Let Σ be some alphabet and R be the set of regular expressions on Σ.

Theorem 1

$\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), (L_1 \cdot L_2^*)^+ = L_1 \cdot (L_1 \cup L_2)^*$

Theorem 2

$\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), (L_1 \cdot L_2^*)^* = L_1 \cdot (L_1 \cup L_2)^* \cup \{\varepsilon\}$

Corollary 1

$\forall r_1, r_2 \in R, L[(r_1 \cdot r_2^*)^+] = L[r_1 \cdot (r_1 + r_2)^*]$

Corollary 2

$\forall r_1, r_2 \in R, L[(r_1 \cdot r_2^*)^*] = L[r_1 \cdot (r_1 + r_2)^* + \varepsilon]$

Proof of ***Theorem 1***

Let $L_1, L_2 \in \mathcal{P}(\Sigma^*)$, we must show that $L_1 \cdot (L_1 \cup L_2)^* = (L_1 \cdot L_2^*)^+$:

First we will prove that $L_1 \cdot (L_1 \cup L_2)^* \subseteq (L_1 \cdot L_2^*)^+$ by proving the following proposition by induction on n : $\forall n \in \mathbb{N}, L_1 \cdot (L_1 \cup L_2)^n \subseteq \bigcup_{i=1}^{n+1} (L_1 \cdot L_2^*)^i$

(**Note**: $\mathbb{N}$ is the set of **non-negative** integers).

Basis:

For $n = 0$ we must show that $L_1 \cdot (L_1 \cup L_2)^0 \subseteq \bigcup_{i=1}^{1} (L_1 \cdot L_2^*)^i$ **:**

Since $\{\varepsilon\} \subseteq L_2^*$ we get that $L_1 \cdot \{\varepsilon\} \subseteq L_1 \cdot L_2^*$, But since $(L_1 \cup L_2)^0 = \{\varepsilon\}$ we get that $L_1 \cdot (L_1 \cup L_2)^0 \subseteq L_1 \cdot L_2^*$, Now since $\bigcup_{i=1}^{1} (L_1 \cdot L_2^*)^i = (L_1 \cdot L_2^*)^1 = L_1 \cdot L_2^*$ we get $L_1 \cdot (L_1 \cup L_2)^0 \subseteq \bigcup_{i=1}^{1} (L_1 \cdot L_2^*)^i$ as was to be shown.

Induction Hypothesis

Suppose that for some $n = k \in \mathbb{N}$ we have $L_1 \cdot (L_1 \cup L_2)^k \subseteq \bigcup_{i=1}^{k+1} (L_1 \cdot L_2^*)^i$.

Induction Step

We must show that $L_1 \cdot (L_1 \cup L_2)^{k+1} \subseteq \bigcup_{i=1}^{k+2} (L_1 \cdot L_2^*)^i$:

Let $w \in L_1 \cdot (L_1 \cup L_2)^{k+1}$, Since $L_1 \cdot (L_1 \cup L_2)^{k+1} = L_1 \cdot (L_1 \cup L_2)^k \cdot (L_1 \cup L_2)$ we get that $\exists u \in L_1 \cdot (L_1 \cup L_2)^k, v \in L_1 \cup L_2, w = u \cdot v$, Now since $u \in L_1 \cdot (L_1 \cup L_2)^k$ we get by the induction hypothesis that $u \in \bigcup_{i=1}^{k+1} (L_1 \cdot L_2^*)^i$,

And so $\exists i \in \{1,...,k+1\}, u \in \left(L_1 \cdot L_2^*\right)^i$.

Now since $v \in L_1 \cup L_2$ there are two cases: $v \in L_1 \vee v \in L_2$

(1) If $v \in L_1$ then since $L_1 \subseteq L_1 \cdot L_2^*$ we get that $v \in L_1 \cdot L_2^*$ and so $w = u \cdot v \in (L_1 \cdot L_2^*)^i \cdot (L_1 \cdot L_2^*)^1 = (L_1 \cdot L_2^*)^{i+1}$, Since $i \in \{1,...,k+1\}$ we get that $i+1 \in \{2,...,k+2\}$, Now if we denote $j = i+1$ we get that $j \in \{2,...,k+2\} \subseteq \{1,...,k+2\}$, Thus we've shown that $\exists j \in \{1,...,k+2\}, w \in (L_1 \cdot L_2^*)^j$ and so $w \in \bigcup_{i=1}^{k+2} (L_1 \cdot L_2^*)^i$.

(2) If $v \in L_2$, Since $1 \leq i$ we get that $(L_1 \cdot L_2^*)^i = (L_1 \cdot L_2^*)^{i-1} \cdot (L_1 \cdot L_2^*)^1$, Now because $u \in (L_1 \cdot L_2^*)^i = (L_1 \cdot L_2^*)^{i-1} \cdot (L_1 \cdot L_2^*)^1$ we get that $\exists x \in (L_1 \cdot L_2^*)^{i-1}, y \in (L_1 \cdot L_2^*)^1, u = x \cdot y$, Now since $y \in L_1 \cdot L_2^*$ and $v \in L_2$ we get that $y \cdot v \in (L_1 \cdot L_2^*) \cdot L_2 = L_1 \cdot (L_2^* \cdot L_2) = L_1 \cdot L_2^+ \subseteq L_1 \cdot L_2^*$ and so $w = u \cdot v = x \cdot y \cdot v \in (L_1 \cdot L_2^*)^{i-1} \cdot (L_1 \cdot L_2^*) = (L_1 \cdot L_2^*)^i$, Now since $i \in \{1,...,k+1\}$ and since $\{1,...,k+1\} \subseteq \{1,...,k+2\}$ we get that $i \in \{1,...,k+2\}$, Thus, we've shown that $\exists i \in \{1,...,k+2\}, w \in (L_1 \cdot L_2^*)^i$ and we can conclude that $w \in \bigcup_{i=1}^{k+2} (L_1 \cdot L_2^*)^i$.

From cases **(1)** and **(2)** we can conclude that it is always the case $w \in \bigcup_{i=1}^{k+2} (L_1 \cdot L_2^*)^i$ as was to be shown.

Now we can prove that $L_1 \cdot (L_1 \cup L_2)^* \subseteq (L_1 \cdot L_2^*)^+$:

Let $w \in L_1 \cdot (L_1 \cup L_2)^*$, Therefore, we get that $\exists n \in \mathbb{N}, w \in L_1 \cdot (L_1 \cup L_2)^n$, Now by the proposition we've just shown we get that

$L_1 \cdot (L_1 \cup L_2)^n \subseteq \bigcup_{i=1}^{n+1} (L_1 \cdot L_2^*)^i \subseteq \bigcup_{i=1}^{\infty} (L_1 \cdot L_2^*)^i = (L_1 \cdot L_2^*)^+$, and so $w \in (L_1 \cdot L_2^*)^+$ as was to be shown.

Now we will prove that $(L_1 \cdot L_2^*)^+ \subseteq L_1 \cdot (L_1 \cup L_2)^*$ by proving the following proposition by induction on n : $\forall n \in \mathbb{Z}^+, (L_1 \cdot L_2^*)^n \subseteq L_1 \cdot (L_1 \cup L_2)^*$
(**Note**: $\mathbb{Z}^+$ is the set of **positive** integers).

Basis:

For $n = 1$ we must show that $(L_1 \cdot L_2^*)^1 \subseteq L_1 \cdot (L_1 \cup L_2)^*$:

Since $L_2 \subseteq L_1 \cup L_2$ we get that $L_2^* \subseteq (L_1 \cup L_2)^*$ and so $L_1 \cdot L_2^* \subseteq L_1 \cdot (L_1 \cup L_2)^*$.

Induction Hypothesis

Suppose that for some $n = k \in \mathbb{Z}^+$ we have $(L_1 \cdot L_2^*)^k \subseteq L_1 \cdot (L_1 \cup L_2)^*$.

Induction Step

We must show that $(L_1 \cdot L_2^*)^{k+1} \subseteq L_1 \cdot (L_1 \cup L_2)^*$:

Since in the basis case we've shown that $L_1 \cdot L_2^* \subseteq L_1 \cdot (L_1 \cup L_2)^*$, And in the induction hypothesis we supposed that $(L_1 \cdot L_2^*)^k \subseteq L_1 \cdot (L_1 \cup L_2)^*$,
We can conclude that :

$$(L_1 \cdot L_2^*)^{k+1} = (L_1 \cdot L_2^*) \cdot (L_1 \cdot L_2^*)^k \subseteq L_1 \cdot (L_1 \cup L_2)^* \cdot L_1 \cdot (L_1 \cup L_2)^* \subseteq$$
$$\subseteq L_1 \cdot (L_1 \cup L_2)^* \cdot (L_1 \cup L_2) \cdot (L_1 \cup L_2)^* = L_1 \cdot (L_1 \cup L_2)^+ \subseteq L_1 \cdot (L_1 \cup L_2)^*$$

as was to be shown.

Now we will show that $(L_1 \cdot L_2^*)^+ \subseteq L_1 \cdot (L_1 \cup L_2)^*$:

Let $w \in (L_1 \cdot L_2^*)^+$, Therefore $\exists n \in \mathbb{Z}^+, w \in (L_1 \cdot L_2^*)^n$, and by the proposition we've just shown we get $w \in L_1 \cdot (L_1 \cup L_2)^*$ as was to be shown.

From the two set inclusions we've just shown we can conclude that $L_1 \cdot (L_1 \cup L_2)^* = (L_1 \cdot L_2^*)^+$ as was to be shown.

Q.E.D.

Proof of ***Theorem 2***

Let $L_1, L_2 \in \mathcal{P}(\Sigma^*)$ **,** we must show that $L_1 \cdot (L_1 \cup L_2)^* \cup \{\varepsilon\} = (L_1 \cdot L_2^*)^*$:

Since $(L_1 \cdot L_2^*)^* = (L_1 \cdot L_2^*)^0 \cup (L_1 \cdot L_2^*)^+ = \{\varepsilon\} \cup (L_1 \cdot L_2^*)^+$ and since $L_1 \cdot (L_1 \cup L_2)^* = (L_1 \cdot L_2^*)^+$ by ***Theorem 1***, we can conclude that $(L_1 \cdot L_2^*)^* = (L_1 \cdot L_2^*)^+ \cup \{\varepsilon\} = L_1 \cdot (L_1 \cup L_2)^* \cup \{\varepsilon\}$ as was to be shown.

Q.E.D.

Proof of ***Corollary 1***

Let $r_1, r_2 \in R$ be some regular expressions, we must show that $L[(r_1 \cdot r_2^*)^+] = L[r_1 \cdot (r_1 + r_2)^*]$:

Since $L[(r_1 \cdot r_2^*)^+] = L[r_1 \cdot r_2^*]^+ = (L[r_1] \cdot L[r_2^*])^+ = (L[r_1] \cdot L[r_2]^*)^+$ and $L[r_1 \cdot (r_1 + r_2)^*] = L[r_1] \cdot L[(r_1 + r_2)^*] = L[r_1] \cdot L[r_1 + r_2]^* = L[r_1] \cdot (L[r_1] \cup L[r_2])^*$,
And since by ***Theorem 1*** we get $(L[r_1] \cdot L[r_2]^*)^+ = L[r_1] \cdot (L[r_1] \cup L[r_2])^*$,
we can conclude that $L[(r_1 \cdot r_2^*)^+] = L[r_1 \cdot (r_1 + r_2)^*]$ as was to be shown.

Q.E.D.

Proof of ***Corollary 2***

Let $r_1, r_2 \in R$ be some regular expressions, we must show that $L[(r_1 \cdot r_2^*)^*] = L[r_1 \cdot (r_1 + r_2)^* + \varepsilon]$:

Since $L[(r_1 \cdot r_2^*)^*] = L[r_1 \cdot r_2^*]^* = (L[r_1] \cdot L[r_2^*])^* = (L[r_1] \cdot L[r_2]^*)^*$ and

$$L[r_1 \cdot (r_1 + r_2)^* + \varepsilon] = L[r_1 \cdot (r_1 + r_2)^*] \cup L[\varepsilon] = L[r_1] \cdot L[(r_1 + r_2)^*] \cup \{\varepsilon\} =$$
$$= L[r_1] \cdot L[r_1 + r_2]^* \cup \{\varepsilon\} = L[r_1] \cdot (L[r_1] \cup L[r_2])^* \cup \{\varepsilon\},$$

And since by ***Theorem 2*** we get $(L[r_1] \cdot L[r_2]^*)^* = L[r_1] \cdot (L[r_1] \cup L[r_2])^* \cup \{\varepsilon\}$,
we can conclude that $L[(r_1 \cdot r_2^*)^*] = L[r_1 \cdot (r_1 + r_2)^* + \varepsilon]$ as was to be shown.

Q.E.D.
