Polar Co-ordinates

Let O be a fixed point and OX be a fixed straight line. The position of any point P on the plane containing the point O. and the line OX in the polar system is determined by the distance OP and the angle which OP makes with the line OX. The distance OP is called the radius vector and is denoted by 'r' and the $X\hat{O}P = \theta$ is angle measured in anticlockwise direction, is called the vectorial angle. Then the co-ordinates (r, θ) are called polar co-ordinates of the point P.

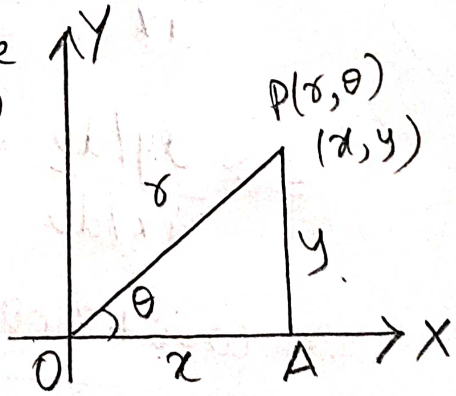
The fixed point O is called Pole and the fixed straight line OX is called the initial line.

Polar curves are always represented in Terms of 'r' and 'θ'. They are written in the form

$$r = f(\theta) \quad \text{or} \quad f(r, \theta) = 0.$$

Relation between the Cartesian and the Polar co-ordinates

Let $P(x, y)$ be any point on a plane with the cartesian co-ordinates (x, y) and the polar co-ordinates (r, θ). Draw PA perpendicular to OX, the from the triangle OPA,



$$OA = x = r\cos\theta \quad \text{—} (1)$$

$$AP = y = r\sin\theta \quad \text{—} (2)$$

Squaring & adding (1) + (2) we get,

$$x^2 + y^2 = r^2.$$

and $r = \sqrt{x^2+y^2}$ is called modulus.

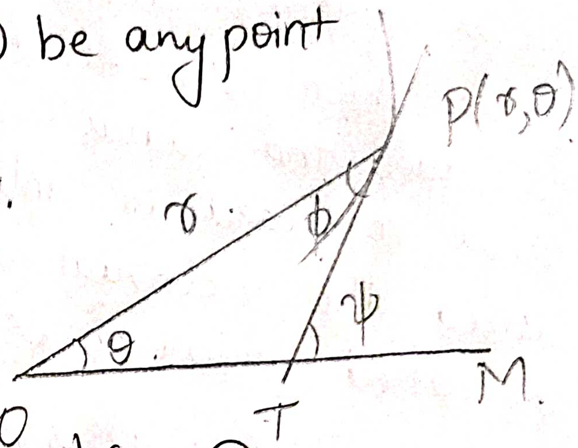
dividing ② by ① we get,

$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called amplitude.

These relations are used to transform the equations from polar form to Cartesian form and vice Versa.

Angle between the Radius Vector and the Tangent

Let O be the pole and $P(r,\theta)$ be any point on the curve.



$\angle OPT = \phi$, $\angle PTM = \psi$, $\angle POT = \theta$.

$\theta + \phi = \psi$ — ①

Let $r = f(\theta)$, $x = r\cos\theta$.

$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta \cdot \frac{dr}{d\theta}$ — ②

$y = r\sin\theta$.

$\frac{dy}{d\theta} = r\cos\theta + \sin\theta \cdot \frac{dr}{d\theta}$.

$\therefore \frac{dy}{dx} = \tan\psi = \tan(\theta+\phi)$

$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi}$.

$\Rightarrow \frac{dy/d\theta}{dx/d\theta} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi}$.

$\frac{r\cdot\cos\theta + \sin\theta \cdot \frac{dr}{d\theta}}{-r\sin\theta + \cos\theta \cdot \frac{dr}{d\theta}} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi}$ (divide LHS by $\frac{dr}{d\theta}.\cos\theta$).

$\frac{\tan\theta + r.\frac{d\theta}{dr}}{1 - \tan\theta \cdot r\cdot\frac{d\theta}{dr}} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \tan\phi}$.

$$\boxed{\tan\phi = r\cdot\frac{d\theta}{dr}}.$$

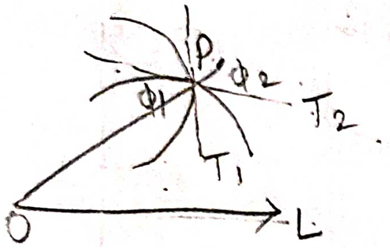
Angle of intersection between the two curves

(3)

Let ϕ_1 and ϕ_2 be the values of ϕ for the two curves at this point, $\phi = \phi_1 - \phi_2$.

$$\tan\phi = \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$$

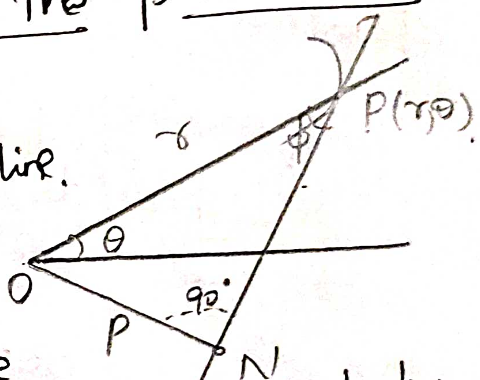
Condition for orthogonal is $\phi = \frac{\pi}{2}$.



$$1 + \tan\phi_1 \cdot \tan\phi_2 = 0.$$

$$\therefore \boxed{\tan\phi_1 \cdot \tan\phi_2 = -1}$$

Length of the perpendicular from the pole to the tangent.



Let O be the pole and OL be the initial line.
Let O be the pole and on the curve &
Let $P(r,\theta)$ be any pt on the curve & $OP = r$ and $L\hat{O}P = \theta$.
hence we have draw $ON = p$ (say) $\perp$lar from the pole onto the tangent at P and let ϕ be the angle made by the radius vector with the tangent and $L\hat{O}P = \theta$,
From the fig $O\hat{N}P = 90°$ and $L\hat{O}P = \theta$,
Now, from the right angled triangle ONP,

$$\sin\phi = \frac{ON}{OP} \Rightarrow \sin\phi = \frac{p}{r} \Rightarrow \boxed{p = r\sin\phi}$$

(This expression is the basic exp for the length of the perpendicular p. we proceed to present the expression for p in terms of θ in two standard forms)

We have $p = r\sin\phi$ — (1)

and $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$ — (2)

Squaring eq ① and taking reciprocal, (1)

$$\frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{1}{\sin^2\phi} \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2} \cdot \operatorname{cosec}^2\phi$$

i.e, $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

Now using eq ② $\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$

or $\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$ — ③

Further, Let $\frac{1}{r} = u$.

Differentiating w.r.t θ we get,

$$-\frac{1}{r^2}\cdot\left(\frac{dr}{d\theta}\right) = \frac{du}{d\theta} \Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2.$$

thus ③ becomes

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2. \quad \text{— ④}$$

Q 1) Prove with usual notations $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

2) Prove that for the curve $r = f(\theta)$

$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ where $u = \frac{1}{r}$.

Method of Solving problems

1. Given the eq in the form $r = f(\theta)$, we prefer to take logarithms first on both sides of the eq and then diff w.r.t 'θ' which always gives the term $\frac{1}{r} \cdot \frac{dr}{d\theta}$ and we directly use $\cot\phi$ for the same.
2. We simply RHS too and try to put it in terms of Cot and so that we obtain ϕ or ϕ_1 & ϕ_2.
3. $|\phi_2 - \phi_1|$ or $|\phi_1 - \phi_2|$ gives the angle of intersection.
4. If this contains θ, then we have to find θ by solving the pair of equations to obtain the angle of intersection independent of θ.
5. Suppose we are not able to obtain ϕ_1 and ϕ_2 explicitly then we have to write the expressions for $\tan\phi_1$, $\tan\phi_2$ and use the formula for $\tan(\phi_1 - \phi_2)$.
6. If $\tan(\phi_1 - \phi_2) = \alpha$ (say) then angle of intersection is equal to $\tan^{-1}\alpha$.
7. If $\tan\phi_1 \cdot \tan\phi_2 = -1$, then $\tan(\phi_1 - \phi_2) = \infty \Rightarrow \phi_1 - \phi_2 = \pi/2$

Imp formulas.

$1 + \cos\theta = 2\cos^2(\theta/2)$

$1 - \cos\theta = 2\sin^2(\theta/2)$

$\sin\theta = 2\sin\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{2}\right)$

$\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$.

Problems

Find the angle between the radius vector and the tangent for the following polar curves

1. $r = a(1-\cos\theta)$

take log on B.S

$\log r = \log[a(1-\cos\theta)]$

$\log r = \log a + \log(1-\cos\theta)$

diff w.r.t 'θ'

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$

i.e $\cot\phi = \frac{2\sin\theta/2\cdot\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$

$\cot\phi = \cot\frac{\theta}{2} \Rightarrow \boxed{\phi = \frac{\theta}{2}}$

2. $r^2\cos 2\theta = a^2$

take log on B.S

$\log(r^2\cos 2\theta) = \log a^2$

$\log r^2 + \log\cos 2\theta = 2\log a$

$2\log r + \log\cos 2\theta = 2\log a$

diff $\frac{2}{r}\cdot\frac{dr}{d\theta} + \frac{(-2\sin 2\theta)}{\cos 2\theta} = 0$

$\tan\phi = r/\left(\frac{dr}{d\theta}\right)$

$\therefore \cot\phi = \frac{1}{r}\left(\frac{dr}{d\theta}\right)$

ie $\frac{1}{r}\cdot\frac{dr}{d\theta} = \tan 2\theta$.

$\cot\phi = \cot\left(\frac{\pi}{2} - 2\theta\right)$

$\boxed{\phi = \frac{\pi}{2} - 2\theta}$

3. $r^m = a^m(\cos m\theta + \sin m\theta)$.

take log on B.S

$m \log r = m \cdot \log a + \log(\cos m\theta + \sin m\theta)$

diff w.r.t 'θ'

$$\frac{m}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{(-m\sin m\theta + m\cos m\theta)}{\cos m\theta + \sin m\theta}.$$

$$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}.$$

$$\cot\phi = \frac{\cancel{\cos m\theta}(1-\tan m\theta)}{\cancel{\cos m\theta}(1+\tan m\theta)}$$

$$\cot\phi = \cot\left(\frac{\pi}{4} + m\theta\right)$$

$$\boxed{\phi = \frac{\pi}{4} + m\theta}$$

4. $\frac{l}{r} = 1 + e\cos\theta$.

take log on B.S

$\log l - \log r = \log(1 + e\cos\theta)$.

diff $0 - \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$.

$$\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}.$$ (this Cannot be simplified.

$$\tan\phi = \frac{1+e\cos\theta}{e\sin\theta}$$

thus, $\phi = \tan^{-1}\left[\frac{1+e\cos\theta}{e\sin\theta}\right]$

5. $r = a(1+\cos\theta)$ at $\theta = \pi/3$.

$\log r = \log a + \log(1+\cos\theta)$

diff w.r.t θ,

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1+\cos\theta} = -\frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$

$\cot\phi = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \Rightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}$.

At $\theta = \pi/3 \Rightarrow \phi = \frac{\pi}{2}+\frac{\pi}{6} \Rightarrow \boxed{\phi = \frac{2\pi}{3} = 120^\circ}$

6. $r\cdot\cos^2\left(\frac{\theta}{2}\right) = a$.

$\log r + 2\cdot\log\cos\left(\frac{\theta}{2}\right) = \log a$.

diff w.r.t 'θ'

$\frac{1}{r}\cdot\frac{dr}{d\theta} + 2\cdot\frac{(-1/2)\cdot\sin(\theta/2)}{\cos(\theta/2)} = 0$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \tan\theta/2$.

$\cot\phi = \cot\left(\frac{\pi}{2}-\theta/2\right) \Rightarrow \phi = \frac{\pi}{2}-\theta/2$

at $\theta = \frac{2\pi}{3}$, $\phi = \frac{\pi}{2}-\frac{\pi}{3} = \frac{\pi}{6} = 30^\circ$

7. $\frac{2a}{r} = 1-\cos\theta$.

$\log 2a - \log r = \log(1-\cos\theta)$

diff w.r.t θ,

$0 - \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot\theta/2$.

$-\cot\phi = \cot(\theta/2)$ or $\cot(-\phi) = \cot\theta/2 \Rightarrow \phi = -\frac{\theta}{2}$.

at $\theta = 2\pi/3$, $\phi = -\pi/3 = -60^\circ \Rightarrow \boxed{\phi = -\pi/3}$

8. $r = a(1+\sin\theta)$

$\log r = \log a + \log(1+\sin\theta)$

diff $\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$ or $\cot\phi = \frac{\cos\theta}{1+\sin\theta}$.

$\theta = \pi/2$, $\cot\phi = \frac{0}{1+1} = 0$ or $\cot\phi = 0 \Rightarrow \boxed{\phi = \pi/2}$

Or method 2

$$\cot\phi = \frac{\cos^2(\theta/2) - \sin^2(\theta/2)}{\cos^2(\theta/2) + \sin^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2)}$$

$$\cot\phi = \frac{(\cos\theta/2 - \sin\theta/2)\cancel{(\cos\theta/2 + \sin\theta/2)}}{(\cos\theta/2 + \sin\theta/2)^{\cancel{2}}}$$

$$\cot\phi = \frac{\cos\theta/2\,[1 - \tan\theta/2]}{\cos\theta/2\,[1 + \tan\theta/2]}$$

$$\cot\phi = \cot\left(\frac{\pi}{4} + \theta/2\right) \Rightarrow \phi = \frac{\pi}{4} + \theta/2.$$

put $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{4} + \frac{\pi}{4} = \frac{2\pi}{4} = \pi/2$

set 3 prob

Show that the foll pairs of curves intersect each other orthogonally.

9. $r = a(1+\cos\theta)$ & $r = b(1-\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$; $\log r = \log b + \log(1-\cos\theta)$

diff $\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1+\cos\theta}$; $\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$

$\cot\phi_1 = \frac{-2\sin\theta/2\cdot\cos\theta/2}{2\cos^2\theta/2}$; $\cot\phi_2 = \frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\sin^2\theta/2}$

$\cot\phi_1 = -\tan(\theta/2) = \cot\left(\frac{\pi}{2} + \theta/2\right)$; $\cot\phi_2 = \cot(\theta/2)$

$\boxed{\phi_1 = \pi/2 + \theta/2}$ $\boxed{\phi_2 = \frac{\theta}{2}}$

$\therefore$ angle of intersection $= |\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}\right| = \frac{\pi}{2}$

thus curves intersect each other orthogonally

10. $r = a(1+\sin\theta)$; $r = a(1-\sin\theta)$ 10

$\log r = \log a + \log(1+\sin\theta)$; $\log r = \log a + \log(1-\sin\theta)$

diff

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta}$; $\therefore \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-\cos\theta}{1-\sin\theta}$

$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$; $\cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$.

It is sufficient that if we can show that

$\tan\phi_1 \cdot \tan\phi_2 = -1$.

$\rightarrow \tan\phi_1 = \frac{1+\sin\theta}{\cos\theta}$ and $\tan\phi_2 = \frac{1-\sin\theta}{-\cos\theta}$

thus, $\tan\phi_1 \cdot \tan\phi_2 = \frac{1-\sin^2\theta}{-\cos^2\theta} = \frac{\cos^2\theta}{-\cos^2\theta} = -1$

thus the curves intersect each other orthogonally

11. $r^n = a^n \cdot \cos n\theta$; $r^n = b^n \sin n\theta$.

$n\log r = n\log a + \log(\cos n\theta)$; $n\log r = n\log b + \log(\sin n\theta)$

diff $\frac{n}{r}\cdot\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta}$; $\frac{n}{r}\cdot\frac{dr}{d\theta} = \frac{n\cos n\theta}{\sin n\theta}$.

$\frac{1}{r}\cdot\frac{dr}{d\theta} = -\tan n\theta$. ; $\frac{1}{r}\cdot\frac{dr}{d\theta} = \cot n\theta$

$\cot\phi_1 = \cancel{\cot}\left(\frac{\pi}{2}+n\theta\right)$; $\phi_2 = n\theta$

$\phi_1 = \frac{\pi}{2} + n\theta$.

$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \cancel{n\theta} - \cancel{n\theta}\right| = \frac{\pi}{2}$.

thus the curves intersect each other orthogonally

12. $r^2 \sin 2\theta = a^2$, $r^2 \cos 2\theta = b^2$

$2\log r + \log(\sin 2\theta) = 2\log a$; $2\log r + \log(\cos 2\theta) = 2b\log b$

diff w.r.t θ

$\frac{2}{r}\cdot\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = -\cot 2\theta$

$\cot\phi_1 = -\cot 2\theta$

$\cot\phi_1 = \cot(-2\theta)$

$\phi_1 = -2\theta.$

$\frac{2}{r}\cdot\frac{dr}{d\theta} - \frac{2\sin 2\theta}{\cos 2\theta} = 0$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \tan 2\theta.$

$\cot\phi_2 = \tan 2\theta$

$\cot\phi_2 = \cot\left(\frac{\pi}{2} - 2\theta\right)$

$\phi_2 = \frac{\pi}{2} - 2\theta.$

$(\phi_1 - \phi_2) = \left|-2\theta - \frac{\pi}{2} + 2\theta\right| - \pi/2$

thus curves intersect each other orthogonally.

13. $r = 4\sec^2(\theta/2)$, $r = 9\cdot\operatorname{cosec}^2(\theta/2)$

$\log r = \log 4 + 2\log\sec(\theta/2)$,

diff $\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{2}{\sec(\theta/2)}\cdot\sec\theta/2\cdot\tan\theta/2\cdot\frac{1}{2}$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \tan\theta/2.$

$\cot\phi_1 = \cot\left(\frac{\pi}{2} - \theta/2\right)$

$\phi_1 = \frac{\pi}{2} - \frac{\theta}{2}.$

$\log r = \log 9 + 2\log\operatorname{cosec}(\theta/2)$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-2\operatorname{cosec}(\theta/2)\cdot\cot\theta/2}{\operatorname{cosec}\theta/2}\cdot\frac{1}{2}$

$\cot\phi_2 = \cot(-\theta/2)$

$= -\cot\theta/2$

$\cot\phi_2 = -\theta/2$

$\therefore |\phi_1 - \phi_2| - \left|\frac{\pi}{2} - \frac{\theta}{2} + \frac{\theta}{2}\right| - \pi/2$

thus the curves intersect each other orthogonally.

14. $r = ae^{\theta}$, $re^{\theta} = b$.

$\log r = \log a + \theta \log e$; $\log r + \theta \log e = \log b$.

But $\log e = 1$ diff w.r.t 'θ'.

$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + 1$; $\frac{1}{r} \cdot \frac{dr}{d\theta} + 1 = 0$.

$\cot\phi_2 = -1$

$\cot\phi_1 = 1$ $\phi_2 = -\pi/4$ or $3\pi/4$.

$\phi_1 = \pi/4$.

$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{4} + \frac{\cancel{8}\pi}{4}\right| = (2\pi/4) = \pi/2$

Thus curves intersect each other orthogonally.

Find the angle between the foll pairs of curves

15 $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$.

$\log r = \log(\sin\theta + \cos\theta)$ $\log r = \log 2 + \log(\sin\theta)$

differentiating these w.r.t θ we get, $\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\cos\theta}{\sin\theta}$.

$\frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$.

$\cot\phi_1 = \frac{\cos\theta(1 - \tan\theta)}{\cos\theta(1 + \tan\theta)}$ $\cot\phi_2 = \cot\theta$

$\Rightarrow \boxed{\phi_2 = \theta.}$

$\cot\phi_1 = \cot\left(\frac{\pi}{4} + \theta\right)$

$\phi_1 = \frac{\pi}{4} + \theta$.

$|\phi_1 - \phi_2| = \left|\frac{\pi}{4} + \theta - \theta\right| = \pi/4$

$\therefore$ angle of intersection is $\pi/4$

16. $r = a\log\theta$, $\qquad r = \dfrac{a}{\log\theta}$. (13)

$\log r = \log a + \log(\log\theta)$; $\quad \log r = \log a - \log(\log\theta)$

diff $\dfrac{1}{r}\cdot\dfrac{dr}{d\theta} = \dfrac{1}{\log\theta\cdot\theta}$ $\qquad \dfrac{1}{r}\cdot\dfrac{dr}{d\theta} = -\dfrac{1}{\theta\cdot\log\theta}$.

$\cot\phi_1 = \dfrac{1}{\theta\cdot\log\theta}$. $\qquad \cot\phi_2 = -\dfrac{1}{\theta\cdot\log\theta}$

we cannot find ϕ_1 and ϕ_2 explicitly

$\tan\phi_1 = \theta\log\theta$ $\qquad \tan\phi_2 = -\theta\log\theta$.

Consider, $\tan(\phi_1-\phi_2) = \dfrac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\cdot\tan\phi_2}$

$$\tan(\phi_1-\phi_2) = \frac{2\theta\cdot\log\theta}{1-(\theta\log\theta)^2} \quad \text{—(1)}$$

We have to find θ by solving the given pair of eq^n $r = a\log\theta$ & $r = \dfrac{a}{\log\theta}$.

Equating the RHS, we have $a\log\theta = \dfrac{a}{\log\theta}$.

i.e., $(\log\theta)^2 = 1$ or $\log\theta = 1$
$\Rightarrow \theta = e$.

Subs, $\theta = e$ in (1) we get

$$\tan(\phi_1-\phi_2) = \frac{2e}{1-e^2} \quad (\because \log_e e = 1)$$

thus angle of intersection:

$$(\phi_1-\phi_2) = \tan^{-1}\left(\frac{2e}{1-e^2}\right).$$

17. $r^2 \sin 2\theta = 4$,

$2\log r + \log(\sin 2\theta) = \log 4$

diff

$$\frac{2}{r}\cdot\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = -\cot 2\theta.$$

$$\cot\phi_1 = \cot(-2\theta)$$

$$\phi_1 = -2\theta.$$

; $r^2 = 16\sin 2\theta$. 14

$2\log r = \log 16 + \log(\sin 2\theta)$

$$\frac{2}{r}\cdot\frac{dr}{d\theta} = \frac{2\cos 2\theta}{\sin 2\theta}.$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \cot 2\theta.$$

$$\cot\phi_2 = \cot 2\theta.$$

$$\phi_2 = 2\theta$$

$$|\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta \quad \text{—} (1)$$

Now consider $r^2 = \dfrac{4}{\sin 2\theta}$ and $r^2 = 16\sin 2\theta$

$$\frac{4}{\sin 2\theta} = 16\sin 2\theta.$$

or $4\sin^2 2\theta = 1$

$\sin^2 2\theta = 1/4$

$\sin 2\theta = 1/2.$

$2\theta = \pi/6$ $\Rightarrow$ $\boxed{\theta = \pi/12}$

Subs $\theta = \pi/12$ in (1) we get $|\phi_1 - \phi_2| = \pi/3$.

thus the angle of intersection is $\pi/3 = 60°$

18. $r = a(1-\cos\theta)$

$\log r = \log a + \log(1-\cos\theta)$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$.

$\cot\phi_1 = \frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\sin^2(\theta/2)}$

$\cot\phi_1 = \cot(\theta/2)$

$\phi_1 = \theta/2$.

$r = 2a\cos\theta$.

$\log r = \log 2a + \log(\cos\theta)$.

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta}$.

$\cot\phi_2 = -\tan\theta$.

$\cot\phi_2 = \cot(\pi/2+\theta)$

$\phi_2 = \pi/2+\theta$.

$|\phi_1-\phi_2| = \left|\frac{\theta}{2} - \frac{\pi}{2} - \theta\right| = \frac{\pi}{2}+\frac{\theta}{2}$ — (1).

Consider, $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$.

$a(1-\cos\theta) = 2a\cos\theta$.

$a - a\cos\theta = 2a\cos\theta$.

$3\cancel{a}\cos\theta = 1$

$\cos\theta = 1/3$

$\theta = \cos^{-1}(1/3)$

$\frac{\theta}{2} = \frac{1}{2}\cdot\cos^{-1}(1/3)$.

subs this value in (1)

thus the angle of intersection is $\frac{\pi}{2} + \frac{1}{2}\cdot\cos^{-1}\left(\frac{1}{3}\right)$

19. $r = 6\cdot\cos\theta$.

$\log r = \log 6 + \log(\cos\theta)$

diff

$\frac{1}{r}\cdot\frac{dr}{d\theta} = -\frac{\sin\theta}{\cos\theta}$.

$\cot\phi_1 = -\tan\theta$.

$\cot\phi_1 = \cot\left(\frac{\pi}{2}+\theta\right)$

$\phi_1 = \pi/2+\theta$.

$|\phi_1-\phi_2| = \theta/2$ —— (1).

$r = 2(1+\cos\theta)$

$\log r = \log 2 + \log(1+\cos\theta)$.

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$

$\cot\phi_2 = \frac{-2\sin(\theta/2)\cdot\cos(\theta/2)}{2\cos^2(\theta/2)}$.

$\phi_2 = \pi/2+\theta/2$

Equating the RHS of the given eq^n (16)

$6\cos\theta = 2(1+\cos\theta)$

$\cos\theta = 1/2 \Rightarrow \cos\theta = \cos\pi/3$

$\theta = \pi/3$

from ①, $|\phi_2 - \phi_1| = \frac{\theta}{2} = \frac{\pi}{2\cdot 3} = \frac{\pi}{6} = 30^\circ$

Thus angle of intersection is 30°

20, $r^n = a^n \cdot \sec(n\theta+\alpha)$, $r^n = b^n \sec(n\theta+\beta)$

$n\log r = n\log a + \log[\sec(n\theta+\alpha)]$; $n\log r = n\log b + \log\sec(n\theta+\beta)$

diff

$\frac{n}{r}\cdot\frac{dr}{d\theta} = \frac{n\sec(n\theta+\alpha)\cdot\tan(n\theta+\alpha)}{\sec(n\theta+\alpha)}$; $\frac{n}{r}\cdot\frac{dr}{d\theta} = \frac{n\sec(n\theta+\beta)\tan(n\theta+\beta)}{\sec(n\theta+\beta)}$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \tan(n\theta+\alpha)$; $\frac{1}{r}\cdot\frac{dr}{d\theta} = \tan(n\theta+\beta)$

$\cot\phi_1 = \cot\left[\frac{\pi}{2} - (n\theta+\alpha)\right]$; $\cot\phi_2 = \cot\left(\frac{\pi}{2} - (n\theta+\beta)\right)$

$\phi_1 = \frac{\pi}{2} - n\theta - \alpha$. ; $\phi_2 = \frac{\pi}{2} - n\theta - \beta$

$\therefore |\phi_1 - \phi_2| = |-\alpha+\beta| = \alpha - \beta$ where $\alpha > \beta$.

Thus angle of intersection is $\alpha - \beta$ where $\alpha > \beta$

17

21) $r = a(1+\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$

$\cot\phi_1 = -\frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\cos^2(\theta/2)}$

$\cot\phi_1 = -\tan(\theta/2)$

$\cot\phi_1 = \cot\left(\frac{\pi}{2}+\theta/2\right)$

$\phi_1 = \frac{\pi}{2}+\theta/2$

$r^2 = a^2\cos 2\theta$

$2\log r = 2\log a + \log(\cos 2\theta)$

$\frac{2}{r}\cdot\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta}$

$\cot\phi_2 = -\tan 2\theta$

$\cot\phi_2 = \cot\left(\frac{\pi}{2}+2\theta\right)$

$\phi_2 = \frac{\pi}{2}+2\theta$

$|\phi_1-\phi_2| = \left|\frac{\pi}{2}+\frac{\theta}{2}-\frac{\pi}{2}-2\theta\right| = \frac{3\theta}{2}$ — (1)

Now, Squaring the first of the given eq^n & then equating with the RHS of the second eq^n, we have.

$a^2(1+\cos\theta)^2 = a^2\cos 2\theta$

$1+2\cos\theta+\cos^2\theta = a^2\cos 2\theta$

$1+2\cos\theta+\cos^2\theta = 2\cos^2\theta-1$

$\cos^2\theta-2\cos\theta-2=0$

$\cos\theta = \frac{2\pm\sqrt{4+8}}{2} = \frac{2\pm2\sqrt{3}}{2}$

$= 1\pm\sqrt{3}$

Sine $\cos\theta$ Cannot exceed 1 ~~numerically~~, we have to take

$\cos\theta = 1-\sqrt{3}$.

$1-2\sin^2(\theta/2) = 1-\sqrt{3}$ or $\sin^2(\theta/2) = \frac{\sqrt{3}}{2} = \sqrt{\frac{3}{4}}$

$\sin\left(\frac{\theta}{2}\right) = \left(\sqrt{\frac{3}{4}}\right)^{1/2} = (3/4)^{1/4}$

$\therefore \theta/2 = \sin^{-1}(3/4)^{1/4}$ Subs this value in (1) angle of intersection $3\sin^{-1}(3/4)^{1/4}$.

22. $r = a\theta$, $r = \frac{a}{\theta}$

$\log r = \log a + \log\theta$ $\quad$ $\log r = \log a - \log\theta$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{1}{\theta}$ $\quad$ $\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{-1}{\theta}$

$\cot\phi_1 = \frac{1}{\theta}$ $\quad$ $\cot\phi_2 = \frac{-1}{\theta}$

$\tan\phi_1 = \theta$ $\quad$ $\tan\phi_2 = -\theta$

Also by equating the RHS of the given equation we have,

$a\theta = \frac{a}{\theta}$ or $\theta^2 = 1 \Rightarrow \theta = \pm 1$

when $\theta = 1$, $\tan\phi_1 = 1$, $\tan\phi_2 = -1$ &

when $\theta = -1$, $\tan\phi_1 = -1$, $\tan\phi_2 = 1$

$\therefore \tan\phi_1 \cdot \tan\phi_2 = -1 \Rightarrow \phi_1 - \phi_2 = \pi/2$

Thus the angle of intersection is $\pi/2$.

H.W.

(1) $r = \frac{a\theta}{1+\theta}$ & $r = \frac{a}{1+\theta^2}$

Ans $\tan^{-1}(3)$

(2) Find the angle b/w the curves $r = a(1+\sin\theta)$ and $r = a(1-\cos\theta)$.

Ans $= \pi/4$.