or $x^2y_{n+2} + (2n+1)xy_{n+1}$

* If $y = e^{m\sin^{-1}x}$ Show that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$.

Sol^n Consider $y = e^{m\sin^{-1}x}$

diff w.r.t x. $y_1 = e^{m\sin^{-1}x}\cdot\frac{m}{\sqrt{1-x^2}}$ [Since $y = e^{m\sin^{-1}x}$]

$\sqrt{1-x^2}\cdot y_1 = my$.

S.B.S $(1-x^2)y_1^2 = m^2y^2$.

diff w.r.t x.

$(1-x^2)2y_1y_2 - 2xy_1^2 = 2m^2yy_1$

$\div$ by $2y_1$.

$(1-x^2)y_2 - xy_1 - m^2y = 0.$

diff each term n times by using Leibnitz's thm

$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - xy_{n+1} - {}^nC_1 y_n - m^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - m^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n+m^2)y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

* If $x = \sin t$ and $y = \cos pt$ show that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-p^2)y_n = 0$

$\rightarrow$ $x = \sin t \Rightarrow t = \sin^{-1}x.$

$y = \cos pt \Rightarrow y = \cos[p\cdot\sin^{-1}x]$ — (1)

diff Both sides w.r.t x we have

$$y_1 = -\sin[p\cdot\sin^{-1}x]\cdot\frac{p}{\sqrt{1-x^2}}.$$

$$\sqrt{1-x^2}\cdot y_1 = -p\sin[p\sin^{-1}x].$$

S.B.S. $(1-x^2)y_1^2 = p^2\sin^2[p\sin^{-1}x]$

$(1-x^2)y_1^2 = p^2[1-\cos^2(p\sin^{-1}x)]$

$(1-x^2)y_1^2 = p^2 - p^2y^2$ (from (1))

diff B.S w.r.t x, we get

$(1-x^2)\cdot 2y_1y_2 - 2xy_1^2 = -2p^2yy_1$

$\div$ by $2y_1$ $(1-x^2)y_2 - xy_1 + p^2y = 0$

diff each term n times, by using Leibnitz's thm

$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - xy_{n+1} - {}^nC_1 y_n + p^2y_n = 0$

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n-p^2)y_n = 0.$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-p^2)y_n = 0$

If $y=(\sin^{-1}x)^2$ Show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$, differentiate the above equation n times w.r.t x, Also find $y_n(0)$.

$\rightarrow$ $y = (\sin^{-1}x)^2$.

diff $y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = 2\sin^{-1}x.$ —— (1)

S.B.S

$(1-x^2)y_1^2 = 4[\sin^{-1}x]^2$.

$(1-x^2)y_1^2 = 4y$.

diff. B.S w.r.t 'x'

$(1-x^2)2y_1y_2 - 2xy_1^2 = 4y_1$

$\div$ by $2y_1$

$(1-x^2)y_2 - xy_1 - 2 = 0$ —(2)

diff each term n times by using Leibnitz's thm we get

$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - xy_{n+1} - {}^nC_1 y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$ —(3)

putting $x=0$ in (1) & (2) we get

$y_1(0) = 0$,

$y_2(0) = 0$.

Again putting $x=0$ in (3)

$y_{n+2}(0) - n^2y_n(0) = 0$

$\Rightarrow y_{n+2}(0) = n^2y_n(0)$.

put $n = 1, 3, 5, \ldots$ we get

$y_3(0) = 1^2\, y_1(0) = 0$ $[\because y_1(0)=0]$

$y_5(0) = 3^2\, y_3(0) = 0$ $[\because y_3(0)=0]$

$y_7(0) = 7^2\, y_5(0) = 0$ $[\because y_5(0)=0]$

$\therefore y_n(0) = 0$ when n is odd

putting $n = 2, 4, 6, \ldots$ we get

$y_4(0) = 2^2\, y_2(0) = 2\cdot 2^2$ $[\because y_2(0)=2]$

$y_6(0) = 4^2\, y_4(0) = 2\cdot 2^2\cdot 4^2$ $[\because y_4(0)=2\cdot 2^2]$

$y_8(0) = 6^2\, y_6(0) = 2\cdot 2^2\cdot 4^2\cdot 6^2$

$\vdots$

$y_n(0) = (n-2)^2\, y_{n-2}(0)$

$y_n(0) = 2\cdot 2^2\cdot 4^2\cdot 6^2 \ldots (n-2)^2$.

Thus when n is even

$y_n(0) = 2\cdot 2^2\cdot 4^2\cdot 6^2 \ldots (n-2)^2$

(23)

If $x = \tan(\log y)$. show that,

$(1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$.

$\Rightarrow$. $x = \tan(\log y)$.

$\tan^{-1}x = \log y$

$y = e^{\tan^{-1}x}$.

diff. $y_1 = e^{\tan^{-1}x} \cdot \frac{1}{1+x^2}$

$(1+x^2)y_1 = e^{\tan^{-1}x}$.

$(1+x^2)y_1 = y$.

diff $(1+x^2)y_2 + 2xy_1 = y_1$

$(1+x^2)y_2 + (2x-1)y_1 = 0$

Diff each term n times by using Leibnitz's thm we get

$(1+x^2)y_{n+2} + {}^nC_1 2x y_{n+1} + {}^nC_2 2 \cdot y_n + (2x-1)y_{n+1} + {}^nC_1 2 \cdot y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2nx y_{n+1} + n(n-1)y_n + (2x-1)y_{n+1} + 2n y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + (n^2 - n + 2n]y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} +$

H.W

Find the n^{th} derivative of

1. $x^2 \log x$
2. $x^3 . \cos x$.
3. $x^2 . e^x . \cos x$.
4. $x^3 . \sin ax$.

If $y = \sin(m . \sin^{-1}x)$. Show that
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$.

If $y = e^{m\cos^{-1}x}$. Show that.
$(1-x^2)y_{n+2} - (2n+1)x . y_{n+1} - (n^2+m^2)y_n = 0$