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Dongfang Solution of Induced Second Order Dirac Equations

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Solving the radial Dirac equation of the hydrogen atom, it usually follows the treatment method of Schrödinger equation of the hydrogen atom and expresses the two-component wave function as two new variables divided by the radial independent variables, thus transforming the equation into the induced first-order Dirac equation system. According to the induced first-order Dirac equations, two induced second order Dirac equations constrained by the same energy parameter can be obtained. Here, I study the eigensolutions of the induced second-order Dirac equation system of the hydrogen atom, and draw several unusual conclusions. The exact solution of the first-order induced Dirac equation system of hydrogen atom satisfies the induced second-order Dirac equation, but the complexity of correlation checking increases with the increase of the radial quantum number, and even the checking process of energy states with small radial quantum numbers is very complicated; The induced second-order Dirac equation of hydrogen atom has an eigensolution, and its energy eigenvalue is the same as that of the induced first-order Dirac equation; Different from the constraint of the coefficients of the two wave function components of the first-order Dirac equation system, the exact solutions of the two induced second-order equations are independent of each other, which means that the coefficients of the two component functions have their own normalized coefficients. The independence of the component function of the second order equation poses a new challenge to the physical meaning of the multi-component wave function of Dirac theory.

Keywords: Dirac equation; Induced first-order Dirac equation; Induced second-order Dirac equation; Dongfang solution; Existence and uniqueness of Solution.

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1 Introduction

How much influence does the Dirac equation[1-7] have in the field of physics and even mathematics? Google academic search Dirac equation obtained more than 1.2 million pieces of information, and search second-order Dirac equation obtained more than 860000 pieces of information, and this data has been increasing. However, the vast majority of articles on the Dirac equation published in various prestigious journals is vague and useless[8-10], or is fundamental errors and cannot be corrected[11-20]. This exposes the defects of the academic system in which a few people control the scientific field.

The original Dirac equation is a matrix equation, that is, a system of first order partial differential equations. However, the Dirac equation is rarely correctly understood[21-26]. The evidence to support this view is that those questions that should have clear answers have been ignored. For example, according to the conditions of the Dirac matrix, using the relativistic momentum energy relationship and the operator principle of quantum mechanics to construct the first order wave equation, is the result unique to the Dirac equation? Does changing the sign of the Dirac matrix affect the physical meaning and the exact solution of the Dirac equation? Is the Darwin[27] and Gordon[28] solutions of the Dirac equation of the Coulomb field, which the textbook advocates, reasonable? Just like Wong and Yeh[16, 29], is it scientific to change the sign of mass in the Dirac equation and create a mixed operator to construct the so-called second-order Dirac equation? Is there any other solution method for the Dirac equation to give different exact solutions[30]? How many processes and conclusions of handling Dirac equation that do not conform to mathematical rules are covered up? Does the radial momentum operator and angular quantum number defined by Dirac's theory satisfy the mathematical and physical inference of causality? From these important problems that have not been

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*The author once published the article "Equivalent solution of radial Dirac-Coulomb equation" signed as Rui Chen, but did not explain the intention of giving the authentic second-order Dirac equation of Coulomb field because of succumbing to the bad hidden rules of theoretical physics that it is not allowed to point out the errors of famous theories and famous papers. Now change the title of the article, modify the abstract and conclusion of the article, indicate the authors point of view, and republish this article.

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paid attention to in the past hundred years, the treatment of the Dirac equation in physics is not reliable, and even there are second-order Dirac equations with different forms. The Dirac equation is also bound to the definition of the Dirac Sea and antimatter, etc.[31-37]. In fact, they are not necessarily causal.

The formal solutions of the first and second order deformed Dirac equations, which are mathematically very simple to deal with, obtained by introducing the function transformation, do not satisfy the first order deformed Dirac and are therefore terminated[38]. This prompted me to examine the theory of Dirac equation in depth. It is difficult to test all kinds of Dirac equation one by one in the research of limited forces. Many theories have been successfully published for recognition only by virtue of the fame of the Dirac equation. In fact, they are not real Dirac equation theories. We know that in mathematics, we can write a partial differential equation or a system of partial differential equations randomly and study its solution. But in physics, we cannot arbitrarily construct a wave equation without physical model and mechanical laws. The construction of any quantum mechanical wave equation that conforms to scientific logic will mean a major revolution or progress in quantum theory. Therefore, those nominal Dirac equations that are not real but just borrowed from Dirac's fame have no physical significance, even though the pieced expected solutions there are often interpreted in a dazzling way.

Because of the characteristics of differential, when dealing with Schr ö dinger equation, Klein-Gordon equation and the Dirac equation of Coulomb-like field, the radial wave function or the radial wave function is usually expressed as a new variable divided by the radial independent variable, thus transforming the wave equation into a new variable differential equation. When dealing with the Dirac equation, we usually imitate this method, and take the quotient of two new variables represented by the upper and lower components of the two-component wave function and the independent variables, thus obtaining the induced first order Dirac equation system. The first order differential equations can be transformed into two second order differential equations, which are the induced second order Dirac equations of hydrogen-like atoms. This paper focuses on the mathematical process, discusses the exact solution of the induced second-order Dirac equation of hydrogen-like atoms, and gives the Dongfang eigensolution with new mathematical meaning.

2 Induced first-order Dirac hydrogen equation

Dongfang unitary principle[39-42] has a very powerful logic verification function. Using the unitary principle to test quantum mechanics, we have found problems and conclusions that have not been found in the past. The variability of the relative speed of light[41], the unsolved morbid equation of quantum number[42], the Com quantum equation of LIGO signal[43-45], the modification of the basic equation of molecular dynamics[46], the operator evolution equation group of angular motion law[47], the end of the Yukawa nuclear force meson theory[48], the end of the Klein-Gordon equation of Coulomb field[49] and the end of the teratogenic simplified Dirac hydrogen equation[38] should have a profound impact on the development of physics.

When solving the radial Schrödinger equation of hydrogen atom, the wave function is usually expressed as a new unknown function divided by the radial independent variable, so that the second order differential equation for solving the original radial wave function is transformed into solving the second order equation for the new unknown function. This seems to be for convenience, but it doesn't bring much convenience. However, this habit has been followed by the radial Dirac equation used to deal with hydrogen model. Generally, the two components of the Dirac wave function are expressed as two new variables divided by the radial independent variables, so that the original radial Dirac equation of the hydrogen atom is transformed into a first-order equation group about two new variables, which is called the induced first-order Dirac equation group. The corresponding two second-order differential equations are called induced second-order Dirac differential equations. The purpose of naming "inducted equation" is to avoid confusing similar equations with different physical meanings.

Many documents claim to have constructed a unique second-order Dirac hydrogen equation and obtained an ideal solution. The construction of the so-called second-order Dirac equation raises the puzzling question: why claim to construct the second-order Dirac equation but discard the second-order Dirac equation derived from the Dirac equation system? I have treated the induced second-order Dirac differential equation as the original second-order Dirac equation for a long time. In fact, there are some differences between them. From a mathematical point of view, the original Dirac equation and the induced second-order Dirac equation of hydrogen can be selected as two metrics respectively. According to Dongfang unitary principle, the quotient of the wave function divided by the radial independent variable as the exact solution of the induced second-order Dirac equation should be reduced to the exact solution of the Dirac equation. We need to give the exact solution of the induced second-order Dirac equation to confirm this conclusion. For this reason, here we first derive the induced second-order Dirac hydrogen equation.

In the Dirac electron theory, the four-order matrix Dirac equation of the original four-component wave function is applied to the hydrogen atom. After a series of logical processing that cannot be proved by mathematical theory, it is transformed into the second-order radial Dirac equation of the two-component wave function,

$$\left[-i\hbar c\left(\frac{d}{dr}+\frac{1}{r}\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}-\frac{\hbar c\kappa}{r}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+mc^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\left(E+\frac{\alpha\hbar c}{r}\right)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{1}$$

Where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant, m is the static mass of the electron, c is the speed of light in vacuum, $\hbar = h/2\pi$ is the reduced Planck constant, and E is the energy eigenvalue parameter, $\kappa = \pm 1, \ \pm 2, \ \cdots$ is the angular quantum number defined by Dirac's hydrogen atom theory through a set of operation rules independent of mathematics, r is the size of the radial vector and the independent variable in the equation, and ψ_1 and ψ_2 are the two unknown components of the two-component wave function. Expand equation (1)

$$\begin{aligned}&\frac{d\psi_2}{dr}+\frac{1+\kappa}{r}\psi_2-\left(\frac{mc^2-E}{\hbar c}-\frac{\alpha}{r}\right)\psi_1=0\\ &\frac{d\psi_1}{dr}+\frac{1-\kappa}{r}\psi_1-\left(\frac{mc^2+E}{\hbar c}+\frac{\alpha}{r}\right)\psi_2=0\end{aligned} \tag{2}$$

Usually, two new functions are introduced to replace the two components of the wave function,

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}r^{-1}F(r)\\ r^{-1}G(r)\end{pmatrix} \tag{3}$$

Its first derivative is

$$\frac{d\psi_1}{dr}=\frac{1}{r}\frac{dF}{dr}-\frac{1}{r^2}F,\quad \frac{d\psi_2}{dr}=\frac{1}{r}\frac{dG}{dr}-\frac{1}{r^2}G \tag{4}$$

Substitute (3) and (4) for equation (2) to obtain the first-order differential equations[1-3, 5, 6] for$F$ and G,

$$\begin{aligned}&\frac{dG}{dr}+\frac{\kappa}{r}G-\left(\frac{mc^2-E}{\hbar c}-\frac{\alpha}{r}\right)F=0\\ &\frac{dF}{dr}-\frac{\kappa}{r}F-\left(\frac{mc^2+E}{\hbar c}+\frac{\alpha}{r}\right)G=0\end{aligned} \tag{5}$$

This system of equations is not the original wave function components ψ_1 and ψ_2, but the differential equations of the new variables $F(r)$ and $G(r)$. It is called the induced first-order Dirac radial equations of the hydrogen atom. For the convenience of calculation, parameter c_1, c_2, a and dimensionless independent variable parameter ρ are usually introduced. The definition is as follows,

$$c_1=\frac{mc^2+E}{\hbar c},\ c_2=\frac{mc^2-E}{\hbar c},\ a=\sqrt{c_1c_2},\ \rho=ar \tag{6}$$

Substitute (6) into (5), and the two equations become,

$$\begin{aligned}&\frac{dG}{d\rho}+\frac{\kappa}{\rho}G-\left(\frac{c_2}{a}-\frac{\alpha}{\rho}\right)F=0\\ &\frac{dF}{d\rho}-\frac{\kappa}{\rho}F-\left(\frac{c_1}{a}+\frac{\alpha}{\rho}\right)G=0\end{aligned} \tag{7}$$

This is the abbreviation of the induced first-order Dirac radial equations of the hydrogen atom.

3 Looking back on the solution of the first order Dirac equation system

This section re-solves the induced first-order Dirac equations (7) to get rid of the bad habit of relying too much on literature search, citing classics, showing erudition and bypassing the necessary calculations. First find the general solution of equation (7), and then use the definite solution condition to determine the special solution. The usual way is to transplant the definite solution condition of Schrödinger equation to the Dirac equation. The definite solution condition is selected as the natural boundary condition that the wave function is bounded in the whole space, so the specific form of the natural boundary condition of Dirac radial wave function is

$$\lim_{\rho\to 0}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\ \left|\begin{matrix}\psi_1(0<\rho<\infty)\\ \psi_2(0<\rho<\infty)\end{matrix}\right|\neq\left|\begin{matrix}\pm\infty\\ \pm\infty\end{matrix}\right|,\ \lim_{\rho\to\infty}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{8}$$

From this, it can be inferred that the conditions for determining the solution of the induced first-order Dirac radial equations (7) are

$$\lim_{\rho\to 0}\begin{pmatrix}F\\ G\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\ \left|\begin{matrix}F(0<\rho<\infty)\\ G(0<\rho<\infty)\end{matrix}\right|\neq\left|\begin{matrix}\pm\infty\\ \pm\infty\end{matrix}\right|,\ \lim_{\rho\to\infty}\begin{pmatrix}F\\ G\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{9}$$

Referring to the solution of Schrödinger equation of hydrogen atom, take the solution $F\to e^{-\rho}$ and $G\to e^{-\rho}$ of the second-order asymptotic equation $d^2F/d\rho^2-F\to 0$ and $d^2G/d\rho^2-G\to 0$ when $\rho\to\infty$ as the weighting function, and make the solution of equation (7) in the form of,

$$F=e^{-\rho}f(\rho),\quad G=e^{-\rho}g(\rho) \tag{10}$$

Substitute (10) into equation (7) to obtain the first-order differential equations of f and g,

$$\begin{aligned}&\frac{dg}{d\rho}-\left(1-\frac{\kappa}{\rho}\right)g-\left(\frac{c_2}{a}-\frac{\alpha}{\rho}\right)f=0\\ &\frac{df}{d\rho}-\left(1+\frac{\kappa}{\rho}\right)f-\left(\frac{c_1}{a}+\frac{\alpha}{\rho}\right)g=0\end{aligned} \tag{11}$$

Find the series solution of f and g, but the boundary condition (11) requires that both series be interrupted as polynomials,

$$f=\sum_{\nu=0}^{n} b_\nu \rho^{s+\nu}, \quad g=\sum_{\nu=0}^{n} d_\nu \rho^{s+\nu} \tag{12}$$

Substitute (12) into (10) to get,

$$\sum_{\nu=0}^{n}\left[\alpha b_\nu+(s+\nu+\kappa)\, d_\nu-d_{\nu-1}-\frac{c_2}{a} b_{\nu-1}\right]\rho^{s+\nu-1}=0$$
$$\sum_{\nu=0}^{n}\left[(s+\nu-\kappa)\, b_\nu-\alpha d_\nu-\frac{c_1}{a} d_{\nu-1}-b_{\nu-1}\right]\rho^{s+\nu-1}=0$$

Therefore, the coefficients of two polynomials (12) satisfy the recurrence relationship group,

$$\begin{aligned}&\alpha b_\nu+(s+\nu+\kappa)\, d_\nu-\frac{c_2}{a} b_{\nu-1}-d_{\nu-1}=0\\&(s+\nu-\kappa)\, b_\nu-\alpha d_\nu-b_{\nu-1}-\frac{c_1}{a} d_{\nu-1}=0\end{aligned} \tag{13}$$

Let $\nu=0$, notice $b_{-1}=d_{-1}=0$, and get the linear homogeneous equations for b_0 and d_0,

$$\begin{aligned}&\alpha b_0+(s+\kappa)\, d_0=0\\&(s-\kappa)\, b_0-\alpha d_0=0\end{aligned}$$

The necessary and sufficient condition for the system of the above linear homogeneous equations to have nontrivial solutions is that the coefficient determinant is zero, so the index value satisfying the boundary condition (11) is obtained,

$$s=\sqrt{\kappa^2-\alpha^2} \tag{14}$$

Let $\nu=n$ and $\nu=n+1$ respectively. Noting that$b_{n+1}=d_{n+1}=0$, the last four equations are obtained from (13),

$$\begin{aligned}&\alpha b_n+(s+n+\kappa)\, d_n-\frac{c_2}{a} b_{n-1}-d_{n-1}=0\\&(s+n-\kappa)\, b_n-\alpha d_n-b_{n-1}-\frac{c_1}{a} d_{n-1}=0\\&-\frac{c_2}{a} b_n-d_n=0\\&-b_n-\frac{c_1}{a} d_n=0\end{aligned}$$

According to the parameter definition in (6), $a^2=c_1c_2$, the last two equations are linearly related. The first two equations must be equivalent. From these four equations, the eigenvalue equation for determining the quantized energy formula is obtained,

$$a=\frac{\alpha\,(c_1-c_2)}{2\,(s+n)} \tag{15}$$

Where $n=0,1,2,\cdots$ is the radial quantum number, and $\kappa=\pm1,\pm2,\cdots$ is the angular quantum number defined by Dirac's hydrogen atom theory. Substitute the parameter definition in (6), and finally obtain the Dirac energy level formula of hydrogen atom

$$E=\frac{mc^2}{\sqrt{1+\frac{\alpha^2}{\left(n+\sqrt{\kappa^2-\alpha^2}\right)^2}}} \tag{16}$$

The above reproduces the calculation process of the Dirac energy level formula of the hydrogen atom, rather than just citing the existing conclusions of a large number of documents. It is generally considered that doing so in the paper is unnecessary because it is a repetition of the existing work. However, the fact is that only by repeatedly and independently deducing the existing theory can we truly grasp the essence of this theory and discover major issues that have not been noticed in history. The Dirac energy level formula has been rapidly recognized and widely spread because it predicts the fine structure of the hydrogen atom spectrum. Combining the above steps, the exact solution of the induced first-order Dirac radial equations (7) for hydrogen is determined by (10), (12), (13), (14) and (16). If the characteristic quantity of the hydrogen atom is used to represent the defined parameter (6), the energy eigenvalue needs to be substituted into (6), but the expression form of each parameter thus obtained is not concise. It is advisable to retain the energy parameter E and eliminate the parameters c_1, c_2 and a. Here, the exact solution of the first-order Dirac induced by hydrogen is merged as follows,

$$\begin{aligned}&F=e^{-\rho}\sum_{\nu=0}^{n} b_\nu \rho^{\sqrt{\kappa^2-\alpha^2}+\nu}, \quad G=e^{-\rho}\sum_{\nu=0}^{n} d_\nu \rho^{\sqrt{\kappa^2-\alpha^2}+\nu}\\&\alpha b_\nu+\left(\sqrt{\kappa^2-\alpha^2}+\kappa+\nu\right) d_\nu-\sqrt{\frac{mc^2-E}{mc^2+E}}\, b_{\nu-1}-d_{\nu-1}=0\\&\left(\sqrt{\kappa^2-\alpha^2}-\kappa+\nu\right) b_\nu-\alpha d_\nu-b_{\nu-1}-\sqrt{\frac{mc^2+E}{mc^2-E}}\, d_{\nu-1}=0\\&E=\frac{mc^2}{\sqrt{1+\frac{\alpha^2}{\left(n+\sqrt{\kappa^2-\alpha^2}\right)^2}}}, \quad (n=0,1,\cdots,\ \kappa=\pm1,\pm2,\cdots)\end{aligned} \tag{17}$$

4 Induced second-order Dirac hydrogen equation

The Dirac equation has great influence, which has led to the emergence of many second-order Dirac equations. We have ended the transformation Dirac equation that decomposes the wave function, and the corresponding transformation second order Dirac equation is also naturally ended. It is puzzling that for nearly 100 years, the second order differential equation transformed from the first order differential equation system has been avoided. There may be some reasons behind this.

Now consider the second-order Dirac equation transformed by the induced first-order Dirac equation system (7) for the hydrogen. The expressions for F and G are obtained from the first and second equations of (7) respectively,

$$\begin{aligned}F&=\frac{a\rho}{c_2\rho-\alpha a}\frac{dG}{dr}+\frac{\kappa a}{c_2\rho-\alpha a}G\\G&=\frac{a\rho}{c_1\rho+\alpha a}\frac{dF}{d\rho}-\frac{\kappa a}{c_1\rho+\alpha a}F\end{aligned} \tag{18}$$

Their first derivatives are,

$$
\begin{aligned}
\frac{dF}{d\rho} &= \frac{a\rho}{c_2\rho-\alpha a}\frac{d^2G}{d\rho^2}+\left(\frac{d}{d\rho}\frac{a\rho}{c_2\rho-\alpha a}+\frac{\kappa a}{c_2\rho-\alpha a}\right)\frac{dG}{d\rho}+\left(\frac{d}{d\rho}\frac{\kappa a}{c_2\rho-\alpha a}\right)G\\
&= \frac{a\rho}{c_2\rho-\alpha a}\frac{d^2G}{d\rho^2}+\frac{\left[\kappa c_2\rho-\alpha\left(1+\kappa\right)a\right]a}{\left(\rho c_2-\alpha a\right)^2}\frac{dG}{d\rho}-\frac{\kappa c_2 a}{\left(\rho c_2-a\alpha\right)^2}G\\
\frac{dG}{d\rho} &= \frac{a\rho}{c_1\rho+\alpha a}\frac{d^2F}{d\rho^2}+\left(\frac{d}{d\rho}\frac{a\rho}{c_1\rho+\alpha a}-\frac{\kappa a}{c_1\rho+\alpha a}\right)\frac{dF}{d\rho}-\left(\frac{d}{d\rho}\frac{\kappa a}{c_1\rho+\alpha a}\right)F\\
&= \frac{a\rho}{c_1\rho+\alpha a}\frac{d^2F}{d\rho^2}-\frac{\left[\kappa c_1\rho-\alpha\left(1-\kappa\right)a\right]a}{\left(a\alpha+c_1\rho\right)^2}\frac{dF}{d\rho}+\frac{\kappa c_1 a}{\left(c_1\rho+\alpha a\right)^2}F
\end{aligned}
\tag{19}
$$

Use (18) and (19) to eliminate the term containing G in (7) the first equation and the term containing F in the second equation, one gets,

$$
\begin{aligned}
&\frac{a\rho}{a\alpha+\rho c_1}\frac{d^2F}{d\rho^2}+\frac{a^2\alpha}{\left(a\alpha+\rho c_1\right)^2}\frac{dF}{d\rho}+\left(\frac{\alpha}{\rho}+\frac{a\kappa c_1}{\left(a\alpha+\rho c_1\right)^2}-\frac{a\kappa^2}{a\alpha\rho+\rho^2c_1}-\frac{c_2}{a}\right)F=0\\
&\frac{a\rho}{c_2\rho-\alpha a}\frac{d^2G}{d\rho^2}-\frac{a^2\alpha}{\left(a\alpha-\rho c_2\right)^2}\frac{dG}{d\rho}+\left(-\frac{\alpha}{\rho}-\frac{c_1}{a}-\frac{a\kappa c_2}{\left(a\alpha-\rho c_2\right)^2}+\frac{a\kappa^2}{a\alpha\rho-\rho^2c_2}\right)G=0
\end{aligned}
\tag{20}
$$

This is a system of second-order differential equations constrained by the energy parameter E written in c_1, c_2 and A. Multiply the first equation of equation (20) with $\left(a\alpha+\rho c_1\right)^2$ and the second equation of equation (20) with $\left(a\alpha-\rho c_2\right)^2$, and get the following form

$$
\begin{aligned}
&\left(\alpha\rho^2+\frac{c_1}{a}\rho^3\right)\frac{d^2F}{d\rho^2}+\alpha\rho\frac{dF}{d\rho}+\left[\frac{\alpha^3-\alpha\kappa^2}{\rho}\rho-\frac{\left(\kappa^2-\kappa-2\alpha^2\right)c_1+\alpha^2c_2}{a}\rho+\left(\frac{\alpha c_1^2}{a^2}-\frac{2\alpha c_1c_2}{a^2}\right)\rho^2-\frac{c_1^2c_2}{a^3}\rho^3\right]F=0\\
&\left(\alpha\rho^2-\frac{c_2}{a}\rho^3\right)\frac{d^2G}{d\rho^2}+\alpha\rho\frac{dG}{d\rho}+\left[\frac{\alpha^3-\alpha\kappa^2}{\rho}\rho+\frac{\alpha^2c_1+\left(\kappa^2+\kappa-2\alpha^2\right)c_2}{a}\rho-\left(\frac{2\alpha c_1c_2}{a^2}-\frac{\alpha c_2^2}{a^2}\right)\rho^2+\frac{c_1c_2^2}{a^3}\rho^3\right]G=0
\end{aligned}
\tag{21}
$$

Then use the relation $a=\sqrt{c_1c_2}$ of the parameters defined in formula (6) to simplify, and obtain the second-order induced Dirac equation of the hydrogen atom:

$$
\begin{aligned}
&\left(\alpha\rho^2+\frac{c_1}{a}\rho^3\right)\frac{d^2F}{d\rho^2}+\alpha\rho\frac{dF}{d\rho}+\left[\alpha\left(\alpha^2-\kappa^2\right)+\frac{\left(\kappa-\kappa^2+2\alpha^2\right)c_1-\alpha^2c_2}{a}\rho+\alpha\left(\frac{c_1^2}{a^2}-2\right)\rho^2-\frac{c_1}{a}\rho^3\right]F=0\\
&\left(\alpha\rho^2-\frac{c_2}{a}\rho^3\right)\frac{d^2G}{d\rho^2}+\alpha\rho\frac{dG}{d\rho}+\left[\alpha\left(\alpha^2-\kappa^2\right)+\frac{\left(\kappa^2+\kappa-2\alpha^2\right)c_2+\alpha^2c_1}{a}\rho+\alpha\left(\frac{c_2^2}{a^2}-2\right)\rho^2+\frac{c_2}{a}\rho^3\right]G=0
\end{aligned}
\tag{22}
$$

5 The S state Dirac wave function satisfies the induced second-order Dirac equation

The induced second order Dirac equation (22) of the hydrogen atom is transformed from the induced first order Dirac equation (7) of the hydrogen atom. The solution of the first order equation (23) should satisfy the second order equation (22). However, it is too troublesome to determine the coefficients of the series one by one to give the specific wave function. This may be the main reason why Dirac quantum theory and all the literature on the Dirac equation only focus on the quantized energy formula and avoid the wave function.

Here we write the specific form of the so-called S state wave function when the series is interrupted to the radial quantum number $n=0$, and explain that the solution of the induced first-order Dirac equation system satisfies the induced second-order Dirac equation. When $n=0$, the simplest form of Dirac energy level formula of S state is obtained from energy level formula (16),

$$
E_0=mc^2\sqrt{1-\frac{\alpha^2}{\kappa^2}}
\tag{23}
$$

Substituting (23) into (6) gives the defined parameters. Take $n=0$, substitute (14) into (12) and then into (10), and the specific forms of the components of the two wave functions are only different coefficients. The results are as following,

$$
\begin{aligned}
&\frac{c_{01}}{a_0}=\frac{\kappa}{\alpha}\left(1+\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right), \quad \frac{c_{02}}{a_0}=\frac{\kappa}{\alpha}\left(1-\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right)\\
&\frac{c_{01}^2}{a_0^2}=\frac{\alpha^2}{\kappa^2}\left(1+\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right)^2, \quad \frac{c_{02}^2}{a_0^2}=\frac{\kappa^2}{\alpha^2}\left(\sqrt{1-\frac{\alpha^2}{\kappa^2}}-1\right)^2\\
&F_0=b_0\rho^{\sqrt{\kappa^2-\alpha^2}}e^{-\rho}, \quad G_0=d_0\rho^{\sqrt{\kappa^2-\alpha^2}}e^{-\rho}
\end{aligned}
\tag{24}
$$

The S state solution of the first order induced Dirac equation (24) must satisfy the S state form of the second order induced Dirac equation

$$
\begin{aligned}
&\left(\alpha\rho^2+\frac{c_{01}}{a_0}\rho^3\right)\frac{d^2F_0}{d\rho^2}+\alpha\rho\frac{dF_0}{d\rho}+\left[\alpha^3-\alpha\kappa^2-\frac{\left(\kappa^2-\kappa-2\alpha^2\right)c_{01}+\alpha^2c_{02}}{a_0}\rho+\left(\frac{\alpha c_{01}^2}{a_0^2}-2\alpha\right)\rho^2-\frac{c_{01}}{a_0}\rho^3\right]F_0=0\\
&\left(\alpha\rho^2-\frac{c_{02}}{a_0}\rho^3\right)\frac{d^2G_0}{d\rho^2}+\alpha\rho\frac{dG_0}{d\rho}+\left[\alpha^3-\alpha\kappa^2+\frac{\alpha^2c_{01}+\left(\kappa^2+\kappa-2\alpha^2\right)c_{02}}{a_0}\rho-\left(2\alpha-\frac{\alpha c_{02}^2}{a_0^2}\right)\rho^2+\frac{c_{02}}{a_0}\rho^3\right]G_0=0
\end{aligned}
\tag{25}
$$

Calculate the derivatives of the S state wave function in (24), and the results are listed as follows

$$
\begin{aligned}
&\frac{dF_0}{d\rho}=e^{-\rho}\left(\sqrt{\kappa^2-\alpha^2}-\rho\right)\rho^{\sqrt{\kappa^2-\alpha^2}-1}b_0\\
&\frac{dG_0}{d\rho}=e^{-\rho}\left(\sqrt{\kappa^2-\alpha^2}-\rho\right)\rho^{\sqrt{\kappa^2-\alpha^2}-1}d_0\\
&\frac{d^2F_0}{d\rho^2}=e^{-\rho}\rho^{\sqrt{\kappa^2-\alpha^2}-2}\left(\kappa^2-\alpha^2-\sqrt{\kappa^2-\alpha^2}-2\sqrt{\kappa^2-\alpha^2}\rho+\rho^2\right)b_0\\
&\frac{d^2G_0}{d\rho^2}=e^{-\rho}\rho^{\sqrt{\kappa^2-\alpha^2}-2}\left(\kappa^2-\alpha^2-\sqrt{\kappa^2-\alpha^2}-2\sqrt{\kappa^2-\alpha^2}\rho+\rho^2\right)d_0
\end{aligned}
\tag{26}
$$

Substitute (24) and (26) into the left side of the first equation and the second equation of (25) respectively, and get

$$
\begin{aligned}
&\left(\alpha\rho^2+\frac{c_{01}}{a_0}\rho^3\right)\frac{d^2F_0}{d\rho^2}+\alpha\rho\frac{dF_0}{d\rho}+\left[\alpha^3-\alpha\kappa^2-\frac{\left(\kappa^2-\kappa-2\alpha^2\right)c_{01}+\alpha^2c_{02}}{a_0}\rho+\left(\frac{\alpha c_{01}^2}{a_0^2}-2\alpha\right)\rho^2-\frac{c_{01}}{a_0}\rho^3\right]F_0\\
&=\left\{\begin{array}{l}
\left(\alpha\rho^2+\frac{\kappa}{\alpha}\left(1+\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right)\rho^3\right)e^{-\rho}\rho^{\sqrt{\kappa^2-\alpha^2}-2}\begin{pmatrix}\kappa^2-\alpha^2-\sqrt{\kappa^2-\alpha^2}\\-2\sqrt{\kappa^2-\alpha^2}\rho+\rho^2\end{pmatrix}+\alpha\rho e^{-\rho}\left(\sqrt{\kappa^2-\alpha^2}-\rho\right)\rho^{\sqrt{\kappa^2-\alpha^2}-1}\\
+\left[\begin{array}{l}\alpha^3-\alpha\kappa^2-\left(\left(\kappa^2-\kappa-2\alpha^2\right)\frac{mc}{\hbar}\left(1+\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right)+\alpha^2\left(1-\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right)\right)\rho\frac{\kappa}{\alpha}\\
+\left(\alpha\frac{\kappa^2}{\alpha^2}\left(1+\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right)^2-2\alpha\right)\rho^2-\frac{\kappa}{\alpha}\left(1+\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right)\rho^3\end{array}\right]\rho^{\sqrt{\kappa^2-\alpha^2}}e^{-\rho}
\end{array}\right\}b_0\\
&=-b_0e^{-\rho}\rho^{1+\sqrt{\kappa^2-\alpha^2}}\left(\left(1+\sqrt{1-\frac{\alpha^2}{\kappa^2}}\right)\frac{\kappa}{\alpha}\left(\sqrt{\kappa^2-\alpha^2}-\kappa\right)(1+2\rho)+\alpha\left(1-2\sqrt{1-\frac{\alpha^2}{\kappa^2}}\kappa+2\sqrt{\kappa^2-\alpha^2}+2\rho\right)\right)\\
&=-b_0e^{-\rho}\rho^{1+\sqrt{\kappa^2-\alpha^2}}\left(\left(\kappa+\sqrt{\kappa^2-\alpha^2}\right)\frac{\sqrt{\kappa^2-\alpha^2}-\kappa}{\alpha}(1+2\rho)+\alpha\left(1-2\sqrt{\kappa^2-\alpha^2}+2\sqrt{\kappa^2-\alpha^2}+2\rho\right)\right)\\
&=0
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\left(\alpha\rho^2-\frac{c_{02}}{a_0}\rho^3\right)\frac{d^2G_0}{d\rho^2}+\alpha\rho\frac{dG_0}{d\rho}+\left[\alpha^3-\alpha\kappa^2+\frac{\alpha^2c_{01}+\left(\kappa^2+\kappa-2\alpha^2\right)c_{02}}{a_0}\rho-\left(2\alpha-\frac{\alpha c_{02}^2}{a_0^2}\right)\rho^2+\frac{c_{02}}{a_0}\rho^3\right]G_0\\
&=\left\{\begin{array}{l}\left(\alpha\rho^2-\dfrac{\kappa}{\alpha}\left(1-\sqrt{1-\dfrac{\alpha^2}{\kappa^2}}\right)\rho^3\right)e^{-\rho}\rho^{\sqrt{\kappa^2-\alpha^2}-2}\begin{pmatrix}\kappa^2-\alpha^2-\sqrt{\kappa^2-\alpha^2}\\-2\sqrt{\kappa^2-\alpha^2}\rho+\rho^2\end{pmatrix}+\alpha\rho e^{-\rho}\left(\sqrt{\kappa^2-\alpha^2}-\rho\right)\rho^{\sqrt{\kappa^2-\alpha^2}-1}\\
+\left[\begin{array}{l}\alpha^3-\alpha\kappa^2+\left(\alpha^2\dfrac{mc}{\hbar}\left(1+\sqrt{1-\dfrac{\alpha^2}{\kappa^2}}\right)+\left(\kappa^2+\kappa-2\alpha^2\right)\left(1-\sqrt{1-\dfrac{\alpha^2}{\kappa^2}}\right)\right)\dfrac{\kappa}{\alpha}\rho\\
-\left(2\alpha-\alpha\left(\sqrt{1-\dfrac{\alpha^2}{\kappa^2}}-1\right)^2\dfrac{\kappa^2}{\alpha^2}\right)\rho^2+\left(1-\sqrt{1-\dfrac{\alpha^2}{\kappa^2}}\right)\dfrac{\kappa}{\alpha}\rho^3\end{array}\right]\rho^{\sqrt{\kappa^2-\alpha^2}}e^{-\rho}\end{array}\right\}d_0\\
&=-d_0e^{-\rho}\rho^{1+\sqrt{\kappa^2-\alpha^2}}\left(\left(\sqrt{1-\frac{\alpha^2}{\kappa^2}}-1\right)\frac{\kappa}{\alpha}\left(\kappa+\sqrt{\kappa^2-\alpha^2}\right)(1+2\rho)+\alpha\left(1-2\sqrt{1-\frac{\alpha^2}{\kappa^2}}\kappa+2\sqrt{\kappa^2-\alpha^2}+2\rho\right)\right)\\
&=-d_0e^{-\rho}\rho^{1+\sqrt{\kappa^2-\alpha^2}}\left(\left(\sqrt{\kappa^2-\alpha^2}-\kappa\right)\frac{\kappa+\sqrt{\kappa^2-\alpha^2}}{\alpha}(1+2\rho)+\alpha\left(1-2\sqrt{\kappa^2-\alpha^2}+2\sqrt{\kappa^2-\alpha^2}+2\rho\right)\right)\\
&=0
\end{aligned}
\tag{28}
$$

The above calculation process is reserved to facilitate the reader to test the mathematical logic. Scientists should often calculate famous theories independently, rather than go out of the mode that has always relied on a large number of interpretations based on memory and replication. Although the traditional mode can make individuals quickly succeed, imperfect and even erroneous theories continue to continue, eventually making some scientific theories develop into religions.

When the radial quantum number $n=1$, the exact solution of the induced first-order Dirac equation system (7) also satisfies the second-order induced Dirac equation (22), but the calculation process of the test is lengthy and cumbersome, which is omitted here. The larger the radial quantum number, the more cumbersome the calculation. I have never tested whether the exact solution of the first-order Dirac equation system (7) satisfies the second-order induced Dirac equation (22) when $n=2,\ 3,\ 4,\ \cdots$, but I am sure that there will be no negative conclusion. Readers, especially physicists who rely too much on reciting conclusions and the qualitative logic of modern physics and rarely calculate independently, can try to verify. Every time such verification is completed, there will be new gains.

6 Dongfang solution of induced second-order Dirac hydrogen equation

If only the energy eigenvalues is concerned, it is possible to treat the formal solution of the wave equation that cannot be tested as the true solution, and the formal solution may be the false solution of the equation. The coefficients of equation (22) are all polynomials, and the results obtained by using the approximate solution to construct the weighted function are consistent with those obtained by using the generalized optimal differential equation theorem[50, 51]. The former is familiar to us. When $\rho\to\infty$, the approximate second order equation derived from the approximate equations of the two equations in (22) is the same, $d^2F\big/d\rho^2-F\to 0,\quad d^2G\big/d\rho^2-G\to 0$. The solution of the approximate equation satisfying the boundary condition is $F\to e^{-\rho},\quad G\to e^{-\rho}$. It is inferred that the solution of equation (22) has the same form as that of equation (10),

$$F=e^{-\rho}f(\rho),\quad G=e^{-\rho}g(\rho)\tag{29}$$

Their first and second derivative is respectively,

$$
\begin{aligned}
\frac{dF}{d\rho}&=e^{-\rho}\left(\frac{df}{d\rho}-f\right),\quad &\frac{d^2F}{d\rho^2}&=e^{-\rho}\left(\frac{d^2f}{d\rho^2}-2\frac{df}{d\rho}+f\right)\\
\frac{dG}{d\rho}&=e^{-\rho}\left(\frac{dg}{d\rho}-g\right),\quad &\frac{d^2G}{d\rho^2}&=e^{-\rho}\left(\frac{d^2g}{d\rho^2}-2\frac{dg}{d\rho}+g\right)
\end{aligned}
$$

Substitute them into equation (22) to obtain the second order differential equation of f and g,

$$\begin{aligned}
&\left(\alpha\rho^2+\frac{c_1}{a}\rho^3\right)e^{-\rho}\left(\frac{d^2f}{d\rho^2}-2\frac{df}{d\rho}+f\right)+\alpha\rho e^{-\rho}\left(\frac{df}{d\rho}-f\right)\\
&+\left\{\alpha\left(\alpha^2-\kappa^2\right)+\left[\left(\kappa-\kappa^2+2\alpha^2\right)\frac{c_1}{a}-\alpha^2\frac{c_2}{a}\right]\rho+\alpha\left(\frac{c_1^2}{a^2}-2\right)\rho^2-\frac{c_1}{a}\rho^3\right\}e^{-\rho}f=0\\
&\left(\alpha\rho^2-\frac{c_2}{a}\rho^3\right)e^{-\rho}\left(\frac{d^2g}{d\rho^2}-2\frac{dg}{d\rho}+g\right)+\alpha\rho e^{-\rho}\left(\frac{dg}{d\rho}-g\right)\\
&+\left\{\alpha\left(\alpha^2-\kappa^2\right)+\left[\left(\kappa^2+\kappa-2\alpha^2\right)\frac{c_2}{a}+\alpha^2\frac{c_1}{a}\right]\rho+\alpha\left(\frac{c_2^2}{a^2}-2\right)\rho^2+\frac{c_2}{a}\rho^3\right\}e^{-\rho}g=0
\end{aligned}\tag{30}$$

By combining similar terms, the above equations are simplified as following,

$$\begin{aligned}
&\left(\alpha a^2+ac_1\rho\right)\frac{d^2f}{d\rho^2}+\left(\frac{\alpha a^2}{\rho}-2\alpha a^2-2ac_1\rho\right)\frac{df}{d\rho}\\
&+\left\{\frac{\alpha\left(\alpha^2-\kappa^2\right)a^2}{\rho^2}-\frac{\alpha a^2-\left(2\alpha^2+\kappa-\kappa^2\right)ac_1+\alpha^2ac_2}{\rho}+\alpha\left(c_1^2-a^2\right)\right\}f=0\\
&\left(\alpha a^2-ac_2\rho\right)\frac{d^2g}{d\rho^2}+\left(\frac{\alpha a^2}{\rho}-2\alpha a^2+2ac_2\rho\right)\frac{dg}{d\rho}\\
&+\left\{\frac{\alpha\left(\alpha^2-\kappa^2\right)a^2}{\rho^2}-\frac{a\left[\alpha a+\left(2\alpha^2-\kappa-\kappa^2\right)c_2-\alpha^2c_1\right]}{\rho}+\alpha\left(c_2^2-a^2\right)\right\}g=0
\end{aligned}\tag{31}$$

It should be noted that the two equations of equation (31) are constrained by the same energy parameter, so they are not completely independent of each other, but are implicit second-order differential equations. Let the form of the interrupted series solution of the two function components $F=e^{-\rho}f(\rho)$ and $G=e^{-\rho}g(\rho)$ satisfying the boundary condition (9) to be,

$$F=e^{-\rho}\sum_{\nu=0}^{n}b_\nu\rho^{\sqrt{\kappa^2-\alpha^2}+\nu},\quad G=e^{-\rho}\sum_{\nu=0}^{n}d_\nu\rho^{\sqrt{\kappa^2-\alpha^2}+\nu}\tag{32}$$

Substitute it into equation (31) to get,

$$\begin{aligned}
&\left(\alpha+\frac{c_1}{a}\rho\right)\sum_{\nu=0}^{n}(s+\nu)(s+\nu-1)b_\nu\rho^{s+\nu-2}+\left(\frac{\alpha}{\rho}-2\alpha-2\frac{c_1}{a}\rho\right)\sum_{\nu=0}^{n}(s+\nu)b_\nu\rho^{s+\nu-1}\\
&+\left\{\frac{\alpha\left(\alpha^2-\kappa^2\right)}{\rho^2}-\frac{1}{\rho}\left[\alpha-\left(2\alpha^2+\kappa-\kappa^2\right)\frac{c_1}{a}+\alpha^2\frac{c_2}{a}\right]+\alpha\left(\frac{c_1^2}{a^2}-1\right)\right\}\sum_{\nu=0}^{n}b_\nu\rho^{s+\nu}=0\\
&\left(\alpha-\frac{c_2}{a}\rho\right)\sum_{\nu=0}^{n}(s+\nu)(s+\nu-1)d_\nu\rho^{s+\nu-2}+\left(\frac{\alpha}{\rho}-2\alpha+2\frac{c_2}{a}\rho\right)\sum_{\nu=0}^{n}(s+\nu)d_\nu\rho^{s+\nu-1}\\
&+\left\{\frac{\alpha\left(\alpha^2-\kappa^2\right)}{\rho^2}-\frac{1}{\rho}\left[\alpha+\left(2\alpha^2-\kappa-\kappa^2\right)\frac{c_2}{a}-\alpha^2\frac{c_1}{a}\right]-\alpha\left(1-\frac{c_2^2}{a^2}\right)\right\}\sum_{\nu=0}^{n}d_\nu\rho^{s+\nu}=0
\end{aligned}\tag{33}$$

Combine similar terms to obtain the recurrence relationship satisfied by the coefficients of the two series,

$$\begin{aligned}
&\sum_{\nu=0}^{n}\left\{\begin{aligned}&\alpha\left[(s+\nu)^2+\left(\alpha^2-\kappa^2\right)\right]b_\nu-\left[2(s+\nu-2)\frac{c_1}{a}-\alpha\left(\frac{c_1^2}{a^2}-1\right)\right]b_{\nu-2}\\&+\left[\left(\left(2\alpha^2+\kappa-\kappa^2\right)+(s+\nu-2)(s+\nu-1)\right)\frac{c_1}{a}-\alpha^2\frac{c_2}{a}-\alpha(2s+2\nu-1)\right]b_{\nu-1}\end{aligned}\right\}\rho^{s+\nu-2}=0\\
&\sum_{\nu=0}^{n}\left\{\begin{aligned}&\alpha\left[(s+\nu)^2+\left(\alpha^2-\kappa^2\right)\right]a^2d_\nu+\left[2(s+\nu-2)\frac{c_2}{a}-\alpha\left(1-\frac{c_2^2}{a^2}\right)\right]d_{\nu-2}\\&-\left[\left(\left(2\alpha^2-\kappa-\kappa^2\right)+(s+\nu-2)(s+\nu-1)\right)\frac{c_2}{a}-\alpha^2c_1+\alpha(2s+2\nu-1)\right]d_{\nu-1}\end{aligned}\right\}\rho^{s+\nu-2}=0
\end{aligned}\tag{34}$$

Let $\nu=0$, note that $b_{-1}=b_{-2}=\cdots=0$, $d_{-1}=d_{-2}=\cdots=0$, and $b_0\neq 0$, $d_0\neq 0$. The above two recurrence relations give the same index equation,

$$\alpha\left[s^2+\left(\alpha^2-\kappa^2\right)\right]a^2=0$$

Solve this equation, eliminate the negative roots that do not meet the boundary conditions, and take the positive root. The result is consistent with (14),

$$s=\sqrt{\kappa^2-\alpha^2} \tag{35}$$

Substitute this result into (34), and the recurrence relationship group satisfied by the coefficients of the two series is specifically expressed as,

$$\begin{aligned}
&\left\{\begin{aligned}&\alpha\left(2\nu\sqrt{\kappa^2-\alpha^2}+\nu^2\right)b_\nu-\left[2\left(\sqrt{\kappa^2-\alpha^2}+\nu-2\right)\frac{c_1}{a}-\alpha\left(\frac{c_1^2}{a^2}-1\right)\right]b_{\nu-2}\\&+\left[\left(2+\alpha^2+\kappa-3\nu+(2\nu-3)\sqrt{\kappa^2-\alpha^2}+\nu^2\right)\frac{c_1}{a}-\alpha^2\frac{c_2}{a}-\alpha\left(2\nu-1+2\sqrt{\kappa^2-\alpha^2}\right)\right]b_{\nu-1}\end{aligned}\right\}=0\\
&\left\{\begin{aligned}&\alpha\left[2\nu\sqrt{\kappa^2-\alpha^2}+\nu^2\right]d_\nu+\left[2\left(\sqrt{\kappa^2-\alpha^2}+\nu-2\right)\frac{c_2}{a}-\alpha\left(1-\frac{c_2^2}{a^2}\right)\right]d_{\nu-2}\\&-\left[\left(2+\alpha^2-\kappa-3\nu+(2\nu-3)\sqrt{\kappa^2-\alpha^2}+\nu^2\right)\frac{c_2}{a}-\alpha^2\frac{c_1}{a}+\alpha\left(2\sqrt{\kappa^2-\alpha^2}+2\nu-1\right)\right]d_{\nu-1}\end{aligned}\right\}=0
\end{aligned} \tag{36}$$

In (36), let $\nu=n+2$, note that the expected solution requires $p_{n+1}=p_{n+2}=\cdots=0$, $q_{n+1}=q_{n+2}=\cdots=0$, and $p_n\neq 0$, $q_n\neq 0$, so two eigenvalue equations are obtained, which should be equivalent because they describe the same energy parameter,

$$\begin{aligned}\left[2\left(s+n\right)ac_1-\alpha\left(c_1^2-a^2\right)\right]b_n=0\\ \left[2\left(s+n\right)ac_2-\alpha\left(a^2-c_2^2\right)\right]d_n=0\end{aligned}$$

Restore with defined parameters (6),

$$\begin{aligned}&\frac{2\left(s+n\right)\sqrt{m^2c^4-E^2}}{\hbar c}\frac{mc^2+E}{\hbar c}-\alpha\left[\left(\frac{mc^2+E}{\hbar c}\right)^2-\frac{\left(mc^2-E\right)\left(mc^2+E\right)}{\hbar^2c^2}\right]=0\\&\frac{2\left(s+n\right)\sqrt{m^2c^4-E^2}}{\hbar c}\frac{mc^2-E}{\hbar c}-\alpha\left[\frac{\left(mc^2-E\right)\left(mc^2+E\right)}{\hbar^2c^2}-\left(\frac{mc^2-E}{\hbar c}\right)^2\right]=0\end{aligned}$$

After simplification, the two eigenvalue equations are the same, namely,

$$\left(s+n\right)\sqrt{m^2c^4-E^2}-\alpha E=0$$

Solve this equation and use (35) or (14) to obtain the Dirac energy level formula shown in (16),

$$E=\frac{mc^2}{\sqrt{1+\frac{\alpha^2}{\left(n+\sqrt{\kappa^2-\alpha^2}\right)^2}}} \tag{37}$$

Where $n=0,1,2,\cdots$ is the radial quantum number, and $\kappa=\pm1,\pm2,\cdots$ is the angular quantum number defined by Dirac's hydrogen atom theory.

The induced second-order Dirac equation (22) of hydrogen atom is actually a set of second-order equations constrained by the same energy parameter, and its eigensolutions are determined by (29), (32), (35), (36) and (37). The exact solutions of the induced second-order Dirac equations for the hydrogen with the retention of energy parameter

E and elimination of parameters c_1, c_2 and a are summarized as follows

$$
\begin{aligned}
&F=e^{-\rho}\sum_{\nu=0}^{n} b_\nu \rho^{\sqrt{\kappa^2-\alpha^2}+\nu}, \quad G=e^{-\rho}\sum_{\nu=0}^{n} d_\nu \rho^{\sqrt{\kappa^2-\alpha^2}+\nu}\\
&\left\{\begin{aligned}&\alpha\left(2\nu\sqrt{\kappa^2-\alpha^2}+\nu^2\right)b_\nu-\left[2\left(\sqrt{\kappa^2-\alpha^2}+\nu-2\right)\sqrt{\frac{mc^2+E}{mc^2-E}}-\frac{2\alpha E}{mc^2-E}\right]b_{\nu-2}\\
&+\left[\left(2+\alpha^2+\kappa-3\nu+(2\nu-3)\sqrt{\kappa^2-\alpha^2}+\nu^2\right)\sqrt{\frac{mc^2+E}{mc^2-E}}-\alpha^2\sqrt{\frac{mc^2-E}{mc^2+E}}-\alpha\left(2\nu-1+2\sqrt{\kappa^2-\alpha^2}\right)\right]b_{\nu-1}\end{aligned}\right\}=0\\
&\left\{\begin{aligned}&\alpha\left[2\nu\sqrt{\kappa^2-\alpha^2}+\nu^2\right]d_\nu+\left[2\left(\sqrt{\kappa^2-\alpha^2}+\nu-2\right)\sqrt{\frac{mc^2-E}{mc^2+E}}-\frac{2\alpha E}{mc^2+E}\right]d_{\nu-2}\\
&-\left[\left(2+\alpha^2-\kappa-3\nu+(2\nu-3)\sqrt{\kappa^2-\alpha^2}+\nu^2\right)\sqrt{\frac{mc^2-E}{mc^2+E}}-\alpha^2\sqrt{\frac{mc^2+E}{mc^2-E}}+\alpha\left(2\sqrt{\kappa^2-\alpha^2}+2\nu-1\right)\right]d_{\nu-1}\end{aligned}\right\}=0\\
&E=\frac{mc^2}{\sqrt{1+\frac{\alpha^2}{\left(n+\sqrt{\kappa^2-\alpha^2}\right)^2}}}, \quad (n=0,1,\cdots,\ \kappa=\pm1,\pm2,\cdots)
\end{aligned}
\tag{38}
$$

This is an expression group, named Dongfang solution of the induced second-order Dirac equation of the hydrogen atom.

How to name the research results is a topic beyond the topic. Historically, after independent research results of professional researchers are published, their students or friends or followers may name them in the newly published promotion articles. I have made a lot of breakthroughs. In the past 40 years, I have continuously submitted contributions to famous journals at home and abroad, but few journals have published these papers. Later, when I checked the literature, I found that other authors later published the same results in influential journals as in the manuscript. The usual saying is that many people around the world are studying the same breakthrough topic, and others have published it, but you have not published it. However, the physical and mathematical logic of the articles that published the same results violated the unitary principle. Over the past few decades, many manuscripts submitted to famous academic journals such as Nature and Physical Review have been stifled. This long experience makes it impossible for me to continue to listen to the so-called academic ethics of mainstream academic journals. When I propagandize breakthrough discoveries, I often get slandered and abused. Some mainstream scholars even joined forces to write lengthy libel letters with online signatures and send them to the departments where I work, harassing my work and threatening my life. Every researcher hopes that his hard research achievements of decades will be recognized. Since no prestigious academic journal has accepted any of my breakthrough research achievements for decades, how to name my research achievements now is naturally not bound by the hidden rules of mainstream academia. Of course, attackers can also name each research result according to their wishes.

The spread of truth, of course, depends on the attention, reproduction, interpretation and development of theories by many researchers. However, it is a virtue to respect the work of the original creator. A real researcher, a researcher who loves the truth, and a researcher who really adheres to the moral code will not concentrate on blaming the original author for naming the research results. When reading the expression group (38), his attention will be attracted to the question of whether the exact solution (17) of the first order induced Dirac equation group (7) of the hydrogen atom is equivalent to the exact solution of the second order induced Dirac equation group (22), thus meeting the Dongfang normalization principle. There are great differences between them in form. Section 5 tests the consistency of the two when $n=0$ and introduces the positive results when $n=1$. So, what is the process and result of the test for any radial quantum number $n \geqslant 2$?

7 Conclusions and comments

The Dirac equation has great mathematical charm, resulting in the generation of many second-order Dirac equations. But some are unreasonable, and some are even wrong. It takes too much time and energy to test various theories one by one. The teratogenic simplified Dirac equation has been ended, including the corresponding second-order Dirac equation[11-15]. Here, the induced second-order Dirac equation of the hydrogen atom is derived, and the solution of the induced second-order Dirac equation is given, which is a neutral result. The isomorphic second-order Dirac equation and the isomeric second-order Dirac equation will be discussed later, and then the first-order Dirac equation will be reprocessed from a new perspective. We are opening a bright window to let the world gradually see the right and wrong of the whole Dirac quantum theory and its derivative theory.

It is not easy to test the exact solution (17) of the first order Dirac equation system (7) or the exact solution of the second order induced Dirac equation system (22). It is complex enough to just test whether the solution of the induced first order Dirac equation system satisfies the two induced second order differential equations. For each kind of Dirac equation, it is very important to distinguish those subtle differences that are imperceptible. Although some differences are subtle and even difficult to find, they may contain principled problems. In fact, they cause the theory to deviate too far from the natural law and have to distort the logical reorganization to make the inference conform to the expectation. For example, construct the so-called complete conservative exchange observable value[7]. There are essential differences in the theories of various named Dirac hydrogen equations, but all claim to obtain the same expected solution. Have we ever thought about studying the causal relationship between them? On the surface, completely different wave equations of the same physical model can have the same energy eigenvalue set. This seems to partially conform to Dongfang unitary principle. However, conforming to the unitary principle is only a necessary condition, not a sufficient condition, for the theory to hold.

The test of the unitary principle of the induced second-order Dirac equation has two meanings: 1) The two second-order differential equations form a system of equations subject to the constraints of the common energy parameters, so they are not completely independent. The calculation results of the energy eigensolutions are consistent with the unitary principle, but this does not mean that the problem is over; 2) After the energy eigensolutions of the two second-order differential equations are proved to be the same, the wave functions as the solutions of the respective equations show independence. They can have independent normalization coefficients. Compared with the two-component wave functions of the original Dirac equation, the conclusion does not conform to the unitary principle, which poses a challenge to the physical meaning of the multi-component wave functions of Dirac. The meaning of Dirac multi-component wave function is not clear. The traditional normalized definition of wave function is only a guess that has not been strictly proved. Is the Bonn statistical interpretation of wave function unique and reasonable? Finding the ultimate answer to such questions will promote the great changes in physical theory and the development of mathematical theory.

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