n^{th} derivative of Standard functions.

1. To find the n^{th} derivative of x^m, where m is a positive integer.

Let $y = x^m$

$y_1 = m \cdot x^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(m-1)(m-2)x^{m-3}$

$\cdots\cdots$

$y_n = m(m-1)(m-2)\cdots\cdots(m-n+1) \cdot x^{m-n}$

$$y_n = \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)\cdots 2 \cdot 1}{(m-n)(m-n-1)\cdots 2 \cdot 1} \cdot x^{m-n}$$

$$\boxed{y_n = \frac{m!}{(m-n)!} x^{m-n}}$$ $n \le m$

Note.

1. If $y = x^m$, m is a positive integer and $n = m \Rightarrow y_n = n!$
2. If $y = x^m$, m is a positive integer and $n > m \Rightarrow y_n = 0$.

2. To find the n^{th} derivative of $(ax+b)^m$, where m is a positive integer and $n \le m$.

Let $y = (ax+b)^m$

$y_1 = m(ax+b)^{m-1} \cdot a$

$y_2 = m(m-1)(ax+b)^{m-2} \cdot a^2$

$\cdots\cdots$

$y_n = m(m-1)(m-2)\cdots\cdots(m-n+1)(ax+b)^{m-n}$.

$$\boxed{y_n = \frac{m!}{(m-n)!}(ax+b)^{m-n} \cdot a^n}$$ m is a +ve int. & $n \le m$.

Note:

1. If m is a positive integer and $y = (ax+b)^m$, then $\boxed{y_m = m!\,a^m}$ when $m = n$.

2. If m is a +ve int $m < n$ and $y = (ax+b)^m$, then $y_n = 0$.

3. If $m = -1$ and $y = (ax+b)^m$, then $\boxed{y_n = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}}$

i.e $D^n\left(\frac{1}{(ax+b)}\right) = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$.

3. To find the n^{th} derivative of $\log(ax+b)$.

Let $y = \log(ax+b) \Rightarrow y_1 = \frac{1}{(ax+b)} \cdot a$.

We know that, $D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$.

Thus the n^{th} derivative of $y = \log(ax+b)$ is $(n-1)^{th}$ derivative of $\frac{1}{(ax+b)}$

$$\therefore D^n[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)!\,a^{n-1}}{(ax+b)^n}.$$

4. To find the n^{th} derivative of $\frac{1}{(ax+b)^m}$.

We have. $D^n(ax+b)^m = m(m-1)(m-2)\ldots(m-n+1)a^n.(ax+b)^{m-n}$

Let m be negative, say $m = -p$, where p is positive.

then we have,

$$D^n\left[\frac{1}{(ax+b)^p}\right] = \frac{(-1)^n p(p-1)(p-2)\ldots(p+n-1).a^n}{(ax+b)^{p+n}}.$$

$$= \frac{(-1)^n (p+n-1)!}{(p-1)!} \cdot \frac{1}{(ax+b)^{p+n}} \cdot a^n.$$

$$D^n\left[\frac{1}{(ax+b)^m}\right] = \frac{(-1)^n (m+n-1)!}{(m-1)!} \cdot \frac{1}{(ax+b)^{m+n}} \cdot a^n.$$

5. To find the n^{th} derivative of e^{mx}

Let $y = e^{mx} \Rightarrow y_1 = m \cdot e^{mx}$.

$y_2 = m \cdot m e^{mx} = m^2 e^{mx}$

$y_3 = m^3 e^{mx}$ ⋮

$$\boxed{y_n = m^n \cdot e^{mx}}$$

Note: if $y = e^x$ then $y_n = e^x$.

6. To find the n^{th} derivative of $\sin(ax+b)$

$\therefore \sin(90+\theta) = \cos\theta$

Let $y = \sin(ax+b)$

$y_1 = a \cdot \cos(ax+b) = a \sin\left[ax+b+\frac{\pi}{2}\right]$

$y_2 = a^2 \cdot \cos\left(ax+b+\frac{\pi}{2}\right) = a^2 \sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$

$y_3 = a^3 \cos\left(ax+b+2\cdot\frac{\pi}{2}\right) = a^3 \sin\left(ax+b+\frac{3\pi}{2}\right)$.

⋮

$$\boxed{y_n = a^n \cdot \sin\left(ax+b+\frac{n\pi}{2}\right)}$$

Note: if $y = \sin x$, then $y_n = \sin\left(x+\frac{n\pi}{2}\right)$.

7. To find the n^{th} derivative of $\cos(ax+b)$

$\cos(90+\theta) = -\sin\theta$.

Let $y = \cos(ax+b)$

$y_1 = -a\sin(ax+b) = a \cdot \cos\left(ax+b+\frac{\pi}{2}\right)$

$y_2 = -a^2 \sin\left(ax+b+\frac{\pi}{2}\right) = -a^2 \cdot \cos\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$

$y_3 = -a^3 \sin\left(ax+b+2\cdot\frac{\pi}{2}\right) = a^3 \cdot \cos\left(ax+b+3\cdot\frac{\pi}{2}\right)$.

⋮

$y_n = a^n \cdot \cos\left(ax+b+\frac{n\pi}{2}\right)$

Note: If $y = \cos(x)$ then $y_n = \cos\left(x+\frac{n\pi}{2}\right)$.

8. To find the n^{th} derivative of $e^{ax}\sin(bx+c)$ (4)

Let $y = e^{ax}\sin(bx+c)$

$y_1 = e^{ax}\cos(bx+c)\cdot b + a\cdot e^{ax}\sin(bx+c)$

or $y_1 = e^{ax}[a\sin(bx+c) + b\cdot\cos(bx+c)]$.

put $a = r\cos\theta$ and $b = r\sin\theta$.

then $r^2 = a^2+b^2$, and $\tan\theta = \frac{b}{a} \Rightarrow \tan^{-1}\left(\frac{b}{a}\right) = \theta$.

$\Rightarrow r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

thus, we have,

$\cos A \sin B + \sin A \cos B = \sin(A+B)$

$y_1 = e^{ax}[r\cos\theta\cdot\sin(bx+c) + r\sin\theta\cdot\cos(bx+c)]$

$y_1 = r\cdot e^{ax}\sin(bx+c+\theta)$.

This result implies that, the derivative y_1 can be obtained from the function of y, on multiplying by r and increasing by angle θ. and also by repeating the same procedure we have

$y_2 = r^2 e^{ax}\sin(bx+c+2\theta)$.

thus $\boxed{y_n = r^n\cdot e^{ax}\cdot\sin(bx+c+n\theta)}$

where $r = \sqrt{a^2+b^2}$, and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

Similarly if $\boxed{y = e^{ax}\cdot\cos(bx+c)}$

then $\boxed{y_n = r^n\cdot e^{ax}\cdot\cos(bx+c+n\theta)}$

where $r = \sqrt{a^2+b^2}$. and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

H.W.

9. To find the n^{th} derivative of a^{mx}.

Let, $\boxed{y = a^{mx}} \Rightarrow$

$$y_1 = m \cdot a^{mx} \cdot \log a$$
$$y_2 = m^2 \cdot a^{mx} (\log a)^2.$$
$$y_3 = m^3 \cdot a^{mx} (\log a)^3.$$
$$\cdots$$
$$\boxed{y_n = m^n \cdot a^{mx} (\log a)^n}$$

Note: If $y = a^x$, then $m=1 \Rightarrow$

$$y_n = m^n \cdot a^{mx} (\log a)^n$$
$$y_n = (1)^n \cdot a^x (\log a)^n$$
$$y_n = a^x (\log a)^n.$$

Problems.

I. Find the n^{th} derivatives of

$\rightarrow$ P.F.

1. $\dfrac{1}{(x+2)(x-1)}$.

Let $y = \dfrac{1}{(x+2)(x-1)}$

Resolving into partial fractions,

$$\frac{y \Rightarrow 1}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

$$1 = A(x-1) + B(x+2)$$

if $x = -2$ | $1 = A(-2-1) + B(0)$
$1 = A(-3) \Rightarrow \boxed{A = \frac{-1}{3}}$

if $x = 1$ | $1 = A\cancel{(1-1)}^{0} + B(1+2)$
$1 = B(3) \Rightarrow \boxed{B = \frac{1}{3}}$

$$y = \left(\frac{-1}{3}\right)\frac{1}{(x+2)} + \left(\frac{1}{3}\right)\frac{1}{(x-1)}$$
$$= \frac{1}{3}\left[\frac{1}{x-1} - \frac{1}{x+2}\right]$$
$$\Rightarrow y_n = \frac{1}{3}\left[D^n\left(\frac{1}{x-1}\right) - D^n\left(\frac{1}{x+2}\right)\right]$$
$$= \frac{1}{3}\left[\frac{(-1)^n \cdot n!}{(x-1)^{n+1}} - \frac{(-1)^n \cdot n!}{(x+2)^{n+1}}\right]$$
$$= \frac{(-1)^n \cdot n!}{3}\left[\frac{1}{(x-1)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right]$$

2. $\dfrac{1}{6x^2-5x+1} = \dfrac{1}{(3x-1)(2x-1)}$

Resolving into partial fractions, we get

$$y = \frac{1}{(3x-1)(2x-1)} = \frac{1}{(3x-1)(\frac{2}{3}-1)} + \frac{1}{(\frac{3}{2}-1)(2x-1)}$$

$$\Rightarrow y = \frac{-3}{3x-1} + \frac{2}{2x-1}$$

$$\Rightarrow y_n = 2 \cdot D^n\left(\frac{1}{2x-1}\right) - 3 \cdot D^n\left(\frac{1}{3x-1}\right)$$

$$= \frac{2 \cdot (-1)^n \cdot n!\, 2^n}{(2x-1)^{n+1}} - \frac{3 \cdot (-1)^n \cdot n!\, 3^n}{(3x-1)^{n+1}}$$

$$= (-1)^n \cdot n! \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right].$$

3. $\dfrac{x^2}{(x-1)^2(x-2)}$

Let $y = \dfrac{x^2}{(x-1)^2(x-2)}$

Resolving into partial fractions we get,

$$y = \frac{A}{(x-1)} + \frac{(1)^2}{(x-1)^2(1-2)} + \frac{2^2}{(2-1)^3(x-2)}$$

$$y = \frac{A}{x-1} - \frac{1}{(x-1)^2} + \frac{4}{(x-2)}$$

put $x=0$, $\Rightarrow 0 = -A - 1 - 2 \Rightarrow A = -3$

thus we have, $y = \dfrac{-3}{x-1} - \dfrac{1}{(x-1)^2} + \dfrac{4}{x-2}$

$$y_n = -3 \cdot D^n\left(\frac{1}{x-1}\right) - D^n\left(\frac{1}{(x-1)^2}\right) + 4 \cdot D^n\left(\frac{1}{x-2}\right)$$

$$= (-3) \cdot \frac{(-1)^n n!}{(x-1)^{n+1}} - \frac{(-1)^n \cdot (2+n-1)!}{(2-1)!\,(x-1)^{n+2}} + 4 \cdot \frac{(-1)^n \cdot n!}{(x-2)^{n+1}}$$

$$= (-1)^n \cdot n! \left[\frac{3}{(x-1)^{n+1}} - \frac{n+1}{(x-1)^{n+2}} + \frac{1}{(x-2)^{n+1}}\right).$$

4. $\log\sqrt{\dfrac{2x+1}{x-2}}$.

Let $y = \log\sqrt{\dfrac{2x+1}{x-2}}$.

$$y = \frac{1}{2}\cdot\log\left(\frac{2x+1}{x-2}\right)$$

$$y = \frac{1}{2}\left[\log(2x+1) - \log(x-2)\right]$$

$$y_n = \frac{1}{2}\left[D^n\{\log(2x+1)\} - D^n\{\log(x-2)\}\right]$$

$$= \frac{1}{2}\left[\frac{(-1)^{n-1}\cdot(n-1)!\,2^n}{(2x+1)^n} - \frac{(-1)^{n-1}\cdot(n-1)!}{(x-2)^n}\right]$$

$$= \frac{(-1)^{n-1}\cdot(n-1)!}{2}\left[\frac{2^n}{(2x+1)^n} - \frac{1}{(x-2)^n}\right].$$

5. $\log(x^2-4)$

$$y = \log(x^2-4)$$

$$= \log(x-2) + \log(x+2)$$

$$y_n = D^n\left[\log(x-2)\right] + D^n\left[\log(x+2)\right].$$

$$y_n = \frac{(-1)^{n-1}\cdot(n-1)!}{(x-2)^n} + \frac{(-1)^{n-1}\cdot(n-1)!}{(x+2)^n}.$$

$$y_n = (-1)^{n-1}\cdot(n-1)!\left[\frac{1}{(x-2)^n} + \frac{1}{(x+2)^n}\right]$$

6. $y = \sin^2 x$.

$$y = \frac{1}{2}(1-\cos 2x)$$

$$y_n = \frac{1}{2}D^n(1-\cos 2x)$$

$$y_n = -\frac{1}{2}\cdot D^n\cdot\cos 2x = \frac{-1}{2}\,2^n\cdot\cos\left(2x+\frac{n\pi}{2}\right)$$

$$= -2^{n-1}\cdot\cos\left(2x+\frac{n\pi}{2}\right).$$

7. $\cos^4 x$

Let $y = \cos^4 x$.

$y = (\cos^2 x)^2$

$y = \frac{1}{4}(1+\cos 2x)^2$

$y = \frac{1}{4}\left[1+2\cdot\cos 2x+\cos^2 2x\right]$

$y = \frac{1}{4}\left[1+2\cdot\cos 2x+\frac{1}{2}(1+\cos 4x)\right]$

$y = \frac{1}{4}+\frac{1}{8}+\frac{1}{2}\cdot\cos 2x+\frac{1}{8}\cdot\cos 4x$.

$y_n = D^n\left[\frac{1}{2}\cos 2x+\frac{1}{8}\cdot\cos 4x\right]$

$y_n = \frac{1}{2}\cdot 2^n\cdot\cos\left(2x+\frac{n\pi}{2}\right)+\frac{1}{2}\cdot 4^n\cos\left(4x+\frac{n\pi}{2}\right)$

$y_n = 2^{n-1}\cdot\cos\left(2x+\frac{n\pi}{2}\right)+2^{2n-3}\cos\left(4x+\frac{n\pi}{2}\right)$

8. $\sin^2 x\cdot\cos^3 x$.

$y = \sin^2 x\cdot\cos^3 x$.

$y = \sin^2 x\cdot\cos^2 x\cdot\cos x$.

$y = \frac{1}{4}\sin^2 2x\cdot\cos x$

$y = \frac{1}{4}\left[\frac{1-\cos 4x}{2}\right]\cdot\cos x$.

$y = \frac{1}{8}\left[\cos x-\cos 4x\cdot\cos x\right]$

$y = \frac{1}{8}\left[\cos x-\frac{1}{2}(\cos 5x+\cos 3x)\right]$

$y = \frac{1}{16}\left[2\cos x-\cos 5x-\cos 3x\right]$

$y_n = \frac{1}{16}\left[D^n(2\cos x)-D^n(\cos 5x)-D^n(\cos 3x)\right]$.

$y_n = \frac{1}{16}\left[2\cdot\cos\left(x+\frac{n\pi}{2}\right)-5^n\cdot\cos\left(5x+\frac{n\pi}{2}\right)-3^n\cdot\cos\left(3x+\frac{n\pi}{2}\right)\right]$

9. $y = \cos x \cdot \cos 2x \cdot \cos 3x$.

$y = (\cos 3x \cdot \cos x) \cdot \cos 2x$.

$y = \frac{1}{2}[\cos 4x + \cos 2x] \cdot \cos 2x$.

$= \frac{1}{2}[\cos 4x \cdot \cos 2x + \cos^2 2x]$

$= \frac{1}{2}\left[\frac{1}{2}(\cos 6x + \cos 2x) + \frac{1+\cos 4x}{2}\right]$

$= \frac{1}{4}[\cos 6x + \cos 2x + 1 + \cos 4x]$

$= \frac{1}{4}(D^n[\cos 6x) + D^n(\cos 2x) + D^n(\cos 4x))]$

$= \frac{1}{4}\left[6^n \cdot \cos\left(6x + \frac{n\pi}{2}\right) + 2^n \cdot \cos\left(2x + \frac{n\pi}{2}\right) + 4^n \cdot \cos\left(4x + \frac{n\pi}{2}\right)\right]$

10. $e^x \sin x \cdot \cos 2x$.

$y = e^x \cdot \sin x \cdot \cos 2x$.

$= e^x \cdot \frac{1}{2}[\sin 3x - \sin x]$.

$y^n = \frac{1}{2} \cdot D^n[e^x \sin 3x) - \frac{1}{2} D^n(e^x \sin x)$

$y_n = \frac{1}{2}\left[(1^2+3^2)^{n/2} \cdot e^x \cdot \sin(3x + n \cdot \tan^{-1} 3)\right]$

$- \frac{1}{2}\left[(1^2+1^2)^{n/2} \cdot e^x \cdot \sin(x + n \tan^{-1} 1)\right]$.

$= \frac{1}{2} \cdot e^x \left[10^{n/2} \sin(3x + n \cdot \tan^{-1} 3) - 2^{n/2} \cdot \sin\left(x + \frac{n\pi}{4}\right)\right]$.

11. $y = \sinh 2x \cdot \sin 4x.$

$$y = \left(\frac{e^{2x} - e^{-2x}}{2}\right) \cdot \sin 4x.$$

$$= \frac{1}{2}\left[e^{2x} \cdot \sin 4x - e^{-2x} \cdot \sin 4x\right]$$

$$y_n = \frac{1}{2}\left[D^n(e^{2x} \cdot \sin 4x) - D^n(e^{-2x} \cdot \sin 4x)\right]$$

$$y_n = \frac{1}{2}\left[(2^2+4^2)^{n/2} \cdot e^{2x} \cdot \sin(4x + n \cdot \tan^{-1} 2)\right]$$

$$- \frac{1}{2}\left[(2^2+4^2)^{n/2} e^{-2x} \sin(4x + n\tan^{-1}(-2))\right]$$

$$y_n = \frac{1}{2}(20)^{n/2} \cdot \left[e^{2x} \cdot \sin(4x + n\tan^{-1} 2) - e^{-2x} \sin(4x - n\tan^{-1} 2)\right]$$

12. Find the n^{th} derivative of $\frac{1}{x^2+a^2}$.

Let $y = \frac{1}{x^2+a^2}$.

put $x = a\cot\theta$, $\Rightarrow \theta = \cot^{-1}\left(\frac{x}{a}\right) \Rightarrow \frac{dx}{d\theta} = \frac{-a}{\sin^2\theta}$.

$$\therefore y = \frac{1}{x^2+a^2} \Rightarrow y = \frac{1}{a^2\cot^2\theta + a^2}$$

$$= \frac{1}{a^2(1+\cot^2\theta)} = \frac{1}{a^2 \operatorname{cosec}^2\theta}$$

$$y = \frac{1}{a^2}\sin^2\theta.$$

diff

$$y_1 = \frac{1}{a^2} \, 2\sin\theta \cdot \cos\theta \cdot \frac{d\theta}{dx}.$$

$$= \frac{1}{a^2}\sin 2\theta \cdot \left(\frac{-\sin^2\theta}{a}\right)$$

$$y_1 = -\frac{1}{a^3} \cdot \sin 2\theta \cdot \sin^2\theta.$$

Now diff again

$$y_2 = -\frac{1}{a^3}\left[\sin 2\theta \cdot 2\sin\theta \cdot \cos\theta + 2\cos 2\theta \cdot \sin^2\theta\right] \cdot \left[-\frac{\sin^2\theta}{2}\right]$$

$$= (-1)^2 \cdot \frac{1}{a^4} \cdot 2\sin 2\theta\left[\sin\theta \cdot \cos 2\theta + \cos\theta \cdot \sin 2\theta\right] \cdot \sin^2\theta$$

$$y_2 = (-1)^2 \cdot \frac{1}{a^4}(2!) \cdot \sin^3\theta \cdot \sin 3\theta.$$

Similarly,

$$y_3 = (-1)^3 \cdot \frac{1}{a^5} \cdot (3!) \cdot \sin^4\theta \cdot \sin 4\theta.$$

Generalizing we get,

$$y_n = (-1)^n \cdot \frac{1}{a^{n+2}} \cdot (n!) \cdot \sin^{n+1}\theta \cdot \sin(n+1)\theta.$$

where $\theta = \cot^{-1}\left(\frac{x}{a}\right)$

13. Find the n^{th} derivative of $\tan^{-1}\left[\frac{1+x}{1-x}\right]$.

Sol^n: Let $y = \tan^{-1}\left[\frac{1+x}{1-x}\right]$

put $x = \tan\theta$, $\Rightarrow$ $\theta = \tan^{-1}x$.

$$y = \tan^{-1}\left(\frac{1+x}{1-x}\right) \Rightarrow y = \tan^{-1}\left[\tan\left(\theta + \frac{\pi}{4}\right)\right].$$

$$y = \theta + \frac{\pi}{4}.$$

$$y = \frac{\pi}{4} + \tan^{-1}x.$$

diff $y_1 = \frac{1}{1+x^2}$

Differentiating $(n-1)$ times, we get.

$$y_n = D^{n-1}\left(\frac{1}{1+x^2}\right)$$

$$y_n = (-1)^{n-1} \cdot \frac{1}{1^{n+1}} \cdot (n-1)!\, \sin n\alpha \cdot \sin^n\alpha.$$

$$y_n = (-1)^{n-1} \cdot (n-1)!\, \sin n\alpha \cdot \sin^n\alpha.$$

where $\alpha = \cot^{-1}x$.

Q. If $y = \sin mx + \cos mx$, show that

$$y_r = m^r\left[1 + (-1)^r \cdot \sin 2mx\right]^{1/2}$$

→ Let $y = \sin mx + \cos mx$.

$$y_r = m^r\left[\sin\left(mx + \frac{r\pi}{2}\right) + \cos\left(mx + \frac{r\pi}{2}\right)\right]$$

× & ÷ by $\sqrt{2}$.

$$y_r = m^r\left[(\sin\theta + \cos\theta)^2\right]^{1/2} \quad \text{when } \theta = mx + \frac{r\pi}{2}$$

expand $(a+b)^2$

$$y_r = m^r\left[1 + 2\sin\theta \cdot \cos\theta)\right]^{1/2}$$

$$y_r = m^r\left[(1 + \sin 2\theta)\right]^{1/2}$$

$$y_r = m^r\left[1 + \sin(2mx + r\pi)\right]^{1/2}$$

$$y_r = m^r\left[1 + (-1)^r \cdot \sin 2mx\right]^{1/2}$$

~~where~~ $\left[\because \sin(n\pi + \theta) = (-1)^n \cdot \sin\theta\right]$

H.W

Find the n^{th} derivatives of the foll

1. $\dfrac{2x-1}{(x-2)(x+1)}$.
2. $\dfrac{x}{(x-3)^2(2x-1)}$
3. $\dfrac{x+1}{6x^2 - 7x - 3}$.
4. $\log(x^2 - a^2)$.
5. $\cos^2 x \cdot \sin^3 x$
6. $\cosh 4x \cdot \cos 3x$.
7. $\cos 5x \cdot \cos 3x$.
8. $\sin x \cdot \sin 2x \cdot \sin 3x$.
9. $e^{3x} \cdot \sin^2 x$.
10. $e^{-3x} \cdot \cos^3 x$.
11. $e^{ax} \cdot \sin^2 x \cdot \sin 2x$.
12. $\tan^{-1}\left[\dfrac{2x}{1-x^2}\right]$.

Leibnitz's Theorem:

Here, we shall see a formula for finding the n^{th} derivative of the product of two functions in the form of a theorem known as Leibnitz's theorem.

Theorem: If u and v are two functions of x having derivatives of n^{th} order and $y = u \cdot v$, then

$$y_n = {}^nC_0 \cdot u_n v + {}^nC_1 \cdot u_{n-1} v_1 + \dots + {}^nC_r \cdot u_{n-r} v_r + \dots + {}^nC_n u_1 v_n.$$

Thm is proved using M.I.

Problems

Find the n^{th} derivative of $x^3 \cdot e^{ax}$.

Let $y = x^3 \cdot e^{ax}$ and ~~$u = x^3$~~ . ~~$v = e^{ax}$~~.

$u = e^{ax}$. $\qquad v = x^3$.

$u_n = a^n \cdot e^{ax}$ $\qquad v_1 = 3x^2$

$u_{n-1} = a^{n-1} \cdot e^{ax}$. $\qquad v_2 = 6x$

$u_{n-2} = a^{n-2} \cdot e^{ax}$ $\qquad v_3 = 6$

$u_{n-3} = a^{n-3} \cdot e^{ax}$. $\qquad v_4 = 0$.

By Leibnitz's thm we have,

$$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3.$$

$$y_n = a^n \cdot e^{ax} \cdot x^3 + n \cdot a^{n-1} \cdot e^{ax} \cdot 3x^2 + \frac{n(n-1)}{2} \cdot a^{n-2} \cdot e^{ax} \cdot 6x + \frac{n(n-1)(n-2)}{3!} a^{n-3} \cdot e^{ax} \cdot 6.$$

$$y_n = a^{n-3} \cdot e^{ax} \left[a^3x^3 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2) \right]$$

Find the n^{th} derivative of $e^x \cdot \log x$.

Let $y = e^x \cdot \log x$.

$u = e^x$, $\quad v = \log x$.

$u_1 = e^x$, $\quad v_1 = \frac{1}{x}$

$u_2 = e^x$ $\quad v_2 = -\frac{1}{x^2}$

$\vdots \qquad \vdots$

$u_n = e^x$. $\quad v_n = \frac{(-1)^n (n-1)!}{x^n}$.

By Leibnitz's theorem, we have.

$y_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1} v_1 + \dots + {}^nC_n\, u\, v_n$.

$= e^x \cdot \log x + n \cdot e^x \cdot \left(\frac{1}{x}\right) + \frac{n(n-1)}{2} \cdot e^x \left(\frac{-1}{x^2}\right) + \dots + \frac{e^x (-1)^{n-1} (n-1)!}{x^n}$.

$= e^x \left[\log x + \frac{n}{x} - \frac{n(n-1)}{2x^2} + \dots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right]$.

3. If $y = x^n \cdot \log x$ show that $y_{n+1} = \frac{n!}{x}$.

$y = x^n \cdot \log x$

$y_1 = x^n \cdot \frac{1}{x} + n \cdot x^{n-1} \cdot \log x$

$x y_1 = x^n + n \cdot \underline{x^n \cdot \log x}$ $\quad (\because y = x^n \log x)$

$x y_1 = x^n + n y$.

Differentiating n times. using Leibnitz's theorem, we have $\quad (\because D^n(x^n) = n!)$

$x y_{n+1} + \underline{n \cdot 1 \cdot y_n} = n! + \underline{n y_n}$.

$x y_{n+1} = n!$

$\Rightarrow y_{n+1} = \frac{n!}{x}$.

4. Find the n^{th} derivative of $x^2 \cdot e^x \cdot \cos x$.

Let $y = x^2 \cdot e^x \cdot \cos x$. and

$u = e^x \cdot \cos x$ $\quad V = x^2$.

$u_n = 2^{n/2}[e^x \cos(x + n\tan^{-1} 1)]$ $\quad V_1 = 2x$

$= 2^{n/2} \cdot e^x \cdot \cos(x + \frac{n\pi}{4})$ $\quad V_2 = 2$

$V_3 = 0$.

$u_{n-1} = 2^{n-1/2} \cdot e^x \cdot \cos[x + (n-1) \cdot \frac{\pi}{4})$

$u_{n-2} = 2^{n-2/2} \cdot e^x \cdot \cos[x + (n-2)\pi/4]$.

By Leibnitz's theorem we have,

$$y_n = 2^{n/2} \cdot e^x \cos(x + \frac{n\pi}{4}) \cdot x^2 + 2^{(n-1)/2} \cdot e^x \cdot \cos(x + \frac{(n-1)\pi}{4}) \cdot 2x + 2^{(n-2)/2} \cdot e^x \cdot \cos[x + \frac{(n-2)\pi}{4}]$$

$$y_n = 2^{\frac{(n-2)}{2}} e^x \{2 \cdot x^2 \cdot \cos[x + \frac{n\pi}{4}] + n \cdot 2^{3/2} \cdot \cos[x + \frac{(n-1)\pi}{4}]\}$$

$$+ 2^{(n-2)/2} \cdot e^x [n(n-1) \cdot \cos[x + \frac{(n-2)\pi}{4}] \cdot \}$$

5. Show that,

$$D^n[\frac{\log x}{x}] = \frac{(-1)^n \cdot n!}{x^{n+1}}[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}].$$

Let $y = \frac{\log x}{x} = \log x \cdot \frac{1}{x}$.

& $u = \log x$. $v = \frac{1}{x}$.

$y_n = \frac{(-1)^{n-1} \cdot (n-1)!}{x^n}$ $\quad V_n = \frac{(-1)^n \cdot n!}{x^{n+1}}$.

By Leibnitz's rule, we have

$$y_n = u_n v + {}^nC_1 u_{n-1} V_1 + {}^nC_2 u_{n-2} V_2 + \cdots + u V_n.$$

$$= \frac{(-1)^{n-1} \cdot (n-1)!}{x^n} \cdot \frac{1}{x} + \frac{n \cdot (-1)^{n-2} \cdot (n-2)!}{x^{n-1}} \cdot (\frac{-1}{x^2}) + \cdots + \log x \cdot \frac{(-1)^n \cdot n!}{x^{n+1}}$$

$$= \frac{(-1)^{n-1} \cdot n!}{x^{n+1}} \cdot [\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}]$$