Let **G = (V,T,P,S)** be a context-free grammar.
Prove that there exists a context-free grammar **G'** such that **L(G') = L(G) - {ε}** and **G'** does not contain redundant symbols and ε-rules.

Lets define the set **E(G)** as follows:

$E_0 = \{ A \in V \mid (A \longrightarrow \varepsilon) \in P \}$

$\forall n \in \mathbb{N},\ E_{n+1} = E_n \cup \{ A \in V \mid \exists\ t \in \mathbb{Z}^+,\ \exists\ X_1, \dots, X_t \in E_n,\ (A \longrightarrow X_1 \dots X_t) \in P \}$

Since the E_i's form an ascending chain (I.e. $E_0 \subseteq E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots \subseteq V$) and since **V** is finite, there is a least **i**, say i_0 such that $E_{i_0} = E_{i_0+1}$, Now we'll define $E(G) = E_{i_0}$, It can be shown that $E(G) = \{ A \in V \mid A \Longrightarrow_G^* \varepsilon \}$.

The set **E(G)** will be called: **The set of erasable variables,**
or alternatively: **The set of nullable variables.**

Now we'll define the set of production rules **P'** of the new grammar **G'** as follows:

First we'll define a function $F : V \cup T \longrightarrow \mathscr{P}(V \cup T \cup \{\varepsilon\})$ as follows:

$$\forall X \in V \cup T,\ F(X) = \begin{cases} \{X\} & X \notin E(G) \\ \{\varepsilon, X\} & X \in E(G) \end{cases}$$

(It is clear that $\forall a \in T,\ F(a) = \{a\}$ since it cannot be the case that $a \in E(G)$ since $E(G) \subseteq V$)

Now we can define **P'** as follows:

$$P' = \{ A \longrightarrow \alpha_1 \dots \alpha_t \mid \exists\ t \in \mathbb{Z}^+,\ \exists\ X_1, \dots, X_t \in V \cup T,\ (A \longrightarrow X_1 \dots X_t) \in P \wedge (\forall\ i \in \{1, \dots, t\},\ \alpha_i \in F(X_i)) \wedge (\exists\ i \in \{1, \dots, t\},\ \alpha_i \neq \varepsilon) \}$$

Now we can define the grammar **G'** as: **G' = (V, T, P', S)**

We'll prove that $\forall A \in V,\ w \in T^*,\ A \Longrightarrow_G^* w \wedge w \neq \varepsilon \longrightarrow A \Longrightarrow_{G'}^* w \equiv$

$\forall n \in \mathbb{Z}^+,\ \forall A \in V,\ w \in T^*,\ A \Longrightarrow_G^n w \wedge w \neq \varepsilon \longrightarrow A \Longrightarrow_{G'}^* w$ by induction on **n**:

Basis

For $\mathbf{n = 1}$ we must show that
$\forall \mathbf{A} \in \mathbf{V}, \mathbf{w} \in \mathbf{T}^*, \mathbf{A} \Longrightarrow_G{}^1 \mathbf{w} \wedge \mathbf{w} \neq \varepsilon \longrightarrow \mathbf{A} \Longrightarrow_{G'}{}^* \mathbf{w}$:

Let $\mathbf{A} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{T}^*$ be such that $\mathbf{A} \Longrightarrow_G{}^1 \mathbf{w}$ and $\mathbf{w} \neq \varepsilon$.
Since $\mathbf{A} \Longrightarrow_G{}^1 \mathbf{w}$ we get that there must be some production rule of the form $(\mathbf{A} \longrightarrow \mathbf{w}) \in \mathbf{P}$, since $\mathbf{w} \neq \varepsilon$ we get that $|\mathbf{w}| > 0$, Now if we let $\mathbf{t} = |\mathbf{w}|$ then we we'll get that $\mathbf{t} > 0$ and $\exists\ \mathbf{X}_1, \dots, \mathbf{X}_t \in \mathbf{T}, \mathbf{w} = \mathbf{X}_1 \dots \mathbf{X}_t$, and so $(\mathbf{A} \longrightarrow \mathbf{X}_1 \dots \mathbf{X}_t) \in \mathbf{P}$, Now since $\forall \mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{t}\}, \mathbf{X}_i \in \mathbf{T}$ we get that $\forall \mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{t}\}, \mathbf{F}(\mathbf{X}_i) = \{\mathbf{X}_i\}$ by definition of $\mathbf{F}$, Now it is clear from the definition of the set $\mathbf{P'}$ that $(\mathbf{A} \longrightarrow \mathbf{X}_1 \dots \mathbf{X}_t) \in \mathbf{P'}$ which means that $(\mathbf{A} \longrightarrow \mathbf{w}) \in \mathbf{P'}$ and so $\mathbf{A} \Longrightarrow_{G'}{}^1 \mathbf{w}$ which implies that $\mathbf{A} \Longrightarrow_{G'}{}^* \mathbf{w}$ as was to be shown.

Induction hypothesis

Suppose that for some $\mathbf{n = k \geq 1}$ we have:
$\forall \mathbf{j} \in \{\mathbf{1}, \dots, \mathbf{k}\}, \forall \mathbf{A} \in \mathbf{V}, \mathbf{w} \in \mathbf{T}^*, \mathbf{A} \Longrightarrow_G{}^j \mathbf{w} \wedge \mathbf{w} \neq \varepsilon \longrightarrow \mathbf{A} \Longrightarrow_{G'}{}^* \mathbf{w}$

Induction step

We must prove that $\forall \mathbf{A} \in \mathbf{V}, \mathbf{w} \in \mathbf{T}^*, \mathbf{A} \Longrightarrow_G{}^{k+1} \mathbf{w} \wedge \mathbf{w} \neq \varepsilon \longrightarrow \mathbf{A} \Longrightarrow_{G'}{}^* \mathbf{w}$:

Let $\mathbf{A} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{T}^*$ be such that $\mathbf{A} \Longrightarrow_G{}^{k+1} \mathbf{w}$ and $\mathbf{w} \neq \varepsilon$. Since $\mathbf{A} \Longrightarrow_G{}^{k+1} \mathbf{w}$ we get that $\exists \beta \in (\mathbf{V} \cup \mathbf{T})^*, \mathbf{A} \Longrightarrow_G{}^1 \beta \Longrightarrow_G{}^k \mathbf{w}$ and so $(A \longrightarrow \beta) \in \mathbf{P}$, Since $\mathbf{k \geq 1}$ and since $\mathbf{G}$ is a context-free grammar we get that $\beta \in (\mathbf{V} \cup \mathbf{T})^* \mathbf{V} (\mathbf{V} \cup \mathbf{T})^*$ and so $|\beta| \geq \mathbf{1}$. Now we'll denote $\mathbf{t} = |\beta|$ and we get that
$\exists\ \mathbf{X}_1, \dots, \mathbf{X}_t \in \mathbf{V} \cup \mathbf{T}, \beta = \mathbf{X}_1 \dots \mathbf{X}_t$ and thus $(\mathbf{A} \longrightarrow \mathbf{X}_1 \dots \mathbf{X}_t) \in \mathbf{P}$,
Now we get that $\mathbf{A} \Longrightarrow_G{}^1 \mathbf{X}_1 \dots \mathbf{X}_t \Longrightarrow_G{}^k \mathbf{w}$
In particular we get $\mathbf{X}_1 \dots \mathbf{X}_t \Longrightarrow_G{}^k \mathbf{w}$ and thus
$\forall \mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{t}\}, \exists\ \mathbf{w}_i \in \mathbf{T}^*, (\exists \mathbf{j} \in \{\mathbf{1}, \dots, \mathbf{k}\}, \mathbf{X}_i \Longrightarrow_G{}^j \mathbf{w}_i) \vee \mathbf{X}_i = \mathbf{w}_i$
and $\mathbf{w} = \mathbf{w}_1 \dots \mathbf{w}_t$

Now we will show that **(∗)** $\forall i \in \{1, \dots, t\}, \alpha_i \Longrightarrow_{G'}^{*} w_i$ where

$$\forall i \in \{1, \dots, t\}, \alpha_i = \begin{cases} X_i & w_i \neq \varepsilon \\ \varepsilon & w_i = \varepsilon \end{cases}$$

Let $i \in \{1, \dots, t\}$, there are two cases: $w_i \neq \varepsilon \vee w_i = \varepsilon$

(§1) If $w_i \neq \varepsilon$ we get that $\alpha_i = X_i$, and now there are two cases: $(\exists j \in \{1, \dots, k\}, X_i \Longrightarrow_{G}^{j} w_i) \vee X_i = w_i$

(§1.1) If $\exists j \in \{1, \dots, k\}, X_i \Longrightarrow_{G}^{j} w_i$ then we get by the induction hypothesis that $X_i \Longrightarrow_{G'}^{*} w_i$ and so $\alpha_i \Longrightarrow_{G'}^{*} w_i$

(§1.2) If $X_i = w_i$, Since $w_i \Longrightarrow_{G'}^{*} w_i$ we get $X_i \Longrightarrow_{G'}^{*} w_i$ and so $\alpha_i \Longrightarrow_{G'}^{*} w_i$

(§2) If $w_i = \varepsilon$ we get that $\alpha_i = \varepsilon$ and since $\varepsilon \Longrightarrow_{G'}^{*} \varepsilon$ we get that $\alpha_i \Longrightarrow_{G'}^{*} w_i$

From cases **(§1)** and **(§2)** we can conclude that it is always true that $\alpha_i \Longrightarrow_{G'}^{*} w_i$ as was to be shown.

Now we'll show that $A \Longrightarrow_{G'}^{*} w$.

It is clear that $(A \longrightarrow \alpha_1 \dots \alpha_t) \in P'$ by definition of **P'** (Since $w \neq \varepsilon$ and since $w = w_1 \dots w_t$ we get that $\exists\ i \in \{1, \dots, t\}, w_i \neq \varepsilon$ and so $\exists\ i \in \{1, \dots, t\}, \alpha_i \neq \varepsilon$, also we know that $(A \longrightarrow X_1 \dots X_t) \in P$ and we know that $\forall\ i \in \{1, \dots, t\},\ \alpha_i \in F(X_i)$ and so $(A \longrightarrow \alpha_1 \dots \alpha_t) \in P'$).

Now we can get by applying **(∗)** multiple times: $A \Longrightarrow_{G'}^{1} \alpha_1\alpha_2 \dots \alpha_t \Longrightarrow_{G'}^{*} w_1\alpha_2 \dots \alpha_t \Longrightarrow_{G'}^{*} w_1w_2 \dots \alpha_t \Longrightarrow_{G'}^{*} \dots \Longrightarrow_{G'}^{*} w_1w_2 \dots w_t = w$ and so $A \Longrightarrow_{G'}^{*} w$ as was to be shown.

Now we'll prove that $\forall \mathbf{A}\in\mathbf{V}, \mathbf{w}\in\mathbf{T}^*, \mathbf{A} \Longrightarrow_{G'}^* \mathbf{w} \longrightarrow \mathbf{A} \Longrightarrow_{G}^* \mathbf{w} \wedge \mathbf{w} \neq \varepsilon$ in two parts: First we'll prove that $\forall \mathbf{A}\in\mathbf{V}, \mathbf{w}\in\mathbf{T}^*, \mathbf{A} \Longrightarrow_{G'}^* \mathbf{w} \longrightarrow \mathbf{w} \neq \varepsilon$ and then we'll prove that $\forall \mathbf{A}\in\mathbf{V}, \mathbf{w}\in\mathbf{T}^*, \mathbf{A} \Longrightarrow_{G'}^* \mathbf{w} \longrightarrow \mathbf{A} \Longrightarrow_{G}^* \mathbf{w}$.

We'll prove that $\forall \mathbf{A}\in\mathbf{V}, \mathbf{w}\in\mathbf{T}^*, \mathbf{A} \Longrightarrow_{G'}^* \mathbf{w} \longrightarrow \mathbf{w} \neq \varepsilon$: by proving the following propositions in order:

(§1) $\forall \mathbf{A}\in\mathbf{V}, \alpha\in(\mathbf{V}\cup\mathbf{T})^*, (\mathbf{A} \longrightarrow \alpha)\in\mathbf{P'} \longrightarrow \alpha \neq \varepsilon$

(§2) $\forall\, \alpha,\beta\in(\mathbf{V}\cup\mathbf{T})^*, \alpha \Longrightarrow_{G'} \beta \longrightarrow \beta \neq \varepsilon$

(§3) $\forall \mathbf{A}\in\mathbf{V}, \mathbf{w}\in\mathbf{T}^*, \mathbf{A} \Longrightarrow_{G'}^* \mathbf{w} \longrightarrow \mathbf{w} \neq \varepsilon$

Proof of **(§1)**:

Let $\mathbf{A}\in\mathbf{V}$ and $\alpha\in(\mathbf{V}\cup\mathbf{T})^*$ be such that $(\mathbf{A} \longrightarrow \alpha)\in\mathbf{P'}$,

(we must show that $\alpha \neq \varepsilon$)

Now we get by definition of **P'** that

$\exists\ \mathbf{t}\in\mathbb{Z}^+, \exists\ \alpha_1, \ldots, \alpha_t \in \mathbf{V} \cup \mathbf{T} \cup \{\varepsilon\}, \alpha = \alpha_1 \ldots \alpha_t$ and so

$(\mathbf{A} \longrightarrow \alpha_1 \ldots \alpha_t)\in\mathbf{P'}$, Again, by definition of **P'** we get that

$\exists\ \mathbf{i}\in\{1, \ldots, \mathbf{t}\}, \alpha_i \neq \varepsilon$ and so $|\alpha| = |\alpha_1 \ldots \alpha_t| = |\alpha_1| + \ldots + |\alpha_i| + \ldots + |\alpha_t| > 0$

and so $\alpha \neq \varepsilon$ as was to be shown.

Proof of **(§2)**:

Let $\alpha,\beta\in(\mathbf{V}\cup\mathbf{T})^*$ be such that $\alpha \Longrightarrow_{G'} \beta$, (We must show that $\beta \neq \varepsilon$)

Since $\alpha \Longrightarrow_{G'} \beta$ we get by definition of the $\Longrightarrow_{G'}$ relation that

$\exists\ \psi,\chi,\gamma\in(\mathbf{V}\cup\mathbf{T})^*, \mathbf{A}\in\mathbf{V}, \alpha = \psi\mathbf{A}\chi \wedge \beta = \psi\gamma\chi \wedge (\mathbf{A}\longrightarrow\gamma)\in\mathbf{P'}$

Now since $\mathbf{A}\in\mathbf{V}, \gamma\in(\mathbf{V}\cup\mathbf{T})^*$ and $(\mathbf{A}\longrightarrow\gamma)\in\mathbf{P'}$ we get by **(§1)** that $\gamma \neq \varepsilon$ and so $|\beta| = |\psi\gamma\chi| = |\psi| + |\gamma| + |\chi| \geq |\gamma| > 0$ and so $\beta \neq \varepsilon$ as was be shown.

Proof of **(§3)**

Let $\mathbf{A}\in\mathbf{V}$ and $\mathbf{w}\in\mathbf{T}^*$ be such that $\mathbf{A} \Longrightarrow_{G'}^* \mathbf{w}$ (we must show that $\mathbf{w} \neq \varepsilon$)

It is clear that $\mathbf{A} \Longrightarrow_{G'}^{+} \mathbf{w}$ and so $\exists\ \mathbf{j} \in \mathbb{Z}^{+}$, $\mathbf{A} \Longrightarrow_{G'}^{j} \mathbf{w}$ and thus

$\exists\ \boldsymbol{\beta} \in (\mathbf{V} \cup \mathbf{T})^{*}$, $\mathbf{A} \Longrightarrow_{G'}^{j-1} \boldsymbol{\beta} \Longrightarrow_{G'} \mathbf{w}$, in particular $\boldsymbol{\beta} \Longrightarrow_{G'} \mathbf{w}$ and by **(§2)** we get that $\mathbf{w} \neq \varepsilon$ as was to be shown.

Now we'll prove that **(§4)** $\forall \mathbf{A} \in \mathbf{V}, \mathbf{w} \in \mathbf{T}^{*}, \mathbf{A} \Longrightarrow_{G'}^{*} \mathbf{w} \longrightarrow \mathbf{A} \Longrightarrow_{G}^{*} \mathbf{w} \equiv$

$\forall \mathbf{n} \in \mathbb{Z}^{+}, \forall \mathbf{A} \in \mathbf{V}, \mathbf{w} \in \mathbf{T}^{*}, \mathbf{A} \Longrightarrow_{G'}^{n} \mathbf{w} \longrightarrow \mathbf{A} \Longrightarrow_{G}^{*} \mathbf{w}$ by induction on **n**:

<u>Basis</u>

For **n = 1** we must show that
$\forall \mathbf{A} \in \mathbf{V}, \mathbf{w} \in \mathbf{T}^{*}, \mathbf{A} \Longrightarrow_{G'}^{1} \mathbf{w} \longrightarrow \mathbf{A} \Longrightarrow_{G}^{*} \mathbf{w}$:

Let $\mathbf{A} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{T}^{*}$ be such that $\mathbf{A} \Longrightarrow_{G'}^{1} \mathbf{w}$, Since $\mathbf{A} \Longrightarrow_{G'}^{1} \mathbf{w}$ we get that $(\mathbf{A} \longrightarrow \mathbf{w}) \in \mathbf{P'}$ and $\mathbf{A} \Longrightarrow_{G'}^{*} \mathbf{w}$, Now by **(§3)** we get that $\mathbf{w} \neq \varepsilon$.

Let's denote $\mathbf{t} = |\mathbf{w}|$ and we will get that $\exists\ \alpha_1, \ldots, \alpha_t \in \mathbf{T}, \mathbf{w} = \alpha_1 \ldots \alpha_t$,

thus $(\mathbf{A} \longrightarrow \alpha_1 \ldots \alpha_t) \in \mathbf{P'}$, and by definition of **P'** we will get that

$\exists\ \mathbf{X}_1, \ldots, \mathbf{X}_t \in \mathbf{V} \cup \mathbf{T}, (\mathbf{A} \longrightarrow \mathbf{X}_1 \ldots \mathbf{X}_t) \in \mathbf{P}$ and $\forall\ \mathbf{i} \in \{1, \ldots, t\}$, $\boldsymbol{\alpha}_i \in \mathbf{F}(\mathbf{X}_i)$ and

$\exists\ \mathbf{i} \in \{1, \ldots, t\}, \boldsymbol{\alpha}_i \neq \varepsilon$.

We'll prove that **(§4.1)** $\forall\ \mathbf{i} \in \{1, \ldots, t\}$, $\mathbf{X}_i \in \mathbf{T}$ by contradiction:

Suppose that $\exists \mathbf{i} \in \{1, \ldots, t\}, \mathbf{X}_i \notin \mathbf{T}$, Since $\mathbf{X}_i \in \mathbf{V} \cup \mathbf{T}$ we get that it must be the case $\mathbf{X}_i \in \mathbf{V}$, Now since $\mathbf{E}(\mathbf{G}) \subseteq \mathbf{V}$ we get that there are two cases:
$\mathbf{X}_i \in \mathbf{E}(\mathbf{G}) \vee \mathbf{X}_i \notin \mathbf{E}(\mathbf{G})$:

(§4.1.1) If $\mathbf{X}_i \in \mathbf{E}(\mathbf{G})$ we get that $\mathbf{F}(\mathbf{X}_i) = \{\varepsilon, \mathbf{X}_i\}$ and since $\boldsymbol{\alpha}_i \in \mathbf{F}(\mathbf{X}_i)$ we get that $\boldsymbol{\alpha}_i \in \{\varepsilon, \mathbf{X}_i\}$, Now there are two additional cases: $\boldsymbol{\alpha}_i = \varepsilon \vee \boldsymbol{\alpha}_i = \mathbf{X}_i$:

> **(§4.1.1.1)** If $\boldsymbol{\alpha}_i = \varepsilon$, Since $\boldsymbol{\alpha}_i \in \mathbf{T}$ we get that $\varepsilon \in \mathbf{T}$ which contradicts the fact that $\varepsilon \notin \mathbf{T}$.

(§4.1.1.2) If $\alpha_i = X_i$, Since $X_i \in V$ we get that $\alpha_i \in V$ and since $\alpha_i \in T$ we get that $\alpha_i \in V \cap T$ and so $V \cap T \neq \emptyset$ which contradicts the fact that $V \cap T = \emptyset$.

(§4.1.2) If $X_i \notin E(G)$ we get that $F(X_i) = \{X_i\}$ and so $\alpha_i \in \{X_i\}$ which implies that $\alpha_i = X_i$, Since $X_i \in V$ we get that $\alpha_i \in V$ and since $\alpha_i \in T$ we get that $\alpha_i \in V \cap T$ and so $V \cap T \neq \emptyset$ which contradicts the fact that $V \cap T = \emptyset$.

From cases **(§4.1.1)** and **(§4.1.2)** we get a contradiction, And so **(§4.1)** must be true.

We'll prove that **(§4.2)** $\forall\ i \in \{1 , \dots , t\}, \alpha_i = X_i$:

Let $i \in \{1 , \dots , t\}$, Since $\alpha_i \in F(X_i)$ and since by **(§4.1)** $X_i \in T$ we get that $F(X_i) = \{X_i\}$ and so $\alpha_i \in \{X_i\}$ which implies that $\alpha_i = X_i$ as was to be shown.

Now we'll show that $A \Longrightarrow_G^* w$:

Since $(A \longrightarrow X_1 \dots X_t) \in P$ we get by **(§4.2)** that $(A \longrightarrow \alpha_1 \dots \alpha_t) \in P$ and so $(A \longrightarrow w) \in P$ which implies that $A \Longrightarrow_G^1 w$ and so $A \Longrightarrow_G^* w$ as was to be shown.

<u>Induction hypothesis</u>

Suppose that for some $n = k \geq 1$ we have:
$\forall j \in \{ 1 , \dots , k \}, \forall A \in V, w \in T^*, A \Longrightarrow_{G'}^j w \longrightarrow A \Longrightarrow_G^* w$

Induction step

We must prove that $\forall \mathbf{A} \in \mathbf{V}, \mathbf{w} \in \mathbf{T}^*, \mathbf{A} \Longrightarrow_{G'}^{k+1} \mathbf{w} \longrightarrow \mathbf{A} \Longrightarrow_{G}^{*} \mathbf{w}$:

Let $\mathbf{A} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{T}^*$ be such that $\mathbf{A} \Longrightarrow_{G'}^{k+1} \mathbf{w}$, Therefore we get that

$\exists \beta \in (\mathbf{V} \cup \mathbf{T})^*$, $\mathbf{A} \Longrightarrow_{G'}^{1} \beta \Longrightarrow_{G'}^{k} \mathbf{w}$ and $(\mathbf{A} \longrightarrow \beta) \in \mathbf{P'}$, Therefore we get that

$\exists\ t \in \mathbb{Z}^+$,

$\exists\ \alpha_1 , \dots , \alpha_t \in \mathbf{V} \cup \mathbf{T} \cup \{\varepsilon\},\ \beta = \alpha_1 \dots \alpha_t \qquad \wedge$

$\exists\ \mathbf{X}_1 , \dots , \mathbf{X}_t \in \mathbf{V} \cup \mathbf{T}, (\mathbf{A} \longrightarrow \mathbf{X}_1 \dots \mathbf{X}_t) \in \mathbf{P} \qquad \wedge$

$\forall\ i \in \{1 , \dots , t \},\ \alpha_i \in F(\mathbf{X}_i) \qquad \wedge$

$\exists\ i \in \{1 , \dots , t \},\ \alpha_i \neq \varepsilon$

Therefore we get that $\mathbf{A} \Longrightarrow_{G}^{1} \mathbf{X}_1 \dots \mathbf{X}_t$ and $\alpha_1 \dots \alpha_t \Longrightarrow_{G'}^{k} \mathbf{w}$ and so

$\forall i \in \{ 1 , \dots , t \}, \exists\ \mathbf{w}_i \in \mathbf{T}^*, (\exists j \in \{1 , \dots , k\}, \alpha_i \Longrightarrow_{G'}^{j} \mathbf{w}_i) \vee \alpha_i = \mathbf{w}_i$

We'll prove that **(§5)** $\forall\ i \in \{1 , \dots , t \},\ \alpha_i \Longrightarrow_{G}^{*} \mathbf{w}_i$:

Let $i \in \{1 , \dots , t \}$, There are two cases $(\exists j \in \{1 , \dots , k\}, \alpha_i \Longrightarrow_{G'}^{j} \mathbf{w}_i) \vee \alpha_i = \mathbf{w}_i$

(§5.1) If $\exists j \in \{1 , \dots , k\}, \alpha_i \Longrightarrow_{G'}^{j} \mathbf{w}_i$ then it must be the case that $\alpha_i \in \mathbf{V}$, Thus we get by the induction hypothesis that $\alpha_i \Longrightarrow_{G}^{*} \mathbf{w}_i$.

(§5.2) If $\alpha_i = \mathbf{w}_i$ we get that $\alpha_i \Longrightarrow_{G}^{0} \mathbf{w}_i$ and so $\alpha_i \Longrightarrow_{G}^{*} \mathbf{w}_i$.

Therefore, by **(§5.1)** and **(§5.2)** we get that it is always the case that
$\alpha_i \Longrightarrow_{G}^{*} \mathbf{w}_i$ as was to be shown.

Now we'll prove that **(§6)** $\forall\ i \in \{1 , \dots , t \},\ \mathbf{X}_i \Longrightarrow_{G}^{*} \mathbf{w}_i$:

Let $i \in \{1 , \dots , t \}$, Therefore $\alpha_i \in F(\mathbf{X}_i)$, By **(§5)** we get that $\alpha_i \Longrightarrow_{G}^{*} \mathbf{w}_i$, Now since $\mathbf{X}_i \in \mathbf{V} \cup \mathbf{T}$ there are two cases: $\mathbf{X}_i \in \mathbf{V} \vee \mathbf{X}_i \in \mathbf{T}$

(§6.1) If $\mathbf{X_i} \in \mathbf{V}$, There are two cases: $\mathbf{X_i} \in \mathbf{E(G)} \vee \mathbf{X_i} \notin \mathbf{E(G)}$

(§6.1.1) If $\mathbf{X_i} \in \mathbf{E(G)}$ then $\mathbf{F(X_i) = \{\varepsilon, X_i\}}$, Now since $\boldsymbol{\alpha_i} \in \mathbf{F(X_i)}$ we get that $\boldsymbol{\alpha_i} \in \{\varepsilon, \mathbf{X_i}\}$, Now there are two cases: $\boldsymbol{\alpha_i} = \varepsilon \vee \boldsymbol{\alpha_i} = \mathbf{X_i}$

(§6.1.1.1) If $\boldsymbol{\alpha_i} = \varepsilon$ then we get that $\varepsilon \Longrightarrow_G^* \mathbf{w_i}$ and now it must be the case that $\varepsilon \Longrightarrow_G^0 \mathbf{w_i}$ and so $\varepsilon = \mathbf{w_i}$, Now since $\mathbf{X_i} \in \mathbf{E(G)}$ we get that $\mathbf{X_i} \Longrightarrow_G^* \varepsilon$ and so $\mathbf{X_i} \Longrightarrow_G^* \mathbf{w_i}$.

(§6.1.1.2) If $\boldsymbol{\alpha_i} = \mathbf{X_i}$, Since $\boldsymbol{\alpha_i} \Longrightarrow_G^* \mathbf{w_i}$ we get that $\mathbf{X_i} \Longrightarrow_G^* \mathbf{w_i}$.

(§6.1.2) If $\mathbf{X_i} \notin \mathbf{E(G)}$ then $\mathbf{F(X_i) = \{X_i\}}$, Now since $\boldsymbol{\alpha_i} \in \mathbf{F(X_i)}$ we get that $\boldsymbol{\alpha_i} \in \mathbf{\{X_i\}}$ and so $\boldsymbol{\alpha_i} = \mathbf{X_i}$, Now since $\boldsymbol{\alpha_i} \Longrightarrow_G^* \mathbf{w_i}$
we get that $\mathbf{X_i} \Longrightarrow_G^* \mathbf{w_i}$.

(§6.2) If $\mathbf{X_i} \in \mathbf{T}$ then we get that $\mathbf{F(X_i) = \{X_i\}}$, Now since $\boldsymbol{\alpha_i} \in \mathbf{F(X_i)}$ we get that $\boldsymbol{\alpha_i} \in \mathbf{\{X_i\}}$ and so $\boldsymbol{\alpha_i} = \mathbf{X_i}$, Now since $\boldsymbol{\alpha_i} \Longrightarrow_G^* \mathbf{w_i}$ we get that $\mathbf{X_i} \Longrightarrow_G^* \mathbf{w_i}$.

Therefore we got that $\mathbf{X_i} \Longrightarrow_G^* \mathbf{w_i}$ as was to be shown.

Now we'll show that $\mathbf{A} \Longrightarrow_G^* \mathbf{w}$:

$$\mathbf{A} \Longrightarrow_G^1 \mathbf{X_1 \ldots X_t} \Longrightarrow_G^* \mathbf{w_1 \ldots w_t = w}$$

And so $\mathbf{A} \Longrightarrow_G^* \mathbf{w}$ as was to be shown.

Now by combining **(§3)** and **(§4)** we get :

$$\forall \mathbf{A} \in \mathbf{V}, \mathbf{w} \in \mathbf{T^*}, \mathbf{A} \Longrightarrow_{G'}^* \mathbf{w} \longrightarrow \mathbf{A} \Longrightarrow_G^* \mathbf{w} \wedge \mathbf{w} \neq \varepsilon$$

as was to be shown.

From the two proofs we get :

$$\forall A \in V, w \in T^*, A \Longrightarrow_{G'}^* w \longleftrightarrow A \Longrightarrow_{G}^* w \wedge w \neq \varepsilon$$

Now we'll show that $\mathbf{L(G') = L(G) - \{\varepsilon\}}$:

Let $\mathbf{w \in T^*}$:

$\mathbf{w \in L(G') \Longleftrightarrow S \Longrightarrow_{G'}^* w \Longleftrightarrow S \Longrightarrow_{G}^* w \wedge w \neq \varepsilon \Longleftrightarrow w \in L(G) - \{\varepsilon\}}$.

and so $\mathbf{L(G') = L(G) - \{\varepsilon\}}$.

Now we can build a new grammar **G'' = (V'',T'',P'',S)** by using **Theorem 7.3** on the grammar **G'**. The grammar **G''** does not contain redundant symbols and **L(G'') = L(G') = L(G) - {ε}**, Now since the grammar **G''** is a sub grammar of the grammar **G'** and since **G'** does not contain ε-rules we get that the grammar **G''** does not contain ε-rules and so:

There exists grammar **G''** that satisfies **L(G'') = L(G) - {ε}** and **G''** does not contain redundant symbols and ε-rules as was to be shown.

Q.E.D.
