Let $\Sigma = \{a,b,c\}$ and let $L_{mp} = \left\{ wcw^R \mid w \in \{a,b\}^* \right\}$.
We will prove that the pushdown automaton M defined as follows satisfies $L_e(M) = L_{mp}$.

$M = \left(\{q_0, q_1\}, \{a,b,c\}, \{\dashv, A, B\}, \delta, q_0, \dashv, \varnothing \right)$, Where
$\delta : Q \times \Sigma \times \Gamma \to 2^{Q\times\Gamma^*}_{FIN}$ defined as:

δ / Γ	$\dashv$			A			B		
$Q \setminus \Sigma$	a	b	c	a	b	c	a	b	c
q_0	(q_0, A)	(q_0, B)	(q_1, ε)	(q_0, AA)	(q_0, BA)	(q_1, A)	(q_0, AB)	(q_0, BB)	(q_1, B)
q_1	$\varnothing$	$\varnothing$	$\varnothing$	(q_1, ε)	$\varnothing$	$\varnothing$	$\varnothing$	(q_1, ε)	$\varnothing$

First let's define a function $\Delta : \{a,b\} \to \{A,B\}$ as follows:

$$\forall x \in \{a,b\}, \Delta(x) = \left\{ \begin{array}{ll} A & x = a \\ B & x = b \end{array} \right\}$$

Now we'll extend Δ to $\{a,b\}^*$ by defining a new function
$\widehat{\Delta} : \{a,b\}^* \to \{A,B\}^*$ as follows:

$\widehat{\Delta}(\varepsilon) = \varepsilon$

$\forall w \in \{a,b\}^*, \sigma \in \{a,b\}, \widehat{\Delta}(w \cdot \sigma) = \widehat{\Delta}(w) \cdot \Delta(\sigma)$

The following facts can be proved:

$\forall x, y \in \{a,b\}^*, \widehat{\Delta}(x \cdot y) = \widehat{\Delta}(x) \cdot \widehat{\Delta}(y)$

$\forall x \in \{a,b\}^*, \widehat{\Delta}(x^R) = \widehat{\Delta}(x)^R$

We'll prove by induction on n that **(§)**

$$\forall w \in \{a,b\}^+, (q_0, w, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(w)^R) \equiv$$

$$\equiv \forall n \in \mathbb{Z}^+, \forall w \in \{a,b\}^n, (q_0, w, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(w)^R):$$

Basis:

For $n=1$ we must show that $\forall w \in \{a,b\}^1, (q_0, w, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(w)^R)$:

Let $w \in \{a,b\}$, There are two cases: $w = a \vee w = b$

(1) If $w = a$ then we get that $(q_0, w, \dashv) = (q_0, a, \dashv) \vdash_M (q_0, \varepsilon, A)$,

Now $\widehat{\Delta}(w)^R = \widehat{\Delta}(a)^R = \widehat{\Delta}(\varepsilon \cdot a)^R = \left(\widehat{\Delta}(\varepsilon) \cdot \Delta(a)\right)^R = (\varepsilon \cdot A)^R = A^R = A$,

And so $(q_0, \varepsilon, A) = (q_0, \varepsilon, \widehat{\Delta}(w)^R)$, Thus we can conclude that

$(q_0, w, \dashv) \vdash_M (q_0, \varepsilon, \widehat{\Delta}(w)^R)$, And so $(q_0, w, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(w)^R)$.

(2) If $w = b$ then we get that $(q_0, w, \dashv) = (q_0, b, \dashv) \vdash_M (q_0, \varepsilon, B)$,

Now $\widehat{\Delta}(w)^R = \widehat{\Delta}(b)^R = \widehat{\Delta}(\varepsilon \cdot b)^R = \left(\widehat{\Delta}(\varepsilon) \cdot \Delta(b)\right)^R = (\varepsilon \cdot B)^R = B^R = B$,

And so $(q_0, \varepsilon, B) = (q_0, \varepsilon, \widehat{\Delta}(w)^R)$, Thus we can conclude that

$(q_0, w, \dashv) \vdash_M (q_0, \varepsilon, \widehat{\Delta}(w)^R)$, And so $(q_0, w, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(w)^R)$.

From **(1)** and **(2)** we can conclude that it is always the case

$(q_0, w, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(w)^R)$ as was to be shown.

Induction Hypothesis:

Suppose that for some $n = k \in \mathbb{Z}^+$ we have

$\forall w \in \{a,b\}^k, (q_0, w, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(w)^R)$

Induction Step:

We must show that for $n = k+1$ we have

$\forall w \in \{a,b\}^{k+1}, (q_0, w, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(w)^R)$:

Let $w \in \{a,b\}^{k+1}$, Therefore $\exists \tilde{w} \in \{a,b\}^k, \sigma \in \{a,b\}, w = \tilde{w} \cdot \sigma$, Since $\tilde{w} \in \{a,b\}^k$, We get by the induction hypothesis that

$(q_0,\tilde{w},\dashv) \vdash_M^* (q_0,\varepsilon,\widehat{\Delta}(\tilde{w})^R)$, And so

$(q_0,w,\dashv) = (q_0,\tilde{w}\sigma,\dashv) \vdash_M^* (q_0,\sigma,\widehat{\Delta}(\tilde{w})^R)$, Now there are two cases: $\sigma = a \vee \sigma = b$

(1) If $\sigma = a$ then we get that

$(q_0,\sigma,\widehat{\Delta}(\tilde{w})^R) = (q_0,a,\widehat{\Delta}(\tilde{w})^R) \vdash_M (q_0,\varepsilon,A \cdot \widehat{\Delta}(\tilde{w})^R)$,

Now $\widehat{\Delta}(\sigma^R) = \widehat{\Delta}(\sigma) = \widehat{\Delta}(a) = \widehat{\Delta}(\varepsilon \cdot a) = \widehat{\Delta}(\varepsilon) \cdot \Delta(a) = \varepsilon \cdot A = A$,

And so:

$$A \cdot \widehat{\Delta}(\tilde{w})^R = \widehat{\Delta}(\sigma^R) \cdot \widehat{\Delta}(\tilde{w})^R = \widehat{\Delta}(\sigma^R) \cdot \widehat{\Delta}(\tilde{w}^R) = \widehat{\Delta}(\sigma^R \cdot \tilde{w}^R) =$$
$$= \widehat{\Delta}\Big((\tilde{w} \cdot \sigma)^R\Big) = \widehat{\Delta}(w^R) = \widehat{\Delta}(w)^R,$$

Thus we can conclude that $(q_0,\varepsilon,A \cdot \widehat{\Delta}(\tilde{w})^R) = (q_0,\varepsilon,\widehat{\Delta}(w)^R)$, And so $(q_0,\sigma,\widehat{\Delta}(\tilde{w})^R) \vdash_M (q_0,\varepsilon,\widehat{\Delta}(w)^R)$.

(2) If $\sigma = b$ then we get that

$(q_0,\sigma,\widehat{\Delta}(\tilde{w})^R) = (q_0,b,\widehat{\Delta}(\tilde{w})^R) \vdash_M (q_0,\varepsilon,B \cdot \widehat{\Delta}(\tilde{w})^R)$,

Now $\widehat{\Delta}(\sigma^R) = \widehat{\Delta}(\sigma) = \widehat{\Delta}(b) = \widehat{\Delta}(\varepsilon \cdot b) = \widehat{\Delta}(\varepsilon) \cdot \Delta(b) = \varepsilon \cdot B = B$,

And so:

$$B \cdot \widehat{\Delta}(\tilde{w})^R = \widehat{\Delta}(\sigma^R) \cdot \widehat{\Delta}(\tilde{w})^R = \widehat{\Delta}(\sigma^R) \cdot \widehat{\Delta}(\tilde{w}^R) = \widehat{\Delta}(\sigma^R \cdot \tilde{w}^R) =$$
$$= \widehat{\Delta}\Big((\tilde{w} \cdot \sigma)^R\Big) = \widehat{\Delta}(w^R) = \widehat{\Delta}(w)^R,$$

Thus we can conclude that $(q_0,\varepsilon,B \cdot \widehat{\Delta}(\tilde{w})^R) = (q_0,\varepsilon,\widehat{\Delta}(w)^R)$ and so $(q_0,\sigma,\widehat{\Delta}(\tilde{w})^R) \vdash_M (q_0,\varepsilon,\widehat{\Delta}(w)^R)$.

From cases **(1)** and **(2)** we see that it is always the case $(q_0,\sigma,\widehat{\Delta}(\tilde{w})^R) \vdash_M (q_0,\varepsilon,\widehat{\Delta}(w)^R)$.

Now we can conclude that $(q_0,w,\dashv) \vdash_M^* (q_0,\sigma,\widehat{\Delta}(\tilde{w})^R) \vdash_M (q_0,\varepsilon,\widehat{\Delta}(w)^R)$,

And so $(q_0,w,\dashv) \vdash_M^* (q_0,\varepsilon,\widehat{\Delta}(w)^R)$ as was to be shown.

Now we'll show that $L_{mp} \subseteq L_e(M)$:

Let $w \in L_{mp}$, Thus $\exists x \in \{a,b\}^*, w = xcx^R$, Now there are two cases: $x = \varepsilon \vee x \neq \varepsilon$

(1) If $x = \varepsilon$, Then $w = c$ and so $(q_0, w, \dashv) = (q_0, c, \dashv) \vdash_M (q_1, \varepsilon, \varepsilon)$, Thus, we can conclude that $w \in L_e(M)$.

(2) If $x \neq \varepsilon$, Then $x \in \{a,b\}^+$ and we can conclude by using **(§)** that $(q_0, x, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(x)^R)$, Thus, $(q_0, w, \dashv) = (q_0, xcx^R, \dashv) \vdash_M^* (q_0, cx^R, \widehat{\Delta}(x)^R) \vdash_M (q_1, x^R, \widehat{\Delta}(x)^R)$, Now, Since M is in state q_1, We are currently emptying the stack, And for each a in x^R there is a corresponding A in $\widehat{\Delta}(x)^R$, Similarly, for each b in x^R there is a corresponding B in $\widehat{\Delta}(x)^R$, Thus $(q_1, x^R, \widehat{\Delta}(x)^R) \vdash_M^* (q_1, \varepsilon, \varepsilon)$, And we can conclude that $(q_0, w, \dashv) \vdash_M^* (q_1, \varepsilon, \varepsilon)$, And so $w \in L_e(M)$.

From cases **(1)** and **(2)** we see that it is always the case that $w \in L_e(M)$ as was to be shown.

Now we'll show that $L_e(M) \subseteq L_{mp}$:

Let $w \in L_e(M)$, Thus $(q_0, w, \dashv) \vdash_M^* (q_1, \varepsilon, \varepsilon)$. Since M reaches state q_1, We get that there must be some c in the input string that took M from state q_0 to state q_1. Now, because the first time M encounters c, M changes its state from q_0 to q_1 we can conclude that $\exists x \in \{a,b\}^*, y \in \{a,b,c\}^*, w = xcy$, Now there are two cases: $x = \varepsilon \vee x \neq \varepsilon$

(1) If $x = \varepsilon$ then $w = cy$ and so $(q_0, w, \dashv) = (q_0, cy, \dashv) \vdash_M (q_1, y, \varepsilon)$. Now since $(q_0, w, \dashv) \vdash_M^* (q_1, \varepsilon, \varepsilon)$, We get that it must be the case that $(q_1, y, \varepsilon) \vdash_M^* (q_1, \varepsilon, \varepsilon)$. Now since the stack is empty, M cannot do any move, And so, It must be the case that $(q_1, y, \varepsilon) \vdash_M^0 (q_1, \varepsilon, \varepsilon)$, Therefore $y = \varepsilon$, And we get that $w = c$. Now since $c \in L_{mp}$, We can conclude that $w \in L_{mp}$.

(2) If $x \neq \varepsilon$ then $x \in \{a,b\}^+$ and by **(§)** we can conclude that $(q_0, x, \dashv) \vdash_M^* (q_0, \varepsilon, \widehat{\Delta}(x)^R)$, Thus, $(q_0, w, \dashv) = (q_0, xcy, \dashv) \vdash_M^* (q_0, cy, \widehat{\Delta}(x)^R)$, Now since the first character in x must be one of a or b, We get that the character at the top of the stack must be one of A or B and it cannot be the character $\dashv$. Therefore, the first character in $\widehat{\Delta}(x)^R$ must be one of A or B, And so $(q_0, cy, \widehat{\Delta}(x)^R) \vdash_M (q_1, y, \widehat{\Delta}(x)^R)$, Thus $(q_0, w, \dashv) \vdash_M^* (q_1, y, \widehat{\Delta}(x)^R)$.

Now since $(q_0, w, \dashv) \vdash_M^* (q_1, \varepsilon, \varepsilon)$, We get that it must be the case that $(q_1, y, \widehat{\Delta}(x)^R) \vdash_M^* (q_1, \varepsilon, \varepsilon)$. Now, Since M is in the state q_1 we get that M is currently emptying the stack and for each a in y there must be a corresponding A in $\widehat{\Delta}(x)^R$, Similarly, for each b in y there is must be a corresponding B in $\widehat{\Delta}(x)^R$, Thus we can conclude that $\widehat{\Delta}(y) = \widehat{\Delta}(x)^R = \widehat{\Delta}(x^R)$.

It can be shown that $\widehat{\Delta}$ is a one-to-one function and from this fact we can conclude that $y = x^R$. Thus $w = xcx^R$ where $x \in \{a,b\}^+$. Now by definition of L_{mp} we can conclude that $w \in L_{mp}$.

From cases **(1)** and **(2)** we can conclude that it is always the case that $w \in L_{mp}$ as was to be shown.

From the two set inclusions we've just shown, We can conclude that $L_e(M) = L_{mp}$ as was to be shown.

Q.E.D.