MATHEMATICS & NATURE
Mathematics,Physics, Mechanics & Astronomy

· **Article** ·
Mathematics and Physics

Random Vol. **2** Sn: 202206
https://mathnature.github.io

# Dongfang Challenge Solution of Dirac Hydrogen Equation

X. D. Dongfang

*Orient Research Base of Mathematics and Physics,*
*Wutong Mountain National Forest Park, Shenzhen, China*

In quantum mechanics, the Schrödinger equation is used to describe the bound state system, and the exact solution of the equation needs to be determined by boundary conditions. However, the size of an atomic nucleus is usually not considered, and the boundary condition that the proposed wave function should meet is only a rough form. The rough boundary condition causes the S-state wave function of the exact solution of the Klein-Gordon equation and Dirac equation of the so-called relativistic quantum mechanics describing the bound state of the Coulomb field to diverge at the coordinate origin and makes the hydrogen-like atom with the nuclear charge number Z>137 appear unreal virtual energy. The divergence of the wave function means that the probability density of the electron or meson appearing near the nucleus will increase rapidly. What it predicts is the untrue conclusion that the hydrogen-like atom in S-state will collapse rapidly into a neutron-like atom. Considering that the atomic nucleus has a certain radius, the exact boundary conditions that the wave function should satisfy are given here, and then the Dirac equation of hydrogen-like atom is solved again, and a new exact solution without wave function divergence and virtual energy is obtained. Surprisingly, the exact boundary condition makes the angular quantum number naturally regress to the eigenvalue determined by the exact solution of the equation. It has no contribution to the quantized energy, which means that the angular quantum number constructed by Dirac electron theory is denied. Unlike the Dirac energy level formula, the new energy level formula corresponding to the solution of the Dirac equation for the hydrogen-like atom satisfying the exact boundary conditions has no so-called fine structure, and its accuracy is equivalent to the accuracy of the Bohr energy level. The exact boundary condition solution poses a serious challenge to the Dirac relativistic quantum mechanics, which is called the challenging solution of the Dirac equation for the hydrogen atom.

**Keywords:** Dirac equation; boundary condition; energy eigenvalues; uniqueness of solution.

**PACS number(s):** 03.65.Pm—Relativistic wave equations; 03.65.Ge—Solutions of wave equations: bound states; 02.30.Gp—Special functions; 02.30.Hq—Ordinary differential equations; 32.10.Fn—Fine and hyperfine structure.

## 1 Introduction

In history, it was not meeting expectations to use the Klein-Gordon equation[1, 2] to describe the hydrogen atom because its eigenvalues of the quantum energy is incompletely agrees accurately with the experimentally observed hydrogen spectra. Considering the relativistic effect, Dirac introduced his relativistic wave equation for the single electron[3] in 1928. Darwin[4] and Gordon[5] first obtain the exact solution of the Dirac equation with a Coulomb potential. Biedenharm[6], Wong, Yeh[7, 8] and Su[9] etc. also constructed the different second order Dirac equation and obtain the different form of solutions. Nenciu[10], Kalus and Wüst[11] investigated the different construction methods of self-adjoint extension of the Dirac operators with coulomb potential, and it is also showed that the distinguished self-adjoint extensions given by the two methods are identical. It should be noted that, there is no essential difference in various treatment methods of the Dirac equation in modern mathematical physics textbooks[12]. Formally, the Dirac equation combines quantum mechanics and relativity to describe the spin and magnetic moment of electrons in a completely natural way. Especially, the distinguished Dirac formula of energy levels in a Coulomb field can explain the fine-structure of the hydrogen atom. This is considered to be one of the important indicates to the achievements of the Diac theory.

The Dirac equation is generally considered to be successful in many ways. However, the results of the systematic examination of Dirac's electronic theory and its various development theories by using Dongfang's unitary principle show that all kinds of extremely exaggerated beautification descriptions of Dirac's equation are basically not true, and there are many false calculations and distorted calculations purely for the purpose of piecing together the expected conclusions[13-17]. Although physicists have been unable to face a large num-

**Citation**: Dongfang, X. D. Dongfang Challenge Solution of Dirac Hydrogen Equation. *Mathematics & Nature* **2**, 202206 (2022).

*This article revises the author's article entitled New Exact Solution of Dirac-Coulomb Equation with Exact Boundary Condition signed Ruida Chen published in Int J Theor Phys in 2008 to express the author's clear scientific position.

 https://orcid.org/0000-0002-3644-5170

ber of major quantum mechanical problems that will overturn traditional cognition, such as the morbid equations of quantum numbers[18], the operator equations of angular motion laws[19], the macroscopic com quantum equations[20-22], and the Yukawa nuclear force meson theory[23] distorting mathematics[24, 25], and so on. Perhaps because of the complex mathematical process in form, including the introduction to Dirac's achievements cited above, physicists always have a preconceived understanding of Dirac's equation, and few have tested the authenticity of various conclusions through further in-depth mathematical calculations[26, 27]. Just say that the so-called exact solution of the Dirac equation for the hydrogen atom, which is used to fit the energy level formula of fine structure, has enough mathematical problems to need clear answers.

In order to facilitate the distinction, the recognized solution of Dirac equation respected in the textbook is called the traditional solution of the Dirac equation. In fact, it belongs to one of the formal solutions of Dirac equation that do not conform to the mathematical operation rules[14], which means that there are other completely different formal solutions, but it has not been found by theoretical physics scholars, the Dirac equation itself has many mathematical difficulties that are more difficult to be noticed by theoretical physics scholars and can not be solved at present. Dirac equation combines quantum mechanics with special relativity[28], and the difficulty of negative kinetic energy appeared at the beginning. The negative kinetic energy solution was originally a veto solution, but it was interpreted as that the interaction may lead to the transition to the negative energy state, making the matter in this framework impossible to be stable, so a veto solution was turned into a new progress. This logic of modern physics is difficult to reverse. However, it seems that only Dirac equation can be used to fit the so-called fine energy level structure formula of hydrogen atom, so Dirac equation has always been very attractive. There is a practical problem: if the explanation of the abnormal inference of quantum mechanics deviates too far from the mathematical logic of quantum mechanics itself, human understanding of it will be very unclear, so quantum mechanics will appear very mysterious, so that more people will stagnate in the stage of popular science description, which will inevitably lead to dogmatism. As we all know, there is still no clear solution to the localization problem and Klein paradox in quantum electrodynamics. Therefore, when describing the achievements of the Dirac equation, we'd better not go too far from Dirac's point of view: "... a book on the new physics, if not purely descriptive of experimental work, must be essentially mathematical." This means that the logical test of theoretical physics from the perspective of mathematics is not only reliable, but also necessary.

When I promote the optimization differential theory[29, 30], I hope to find practical examples of generalized optimization differential equations, and my attention is attracted to the second order Dirac equation. When solving the real second order Dirac hydrogen equation, it was unexpectedly found that the result of determining the exact solution of the equation with boundary conditions did not completely meet the boundary conditions. This cannot be accepted by mathematical logic, especially the distortion of the description of the universe structure constitutes a serious physical logic paradox. Now we focus on the divergence of the S-state wave function of the original Dirac equation of the hydrogen atom traditional solution and the problem of the virtual energy in the Dirac energy level formula when the nuclear charge number $Z > 137$[31]. It is one of the mathematical difficulties implied in Dirac theory. From a mathematical point of view, the divergence of the wave function as the traditional solution of Dirac equation at the coordinate origin means that the traditional solution of the equation does not meet the definite solution conditions, which shows that the logic of Dirac traditional solution is not self consistent. From a physical point of view, the wave function of hydrogen atom represents the electron probability density, and the wave function as the traditional solution of Dirac equation diverges at the coordinate origin, which means that the atom collapses into neutrons or neutron-like, indicating that the Dirac traditional solution is contrary to the fact of the cosmic structure. The most concise is that the appearance of virtual energy in Dirac energy level formula is a direct veto solution. If the solution of a wave equation hides some mathematical or physical logical difficulties or even errors, it must mean some new and important events, including new mathematical problems to be found and solved.

As we all know, the most attractive part of quantum mechanics is to find the solution of the Schrödinger equation[32, 33] of the hydrogen atom satisfying the natural boundary conditions, and the Bohr energy level formula is naturally obtained[34]. This seems to be the most difficult mystery of mathematical physics. Although we can write wave equations that are far from known forms (which will be introduced one by one later) and obtain different forms of energy level formulas with the same accuracy, at present, we can not find the reason why various wave equations of quantum mechanics hide similar energy level formulas. According to the results of all aspects of analysis, just talking about the logic of mathematics, the traditional solution of Dirac equation implies irreconcilable logical contradictions, which is attributed to the fact that the definite solution conditions of the equation directly misappropriate the rough boundary conditions of Schrödinger equation without considering the size of an atomic nucleus. Therefore, considering the fact that the nucleus has a certain size, the exact boundary conditions of the hydrogen atom can be written. It is necessary to generalize the analytical theory of optimization differential equations in order to re solve the Schrödinger equation with exact boundary conditions. However, there seems to be no difficulty in re solving the Dirac equation with exact boundary conditions, and we get the bounded wave function in the whole space, and the corresponding energy level formula does not appear the virtual energy when the nuclear charge number$Z > 137$. However, the energy level formula determined by the exact boundary has the same accuracy as the Bohr energy formula, and there is no so-called fine structure.

Therefore, Dirac equation, as the core principle of relativistic quantum mechanics[35-41], faces many difficult choices. Or hunting for ingenious explanations to dilute the contradiction between mathematical and physical logic, and choosing rough boundary to solve the equation to obtain the expected traditional solution; Or follow the logic rules of mathematics and physics and choose the exact boundary to solve the equation and accept the regular solution that does not meet the expectation; Or find a new wave equation to establish a better theory. In fact, Dirac equation hides more logical difficulties than we know or even can imagine. The exact boundary condition solution may not be the final answer, but it can be accepted by some physicists. The mathematical inference of exact boundary conditions is at least better than the cover up of logical difficulties by qualitative explanations. All solutions of the Dirac equation of the hydrogen atom under precise boundary conditions, regardless of the relationship with experimental observations, pose a challenge to the relativistic quantum mechanics, which is called the challenging solution of the Dirac equation for the hydrogen atom.

## 2 Rough boundary condition and divergence of Dirac function

In 1928, Dirac constructed the wave equation of quantum system in line with the meaning of special relativity, called Dirac equation, which opened the era of relativistic quantum mechanics. This section only introduces the main conclusions of the Dirac theory of hydrogen like atoms, including the Dirac energy level formula and Dirac wave function, to illustrate the contradiction problems such as divergence and virtual energy implied in the traditional solution of the Dirac equation determined by rough boundary. For the detailed process of solving the Dirac equation of hydrogen like atom to obtain the Dirac energy level formula and Dirac wave function, you can read the relevant chapters of various quantum mechanics textbooks[42-44]. The dirac equation is used to describe hydrogen like atom. In order to obtain the expected energy level formula, the problem is finally transformed into solving the Dirac radial equation,

$$\left[c\hat{p}_r\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}-\frac{\hbar c\kappa}{r}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+mc^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]R=\left(E+\frac{Ze^2}{4\pi\varepsilon_0 r}\right)R$$

Boundary conditions play a decisive role in solving wave equations[45-52]. In quantum mechanics, the Schrödinger equation is used to describe the hydrogen atomic system, and the size of an atomic nucleus is usually not considered in the determination of boundary conditions. The boundary conditions of the atom like the Schrödinger equation with nuclear charge number Z are written in the following rough form

$$\begin{aligned}&\lim_{r\to 0}\psi=0\\ &\psi\left(0<r<\infty\right)\neq\pm\infty\\ &\lim_{r\to\infty}\psi=0\end{aligned}\tag{1}$$

where $R$ is the radial wave function. Solving the Schrödinger equation by using this rough boundary condition, the Bohr formula of the energy levels is naturally obtained. It is a landmark work to open the era of quantum mechanics and is considered to be perfect in mathematics. Relativistic quantum mechanics still uses the above rough boundary conditions to solve the Dirac equation, and obtains the Dirac energy level formula of the energy level in the Coulomb field, which is consistent with the expectation[42-44]

$$E=\frac{mc^2}{\sqrt{1+\frac{Z^2\alpha^2}{\left(n_r+\sqrt{\kappa^2-Z^2\alpha^2}\right)^2}}}\tag{2}$$

where $n_r=0,1,2,\cdots$, $\alpha$ is the fine structure constant, while $\kappa=\pm1,\ \pm2,\ \pm3,\ \cdots$, is an artificial constant constructed by the so-called angular momentum coupling theory that bypasses the basic mathematical operation rules. In fact, it does not accord with the physical meaning. Taking a few special cases as implicit axioms and defining a set of calculation methods to achieve the expected purpose is a common writing technique in morden physics. Its interference with mathematics will inevitably bring irreconcilable logical contradictions. Saying alone the Dirac ground state or Dirac $S$ state of the hydrogen atom, because $n=0$, $\kappa=\pm1$, when z $> 137$, the energy of the system become an imaginary number. This is a purely mathematical problem. Of course, on this issue, morden physics has a set of logic of causal inversion, which connects the maximum number of nuclear charges with the speed of light. The specific reason is that no particle with mass, including electrons, can reach or exceed the speed of light, and it gives the reader a formal logic of non inevitable causality: the atomic number of the heaviest element in the universe must therefore be less than 137. Therefore, the aforementioned logical contradiction was completely covered up by this illusion. Later, few people found implicit mathematical difficulties of Dirac equation from the perspective of mathematical and physical logic, and could not fundamentally solve these difficulties. Part of physics has been being in such a logical cycle of causal inversion.

Eloquence, which deviates far from the essence of mathematics and physics, is often widely accepted as the symbol of the wisdom of mankind, and has a profound impact on the thinking of later physics scholars. However, an explanation of cause and effect inversion can only cover up a single logical contradiction, and the theory presenting a certain logical contradiction must imply many logical contradictions. Now let's discuss the implied divergence of Dirac wave function of hydrogen-like atom. The two-component Dirac wave function of a hydrogen-like atom is

$$R(r)=\begin{pmatrix} e^{-ar}\sum\limits_{\nu=0}^{n_r} b_\nu (ar)^{\sqrt{\kappa^2-Z^2\alpha^2}+\nu-1} \\ e^{-ar}\sum\limits_{\nu=0}^{n_r} d_\nu (ar)^{\sqrt{\kappa^2-Z^2\alpha^2}+\nu-1} \end{pmatrix} \tag{3}$$

where $a=\sqrt{m^2c^4-E^2}\big/\hbar c$ and the coefficients of the polynomial satisfy the system of the following recurrence relations

$$\begin{aligned} &\sqrt{\frac{mc^2+E}{mc^2-E}}b_{\nu-1}+d_{\nu-1}-Z\alpha b_\nu-\left(\kappa+\sqrt{\kappa^2-Z^2\alpha^2}+\nu\right)d_\nu=0 \\ &b_{\nu-1}+\sqrt{\frac{mc^2-E}{mc^2+E}}d_{\nu-1}+\left(\kappa-\sqrt{\kappa^2-Z^2\alpha^2}-\nu\right)b_\nu+Z\alpha d_\nu=0 \end{aligned} \tag{4}$$

This is a mixed recurrence system of equations that cannot be transformed into a single series (polynomial) coefficient recurrence relationship. However, the Dirac wave function for the $S$ state of the hydrogen-like atom is divergent as $\kappa=\pm1$ ($S$ state) . It can be seen that whatever the radial quantum number $n_r$ takes any value, the first term of the two-component Dirac wave function (3) is the same. We have,

$$\lim_{r\to0}|R|=\lim_{r\to0}\left|\begin{pmatrix} e^{-ar}\left[b_0(ar)^{\sqrt{1-Z^2\alpha^2}-1}+\sum\limits_{\nu=1}^{n_r} b_\nu (ar)^{\sqrt{1-Z^2\alpha^2}+\nu-1}\right] \\ e^{-ar}\left[d_0(ar)^{\sqrt{1-Z^2\alpha^2}-1}+\sum\limits_{\nu=1}^{n_r} d_\nu (ar)^{\sqrt{1-Z^2\alpha^2}+\nu-1}\right] \end{pmatrix}\right|=\begin{pmatrix}\infty\\ \infty\end{pmatrix} \tag{5}$$

It can be seen that the traditional solution of the Dirac equation does not meet the definite solution conditions of the equation. This is a typical logical contradiction in a pure mathematical sense, which will not disappear because of the defining relationship between the nuclear charge limit and the speed of light limit.

For the time being, we don't discuss those problems that the abnormal solutions of some equations in modern physics often turn into the problem of how strange inferences are generally accepted. When looking from a physical point of view, the divergence of the Dirac wave function for $S$ state implies that the probability density of the electron around the nucleus rapidly increases as it close to the atomic nucleus. The probability density corresponding to the wave function of two components is defined as

$$\rho(r,t)=R^*(r,t)\,R(r,t) \tag{6}$$

According to (3), we have

$$R^*(r)=e^{-ar}\begin{pmatrix} \sum\limits_{\nu=0}^{n_r} b_\nu (ar)^{\nu+\sqrt{\kappa^2-Z^2\alpha^2}-1} & \sum\limits_{\nu=0}^{n_r} d_\nu (ar)^{\nu+\sqrt{\kappa^2-Z^2\alpha^2}-1} \end{pmatrix} \tag{7}$$

In this case the radial probability density for the electron of the so called relativistic hydrogen atom is as follows

$$\rho=\left[e^{-ar}\sum_{\nu=0}^{n_r} b_\nu (ar)^{\nu+\sqrt{\kappa^2-Z^2\alpha^2}-1}\right]^2+\left[e^{-ar}\sum_{\nu=0}^{n_r} d_\nu (ar)^{\nu+\sqrt{\kappa^2-Z^2\alpha^2}-1}\right]^2 \tag{8}$$

For S-state which implies $\kappa=\pm1$, as $r\to0$, the above formula becomes

$$\lim_{r\to0}\rho=\lim_{r\to0}\left\{\left[e^{-ar}\left(\frac{b_0}{(ar)^{1-\sqrt{1-Z^2\alpha^2}}}+\cdots\right)\right]^2+\left[e^{-ar}\left(\frac{d_0}{(ar)^{1-\sqrt{1-Z^2\alpha^2}}}+\cdots\right)\right]^2\right\}=\infty \tag{9}$$

What this result predicts should be that the hydrogen and hydrogen-like atom in the ground state must rapidly collapse to the neutron-like. However the fact is not thusness. That is to say, the original solution of the Dirac equation for the hydrogen and hydrogen-like atom neither agrees with the mathematical principle nor agrees with the physical signification. Unexpectedly, such divergence was defined as so-called "mild divergence"[53-55]so that hardly might one open out its actual meaning, and the correct deduction has been buried. We know that the Klein-Gordon for the meson without spin has the same divergence, but the Klein-Gordon divergence in the Coulomb field can be eliminated by the suitable mathematical method. This part will be introduced in detail after expounding the relevant new mathematical theorems.

Using a cut-off procedure for the wave function that is similar to the case of considering an extended nucleus to blench the divergence should be independent of the exact solution for the Dirac equation for the hydrogen-like atom, otherwise it may oppresses the correct wave equation and it real solution. For the traditional solution of the Dirac equation for the hydrogen-like atom, why coming forth the mathematical difficulty such as the expression (5) and (9) and the virtual energies is that the size of the nucleus of the hydrogen-like atom is not considered in the rough boundary condition (1), and the nuclear are regarded as the point in geometrical meaning. The point in geometrical meaning falls short of the actual case of the atomic nucleus. In fact, the necessary of the normalizable wave function of the hydrogen atom has been discussed home and widely in some modern physics textbook[42-44]. Now one should consider the actual size of the atomic nucleus to rewrite the boundary condition then find the eigensolution of the Dirac equation for the hydrogen and hydrogen-like atom.

## 3 Exact boundary condition and challenge solution of Dirac equation

If the wave equation theory of quantum mechanics itself will not be challenged, then boundary conditions for various wave equations of quantum mechanics can be written out based on the structure of the physical model and distributing character of the physical quantity. There are two basic facts, one is that the atomic nucleus has definite size, we suppose its equivalent radius or barrier width is $\delta$, and another is that the electron does not enter the inside of the atomic nucleus, and does not collide to and rub with the atomic nucleus. In this way, any wave equation that describes the atom has the same exact boundary condition

$$\begin{aligned} &R\left(r=\delta\right)\neq\pm\infty, \\ &R\left(r\to\infty\right)=0, \\ &-\infty<R\left(\delta<r<\infty\right)<\infty \end{aligned} \tag{10}$$

One can use this exact condition to solve the Schrödinger equation for the hydrogen atom so as to recover the Bohr formula of the energy levels. It is well known that the Schrödinger equation of the hydrogen-like atom has no virtual energy solution, and there is no mathematical contradiction that the solution of the equation does not meet the conditions of a definite solution. Then, in order to solve the problem that the virtual energy of Dirac's solution under rough boundary conditions and the divergence of the solution of the equation do not meet the boundary condition, what is the result of finding the solution of Dirac's equation under accurate boundary conditions? Although formally one would obtain the satisfying formula that is as exact as the distinguished Dirac formula when considering the spin-orbit coupling in the Dirac equation, and further solutions can also be made from the point of view that the boundary conditions are related to the self adjoint of the operator, so as to give a solution to avoid the divergence of Dirac function. We hope to jump out of all kinds of logical thinking of non inevitable causality and reprocess Dirac equation with accurate boundary conditions from the perspective of pure mathematics[56]. This is because there are similar problems that need to develop relevant mathematical theories.

It must be reiterated, as described in the text after equation (2), the Dirac theory of hydrogen atom uses an unconventional operation method to determine the angular quantum number $\kappa=\pm1,\pm2,\cdots$, so as to obtain the radial Dirac equation of the hydrogen atom or hydrogen-like atom

$$\left[c\hat{p}_r\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}-\frac{\hbar c\kappa}{r}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+mc^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]R=\left(E+\frac{e^2}{4\pi\varepsilon_0 r}\right)R \tag{11}$$

Where the radial momentum operator is controversially defined as $\hat{p}_r=-i\hbar\left(\partial/\partial r+1/r\right)$, whether it is reasonable or not has been discussed in detail and clearly concluded[19]. This means that it is necessary to understand the solution of the regular radial Dirac equation derived from the radial momentum operator that conforms to the laws of mathematical operation rather than man-made definition. This part is left to the reader to solve. Following the radial momentum operator defined by the Dirac theory, this paper focuses on the solution of the recognized radial Dirac equation under accurate boundary conditions. It usually introduces a mathematical transformation

$$R=\begin{pmatrix}\dfrac{1}{r}F\left(r\right)\\ \dfrac{1}{r}G\left(r\right)\end{pmatrix} \tag{12}$$

and translate the radial Dirac equation for the hydrogen atom[42-44, 57] into the following form

$$\begin{aligned} &\left(\frac{E-mc^2}{\hbar c}+\frac{\alpha}{r}\right)F+\left(\frac{\kappa}{r}+\frac{d}{dr}\right)G=0 \\ &\left(\frac{E+mc^2}{\hbar c}+\frac{\alpha}{r}\right)G+\left(\frac{\kappa}{r}-\frac{d}{dr}\right)F=0 \end{aligned} \tag{13}$$

Considering the exact boundary condition (10), introduce the transform

$$\xi=r-\delta \quad (\xi\geqslant 0) \tag{14}$$

the boundary condition (10) can be overwritten as follows

$$\begin{aligned} &R(\xi\to 0)\neq\pm\infty \\ &R(\xi\to\infty)=0 \\ &-\infty<R(0<\xi<\infty)<\infty \end{aligned} \tag{15}$$

then $r=\xi+\delta$, substituting it into (13), one obtains

$$\begin{aligned} &\left(\frac{E-mc^2}{\hbar c}+\frac{\alpha}{\xi+\delta}\right)F+\left(\frac{\kappa}{\xi+\delta}+\frac{d}{d\xi}\right)G=0 \\ &\left(\frac{E+mc^2}{\hbar c}+\frac{\alpha}{\xi+\delta}\right)G+\left(\frac{\kappa}{\xi+\delta}-\frac{d}{d\xi}\right)F=0 \end{aligned} \tag{16}$$

For this kind of variable coefficient differential equation, the undetermined form of the exact solution of the equation is usually written by using the asymptotic solution of the differential equation, and then the equation is solved. The asymptotic solution comes from the solution of the asymptotic differential equation with $\xi\to\infty$.

When $\xi\to\infty$, the asymptotic form of equation (16) is

$$\begin{aligned} &\frac{dG}{d\xi}-\frac{mc^2-E}{\hbar c}F\approx 0 \\ &\frac{dF}{d\xi}-\frac{mc^2+E}{\hbar c}G\approx 0 \end{aligned}$$

From this, two second-order asymptotic second-order differential equations are obtained

$$\begin{aligned} &\frac{dF^2}{d\xi^2}-\frac{m^2c^4-E^2}{\hbar^2c^2}F\approx 0 \\ &\frac{dG^2}{d\xi^2}-\frac{m^2c^4-E^2}{\hbar^2c^2}G\approx 0 \end{aligned}$$

Their asymptotic solutions are the same,

$$F\approx e^{-a\xi},\quad G\approx e^{-a\xi}$$

Where $a=\sqrt{m^2c^4-E^2}/\hbar c$. the exact solution form of equation (16) takes its asymptotic solution as the weight function, and let the specific form of the exact solution be

$$F=e^{-a\xi}f(\xi),\quad G=e^{-a\xi}g(\xi) \tag{17}$$

By transforming the first-order differential equations into second-order differential equations, it can be proved that the asymptotic solution method of differential equations with variable coefficients is reasonable. Substituting (17) into equation (16), one then obtains

$$\begin{aligned} &\left[\frac{E-mc^2}{\hbar c}(\xi+\delta)+\alpha\right]f+(\xi+\delta)\frac{dg}{d\xi}+[\kappa-a(\xi+\delta)]g=0 \\ &\left[\frac{E+mc^2}{\hbar c}(\xi+\delta)+\alpha\right]g-(\xi+\delta)\frac{df}{d\xi}+[\kappa+(a\xi+\delta)]f=0 \end{aligned} \tag{18}$$

the eigensolutions of equations (18) correspond to quantum energy are two interrupted series, which the number of terms is determined by the eigenvalues.

In order to find the general series solutions for equations (18), it is assumed that the formal solutions are

$$f(\xi)=\sum_{\nu=0}^{\infty}b_\nu\xi^{\sigma+\nu},\quad g(\xi)=\sum_{\nu=0}^{\infty}d_\nu\xi^{\sigma+\nu} \tag{19}$$

Substituting into equations (18), one obtains the linear system of recurrence relations

$$\begin{aligned} &\sum_{\nu=0}^{\infty}\left[\frac{E-mc^2}{\hbar c}b_{\nu-1}+\frac{E-mc^2}{\hbar c}\delta b_\nu+\alpha b_\nu+Kd_\nu+(\sigma+\nu)d_\nu+\delta(\sigma+\nu+1)d_{\nu+1}-ad_{\nu-1}-\delta ad_\nu\right]\xi^{\sigma+\nu}=0 \\ &\sum_{\nu=0}^{\infty}\left[\frac{E+mc^2}{\hbar c}d_{\nu-1}+\frac{E+mc^2}{\hbar c}\delta d_\nu+\alpha d_\nu+Kb_\nu-(\sigma+\nu)b_\nu-\delta(\sigma+\nu+1)b_{\nu+1}+ab_{\nu-1}+\delta ab_\nu\right]\xi^{\sigma+\nu}=0 \end{aligned} \tag{20}$$

hence the coefficient of the power series satisfy the following system of recurrence relations

$$\begin{aligned} &\frac{E-mc^2}{\hbar c}b_{\nu-1}+\left(\frac{E-mc^2}{\hbar c}\delta+\alpha\right)b_\nu-ad_{\nu-1}+\delta(\sigma+\nu+1)d_{\nu+1}+(\kappa+\sigma+\nu-\delta a)d_\nu=0 \\ &\frac{E+mc^2}{\hbar c}d_{\nu-1}+\left(\frac{E+mc^2}{\hbar c}\delta+\alpha\right)d_\nu+ab_{\nu-1}-\delta(\sigma+\nu+1)b_{\nu+1}+(\kappa-\sigma-\nu+\delta a)b_\nu=0 \end{aligned} \tag{21}$$

Corresponding to $\nu=-1$ the indicial equations are given that $\delta\sigma b_0=0$ and $\delta\sigma d_0=0$. Because $\delta\neq 0$, $b_0\neq 0$ and $d_0\neq 0$, one obtains

$$\sigma=0 \tag{22}$$

so that the wave functions satisfy the boundary condition at $r\to\delta$ namely $\xi\to 0$, the above equations reduce to

$$\begin{aligned}
&\frac{E-mc^2}{\hbar c}b_{\nu-1}+\left(\frac{E-mc^2}{\hbar c}\delta+\alpha\right)b_\nu-ad_{\nu-1}+\delta\left(\nu+1\right)d_{\nu+1}+\left(\kappa+\nu-\delta a\right)d_\nu=0\\
&\frac{E+mc^2}{\hbar c}d_{\nu-1}+\left(\frac{E+mc^2}{\hbar c}\delta+\alpha\right)d_\nu+ab_{\nu-1}-\delta\left(\nu+1\right)b_{\nu+1}+\left(\kappa-\nu+\delta a\right)b_\nu=0
\end{aligned} \tag{23}$$

Respectively evaluate for $\nu=0,1,2,\cdots,n_r$, $b_{n_r+1}=d_{n_r+1}=0$, make use of that $b_{-2}=d_{-2}=0$ and $b_{-1}=d_{-1}=0$, equations (23) give

$$\begin{aligned}
&\left(\frac{E-mc^2}{\hbar c}\delta+\alpha\right)b_0+\left(\kappa-\delta a\right)d_0+\delta d_1=0\\
&\left(\kappa+\delta a\right)b_0+\left(\frac{E+mc^2}{\hbar c}\delta+\alpha\right)d_0-\delta b_1=0\\
&\frac{E-mc^2}{\hbar c}b_0+\left(\frac{E-mc^2}{\hbar c}\delta+\alpha\right)b_1-ad_0+\left(\kappa+1-\delta a\right)d_1+2\delta d_2=0\\
&ab_0+\left(\kappa-1+\delta a\right)b_1+\frac{E+mc^2}{\hbar c}d_0+\left(\frac{E+mc^2}{\hbar c}\delta+\alpha\right)d_1-2\delta b_2=0\\
&\vdots\\
&\frac{E-mc^2}{\hbar c}b_{n_r-1}+\left(\frac{E-mc^2}{\hbar c}\delta+\alpha\right)b_{n_r}-ad_{n_r-1}+\left(\kappa+n_r-\delta a\right)d_{n_r}=0\\
&ab_{n_r-1}+\left(\kappa-n_r+\delta a\right)b_{n_r}+\frac{E+mc^2}{\hbar c}d_{n_r-1}+\left(\frac{E+mc^2}{\hbar c}\delta+\alpha\right)d_{n_r}=0\\
&\frac{E-mc^2}{\hbar c}b_{n_r}-ad_{n_r}=0\\
&ab_{n_r}+\frac{E+mc^2}{\hbar c}d_{n_r}=0
\end{aligned} \tag{24}$$

The last two formulas are linearly dependent. Note that $a=\sqrt{m^2c^4-E^2}\big/\hbar c$ and use $\left(E+mc^2\right)\big/\hbar c$ to multiply the third formula from bottom and use $\sqrt{m^2c^4-E^2}\big/\hbar c$ to multiply the fourth formula from bottom, and then add the two new formulas, it is given as follows

$$\left[\alpha\left(E+mc^2\right)+\left(\kappa-n_r\right)\sqrt{m^2c^4-E^2}\right]b_{n_r}+\left[\left(\kappa+n_r\right)\left(E+mc^2\right)+\alpha\sqrt{m^2c^4-E^2}\right]d_{n_r}=0 \tag{25}$$

Substituting for the second formula from the bottom, one will obtain a new formula of the energy levels for the hydrogen atom

$$E=\frac{mc^2}{\sqrt{1+\frac{\alpha^2}{n_r^2}}},\quad \left(n_r=0,\ 1,\ 2,\ 3,\ \cdots\right) \tag{26}$$

It is different from the Dirac formula of the energy levels for the hydrogen atom. This result is the inevitable deduction of the Dirac equation with the exact boundary condition for the hydrogen atom. With the exact boundary condition (10) and the new formula of the energy levels (26), all of the corresponding wave functions satisfy the boundary conditions and there is not any virtual energy.

According to (11), (14), (16), (19), (23), the whole wave function with the exact boundary condition is as follows

$$R=\begin{pmatrix}\frac{e^{-a\xi}}{\xi+\delta}\sum\limits_{\nu=0}^{n_r}b_\nu\xi^\nu\\ \frac{e^{-a\xi}}{\xi+\delta}\sum\limits_{\nu=0}^{n_r}d_\nu\xi^\nu\end{pmatrix}=\begin{pmatrix}e^{-\frac{\sqrt{m^2c^4-E^2}}{\hbar c}(r-\delta)}\frac{1}{r}\sum\limits_{\nu=0}^{n_r}b_\nu(r-\delta)^\nu\\ e^{-\frac{\sqrt{m^2c^4-E^2}}{\hbar c}(r-\delta)}\frac{1}{r}\sum\limits_{\nu=0}^{n_r}d_\nu(r-\delta)^\nu\end{pmatrix},\quad \left(n_r=0,\ 1,\ 2,\ \cdots\right) \tag{27}$$

the coefficients of the corresponding polynomial are determined by the system of recurrence relations (23). All appearance, at the boundary of the hydrogen atom

$$\begin{aligned}\lim_{r\to\delta} R = \lim_{\xi\to 0} R = \begin{pmatrix}\text{Constant}\\ \text{Constant}\end{pmatrix}\\ \lim_{r\to\infty} R = \lim_{\xi\to\infty} R = \begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned} \tag{28}$$

Make use of the definition (6), the probability density of the electron appearing outside the nucleus of the hydrogen atom takes the form

$$\rho = \left(\frac{e^{-a\xi}}{\xi+\delta}\sum_{\nu=0}^{n_r} b_\nu \xi^\nu\right)^2 + \left(\frac{e^{-a\xi}}{\xi+\delta}\sum_{\nu=0}^{n_r} d_\nu \xi^\nu\right)^2$$

Namely,

$$\rho = \left(e^{-\frac{\sqrt{m^2c^4-E^2}}{\hbar c}(r-\delta)}\frac{1}{r}\sum_{\nu=0}^{n_r} b_\nu (r-\delta)^\nu\right)^2 + \left(e^{-\frac{\sqrt{m^2c^4-E^2}}{\hbar c}(r-\delta)}\frac{1}{r}\sum_{\nu=0}^{n_r} d_\nu (r-\delta)^\nu\right)^2 \tag{29}$$

homoplastically, one has

$$\begin{aligned}\lim_{r\to\delta} \rho = \lim_{\xi\to 0} \rho = \text{Constant}\\ \lim_{r\to\infty} \rho = \lim_{\xi\to\infty} \rho = 0\end{aligned} \tag{30}$$

Obviously, the exact solution of the Dirac equation of the hydrogen atom under exact boundary conditions does not show obvious mathematical contradiction. However, the energy level formula thus obtained is no different from Bohr energy level formula. This is obviously not the result expected by quantum mechanics. The exact solution of the Dirac equation of the hydrogen atom under rough boundary conditions is only formal. Although it pieced together the exact energy formula of the hydrogen atom that meets the expectation, there are enough irreconcilable mathematical and physical logical contradictions. In order to obtain the desired formula, it is not a choice that scientists should make at the expense of mathematical rules and objective logic. Therefore, the exact solution of the Dirac equation of the hydrogen atom under the exact boundary poses a severe challenge to relativistic quantum mechanics, which is called the challenging solution of the Dirac equation. Considering the situation how much the problems hidden in the Dirac equation that will be introduced step by step later are beyond our guess, it can be inferred that the Dirac equation may not be the best choice to describe the fine structure of the hydrogen atom. If readers think that this conclusion damages the reputation of celebrities and produces resistance or even disgust, it will not be conducive to the progress of scientific theory. You can find a better wave equation to describe the fine structure of the hydrogen atom and obtain the ideal energy level formula. Although in the author's opinion, it may still not be the best solution, it is always better than the theory full of contradictions.

## 4 Conclusions and comments

Famous theories about natural phenomena as well as incredible fairy tales that are wildly under the aura of science can always more effectively stimulate the enthusiasm of those who pursue truth to test those logic and inferences. In this paper we expatiated that the divergences of the Dirac function and the virtual energy of the Dirac formula of the energy levels for the hydrogen and hydrogen-like atom are due to the traditional rough boundary condition. By using the exact boundary condition one can obtain a new solution of the Dirac equation in the Coulomb field. The new solution without any mathematical difficulty gives a new formula of the energy levels which is different from the distinguished Dirac formula. One can find that in the new solution the values constructed by Dirac have been written in the radius of the atomic nucleus and they are independent of the new formula for the energy levels. Only when looking from the point of view of the formula of the energy levels, the new formula is not as exact as the Dirac formula. However, considering the spin-orbit coupling or some new potential parameters, one will obtain the more exact formula.

It generally believes that the Dirac equation has been successful in many important predictions[58]. The Dirac equation has an infinite number of negative energy solutions, which are considered to lead to the discovery of the existence of positrons and interpreted as an unexpected relativistic interaction between the translational motion of electrons and spin, resulting in violent oscillations of particles at very high frequencies and a distance of about one Compton wavelength. This leads to the concept of Dirac electron. The further description is that as long as the electron wave function contains positive energy and negative energy components, and zeitbergen will occur. Both sets of states need to establish an arbitrary electronic state. As we all know, the wonderful part of quantum mechanics is to naturally obtain the energy level formula of bound states by solving the wave equation with boundary conditions. All other follow-up products are just academic whitewash. Entanglement of quantum states leads to the birth of quantum computer? So what is the real meaning of wave function? Let's look forward to the wonderful ending of the story. Since the exact solution of the Dirac equation that meets the expectation can only be obtained under the rough boundary conditions that do not meet the atomic structure, the Dirac exact energy level formula should be treated again. This paper reveals the hidden mathematical and physical difficulties in the Dirac equation of Coulomb potential, and gives the exact solution of the Dirac equation of the hydrogen atom under the exact boundary conditions. The purpose is to accurately present the inevitable mathematical logic of different wave equations. At least the divergence can be eliminated by using the exact boundary conditions and abandoning the rough boundary conditions, although this may not be the ultimate answer. The solution of the Dirac equation, which ostensibly eliminates the mathematical contradiction, can not give the formula of fine structure energy level of the hydrogen atom. Failure to meet the expectation is the driving force to find the ultimate answer.

No matter which wave equation is used to describe the quantum system of the bound statge, their eigenvalues set and eigensolutions set must be in agreement with the uniqueness, and the solution must accord with the conditions for the exact solution but not any approximate solution such as cut off potential. In principle, we cannot immolate the mathematical rule to obtain a formula only for agreeing with the experimentally observed hydrogen spectra. Actually the divergence of the original solution for the Dirac equation in the Coulomb field is nonexistent. The Dirac equation is more and more widely applied for various models. One should note that using different mathematical methods and different boundary condition to solve the differential equation will obtain different results[59, 60]. The rough boundary condition for the Dirac equation with the Coulomb potential brings on so mathematical contradictions. We need to revise all incorrect mathematical methods and search the correct mathematical methods to find the correct and exact solution of the various wave equations. One can find that the classical solution of the plane transverse electromagnetic mode of the Maxwell equation is also incorrect[61, 62], because of the incorrect mathematical method for solving the wave equation. For quantum mechanics, only when the solution method of the wave equation conforms to the law of mathematical operation and the exact solution does not violate the natural law, the corresponding energy level formula is meaningful. Further research should explore the exacter formula of the energy levels for the hydrogen and hydrogen-like atom by using the exact boundary condition that accords with the structure of the atoms. Here it should be pointed out that the Schrödinger equation and the Klein-Gordon equation in the Coulomb field with the exact boundary condition have the same corresponding formula of the energy levels.

Usually we can try to maintain famous theories. However, when irreconcilable mathematical irreconcilable contradictions and physical and logical contradictions implied in the famous theory are found, and better treatment methods are presented, we should make some changes. The author's research does not mean a complete denial of the achievements of his predecessors. Without the brilliant footprints of predecessors, the later researchers can not make any breakthrough in mathematics and physics. Correcting the mistakes of predecessors is an important link in the development of science. From the perspective of mathematics, correctly dealing with logical contradictions implied in the Dirac equation will bring not only the development of mathematics, but also the profound reform of quantum mechanics, and even the great reform of the whole theoretical physics[14-17, 19, 20]. Isn't it a blessing that scientific theory is facing a great opportunity for profound change?


1 Gordon, W. Der comptoneffekt nach der Schrödingerschen theorie. *Zeitschrift für Physik* **40**, 117-133 (1926).

2 Klein, O. Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift für Physik* **37**, 895-906 (1926).

3 Dirac, P. A. M. The quantum theory of the electron. Part II. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **118**, 351-361 (1928).

4 Darwin, C. G. The wave equations of the electron. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **118**, 654-680 (1928).

5 Gordon, W. Die energieniveaus des wasserstoffatoms nach der diracschen quantentheorie des elektrons. *Zeitschrift für Physik* **48**, 11-14 (1928).

6 Biedenharn, L. C. Remarks on the relativistic Kepler problem. *Physical Review* **126**, 845 (1962).

7 Wong, M. & Yeh, H.-Y. Simplified solution of the Dirac equation with a Coulomb potential. *Physical Review D* **25**, 3396 (1982).

8 Wong, M. & Yeh, H.-Y. Exact solution of the Dirac-Coulomb equation and its application to bound-state problems. II. Interaction with radiation. *Physical Review A* **27**, 2305 (1983).

9 Su, J.-Y. Simplified solutions of the Dirac-Coulomb equation. *Physical Review A* **32**, 3251 (1985).

10 Nenciu, G. Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms. *Communications in Mathematical Physics* **48**, 235-247 (1976).

11 Klaus, M. & Wüs, R. Characterization and uniqueness of dis-

tinguished self-adjoint extensions of Dirac operators. *Communications in Mathematical Physics* **64**, 171-176 (1979).
12 B. Thaller, *The Dirac Equation*, Springer, New York, 1992. Preface.
13 Dongfang, X. D. The End of Klein-Gordon Equation for Coulomb Field. Mathematics & Nature 2, 202201 (2022).
14 Dongfang, X. D. The End of Teratogenic Simplified Dirac Hydrogen Equations. Mathematics & Nature 2, 202202 (2022).
15 Dongfang, X. D. Dongfang Solution of Induced Second Order Dirac Equations. Mathematics & Nature 2, 202203 (2022).
16 Dongfang, X. D. The End of Isomeric Second Order Dirac Hydrogen Equations. Mathematics & Nature 2, 202204 (2022).
17 Dongfang, X. D. The End of Expectation for True Second Order Dirac Equation. Mathematics & Nature 2, 202205 (2022).
18 Dongfang, X. D. The Morbid Equation of Quantum Numbers. Mathematics & Nature 1, 202102 (2021).
19 Dongfang, X. D. Dongfang Angular Motion Law and Operator Equations. Mathematics & Nature 1, 202105 (2021).
20 Dongfang, X. D. Relativistic Equation Failure for LIGO Signals. Mathematics & Nature 1, 202103 (2021).
21 Dongfang, X. D. Dongfang Com Quantum Equations for LIGO Signal. Mathematics & Nature 1, 202106 (2021).
22 Dongfang, X. D. Com Quantum Proof of LIGO Binary Mergers Failure. Mathematics & Nature 1, 202107 (2021).
23 Yukawa, H. On the Interaction of Elementary Particles I. *Proc. Phys. Math. Soc. Jpn.* **17**, 48 (1935).
24 Dongfang, X. D. Nuclear Force Constants Mapped by Yukawa Potential. Mathematics & Nature 1, 202109 (2021).
25 Dongfang, X. D. The End of Yukawa Meson Theory of Nuclear Forces. Mathematics & Nature 1, 202110(2021).
26 Dongfang, X. D. Dongfang Modified Equations of Molecular Dynamics. Mathematics & Nature 1, 202104 (2021).
27 Dongfang, X. D. Dongfang Modified Equations of Electromagnetic Wave. Mathematics & Nature 1, 202108 (2021).
28 Dongfang, X. D. On the Relativity of the Speed of Light. Mathematics & Nature, **1**, 202101 (2021).
29 Chen, R. The Optimum Differential Equations. Chinese Journal of Engineering Mathematics **17**, 82-86 (2000).
30 Chen, R. The Uniqueness of the Eigenvalue Assemblage for Optimum Differential Equations, Chinese Journal of Engineering Mathematics **20**, 121-124(2003).
31 Chen, R. New exact solution of Dirac-Coulomb equation with exact boundary condition. International Journal of Theoretical Physics **47**, 881-890 (2008).
32 Schrödinger, E. An undulatory theory of the mechanics of atoms and molecules. *Physical review* **28**, 1049 (1926)..
33 Schrödinger, E. Quantisierung als eigenwertproblem. *Annalen der physik* **385**, 437-490 (1926).
34 Bohr, N. On the constitution of atoms and molecules. Par III. Systems containing several nuclei. *Phil. Mag.* **6**, p. 875 (1913).
35 Villalba, V. M. Bound states of the hydrogen atom in the presence of a magnetic monopole field and an Aharonov-Bohm potential. *Physics Letters A* **193**, 218-222 (1994).
36 Dominguez-Adame, F. & Rodriguez, A. A one-dimensional relativistic screened Coulomb potential. *Physics Letters A* **198**, 275-278 (1995).
37 Horbatsch, M. & Shapoval, D. Variational solution of the two-center Dirac-Coulomb problem in the free-particle representation. *Physics Letters A* **209**, 63-68 (1995).
38 Gu, Y. Repulsive bound states produced by momentum cutoff in Coulomb systems. *Physics Letters A* **295**, 81-86 (2002).
39 Dong, S.-H. & Lozada-Cassou, M. Scattering of the Dirac particle by a Coulomb plus scalar potential in two dimensions. *Physics Letters A* **330**, 168-172 (2004).
40 Marques, G. d. A., Bezerra, V. B. & Fernandes, S. G. Exact solution of the Dirac equation for a Coulomb and scalar potentials in the gravitational field of a cosmic string. *Physics Letters A* **341**, 39-47 (2005).
41 Ritchie, B. Relativistic bound states. *Physics Letters A* **352**, 239-241 (2006).
42 Dirac, P. A. M. *The principles of Quantum Mechanics*, Clarendon Press, Oxford, 1958, pp268-272
43 Greiner, W. *Relativistic Quantum Mechanics: Wave Equations*, Springer, 2000, 3rd Edition, pp225-260.
44 Schiff, L. I. Quantum Mechanics. *M cG raw-H ill, New York* (1968).
45 Ciftci, H., Hall, R. L. & Saad, N. Iterative solutions to the Dirac equation. *Physical Review A* **72**, 022101 (2005).
46 Nakatsuji, H. & Nakashima, H. Analytically solving the relativistic Dirac-Coulomb equation for atoms and molecules. *Physical review letters* **95**, 050407 (2005).
47 Gavrilov, S. P. & Gitman, D. M. Vacuum instability in external fields. *Physical Review D* **53**, 7162 (1996).
48 Vakarchuk, I. On Dirac theory in the space with deformed Heisenberg algebra: exact solutions. *Journal of Physics A: Mathematical and General* **38**, 7567 (2005).
49 Villalba, V. M. in *Journal of Physics: Conference Series.* 016 (IOP Publishing).
50 Torrente-Lujan, E. Baryon asymmetry at the weak phase transition in the presence of arbitrary CP violation. *Physics of Atomic Nuclei* **69**, 147-158 (2006).
51 Berkdemir, C., Berkdemir, A. & Sever, R. Systematical approach to the exact solution of the Dirac equation for a deformed form of the Woods–Saxon potential. *Journal of Physics A: Mathematical and General* **39**, 13455 (2006).
52 Morsink, S. M. & Mann, R. B. Black hole radiation of Dirac particles in 1+ 1 dimensions. *Classical and Quantum Gravity* **8**, 2257 (1991).
53 Taub, A. A special method for solving the dirac equations. *Reviews of Modern Physics* **21**, 388 (1949).
54 Yennie, D., Ravenhall, D. G. & Wilson, R. Phase-shift calculation of high-energy electron scattering. *Physical Review* **95**, 500 (1954).
55 Bethe, H. & Salpeter, E. in *Handbuch der Physik* Vol. 35 (Springer-Verlag Berlin, 1957)..
56 Deck, R., Amar, J. G. & Fralick, G. Nuclear size corrections to the energy levels of single-electron and-muon atoms. *Journal of Physics B: Atomic, Molecular and Optical Physics* **38**, 2173 (2005).
57 Bjorken, J. D. & Drell, S. D. *Relativistic quantum fields.* (McGraw-Hill, 1965).
58 Buchanan, M. Dirac gets the jitters. *Nature Physics* **1**, 5-5 (2005).
59 Dong, S.-H. & Ma, Z.-Q. Exact solutions to the Dirac equation with a Coulomb potential in 2+ 1 dimensions. *Physics Letters A* **312**, 78-83 (2003)..
60 Alhaidari, A. Solution of the Dirac equation with position-dependent mass in the Coulomb field. *Physics Letters A* **322**, 72-77 (2004)..
61 Chen, R. The Problem of Initial Value for the Plane Transverse Electromagnetic Mode. Acta Phys. Sin-CH ED **49**, 2518 (2000).
62 Chen, R., Li, X. J. Achievable Transverse Cylindrical Electromagnetic Mode, Progress In Electromagnetics Research Letters **24**, 59-68(2011).