3 mon

Vibrations

The to and fro motion of a body about its mean position is called oscillations or Vibrations.

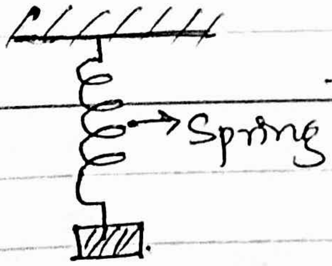
There are three parts of oscillations.

i) Free Oscillation
ii) Damped "
iii) Forced "

4 tue

i) Free Oscillation / vibration is defined as the body which vibrates with definite interval and with natural frequency.

Ex:- SHO. – Simple Harmonic Oscillation.



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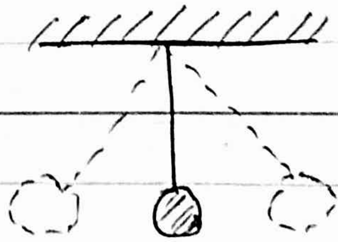

Simple pendulum.

A particle is said to execute SHM when it vibrates periodically in such a way that at any instant, the restoring force acting on it is proportional to its displacement from mean position and oppositely directed.

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Notes

6 thu

$\therefore F_{Restoring} \propto -x \longrightarrow (1)$

Where F is the restoring force acting on the oscillator when its displacement from equilibrium position is 'x'.

$$F_R = -kx. \longrightarrow (2)$$

7 fri

Now according to Newton's 2nd law.

$$F = ma. \longrightarrow (3).$$

From eqn (2) and (3).

$$ma = -kx.$$

8 sat

$$ma + kx = 0 \longrightarrow (4)$$

9 sun

We know that, $a = \frac{d^2x}{dt^2}$

Eqn (4) $\Rightarrow$ $m\frac{d^2x}{dt^2} + kx = 0.$

Divide by 'm' to above eqn.

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0.$$

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Notes

Let us substitute $\frac{K}{m} = \omega^2$ in above eqn.

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \longrightarrow (5).$$

$\omega = \sqrt{\frac{K}{m}}$. Angular freq.

Eqn (5) represents the differential of Simple Harmonic oscillator / vibration.

Solution of differential equation can be given as

$x = e^{\alpha t}$ $\longrightarrow$ (6) where x is the displacement or amplitude

Differentiating above 'x'

$$\frac{dx}{dt} = \alpha . e^{\alpha t} \longrightarrow (7) \Rightarrow \frac{d^2x}{dt^2} = \alpha^2 . e^{\alpha t} \longrightarrow (8)$$

Substituting (7) and (8) in eqn (5).

$$\alpha^2 . e^{\alpha t} + \omega^2 e^{\alpha t} = 0.$$

$$e^{\alpha t} (\alpha^2 + \omega^2) = 0.$$

$$e^{\alpha t} \neq 0 \; ; \; \therefore \alpha^2 + \omega^2 = 0.$$

13 thu

$$\alpha^2 = -\omega^2.$$

$$\therefore \alpha = \pm i\omega.$$

General solution of eqn ⑤ is given as

$$x = A.e^{i\omega t} + Be^{-i\omega t} \longrightarrow (9)$$

where A and B are constants

14 fri

Now $e^{i\omega t} = \cos\omega t + i\sin\omega t.$

$e^{-i\omega t} = \cos\omega t - i\sin\omega t.$

$$(9) \Rightarrow x = A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t).$$

$$\Rightarrow x = A\cos\omega t + iA\sin\omega t + B\cos\omega t - Bi\sin\omega t.$$

15 sat

$$x = \cos\omega t\,(A+B) + i\sin\omega t\,(A-B). \longrightarrow (10)$$

Replacing $(A+B)$ and $i(A-B)$ by some constants

$$A+B = R\sin\phi \quad \text{and} \quad i(A-B) = R\cos\phi.$$

$$(10) \Rightarrow x = \cos\omega t\,.\,R\sin\phi + \sin\omega t\,.\,R\cos\phi.$$

$$x = R(\cos\omega t\,.\,\sin\phi + \sin\omega t\,.\,\cos\phi).$$

$$\boxed{x = R\sin(\omega t+\phi)} \longrightarrow (11).$$

Eqn (11) represents the final solⁿ of SHM.

where $R \rightarrow$ Amplitude of the oscillatory system.
$(\phi+\omega t) \rightarrow$ Phase of vibration.
$x = R\sin(\omega t+\phi) \rightarrow$ represents the displacements of SHM. (amplitude)

We now calculate the time period and frequency of SHM.

Time period is defined as the time required for one complete oscillation.

$$t = 2\pi/\omega \Rightarrow 2\pi/\sqrt{k/m}$$

$\because \omega^2 = \frac{k}{m}$.
$\omega = \sqrt{k/m}$

Frequency of SHM:

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{\sqrt{k/m}}{2\pi}.$$

Significance of 'ω' in eqn (11).

If 't' is increased to $2\pi/\omega$

ie $t = t + \frac{2\pi}{\omega}$.

$\therefore$ (11) $\Rightarrow$ $x = R\sin\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right)$.

$= R\sin(\omega t + 2\pi + \phi)$.

$x = R\sin(2\pi + (\omega t + \phi))$

$\boxed{x = R\sin(\omega t + \phi)}$ $\because \sin(2\pi + \theta) = \sin\theta$.

which means displacement is same after time $t = 2\pi/\omega$.

So, the period $\boxed{t = \frac{2\pi}{\omega}}$ is the time period.

ii) Damped oscillation / vibration — For a free oscillation the energy remains constant and hence oscillator continues indefinitely.

But in reality, the amplitude of the oscillatory system gradually decreases due to the damping forces like friction (air) and resistance of the media.

"The oscillator whose amplitude, in successive oscillations goes on decreasing due to the presence of resistive forces are called

25 tue

as damped oscillators and oscillations are called damped oscillations (DHO)."

The damping force always act in a opposite directions to the motion of oscillatory body and velocity dependent.

$F_{damping} \propto -v. \Rightarrow F_{damping} = -\gamma v.$

26 wed

$\therefore F_{damping} = -\gamma v. \rightarrow (1)$ $\gamma \rightarrow$ Damping constant. $v \rightarrow$ Damping force velocity.

Now $F_{restoring} = -kx. \rightarrow (2)$

According to Newton's 2nd law of motion.

$F = ma. \rightarrow (3)$.

Total Force, $F = F_{Restoring} + F_{Damping}$.

$\therefore \; ma = -kx - r\frac{dx}{dt}$.

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0.$$

Divide by 'm' in the above eqn.

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = 0.$$

Substitute $r/m = 2k$ and $k/m = \omega^2$ in above eqn.

$$\therefore \boxed{\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0.} \longrightarrow (4).$$

where "2k" is the damping coefficient and "ω" is the natural freq. of oscillating body.

Eqn (4) represents the differential eqn of Damped Harmonic Oscillator.

This is second order linear differential Homogeneous equation in x.

The general solution of the above eqn can be written as.

$x = A.e^{\alpha t}$. x is the amplitude

$\frac{dx}{dt} = A.\alpha.e^{\alpha t} \Rightarrow \frac{d^2x}{dt^2} = A.\alpha^2 e^{\alpha t}$.

④ $\Rightarrow A.\alpha^2.e^{\alpha t} + 2k.A\alpha e^{\alpha t} + w^2 A e^{\alpha t} = 0$.

$\Rightarrow Ae^{\alpha t}\left[\alpha^2 + 2k\alpha + w^2\right] = 0$.

Since $Ae^{\alpha t} \neq 0$ but $\alpha^2 + 2k\alpha + w^2 = 0$

Quadratic eqn.

The two roots of the Quad. eqn. are.

$\alpha = \frac{-2k \pm \sqrt{4k^2 - 4.1.w^2}}{2.1} = \frac{-2k \pm 2\sqrt{k^2 - w^2}}{2}$

$\alpha = -k \pm \sqrt{k^2 - w^2} \longrightarrow$ ⑤.

$\therefore$ α in eqn ⑤ has two values.

$\alpha_1 = -k + \sqrt{k^2 - w^2}$ and $\alpha_2 = -k - \sqrt{k^2 - w^2}$

Therefore, general soln of differential eqn is.

$x = A.e^{\alpha_1 t} + B.e^{\alpha_2 t}$.

$x = A.e^{(-k + \sqrt{k^2 - w^2})t} + B.e^{(-k - \sqrt{k^2 - w^2})t} \longrightarrow$ ⑥

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Notes

Eqn (6) represents the amplitude of D.H.O.

The actual form of eqn (6) depends on whether

(a) $k^2 > \omega^2$ Heavy damping

(b) $k^2 < \omega^2$ Low "

(c) $k^2 = \omega^2$ Critical "

(a) Case-I $k^2 > \omega^2$ (Heavy damping)

If the damping is high, then $\sqrt{k^2-\omega^2} = \beta$ is a real quantity and positive.

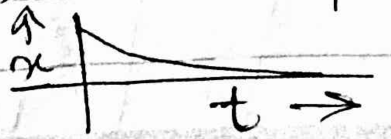
Then eqn (6) $\Rightarrow$

$$x = e^{-kt}\left[A.e^{(\sqrt{k^2-\omega^2})t} + B.e^{(-\sqrt{k^2-\omega^2})t}\right] \longrightarrow (7)$$

$$x = e^{-kt}\left[A.e^{\beta t} + B.e^{-\beta t}\right]$$

$$x = A.e^{-(k-\beta)t} + B.e^{-(k+\beta)t}.$$

Since both the exponents are negative then the displacement 'x' decreases continuously with the time. That is if the particle when once displaced return to its equilibrium position quite slowly.



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Notes Non oscillatory or aperiodic motion.

Ex:- Pendulum in viscous medium.

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(b) Case - II $k^2 < w^2$ — Under damping / Low damping.
there $\sqrt{k^2-w^2}$ is imaginary.

So. eqn (6) $\sqrt{k^2-w^2} = \sqrt{-(w^2-k^2)} = i\sqrt{w^2-k^2} = i\omega$
where $\sqrt{-1} = i$ and $\omega = \sqrt{w^2-k^2}$

$\therefore$ Eqn (6) $\Rightarrow x = e^{-kt}\left[A e^{i\omega t} + B e^{-i\omega t}\right]$

$x = e^{-kt}\left[A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)\right]$

$x = e^{-kt}\left[(A+B)\cos\omega t + i(A-B)\sin\omega t\right]$

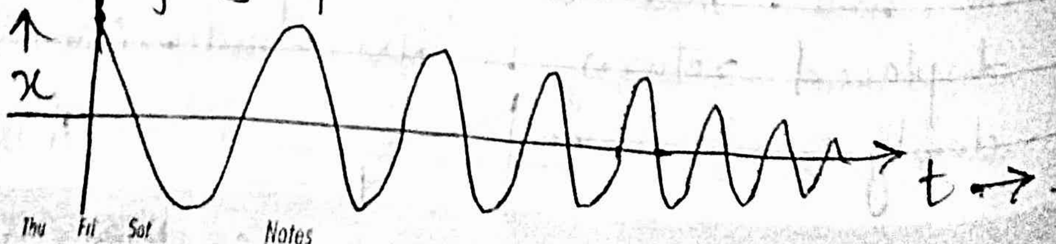
$\Longrightarrow A+B = a_0 \sin\phi \quad ; \quad i(A-B) = a_0\cos\phi.$

$\therefore x = e^{-kt}\left[a_0 \sin\phi\cos\omega t + a_0\cos\phi\sin\omega t\right].$

$x = a_0 e^{-kt}\left[\sin\phi\cos\omega t + \cos\phi\sin\omega t\right]$

$\boxed{x = a_0 e^{-kt}\sin(\omega t+\phi).} \longrightarrow (8).$

$a_0 e^{-kt}$ — represents amplitude of oscillation and it decays exponentially with time due to e^{-kt}.
e^{-kt} — Damping factor.

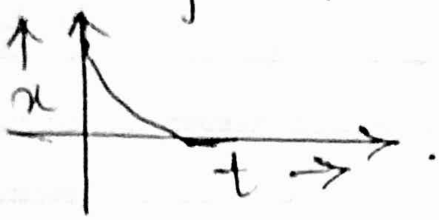


(c) Case III $k^2 = \omega^2$ (critical damping)

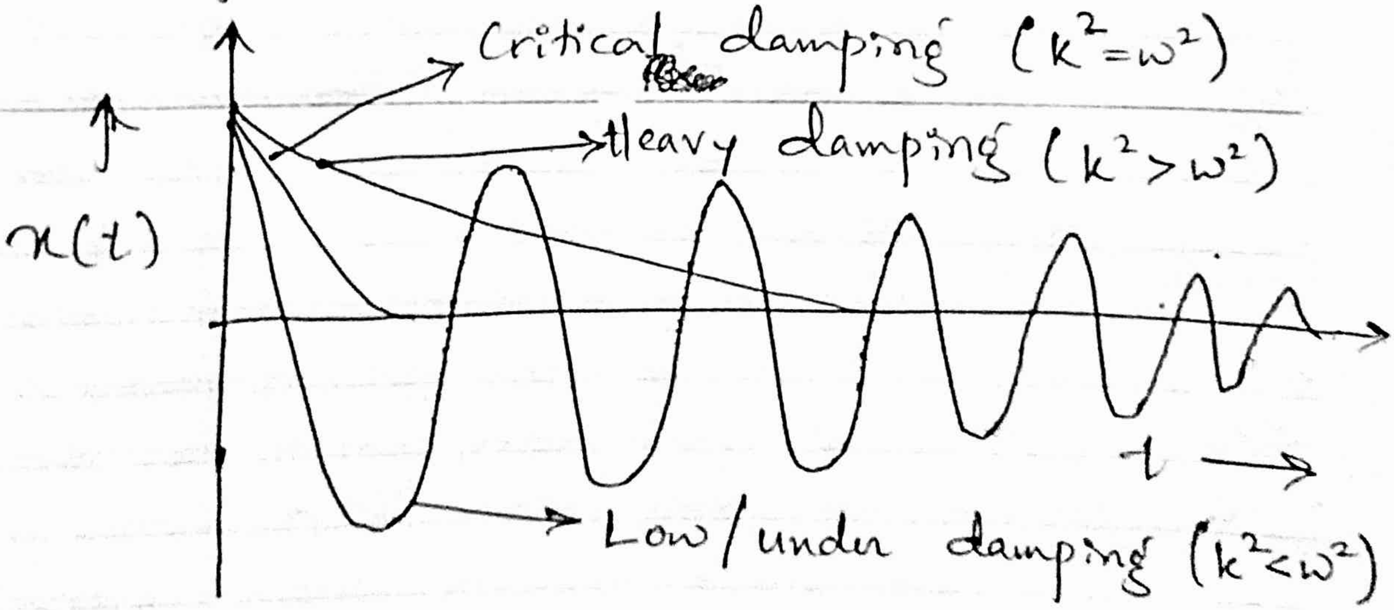
Eqn (6) will be like $x = (A + B) e^{-kt}$.

Therefore, the displacement approaches to zero asymptotically for given value of initial possition.

Here, the critically damped oscillator approaches to the equilibrium possition more rapidly than compared to heavy damped.



Curves of all the three cases:



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Notes

iii) Forced/Driven Oscillator (FHO):

When oscillator is subjected to external periodic force and it oscillates with the frequency of external periodic force, it is called as Forced Harmonic Oscillator (FHO).

15 tue

$$F = F_{Restoring} + F_{damping} + F_{ext\text{-}periodic\ force}$$

$$= -kx - \gamma v + F_0 \sin pt.$$

$$ma = -kx - \gamma \frac{dx}{dt} + F_0 \sin pt.$$

$$m\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = F_0 \sin pt.$$

16 wed Divide the above eqn by m.

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \left(\frac{k}{m}\right)x = \frac{F_0}{m} \sin pt.$$

$$\boxed{\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega^2 x = f_0 \sin pt} \longrightarrow (1).$$

Eqn (1) is the differential eqn of FHO.

where $\frac{\gamma}{m} = 2K$; $\frac{k}{m} = \omega^2$; $\frac{F_0}{m} = f_0$.

17 thu

Let us assume the solution of eqn (1) as.

$$x = A \sin(pt - \theta)$$

where p is the driving freqnency.

$$\frac{dx}{dt} = Ap\cos(pt-\theta) \qquad \frac{d^2x}{dt^2} = -Ap^2\sin(pt-\theta).$$

18 fri Substituting above values in eqn (1).

$$-Ap^2\sin(pt-\theta) + 2KAp\cos(pt-\theta) + w^2 A\sin(pt-\theta) = f_0 \sin((pt-\theta)+\theta).$$

$\to \sin(A+B)$

$$\Rightarrow \sin(pt-\theta)(w^2 A - Ap^2) + 2KAp\cos(pt-\theta) = f_0 \sin(pt-\theta)\cos\theta + f_0\cos(pt-\theta)\sin\theta.$$

$$\Rightarrow A(w^2-p^2)\sin(pt-\theta) + 2KAp\cos(pt-\theta) = f_0\sin(pt-\theta)\cos\theta + f_0\sin\theta\cos(pt-\theta) \to (2)$$

Now case-1 If $(pt-\theta)=0$ then $\because \sin 0 = 0$

eqn (2) $\Rightarrow 2KAp = f_0\sin\theta \to (3)$ $\quad \cos 0 = 1$

Case-2 If $(pt-\theta) = \pi/2$ then $\because \sin\pi/2 = 1$

eqn (2) $\Rightarrow (w^2-p^2)A = f_0\cos\theta \to (4)$ $\quad \cos\pi/2 = 0.$

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Notes

Squaring and adding eqn (3) and (4) we get

$$4K^2p^2A^2 + A^2(\omega^2 - p^2)^2 = f_0^2(\sin^2\theta + \cos^2\theta)$$

$$A^2\left[4K^2p^2 + (\omega^2 - p^2)^2\right] = f_0^2.$$

$$\boxed{A = \frac{f_0}{\sqrt{(\omega^2 - p^2)^2 + 4K^2p^2}}} \rightarrow (5)$$

22 tue

Now let us substitute eqn (5) in $x = A\sin(pt - \theta)$

$$\boxed{x = \frac{f_0}{\sqrt{(\omega^2 - p^2) + 4K^2p^2}} \sin(pt - \theta)} \rightarrow (6)$$

Eqn (6) is the General Solution eqn of FHO.

23 wed

(a) <u>Case - I</u> When $\omega >> p$ and damping is low

ie., Driving frequency is very less than the natural frequency.

Eqn (5) $\Rightarrow$ $A = \frac{f_0}{\omega^2} = \frac{F_0/m}{k/m} = \frac{F_0}{k}.$

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Notes

24 thu

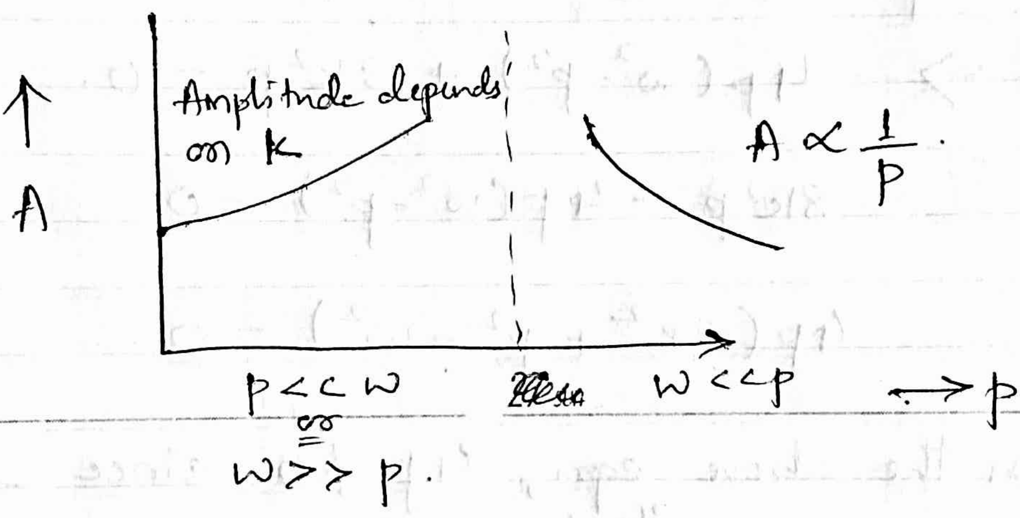
In this case amplitude depends on 'k' force constant only but not on the mass, damping and frequency of driving force.

(b) <u>Case - II</u> When $\omega << p$ and damping is low.

then Eqn (5) $\Rightarrow$ $A \cong \frac{f_0}{p^2}$

25 fri

$\Rightarrow$ If 'p' increases then amplitude 'A' will decrease



26 sat

(c) <u>Case - III</u> When $\omega = p$ (Resonance case).

In this case the amplitude will be maximum.

Eqn (5) $\Rightarrow$ $A = \frac{f_0}{\sqrt{4k^2p^2}}$ $\Rightarrow A = \frac{f_0}{2kp}$.

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but in order to get 'A' maximum then we should get denominator of eqn ⑤ be minimum.

Consider $U = (\omega^2 - p^2)^2 + 4k^2p^2$ should be minimum.

also $\frac{du}{dp}$ should be minimum.

1 tue

$$\therefore \frac{dU}{dp} = 2(\omega^2 - p^2)(-2p) + 8k^2p = 0.$$

$$\Rightarrow -4p(\omega^2 - p^2) + 8k^2p = 0.$$

$$8k^2p - 4p(\omega^2 - p^2) = 0.$$

$$4p(2k^2 + p^2 - \omega^2) = 0.$$

2 wed

In the above eqn, $4p \neq 0$ since driving frequency can't be zero.

$$\therefore p^2 - \omega^2 + 2k^2 = 0$$

$$p^2 = \omega^2 - 2k^2$$

$$p = \sqrt{\omega^2 - 2k^2} = p_r \rightarrow \text{Resonant frequency}$$

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At this frequency 'p_r' the amplitude will be maximum.

Substitute 'p_r' in the above expression for amplitude

$$A = \frac{f_0}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}$$

4 fri

Replace $p \longrightarrow$ as $p_r = \sqrt{\omega^2 - 2k^2}$

$$\therefore A = \frac{f_0}{\sqrt{(\omega^2 - \{\omega^2 - 2k^2\}^2)^2 + 4k^2(\sqrt{\omega^2 - 2k^2})^2}}$$

$$A_{max} = \frac{f_0}{\sqrt{(2k^2)^2 + 4k^2(\omega^2 - 2k^2)}}$$

5 sat

$$\boxed{A_{max} = \frac{f_0}{2k\sqrt{\omega^2 - k^2}}}$$

For a certain value of driving frequency the amplitude will be maximum which is called as amplitude maximum Resonance.

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