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Dongfang Special Entangled Spherical Harmonic Functions

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Abstract: The spherical harmonic partial differential equation is usually solved by variable separation. When the magnetic quantum number m of the obtained Legendre spherical harmonic function is 0, the Legendre spherical harmonic function degenerates into the Legendre function of angle θ. This is a deduction that may seem rigorous in mathematics but is actually not true. Replace the cosine function of angle θ in the Legendre function with the product of the sine function of angle θ and the sine function or cosine function of angle φ, and obtain two special binary trigonometric series called special entanglement functions that satisfy the spherical harmonic partial differential equation. According to the basic principles of differential equation, the linear combination of three special spherical harmonic functions is a local special general solution of the spherical harmonic partial differential equation with a magnetic quantum number m of 0. However, the normalization condition cannot determine the three undetermined coefficients and therefore cannot determine the specific spherical harmonic function. The established physical conclusions based on spherical harmonic functions, especially the mathematical principles of quantum mechanics, need to supplement the definite solution conditions of spherical harmonic partial differential equations and rewrite them after determining specific exact solutions.

Keywords: Direct product function, Entanglement function, Laplace equation, spherical harmonic partial differential equation, spherical harmonic function, wave function, probability density.

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1 Legendre direct product spherical harmonic function

Definition 1 Direct Product Function: In a generalized space with n coordinate parameters $q_1,\ q_2,\ \cdots,\ q_n$, a multivariate function $\psi(q_1,\ q_2,\ \cdots,\ q_n)$ can be decomposed into the product of several functions represented only by partial coordinate parameters, and the intersection of the

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coordinate parameter sets of these functions is empty. This multivariate function is called a direct product function, and the direct product function in the form of $\prod_{j=1}^{n} \psi(q_j)$ is a complete direct product function.

Definition 2 Entangled Function: In a generalized space with n coordinate parameters $q_1,\ q_2,\ \cdots,\ q_n$, a multivariate function $\psi(q_1,\ q_2,\ \cdots,\ q_n)$ cannot be decomposed into the product of several functions represented only by partial coordinate parameters, and the intersection of the coordinate parameter sets of these functions is empty. This multivariate function is called an entangled function.

By using the method of variable separation to solve the Laplace equation[1-5], various Schrödinger equations[6-8], etc. in a spherical coordinate system, second-order partial differential equations with respect to two angles θ and φ can be obtained, which are called spherical harmonic partial differential equations,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+l\left(l+1\right)Y=0 \tag{1.1}$$

Among them, the variable separation universal constant λ has been written as $\lambda = l\left(l+1\right)$. The spherical harmonic partial differential equation[9] is consistent with the quantum mechanics angular momentum square operator equation. Further use the method of variable separation to solve the spherical harmonic partial differential equation, obtaining the general analytical solution of the spherical harmonic partial differential equation, which includes bounded functions with rotational symmetry and unbounded functions without specific rotational symmetry.

Lemma 1 The direct product analytical function

$$Y_l^m\left(\theta,\varphi\right)=\left\{\begin{array}{l}\left(\alpha_l^{\langle m\rangle}\sin m\varphi+a_l^{\langle m\rangle}\cos m\varphi\right)\sin^m\theta\\ \times\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{\left(2k-l+m-2\right)\left(2k+l+m-1\right)}{2k\left(2k-1\right)}\right)\cos^{2n}\theta\\ +\left(\beta_l^{\langle m\rangle}\sin m\varphi+b_l^{\langle m\rangle}\cos m\varphi\right)\sin^m\theta\\ \times\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{\left(2k-l+m-1\right)\left(2k+l+m\right)}{\left(2k+1\right)2k}\right)\cos^{2n+1}\theta\end{array}\right\} \tag{1.2}$$

with four undetermined coefficients $\alpha_l^{\langle m\rangle}$, $a_l^{\langle m\rangle}$, $\beta_l^{\langle m\rangle}$, and $b_l^{\langle m\rangle}$ satisfies the spherical harmonic partial differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+l\left(l+1\right)Y=0$$

for any constant m and l.

The analytical part of the direct product analytic function (1.2) is the usual expression obtained by solving the associated Legendre equation, so Lemma 1 does not require a proof to be written.

The analytical function that satisfies the spherical harmonic partial differential equation may not necessarily be the expected solution. Scientific theories such as mathematics and physics often develop according to expectations, which may lead to certain biases. Usually, it is agreed to take integers $m = 0,\ \pm1,\ \pm2,\ \cdots$, and $l = 0,\ 1,\ 2,\ \cdots$, so that the spherical harmonic function satisfies the natural period and bounded function conditions, and m is called the magnetic quantum number and l is the orbital angular momentum quantum number. One term in the analytical function (1.2) of this condition is interrupted as a polynomial, which is called the associated Legendre polynomial[10], represented by $P_l^m(\cos\theta)$,

$$P_l^m(\cos\theta) = \begin{cases} \underbrace{\chi_l^{\langle m\rangle}\sin^m\theta\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l+m-2)(2k+l+m-1)}{2k(2k-1)}\right)\cos^{2n}\theta}_{l-m=0,\,2,\,4,\,\cdots} \\ \underbrace{\eta_l^{\langle m\rangle}\sin^m\theta\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l+m-1)(2k+l+m)}{(2k+1)2k}\right)\cos^{2n+1}\theta}_{l-m=1,\,3,\,5,\,\cdots} \end{cases} \tag{1.3}$$

The undetermined coefficients $\chi_l^{\langle m\rangle}$ and $\eta_l^{\langle m\rangle}$ are usually represented by normalization coefficients, which are only operational but not necessarily logical inferences. The solution of the spherical harmonic partial differential equation is limited to a bounded function with rotational symmetry expressed by the associated Legendre function,

$$\mathscr{Y}_l^{\langle m\rangle}(\theta,\varphi) = \left(A_l^{\langle m\rangle}\sin m\varphi + B_l^{\langle m\rangle}\cos m\varphi\right)P_l^m(\cos\theta) \tag{1.4}$$

This is the real number expression for the Legendre direct product spherical harmonic functions [?, 11, 13]. Real number expressions have two undetermined coefficients A_m and B_m, and cannot be directly represented by normalization coefficients. Quantum mechanics writes the spherical harmonic function in the form of a complex function, providing a unique normalization coefficient, which is intriguing. More related issues will be gradually introduced later to illustrate that normalization coefficients are not always effective.

The expression of Legendre's direct product spherical harmonic function is concise, but a simplified expression may lead us to overlook certain important principles. Let's return to the universal analytical expression and use (1.3) to represent the Legendre direct product spherical harmonic function.

Lemma 2 For integers m and non negative integers l that satisfy the condition $m \leqslant l$, the Legendre direct product spherical harmonic function

$$\mathscr{Y}_l^{\langle m\rangle}(\theta,\varphi) = \begin{cases} \underbrace{\left\{\begin{aligned}&\left(\alpha_l^{\langle m\rangle}\sin m\varphi + a_l^{\langle m\rangle}\cos m\varphi\right)\sin^m\theta \\ &\times\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l+m-2)(2k+l+m-1)}{2k(2k-1)}\right)\cos^{2n}\theta\end{aligned}\right\}}_{l-m=0,\,2,\,4,\,\cdots} \\ \underbrace{\left\{\begin{aligned}&\left(\beta_l^{\langle m\rangle}\sin m\varphi + b_l^{\langle m\rangle}\cos m\varphi\right)\sin^m\theta \\ &\times\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l+m-1)(2k+l+m)}{(2k+1)2k}\right)\cos^{2n+1}\theta\end{aligned}\right\}}_{l-m=1,\,3,\,5,\,\cdots} \end{cases} \tag{1.5}$$

containing two undetermined coefficients $\alpha_l^{\langle m\rangle}$ and $a_l^{\langle m\rangle}$ or $\beta_l^{\langle m\rangle}$ and $b_l^{\langle m\rangle}$, or written in polynomial form

$$\mathscr{Y}_l^{\langle m\rangle}(\theta,\varphi)=\begin{cases}\underbrace{\left\{\begin{array}{l}\left(\alpha_l^{\langle m\rangle}\sin m\varphi+a_l^{\langle m\rangle}\cos m\varphi\right)\sin^m\theta\\ \times\displaystyle\sum_{n=0}^{(l-m+2)/2}\left(\prod_{k=1}^{n}\frac{(2k-l+m-2)(2k+l+m-1)}{2k(2k-1)}\right)\cos^{2n}\theta\end{array}\right\}}_{l-m=0,2,4,\cdots}\\ \underbrace{\left\{\begin{array}{l}\left(\beta_l^{\langle m\rangle}\sin m\varphi+b_l^{\langle m\rangle}\cos m\varphi\right)\sin^m\theta\\ \times\displaystyle\sum_{n=0}^{(l-m+1)/2}\left(\prod_{k=1}^{n}\frac{(2k-l+m-1)(2k+l+m)}{(2k+1)2k}\right)\cos^{2n+1}\theta\end{array}\right\}}_{l-m=1,3,5,\cdots}\end{cases}\tag{1.6}$$

satisfies the spherical harmonic partial differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathscr{Y}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\mathscr{Y}}{\partial\varphi^2}+l(l+1)\mathscr{Y}=0$$

Now let's focus on an issue that has been overlooked. From a purely deductive perspective, if $m=0$, the spherical harmonic function degenerates into a Legendre polynomial, known as a Legendre special spherical harmonic function. The Legendre special spherical harmonic function is independent of angle φ and is a cosine series of angle θ [14], which is a univariate function. But in reality, there is no integer m in the spherical harmonic partial differential equation. This leads to a new problem: Is there any missing spherical harmonic function with angle φ in the solution set of the spherical harmonic partial differential equation given by the variable separation method with $m=0$?

Problem 1 When the magnetic quantum number $m=0$, the Legendre spherical harmonic functions degenerate into the one-dimensional Legendre special spherical harmonic function with the angle θ

$$\mathscr{Y}_l^{0}(\theta,\varphi)=\begin{cases}\underbrace{a_l\displaystyle\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)\cos^{2n}\theta}_{l=0,\,2,\,4,\,\cdots}\\ \underbrace{b_l\displaystyle\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)\cos^{2n+1}\theta}_{l=1,\,3,\,5,\,\cdots}\end{cases}\tag{1.7}$$

and without regard to angle φ, or written in polynomial form

$$\mathscr{Y}_l^{\langle 0\rangle}(\theta,\varphi)=\begin{cases}\underbrace{a_l\displaystyle\sum_{n=0}^{(l+2)/2}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)\cos^{2n}\theta}_{l=0,2,4,\cdots}\\ \underbrace{b_l\displaystyle\sum_{n=0}^{(l+1)/2}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)\cos^{2n+1}\theta}_{l=1,3,5,\cdots}\end{cases}\tag{1.8}$$

So, is there a binary special spherical harmonic function with angles θ and φ that satisfies the spherical harmonic partial differential equation,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathscr{Y}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\mathscr{Y}}{\partial\varphi^2}+l\left(l+1\right)\mathscr{Y}=0$$

corresponding to a magnetic quantum number $m=0$?

Table 1.1 lists the specific forms of Legendre special unary spherical harmonic functions corresponding to different integers for l in (1.7) and (1.8) when magnetic quantum number $m=0$.

Table 1.1 Some Legendre special univariate spherical harmonic functions

$$\mathscr{Y}_0^0\left(\theta,\varphi\right)=a_0$$

$$\mathscr{Y}_1^0\left(\theta,\varphi\right)=b_1\cos\theta$$

$$\mathscr{Y}_2^0\left(\theta,\varphi\right)=a_2\left(1-3\cos^2\theta\right)$$

$$\mathscr{Y}_3^0\left(\theta,\varphi\right)=b_3\left(\cos\theta-\frac{5}{3}\cos^3\theta\right)$$

$$\mathscr{Y}_4^0\left(\theta,\varphi\right)=a_4\left(1-10\cos^2\theta+\frac{35}{3}\cos^4\theta\right)$$

$$\mathscr{Y}_5^0\left(\theta,\varphi\right)=b_5\left(\cos\theta-\frac{14}{3}\cos^3\theta+\frac{21}{5}\cos^5\theta\right)$$

$$\mathscr{Y}_6^0\left(\theta,\varphi\right)=a_6\left(1-21\cos^2\theta+63\cos^4\theta-\frac{231}{5}\cos^6\theta\right)$$

$$\mathscr{Y}_7^0\left(\theta,\varphi\right)=b_7\left(\cos\theta-9\cos^3\theta+\frac{99}{5}\cos^5\theta-\frac{429}{35}\cos^7\theta\right)$$

$$\mathscr{Y}_8^0\left(\theta,\varphi\right)=a_8\left(1-36\cos^2\theta+198\cos^4\theta-\frac{1716}{5}\cos^6\theta+\frac{1287}{7}\cos^8\theta\right)$$

$$\mathscr{Y}_9^0\left(\theta,\varphi\right)=b_9\left(\cos\theta-\frac{44}{3}\cos^3\theta+\frac{286}{5}\cos^5\theta-\frac{572}{7}\cos^7\theta+\frac{2431}{63}\cos^9\theta\right)$$

$$\mathscr{Y}_{10}^0\left(\theta,\varphi\right)=a_{10}\left(\begin{array}{l}1-55\cos^2\theta+\frac{1430}{3}\cos^4\theta-1430\cos^6\theta\\+\frac{12155}{7}\cos^8\theta-\frac{46189}{63}\cos^{10}\theta\end{array}\right)$$

2 Special entangled spherical harmonic function of the first kind

The answer to question 1 is yes. When the magnetic quantum number $m=0$, replace $\cos\theta$ in (1.2) and (1.7) with $\sin\varphi\sin\theta$ and $\cos\varphi\sin\theta$, respectively, to obtain special binary entanglement analytic functions $X_l^{\langle 0\rangle}\left(\theta,\varphi\right)$ and $D_l^{\langle 0\rangle}\left(\theta,\varphi\right)$ that satisfy the spherical harmonic partial differential equation. Extract the spherically symmetric parts, which are respectively called the first kind of special entangled spherical harmonic function $\mathscr{X}_l^{\langle 0\rangle}\left(\theta,\varphi\right)$ with zero magnetic quantum number and the second kind of special entangled spherical harmonic function $\mathscr{D}_l^{\langle 0\rangle}\left(\theta,\varphi\right)$, which correspond one-to-one with the Legendre special spherical harmonic function.

Theorem 1 For any constant l, the first kind of special entanglement analytic function

$$X_l^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l} c_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\sin\varphi\sin\theta)^{2n}\\ +d_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(\sin\varphi\sin\theta)^{2n+1}\end{array}\right\} \tag{2.1}$$

with two undetermined coefficients c_l and d_l satisfies the spherical harmonic partial differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial X}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\varphi^2}+l(l+1)X=0 \tag{2.2}$$

with magnetic quantum number $m=0$.

Proof: The spherical harmonic partial differential equation (2.2) has the following equivalent form

$$\frac{\partial^2 X}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial X}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\varphi^2}+l(l+1)X=0 \tag{2.3}$$

The first and second partial derivatives of $X_l^0(\theta,\varphi)$ with respect to angles θ and φ are as follows

$$\begin{array}{l}
\dfrac{\partial X}{\partial\theta}=\left\{\begin{array}{l} c_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\sin^{2n}\varphi\sin^{2n-1}\theta\cos\theta\\ +d_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\sin^{2n+1}\varphi\sin^{2n}\theta\cos\theta\end{array}\right\}\\
\dfrac{\partial X}{\partial\varphi}=\left\{\begin{array}{l} c_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\sin^{2n-1}\varphi\sin^{2n}\theta\cos\varphi\\ +d_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\sin^{2n}\varphi\sin^{2n+1}\theta\cos\varphi\end{array}\right\}\\
\dfrac{\partial^2 X}{\partial\theta^2}=\left\{\begin{array}{l} c_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\left[(2n-1)\cot^2\theta-1\right](\sin\theta\sin\varphi)^{2n}\\ +d_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\left(2n\cot^2\theta-1\right)(\sin\theta\sin\varphi)^{2n+1}\end{array}\right\}\\
\dfrac{\partial^2 X}{\partial\varphi^2}=\left\{\begin{array}{l} c_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\left[(2n-1)\cot^2\varphi-1\right](\sin\theta\sin\varphi)^{p}\\ +d_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\left(2n\cot^2\varphi-1\right)(\sin\theta\sin\varphi)^{2n+1}\end{array}\right\}
\end{array} \tag{2.4}$$

Substitute (2.1) and (2.4) into the left side of (2.3) to obtain

$$\frac{\partial^2 X}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial X}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\varphi^2}+l(l+1)X$$

$$= \left\{ \begin{array}{l} c_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n \left[(2n-1)\cot^2\theta - 1 \right] (\sin\theta \sin\varphi)^{2n} \\ +d_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1) \left(2n\cot^2\theta - 1 \right) (\sin\theta \sin\varphi)^{2n+1} \end{array} \right\}$$

$$+ \frac{\cos\theta}{\sin\theta} \left\{ \begin{array}{l} c_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n\sin^{2n}\varphi\sin^{2n-1}\theta\cos\theta \\ +d_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1)\sin^{2n+1}\varphi\sin^{2n}\theta\cos\theta \end{array} \right\}$$

$$+ \frac{1}{\sin^2\theta} \left\{ \begin{array}{l} c_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n \left[(2n-1)\cot^2\varphi - 1 \right] (\sin\theta \sin\varphi)^{p} \\ +d_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1) \left(2n\cot^2\varphi - 1 \right) (\sin\theta \sin\varphi)^{2n+1} \end{array} \right\}$$

$$+ l(l+1) \left\{ \begin{array}{l} c_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) (\sin\varphi \sin\theta)^{2n} \\ +d_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (\sin\varphi \sin\theta)^{2n+1} \end{array} \right\}$$

$$= c_l \left\{ \begin{array}{l} \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n \left[(2n-1)\cot^2\theta - 1 \right] (\sin\theta \sin\varphi)^{2n} \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n\sin^{2n}\varphi\sin^{2n-2}\theta\cos^2\theta \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n \left[(2n-1)\cot^2\varphi - 1 \right] \sin^{2n-2}\theta\sin^{2n}\varphi \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) l(l+1)(\sin\varphi \sin\theta)^{2n} \end{array} \right\}$$

$$+ d_l \left\{ \begin{array}{l} \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1) \left(2n\cot^2\theta - 1 \right) (\sin\theta \sin\varphi)^{2n+1} \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1)\sin^{2n+1}\varphi\sin^{2n-1}\theta\cos^2\theta \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1) \left(2n\cot^2\varphi - 1 \right) \sin^{2n-1}\theta\sin^{2n+1}\varphi \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) l(l+1)(\sin\varphi \sin\theta)^{2n+1} \end{array} \right\} \tag{2.5}$$

Thefore,

$$\frac{\partial^2 X}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial X}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\varphi^2} + l(l+1)X$$

$$
\begin{aligned}
&= c_l \left\{ \begin{aligned}
&\sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) \left\{ \begin{aligned} &\left[(2n)^2 - 2n \right] \sin^{2n-2}\theta \sin^{2n}\varphi \\ &-(2n)^2 (\sin\theta \sin\varphi)^{2n} \end{aligned} \right\} \\
&+ \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) \left[\begin{aligned} &2n \sin^{2n}\varphi \sin^{2n-2}\theta \\ &-2n(\sin\varphi \sin\theta)^{2n} \end{aligned} \right] \\
&+ \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) \left[\begin{aligned} &2n(2n-1)(\sin\theta \sin\varphi)^{2n-2} \\ &-(2n)^2 \sin^{2n-2}\theta \sin^{2n}\varphi \end{aligned} \right] \\
&+ \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) l(l+1)(\sin\varphi \sin\theta)^{2n}
\end{aligned} \right\} \\
&+ d_l \left\{ \begin{aligned}
&\sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) \left\{ \begin{aligned} &\left[(2n+1)^2 - (2n+1) \right] \sin^{2n-1}\theta \sin^{2n+1}\varphi \\ &-(2n+1)^2 (\sin\theta \sin\varphi)^{2n+1} \end{aligned} \right\} \\
&+ \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) \left[\begin{aligned} &(2n+1) \sin^{2n-1}\theta \sin^{2n+1}\varphi \\ &-(2n+1)(\sin\theta \sin\varphi)^{2n+1} \end{aligned} \right] \\
&+ \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) \left[\begin{aligned} &(2n+1)2n(\sin\varphi)^{2n-1} \\ &-(2n+1)^2 \sin^{2n-1}\theta \sin^{2n+1}\varphi \end{aligned} \right] \\
&+ \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) l(l+1)(\sin\varphi \sin\theta)^{2n+1}
\end{aligned} \right\} \\
&= c_l \sum_{n=0}^{\infty} \left\{ \begin{aligned}
&\left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n(2n-1)(\sin\theta \sin\varphi)^{2n-2} \\
&- \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) (2n-l)(2n+l+1)(\sin\varphi \sin\theta)^{2n}
\end{aligned} \right\} \\
&+ d_l \sum_{n=0}^{\infty} \left\{ \begin{aligned}
&\left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1)2n(\sin\varphi)^{2n-1} \\
&- \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n-l+1)(2n+l+2)(\sin\varphi \sin\theta)^{2n+1}
\end{aligned} \right\}
\end{aligned} \tag{2.6}
$$

The above calculation uses the identity equation

$$
\begin{aligned}
&(2n)^2 + 2n - l(l+1) = (2n-l)(2n+l+1) \\
&(2n+1)^2 + (2n+1) - l(l+1) = (2n-l+1)(2n+l+2)
\end{aligned} \tag{2.7}
$$

Merge similar items in (2.6) and use the relationship equation

$$
\begin{aligned}
&\prod_{k=1}^{n+1} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} = \frac{(2n-l)(2n+l+1)}{2(n+1)(2n+1)} \prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \\
&\prod_{k=1}^{n+1} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} = \frac{(2n-l+1)(2n+l+2)}{(2n+3)2(n+1)} \prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k}
\end{aligned} \tag{2.8}
$$

One obtains,

$$
\begin{aligned}
&\frac{\partial^2 X}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial X}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\varphi^2}+l\left(l+1\right)X\\
&=c_l\sum_{n=0}^{\infty}\left\{\begin{array}{l}\left(\prod\limits_{k=1}^{n+1}\dfrac{\left(2k-l-2\right)\left(2k+l-1\right)}{2k\left(2k-1\right)}\right)2\left(n+1\right)\left(2n+1\right)\\ -\left(\prod\limits_{k=1}^{n}\dfrac{\left(2k-l-2\right)\left(2k+l-1\right)}{2k\left(2k-1\right)}\right)\left(2n-l\right)\left(2n+l+1\right)\end{array}\right\}\left(\sin\varphi\sin\theta\right)^{2n}\\
&+d_l\sum_{n=0}^{\infty}\left\{\begin{array}{l}\left(\prod\limits_{k=1}^{n+1}\dfrac{\left(2k-l-1\right)\left(2k+l\right)}{\left(2k+1\right)2k}\right)\left(2n+3\right)2\left(n+1\right)\\ -\left(\prod\limits_{k=1}^{n}\dfrac{\left(2k-l-1\right)\left(2k+l\right)}{\left(2k+1\right)2k}\right)\left(2n-l+1\right)\left(2n+l+2\right)\end{array}\right\}\left(\sin\varphi\sin\theta\right)^{2n+1}\\
&=0
\end{aligned}
\tag{2.9}
$$

So Theorem 1 holds. End of proof.

If the unbounded function term in the first kind of entangled analytical solution of the spherical harmonic equation is removed, the result is a bounded polynomial function, that is, the first kind of special entangled spherical harmonic function.

Inference 1 For any non negative integer l, a special class of entangled spherical harmonic functions

$$
\mathscr{X}_l^{\langle 0\rangle}\left(\theta,\varphi\right)=\left\{\begin{array}{l}\underbrace{c_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{\left(2k-l-2\right)\left(2k+l-1\right)}{2k\left(2k-1\right)}\right)\left(\sin\varphi\sin\theta\right)^{2n}}_{l=0,\,2,\,4,\,\cdots}\\ \underbrace{d_l\sum\limits_{n=0}^{\infty}\left(\prod\limits_{k=1}^{n}\dfrac{\left(2k-l-1\right)\left(2k+l\right)}{\left(2k+1\right)2k}\right)\left(\sin\varphi\sin\theta\right)^{2n+1}}_{l=1,\,3,\,5,\,\cdots}\end{array}\right.
\tag{2.10}
$$

containing only one undetermined coefficient c_l or d_l, or written as polynomial form

$$
\mathscr{X}_l^{\langle 0\rangle}\left(\theta,\varphi\right)=\left\{\begin{array}{l}\underbrace{c_l\sum\limits_{n=0}^{(l+2)/2}\left(\prod\limits_{k=1}^{n}\dfrac{\left(2k-l-2\right)\left(2k+l-1\right)}{2k\left(2k-1\right)}\right)\left(\sin\varphi\sin\theta\right)^{2n}}_{l=0,\,2,\,4,\,\cdots}\\ \underbrace{d_l\sum\limits_{n=0}^{(l+1)/2}\left(\prod\limits_{k=1}^{n}\dfrac{\left(2k-l-1\right)\left(2k+l\right)}{\left(2k+1\right)2k}\right)\left(\sin\varphi\sin\theta\right)^{2n+1}}_{l=1,\,3,\,5,\,\cdots}\end{array}\right.
\tag{2.11}
$$

satisfies the spherical harmonic partial differential equation

$$
\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial X}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\varphi^2}+l\left(l+1\right)X=0
\tag{2.12}
$$

with magnetic quantum number $m=0$.

Polynomials (2.10) and (2.11) are called the first kind of special spherical harmonic functions. Table 2.1 lists the specific forms of the first kind of special entangled spherical harmonic functions corresponding to different integers of l in (2.10) and (2.11) when the magnetic quantum number

$m = 0$.

Table 2.1 Some special entangled spherical harmonic functions of the first kind

$$\mathscr{X}_0^{\langle 0\rangle}(\theta,\varphi) = c_0$$
$$\mathscr{X}_1^{\langle 0\rangle}(\theta,\varphi) = d_1 \sin\theta \sin\varphi$$
$$\mathscr{X}_2^{\langle 0\rangle}(\theta,\varphi) = c_2 \left(1 - 3\sin^2\theta \sin^2\varphi\right)$$
$$\mathscr{X}_3^{\langle 0\rangle}(\theta,\varphi) = d_3 \left(\sin\theta\sin\varphi - \frac{5}{3}\sin^3\theta\sin^3\varphi\right)$$
$$\mathscr{X}_4^{\langle 0\rangle}(\theta,\varphi) = c_4 \left(1 - 10\sin^2\theta\sin^2\varphi + \frac{35}{3}\sin^4\theta\sin^4\varphi\right)$$
$$\mathscr{X}_5^{\langle 0\rangle}(\theta,\varphi) = d_5 \left(\sin\theta\sin\varphi - \frac{14}{3}\sin^3\theta\sin^3\varphi + \frac{21}{5}\sin^5\theta\sin^5\varphi\right)$$
$$\mathscr{X}_6^{\langle 0\rangle}(\theta,\varphi) = c_6 \left(1 - 21\sin^2\theta\sin^2\varphi + 63\sin^4\theta\sin^4\varphi - \frac{231}{5}\sin^6\theta\sin^6\varphi\right)$$
$$\mathscr{X}_7^{\langle 0\rangle}(\theta,\varphi) = d_7 \left(\sin\theta\sin\varphi - 9\sin^3\theta\sin^3\varphi + \frac{99}{5}\sin^5\theta\sin^5\varphi - \frac{429}{35}\sin^7\theta\sin^7\varphi\right)$$
$$\mathscr{X}_8^{\langle 0\rangle}(\theta,\varphi) = c_8 \begin{pmatrix} 1 - 36\sin^2\theta\sin^2\varphi + 198\sin^4\theta\sin^4\varphi \\ -\frac{1716}{5}\sin^6\theta\sin^6\varphi + \frac{1287}{7}\sin^8\theta\sin^8\varphi \end{pmatrix}$$
$$\mathscr{X}_9^{\langle 0\rangle}(\theta,\varphi) = d_9 \begin{pmatrix} \sin\theta\sin\varphi - \frac{44}{3}\sin^3\theta\sin^3\varphi + \frac{286}{5}\sin^5\theta\sin^5\varphi \\ -\frac{572}{7}\sin^7\theta\sin^7\varphi + \frac{2431}{63}\sin^9\theta\sin^9\varphi \end{pmatrix}$$
$$\mathscr{X}_{10}^{\langle 0\rangle}(\theta,\varphi) = c_{10} \begin{pmatrix} 1 - 55\sin^2\theta\sin^2\varphi + \frac{1430}{3}\sin^4\theta\sin^4\varphi - 1430\sin^6\theta\sin^6\varphi \\ +\frac{12155}{7}\sin^8\theta\sin^8\varphi - \frac{46189}{63}\sin^{10}\theta\sin^{10}\varphi \end{pmatrix}$$

3 Special entangled spherical harmonic function of the second kind

Theorem 2 For any constant l, the first kind of special entanglement analytic function

$$D_l^{\langle 0\rangle}(\theta,\varphi) = \left\{ \begin{aligned} & f_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) (\cos\varphi\sin\theta)^{2n} \\ & + g_l \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)\,2k} \right) (\cos\varphi\sin\theta)^{2n+1} \end{aligned} \right\} \tag{3.1}$$

with two undetermined coefficients f_l and g_l satisfies the spherical harmonic partial differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial D}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 D}{\partial\varphi^2} + l(l+1)D = 0 \tag{3.2}$$

with magnetic quantum number $m = 0$.

Proof: The spherical harmonic partial differential equation (3.2) has the following equivalent form

$$\frac{\partial^2 D}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial D}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 D}{\partial\varphi^2} + l(l+1)D = 0 \tag{3.3}$$

The first and second partial derivatives of $X_l^0(\theta,\varphi)$ with respect to angles θ and φ are as follows

$$\frac{\partial D}{\partial \theta}=\left\{\begin{aligned}&f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\cos^{2n}\varphi\sin^{2n-1}\theta\cos\theta\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\cos^{2n+1}\varphi\sin^{2n}\theta\cos\theta\end{aligned}\right\}$$

$$\frac{\partial D}{\partial \varphi}=\left\{\begin{aligned}&f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\left(-\cos^{2n-1}\varphi\sin^{2n}\theta\sin\varphi\right)\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\left(-\cos^{2n}\varphi\sin^{2n+1}\theta\sin\varphi\right)\end{aligned}\right\}$$

$$\frac{\partial^2 D}{\partial \theta^2}=\left\{\begin{aligned}&f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\left[(2n-1)\cot^2\theta-1\right]\cos^{2n}\varphi\sin^{2n}\theta\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\left(2n\cot^2\theta-1\right)\cos^{2n+1}\varphi\sin^{2n+1}\theta\end{aligned}\right\}$$

$$\frac{\partial^2 D}{\partial \varphi^2}=\left\{\begin{aligned}&f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\left[(2n-1)\tan^2\varphi-1\right]\cos^{2n}\varphi\sin^{2n}\theta\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\left(2n\tan^2\varphi-1\right)\cos^{2n+1}\varphi\sin^{2n+1}\theta\end{aligned}\right\}\tag{3.4}$$

Substitute (3.1) and (3.4) into the left side of (3.3) to obtain

$$\begin{aligned}&\frac{\partial^2 D}{\partial \theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial D}{\partial \theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 D}{\partial \varphi^2}+l(l+1)D\\&=\left\{\begin{aligned}&f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\left[(2n-1)\cot^2\theta-1\right]\cos^{2n}\varphi\sin^{2n}\theta\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\left(2n\cot^2\theta-1\right)\cos^{2n+1}\varphi\sin^{2n+1}\theta\end{aligned}\right\}\\&+\frac{\cos\theta}{\sin\theta}\left\{\begin{aligned}&f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\cos^{2n}\varphi\sin^{2n-1}\theta\cos\theta\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\cos^{2n+1}\varphi\sin^{2n}\theta\cos\theta\end{aligned}\right\}\\&+\frac{1}{\sin^2\theta}\left\{\begin{aligned}&f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n\left[(2n-1)\tan^2\varphi-1\right]\cos^{2n}\varphi\sin^{2n}\theta\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)\left(2n\tan^2\varphi-1\right)\cos^{2n+1}\varphi\sin^{2n+1}\theta\end{aligned}\right\}\\&+l(l+1)\left\{\begin{aligned}&f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\cos\varphi\sin\theta)^{2n}\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(\cos\varphi\sin\theta)^{2n+1}\end{aligned}\right\}\end{aligned}$$

$$
= f_l \left\{ \begin{array}{l} \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n \left[(2n-1)\cot^2\theta - 1 \right] \cos^{2n}\varphi \sin^{2n}\theta \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n\cos^{2n}\varphi \sin^{2n-2}\theta \cos^2\theta \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) 2n \left[(2n-1)\tan^2\varphi - 1 \right] \cos^{2n}\varphi \sin^{2n-2}\theta \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) l(l+1)(\cos\varphi\sin\theta)^{2n} \end{array} \right\}
$$

$$
+ g_l \left\{ \begin{array}{l} + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1)\left(2n\cot^2\theta - 1\right) \cos^{2n+1}\varphi \sin^{2n+1}\theta \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1)\cos^{2n+1}\varphi \sin^{2n-1}\theta \cos^2\theta \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) (2n+1)\left(2n\tan^2\varphi - 1\right) \cos^{2n+1}\varphi \sin^{2n-1}\theta \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) l(l+1)(\cos\varphi\sin\theta)^{2n+1} \end{array} \right\} \tag{3.5}
$$

Thefore,

$$
\frac{\partial^2 D}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial D}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 D}{\partial\varphi^2} + l(l+1)D
$$

$$
= f_l \left\{ \begin{array}{l} \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) \left\{ \begin{array}{l} \left[(2n)^2 - 2n\right] \cos^{2n}\varphi \sin^{2n-2}\theta \\ -(2n)^2\cos^{2n}\varphi \sin^{2n}\theta \end{array} \right\} \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) \left(2n\cos^{2n}\varphi \sin^{2n-2}\theta - 2n\cos^{2n}\varphi \sin^{2n}\theta\right) \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) \left\{ \begin{array}{l} \left[(2n)^2 - 2n\right] \cos^{2n-2}\varphi \sin^{2n-2}\theta \\ -(2n)^2\cos^{2n}\varphi \sin^{2n-2}\theta \end{array} \right\} \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \right) l(l+1)(\cos\varphi\sin\theta)^{2n} \end{array} \right\}
$$

$$
+ g_l \left\{ \begin{array}{l} + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) \left\{ \begin{array}{l} \left[(2n+1)^2 - (2n+1)\right] \cos^{2n+1}\varphi \sin^{2n-1}\theta \\ -(2n+1)^2\cos^{2n+1}\varphi \sin^{2n+1}\theta \end{array} \right\} \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) \left[\begin{array}{l} (2n+1)\cos^{2n+1}\varphi \sin^{2n-1}\theta \\ -(2n+1)\cos^{2n+1}\varphi \sin^{2n+1}\theta \end{array} \right] \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) \left\{ \begin{array}{l} \left[(2n+1)^2 - (2n+1)\right] \cos^{2n-1}\varphi \sin^{2n-1}\theta \\ -(2n+1)^2\cos^{2n+1}\varphi \sin^{2n-1}\theta \end{array} \right\} \\ + \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{(2k-l-1)(2k+l)}{(2k+1)2k} \right) l(l+1)(\cos\varphi\sin\theta)^{2n+1} \end{array} \right\}
$$

$$
\begin{aligned}
&= f_l \left\{ \begin{aligned} &\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)2n(2n-1)\cos^{2n-2}\varphi\sin^{2n-2}\theta \\ &-\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(2n-l)(2n+l+1)(\cos\varphi\sin\theta)^{2n} \end{aligned}\right\} \\
&+ g_l \left\{ \begin{aligned} &+\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+1)2n\cos^{2n-1}\varphi\sin^{2n-1}\theta \\ &-\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n-l+1)(2n+l+2)(\cos\varphi\sin\theta)^{2n+1} \end{aligned}\right\}
\end{aligned}
\tag{3.6}
$$

The above calculation uses the identity equation

$$
\begin{aligned}
&(2n)^2+2n-l(l+1)=(2n-l)(2n+l+1) \\
&(2n+1)^2+(2n+1)-l(l+1)=(2n-l+1)(2n+l+2)
\end{aligned}
\tag{3.7}
$$

Merge similar items in (3.6) and use the relationship equation

$$
\begin{aligned}
&\prod_{k=1}^{n+1}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}=\frac{(2n-l)(2n+l+1)}{2(n+1)(2n+1)}\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)} \\
&\prod_{k=1}^{n+1}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}=\frac{(2n-l+1)(2n+l+2)}{(2n+3)2(n+1)}\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}
\end{aligned}
\tag{3.8}
$$

One obtains,

$$
\begin{aligned}
&\frac{\partial^2 D}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial D}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 D}{\partial\varphi^2}+l(l+1)D \\
&= f_l \left\{ \begin{aligned} &\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n+1}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(2n+2)(2n+1) \\ &-\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(2n-l)(2n+l+1) \end{aligned}\right\}(\cos\varphi\sin\theta)^{2n} \\
&+ g_l \left\{ \begin{aligned} &+\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n+1}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n+3)(2n+2) \\ &-\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(2n-l+1)(2n+l+2) \end{aligned}\right\}(\cos\varphi\sin\theta)^{2n+1} \\
&= 0
\end{aligned}
\tag{3.9}
$$

So Theorem 1 holds. End of proof.

If the unbounded function term in the second kind of entangled analytical solution of the spherical harmonic equation is removed, the result is a bounded polynomial function, that is, the second kind of special entangled spherical harmonic function.

Inference 2 For any non negative integer l, a special class of entangled spherical harmonic functions

$$\mathscr{D}_l^0(\theta,\varphi)=\begin{cases}\underbrace{f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\cos\varphi\sin\theta)^{2n}}_{l=0,\,2,\,4,\,\cdots}\\ \underbrace{g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)\,2k}\right)(\cos\varphi\sin\theta)^{2n+1}}_{l=1,\,3,\,5,\,\cdots}\end{cases}\tag{3.10}$$

containing only one undetermined coefficient f_l or g_l, or written as polynomial form

$$\mathscr{D}_l^{\langle 0\rangle}(\theta,\varphi)=\begin{cases}\underbrace{f_l\sum_{n=0}^{(l+2)/2}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\cos\varphi\sin\theta)^{2n}}_{l=0,\,2,\,4,\,\cdots}\\ \underbrace{g_l\sum_{n=0}^{(l+1)/2}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)\,2k}\right)(\cos\varphi\sin\theta)^{2n+1}}_{l=1,\,3,\,5,\,\cdots}\end{cases}\tag{3.11}$$

satisfies the spherical harmonic partial differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathscr{D}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\mathscr{D}}{\partial\varphi^2}+l(l+1)\mathscr{D}=0\tag{3.12}$$

with magnetic quantum number $m=0$.

Polynomials (3.10) and (3.11) are called the second kind of special spherical harmonic functions. Table 3.1 lists the specific forms of the second kind of special entangled spherical harmonic functions corresponding to different integers of l in (3.11) when the magnetic quantum number $m=0$.

Table 3.1 Some special entangled spherical harmonic functions of the second kind

$$\mathscr{D}_0^{\langle 0\rangle}(\theta,\varphi)=f_0$$
$$\mathscr{D}_1^{\langle 0\rangle}(\theta,\varphi)=g_1\cos\varphi\sin\theta$$
$$\mathscr{D}_2^{\langle 0\rangle}(\theta,\varphi)=f_2\left(1-3\cos^2\varphi\sin^2\theta\right)$$
$$\mathscr{D}_3^{\langle 0\rangle}(\theta,\varphi)=g_3\left(\cos\varphi\sin\theta-\frac{5}{3}\cos^3\varphi\sin^3\theta\right)$$
$$\mathscr{D}_4^{\langle 0\rangle}(\theta,\varphi)=f_4\left(1-10\cos^2\varphi\sin^2\theta+\frac{35}{3}\cos^4\varphi\sin^4\theta\right)$$
$$\mathscr{D}_5^{\langle 0\rangle}(\theta,\varphi)=g_5\left(\cos\varphi\sin\theta-\frac{14}{3}\sin^3\theta\sin^3\varphi+\frac{21}{5}\cos^5\varphi\sin^5\theta\right)$$
$$\mathscr{D}_6^{\langle 0\rangle}(\theta,\varphi)=f_6\left(1-21\cos^2\varphi\sin^2\theta+63\cos^4\varphi\sin^4\theta-\frac{231}{5}\cos^6\varphi\sin^6\theta\right)$$
$$\mathscr{D}_7^{\langle 0\rangle}(\theta,\varphi)=g_7\left(\cos\varphi\sin\theta-9\cos^3\varphi\sin^3\theta+\frac{99}{5}\cos^5\varphi\sin^5\theta-\frac{429}{35}\cos^7\varphi\sin^7\theta\right)$$
$$\mathscr{D}_8^{\langle 0\rangle}(\theta,\varphi)=f_8\begin{pmatrix}1-36\cos^2\varphi\sin^2\theta+198\cos^4\varphi\sin^4\theta\\ -\dfrac{1716}{5}\cos^6\varphi\sin^6\theta+\dfrac{1287}{7}\cos^8\varphi\sin^8\theta\end{pmatrix}$$

$$\mathscr{D}_9^{\langle 0\rangle}(\theta,\varphi)=g_9\begin{pmatrix}\cos\varphi\sin\theta-\dfrac{44}{3}\cos^3\varphi\sin^3\theta+\dfrac{286}{5}\cos^5\varphi\sin^5\theta\\-\dfrac{572}{7}\cos^7\varphi\sin^7\theta+\dfrac{2431}{63}\cos^9\varphi\sin^9\theta\end{pmatrix}$$

$$\mathscr{D}_{10}^{\langle 0\rangle}(\theta,\varphi)=f_{10}\begin{pmatrix}1-55\cos^2\varphi\sin^2\theta+\dfrac{1430}{3}\cos^4\varphi\sin^4\theta-1430\cos^6\varphi\sin^6\theta\\+\dfrac{12155}{7}\cos^8\varphi\sin^8\theta-\dfrac{46189}{63}\cos^{10}\varphi\sin^{10}\theta\end{pmatrix}$$

4 Entangled wave function and probability density diagram

The spherical harmonic function is the solution of the spherical harmonic partial differential equation, which is the arbitrary specified radius solution of the Laplace equation and the quantum mechanics angular momentum square operator equation in the spherical coordinate system. Usually, a bounded spherical harmonic function is taken, and the accuracy of the results is verified through experimental observations. Some steady-state physical quantities represented by spherical harmonics can be accurately experimentally plotted for their spatial distribution, while others represented by spherical harmonics cannot be experimentally plotted for their spatial distribution. The three-dimensional diagram of spherical harmonic functions can be used to theoretically and intuitively analyze the credibility of describing physical phenomena using spherical harmonic partial differential equations.

The Laplace equation is used to describe electrostatic fields, static magnetic fields, stable temperature fields, and fluid fields. The spherical harmonic function is convenient for describing the distribution of physical quantities such as electrostatic fields and static magnetic fields. This type of application is theoretically perfect, so is there strict consistency between the actual distribution map of the electromagnetic field measured experimentally and the function map of the spherical harmonic function? Here, we can draw a three-dimensional diagram of the Legendre special spherical harmonic function, the first and second types of special entangled spherical harmonic functions, in order to have a clear and intuitive understanding of the solution to the Dirichlet problem of the Laplace equation. In the past, the solution to the spherical harmonic partial differential equation with magnetic quantum numbers was limited to the Legendre special spherical harmonic function. The first and second types of special entangled spherical harmonic functions discovered now allow us to understand that different types of solutions to spherical harmonic partial differential equations are not consistent. The established conclusion of using spherical harmonic functions to describe the angular distribution of various stable fields is theoretically challenged. We hope that experimental physicists can provide clear conclusions in a timely manner to indicate the direction of theory.

Now focus on the difference between the wave function and probability density definition in quantum mechanics. The wave equation satisfied by the angular wave function is exactly the spherical harmonic partial differential equation. The angular wave function with magnetic quantum number $m=0$ is either a real function or an imaginary function. Quantum mechanics defines the square of the wave function modulus as the probability density of a particle appearing in space, so the square of the angular wave function modulus means the probability density of a particle appearing in the angular direction. Because experiments cannot observe the probability density of particles appearing in space or determining their orientation, the description of quantum mechanics largely belongs to the realm of consciousness, as evidenced by the inconsistency between the distribution space of any wave function and the square distribution space of the corresponding wave function modulus representing probability density.

Referring to Table 4.1, when the orbital angular momentum quantum number $l \geqslant 2$, comparing the three-dimensional diagram of the modulus of the special spherical harmonic function with $m = 0$and the three-dimensional diagram of the square of the modulus, the two are clearly inconsistent. The early definition of particle probability density using the square of the wave function modulus was only proposed to eliminate imaginary numbers, but it did not take into account that the fundamental change in the properties of the function had occurred from the wave function modulus to the square of the wave function modulus.

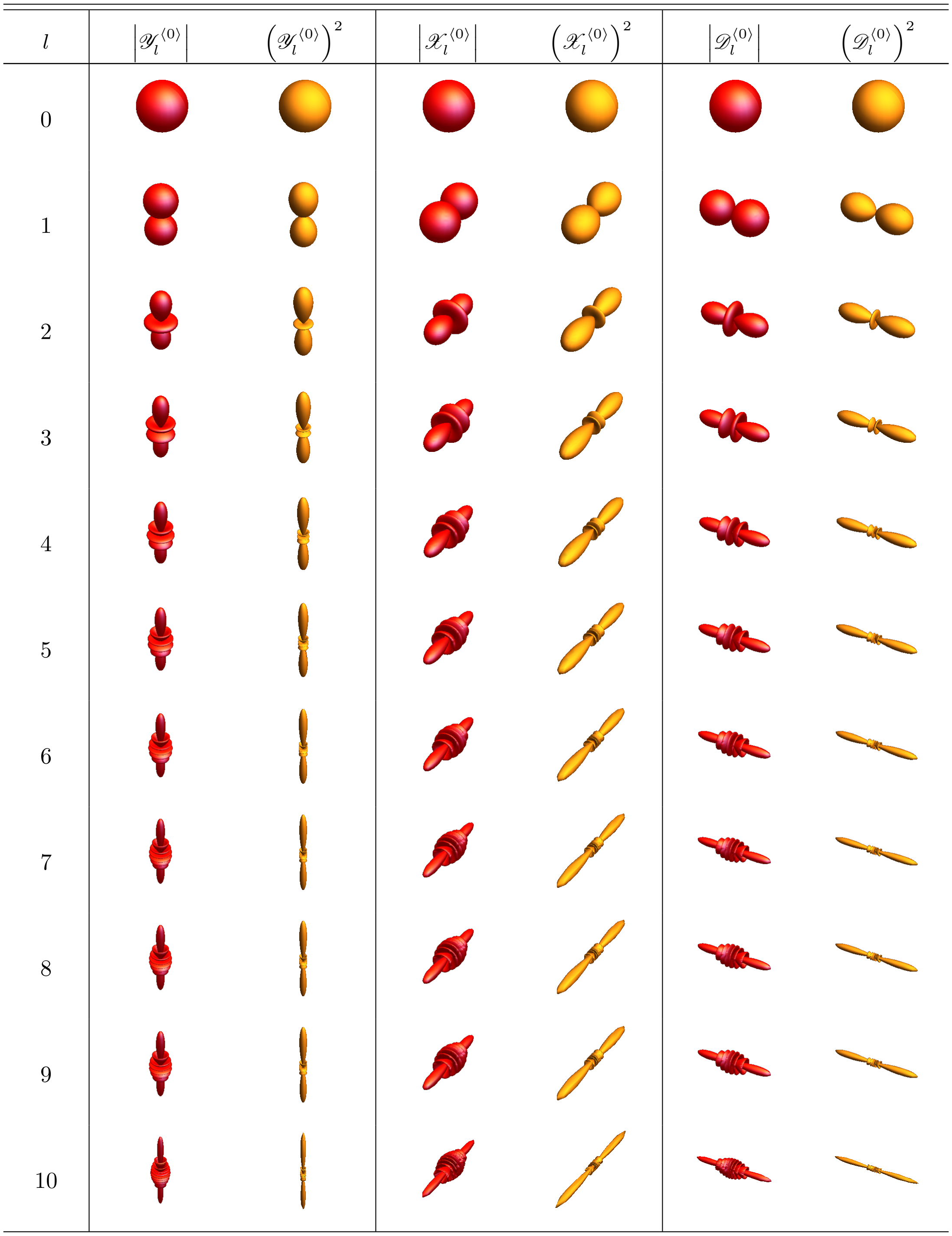
Mathematics can freely define functions and assign them specific meanings. But physics uses functions to describe natural laws, and functions cannot be defined arbitrarily. There is no logical basis for defining the probability density function $\rho(\theta, \varphi)$ as the square of the wave function $\psi(\theta, \varphi)$. This can be proven by the method of proof to the contrary. If the square of the wave function modulus is defined as the probability density because it has the same extreme point as the wave function modulus, then because any even power of the wave function can ensure that the zero and extreme points of the wave function are consistent, it cannot be ruled out to define a probability function for any positive integer n

$$\rho(\theta, \varphi) = [\psi(\theta, \varphi)]^{2n}$$

But in reality, the specific value of n cannot be determined, and the probability density function is uncertain. Therefore, the definition of probability density function is not an inevitable logical inference. The excessive abstraction of quantum mechanics theory in describing implicit fuzzy logic problems should also be perfectly solved from a mathematical perspective. Admittedly, discussing special spherical harmonic functions does not require addressing this issue at the moment.

The theory described by Legendre's special spherical harmonic function in the past is about to be completely changed due to the discovery of entangled spherical harmonic functions. Even in the special case of magnetic quantum number $m = 0$, for any orbital angular momentum quantum number LL, there are three special angular entanglement functions that can be used to describe the solutions of the Laplace equation or wave equation. The three-dimensional diagram shows that the distribution of physical quantities corresponding to the probability density of quantum mechanics for the three special angular wave functions is completely different, while the solutions of Schrödinger equation and Klein Gordon equation depend on the solutions of spherical harmonic partial differential equations. How can quantum mechanics embrace these new mathematical challenges and more undiscovered ones?

Table 4.1 A three-dimensional diagram of some special spherical harmonic functions and their squared modulus



5 Blending special entangled spherical harmonic functions

The Legendre special bounded spherical harmonic function from a single angle, the first kind of special bounded entangled spherical harmonic function, and the second kind of special bounded entangled spherical harmonic function all have only one undetermined coefficient. Normalization conditions can be used to obtain the so-called normalization coefficient for the coefficients. But the previous list of special bounded spherical harmonic functions and the drawing of special bounded spherical harmonic functions did not provide the so-called normalization coefficients. There are two considerations for this. Firstly, for a spherical harmonic function with only one undetermined coefficient, the function graph drawn by arbitrarily assigning undetermined coefficients is a completely similar graph, without the need for normalization coefficients; Secondly, for any principal quantum l, there are three different special spherical harmonic functions with $m = 0$ that satisfy the spherical harmonic partial differential equation. Using normalization conditions to obtain the coefficients of any special spherical harmonic function has neither mathematical nor physical significance.

The basic principle of the solution of ordinary differential equations is that the linear combination of all solutions that satisfy the same ordinary differential equation is the general solution of this ordinary differential equation. This principle also holds true when applied to partial differential equations. However, the solutions of partial differential equations are relatively complex, and different solution methods give different solutions. Usually, the variable separation method is used to construct solutions for some linear partial differential equations, which belong to special general solutions that conform to the construction characteristics. The Legendre spherical harmonic function is a bounded function obtained by constructing solutions to spherical harmonic partial differential equations using the method of separating variables. Since two special types of entangled spherical harmonic functions can be constructed to satisfy the spherical harmonic partial differential equation for the case of magnetic quantum number $m = 0$, the question arises whether there exists a general entangled spherical harmonic function that satisfies the spherical harmonic partial differential equation. How many undiscovered solutions satisfy the same definite solution conditions for the Laplace equation of the same physical model. The answers to these questions will have more impact on the established conclusions of physics and also have a positive impact on mathematical theory. It can now be clarified that in the construction method of known partial differential equation solutions, the linear combination of all solution sets of the same partial differential equation is the local general solution of this linear partial differential equation.

The linear combination of the Legendre special unary analytic function with magnetic quantum number $m = 0$, the first kind of special entanglement analytic function, and the second kind of special entanglement analytic function is the local general solution of the spherical harmonic partial differential equation. The Legendre special bounded spherical harmonic function with magnetic quantum number $m = 0$, the linear combination of the first kind of special bounded entangled spherical harmonic function and the second kind of special bounded entangled spherical harmonic function form a special local universal spherical harmonic function. From any perspective, physical and mathematical conclusions based on spherical harmonic partial differential equations need to be rewritten, and today's scientific theories cannot absolutely dominate the future scientific world, and only the correct parts can sustain development.

Theorem 3 The special mixed entanglement analytic function

$$\Omega_l^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{aligned}&a_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)\cos^{2n}\theta\\&+c_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\sin\varphi\sin\theta)^{2n}\\&+f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\cos\varphi\sin\theta)^{2n}\\&+b_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)\cos^{2n+1}\theta\\&+d_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(\sin\varphi\sin\theta)^{2n+1}\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(\cos\varphi\sin\theta)^{2n+1}\end{aligned}\right\}\tag{5.1}$$

with 6 undetermined coefficients $a_l b_l c_l d_l f_l$, and g_l for any constant l and magnetic quantum number $m=0$ satisfies the spherical harmonic partial differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Omega}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Omega}{\partial\varphi^2}+l(l+1)\Omega=0\tag{5.2}$$

Theorem 4 For any non negative integer l and magnetic quantum number $m=0$, a special mixed bounded angled spherical harmonic function

$$\Upsilon_l^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{aligned}&\underbrace{\left\{\begin{aligned}&a_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)\cos^{2n}\theta\\&+c_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\sin\varphi\sin\theta)^{2n}\\&+f_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\cos\varphi\sin\theta)^{2n}\end{aligned}\right\}}_{l=0,\,2,\,4,\,\cdots}\\&\underbrace{\left\{\begin{aligned}&b_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)\cos^{2n+1}\theta\\&+d_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(\sin\varphi\sin\theta)^{2n+1}\\&+g_l\sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(\cos\varphi\sin\theta)^{2n+1}\end{aligned}\right\}}_{l=1,\,3,\,5,\,\cdots}\end{aligned}\right.\tag{5.3}$$

with three undetermined coefficients $a_l c_l$, and f_l (or $b_l d_l$, and g_l), or written in polynomial form

$$\Upsilon_l^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l}\underbrace{\left\{\begin{array}{l}a_l\sum\limits_{n=0}^{(l+2)/2}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)\cos^{2n}\theta\\+c_l\sum\limits_{n=0}^{(l+2)/2}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\sin\varphi\sin\theta)^{2n}\\+f_l\sum\limits_{n=0}^{(l+2)/2}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-2)(2k+l-1)}{2k(2k-1)}\right)(\cos\varphi\sin\theta)^{2n}\end{array}\right\}}_{l=0,\,2,\,4,\,\cdots}\\\underbrace{\left\{\begin{array}{l}b_l\sum\limits_{n=0}^{(l+1)/2}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)\cos^{2n+1}\theta\\+d_l\sum\limits_{n=0}^{(l+1)/2}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(\sin\varphi\sin\theta)^{2n+1}\\+g_l\sum\limits_{n=0}^{(l+1)/2}\left(\prod\limits_{k=1}^{n}\dfrac{(2k-l-1)(2k+l)}{(2k+1)2k}\right)(\cos\varphi\sin\theta)^{2n+1}\end{array}\right\}}_{l=1,\,3,\,5,\,\cdots}\end{array}\right. \tag{5.4}$$

satisfies the spherical harmonic partial differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Upsilon}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Upsilon}{\partial\varphi^2}+l(l+1)\Upsilon=0 \tag{5.5}$$

Theorem 3 is a summary of Lemma 1, Theorem 1, and Theorem 2, while Theorem 4 is a synthesis of Problem 1, Corollary 1, and Corollary 2, and does not require re proof.

The special mixed bounded entangled spherical harmonic function has three undetermined coefficients, and the normalization condition alone cannot determine these undetermined coefficients, that is, the normalization condition cannot determine the solution of the spherical harmonic partial differential equation. The normalization coefficient in the past was a theoretical bias. Two more definite solution conditions are needed to determine the special mixed bounded entangled spherical harmonic function. Therefore, the key to future research on the solutions of partial differential equations such as quantum mechanics wave equations and Laplace equations is to provide at least two new definite solution conditions for spherical partial differential equations.

Table 5.1 Some special mixed bounded entangled spherical harmonic functions

$$\Upsilon_0^{\langle 0\rangle}(\theta,\varphi)=a_0+c_0+f_0$$
$$\Upsilon_1^{\langle 0\rangle}(\theta,\varphi)=b_1\cos\theta+d_1\sin\theta\sin\varphi+g_1\cos\varphi\sin\theta$$
$$\Upsilon_2^{\langle 0\rangle}(\theta,\varphi)=a_2\left(1-3\cos^2\theta\right)+c_2\left(1-3\sin^2\theta\sin^2\varphi\right)+f_2\left(1-3\cos^2\varphi\sin^2\theta\right)$$
$$\Upsilon_3^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l}b_3\left(\cos\theta-\dfrac{5}{3}\cos^3\theta\right)+d_3\left(\sin\theta\sin\varphi-\dfrac{5}{3}\sin^3\theta\sin^3\varphi\right)\\+g_3\left(\cos\varphi\sin\theta-\dfrac{5}{3}\cos^3\varphi\sin^3\theta\right)\end{array}\right\}$$

$$\Upsilon_4^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l} a_4\left(1-10\cos^2\theta+\dfrac{35}{3}\cos^4\theta\right)+c_4\left(1-10\sin^2\theta\sin^2\varphi+\dfrac{35}{3}\sin^4\theta\sin^4\varphi\right)\\ +f_4\left(1-10\cos^2\varphi\sin^2\theta+\dfrac{35}{3}\cos^4\varphi\sin^4\theta\right)\end{array}\right\}$$

$$\Upsilon_5^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l} b_5\left(\cos\theta-\dfrac{14}{3}\cos^3\theta+\dfrac{21}{5}\cos^5\theta\right)\\ +d_5\left(\sin\theta\sin\varphi-\dfrac{14}{3}\sin^3\theta\sin^3\varphi+\dfrac{21}{5}\sin^5\theta\sin^5\varphi\right)\\ +g_5\left(\cos\varphi\sin\theta-\dfrac{14}{3}\sin^3\theta\sin^3\varphi+\dfrac{21}{5}\cos^5\varphi\sin^5\theta\right)\end{array}\right\}$$

$$\Upsilon_6^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l} a_6\left(1-21\cos^2\theta+63\cos^4\theta-\dfrac{231}{5}\cos^6\theta\right)\\ +c_6\left(1-21\sin^2\theta\sin^2\varphi+63\sin^4\theta\sin^4\varphi-\dfrac{231}{5}\sin^6\theta\sin^6\varphi\right)\\ +f_6\left(1-21\cos^2\varphi\sin^2\theta+63\cos^4\varphi\sin^4\theta-\dfrac{231}{5}\cos^6\varphi\sin^6\theta\right)\end{array}\right\}$$

$$\Upsilon_7^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l} b_7\left(\cos\theta-9\cos^3\theta+\dfrac{99}{5}\cos^5\theta-\dfrac{429}{35}\cos^7\theta\right)\\ +d_7\left(\sin\theta\sin\varphi-9\sin^3\theta\sin^3\varphi+\dfrac{99}{5}\sin^5\theta\sin^5\varphi-\dfrac{429}{35}\sin^7\theta\sin^7\varphi\right)\\ +g_7\left(\cos\varphi\sin\theta-9\cos^3\varphi\sin^3\theta+\dfrac{99}{5}\cos^5\varphi\sin^5\theta-\dfrac{429}{35}\cos^7\varphi\sin^7\theta\right)\end{array}\right\}$$

$$\Upsilon_8^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l} a_8\left(1-36\cos^2\theta+198\cos^4\theta-\dfrac{1716}{5}\cos^6\theta+\dfrac{1287}{7}\cos^8\theta\right)\\ +c_8\left(\begin{array}{l}1-36\sin^2\theta\sin^2\varphi+198\sin^4\theta\sin^4\varphi\\ -\dfrac{1716}{5}\sin^6\theta\sin^6\varphi+\dfrac{1287}{7}\sin^8\theta\sin^8\varphi\end{array}\right)\\ +f_8\left(\begin{array}{l}1-36\cos^2\varphi\sin^2\theta+198\cos^4\varphi\sin^4\theta\\ -\dfrac{1716}{5}\cos^6\varphi\sin^6\theta+\dfrac{1287}{7}\cos^8\varphi\sin^8\theta\end{array}\right)\end{array}\right\}$$

$$\Upsilon_9^{\langle 0\rangle}(\theta,\varphi)=\left\{\begin{array}{l} b_9\left(\cos\theta-\dfrac{44}{3}\cos^3\theta+\dfrac{286}{5}\cos^5\theta-\dfrac{572}{7}\cos^7\theta+\dfrac{2431}{63}\cos^9\theta\right)\\ +d_9\left(\begin{array}{l}\sin\theta\sin\varphi-\dfrac{44}{3}\sin^3\theta\sin^3\varphi+\dfrac{286}{5}\sin^5\theta\sin^5\varphi\\ -\dfrac{572}{7}\sin^7\theta\sin^7\varphi+\dfrac{2431}{63}\sin^9\theta\sin^9\varphi\end{array}\right)\\ +g_9\left(\begin{array}{l}\cos\varphi\sin\theta-\dfrac{44}{3}\cos^3\varphi\sin^3\theta+\dfrac{286}{5}\cos^5\varphi\sin^5\theta\\ -\dfrac{572}{7}\cos^7\varphi\sin^7\theta+\dfrac{2431}{63}\cos^9\varphi\sin^9\theta\end{array}\right)\end{array}\right\}$$

$$\Upsilon_{10}^{\langle 0 \rangle}(\theta,\varphi) = \left\{ \begin{array}{l} a_{10}\left(\begin{array}{l} 1 - 55\cos^2\theta + \dfrac{1430}{3}\cos^4\theta - 1430\cos^6\theta \\ + \dfrac{12155}{7}\cos^8\theta - \dfrac{46189}{63}\cos^{10}\theta \end{array} \right) \\ + c_{10}\left(\begin{array}{l} 1 - 55\sin^2\theta\sin^2\varphi + \dfrac{1430}{3}\sin^4\theta\sin^4\varphi - 1430\sin^6\theta\sin^6\varphi \\ + \dfrac{12155}{7}\sin^8\theta\sin^8\varphi - \dfrac{46189}{63}\sin^{10}\theta\sin^{10}\varphi \end{array} \right) \\ + f_{10}\left(\begin{array}{l} 1 - 55\cos^2\varphi\sin^2\theta + \dfrac{1430}{3}\cos^4\varphi\sin^4\theta - 1430\cos^6\varphi\sin^6\theta \\ + \dfrac{12155}{7}\cos^8\varphi\sin^8\theta - \dfrac{46189}{63}\cos^{10}\varphi\sin^{10}\theta \end{array} \right) \end{array} \right\}$$

How abstract conclusions describing natural phenomena based on imperfect mathematical deductions have passed experimental tests is a meaningful topic. For the special case of magnetic quantum number $m = 0$, the solution of the spherical harmonic partial differential equation, also known as the angular momentum square operator equation, is a special mixed spherical harmonic function. The special mixed spherical harmonic function has three undetermined coefficients, which can not be determined only by the normalization condition. Describing the distribution of various fields using the Laplace equation and the probability of microscopic particles appearing in space using the Schrödinger equation are both based on treating a single type of solution set constructed using the variable separation method as a general solution and drawing one-sided conclusions that cannot be truly confirmed by experimental observations. The theory that hides irreconcilable contradictions often claims to be widely experimentally confirmed, which is one of the important forms of public opinion orientation. The analysis of spherical harmonic functions [15-17] in physics and applied sciences needs to be reconstructed based on further improved spherical harmonic function theory [18-22].

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Notes

This groundbreaking mathematical paper has been reviewed by nearly ten journals for about a year and has not been published, with the longest review time being 50 days. It is unknown whether it has been plagiarized and rewritten like pioneering physics papers in the past. In today's world, mathematics and physics are monopolized by ignorant and immoral despicable people, and great discoveries have nowhere to be published. Free publication on the internet often suffers from their attacks and slander.

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