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The End of Klein-Gordon Equation for Coulomb Field

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The unitary principle is used to test the mathematical procedures and conclusions of the standard theory of the Klein-Gordon equation in a Coulomb field, and it is revealed that the exact solution of the Klein-Gordon equation in a Coulomb field hides the inexorable wave function divergence and virtual energy difficulty. The divergence was not found in the past because it was concealed by an unnecessary function transformation introduced in the process of solving differential equations. Since the expected solution of the Klein-Gordon equation for a Coulomb field does not meet the boundary conditions, but is only a pseudo solution, and there is no other exact solution of the Klein-Gordon equation for the Coulomb field that meets the expectation of energy quantization, which declares the end of the Klein-Gordon equation for the Coulomb field. This conclusion is irreversible. The Klein-Gordon equation is constructed by the evolution of the operator of relativistic momentum energy equation. It is the foundation equation of relativistic quantum force. The end of the desired solution indicates that relativistic quantum mechanics and even relativistic mechanics are facing severe challenges.

Keywords: Unitary principle, Klein-Gordon equation, Inevitable solution, Pseudo solution, Wave function divergence, Energy of imaginary number.

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1 Introduction

Dongfang unitary principle[1, 1] is a basic principle that is universally applicable to the logical test of natural and social sciences. Dongfang's unitary principle is as follows: there is a certain transformation relationship between different metrics, and the natural law itself does not change because of different metrics. If the mathematical form of natural law under different metrics is transformed to one metrics, the result must be the same as the inherent form under this metrics, 1=1, and the transformation is unitary. The theory of natural science must conform to the unitary principle. Although the theory conforming to the unitary principle may not be able to correctly describe the natural law, the assumption, inference and even the whole theory that do not conform to the unitary principle must be wrong.

For a theory or experiment report, different metrics are selected for reasoning. If the conclusions drawn are inconsistent, it is exposed that this theoretical or experimental report is illogical or inconsistent with the facts. The assumption that the speed of light is constant[2-5], the quantum number of quantum mechanics[6-9, 11], the screw double black hole gravitational wave theory of LIGO signal wave[2, 10], Yukawa's nucleon meson theory[6], etc. do not conform to the unitary principle. End of Yukawa's nuclear meson theory is the result of the unitary principle test. Yukawa meson theory repeatedly expresses the same equation in various patterns mathematically, thus concealing the trace of its theory of dismembering the operator wave equation evolved by the relativistic momentum and energy relationship. Then, it defines the radial solution of the dismembered partial differential equation as the nuclear force potential energy function and defines the mass parameter of the radial solution as the mass of meson. Its logic is against the spirit of science. The so-called nuclear force meson theory actually belongs to the category of relativistic quantum mechanics. Does relativistic quantum mechanics also hide mathematical or physical logic problems that can be discovered through the unitary principle test?

The Klein-Gordon equations[15-21] and the Dirac equations[22-24] of relativistic quantum mechanics[25-27] constitute two models of relativistic quantum mechanics, but their inferences are inconsistent, which is inconsistent with the basic principle that the natural law itself does not change because of different selectivity metrics, which means that relativistic quantum mechanics may hide undiscovered logic problems. The exact solution of the Klein-Gordon equation for the hydrogen atom does not meet the expectation, while the solution of the Dirac equation is consistent with the expectation. Because of this, it is generally defined that the Klein-Gordon equa-

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tion is suitable for zero spin particles and the Dirac equation is suitable for $1/2$ spin particles. However, the Dirac equation does not come from the relativistic mechanical law of spin particles, and this definition of the scope of application is farfetched. It does not conform to the unitary principle to define the scope of application of the equation according to different particles after the equation is constructed rather than at the beginning of the equation construction. According to the basic principle of quantum mechanics in which mechanical quantities are replaced by operators to construct wave equations, spins should have corresponding mechanical laws and wave equations constructed with operator principles. What are the classical mechanical laws and relativistic mechanical laws of $1/2$ spin particles? The Dirac theory, as the mainstream of relativistic quantum mechanics, has no answer to this question, while relativistic quantum mechanics only defines the scope of application of the Dirac equation and the Klein-Gordon equation. The Klein-Gordon equation is the operator evolution equation of relativistic momentum equation, which should be the foundation equation of relativistic quantum mechanics.

Some theories of modern physics and their development theories often hide distorted mathematical calculations and biased explanations. The unitary principle has strong logic insight, which enables us to constantly find new problems and always find effective methods to deal with problems and draw reliable conclusions. Starting with this article, I will gradually introduce some irreversible conclusions of relativistic quantum mechanics tested by unitary principle.

2 Klein-Gordon equation of Coulomb field

After reading the Schrödinger equation of the hydrogen atom for the first time, we can easily associate the relativistic wave equation of the hydrogen atom, which is exactly the Klein-Gordon equation that will be read later. In the era of paper-based literature, it is not like now that every time you encounter a problem, you can find a pleasing answer in Internet search. It always takes too much time to consult paper documents, so many problems must be reasoned independently. Adhering to this habit can integrate power and enhance confidence to reveal the truth. When I independently solved Klein-Gordon equation in those years, I encountered difficulties in exact solution divergence and imaginary number energy, and believed that this was a new challenge to relativity from the development of theory of relativity. Later, I read the exact solution of the Klein-Gordon equation of zero spin meson in the Coulomb field from Problems in Quantum Mechanics, and learned that standard theory sometimes evades key problems and describes formal calculation results as scientific conclusions. This is also the motivation for further studying the truth of the Dirac equation.

The exact solution of the Schrödinger equation in the π^- meson Coulomb field is highly similar to the exact solution of the Schrödinger equation for the hydrogen atom. The only difference is their masses. The key to accurately solving the Schrödinger equation is to ignore the boundary condition of the atomic nucleus size, which requires that the series solution be interrupted into a polynomial, so that the energy quantized eigenvalue can be obtained naturally. The formal exact solution of the Klein Gordon equation of π^- meson in the Coulomb field belongs to the standard theoretical solution, and unnecessary function transformation is introduced in the process of solving the equation, which is a key point. In a Coulomb field with a nuclear or central charge number of Z, the potential energy of the π^- meson is

$$V = -\frac{\mathrm{A}\hbar c}{r} \tag{1}$$

Where $\mathrm{A} = Z\alpha$, Z is the number of nuclear charges, α is the so-called fine structure constant, and r is the distance from the electron to the nucleus. The relativistic momentum energy relationship of particles in the field is

$$c^2p^2 + m^2c^4 = (E - V)^2 \tag{2}$$

Substitute equation (1), replace the momentum with the momentum operator, and act on the wave function to obtain the stationary Klein-Gordon equation of π^- meson

$$\left(-\hbar^2c^2\nabla^2 + m^2c^4\right)\psi = \left(E - \frac{\mathrm{A}\hbar c}{r}\right)^2\psi \tag{3}$$

The form of this equation in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \left(\frac{\mathrm{A}^2}{r^2} + \frac{2\mathrm{A}E}{\hbar cr} - \frac{m^2c^4 - E^2}{\hbar^2c^2}\right)\psi = 0 \tag{4}$$

Separating variables, it supposes that

$$\psi = R(r)\,Y(\theta,\phi) \tag{5}$$

Substituting it into equation (4) yields

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(\mathrm{A}^2 + \frac{2\mathrm{A}E}{\hbar c}r - \frac{m^2c^4 - E^2}{\hbar^2c^2}r^2\right) = -\frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2Y}{\partial\phi^2}\right) = \lambda \tag{6}$$

The angular equation is Legendre equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}+\lambda Y=0 \tag{7}$$

The exact solution of the equation satisfying the boundary conditions is $Y_{lm}(\theta,\phi)$, and the eigenvalue is

$$\lambda=l(l+1),\ \ l=0,\ 1,\ 2,\ \cdots,\ \ m=0,\ \pm1,\ \pm2,\ \cdots,\ \pm l \tag{8}$$

According to equation (6), the radial equation of the Klein-Gordon equation of Coulomb field is,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(\mathrm{A}^2-l(l+1)+\frac{2\mathrm{A}E}{\hbar c}r-\frac{m^2c^4-E^2}{\hbar^2c^2}r^2\right)R=0 \tag{9}$$

This is a second-order ordinary differential equation.

3 The primitive solution of the radial equation and its termination

Quantum mechanics uses natural boundary conditions to determine the special solutions of Schrödinger equation, Klein-Gordon equation and other wave equations. The boundary conditions for the wave equation of electrons and π^- mesons in the Coulomb field usually do not consider the size of the atomic nucleus or central charge, specifically,

$$\lim_{r\to0}\psi=0,\quad \psi(0<r<\infty)\neq\pm\infty,\quad \lim_{r\to\infty}\psi=0 \tag{10}$$

This boundary condition is actually a rough boundary condition. The Klein Gordon equation of π^- meson treated by the standard theory usually imitates the solution of Schrödinger equation of the hydrogen atom, and the following function transformation is introduced first,

$$R(r)=\frac{\mathrm{H}(r)}{r} \tag{11}$$

Substitute the function R and its corresponding derivative

$$\frac{dR}{dr}=\frac{1}{r}\frac{d\mathrm{H}}{dr}-\frac{\mathrm{H}}{r^2},\quad \frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=r\frac{d^2\mathrm{H}}{dr^2} \tag{12}$$

into equation (9), turning the radial equation into a second order differential equation about H,

$$\frac{d^2\mathrm{H}}{dr^2}-\left(\frac{m^2c^4-E^2}{\hbar^2c^2}-\frac{2\mathrm{A}E}{\hbar c}\frac{1}{r}+\frac{l(l+1)-\mathrm{A}^2}{r^2}\right)\mathrm{H}=0 \tag{13}$$

Then introduce the dimensionless variable

$$\xi=\beta r,\quad \beta=\frac{2}{\hbar c}\sqrt{m^2c^4-E^2},\quad \varepsilon=\frac{\mathrm{A}E}{\sqrt{m^2c^4-E^2}} \tag{14}$$

Thus, equation (13) is reduced to a dimensionless equation

$$\frac{d^2\mathrm{H}}{d\xi^2}-\left(\frac{1}{4}-\frac{\varepsilon}{\xi}+\frac{l(l+1)-\mathrm{A}^2}{\xi^2}\right)\mathrm{H}=0 \tag{15}$$

Finding the solution of equation (15) in the following form

$$\mathrm{H}=e^{-\xi/2}u(\xi) \tag{16}$$

Then the second order optimal differential equation for $u(\xi)$ is obtained

$$\frac{d^2u}{d\xi^2}-\frac{du}{d\xi}+\left(\frac{\varepsilon}{\xi}-\frac{l(l+1)-\mathrm{A}^2}{\xi^2}\right)u=0 \tag{17}$$

The boundary condition (10) of the wave function requires that the series solution of equation (17) be interrupted into polynomials, which is generally understood as that function (16) satisfies the boundary (10). Let the interrupted series solution of equation (17) to be

$$u(\xi)=\sum_{k=0}^{n}b_k\xi^{s+k},\ s>0,\ b_0\neq0 \tag{18}$$

Its first and second derivatives are

$$\frac{du}{d\xi}=\sum_{k=0}^{n}(s+k)\,b_k\xi^{s+k-1}$$
$$\frac{d^2u}{d\xi^2}=\sum_{k=0}^{n}(s+k)(s+k-1)\,b_k\xi^{s+k-2}$$

Substitute these into the optimal differential equation (17) to get

$$\sum_{k=0}^{n}\left\{\left[(s+k+1)(s+k)-l(l+1)+\mathrm{A}^2\right]b_{k+1}-(s+k-\varepsilon)\,b_k\right\}\sum_{k=0}^{n}b_k\xi^{s+k-1}=0 \tag{19}$$

This relation determines the following recursive relations satisfied by the coefficients of the series

$$\begin{aligned}
&\left[s(s-1)-l(l+1)+\mathrm{A}^2\right]b_0=0\\
&\left[(s+1)s-l(l+1)+\mathrm{A}^2\right]b_1-(s-\varepsilon)b_0=0\\
&\vdots\\
&\left[(s+k+1)(s+k)-l(l+1)+\mathrm{A}^2\right]b_{k+1}-(s+k-\varepsilon)b_k=0\\
&\vdots\\
&\left[(s+n)(s+n-1)-l(l+1)+\mathrm{A}^2\right]b_n-(s+n-1-\varepsilon)b_{n-1}=0\\
&(s+n-\varepsilon)b_n=0
\end{aligned}\tag{20}$$

The first equation is the index equation, which has two roots

$$s=\begin{cases}\dfrac{1}{2}-\sqrt{\left(l+\dfrac{1}{2}\right)^2-\mathrm{A}^2}\\[2ex]\dfrac{1}{2}+\sqrt{\left(l+\dfrac{1}{2}\right)^2-\mathrm{A}^2}\end{cases}\tag{21}$$

The negative root obviously does not satisfy the boundary, so it is abandoned. It is generally believed that taking the positive root

$$s=\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\mathrm{A}^2}\tag{22}$$

satisfies the boundary conditions. Then the coefficient recursive relationship of the polynomial is determined as

$$b_{k+1}=\frac{s+k-\varepsilon}{(k+1)(2s+k)}b_k\tag{23}$$

Because the series breaks to the $b_n\xi^{s+n}$ term, that is, $b_{n+1}=0$, and $b_n\neq 0$. The final form of equation (20) yields $s+n-\varepsilon=0$, that is,

$$\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\mathrm{A}^2}+n-\frac{\mathrm{A}E}{\sqrt{m^2c^4-E^2}}=0\tag{24}$$

The quantized energy formula is thus obtained

$$E=\frac{mc^2}{\sqrt{1+\frac{\mathrm{A}^2}{\left(n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\mathrm{A}^2}\right)^2}}}\tag{25}$$

The above mathematical calculation process and conclusion seem to be perfect.

Problems in Quantum Mechanics reviewed a period of history about Klein Gordon. Shortly after Klein-Gordon equation was proposed in 1926, in order to try to explain the fine structure observed in the spectrum of hydrogen like atoms, calculations similar to those given above were made. The calculated results do not agree with the experimental observations; for the fine structure splitting of the energy level with the principal quantum number, the predicted energy broadening value is

$$E_{n,n-1}-E_{n,0}=\frac{mZ^4e^8}{\hbar^4c^2n^3}\frac{n-1}{n-1/2}\tag{26}$$

This is much larger than the experimental value. The results did not meet expectations. Later, it was explained that the Klein-Gordon equation ignored the spin of the electron and could only describe the spin free particles. The formula for the fine structure should be very similar to equation (25) obtained by Dirac equation. Is that really the case?

The solution of the equation can often inspire people to find the deficiency of the equation. However, according to the unitary principle, the scope of application of an equation cannot be defined by the solution of the equation. The quantum mechanical wave equation describing the particle spin must have a corresponding mechanical equation, so as to ensure that the consistency of the operator principle of quantum mechanics is not broken. From a mathematical point of view, the basis for the above standard theory to choose between the two roots (21) of the index equation is to understand the boundary condition (10) on wave function ψ as the boundary condition on $\mathrm{H}(\xi)$,

$$\lim_{\xi\to 0}\mathrm{H}\neq\pm\infty,\ \mathrm{H}(0<\xi<\infty)\neq\pm\infty,\ \lim_{\xi\to\infty}\mathrm{H}=0\tag{27}$$

This is unreasonable. The function substitution (11) conceals that the radial wave function cannot obtain the value satisfying the boundary strip (10). This can be found by writing the complete formal wave function. Note that $\mathrm{A}=Z\alpha$, substitute (16), (18) and (22) into (11) to obtain the complete formal wave function

$$\psi=e^{-\frac{1}{2}\beta r}\sum_{k=0}^{n}\left(b_k\beta^{\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-Z^2\alpha^2}+k}r^{\sqrt{\left(l+\frac{1}{2}\right)^2-Z^2\alpha^2}-\frac{1}{2}+k}\right)Y_l^m(\theta,\phi)\tag{28}$$

Obviously, when $l=0$

$$\psi_{l=0}=e^{-\frac{1}{2}\beta r}\left[b_0\beta^{\frac{1}{2}+\sqrt{\frac{1}{2^2}-Z^2\alpha^2}}r^{\sqrt{\frac{1}{2^2}-Z^2\alpha^2}-\frac{1}{2}}+\sum_{k=1}^{n}\left(b_k\beta^{\frac{1}{2}+\sqrt{\frac{1}{2^2}-Z^2\alpha^2}+k}r^{\sqrt{\frac{1}{2^2}-Z^2\alpha^2}-\frac{1}{2}+k}\right)\right]Y_l^m(\theta,\phi) \tag{29}$$

The solution of the equation diverges at the coordinate origin:

$$\lim_{r\to 0}|\psi|=\infty \tag{30}$$

The standard theoretical solution of the Klein-Gordon equation of Coulomb field does not satisfy the boundary condition (10).

Another key problem is that Klein-Gordon equation of Coulomb field hides imaginary energy of anti-natural law. The quantized energy formula (25) is investigated. Reviewing the quantized energy formula (25), it is clear that to ensure that the radical $\sqrt{(l+1/2)^2-Z^2\alpha^2}$ is a real number, the following conditions must be met

$$\left(l+\frac{1}{2}\right)^2-Z^2\alpha^2\geqslant 0 \tag{31}$$

namely

$$Z\leqslant\frac{1}{\alpha}\left(l+\frac{1}{2}\right) \tag{32}$$

In the case of $l=0$, the specific result of this inequality is $Z\leqslant 1/2\alpha=68.5$, which means that when Z $\geqslant$ 69, there will be imaginary numbers of energy, which is not in line with the fact.

In conclusion, the complete wave function of the Klein-Gordon equation of the Coulomb field is only a formal wave function, which does not meet the boundary conditions used to determine the special solution of the equation. The unnecessary function transformation (14) is introduced in the process of obtaining the wave function of this form, thus covering the divergence of the wave function at the origin of coordinates. In fact, without introducing the function replacement in the form of (14), the second order differential equation (13) about $\mathrm{H}(r)$ can be directly solved according to the above method, and then substituted into (11) to obtain the complete form of formal radial wave function. This formal solution is exactly the expected solution, which is merged as following:

$$\begin{aligned}
&R(r)=\frac{1}{r}e^{-\frac{\mathrm{A}mc}{\hbar\sqrt{(n+s)^2+\mathrm{A}^2}}r}\sum_{\nu=0}^{n}b_\nu r^{s+\nu}\\
&b_\nu=-\frac{2(n-\nu+1)\mathrm{A}mc}{\hbar\left[(s+\nu)(s+\nu-1)-l(l+1)+\mathrm{A}^2\right]\sqrt{(n+s)^2+\mathrm{A}^2}}b_{\nu-1}\\
&E=\frac{mc^2}{\sqrt{1+\frac{\mathrm{A}^2}{(s+n)^2}}},\quad s=\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\mathrm{A}^2}\\
&(l=0,\ 1,\ 2,\ 3,\ \cdots;\quad n=0,\ 1,\ 2,\ 3,\ \cdots)
\end{aligned} \tag{33}$$

When $l=0$, this formal radial wave function diverges at the coordinate origin, so the exact solution cannot meet the boundary conditions, which means that π^- meson must fall into the atomic nucleus or the opposite sign charge moving in the Coulomb field must fall into the charge center of the Coulomb field. This is obviously contrary to the fact of cosmic structure. On the other hand, the appearance of virtual energy of the Klein Gordon equation of Coulomb field requires limiting the number of nuclear charge or central charge, which is not in accordance with scientific logic. Theoretically, there can be a Coulomb field with a large number of central charges, but it is impossible for an odd sign charge to move in such a Coulomb field to generate imaginary energy.

The inexorable divergence of the exact solution of the Klein-Gordon equation and the imaginary quantized energy declares the end of the Klein-Gordon equation of the Coulomb field.

4 The original solution of radial equation and the end of solution

The function transformation (11) introduced by the standard theory of the Klein-Gordon equation in solving the radial equation (9) diverts the reader's attention from logic, thus making the reader prone to an illusion that the boundary conditions of the wave function can be satisfied only by taking the positive root from the two roots of the index equation, so the complete form of the equation solution at the origin of the coordinates is covered up. If we directly solve the radial wave equation (9) without introducing function substitution (11), we will easily find that the root of the index equation cannot make the exact solution meet the boundary conditions. Many equations in modern physics, even if their expressions are slightly different, it is necessary to calculate and solve them in person or try to prove some basic equations independently. From this, we will eventually understand some of the logic and operation of modern physics.

The general theory for solving the radial equation (9) is to first optimize the equation and obtain the optimal differential equation with respect to one factor of the radial wave function, while the other factor is the approximate solution of the equation when $r \to \infty$. The simplified procedure is to set the radial wave function

$$R = e^{-ar}\Omega \tag{34}$$

where

$$a = \frac{\sqrt{m^2c^4 - E^2}}{\hbar c} \tag{35}$$

Calculate the derivative of R and related terms

$$\begin{aligned} &\frac{dR}{dr} = e^{-ar}\frac{d\Omega}{dr} - ae^{-ar}\Omega \\ &\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left(r^2\left(e^{-ar}\frac{d\Omega}{dr} - ae^{-ar}\Omega\right)\right) \\ &= e^{-ar}r\left[r\frac{d^2\Omega}{dr^2} - (2ar-2)\frac{d\Omega}{dr} + a(ar-2)\Omega\right] \end{aligned} \tag{36}$$

Then (34) and (35) are substituted into the radial equation (9) to obtain the optimal differential equation

$$r^2\frac{d^2\Omega}{dr^2} + \left(2r - 2ar^2\right)\frac{d\Omega}{dr} + \left\{\mathrm{A}^2 - l(1+l) + 2\left(\frac{\mathrm{A}E}{\hbar c} - a\right)r + \left(a^2 - \frac{m^2c^4 - E^2}{\hbar^2c^2}\right)r^2\right\}\Omega = 0$$

Using (35), the above equation is simplified as

$$r^2\frac{d^2\Omega}{dr^2} + \left(2r - 2ar^2\right)\frac{d\Omega}{dr} + \left\{\mathrm{A}^2 - l(1+l) + 2\left(\frac{\mathrm{A}E}{\hbar c} - a\right)r\right\}\Omega = 0 \tag{37}$$

Seek the interrupted series solution of this equation

$$\Omega = \sum_{\nu=0}^{n} b_\nu r^{s+\nu}, \quad \frac{d\Omega}{dr} = \sum_{\nu=0}^{n}(s+\nu)\,b_\nu r^{s+\nu-1}, \quad \frac{d^2\Omega}{dr^2} = \sum_{\nu=0}^{n}(s+\nu)(s+\nu-1)\,b_\nu r^{s+\nu-2} \tag{38}$$

Substitute it into equation (37) to obtain

$$\sum_{\nu=0}^{\infty}\left\{\left[(s+\nu)(s+\nu+1) + \mathrm{A}^2 - l(1+l)\right]b_\nu - 2\left[(s+\nu)a - \frac{\mathrm{A}E}{\hbar c}\right]b_{\nu-1}\right\}r^{s+\nu} = 0 \tag{39}$$

Therefore, the coefficients of the series satisfy the recursive relation

$$\begin{aligned} &\left[s(s+1) + \mathrm{A}^2 - l(1+l)\right]b_0 = 0 \\ &\left[(s+1)(s+2) + \mathrm{A}^2 - l(1+l)\right]b_1 - 2\left[(s+1)a - \frac{\mathrm{A}E}{\hbar c}\right]b_0 = 0 \\ &\vdots \\ &\left[(s+\nu)(s+\nu+1) + \mathrm{A}^2 - l(1+l)\right]b_\nu - 2\left[(s+\nu)a - \frac{\mathrm{A}E}{\hbar c}\right]b_{\nu-1} = 0 \\ &\vdots \\ &\left[(s+n)(s+n+1) + \mathrm{A}^2 - l(1+l)\right]b_n - 2\left[(s+n)a - \frac{\mathrm{A}E}{\hbar c}\right]b_{n-1} = 0 \\ &2\left[(s+n+1)a - \frac{\mathrm{A}E}{\hbar c}\right]b_n = 0 \end{aligned} \tag{40}$$

The above recursive relationship includes index equation and energy eigenvalue. The radial wave function can be determined only after the index s and energy eigenvalue are determined. The first equation is the index equation. The first equation is the index equation. Because $b_0 \neq 0$, $s(s+1) + \mathrm{A}^2 - l(1+l) = 0$. The two roots of this equation are

$$s=\begin{cases}-\dfrac{1}{2}-\sqrt{\left(l+\dfrac{1}{2}\right)^2-\mathrm{A}^2}\\-\dfrac{1}{2}+\sqrt{\left(l+\dfrac{1}{2}\right)^2-\mathrm{A}^2}\end{cases}$$

These are two roots of the original index equation, which is obviously different from the two roots (21) of the special-shaped index equation obtained by introducing function transformation to solve the radial equation (9). The boundary condition (10) of the wave function requires that the negative root be discarded. The second root is usually taken without thinking,

$$s=-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\mathrm{A}^2} \tag{41}$$

However, when $l=0$, this root is actually a negative number, so the radial wave function given by (34) take the form

$$R=e^{-\frac{\sqrt{m^2c^4-E^2}}{\hbar c}r}\sum_{\nu=0}^{n}b_\nu r^{-\frac{1}{2}+\sqrt{\frac{1}{4}-\mathrm{A}^2}+\nu}$$

namely

$$R=\frac{e^{-\frac{\sqrt{m^2c^4-E^2}}{\hbar c}r}}{r^{\frac{1}{2}-\sqrt{\frac{1}{4}-\mathrm{A}^2}}}\left(b_0+\sum_{\nu=1}^{n}b_\nu r^\nu\right) \tag{42}$$

The wave function boundary condition (10) cannot be satisfied. This is because

$$\lim_{r\to0}|R|=\lim_{r\to0}\left|\frac{e^{-\frac{\sqrt{m^2c^4-E^2}}{\hbar c}r}}{r^{\frac{1}{2}-\sqrt{\frac{1}{4}-\mathrm{A}^2}}}\left(b_0+\sum_{\nu=1}^{n}b_\nu r^\nu\right)\right|=\infty \tag{43}$$

The exact solution of the Klein-Gordon equation of Coulomb field diverges at the coordinate origin. From the mathematical point of view, the radial equation (9) does not meet the exact solution of the definite solution condition; From a physical point of view, allowing the radial wave function to diverge at the origin of coordinates means that the universe collapses, which does not conform to the fact. It has been explained previously that when $l=0$, if $\mathrm{A}>1/2$, that is, $Z\alpha>1/2$, and the number of nuclear charges $Z>1/2\alpha=68.5$, the index s is an imaginary number, resulting in an imaginary energy, which does not conform to the fact.

The last equation of the recursive relation (40) is the energy eigenvalue equation. Since $2b_n\neq0$, $(s+n+1)\,a-\mathrm{A}E/\hbar c=0$, the formal energy eigenvalue equation is obtained by substituting (35),

$$(s+n+1)\sqrt{m^2c^4-E^2}-\mathrm{A}E=0 \tag{44}$$

Then substitute (41) into the above equation to obtain the positive formal positive definite eigenvalue

$$E=\frac{mc^2}{\sqrt{1+\frac{\mathrm{A}^2}{\left(n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\mathrm{A}^2}\right)^2}}} \tag{45}$$

The result of substituting (41) into equation (45) is equation (25). The non relativistic approximation of this result is consistent with the Bohr energy level formula. But its mathematical process is untenable, and the hidden imaginary number difficulty cannot be solved.

The above formal solution of primitive state directly exposes that the exact solution of the Klein-Gordon equation of Coulomb field hides fatal problems that do not meet the boundary conditions of the wave equation and exist imaginary energy, which seriously violates the mathematical and physical meanings. Such dual mathematical and physical logic difficulties cannot be avoided by new definitions. Therefore, Klein-Gordon equation of the Coulomb field has only formal solution, which cannot be used to describe the quantum state of the Coulomb field. From (34), (35), (a2), (39), (42) and (45), the formal solution of primitive state is merged as follows:

$$\begin{aligned}&R=e^{-\frac{\mathrm{A}mc}{\hbar\sqrt{(n+s+1)^2+\mathrm{A}^2}}r}\sum_{\nu=0}^{n}b_\nu r^{s+\nu}\\&b_\nu=-\frac{2\mathrm{A}mc\,(n-\nu+1)}{\hbar\left[(s+\nu)(s+\nu+1)-l\,(1+l)+\mathrm{A}^2\right]\sqrt{(n+s+1)^2+\mathrm{A}^2}}b_{\nu-1}\\&E=\frac{mc^2}{\sqrt{1+\frac{\mathrm{A}^2}{(s+n+1)^2}}},\quad s=-\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\mathrm{A}^2}\\&(l=0,\ 1,\ 2,\ 3,\ \cdots;\quad n=0,\ 1,\ 2,\ 3,\ \cdots)\end{aligned} \tag{46}$$

Compared with (33), the indexes s in the two solutions are different, but the final results are the same.

(33) and (46) are only the formal solutions of the Klein-Gordon equation of the coulomb field. In addition, Klein-Gordon equation of the coulomb field has no other reasonable solutions. Therefore, Klein-Gordon equation of the coulomb field has no practical significance, and all relevant deformation equations obtained by distorting mathematics are just props for mass production papers, which are meaningless. So, as the main equation of relativistic quantum mechanics, what about the Dirac equation?

5 Conclusions and comments

Based on the idea of unitary principle, the above content tests the relativistic Klein-Gordon equation of the Coulomb field and proves that its exact solution does not meet the boundary conditions, and the energy eigenvalue implies imaginary energy, which is a pseudo quantization energy. Specifically, the wave function follows the statistical significance of Born's interpretation, but the Klein-Gordon ground state wave function of the Coulomb field is divergent, which means that the meson will fall into the atomic nucleus or central charge when entering the Coulomb field, resulting in an increase in the mass of the atomic nucleus or central charge. Applying it to the atomic system means that the universe seems to have collapsed long ago, obviously violate the laws of nature. In the Klein Gordon equation of the Coulomb field, when the number of central charges or nuclear charges exceeds 68, imaginary energy will appear. This requires that the number of central charges or nuclear charges in the Coulomb field model does not exceed 68, which obviously does not conform to the fact of the atomic structure in nature. According to the unitary principle, the Coulomb field is not necessarily limited to the hydrogen like atom model. It can have a model with a large enough central charge number. A particle with a different sign charges move in a Coulomb field with a large central charge number. This model can solve the additional problems caused by the existence of spin or not. It can be seen that there is no scientific basis for specifying the Klein-Gordon equation to describe particles with zero spin. In a word, Klein-Gordon equation of the Coulomb field has no eigen solution conforming to physical meaning. The eigenwave functions and energy eigenvalues of the Klein-Gordon equation of mesons given by the standard theory are pseudo solutions, which determine the end of the Klein-Gordon equation of the Coulomb fields. This irreversible conclusion urges us to further test other relativistic quantization theories of Coulomb field.

From the point of view of mathematical steps, the function transformation introduced in (11) conceals the problem that the exact solution of the Klein Gordon equation of Coulomb field does not meet the wave function boundary conditions. If the wave function is allowed to diverge, the index equation cannot choose between the two roots. The so-called quantized energy has another set of different eigenvalues, which does not conform to the unitary principle, and the imaginary energy cannot be eliminated. Without introducing the radial wave function transformation, the Klein-Gordon equation of the coulomb field is solved directly to obtain the primitive solution. The problem that the boundary conditions cannot be satisfied is obvious. From the perspective of physical logic, if the relativistic momentum energy relationship is regarded as an accurate mechanical law, and the basic principle of quantum mechanics to construct wave equations by replacing mechanics with operators is correct, then according to the unitary principle, the Klein-Gordon equation derived from the combination of special relativity and quantum mechanics becomes the first-choice equation to describe the quantization law of Coulomb field. Furthermore, according to the unitary principle, if relativistic mechanics is regarded as a theory that accurately describes high-speed motion, then the relativistic description of physical laws is only different in accuracy from that of Newtonian mechanics, and there is no essential difference between them. However, from the Schrödinger equation of quantum mechanics to the Klein Gordon equation of relativistic quantum mechanics, the basic physical model of the Coulomb field has changed substantially. The Schrödinger equation has a solution that conforms to the physical meaning, while the Klein-Gordon equation has no solution that conforms to the physical meaning. This shows that the relativistic Klein-Gordon equation does not conform to the unitary principle.

The unitary principle can be used to test enough mathematical paradoxes hidden in modern physics. The reason for the existence of these mathematical paradoxes may be that the basic mathematical equations of modern physics take the speed of light as the singularity. That is, because of the theory of relativity. Time and space are two basic physical concepts. However, modern physics believes that relativistic mechanics is an accurate theory because it reversely modifies the definition of time and space with the relative speed and the speed of light, while Newtonian mechanics follows the natural order that speed can only be defined based on the concept of time and space first, so it can only be regarded as a low-speed similarity theory. In this way, the high-speed quantum theory described by the relativistic Klein-Gordon equation is the advanced theory of the low-speed quantum theory described by the Schrödinger equation, and the Schrödinger equation is regarded as the low-speed approximation of the Klein-Gordon equation. However, the exact solution of the Klein-Gordon equation is subversive compared with the exact solution of Schrödinger equation and subverts the natural world. The divergence of the standard solution of the Klein-

Gordon equation of the π^- meson, which imitates the Schrödinger equation in solving method, means the collapse of the universe. The existence of its imaginary energy requires the existence of an upper limit on the number of nuclear charges. These deductions that violate the laws of nature prove that the Klein-Gordon equation of the Coulomb field violates the unitary principle and can only be terminated. Physical logic cannot be cause and effect inverted because of personal worship, and time and space cannot be different because of people or theories. The so-called Minkowski space-time and Schwarzschild space-time are both artificially defined. When Klein-Gordon equation is written into the so-called Schwarzschild space-time form, it seems infinitely profound and daunting. The college system forces generations of students to recite random concepts, equations and conclusions of modern physics, and eventually become one of the masters in this field of physics. End of Yukawa's nuclear meson theory and Klein-Gordon equation today reflects the serious defects of the college system. Those wrong physical theories are difficult to be discovered or even refused to be discovered because of the product of the college system.

There is a simple truth that the development theory of a theory whose basic principles hide logical contradictions must contain more logical contradictions. According to the definition of relativistic quantum mechanics, the simple harmonic oscillator and the zero-spin particle in the Coulomb field should be described by the Klein-Gordon equation, while the 1/2 spin particle in the Coulomb field should be described by the Dirac equation. This leads to a unitary problem. Does the 1/2 spin simple harmonic oscillator model exist? The focus of systematic examination of relativistic quantum mechanics is to examine the mathematical processing of the Klein-Gordon equation and the Dirac equation. First of all, we will discuss the Klein-Gordon equation of the zero-spin π^- meson in the Coulomb field and prove that the inevitable solution of the Klein-Gordon equation of the π^- meson does not meet the boundary conditions and implies imaginary energy, so it has no physical significance, thus declaring the end of the Klein-Gordon equation of the Coulomb potential. However, a large number of documents[28-43] claiming to have obtained the exact solution of Klein-Gordon equation distort mathematics and cover up the truth, so we will not comment on them one by one.

Physical problems need to have clear conclusions, and they can have clear conclusions. Historically, Klein-Gordon proposed Klein-Gordon equation based on relativistic momentum and energy equations. Klein and Gordon did not benefit from this, and they failed to find the irrationality of the equation at that time only because of the limitations of mathematical treatment of the equation. However, mass producers of later papers benefited a lot from the Klein-Gordon equation. The relativistic momentum energy relationship will prove to be actually incorrect, although it is often claimed to be confirmed by experiments that cannot be repeated. It is not our interest and responsibility to criticize a large number of reputable SCI journals that vigorously promote modern physics papers created by distorting mathematics, and such specific criticism is also likely to cause general hostility. However, there will always be people in the world who can and will be sure to carry out logical tests to expose the distorted mathematics, sophistry and lies of modern physics. Perhaps readers have to wonder whether modern physics is under the control of a different religion. The distortion of mathematics to mass produce papers may be caused by the pressure of work, but these works are defined as excellent scientific theories and breakthrough scientific achievements because they are published by prestigious academic journals, which puts the truth that is constantly discovered today and tomorrow into a hopeless situation.

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