PROOF

Let L be a language on some alphabet T such that $\varepsilon \in L$ and let G=(V,T,P,S) be a grammar such that $L(G)=L-\{\varepsilon\}$.

Prove that the grammar $G'=(V'=V\cup\{S'\}, T, P'=P\cup\{S'\longrightarrow S|\varepsilon\}, S')$ satisfies $L(G')=L$

Proof that $L(G')\subseteq L$:

Let $w\in L(G')$, there are two case: $w=\varepsilon \vee w\neq\varepsilon$

If $w=\varepsilon$:
Since $\varepsilon\in L$ we get that $w\in L$

If $w\neq\varepsilon$:
Since $w\in L(G')$ we get that $S' \Longrightarrow_{G'}^{*} w$ and so: $\exists n\in\mathbb{N}$, $S' \Longrightarrow_{G'}^{n} w$

It must be the case that $n\neq 0$ because if $n = 0$ we get $S' \Longrightarrow_{G'}^{0} w$ and so $w = S'$, Since $|S'|=1$ we get that $w\in T$, Also since $S'\in V'$ we get that $w\in V'$ and so $w\in T\cap V'$ and we get that $T\cap V' \neq\varnothing$ which contradicts the fact $T\cap V' = \varnothing$.

Also it must be the case that $n\neq 1$ because if $n = 1$ we get $S' \Longrightarrow_{G'}^{1} w$,
There are two cases for the production rules used: $(S'\longrightarrow S)$ or $(S'\longrightarrow\varepsilon)$
If the rule used was $S'\longrightarrow S$ then we get that $S' \Longrightarrow_{G'}^{1} S$ and so we get that $w=S$, Since $|S|=1$ we get that $w\in T$, Also since $S\in V$ we get that $w\in V$ and so $w\in T\cap V$ and we get $T\cap V\neq\varnothing$ which contradicts the fact $T\cap V=\varnothing$.
If the rule used was $S'\longrightarrow\varepsilon$ then we get that $S' \Longrightarrow_{G'}^{1} \varepsilon$ and so we get that $w=\varepsilon$ but this contradicts the facts that $w\neq\varepsilon$.

We've shown that $n\neq 0,1$ and so $\exists 2\leq n\in\mathbb{N}$, $S' \Longrightarrow_{G'}^{n} w$.
And so, The first rule applied to S' must be the rule $S'\longrightarrow S$ and we get that $S' \Longrightarrow_{G'}^{1} S \Longrightarrow_{G'}^{n-1} w$, In particular $S \Longrightarrow_{\mathbf{G}'}^{n-1} w$, Since all the production rules used in this derivation sequence must be production rules from the set P we get that $S \Longrightarrow_{\mathbf{G}}^{n-1} w$ and so $S \Longrightarrow_{\mathbf{G}}^{*} w$, Now by definition of the set L(G) we get that $w\in L(G)=L-\{\varepsilon\}$ and so $w\in L$.

PROOF

Proof that $L \subseteq L(G')$:

Let $w \in L$, there are two cases $w=\varepsilon \vee w \neq \varepsilon$.

If $w=\varepsilon$:

Since $(S' \longrightarrow \varepsilon) \in P'$ we get that $S' \Longrightarrow_{G'}^{1} \varepsilon = w$ and so $w \in L(G')$.

If $w \neq \varepsilon$:

We get that $w \in L-\{\varepsilon\}=L(G)$ and so $S \Longrightarrow_{G}^{*} w$,

Now since G is sub grammar of G' we get that $S \Longrightarrow_{G'}^{*} w$,

Also since $S' \Longrightarrow_{G'}^{1} S$ we get $S' \Longrightarrow_{G'}^{1} S \Longrightarrow_{G'}^{*} w$ and so $S' \Longrightarrow_{G'}^{*} w$,

Now by definition of the set $L(G')$ we get that $w \in L(G')$.

Q.E.D.