Let $\mathbf{G = (V, T, P, S)}$ be a context-free grammar in Chomsky normal form. Prove that for each $\mathbf{w \in L(G)}$, the height of its derivation-tree is at least $\lceil \mathbf{\log_2(|w|)} \rceil$.

First we'll prove the following lemma: $\forall\ \mathbf{x \in \mathbb{R}, n \in \mathbb{Z}, x \leq n \longrightarrow \lceil x \rceil \leq n}$.

Let $\mathbf{x}\in\mathbb{R}$ and $\mathbf{n}\in\mathbb{Z}$ be such that $\mathbf{x \leq n}$, It is clear that $\exists\ \mathbf{k}\in\mathbb{Z}$, $\mathbf{k - 1 < x \leq k}$.
We will show that $\mathbf{k \leq n}$ by contradiction:
Suppose that $\mathbf{k > n}$, Therefore we get that $\mathbf{k - 1 > n - 1}$ and so $\mathbf{n - 1 < k - 1 < x \leq n < k}$, In particular we get that $\mathbf{n - 1 < k}$ and $\mathbf{k - 1 < n}$, Thus $\mathbf{n - k < 1}$ and $\mathbf{-1 < n - k}$, and so $-\mathbf{1 < n - k < 1}$, Now since $\mathbf{n, k}\in\mathbb{Z}$ we get that $\mathbf{n - k}\in\mathbb{Z}$, Thus, It must be the case that $\mathbf{n - k = 0}$ and so $\mathbf{n = k}$. But this contradicts the assumption $\mathbf{k > n}$ and we reached a contradiction. Therefore it must be the case that $\mathbf{k \leq n}$, Now by definition of $\lceil \bullet \rceil$ we get that $\mathbf{k} = \lceil \mathbf{x} \rceil$ and so $\lceil \mathbf{x} \rceil \leq \mathbf{n}$, as was to be shown.

Now by **Theorem 7.7** we get that $\mathbf{|w| \leq 2^{i-1}}$ where $\mathbf{i}$ is the height of the derivation-tree of $\mathbf{w}$, Now by taking $\mathbf{\log_2}$ of both side of the inequality we get that $\mathbf{\log_2(|w|) \leq \log_2(2^{i-1}) = i - 1}$ and so $\mathbf{\log_2(|w|) \leq i - 1}$, Now since $\mathbf{i - 1}\in\mathbb{Z}$, We get by the lemma that $\lceil \mathbf{\log_2(|w|)} \rceil \leq \mathbf{i - 1}$ and so $\lceil \mathbf{\log_2(|w|)} \rceil \mathbf{+ 1 \leq i}$ as was to be shown.

Q.E.D.
