Let Σ be some alphabet and $\Re$ be the set of regular expressions on Σ.

Lemma 1

$$\forall A,B,C,D \in \mathcal{P}\left(\Sigma^*\right), A \subseteq C \wedge B \subseteq D \longrightarrow A \cdot B \subseteq C \cdot D$$

Lemma 2

$$\forall A,B,C \in \mathcal{U}, A \subseteq C \wedge B \subseteq C \wedge A \cap B = \varnothing \longrightarrow A \subseteq C - B$$

Theorem 1

$$\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, \forall L \in \mathcal{P}\left(\Sigma^*\right), L^{m+n} = L^m \cdot L^n$$

Corollary 1 for languages

$$\forall L \in \mathcal{P}\left(\Sigma^*\right), L^* = L^* \cdot L^*$$

Corollary 1 for regular expressions

$$\forall r \in \Re, L[r^*] = L[r^* r^*]$$

Note 1

$\exists L \in \mathcal{P}\left(\Sigma^*\right), L^+ = L^+ \cdot L^+$ and $\exists L \in \mathcal{P}\left(\Sigma^*\right), L^+ \neq L^+ \cdot L^+$

Theorem 2

$$\forall n \in \mathbb{Z}^+, \forall L \in \mathcal{P}\left(\Sigma^*\right), L^* = (L^*)^n$$

Corollary 2 for languages

$$\forall L \in \mathcal{P}\left(\Sigma^*\right), L^* = (L^*)^*$$

Corollary 2 for regular expressions

$$\forall r \in \Re, L[r^*] = L[(r^*)^*]$$

Theorem 3

$$\forall n \in \mathbb{Z}^+, \forall L \in \mathcal{P}\left(\Sigma^*\right), (L^+)^n \subseteq L^+$$

Corollary 3 for languages

$$\forall L \in \mathcal{P}\left(\Sigma^*\right), L^+ = (L^+)^+$$

Corollary 3 for regular expressions

$$\forall r \in \Re, L[r^+] = L[(r^+)^+]$$

Theorem 4

$$\forall n \in \mathbb{N}, \forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^n \subseteq L_2^n$$

Corollary 4A

$$\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^* \subseteq L_2^*$$

Corollary 4B

$$\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^+ \subseteq L_2^+$$

Theorem 5A for languages

$\forall L \in \mathcal{P}\left(\Sigma^*\right), \Sigma \subseteq L \longrightarrow L^* = \Sigma^*$

Theorem 5A for regular expressions

$\forall r \in \Re, \Sigma \subseteq L[r] \longrightarrow L[r^*] = L[\Sigma^*]$

Theorem 5B1 for languages

$\forall L \in \mathcal{P}\left(\Sigma^*\right), \Sigma \subseteq L \wedge \varepsilon \notin L \longrightarrow L^+ = \Sigma^+$

Theorem 5B1 for regular expressions

$\forall r \in \Re, \Sigma \subseteq L[r] \wedge \varepsilon \notin L[r] \longrightarrow L[r^+] = L[\Sigma^+]$

Theorem 5B2 for languages

$\forall L \in \mathcal{P}\left(\Sigma^*\right), \Sigma \subseteq L \wedge \varepsilon \in L \longrightarrow L^+ = \Sigma^*$

Theorem 5B2 for regular expressions

$\forall r \in \Re, \Sigma \subseteq L[r] \wedge \varepsilon \in L[r] \longrightarrow L[r^+] = L[\Sigma^*]$

Proof of ***Lemma 1***

We'll prove that $\forall A, B, C, D \in \mathcal{P}\left(\Sigma^*\right), A \subseteq C \wedge B \subseteq D \longrightarrow A \cdot B \subseteq C \cdot D$:

Let $A, B, C, D \in \mathcal{P}\left(\Sigma^*\right)$ such that $A \subseteq C$ and $B \subseteq D$,

Let $w \in A \cdot B$, Therefore, $\exists u \in A, v \in B, w = u \cdot v$, Now since $u \in A$ we get that $u \in C$ and since $v \in B$ we get that $v \in D$, And so $\exists u \in C, v \in D, w = u \cdot v$, Thus, $w \in C \cdot D$ as was to be shown.

Q.E.D.

Proof of ***Lemma 2***

We will show that $\forall A, B, C \in \mathcal{U}, A \subseteq C \wedge B \subseteq C \wedge A \cap B = \varnothing \longrightarrow A \subseteq C - B$:

Let $A, B, C \in \mathcal{U}$ be some sets over some universal set $\mathcal{U}$, such that $A \subseteq C$, $B \subseteq C$ and $A \cap B = \varnothing$, We must show that $A \subseteq C - B$:

Let $x \in A$, Since $A \subseteq C$ we get that $x \in C$, Now we will show that $x \notin B$ by contradiction:

Suppose that $x \in B$, Since $x \in A$ we get that $x \in A \cap B$, And so $A \cap B \neq \varnothing$ which contradicts the fact that $A \cap B = \varnothing$.

Thus $x \notin B$, Since $x \in C$ we get by definition of set difference that $x \in C - B$ as was to be shown.

Q.E.D.

<u>Proof of ***Theorem 1***</u>

We'll prove that $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, \forall L \in \mathcal{P}\left(\Sigma^*\right), L^{m+n} = L^m \cdot L^n$

by induction on n:

<u>Basis:</u>

For $n = 0$ we must show that $\forall m \in \mathbb{N}, \forall L \in \mathcal{P}\left(\Sigma^*\right), L^{m+0} = L^m \cdot L^0$:

Let $m \in \mathbb{N}$ and $L \in \mathcal{P}(\Sigma^*)$, Therefore, $L^m \cdot L^0 = L^m \cdot \{\varepsilon\} = L^m = L^{m+0}$, as was to be shown.

<u>Induction Hypothesis:</u>

Suppose that for some $n = k \in \mathbb{N}$ we have
$\forall m \in \mathbb{N}, \forall L \in \mathcal{P}\left(\Sigma^*\right), L^{m+k} = L^m \cdot L^k$

<u>Induction Step:</u>

We must show that for $n = k+1$ we have
$\forall m \in \mathbb{N}, \forall L \in \mathcal{P}\left(\Sigma^*\right), L^{m+(k+1)} = L^m \cdot L^{k+1}$:

Let $m \in \mathbb{N}$ and $L \in \mathcal{P}(\Sigma^*)$, Therefore, $L^{m+(k+1)} = L^{(m+k)+1} = L^{(m+k)} \cdot L^1$,
By the induction hypothesis we get that $L^{m+k} = L^m \cdot L^k$ and so
$L^{m+(k+1)} = L^{(m+k)} \cdot L^1 = L^m \cdot L^k \cdot L^1 = L^m \cdot L^{k+1}$ as was to be shown.

Q.E.D.

<u>Proof of ***Corollary 1 for languages***</u>

We must show that $\forall L \in \mathcal{P}\left(\Sigma^*\right), L^* = L^* \cdot L^*$:

Let $L \in \mathcal{P}(\Sigma^*)$, First we'll show that $L^* \subseteq L^* \cdot L^*$:

Let $w \in L^*$, Since $\varepsilon \in L^*$ we get that $w \cdot \varepsilon \in L^* \cdot L^*$, But since $w = w \cdot \varepsilon$ we get that $w \in L^* \cdot L^*$.

Now we'll show that $L^* \cdot L^* \subseteq L^*$:

Let $w \in L^* \cdot L^*$, Thus $\exists u \in L^*, v \in L^*, w = u \cdot v$, Now by definition of L^* we get that $\exists m \in \mathbb{N}, u \in L^m$ and $\exists n \in \mathbb{N}, v \in L^n$, Thus $w = u \cdot v \in L^m \cdot L^n$, Now by ***Theorem 1*** we get that $L^m \cdot L^n = L^{m+n}$, And so $w \in L^{m+n}$, Now if we denote $j = m+n$ we will get that $j \in \mathbb{N}$ (since $m,n \in \mathbb{N}$) and $w \in L^j$, Therefore, we've shown that $\exists j \in \mathbb{N}, w \in L^j$, And we can conclude by definition of L^* that $w \in L^*$.

From the two set inclusions we can conclude that $L^* = L^* \cdot L^*$ as was to be shown.

Q.E.D.

Proof of ***Corollary 1 for regular expressions***

We must show that $\forall r \in \Re, L[r^*] = L[r^* r^*]$:

Let $r \in \Re$ be some regular expression, We will show that $L[r^*] = L[r^* r^*]$:

Since $L[r^* r^*] = L[r^*] \cdot L[r^*] = L[r]^* \cdot L[r]^*$ and since by ***Corollary 1 for languages*** we get that $L[r]^* \cdot L[r]^* = L[r]^*$, We can conclude that $L[r^* r^*] = L[r]^* = L[r^*]$ as was to be shown.

Q.E.D.

Proof of ***Note 1***

First we will show that $\exists L \in \mathcal{P}\left(\Sigma^*\right), L^+ = L^+ \cdot L^+$:

Suppose $\Sigma = \{a\}$, We will take $L = \{\varepsilon\} \in \mathcal{P}\left(\Sigma^*\right)$, It is clear that $L^+ = \{\varepsilon\}$, And so $L^+ \cdot L^+ = \{\varepsilon\} \cdot \{\varepsilon\} = \{\varepsilon \cdot \varepsilon\} = \{\varepsilon\} = L^+$.

Now we will show that $\exists L \in \mathcal{P}\left(\Sigma^*\right), L^+ \neq L^+ \cdot L^+$:

Suppose $\Sigma = \{a\}$, We will take $L = \{a\} \in \mathcal{P}\left(\Sigma^*\right)$, Since $L \subseteq L^+$ and since $a \in L$ we can conclude that $a \in L^+$.

Now we will show that $a \notin L^+ \cdot L^+$ by contradiction:

Suppose that $a \in L^+ \cdot L^+$, Thus $\exists u \in L^+, v \in L^+, a = u \cdot v$, And so $1 = |a| = |u \cdot v| = |u| + |v|$, Which implies that $|v| = 1 - |u|$, Since $0 \leq |u|$ we get that $-|u| \leq 0$, And so $1 - |u| \leq 1$, Thus $|v| \leq 1$, And so $0 \leq |v| \leq 1$.

Now since $0 \leq |v| \leq 1$ we get that $-1 \leq -|v| \leq 0$, And so $0 \leq 1 - |v| \leq 1$,
Now since $|u| = 1 - |v|$, We can conclude that $0 \leq |u| \leq 1$.

Thus $0 \leq |u|, |v| \leq 1$.

Now there are two cases: $|u| = 0 \vee |u| = 1$

(1) If $|u| = 0$ then $u = \varepsilon$, Since $u \in L^+$ we get that $\varepsilon \in L^+$, But $\varepsilon \notin L^+$ and we reached a contradiction.

(2) If $|u| = 1$ then $|v| = 1 - |u| = 1 - 1 = 0$, And so $v = \varepsilon$, Since $v \in L^+$ we get that $\varepsilon \in L^+$, But $\varepsilon \notin L^+$ and we reached a contradiction.

From cases **(1)** and **(2)** we see that we always reach a contradiction,
Thus it must be the case that $a \notin L^+ \cdot L^+$.

Now since $a \in L^+$ and $a \notin L^+ \cdot L^+$ we can conclude that $L^+ \neq L^+ \cdot L^+$.

Q.E.D.

<u>Proof of ***Theorem 2***</u>

We will show that $\forall n \in \mathbb{Z}^+, \forall L \in \mathcal{P}\left(\Sigma^*\right), L^* = (L^*)^n$ by induction on n:

<u>Basis:</u>

For $n = 1$ we must show that $\forall L \in \mathcal{P}\left(\Sigma^*\right), L^* = (L^*)^1$:

It is clear that $\forall L \in \mathcal{P}\left(\Sigma^*\right), L^* = (L^*)^1$, Thus there is nothing to show here.

Induction Hypothesis:

Suppose that for some $n = k \in \mathbb{Z}^+$ we have $\forall L \in \mathcal{P}(\Sigma^*), L^* = (L^*)^k$

Induction Step:

We must show that for $n = k+1$ we have $\forall L \in \mathcal{P}(\Sigma^*), L^* = (L^*)^{k+1}$:

Let $L \in \mathcal{P}(\Sigma^*)$, Since $(L^*)^{k+1} = (L^*)^k \cdot L^*$ and since by the induction hypothesis we get that $L^* = (L^*)^k$, we can conclude that $(L^*)^{k+1} = (L^*)^k \cdot L^* = L^* \cdot L^*$, Now by ***Corollary 1*** we get that $L^* \cdot L^* = L^*$, And so $(L^*)^{k+1} = L^*$ as was to be shown.

Q.E.D.

Proof of ***Corollary 2 for languages***

We must show that $\forall L \in \mathcal{P}(\Sigma^*), L^* = (L^*)^*$:

Let $L \in \mathcal{P}(\Sigma^*)$, First we'll show that $L^* \subseteq (L^*)^*$:

Since $L^* \subseteq \bigcup_{i=0}^{\infty} (L^*)^i$ and since $(L^*)^* = \bigcup_{i=0}^{\infty} (L^*)^i$ we can conclude that $L^* \subseteq (L^*)^*$.

Now we'll show that $(L^*)^* \subseteq L^*$:

Let $w \in (L^*)^*$, Therefore $\exists n \in \mathbb{N}, w \in (L^*)^n$,
Now there are two cases: $n = 0 \vee n \neq 0$

(1) If $n = 0$ then we get that $w \in (L^*)^0 = \{\varepsilon\}$, And so $w = \varepsilon$, Now since $\varepsilon \in L^*$, We can conclude that $w \in L^*$.

(2) If $n \neq 0$ then it must be the case that $n \in \mathbb{Z}^+$ and by ***Theorem 2*** we get that $L^* = (L^*)^n$, Now since $w \in (L^*)^n$ we get $w \in L^*$.

From cases **(1)** and **(2)** we can conclude that it is always the case that $w \in L^*$.

From the two set inclusions we just shown, We can conclude that $L^* = (L^*)^*$ as was to be shown.

Q.E.D.

Proof of ***Corollary 2 for regular expressions***

We must show that $\forall r \in \Re, L[r^*] = L[(r^*)^*]$ ***:***

Let $r \in \Re$ be some regular expression, We will show that $L[r^*] = L[(r^*)^*]$:

Since $L[(r^*)^*] = L[r^*]^* = (L[r]^*)^*$ and since by ***Corollary 2 for languages*** we get that $(L[r]^*)^* = L[r]^*$, We can conclude that $L[(r^*)^*] = L[r]^* = L[r^*]$ as was to be shown.

Q.E.D.

Proof of ***Theorem 3***

We will show that $\forall n \in \mathbb{Z}^+, \forall L \in \mathcal{P}\left(\Sigma^*\right), (L^+)^n \subseteq L^+$ by induction on n:

Basis:

For $n = 1$ we must show that $\forall L \in \mathcal{P}\left(\Sigma^*\right), (L^+)^1 \subseteq L^+$:

Let $L \in \mathcal{P}(\Sigma^*)$, Since $(L^+)^1 = L^+$ we get by definition of set equality that $(L^+)^1 \subseteq L^+$ as was to be shown.

Induction Hypothesis:

Suppose that for some $n = k \in \mathbb{Z}^+$ we have $\forall L \in \mathcal{P}\left(\Sigma^*\right), (L^+)^k \subseteq L^+$

Induction Step:

We must show that for $n = k + 1$ we have $\forall L \in \mathcal{P}\left(\Sigma^*\right), (L^+)^{k+1} \subseteq L^+$:

Let $L \in \mathcal{P}(\Sigma^*)$, Since $(L^+)^{k+1} = (L^+)^k \cdot L^+$ and since by the induction hypothesis we get that $(L^+)^k \subseteq L^+$, we can conclude by ***Lemma 1*** that $(L^+)^{k+1} = (L^+)^k \cdot L^+ \subseteq L^+ \cdot L^+$.

Now we will show that $L^+ \cdot L^+ \subseteq L^+$:

Let $w \in L^+ \cdot L^+$, Thus, $\exists u \in L^+, v \in L^+, w = u \cdot v$, Now by definition of L^+ we get that $\exists i \in \mathbb{Z}^+, u \in L^i$ and $\exists j \in \mathbb{Z}^+, v \in L^j$, Thus, $w = u \cdot v \in L^i \cdot L^j$, Now by ***Theorem 1*** we get that $L^i \cdot L^j = L^{i+j}$ and if we denote $k = i + j$, We will get that $k \in \mathbb{Z}^+$ and $L^{i+j} = L^k$, Thus, $w = u \cdot v \in L^i \cdot L^j = L^{i+j} = L^k$ and we've shown that $\exists k \in \mathbb{Z}^+, w \in L^k$, Now by definition of L^+ we get $w \in L^+$, And so $L^+ \cdot L^+ \subseteq L^+$.

Thus, We can conclude that $(L^+)^{k+1} \subseteq L^+$ as was to be shown.

Q.E.D.

Proof of ***Corollary 3 for languages***

We must show that $\forall L \in \mathcal{P}\left(\Sigma^*\right), L^+ = (L^+)^+$:

Let $L \in \mathcal{P}(\Sigma^*)$, First we'll show that $L^+ \subseteq (L^+)^+$:

Since $L^+ \subseteq \bigcup_{i=1}^{\infty} (L^+)^i$ and since $(L^+)^+ = \bigcup_{i=1}^{\infty} (L^+)^i$ we can conclude that $L^+ \subseteq (L^+)^+$.

Now we'll show that $(L^+)^+ \subseteq L^+$:

Let $w \in (L^+)^+$, Therefore $\exists n \in \mathbb{Z}^+, w \in (L^+)^n$,
Now by ***Theorem 3*** we get that $(L^+)^n \subseteq L^+$, And so $w \in L^+$.

From the two set inclusions we can conclude that $L^+ = (L^+)^+$ as was to be shown.

Q.E.D.

Proof of ***Corollary 3 for regular expressions***

We must show that $\forall r \in \mathfrak{R}, L[r^+] = L[(r^+)^+]$:

Let $r \in \mathfrak{R}$ be some regular expression, We will show that $L[r^+] = L[(r^+)^+]$:

Since $L[(r^+)^+] = L[r^+]^+ = (L[r]^+)^+$ and since by <u>***Corollary 3 for languages***</u> we get that $(L[r]^+)^+ = L[r]^+$, We can conclude that $L[(r^+)^+] = L[r]^+ = L[r^+]$ as was to be shown.

Q.E.D.

<u>Proof of ***Theorem 4***</u>

We'll prove that $\forall n \in \mathbb{N}, \forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^n \subseteq L_2^n$ by induction on n :

<u>***Basis:***</u>

For $n = 0$ we must show that $\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^0 \subseteq L_2^0$:

Let $L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right)$ be such that $L_1 \subseteq L_2$.

Since $L_1^0 = \{\varepsilon\}$ and since $L_2^0 = \{\varepsilon\}$ we get that $L_1^0 = L_2^0$, Now by definition of set equality we get that $L_1^0 \subseteq L_2^0$ as was to be shown.

<u>***Induction Hypothesis:***</u>

Suppose that for some $n = k \in \mathbb{N}$ we have
$\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^k \subseteq L_2^k$

<u>***Induction Step:***</u>

We must show that for $n = k + 1$ we have
$\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^{k+1} \subseteq L_2^{k+1}$:

Let $L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right)$ be such that $L_1 \subseteq L_2$.

By the induction hypothesis we get that $L_1^k \subseteq L_2^k$ and by <u>***Lemma 1***</u> we get that $L_1^k \cdot L_1 \subseteq L_2^k \cdot L_2$, Now since $L_1^{k+1} = L_1^k \cdot L_1$ and since $L_2^{k+1} = L_2^k \cdot L_2$ we get that $L_1^{k+1} \subseteq L_2^{k+1}$ as was to be shown.

Q.E.D.

<u>Proof of ***Corollary 4A***</u>

We must show that $\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^* \subseteq L_2^*$:

Let $L_1, L_2 \in \mathcal{P}(\Sigma^*)$ be such that $L_1 \subseteq L_2$, We must show that $L_1^* \subseteq L_2^*$:

Let $w \in L_1^*$, By definition of L_1^* we get that $\exists n \in \mathbb{N}, w \in L_1^n$, Thus,
By ***Theorem 4*** we get that $L_1^n \subseteq L_2^n$, And so $w \in L_2^n$, Therefore, we've shown that $\exists n \in \mathbb{N}, w \in L_2^n$ and by definition of L_2^* we get that $w \in L_2^*$, Thus, $L_1^* \subseteq L_2^*$ as was to be shown.

Q.E.D.

Proof of ***Corollary 4B***

We must show that $\forall L_1, L_2 \in \mathcal{P}\left(\Sigma^*\right), L_1 \subseteq L_2 \longrightarrow L_1^+ \subseteq L_2^+$:

Let $L_1, L_2 \in \mathcal{P}(\Sigma^*)$ be such that $L_1 \subseteq L_2$, We must show that $L_1^+ \subseteq L_2^+$:

Let $w \in L_1^+$, By definition of L_1^+ we get that $\exists n \in \mathbb{Z}^+, w \in L_1^n$, Thus,
By ***Theorem 4*** we get that $L_1^n \subseteq L_2^n$, And so $w \in L_2^n$, Therefore, we've shown that $\exists n \in \mathbb{Z}^+, w \in L_2^n$ and by definition of L_2^+ we get that $w \in L_2^+$, Thus, $L_1^+ \subseteq L_2^+$ as was to be shown.

Q.E.D.

Proof of ***Theorem 5A for languages***

We must show that $\forall L \in \mathcal{P}\left(\Sigma^*\right), \Sigma \subseteq L \longrightarrow L^* = \Sigma^*$:

Let $L \in \mathcal{P}(\Sigma^*)$ be such that $\Sigma \subseteq L$.

Since $\Sigma \in \mathcal{P}(\Sigma^*)$ and since $\Sigma \subseteq L$ we get by ***Corollary 4A*** that $\Sigma^* \subseteq L^*$.

Now since $L \in \mathcal{P}(\Sigma^*)$ we get by definition of $\mathcal{P}(\Sigma^*)$ that $L \subseteq \Sigma^*$,
Now by ***Corollary 4A*** we get $L^* \subseteq (\Sigma^*)^*$,
By using ***Corollary 2 for languages*** we can conclude that $(\Sigma^*)^* = \Sigma^*$,
And so $L^* \subseteq \Sigma^*$.

From the two set inclusions we've just shown, We can conclude that $L^* = \Sigma^*$ as was to be shown.

Q.E.D.

Proof of ***Theorem 5A for regular expressions***

We will show that $\forall r \in \Re, \Sigma \subseteq L[r] \longrightarrow L[r^*] = L[\Sigma^*]$:

Let $r \in \Re$ be some regular expression such that $\Sigma \subseteq L[r]$, We will show that $L[r^*] = L[\Sigma^*]$:

Since $\Sigma \subseteq L[r]$ we get by ***Theorem 5A for languages*** that $L[r]^* = L[\Sigma]^*$, And so $L[r^*] = L[\Sigma^*]$ as was to be shown.

Q.E.D.

Proof of ***Theorem 5B1 for languages***

We must show that $\forall L \in \mathcal{P}\left(\Sigma^*\right), \Sigma \subseteq L \wedge \varepsilon \notin L \longrightarrow L^+ = \Sigma^+$:

Let $L \in \mathcal{P}(\Sigma^*)$ be such that $\Sigma \subseteq L$ and $\varepsilon \notin L$.

Since $\Sigma \in \mathcal{P}(\Sigma^*)$ and since $\Sigma \subseteq L$ we get by ***Corollary 4B*** that $\Sigma^+ \subseteq L^+$.

Now since $L \in \mathcal{P}(\Sigma^*)$ we get that $L^+ \subseteq \Sigma^*$.

Now we'll show that $\varepsilon \notin L^+$ by contradiction:

Suppose that $\varepsilon \in L^+$, Therefore by definition of L^+ we get that $\exists n \in \mathbb{Z}^+, \varepsilon \in L^n$, Now since $1 \leq n$ we get by the recursive definition of L^n that $L^n = L^{n-1} \cdot L$, Thus $\varepsilon \in L^{n-1} \cdot L$, And so $\exists x \in L^{n-1}, y \in L, \varepsilon = x \cdot y$, Now we get $0 = |\varepsilon| = |x \cdot y| = |x| + |y|$, Thus $|y| = -|x|$, Now since $0 \leq |x|$ we can conclude that $-|x| \leq 0$, And so $|y| \leq 0$, But since $0 \leq |y|$ we get that $0 \leq |y| \leq 0$ and so $|y| = 0$, This implies that $y = \varepsilon$, Now since $y \in L$ we get that $\varepsilon \in L$ which contradicts the fact that $\varepsilon \notin L$.

Therefore it must be the case that $\varepsilon \notin L^+$.

Now since $L^+ \subseteq \Sigma^*$, $\{\varepsilon\} \subseteq \Sigma^*$ and $L^+ \cap \{\varepsilon\} = \varnothing$ (Since $\varepsilon \notin L^+$) we get by ***Lemma 2*** that $L^+ \subseteq \Sigma^* - \{\varepsilon\} = \Sigma^+$.

Since we've shown that $\Sigma^+ \subseteq L^+$ and $L^+ \subseteq \Sigma^+$ we can conclude by definition of set equality that $L^+ = \Sigma^+$ as was to be shown.

Q.E.D.

Proof of ***Theorem 5B1 for regular expressions***

We will show that $\forall r \in \Re, \Sigma \subseteq L[r] \wedge \varepsilon \notin L[r] \longrightarrow L[r^+] = L[\Sigma^+]$:

Let $r \in \Re$ be some regular expression such that $\Sigma \subseteq L[r]$ and $\varepsilon \notin L[r]$, We will show that $L[r^+] = L[\Sigma^+]$:

Since $\Sigma \subseteq L[r]$ and $\varepsilon \notin L[r]$ we get by ***Theorem 5B1 for languages*** that $L[r]^+ = L[\Sigma]^+$, And so $L[r^+] = L[\Sigma^+]$ as was to be shown.

Q.E.D.

Proof of ***Theorem 5B2 for languages***

We must show that b $\forall L \in \mathcal{P}\left(\Sigma^*\right), \Sigma \subseteq L \wedge \varepsilon \in L \longrightarrow L^+ = \Sigma^*$:

Let $L \in \mathcal{P}(\Sigma^*)$ be such that $\Sigma \subseteq L$ and $\varepsilon \in L$.
We'll show that $\Sigma^* \subseteq L^+$:

Let $w \in \Sigma^*$, Since $\Sigma^* = \{\varepsilon\} \cup \Sigma^+$, There are two cases: $w \in \{\varepsilon\} \vee w \in \Sigma^+$

(1) If $w \in \{\varepsilon\}$ then $w = \varepsilon$, Now since $L \subseteq L^+$, and since $\varepsilon \in L$ we get that $\varepsilon \in L^+$,And so $w \in L^+$.

(2) If $w \in \Sigma^+$ then since $\Sigma \subseteq L$ we get by ***Corollary 4B*** that $\Sigma^+ \subseteq L^+$, And so $w \in L^+$.

From cases **(1)** and **(2)** we see that it is always the case that $w \in L^+$, Thus, We can conclude that $\Sigma^* \subseteq L^+$.

Since we've shown that $\Sigma^* \subseteq L^+$ and since it is clear that $L^+ \subseteq \Sigma^*$,We can conclude that $L^+ = \Sigma^*$ as was to be shown.

Q.E.D.

Proof of ***Theorem 5B2 for regular expressions***

We will show that $\forall r \in \Re, \Sigma \subseteq L[r] \wedge \varepsilon \in L[r] \longrightarrow L[r^+] = L[\Sigma^*]$:

Let $r \in \Re$ be some regular expression such that $\Sigma \subseteq L[r]$ and $\varepsilon \in L[r]$, We will show that $L[r^+] = L[\Sigma^*]$:

Since $\Sigma \subseteq L[r]$ and $\varepsilon \in L[r]$ we get by ***Theorem 5B2 for languages*** that $L[r]^+ = L[\Sigma]^*$, And so $L[r^+] = L[\Sigma^*]$ as was to be shown.

Q.E.D.