1.24 Pedal equation of a polar curve

In the context of deriving an expression for the length of the perpendicular (p) from the pole to the tangent we obtained the expression in the form $p = r \sin \phi$.

The equation of the given curve $r = f(\theta)$ expressed in terms of p and r is called as the pedal equation or $p-r$ equation of the curve $r = f(\theta)$.

Remark: *Many equations of the standard cartesian curves $y = f(x)$ are expressible in the parametric form $x = x(t)$, $y = y(t)$. Eliminating t we get $y = f(x)$. We have a similar concept in the case of polar curve $r = f(\theta)$.*

Flow chart for solving problems

Given, $r = f(\theta)$ we first obtain ϕ.

↓

We substitute ϕ (usually a function of θ) into the equation $p = r \sin \phi$ so that this equation assumes the form $p = r\, g(\theta)$

↓

We need to eliminate θ between the equations : $r = f(\theta)$ and $p = r\, g(\theta)$ to obtain an equation in p & r being the pedal equation.

Note : *If we are unable to obtain ϕ explicitly in terms of θ, we have to square and take the reciprocal of $p = r \sin \phi$. This will give us :*

$$\frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi \quad or \quad \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \cot^2 \phi\right]$$

We substitute for $\cot \phi$ *itself in terms of θ. Elimination of θ by using the given equation will give us the pedal equation.*

WORKED PROBLEMS

Find the pedal equation of the following curves.

[24] $2a/r = (1 + \cos\theta)$ **[Dec 2016]**

[25] $r(1 - \cos\theta) = 2a$

[26] $r^2 = a^2 \sec 2\theta$

[27] $r^n = a^n \cos n\theta$ **[June 2017]**

[28] $r^m = a^m(\cos m\theta + \sin m\theta)$ **[June 18]**

[29] $r = a(1 + \cos\theta)$ **[June 2018]**

[30] $l/r = 1 + e\cos\theta$

[31] $r^n = a^n \operatorname{sech} n\theta$

☞ **Solutions**

[24] $\dfrac{2a}{r} = (1 + \cos\theta)$

$\Rightarrow \quad \log 2a - \log r = \log\ (1 + \cos\theta)$

Differentiating w.r.t. θ we get,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

ie., $\quad \cot\phi = \cot(\pi/2 + \theta/2) \Rightarrow \phi = \pi/2 + \theta/2$

Consider, $p = r\sin\phi$ and substituting the value of ϕ we have,

$$p = r\sin(\pi/2 + \theta/2) = r\cos(\theta/2)$$

Now we have, $$\frac{2a}{r} = 1 + \cos\theta \quad \ldots (1)$$

$$p = r\cos(\theta/2) \quad \ldots (2)$$

We have to eliminate θ from (1) and (2).

(It will be convenient for elimination if we can have similar functions of θ in the RHS of the two equations)

(1) can be put in the form $\dfrac{2a}{r} = 2\cos^2(\theta/2)$ or $\dfrac{a}{r} = \cos^2(\theta/2)$

Also from (2), $\dfrac{p}{r} = \cos(\theta/2)$

Hence we get, $\dfrac{a}{r} = \left(\dfrac{p}{r}\right)^2$ or $\dfrac{a}{r} = \dfrac{p^2}{r^2}$ or $p^2 = ar$

Thus $\boxed{p^2 = ar}$ is the required pedal equation.

[25] $r(1 - \cos\theta) = 2a$

$\Rightarrow \quad \log r + \log(1 - \cos\theta) = \log 2a$

Differentiating w.r.t. θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0 \text{ or } \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

ie., $$\cot\phi = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -\cot(\theta/2)$$

ie., $\cot\phi = \cot(-\theta/2) \Rightarrow \phi = -(\theta/2)$

Consider, $p = r\sin\phi$

$\therefore \quad p = r\sin(-\theta/2)$ or $p = -r\sin(\theta/2)$

Now we have, $r(1-\cos\theta) = 2a$ . . . (1)

$p = -r\sin(\theta/2)$ . . . (2)

We have to eliminate θ from (1) and (2).

(1) can be put in the form $r \cdot 2\sin^2(\theta/2) = 2a$ or $r\sin^2(\theta/2) = a$.

$\therefore \quad r\left(\dfrac{p^2}{r^2}\right) = a$ or $p^2 = ar$ since $p/-r = \sin(\theta/2)$, from (2).

Thus $\boxed{p^2 = ar}$ is the required pedal equation.

[26] $r^2 = a^2 \sec 2\theta$

$\Rightarrow \quad 2\log r = 2\log a + \log(\sec 2\theta)$

Differentiating w.r.t. θ we get,

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{2\sec 2\theta\tan 2\theta}{\sec 2\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$$

ie., $\cot\phi = \cot(\pi/2 - 2\theta) \Rightarrow \phi = \pi/2 - 2\theta$

Consider, $p = r\sin\phi$

$\therefore p = r\sin(\pi/2 - 2\theta)$ or $p = r\cos 2\theta$

Now we have, $r^2 = a^2\sec 2\theta$ . . . (1)

$p = r\cos 2\theta$ . . . (2)

From (2), $p/r = \cos 2\theta$ or $r/p = \sec 2\theta$

Substituting in (1) we get, $r^2 = a^2\,(r/p)$ or $pr = a^2$

Thus $\boxed{pr = a^2}$ is the required pedal equation.

[27] $r^n = a^n \cos n\theta$

$\Rightarrow \quad n\log r = n\log a + \log(\cos n\theta)$

Differentiating w.r.t. θ we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

ie., $\cot\phi = \cot(\pi/2 + n\theta) \Rightarrow \phi = \pi/2 + n\theta$

Consider, $p = r \sin\phi$

$\therefore \quad p = r\sin(\pi/2 + n\theta)$ or $p = r\cos n\theta$

Now we have, $r^n = a^n \cos n\theta$ . . . (1)

$p = r\cos n\theta$ . . . (2)

Hence, (1) as a consequence of (2) is $r^n = a^n\,(p/r)$

Thus $\boxed{r^{n+1} = p\,a^n}$ is the required pedal equation.

[28] $r^m = a^m(\cos m\theta + \sin m\theta)$

$\Rightarrow \quad m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

Differentiating w.r.t. θ we get,

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{-m\sin m\theta + m\cos m\theta}{\cos m\theta + \sin m\theta}$$

ie.,
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta} = \frac{\cos m\theta(1-\tan m\theta)}{\cos m\theta(1+\tan m\theta)}$$

ie., $\cot\phi = \cot(\pi/4 + m\theta) \Rightarrow \phi = \pi/4 + m\theta$

Consider, $p = r\sin\phi$

$\therefore \quad p = r\sin(\pi/4 + m\theta)$

ie., $p = r[\sin(\pi/4)\cos m\theta + \cos(\pi/4)\sin m\theta]$

ie., $p = \dfrac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta)$

(*We have used the formula of* $\sin(A+B)$ *and also the values* $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$)

Now we have, $r^m = a^m(\cos m\theta + \sin m\theta)$ . . . (1)

$$p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta) \quad \text{. . . (2)}$$

Using (2) in (1) we get,

$$r^m = a^m \cdot \frac{p\sqrt{2}}{r} \quad \text{or} \quad r^{m+1} = \sqrt{2}\,a^m p$$

Thus $\boxed{r^{m+1} = \sqrt{2}\,a^m p}$ is the required pedal equation.

[29] $r = a(1 + \cos\theta)$

$\Rightarrow \quad \log r = \log a + \log(1 + \cos\theta)$

Differentiating w.r.t. θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

ie., $\quad \cot\phi = \cot(\pi/2 + \theta/2) \Rightarrow \phi = \pi/2 + \theta/2$

Consider, $p = r\sin\phi$

$\therefore \quad p = r\sin(\pi/2 + \theta/2)$ or $p = r\cos(\theta/2)$

Now we have, $\quad r = a(1 + \cos\theta)$ . . . (1

$p = r\cos(\theta/2)$ . . . (2

(1) can be put in the form, $r = a \cdot 2\cos^2(\theta/2)$

ie., $\quad r = 2a\cos^2(\theta/2)$

From (2), $p/r = \cos(\theta/2)$ and hence (1) becomes,

$$r = 2a \cdot (p^2/r^2) \quad \text{or} \quad r^3 = 2a\,p^2$$

Thus $\boxed{r^3 = 2a\,p^2}$ is the required pedal equation.

[30] $l/r = 1 + e\cos\theta$

$\Rightarrow \quad \log l - \log r = \log(1 + e\cos\theta)$

Differentiating w.r.t. θ we get,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$$

ie., $\quad \cot\phi = \dfrac{e\sin\theta}{1+e\cos\theta}$ We cannot find ϕ explicitly.

Consider, $p = r\sin\phi$

By squaring and taking the reciprocal we have,

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

Substituting for $\cot\phi$ itself we have,

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right\} \qquad \ldots (1)$$

Also we have, $$\frac{l}{r} = 1+e\cos\theta \qquad \ldots (2)$$

We need to eliminate θ from (1) and (2).

From (2), $$\frac{l}{r}-1 = e\cos\theta \qquad \ldots (3)$$

Also, $$e^2\sin^2\theta = e^2(1-\cos^2\theta) = e^2 - e^2\cos^2\theta$$

By using (3) we have, $$e^2\sin^2\theta = e^2 - \left(\frac{l}{r}-1\right)^2 \qquad \ldots (4)$$

Now substituting (3) and (4) in (1) we have,

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{e^2-\left(\frac{l}{r}-1\right)^2}{\left(\frac{l^2}{r^2}\right)}\right\}$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{l^2}\left\{e^2-\left(\frac{l}{r}-1\right)^2\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{l^2}\left\{e^2-\frac{l^2}{r^2}+\frac{2l}{r}-1\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}+\frac{e^2}{l^2}-\frac{1}{r^2}+\frac{2}{lr}-\frac{1}{l^2}$$

Thus $\boxed{\frac{1}{p^2} = \frac{e^2-1}{l^2}+\frac{2}{lr}}$ is the required pedal equation.

[31] $r^n = a^n \operatorname{sech} n\theta$

$\Rightarrow \quad n\log r = n\log a + \log(\operatorname{sech} n\theta)$

Differentiating w.r.t. θ we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\operatorname{sech} n\theta\tanh n\theta}{\operatorname{sech} n\theta}$$

ie., $\frac{1}{r}\frac{dr}{d\theta} = -n \tanh n\theta$

ie., $\cot\phi = -\tanh n\theta$ and ϕ cannot be found explicitly.

Consider, $p = r \sin\phi$. Squaring and taking the reciprocal, we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left(1+\tanh^2 n\theta\right) \qquad \ldots (1)$$

Also we have, $r^n = a^n \operatorname{sech} n\theta$

$\therefore$ $\frac{r^n}{a^n} = \operatorname{sech} n\theta$ and we have, $1 - \tanh^2 n\theta = \operatorname{sech}^2 n\theta$

$\therefore$ $\tanh^2 n\theta = 1 - \operatorname{sech}^2 n\theta = 1 - \left(\frac{r^n}{a^n}\right)^2$

We substitute this expression in the RHS of (1).

Thus $\boxed{\frac{1}{p^2} = \frac{1}{r^2}\left\{2 - \frac{r^{2n}}{a^{2n}}\right\}}$ is the required pedal equation.

[32] *For the equiangular spiral $r = a\, e^{\theta \cot \alpha}$, a and α are constants show that the tangent is inclined at a constant angle with the radius vector and hence find the pedal equation of the curve.*

☞ we have $r = a\, e^{\theta \cot \alpha}$

$\Rightarrow$ $\log r = \log a + \theta \cot\alpha \log e$, But $\log e = 1$

$\therefore$ $\log r = \log a + \cot\alpha \cdot \theta$

Differentiating w.r.t. θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha \cdot 1$$

ie., $\cot\phi = \cot\alpha \Rightarrow \phi = \alpha =$ **constant**

Thus the tangent is inclined at a constant angle with the radius vector.

Consider, $p = r\sin\phi$. But $\phi = \alpha$

$\therefore$ $p = r \sin\alpha$. This being independent of θ is the pedal equation.

Thus $\boxed{p = r\sin\alpha}$ is the required pedal equation.

[33] *Show that for the curve* $r\cos\left(\sqrt{a^2-b^2}/a\right)\theta = \sqrt{a^2-b^2}$, $p^2(r^2+b^2) = a^2r^2$

☞ We have $r\cos\left(\sqrt{a^2-b^2}/a\right)\theta = \sqrt{a^2-b^2}$

For convenience, let $\sqrt{a^2-b^2}\,/\,a = k$, a constant.

We now have, $r\cos k\theta = k\,a$

$\Rightarrow$ $\log r + \log(\cos k\theta) = \log(ka)$.

Differentiating w.r.t. θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta}+\frac{-k\sin k\theta}{\cos k\theta}=0$$

ie., $\cot\phi = k\tan k\theta$. We cannot find ϕ explicitly.

Consider, $p = r\sin\phi$.

Squaring and taking the reciprocal, we have,

$$\frac{1}{p^2}=\frac{1}{r^2}\operatorname{cosec}^2\phi \quad \text{or} \quad \frac{1}{p^2}=\frac{1}{r^2}(1+\cot^2\phi)$$

ie., $$\frac{1}{p^2}=\frac{1}{r^2}(1+k^2\tan^2 k\theta) \qquad \ldots (1)$$

Also, $$r\cos k\theta = ka \qquad \ldots (2)$$

We need to eliminate θ from (1) and (2).

From (2), $\cos k\theta = \dfrac{ka}{r} \Rightarrow \sec k\theta = \dfrac{r}{ka}$

Now, $\tan^2 k\,\theta = \sec^2 k\theta - 1 = \dfrac{r^2}{k^2a^2}-1$

Substituting this expression in (1) we get,

$$\frac{1}{p^2}=\frac{1}{r^2}\left\{1+k^2\left(\frac{r^2}{k^2a^2}-1\right)\right\}$$

ie., $$\frac{1}{p^2}=\frac{1}{r^2}\left\{1+\frac{r^2}{a^2}-k^2\right\} \quad \text{But, } k^2=\frac{a^2-b^2}{a^2}$$

$$\therefore \quad \frac{1}{p^2}=\frac{1}{r^2}\left\{1+\frac{r^2}{a^2}-\frac{a^2-b^2}{a^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{a^2 + r^2 - a^2 + b^2}{a^2}\right\} \text{ or } \frac{1}{p^2} = \frac{r^2 + b^2}{r^2 a^2}$$

Thus $\boxed{p^2 (r^2 + b^2) = a^2 r^2}$ as required.

[34] *Establish the pedal equation of the curve* $r^n = a^n \sin n\theta + b^n \cos n\theta$ *in the form* $p^2 (a^{2n} + b^{2n}) = r^{2n+2}$

☞ We have, $r^n = a^n \sin n\theta + b^n \cos n\theta$

$\Rightarrow \quad n \log r = \log (a^n \sin n\theta + b^n \cos n\theta)$

Differentiating w.r.t θ we get,

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{na^n \cos n\theta - nb^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

Dividing by n, $$\cot \phi = \frac{a^n \cos n\theta - b^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

Consider, $p = r \sin \phi$

Since ϕ cannot be found, squaring and taking the reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2 \phi \text{ or } \frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2 \phi)$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}\left\{1 + \frac{(a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{(a^n \sin n\theta + b^n \cos n\theta)^2 + (a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

ie., $$\frac{1}{p^2} = \frac{1}{r^2}\left\{\frac{a^{2n}(\sin^2 n\theta + \cos^2 n\theta) + b^{2n}(\cos^2 n\theta + \sin^2 n\theta)}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right\}$$

(Product terms cancels out in the numerator)

ie., $$\frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{a^{2n} + b^{2n}}{(a^n \sin n\theta + b^n \cos n\theta)^2}$$

or $$\frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{a^{2n} + b^{2n}}{(r^n)^2},$$ by using the given equation.

Thus $\boxed{p^2 (a^{2n} + b^{2n}) = r^{2n+2}}$ is the required pedal equation.

[35] *Find the length of the perpendicular from the pole to the tangent at the point $(a, \pi/2)$ on the curve $r = a\,(1 - \cos\theta)$*

☞ We have, $r = a\,(1 - \cos\theta)$

$\Rightarrow$ $\log r = \log a + \log(1 - \cos\theta)$

Differentiating w.r.t θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot(\theta/2)$$

ie., $\cot\phi = \cot(\theta/2) \Rightarrow \phi = \theta/2$

Length of the perpendicular, $p = r\sin\phi$

ie., $p = r\sin(\theta/2)$

Substituting $(r, \theta) = (a, \pi/2)$ we get $p = a\sin(\pi/4) = a/\sqrt{2}$

Thus $\boxed{p = a/\sqrt{2}}$

ASSIGNMENT

Find the angle between the radius vector and the tangent for the following curves.

[1] $r\sec^2(\theta/2) = 2a$

[2] $r = a\,\mathrm{cosec}^2(\theta/2)$

[3] $r^2 = a^2(\cos 2\theta + \sin 2\theta)$

[4] $r^n\,\mathrm{cosec}\,n\theta = a^n$

Show that the following pairs of curves intersect each other orthogonally.

[5] $r\sec^2(\theta/2) = a$ and $r\,\mathrm{cosec}^2(\theta/2) = b$

[6] $r^n\cos n\theta = a^n$ and $r^n\sin n\theta = b^n$

[7] $2a/r = 1 + \cos\theta$ and $2a/r = 1 - \cos\theta$

[8] $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$

Find the angle of intersection for the following pairs of curves.

[9] $r^n = a^n(\sin n\theta + \cos n\theta)$ and $r^n = a^n\sin n\theta$

[10] $r^2\cos(2\theta + \alpha) = a^2$ and $r^2\cos(2\theta + \beta) = b^2$

Obtain the pedal equation of the following curves.

[11] $r^2 \cos 2\theta = a^2$

[12] $r = 2a/1 + \cos\theta$

[13] $r^2 = a^2 \sin 2\theta + b^2 \cos 2\theta$

[14] $r^n \sec n\theta = a^n$

[15] Show that for the curve $r \sin^2 (\theta/2) = a$ the length of the perpendicular from the pole to the tangent at the point $(2a, \pi/2)$ on the curve is equal to $a\sqrt{2}$.

[16] Show that the length of the perpendicular from the pole to the tangent at the point $\theta = \pi/6$ on the curve $r^2 \cos 2\theta = a^2$ is equal to $a/\sqrt{2}$.

ANSWERS

1. $\pi/2 + \theta/2$
2. $-\theta/2$
3. $\pi/4 + 2\theta$
4. $n\theta$
9. $\pi/4$
10. $\alpha - \beta$
11. $pr = a^2$
12. $p^2 = ar$
13. $r^6 = p^2 (a^4 + b^4)$
14. $p\, a^n = r^{n+1}$

1.3 Curvature and Radius of Curvature

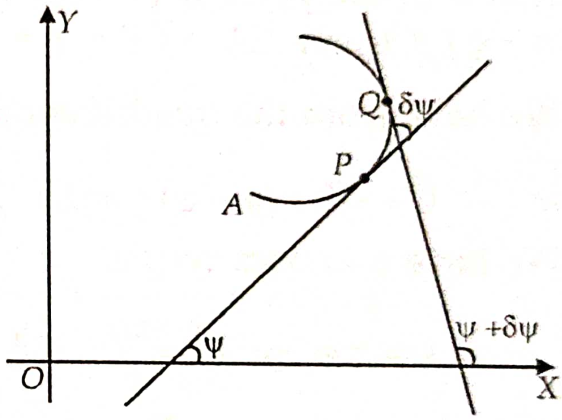
1.31 Introduction

If we traverse in a ghat section (hilly region) where the road is not straight, we often see caution boards "Sharp bend ahead", "hairpin bend ahead" etc. which gives an indication of the difference in the amount of bending of a road at various points which is nothing but *curvature* at various points and we discuss the same in a mathematical way. This aspect is discussed for cartesian and polar form of curves.

1.32 Definition of curvature and radius of curvature

Consider a curve in the XOY plane and let A be a fixed point on it. Let P and Q be two neighbouring points on the curve such that,

$$\overset{\frown}{AP} = s \text{ and } \overset{\frown}{AQ} = s + \delta s \text{ so that } \overset{\frown}{PQ} = \delta s$$



Let ψ and $\psi + \delta\psi$ respectively be the angles made by the tangents at P and Q with the X-axis. The angle $\delta\psi$ between the tangents is called the bending of the curve which depends on δs. $\dfrac{\delta\psi}{\delta s}$ is called as the *mean curvature* of the arc PQ. Also the amount of bending of the curve at P is called as the ***curvature*** of the curve at P and is defined mathematically as

$$\lim_{\substack{\delta s \to 0 \\ (Q \to P)}} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds} \text{ be denoted by } K.$$

That is, ***Curvature*** $= K = \dfrac{d\psi}{ds}$. Further if $K \neq 0$, the reciprocal of the curvature is called as the *radius of curvature* and is denoted by ρ.

That is, ***Radius of curvature*** $= \rho = \dfrac{1}{K} = \dfrac{ds}{d\psi}$

Note :

1. *As it is obvious that ψ depends on s, the relationship between these is called as the intrinsic equation and (s, ψ) are called the intrinsic coordinates of the point P.*

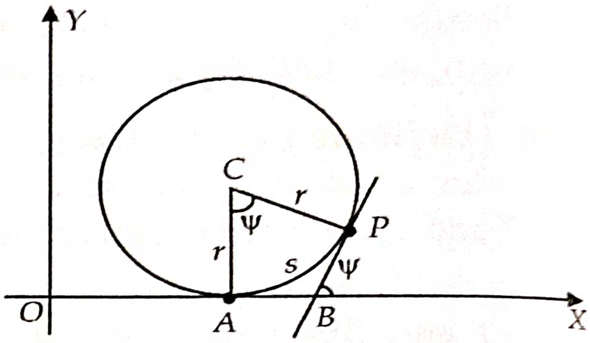
2. *We always take the sign of K and ρ to be positive.*

Remark : *Curvature being the amount of bending is obviously zero for a straight line at all the points on it. It is easy to visualize that the circle has an uniform bending and hence the curvature of a circle is a constant which will be established mathematically.*

A Question Format: *Define curvature and prove that the curvature of a circle is a constant.*

[Definition given already]

Consider a circle of radius r having centre at the point C. Let A be a fixed point on the circle and $P\,(x, y)$ be any point on the the cirlce such that $\overset{\frown}{AP} = s$. Let ψ be the angle made by the tangent at P with the X-axis at the point B (*interior angle being* $\pi - \psi$). Clearly $CA = CP = r =$ radius.



We have from the quadrilateral $CABP$, $\hat{C} + \hat{A} + \hat{B} + \hat{P} = 2\pi$

ie., $\hat{C} + \pi/2 + (\pi - \psi) + \pi/2 = 2\pi$ Hence, $A\hat{C}P = \psi$

We have a known result,

$$s = r\psi \text{ or } \psi = \frac{s}{r} \quad \therefore \quad \frac{d\psi}{ds} = \frac{1}{r} = \text{constant.}$$

Thus the **curvature, $K = 1/r$ = constant.**

- **Expression for the radius of curvature for a cartesian curve $y = f\,(x)$**

Denoting, $y_1 = \dfrac{dy}{dx}$ and $y_2 = \dfrac{d^2y}{dx^2}$, the expression for ρ can be proved in the following form.

$$\boxed{\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}}$$

Note : *Sometimes y_1 at some point on the curve becomes infinity (when the tangent is perpendicular to the x-axis,* $\tan\psi = \tan 90^0 = \infty$ *) in which case we cannot apply the formula for ρ in this form.*

In such a case we have to use the formula in the following alternative form.

$$\boxed{\rho = \frac{(1 + x_1^2)^{3/2}}{x_2}} \quad \text{where } x_1 = \frac{dx}{dy} \text{ and } x_2 = \frac{d^2x}{dy^2}$$

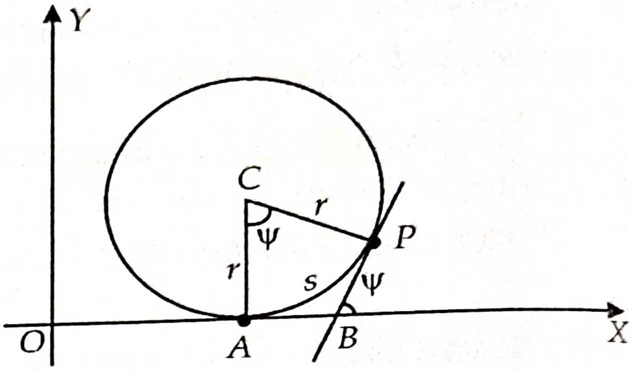
2. *We always take the sign of K and ρ to be positive.*

Remark : *Curvature being the amount of bending is obviously zero for a straight line at all the points on it. It is easy to visualize that the circle has an uniform bending and hence the curvature of a circle is a constant which will be established mathematically.*

A Question Format: *Define curvature and prove that the curvature of a circle is a constant.*

[Definition given already]

Consider a circle of radius r having centre at the point C. Let A be a fixed point on the circle and $P(x, y)$ be any point on the cirlce such that $\overset{\frown}{AP} = s$. Let ψ be the angle made by the tangent at P with the X-axis at the point B (*interior angle being* $\pi - \psi$). Clearly $CA = CP = r =$ radius.



We have from the quadrilateral $CABP$, $\hat{C} + \hat{A} + \hat{B} + \hat{P} = 2\pi$

ie., $\hat{C} + \pi/2 + (\pi - \psi) + \pi/2 = 2\pi$ Hence, $A\hat{C}P = \psi$

We have a known result,

$$s = r\psi \text{ or } \psi = \frac{s}{r} \quad \therefore \quad \frac{d\psi}{ds} = \frac{1}{r} = \text{constant.}$$

Thus the **curvature, $K = 1/r$ = constant.**

- **Expression for the radius of curvature for a cartesian curve $y = f(x)$**

Denoting, $y_1 = \dfrac{dy}{dx}$ and $y_2 = \dfrac{d^2y}{dx^2}$, the expression for ρ can be proved in the following form.

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In such a case we have to use the formula in the following alternative form.

$$\boxed{\rho = \frac{(1 + x_1^2)^{3/2}}{x_2}} \text{ where } x_1 = \frac{dx}{dy} \text{ and } x_2 = \frac{d^2x}{dy^2}$$

WORKED PROBLEMS

[36] *Find the radius of curvature for the curve whose intrinsic equation is* $s = a \log \tan(\pi/4 + \psi/2)$.

☞ $s = a \log \tan(\pi/4+\psi/2)$ and we have $\rho = \dfrac{ds}{d\psi}$

Differentiating w.r.t ψ we have,

$$\frac{ds}{d\psi} = a \cdot \frac{1}{\tan(\pi/4+\psi/2)} \cdot \sec^2(\pi/4+\psi/2) \cdot \frac{1}{2}$$

$$= \frac{a}{2} \cdot \frac{\cos(\pi/4+\psi/2)}{\sin(\pi/4+\psi/2)} \cdot \frac{1}{\cos^2(\pi/4+\psi/2)}$$

$$= \frac{a}{2\sin(\pi/4+\psi/2)\cos(\pi/4+\psi/2)}$$

But, $2\sin\theta\cos\theta = \sin 2\theta$.

$$\therefore \quad \frac{ds}{d\psi} = \frac{a}{\sin[2(\pi/4+\psi/2)]} = \frac{a}{\sin(\pi/2+\psi)} = \frac{a}{\cos\psi} = a\sec\psi$$

Thus, $$\boxed{\rho = a\sec\psi}$$

[37] *Show that the radius of curvature for the catenary of uniform strength* $y = a \log \sec(x/a)$ *is* $a \sec(x/a)$.

☞ We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Consider, $y = a \log \sec(x/a)$

$$\therefore \quad \frac{dy}{dx} = y_1 = \frac{a}{\sec(x/a)} \cdot \sec(x/a)\tan(x/a) \cdot \frac{1}{a}$$

ie., $y_1 = \tan(x/a)$. Also $y_2 = \dfrac{1}{a}\sec^2(x/a)$

Hence, $$\rho = \frac{[1+\tan^2(x/a)]^{3/2} \cdot a}{\sec^2(x/a)} = \frac{a[\sec^2(x/a)]^{3/2}}{\sec^2(x/a)}$$

ie., $\rho = \dfrac{a\sec^3(x/a)}{\sec^2(x/a)} = a\sec(x/a)$

Thus, $\boxed{\rho = a\sec(x/a)}$

[38] *Show that for the catenary $y = c\cosh(x/c)$ the radius of curvature is equal to y^2/c which is also equal to the length of the normal intercepted between the curve and the x-axis.*

☞ We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$y = c\cos h(x/c)$ by data.

$\therefore \quad y_1 = c \cdot \sin h(x/c) \cdot \dfrac{1}{c} = \sin h(x/c)\ ;\ y_2 = \dfrac{1}{c}\cos h(x/c)$

Hence, $\rho = \dfrac{[1+\sin h^2(x/c)]^{3/2}\, c}{\cos h(x/c)} = \dfrac{c[\cos h^2(x/c)]^{3/2}}{\cos h(x/c)}$

ie., $\rho = \dfrac{c\cos h^3(x/c)}{\cos h(x/c)} = c\cos h^2(x/c)$

But $y/c = \cos h(x/c)$ and hence $\rho = c \cdot (y^2/c^2) = y^2/c$

Also we know that the length of the normal (l) is $y\sqrt{1+y_1^2}$

$\therefore \quad l = c\cos h(x/c)\sqrt{1+\sin h^2(x/c)} = c\cos h^2(x/c) = y^2/c$

Thus the required results are proved. $\boxed{\rho = y^2/c = l}$

[39] *Find the radius of curvature for the curve $y = ax^2 + bx + c$ at $x = \dfrac{1}{2a}[\sqrt{a^2-1}-b]$*

☞ $y = ax^2 + bx + c,$ by data.

$\therefore \quad y_1 = 2ax + b, \ y_2 = 2a$

At the given point, $y_1 = 2a \cdot \dfrac{1}{2a}\left[\sqrt{a^2-1}-b\right] + b = \sqrt{a^2-1}$ and $y_2 = 2a$.

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+(a^2-1)]^{3/2}}{2a} = \frac{(a^2)^{3/2}}{2a} = \frac{a^2}{2}$$

Thus, $\boxed{\rho = a^2/2}$

[40] *Find the radius of curvature for the Folium of De-Cartes $x^3 + y^3 = 3a\,xy$ at the point $(3a/2, 3a/2)$ on it.* **[June 2016, 17]**

☞ $x^3 + y^3 = 3a\,xy$, by data.

Differentiating w.r.t x we have,

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y\right)$$

ie., $3(y^2 - ax)\dfrac{dy}{dx} = 3(ay - x^2)$ or $\dfrac{dy}{dx} = y_1 = \dfrac{ay - x^2}{y^2 - ax}$

At $(3a/2, 3a/2)$, $y_1 = \dfrac{3a^2/2 - 9a^2/4}{9a^2/4 - 3a^2/2} = -1$

Next, $\dfrac{d^2y}{dx^2} = y_2 = \dfrac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$

At $(3a/2, 3a/2)$ we note that, $y^2 - ax = 9a^2/4 - 3a^2/2 = 3a^2/4$ and $ay - x^2 = 3a^2/2 - 9a^2/4 = -3a^2/4$.

Hence at $(3a/2, 3a/2)$,

$$y_2 = \frac{(3a^2/4)(-a-3a) - (-3a^2/4)(-3a-a)}{(3a^2/4)^2}$$

ie., $y_2 = \dfrac{-3a^3 - 3a^3}{9a^4/16} = \dfrac{16(-6a^3)}{9a^4} = \dfrac{-32}{3a}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Hence, $\rho = \dfrac{(1+1)^{3/2}}{-32/3a} = \dfrac{2\sqrt{2}\cdot 3a}{-32} = \dfrac{-3\sqrt{2}\,a}{16} = \dfrac{-3a}{8\sqrt{2}}$

Thus $\boxed{|\rho| = 3a/8\sqrt{2}}$

[41] *Find the radius of curvature for the curve* $y^2 = \dfrac{4a^2(2a-x)}{x}$ *where the curve meets the x-axis.*

☞ If the curve meets the x-axis then $y = 0$.

$$\therefore \quad \frac{4a^2(2a-x)}{x} = 0 \Rightarrow 4a^2(2a-x) = 0 \quad \text{or} \quad x = 2a$$

Hence ($2a$, 0) is the point on the curve at which we have to find ρ. The given equation can be put in the form,

$$y^2 = \frac{8a^3}{x} - 4a^2$$

Differentiating w.r.t. x we have, $2yy_1 = \dfrac{-8a^3}{x^2}$ or $y_1 = \dfrac{-4a^3}{x^2 y}$

At ($2a$, 0), y_1 becomes infinity and hence we have to consider dx/dy.

Let, $x_1 = \dfrac{dx}{dy} = \dfrac{-x^2 y}{4a^3}$ and $x_1 = 0$ at $(2a, 0)$.

Now, $x_2 = \dfrac{d^2x}{dy^2} = \dfrac{-1}{4a^3}[x^2\cdot 1 + y\cdot 2x\,x_1]$

$\therefore$ At $(2a,0)$: $x_2 = -4a^2/4a^3 = -1/a$

We have, $\rho = \dfrac{(1+x_1^2)^{3/2}}{x_2}$

$$\rho = \frac{(1+0)^{3/2}}{-1/a} = -a$$

Thus $\boxed{|\rho| = a}$

Note : *A Similar Problem*

Find ρ of the curve $y^2 = \dfrac{a^2(a-x)}{x}$ *at the point* $(a, 0)$ **[Dec 2017, June 18]**

☞ Here we have 'a' instead of '$2a$'. Evidently $\boxed{\rho = a/2}$

[42] *Find the radius of curvature for the curve* $x^2y = a(x^2 + y^2)$ *at* $(-2a, 2a)$.

☞ Consider $x^2 y = a(x^2 + y^2)$ and differentiate w.r.t. x.

$\therefore \quad x^2 y_1 + 2xy = 2a x + 2a yy_1$

ie., $\quad y_1(x^2 - 2ay) = 2ax - 2xy$

or $\quad y_1 = \dfrac{2ax - 2xy}{x^2 - 2ay}$; At$(-2a, 2a)$, y_1 is infinity.

Hence, $x_1 = \dfrac{dx}{dy} = \dfrac{1}{y_1} = \dfrac{x^2 - 2ay}{2ax - 2xy}$ and at $(-2a, 2a)$, we have $x_1 = 0$.

$$\text{Also, } x_2 = \frac{d^2x}{dy^2} = \frac{(2ax-2xy)(2xx_1 - 2a) - (x^2 - 2ay)(2ax_1 - 2x - 2x_1 y)}{(2ax - 2xy)^2}$$

We note that at the point $(-2a, 2a)$,

$(2ax - 2xy) = 4a^2$ and $(x^2 - 2ay) = 0$

$$\therefore \quad (x_2)_{(-2a, 2a)} = \frac{(4a^2)(-2a)}{16a^4} = \frac{-1}{2a}$$

$$\text{We have, } \rho = \frac{(1 + x_1^2)^{3/2}}{x_2} = \frac{(1)^{3/2}}{-1/2a} = -2a$$

Thus, $\boxed{|\rho| = 2a}$

[43] *Find the radius of curvature of the curve* $\sqrt{x} + \sqrt{y} = 4$ *at the point where it cuts the line passing through the origin making an angle 45° with the x-axis.*

☞ The equation of the line is $y = x$ and we shall find the point of intersection of this line with the curve $\sqrt{x} + \sqrt{y} = 4$.

This equation when $y = x$ becomes,

$$\sqrt{x} + \sqrt{x} = 4 \quad \text{or} \quad 2\sqrt{x} = 4 \quad \text{or} \quad \sqrt{x} = 2 \quad \text{or} \quad x = 4$$

$\therefore$ the point of intersection is (4, 4).

Consider, $\sqrt{x}+\sqrt{y} = 4$ and differentiate w.r.t. x

$$\therefore \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y_1 = 0 \text{ or } \frac{y_1}{\sqrt{y}} = \frac{-1}{\sqrt{x}}$$

ie., $y_1 = -\sqrt{y}/\sqrt{x}$. At $(4,4)$ we get $y_1 = -1$

Now, $$y_2 = \frac{d^2y}{dx^2} = \frac{\sqrt{x}\cdot\frac{-1}{2\sqrt{y}}\,y_1-(-\sqrt{y})\cdot\frac{1}{2\sqrt{x}}}{x}$$

$$\therefore \quad \text{At } (4, 4),\ y_2 = \frac{1/2+1/2}{4} = \frac{1}{4}$$

We have, $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(2)^{3/2}}{1/4} = 4\cdot 2\sqrt{2} = 8\sqrt{2}$$

Thus, $$\boxed{\rho = 8\sqrt{2}}$$

[44] *For the curve $y = ax/a+x$, show that* $(2\rho/a)^{2/3} = (x/y)^2+(y/x)^2$. [Jun

☞ $$y = \frac{ax}{a+x}, \text{ by data.}$$

$$\therefore \quad y_1 = \frac{(a+x)a-ax\cdot 1}{(a+x)^2} = \frac{a^2}{(a+x)^2}$$

Also, $$y_2 = \frac{-2a^2}{(a+x)^3}$$

We have, $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Hence, $$\rho = \frac{\left[1+\frac{a^4}{(a+x)^4}\right]^{3/2}\cdot(a+x)^3}{-2a^2}$$

$$\rho = \frac{[(a+x)^4 + a^4]^{3/2} \cdot (a+x)^3}{-2a^2 \left\{ (a+x)^4 \right\}^{3/2}}$$

$$= \frac{[\,(a+x)^4 + a^4\,]^{3/2} \cdot (a+x)^3}{-2a^2\,(a+x)^6}$$

or $$-2\rho = \frac{[\,(a+x)^4 + a^4\,]^{3/2}}{a^2\,(a+x)^3}$$

$$\Rightarrow \quad (-2\rho)^{2/3} = \frac{(a+x)^4 + a^4}{a^{4/3}\,(a+x)^2}$$; We note that $(-2)^{2/3} = 2^{2/3}$

$$\therefore \quad (2\rho)^{2/3} = \frac{1}{a^{4/3}} \left\{ (a+x)^2 + \left(\frac{a^2}{a+x} \right)^2 \right\}$$ But $y = \dfrac{ax}{a+x}$ by data.

$$\therefore \quad (2\rho)^{2/3} = \frac{1}{a^{4/3}} \left\{ \frac{a^2 x^2}{y^2} + \frac{a^2 y^2}{x^2} \right\} = \frac{a^2}{a^{4/3}} \left\{ \frac{x^2}{y^2} + \frac{y^2}{x^2} \right\}$$

ie., $$(2\rho)^{2/3} = a^{2/3} \left\{ \left(\frac{x}{y} \right)^2 + \left(\frac{y}{x} \right)^2 \right\}$$

Thus, $$\boxed{(2\rho/a)^{2/3} = (x/y)^2 + (y/x)^2}$$

[45] *Find the radius of curvature of the curve* $x = a \log(\sec t + \tan t)$, $y = a \sec t$

☞ $x = a \log(\sec t + \tan t)$

$$\frac{dx}{dt} = \frac{a}{\sec t + \tan t} \cdot (\sec t \tan t + \sec^2 t) = \frac{a \sec t(\sec t + \tan t)}{\sec t + \tan t}$$

$$\therefore \quad \frac{dx}{dt} = a \sec t$$

Also, $y = a \sec t$ gives $\dfrac{dy}{dt} = a \sec t \tan t$

Now, $$y_1 = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{a \sec t \tan t}{a \sec t} = \tan t$$

Differentitating w.r.t. x we get, $y_2 = \sec^2 t \dfrac{dt}{dx}$

$$\therefore \quad y_2 = \sec^2 t \cdot \frac{1}{a \sec t} = \frac{\sec t}{a}$$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+\tan^2 t)^{3/2}\, a}{\sec t} = \dfrac{a\sec^3 t}{\sec t} = a\sec^2 t$

Thus

$$\boxed{\rho = a\sec^2 t}$$

[46] *Show that the radius of curvature at any point* θ *on the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *is* $4a\cos(\theta/2)$ **[Dec 2015]**

☞ $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$

$\dfrac{dx}{d\theta} = a(1+\cos\theta)$; $\dfrac{dy}{d\theta} = a\sin\theta$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$$\therefore \quad y_1 = \tan(\theta/2)$$

Differentiating w.r.t. x we get,

$$y_2 = \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{d\theta}{dx}$$

$$= \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{1}{a(1+\cos\theta)} = \frac{\sec^2(\theta/2)}{4a\cos^2(\theta/2)}$$

$$\therefore \quad y_2 = \frac{1}{4a}\sec^4(\theta/2)$$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+\tan^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)}$$

$$\rho = \frac{[\sec^2(\theta/2)]^{3/2} \cdot 4a}{\sec^4(\theta/2)} = \frac{4a\sec^3(\theta/2)}{\sec^4(\theta/2)} = \frac{4a}{\sec(\theta/2)}$$

Thus, $$\boxed{\rho = 4a\cos(\theta/2)}$$

[47] *Find the radius of curvature of the tractrix $x = a\,[\,\cos t + \log\tan(t/2)\,]$, $y = a\sin t$*

☞ For the given curve we have,

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan(t/2)} \cdot \sec^2(t/2) \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\cos(t/2)\sin(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = a\left[\frac{-\sin^2 t + 1}{\sin t}\right] = a \cdot \frac{\cos^2 t}{\sin t}$$

ie., $$\frac{dx}{dt} = a\cos^2 t \operatorname{cosec} t \quad ; \quad \text{Also } \frac{dy}{dt} = a\cos t$$

Now, $$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{a\cos t}{a\cos^2 t \operatorname{cosec} t} = \tan t$$

Further, $$y_2 = \sec^2 t\,\frac{dt}{dx} = \frac{\sec^2 t}{a\cos^2 t\operatorname{cosec} t} = \frac{\sec^4 t\sin t}{a}$$

We have, $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+\tan^2 t)^{3/2} \cdot a}{\sec^4 t\sin t} = \frac{a\sec^3 t}{\sec^4 t\sin t}$$

Thus, $$\boxed{\rho = a\cot t}$$

[48] *Find the radius of curvature of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \pi/4$.*

☞ $x = a\cos^3\theta$; $y = a\sin^3\theta$

$\therefore$ $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$; $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

Now, $y_1 = \dfrac{dy}{dx} = \dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta} = \dfrac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$

Further, $y_2 = -\sec^2\theta\dfrac{d\theta}{dx} = \dfrac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \dfrac{\sec^4\theta\operatorname{cosec}\theta}{3a}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2\theta)^{3/2}\cdot 3a}{\sec^4\theta\operatorname{cosec}\theta} = \frac{3a\sec^3\theta}{\sec^4\theta\operatorname{cosec}\theta} = 3a\cos\theta\sin\theta$$

Thus at $\theta = \pi/4$, $\boxed{\rho = 3a/2}$ since $\cos(\pi/4) = 1/\sqrt{2} = \sin(\pi/4)$.

[49] *Show that the radius of curvature of the curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ is 'at'.*

☞ $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

$\dfrac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t$;

$\dfrac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$

$\therefore$ $y_1 = \dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt} = \dfrac{at\sin t}{at\cos t} = \tan t$

Further, $y_2 = \sec^2 t\,\dfrac{dt}{dx} = \dfrac{\sec^2 t}{at\cos t} = \dfrac{\sec^3 t}{at}$

We have, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+\tan^2 t)^{3/2}}{\sec^3 t}\cdot at = \dfrac{\sec^3 t}{\sec^3 t}\cdot at$

Thus, $\boxed{\rho = at}$

[50] *If ρ be the radius of curvature at any point $P(x, y)$ on the parabola $y^2 = 4ax$, show that ρ^2 varies as $(SP)^3$ where S is the focus of the parabola.*

☞ Consider, $y^2 = 4ax$ and differentiate w.r.t x

$\therefore$ $2yy_1 = 4a$ or $y_1 = 2a/y$

Further, $y_2 = \dfrac{-2a}{y^2} \cdot y_1 = \dfrac{-4a^2}{y^3}$

We have, $\rho = \dfrac{\left(1 + y_1^2\right)^{3/2}}{y_2}$

$$= \frac{\{1 + (4a^2/y^2)\}^{3/2}}{-4a^2/y^3} = \frac{y^3\{(y^2 + 4a^2)/y^2\}^{3/2}}{-4a^2}$$

$$= \frac{y^3}{-4a^2} \cdot \frac{(y^2 + 4a^2)^{3/2}}{(y^2)^{3/2}} = \frac{(y^2 + 4a^2)^{3/2}}{-4a^2}$$

ie., $$\rho = \frac{(4ax + 4a^2)^{3/2}}{-4a^2} = \frac{(4a)^{3/2}(x + a)^{3/2}}{-4a^2}$$

By squaring we have,

$$\rho^2 = \frac{(4a)^3(x + a)^3}{16a^4} = \frac{64a^3(x + a)^3}{16a^4}$$

ie., $$\rho^2 = \frac{4}{a}(x + a)^3 \qquad \ldots (1)$$

The co-ordinates of the focus of the parabola is $S = (a, 0)$ and we have $P = (x, y)$.

$\therefore$ $SP = \sqrt{(x - a)^2 + (y - 0)^2}$ by the distance formula.

$$= \sqrt{x^2 - 2ax + a^2 + y^2} = \sqrt{x^2 - 2ax + a^2 + 4ax}$$

$$= \sqrt{x^2 + 2ax + a^2} = \sqrt{(x + a)^2} = (x + a)$$

Hence, $SP = (x + a)$ and using this result in (1) we have,

$\rho^2 = \dfrac{4}{a}(SP)^3$. That is, $\rho^2 = \text{const.}\,(SP)^3 \Rightarrow \rho^2$ varies as $(SP)^3$

Thus, $$\boxed{\rho^2 \propto (SP)^3}$$

- **Expression for the radius of curvature for a polar curve $r = f(\theta)$**

Denoting, $r_1 = \dfrac{dr}{d\theta}$, $r_2 = \dfrac{d^2r}{d\theta^2}$ the expression for 'ρ' can be proved in the following form.

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}}$$

WORKED PROBLEMS

[51] *Show that for the equiangular spiral $r = a\,e^{\theta \cot \alpha}$ where a and α are constants ρ / r is a constant.*

☞ $r = a\,e^{\theta \cot \alpha}$

$\Rightarrow$ $\log r = \log a + \theta \cot \alpha \log e$. But $\log_e e = 1$

Differentiating w.r.t θ we have,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1 \cdot \cot \alpha \quad \text{or} \quad \frac{dr}{d\theta} = r_1 = r \cot \alpha$$

Hence, $\dfrac{d^2r}{d\theta^2} = r_2 = r_1 \cot \alpha = (r \cot \alpha) \cot \alpha = r \cot^2 \alpha$

We have, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

$\therefore$ $\rho = \dfrac{(r^2 + r^2 \cot^2 \alpha)^{3/2}}{r^2 + 2r^2 \cot^2 \alpha - r^2 \cot^2 \alpha} = \dfrac{(r^2)^{3/2}(\operatorname{cosec}^2 \alpha)^{3/2}}{r^2(1 + \cot^2 \alpha)} = \dfrac{r^3 \operatorname{cosec}^3 \alpha}{r^2 \operatorname{cosec}^2 \alpha}$

Thus $\boxed{\rho/r = \operatorname{cosec} \alpha = \text{constant.}}$

[52] *Show that ρ for the curve $r^n = a^n \cos n\theta$ varies inversely as r^{n-1}*

or

Find the radius of curvature of the curve $r^n = a^n \cos n\theta$ **[Dec 2016]**

☞ $r^n = a^n \cos n\theta$

$\Rightarrow$ $n \log r = n \log a + \log (\cos n\theta)$

Differentiating w.r.t θ we have,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n \sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$\therefore$ $r_1 = -r \tan n\theta$

Further, $r_2 = \dfrac{d^2r}{d\theta^2} = -r_1 \tan n\theta - n\,r \sec^2 n\theta$

We have, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

$$\therefore \quad \rho = \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r(-r_1 \tan n\theta - nr\sec^2 n\theta)}$$

$$= \frac{(r^2)^{3/2}(\sec^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta}$$

$$\rho = \frac{r^3 \sec^3 n\theta}{r^2(1 + \tan^2 n\theta + n\sec^2 n\theta)} = \frac{r\sec^3 n\theta}{\sec^2 n\theta(1+n)} = \frac{r\sec n\theta}{(1+n)}$$

Hence, $\rho = \dfrac{r}{1+n}\sec n\theta$ But $a^n/r^n = \sec n\theta$, by data.

$$\therefore \quad \rho = \frac{r}{1+n}\cdot\frac{a^n}{r^n} \; ; \boxed{\rho = \left[\frac{a^n}{1+n}\right]\frac{1}{r^{n-1}}}$$

That is, $\rho = \text{const} \cdot \dfrac{1}{r^{n-1}}$

Thus $\boxed{\rho \propto 1/r^{n-1}}$

Note : *Similar problem*

Find the radius of curvature for the curve $r^n = a^n \sin n\theta$ **[Dec 2015, 17]**

☞ Proceeding on the same lines we can obtain,

$$\rho = \frac{r}{1+n}\operatorname{cosec} n\theta. \text{ But } a^n/r^n = \operatorname{cosec} n\theta \text{ by data.}$$

Thus, $$\boxed{\rho = \frac{a^n}{1+n}\frac{1}{r^{n-1}}}$$

[53] *Show that for the curve* $r(1-\cos\theta) = 2a$, ρ^2 *varies as* r^3. **[June 2018]**

☞ $r(1-\cos\theta) = 2a$

$\Rightarrow$ $\log r + \log(1-\cos\theta) = \log 2a$

Differentiating w.r.t θ we get,

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0 \quad \text{or} \quad \frac{dr}{d\theta} = \frac{-r\sin\theta}{1-\cos\theta}$$

ie., $\dfrac{dr}{d\theta} = \dfrac{-2r\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -r\cot(\theta/2)$

ie., $r_1 = -r\cot(\theta/2)$

Further, $r_2 = -r \cdot \dfrac{-1}{2}\operatorname{cosec}^2(\theta/2) - r_1\cot(\theta/2)$

ie., $r_2 = \dfrac{r}{2}\operatorname{cosec}^2(\theta/2) + r\cot^2(\theta/2)$

We have, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$

$$\therefore \quad \rho = \frac{\{r^2 + r^2\cot^2(\theta/2)\}^{3/2}}{r^2 + 2r^2\cot^2(\theta/2) - \dfrac{r^2}{2}\operatorname{cosec}^2(\theta/2) - r^2\cot^2(\theta/2)}$$

$$= \frac{(r^2)^{3/2}\{\operatorname{cosec}^2(\theta/2)\}^{3/2}}{r^2\left\{1 + \cot^2(\theta/2) - \dfrac{1}{2}\operatorname{cosec}^2(\theta/2)\right\}}$$

$$= \frac{r\operatorname{cosec}^3(\theta/2)}{1/2 \cdot \operatorname{cosec}^2(\theta/2)} = 2r\operatorname{cosec}(\theta/2)$$

That is, $\rho = 2r\operatorname{cosec}(\theta/2)$

But, $r(1 - \cos\theta) = 2a$, by data.

ie., $r \cdot 2\sin^2(\theta/2) = 2a$ or $\sin^2(\theta/2) = a/r$

$\therefore$ $\operatorname{cosec}(\theta/2) = \sqrt{r/a}$ and hence (1) becomes,

$\rho = 2r \cdot \sqrt{r/a} = 2r^{3/2}/\sqrt{a}$

Hence, $\rho^2 = 4r^3/a = (4/a) \cdot r^3$

Thus, $\boxed{\rho^2 \propto r^3}$

[54] *Find the radius of curvature of the curve $r = a\sin n\theta$ at the pole.*

☞ $r = a\sin n\theta$

$\therefore$ $r_1 = an\cos n\theta$, $r_2 = -an^2\sin n\theta$

At the pole we have, $\theta = 0$. When $\theta = 0$: $r = 0$, $r_1 = an$, $r_2 = 0$

We have, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$

$\therefore \quad \rho = \dfrac{(a^2 n^2)^{3/2}}{2a^2 n^2} = \dfrac{a^3 n^3}{2a^2 n^2} = \dfrac{an}{2}$

Thus $\boxed{\rho = an/2}$ at the pole.

[55] *Show that at the point where the curve $r = a\theta$ intersects the curve $r = a/\theta$, their curvatures are in the ratio 3:1.*

☞ Equating the RHS of the two given equations.

$r = a\theta$ and $r = a/\theta$ we have,

$$a\theta = \frac{a}{\theta} \text{ or } \theta^2 = 1 \qquad \therefore \theta = \pm 1$$

Now, $r = a\theta$, gives $r_1 = a$, $r_2 = 0$

At $\theta = +1$, $r = a$, $r_1 = a$, $r_2 = 0$ . . . (1)

Also $r = a/\theta$ gives $r_1 = -a/\theta^2$, $r_2 = 2a/\theta^3$

At $\theta = +1$, $r = a$, $r_1 = -a$, $r_2 = 2a$ . . . (2)

We have, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$

From (1), $\rho = \dfrac{(a^2 + a^2)^{3/2}}{a^2 + 2a^2} = \dfrac{(2a^2)^{3/2}}{3a^2} = \dfrac{2\sqrt{2}\,a}{3}$ . . . (3)

From (2), $\rho = \dfrac{(a^2 + a^2)^{3/2}}{a^2 + 2a^2 - 2a^2} = \dfrac{(2a^2)^{3/2}}{a^2} = \dfrac{2\sqrt{2}\,a}{1}$ . . . (4)

Hence we have from (3) and (4) the ratio of the corresponding curvatures is

given by $\dfrac{3/2\sqrt{2}\,a}{1/2\sqrt{2}\,a} = \dfrac{3}{1}$, since curvature $K = 1/\rho$.

Thus, the curvatures are in the ratio 3 : 1.

[56] *(a) Show that for the curve $r = a(1 + \cos\theta)$, ρ^2/r is a constant.* [Dec 2017]

(b) If ρ_1 and ρ_2 be the radii of curvatures at the extrimities of the polar chord of the cardioide, show that $\rho_1^2 + \rho_2^2 = 16a^2/9$

☞ (a) $r = a(1 + \cos\theta)$

$\Rightarrow$ $\log r = \log a + \log(1 + \cos\theta)$

Differentiating w.r.t. θ, we have,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan(\theta/2)$$

$\therefore$ $r_1 = -r\tan(\theta/2)$

Further, $r_2 = -\frac{r}{2}\sec^2(\theta/2) - r_1\tan(\theta/2)$

ie., $r_2 = \frac{-r}{2}\sec^2(\theta/2) + r\tan^2(\theta/2)$

We have, $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r\,r_2}$

$$\therefore \quad \rho = \frac{\{r^2 + r^2\tan^2(\theta/2)\}^{3/2}}{r^2 + 2r^2\tan(\theta/2) + \frac{r^2}{2}\sec^2(\theta/2) - r^2\tan^2(\theta/2)}$$

$$= \frac{r^3\{\sec^2(\theta/2)\}^{3/2}}{r^2\left\{1 + \tan^2(\theta/2) + \frac{1}{2}\sec^2(\theta/2)\right\}}$$

$$= \frac{r\sec^3(\theta/2)}{3/2\cdot\sec^2(\theta/2)} = \frac{2r}{3}\sec(\theta/2)$$

$$\rho = \frac{2r}{3}\sec(\theta/2) \qquad \ldots (1$$

But, $r = a(1+\cos\theta) = a\cdot 2\cos^2(\theta/2)$

$\therefore$ $\sec^2(\theta/2) = \dfrac{2a}{r}$ or $\sec(\theta/2) = \dfrac{\sqrt{2a}}{\sqrt{r}}$

Hence (1) becomes $\rho = \dfrac{2r}{3}\dfrac{\sqrt{2a}}{\sqrt{r}}$

That is, $\rho = \dfrac{2}{3}\sqrt{2ar}$

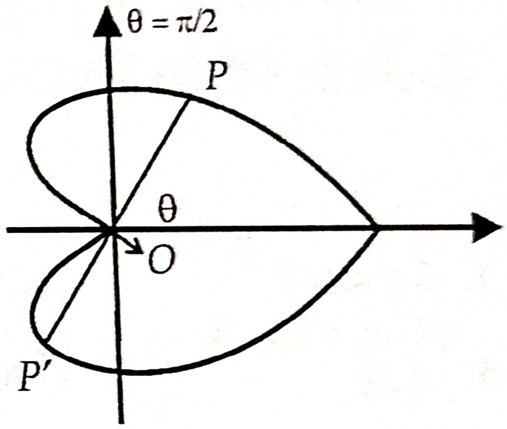
$\therefore \quad \rho^2 = \dfrac{4}{9}(2ar)$ or $\dfrac{\rho^2}{r} = \dfrac{8a}{9}$ = constant.

Thus, $\boxed{\rho^2/r}$ is a constant.

☞ (b) Let POP' be the polar chord (*chord passing through the pole*) of the cardioide $r = a(1+\cos\theta)$. Let ρ_1 and ρ_2 be the radii of curvatures at the point P and P' corresponding to the vectorial angles θ and $(\pi+\theta)$ respectively. We have already obtained,

$$\rho_1 = \frac{2r}{3}\sec(\theta/2)$$

[*first part of this problem*]



$$\therefore \quad \rho_1^2 = \frac{4r^2}{9}\sec^2(\theta/2)$$

But, $r = a(1+\cos\theta) = 2a\cos^2(\theta/2)$

$$\therefore \quad r^2 = 4a^2\cos^4(\theta/2)$$

Hence, $\rho_1^2 = \dfrac{4}{9}\cdot 4a^2\cos^4(\theta/2)\sec^2(\theta/2)$

ie., $\rho_1^2 = \dfrac{16a^2}{9}\cos^2(\theta/2)$ . . . (2)

Now changing θ to $(\pi+\theta)$, we have from (2),

$$\rho_2^2 = \frac{16a^2}{9}\cos^2\left(\frac{\pi+\theta}{2}\right) = \frac{16a^2}{9}\cos^2(\pi/2+\theta/2)$$

ie., $\rho_2^2 = \dfrac{16a^2}{9}\sin^2(\theta/2)$ . . . (3

We shall add (2) and (3).

Thus $\boxed{\rho_1^2+\rho_2^2 = 16a^2/9 = \text{constant.}}$

ASSIGNMENT

Find the radius of curvature for the following curves [1 to 8]

1. $x^{2/3} + y^{2/3} = a^{2/3}$ at any point (x, y)
2. $xy^3 = a^4$ at the point (a, a)
3. $y^2 = x^3 + 8$ at the point (-2, 0)
4. $y = 4 \sin x - \sin 2x$ at the point ($\pi/2$, 4)
5. $y = e^x$ at the point where the curve cuts the y-axis.
6. $x = a \log \sec \theta$, $y = a(\tan \theta - \theta)$
7. $x = a(t - \sin t)$, $y = a(1 - \cos t)$
8. $x = a \cos \theta$, $y = b \sin \theta$ at $(a/\sqrt{2}, b/\sqrt{2})$
9. Show that for the curve $r^2 \sec 2\theta = a^2$, $\rho = a^2/3r$.
10. Show that for the curve $r \cos^2(\theta/2) = a$, ρ^2 varies as r^3.

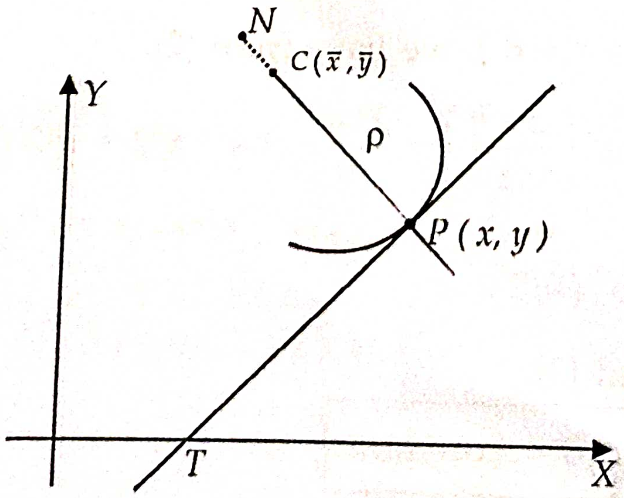
ANSWERS

1. $3(axy)^{1/3}$ 2. $5\sqrt{10}\, a/6$ 3. 6 4. $5\sqrt{5}/4$

5. $2\sqrt{2}$ 6. $a \tan \theta \sec \theta$ 7. $4a \sin(t/2)$ 8. $(a^2 + b^2)^{3/2}/ab$

1.4 Centre of Curvature and Circle of Curvature

The **Centre of curvature** of a point $P(x, y)$ on the given curve is the point $C(\bar{x}, \bar{y})$ which lies in the direction of the inward normal at P and is at distance ρ from it.



PT is the tangent to the curve at P and PN is the inward normal to the curve at P. If $CP = \rho$ then $C(\bar{x}, \bar{y})$ *is the centre of the curvature* to the curve at P.

The coordinates $(\bar{x}, \bar{y})$ of the centre of curvature for the cartesian curve $y = f(x)$ can be proved in the following form.

$$\boxed{\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)\ ;\ \bar{y} = y + \frac{1 + y_1^2}{y_2}}$$

The *Circle of curvature* of the curve at $P(x, y)$ is the circle having centre at the point $C(\bar{x}, \bar{y})$ and radius equal to ρ.

The *equation of the circle of curvature* is given by

$$\boxed{(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2}$$

WORKED PROBLEMS

Find the centre of curvature and the circle of curvature for the following curves.

[57] $y = x + \dfrac{9}{x}$ at $(3,6)$

[58] $y^2 = 12x$ at $(3,6)$

[59] $xy = c^2$ at (c,c)

[60] $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $(a/4, a/4)$

☞ **Solutions**

[57] Consider, $y = x + \dfrac{9}{x}$; $(x,y) = (3,6)$

We shall find y_1 and y_2 and their values at (3, 6).

$$y_1 = 1 - \frac{9}{x^2}\ ,\quad y_2 = 0 - 9(-2x^{-3}) = \frac{18}{x^3}$$

$$\text{At } (3,6):\ y_1 = 1 - \frac{9}{9} = 0\ ,\ y_2 = \frac{18}{27} = \frac{2}{3}$$

Centre of curvature $C = (\bar{x}, \bar{y})$ where,

$$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)\ ;\ \bar{y} = y + \frac{1 + y_1^2}{y_2}$$

$$\bar{x} = 3 - 0(1+0) = 3\ ;\ \bar{y} = 6 + \frac{(1+0)}{(2/3)} = 6 + \frac{3}{2} = \frac{15}{2}$$

Centre of curvature = $(3, 15/2)$

Next, we shall find ρ by using the expression,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\therefore \quad \rho = \frac{(1+0)^{3/2}}{(2/3)} = \frac{3}{2}$$

Circle of curvature : $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

That is, $(x-3)^2 + \left(y - \frac{15}{2}\right)^2 = \left(\frac{3}{2}\right)^2$

Thus the centre and circle of curvature are given by

$$\boxed{(3, 15/2) \text{ and } (x-3)^2 + \left(y - \frac{15}{2}\right)^2 = \frac{9}{4}}$$

[58] Consider $y^2 = 12x$; $(x, y) = (3, 6)$

Differentiating w.r.t x,

$$2yy_1 = 12 \text{ or } y_1 = \frac{6}{y} \text{ ; At } (3,6): y_1 = 1$$

Next, $y_2 = \frac{-6}{y^2} y_1$; At $(3,6)$: $y_2 = \frac{-6}{36} = -\frac{1}{6}$

Substituting these in the formula for the centre of curvature, we have,

$$\bar{x} = 3 - \frac{1}{(-1/6)}(1+1) = 3 + 6(2) = 15$$

$$\bar{y} = 6 + \frac{(1+1)}{(-1/6)} = 6 - 12 = -6$$

Centre of curvature $(\bar{x}, \bar{y}) = (15, -6)$.

Next, we shall substitute the values of y_1 and y_2 in the formula for ρ.

$$\rho = \frac{(1+1)^{3/2}}{(-1/6)} = -6(2\sqrt{2}) \text{ or } \rho = -12\sqrt{2}$$

Circle of curvature : $(x-\overline{x})^2+(y-\overline{y})^2=\rho^2$

That is, $(x-15)^2+(y+6)^2=(-12\sqrt{2})^2=288$

Thus the centre and circle of curvature is given by

$$\boxed{(15,-6) \text{ and } (x-15)^2+(y+6)^2=288}$$

[59] Consider, $xy=c^2$; $(x,y)=(c,c)$

Differentiating w.r.t. x,

$$xy_1+y=0 \text{ or } y_1=\frac{-y}{x} \text{ ; At } (c,c): y_1=-1$$

Next, $$y_2=\frac{x(-y_1)+y(1)}{x^2} \text{ ; At } (c,c): y_2=\frac{c(1)+c}{c^2}=\frac{2}{c}$$

Substituting these in the formula for the centre of curvature, we have

$$\overline{x}=c-\frac{(-1)}{(2/c)}(1+1)=2c$$

$$\overline{y}=c+\frac{(1+1)}{(2/c)}=2c$$

Centre of curvature $(\overline{x},\overline{y})=(2c,2c)$

Next, we shall substitute the values of y_1 and y_2 in the formula for ρ.

$$\rho=\frac{(1+1)^{3/2}}{(2/c)}=\frac{c(2\sqrt{2})}{2} \text{ or } \rho=c\sqrt{2}$$

Circle of curvature : $(x-\overline{x})^2+(y-\overline{y})^2=\rho^2$

That is $(x-2c)^2+(y-2c)^2=\left(c\sqrt{2}\right)^2$

Thus the centre and circle of curvature s given by,

$$\boxed{(2c,2c) \text{ and } (x-2c)^2+(y-2c)^2=2c^2}$$

[60] Consider $\sqrt{x}+\sqrt{y}=\sqrt{a}$; $(x,y)=(a/4,\ a/4)$

Differentiating w.r.t. x,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y_1=0 \text{ or } y_1=-\frac{\sqrt{y}}{\sqrt{x}} \text{ ; At } (a/4,\ a/4): y_1=-1$$

Next, $y_2 = \dfrac{\sqrt{x}\left(\dfrac{-1}{2\sqrt{y}}y_1\right)+\sqrt{y}\left(\dfrac{1}{2\sqrt{x}}\right)}{x}$

At $(a/4, a/4)$: $y_2 = \dfrac{\dfrac{1}{2}+\dfrac{1}{2}}{a/4} = \dfrac{4}{a}$

Substituting these in the formula for the centre of curvature we have,

$$\bar{x} = \frac{a}{4}+\frac{(1+1)}{(4/a)} = \frac{a}{4}+\frac{2a}{4} = \frac{3a}{4}$$

$$\bar{y} = \frac{a}{4}+\frac{(1+1)}{(4/a)} = \frac{a}{4}+\frac{2a}{4} = \frac{3a}{4}$$

Centre of curvature : $(\bar{x},\bar{y}) = (3a/4, 3a/4)$

Next, we shall substitute the values of y_1 and y_2 in the formula for ρ.

$$\rho = \frac{(1+1)^{3/2}}{(4/a)} = \frac{2\sqrt{2}a}{4} = \frac{a}{\sqrt{2}}$$

Circle of curvature $(x-\bar{x})^2+(y-\bar{y})^2 = \rho^2$

That is, $\left(x-\dfrac{3a}{4}\right)^2+\left(y-\dfrac{3a}{4}\right)^2 = \left(\dfrac{a}{\sqrt{2}}\right)^2 = \dfrac{a^2}{2}$

Thus the centre and circle of curvature is given by

$$\boxed{\left(\frac{3a}{4},\frac{3a}{4}\right) \text{ and } \left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2 = \frac{a^2}{2}}$$

1.5 Evolute and Involute

The locus of the centre of curvature of the given curve is called the *Evolute* of the given curve. The given curve is called the *Involute* of its evolute.

Working procedure for problems to find the evolute of the given $y = f(x)$

Step - 1 : We prefer to consider the curve $y = f(x)$ in the parametric form $x = x(t)$ and $y = y(t)$.

Step - 2 : We compute y_1 and y_2 in terms of t.

Step - 3 : We compute the centre of curvature $(\bar{x}, \bar{y})$ for the point (x, y) which will be in terms of t.

Step - 4 : $\bar{x} = \bar{x}(t)$.......(1) $\bar{y} = \bar{y}(t)$..... (2)

We eliminate t from (1) and (2) which results in an expression of the form $F(\bar{x}, \bar{y}) = k$, k being a constant.

Step - 5 : Taking the locus of $(\bar{x}, \bar{y})$ in $F(\bar{x}, \bar{y}) = k$ (*replacing* $\bar{x}, \bar{y}$ *by* x, y) $F(x, y) = k$ will be the required evolute of the given curve $y = f(x)$, which is being referred to as the involute.

Remark : *Standard curves with their name, cartesian form* $y = f(x)$, *the associated parametric from* $x = x(t)$, $y = y(t)$ *is presented in the following table for ready reference.*

Sl. No.	Name of the curve	Cartesian form	Parametric form
1.	(i) Parabola (ii) Parabola	$y^2 = 4ax$ $x^2 = 4ay$	$x = at^2,\ y = 2at$ $x = 2at,\ y = at^2$
2.	Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t,\ y = b\sin t$
3.	Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec t, y = b\tan t$
4.	Rectangular Hyperbola	$xy = c^2$	$x = ct,\ y = \frac{c}{t}$
5.	Astroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3 t,\ y = a\sin^3 t$

WORKED PROBLEMS

[61] *Find the evolute of the parabola* $y^2 = 4ax$.

☞ We shall consider the parametric equation of $y^2 = 4ax$.

$$x = at^2 \quad ; \quad y = 2at$$

$$\therefore \quad \frac{dx}{dt} = 2at \; ; \; \frac{dy}{dt} = 2a$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right) = -\frac{1}{t^2}\frac{dt}{dx} = \frac{-1}{t^2}\cdot\frac{1}{2at} = \frac{-1}{2at^3}$$

We have, $\overline{x} = x - \frac{y_1}{y_2}\left(1+y_1^2\right)$ and $\overline{y} = y + \frac{1+y_1^2}{y_2}$ being the coordinates centre of curvature.

$$\overline{x} = at^2 - \frac{1/t}{-1/2at^3}\left(1+\frac{1}{t^2}\right)$$

$$= at^2 + 2at^2\left(1+\frac{1}{t^2}\right) = at^2 + 2at^2 + 2a$$

That is, $\overline{x} = 3at^2 + 2a$ ...

Next, $\overline{y} = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{\left(-1/2at^3\right)}$

$$\overline{y} = 2at - 2at^3\left(1+\frac{1}{t^2}\right) = 2at - 2at^3 - 2at$$

That is, $\overline{y} = -2at^3$ ...

We shall eliminate t from (1) and (2).

From (1), $t^2 = \frac{\overline{x}-2a}{3a}$

Rising to the power 3/2 on both sides of the above we have,

$$(t^2)^{3/2} = \left(\frac{\overline{x}-2a}{3a}\right)^{3/2} \quad \text{or} \quad t^3 = \left(\frac{\overline{x}-2a}{3a}\right)^{3/2}$$

Using the expression of t^3 in the RHS of (2), we have,

$$\overline{y} = -2a\left(\frac{\overline{x}-2a}{3a}\right)^{3/2}$$

Squaring on both sides we get,

$$(\bar{y})^2 = 4a^2\left(\frac{\bar{x}-2a}{3a}\right)^3 \quad \text{or} \quad (\bar{y})^2 = \frac{4a^2}{27a^3}(\bar{x}-2a)^3$$

ie., $\quad 27a(\bar{y})^2 = 4(\bar{x}-2a)^3$

Now by taking the locus of $(\bar{x}, \bar{y})$ we obtain the required evolute.

Thus the evolute is given by,

$$\boxed{27ay^2 = 4(x-2a)^3}$$

Note : $y^2 = 4ax$ *is a parabola symmetrical about the X - axis.*

$x^2 = 4ay$ *is also a parabola symmetrical about the Y - axis.*

Proceeding on the same lines, the evolute of $x^2 = 4ay$ *works out to be*

$$\boxed{27ax^2 = 4(y-2a)^3}$$

[62] *Find the evolute of the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$.

☞ We shall consider the parametric equation of the given ellipse.

$$x = a\cos t \quad ; \quad y = b\sin t$$

$$\therefore \quad \frac{dx}{dt} = -a\sin t \ ; \quad \frac{dy}{dt} = b\cos t$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{b\cos t}{-a\sin t} = \frac{-b}{a}\cot t$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-b}{a}\cot t\right) = \frac{-b}{a}(-\operatorname{cosec}^2 t)\frac{dt}{dx}$$

ie., $\quad y_2 = \dfrac{b}{a}\operatorname{cosec}^2 t \cdot \dfrac{1}{-a\sin t} \quad$ or $\quad y_2 = \dfrac{-b}{a^2}\operatorname{cosec}^3 t$

We have, $\bar{x} = x - \dfrac{y_1}{y_2}\left(1+y_1^2\right)$ and $\bar{y} = y + \dfrac{1+y_1^2}{y_2}$

$$\bar{x} = a\cos t - \frac{(-b/a)\cot t}{(-b/a^2)\operatorname{cosec}^3 t}\left(1+\frac{b^2}{a^2}\cot^2 t\right)$$

Squaring on both sides we get,

$$(\bar{y})^2 = 4a^2\left(\frac{\bar{x}-2a}{3a}\right)^3 \quad \text{or} \quad (\bar{y})^2 = \frac{4a^2}{27a^3}(\bar{x}-2a)^3$$

ie., $\quad 27a(\bar{y})^2 = 4(\bar{x}-2a)^3$

Now by taking the locus of $(\bar{x},\bar{y})$ we obtain the required evolute.

Thus the evolute is given by,

$$\boxed{27ay^2 = 4(x-2a)^3}$$

Note : $y^2 = 4ax$ *is a parabola symmetrical about the X - axis.*
$x^2 = 4ay$ *is also a parabola symmetrical about the Y - axis.*
Proceeding on the same lines, the evolute of $x^2 = 4ay$ *works out to be*

$$\boxed{27ax^2 = 4(y-2a)^3}$$

[62] *Find the evolute of the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$.

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$$x = a\cos t \quad ; \quad y = b\sin t$$

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$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{b\cos t}{-a\sin t} = \frac{-b}{a}\cot t$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-b}{a}\cot t\right) = \frac{-b}{a}(-\operatorname{cosec}^2 t)\frac{dt}{dx}$$

ie., $\quad y_2 = \dfrac{b}{a}\operatorname{cosec}^2 t\cdot\dfrac{1}{-a\sin t} \quad \text{or} \quad y_2 = \dfrac{-b}{a^2}\operatorname{cosec}^3 t$

We have, $\bar{x} = x - \dfrac{y_1}{y_2}\left(1+y_1^2\right)$ and $\bar{y} = y + \dfrac{1+y_1^2}{y_2}$

$$\bar{x} = a\cos t - \frac{(-b/a)\cot t}{(-b/a^2)\operatorname{cosec}^3 t}\left(1+\frac{b^2}{a^2}\cot^2 t\right)$$

$$\bar{x} = a\cos t - a\sin^3 t \cdot \frac{\cos t}{\sin t}\left(1 + \frac{b^2}{a^2}\frac{\cos^2 t}{\sin^2 t}\right)$$

$$= a\cos t - a\sin^2 t\cos t\left(1 + \frac{b^2}{a^2}\frac{\cos^2 t}{\sin^2 t}\right)$$

$$= a\cos t - a\sin^2 t\cos t - \frac{b^2}{a}\cos^3 t$$

$$= a\cos t(1 - \sin^2 t) - \frac{b^2}{a}\cos^3 t$$

$$= a\cos^3 t - \frac{b^2}{a}\cos^3 t = \cos^3 t\left(a - \frac{b^2}{a}\right)$$

That is, $$\bar{x} = \left(\frac{a^2 - b^2}{a}\right)\cos^3 t \quad \ldots$$

Next, $$\bar{y} = b\sin t + \frac{\left(1 + \frac{b^2}{a}\cot^2 t\right)}{(-b/a^2)\operatorname{cosec}^3 t}$$

$$\bar{y} = b\sin t - \frac{a^2}{b}\sin^3 t\left(1 + \frac{b^2}{a^2}\frac{\cos^2 t}{\sin^2 t}\right)$$

$$= b\sin t - \frac{a^2}{b}\sin^3 t - b\sin t\cos^2 t$$

$$= b\sin t(1 - \cos^2 t) - \frac{a^2}{b}\sin^3 t$$

$$= b\sin^3 t - \frac{a^2}{b}\sin^3 t = \sin^3 t\left(b - \frac{a^2}{b}\right)$$

That is, $$\bar{y} = -\left(\frac{a^2 - b^2}{b}\right)\sin^3 t \quad \ldots$$

We shall eliminate t from (1) and (2).

(**Note :** *We aim to have* $\cos^2 t$, $\sin^2 t$ *and subsequently add*).

Squaring & rising to the power 1/3 on both sides of (1) and (2) we have,

$$(\bar{x})^{2/3} = \left(\frac{a^2-b^2}{a}\right)^{2/3} \cos^2 t \text{ or } (a\bar{x})^{2/3} = (a^2-b^2)^{2/3} \cos^2 t$$

$$(\bar{y})^{2/3} = \left(\frac{a^2-b^2}{b}\right)^{2/3} \sin^2 t \text{ or } (b\bar{y})^{2/3} = (a^2-b^2)^{2/3} \sin^2 t$$

$$\therefore \quad (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}(\cos^2 t + \sin^2 t) = (a^2-b^2)^{2/3}$$

Now, by taking the locus of $(\bar{x}, \bar{y})$ we obtain the required evolute.

Thus the evolute is given by

$$\boxed{(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}}$$

[63] *Find the evolute of the hyperbola* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

☞ We shall consider the parametric equation of the given hyperbola.

$$x = a\sec t \;;\; y = b\tan t$$

$$\therefore \quad \frac{dx}{dt} = a\sec t\tan t \;:\; \frac{dy}{dt} = b\sec^2 t$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b}{a}\cdot\frac{1}{\cos t}\cdot\frac{\cos t}{\sin t} = \frac{b}{a}\operatorname{cosec} t$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a}\operatorname{cosec} t\right) = \frac{b}{a}(-\operatorname{cosec} t\cot t)\frac{dt}{dx}$$

$$ie., \quad y_2 = -\frac{b}{a}\operatorname{cosec} t\cot t\cdot\frac{1}{a\sec t\tan t} = -\frac{b}{a^2}\cdot\frac{1}{\sin t}\cot t\cdot\cos t\cdot\cot t$$

$$ie., \quad y_2 = -\frac{b}{a^2}\cot^3 t$$

We have, $\bar{x} = x - \dfrac{y_1}{y_2}(1+y_1^2)$ and $\bar{y} = y + \dfrac{1+y_1^2}{y_2}$

$$\bar{x} = a\sec t - \frac{(b/a)\operatorname{cosec} t}{(-b/a^2)\cot^3 t}\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2 t\right)$$

$$= a\sec t + a\frac{1}{\sin t}\cdot\frac{\sin^3 t}{\cos^3 t}\left(1+\frac{b^2}{a^2}\frac{1}{\sin^2 t}\right)$$

$$= a\sec t + \frac{a\sin^2 t}{\cos^3 t}\left(1+\frac{b^2}{a^2}\frac{1}{\sin^2 t}\right)$$

$$= a\sec t + \frac{a\sin^2 t}{\cos^3 t} + \frac{b^2}{a}\cdot\frac{1}{\cos^3 t}$$

$$= a\sec t + \frac{a(1-\cos^2 t)}{\cos^3 t} + \frac{b^2}{a}\sec^3 t$$

$$= a\sec t + a\sec^3 t - a\sec t + \frac{b^2}{a}\sec^3 t$$

$$\bar{x} = \sec^3 t\left(a+\frac{b^2}{a}\right) = \left(\frac{a^2+b^2}{a}\right)\sec^3 t$$

That is, $$\bar{x} = \left(\frac{a^2+b^2}{a}\right)\sec^3 t$$

Next, $$\bar{y} = b\tan t + \frac{\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2 t\right)}{(-b/a^2)\cot^3 t}$$

$$= b\tan t - \frac{a^2}{b}\frac{\sin^3 t}{\cos^3 t}\left(1+\frac{b^2}{a^2}\frac{1}{\sin^2 t}\right)$$

$$= b\tan t - \frac{a^2}{b}\frac{\sin^3 t}{\cos^3 t} - b\frac{\sin t}{\cos^3 t}$$

$$= b\tan t - \frac{a^2}{b}\tan^3 t - b\tan t\sec^2 t$$

$$= b\tan t - \frac{a^2}{b}\tan^3 t - b\tan t(1+\tan^2 t)$$

$$= b\tan t - \frac{a^2}{b}\tan^3 t - b\tan t - b\tan^3 t$$

$$= -\left(\frac{a^2}{b}+b\right)\tan^3 t = -\left(\frac{a^2+b^2}{b}\right)\tan^3 t$$

That is, $$\bar{y} = -\left(\frac{a^2+b^2}{b}\right)\tan^3 t \quad \ldots (2)$$

We shall eliminate t from (1) and (2)

Note : *We aim to have,* $\sec^2 t$, $\tan^2 t$ *and subsequently subtract.*

Squaring & rising to the power 1/3 on both sides of (1) and (2) we have,

$$(\bar{x})^{2/3} = \left(\frac{a^2+b^2}{a}\right)^{2/3}\sec^2 t \text{ or } (a\bar{x})^{2/3} = (a^2+b^2)^{2/3}\sec^2 t$$

$$(\bar{y})^{2/3} = \left(\frac{a^2+b^2}{b}\right)^{2/3}\tan^2 t \text{ or } (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}\tan^2 t$$

$$\therefore \quad (a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}(\sec^2 t - \tan^2 t) = (a^2+b^2)^{2/3}$$

Now, by taking the locus of $(\bar{x},\bar{y})$ we obtain the required evolute.

Thus the evolute is given by,

$$\boxed{(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}}$$

[64] *Find the evolute of the curve $xy = c^2$.*

☞ We shall consider the parametric equation of the given curve.

$$x = ct \quad ; \quad y = \frac{c}{t}$$

$$\therefore \quad \frac{dx}{dt} = c \quad ; \quad \frac{dy}{dt} = -\frac{c}{t^2}$$

$$y_1 = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{-c/t^2}{c} = \frac{-1}{t^2}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-1}{t^2}\right) = \frac{2}{t^3}\frac{dt}{dx} = \frac{2}{t^3}\cdot\frac{1}{c} = \frac{2}{ct^3}$$

We have, $\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right)$ and $\bar{y} = y + \frac{1 + y_1^2}{y_2}$

$$\bar{x} = ct - \frac{(-1/t^2)}{(2/ct^3)}\left(1 + \frac{1}{t^4}\right)$$

$$= ct + \frac{ct}{2}\left(1 + \frac{1}{t^4}\right) = ct + \frac{ct}{2} + \frac{c}{2t^3}$$

That is, $$\bar{x} = \frac{3ct}{2} + \frac{c}{2t^3}$$

Next, $$\bar{y} = \frac{c}{t} + \frac{(1 + 1/t^4)}{(2/ct^3)}$$

$$= \frac{c}{t} + \frac{ct^3}{2}\left(1 + \frac{1}{t^4}\right) = \frac{c}{t} + \frac{ct^3}{2} + \frac{c}{2t}$$

That is, $$\bar{y} = \frac{3}{2}\frac{c}{t} + \frac{ct^3}{2}$$

We shall eliminate t from (1) and (2).

Note : *We have to adopt a different technique.*

We add and also subtract (1) and (2).

$$\bar{x} + \bar{y} = \frac{3ct}{2} + \frac{c}{2t^3} + \frac{3}{2}\frac{c}{t} + \frac{ct^3}{2}$$

$$= \frac{c}{2}\left[t^3 + \frac{1}{t^3} + 3\left(t + \frac{1}{t}\right)\right]$$

or $$\bar{x} + \bar{y} = \frac{c}{2}\left[t + \frac{1}{t}\right]^3$$

Also, $$\bar{x} - \bar{y} = \frac{3ct}{2} + \frac{c}{2t^3} - \frac{3}{2}\frac{c}{t} - \frac{ct^3}{2}$$

$$= \frac{-c}{2}\left[t^3 - \frac{1}{t^3} - 3\left(t - \frac{1}{t}\right)\right]$$

or $$(\bar{x}-\bar{y}) = \frac{-c}{2}\left[t-\frac{1}{t}\right]^3 \qquad \ldots (4)$$

We shall square & rise to the power 1/3 on both sides of (3) and (4).

$\therefore$ $$(\bar{x}+\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[t+\frac{1}{t}\right]^2 \qquad \ldots (5)$$

and $$(\bar{x}-\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[t-\frac{1}{t}\right]^2 \qquad \ldots (6)$$

Now (5) – (6) yields,

$$(\bar{x}+\bar{y})^{2/3}-(\bar{x}-\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[t^2+\frac{1}{t^2}+2-t^2-\frac{1}{t^2}+2\right]$$

ie., $$(\bar{x}+\bar{y})^{2/3}-(\bar{x}-\bar{y})^{2/3} = \frac{c^{2/3}}{2^{2/3}}(4) = \frac{c^{2/3}}{2^{2/3}}(2^2) = 2^{4/3}c^{2/3} = (2^2)^{2/3}(c)^{2/3}$$

or $$(\bar{x}+\bar{y})^{2/3}-(\bar{x}-\bar{y})^{2/3} = (4c)^{2/3}$$

Now, by taking the locus of $(\bar{x}, \bar{y})$ we obtain the required evolute.

Thus the evolute is given by,

$$\boxed{(x+y)^{2/3}-(x-y)^{2/3} = (4c)^{2/3}}$$

[65] *Find the evolute of the curve* $x^{2/3}+y^{2/3} = a^{2/3}$

☞ We shall consider the parametric equation of the given curve.

$$x = a\cos^3 t \quad ; \quad y = a\sin^3 t$$

$\therefore$ $$\frac{dx}{dt} = -3a\cos^2 t\sin t \quad ; \quad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\tan t) = -\sec^2 t\frac{dt}{dx}$$

ie., $$y_2 = -\sec^2 t\cdot\frac{1}{-3a\cos^2 t\sin t} = \frac{1}{3a\sin t\cos^4 t}$$

We have, $\bar{x} = x-\dfrac{y_1}{y_2}(1+y_1^2)$ and $\bar{y} = y+\dfrac{1+y_1^2}{y_2}$

$$\bar{x} = a\cos^3 t - \frac{(-\tan t)}{1/3a\sin t\cos^4 t}(1+\tan^2 t)$$

$$= a\cos^3 t + 3a\sin t\cos^4 t \cdot \tan t(\sec^2 t)$$

$$= a\cos^3 t + 3a\sin t\cos^4 t \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos^2 t}$$

That is, $\bar{x} = a(\cos^3 t + 3\sin^2 t\cos t)$ ... (1)

Next, $\bar{y} = a\sin^3 t + \dfrac{(1+\tan^2 t)}{1/3a\sin t\cos^4 t}$

$$= a\sin^3 t + 3a\sin t\cos^4 t \cdot \sec^2 t$$

$$= a\sin^3 t + 3a\sin t\cos^2 t$$

That is, $\bar{y} = a(\sin^3 t + 3\sin t\cos^2 t)$ ... (2)

We shall eliminate t from (1) and (2).

Note : *We adopt the same technique as in the previous problem*

$$\bar{x}+\bar{y} = a(\cos^3 t + \sin^3 t + 3\cos t\sin^2 t + 3\cos^2 t\sin t)$$

and $\bar{x}-\bar{y} = a(\cos^3 t - \sin^3 t + 3\cos t\sin^2 t - 3\cos^2 t\sin t)$

or $\bar{x}+\bar{y} = a(\cos t + \sin t)^3$ ... (3)

and $\bar{x}-\bar{y} = a(\cos t - \sin t)^3$ ... (4)

[*Recall*, $(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$ *and* $(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$]

We shall square & rise to the power 1/3 on both sides of (3) and (4).

$\therefore$ $(\bar{x}+\bar{y})^{2/3} = a^{2/3}(\cos t + \sin t)^2$ ... (5)

and $(\bar{x}-\bar{y})^{2/3} = a^{2/3}(\cos t - \sin t)^2$ ... (6)

Now, (5) + (6) yields,

$$(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = a^{2/3}(1 + 2\cos t\sin t + 1 - 2\cos t\sin t)$$

by expanding and using $\cos^2 t + \sin^2 t = 1$ in RHS.

ie., $(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = 2a^{2/3}$

Now, by taking the locus of $(\bar{x}, \bar{y})$ we obtain the required evolute.

Thus the evolute is given by,

$$\boxed{(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}}$$