Let $\mathbf{\Sigma}$ be some alphabet.

Prove that:

$\forall$ **S,T**$\in \mathscr{P}(\mathbf{\Sigma^*})$**,** $\mathbf{S \neq \varnothing \land S \subseteq T \land (\forall\, t \in T, \exists\, s \in S, s \leq^s t) \longrightarrow}$

$\mathbf{\Sigma^* T \Sigma^* = \Sigma^* S \Sigma^*.}$

Let **S,T**$\in \mathscr{P}(\mathbf{\Sigma^*})$ be such that $\mathbf{S \neq \varnothing}$, $\mathbf{S \subseteq T}$ and $\forall$ $\mathbf{t \in T, \exists\, s \in S, s \leq^s t}$.

(We must show that $\mathbf{\Sigma^* T \Sigma^* = \Sigma^* S \Sigma^*}$)

First we'll show that: $\mathbf{\Sigma^* T \Sigma^* \subseteq \Sigma^* S \Sigma^*}$

Let $\mathbf{w} \in \mathbf{\Sigma^* T \Sigma^*}$, By definition of $\mathbf{\Sigma^* T \Sigma^*}$ we get that

$\mathbf{\exists\, w_1, w_2 \in \Sigma^*, t \in T, w = w_1\, t\, w_2.}$

Now since $\mathbf{t} \in \mathbf{T}$ we get that $\exists$ $\mathbf{s \in S, s \leq^s t}$, By definition of the $\mathbf{\leq^s}$ relation we get that $\exists$ $\mathbf{\psi,\chi \in \Sigma^*, t = \psi\, s\, \chi}$, Thus $\mathbf{w = w_1\, \psi\, s\, \chi\, w_2}$. Lets denote $\mathbf{u_1} = \mathbf{w_1\, \psi}$ and $\mathbf{u_2} = \mathbf{\chi\, w_2}$, We get that $\mathbf{w = u_1\, s\, u_2}$, Since $\mathbf{u_1,u_2} \in \mathbf{\Sigma^*}$ and $\mathbf{s} \in \mathbf{S}$ we get that $\mathbf{u_1\, s\, u_2} \in \mathbf{\Sigma^* S \Sigma^*}$, and so $\mathbf{w} \in \mathbf{\Sigma^* S \Sigma^*}$. Thus $\mathbf{\Sigma^* T \Sigma^* \subseteq \Sigma^* S \Sigma^*}$

Now we'll show that: $\mathbf{\Sigma^* S \Sigma^* \subseteq \Sigma^* T \Sigma^*}$

Let $\mathbf{w} \in \mathbf{\Sigma^* S \Sigma^*}$, By definition of $\mathbf{\Sigma^* S \Sigma^*}$ we get that

$\mathbf{\exists\, w_1, w_2 \in \Sigma^*, s \in S, w = w_1\, s\, w_2.}$

Now since $\mathbf{s} \in \mathbf{S}$ and $\mathbf{S \subseteq T}$ we get that $\mathbf{s} \in \mathbf{T}$ and we get that

$\mathbf{\exists\, w_1, w_2 \in \Sigma^*, s \in T, w = w_1\, s\, w_2.}$ Now by definition of $\mathbf{\Sigma^* T \Sigma^*}$

we get that $\mathbf{w} \in \mathbf{\Sigma^* T \Sigma^*}$. Thus $\mathbf{\Sigma^* S \Sigma^* \subseteq \Sigma^* T \Sigma^*}$

From the two set inclusions we get that $\mathbf{\Sigma^* T \Sigma^* = \Sigma^* S \Sigma^*}$ as was to be shown.

Q.E.D.
