Let Σ be some alphabet and R be the set of regular expressions on Σ.

<u>Theorem 1</u>

$$\forall L_1, L_2 \in \mathcal{P}(\Sigma^*), L_1 \subseteq L_2 \longrightarrow L_1^* \cdot L_2^* = L_2^*$$

<u>Theorem 2</u>

$$\forall L_1, L_2 \in \mathcal{P}(\Sigma^*), L_1 \subseteq L_2 \longrightarrow L_2^* \cdot L_1^* = L_2^*$$

<u>Corollary 1</u>

$$\forall r_1, r_2 \in R, L[r_1] \subseteq L[r_2] \longrightarrow L[r_1^* r_2^*] = L[r_2^*]$$

<u>Corollary 2</u>

$$\forall r_1, r_2 \in R, L[r_1] \subseteq L[r_2] \longrightarrow L[r_2^* r_1^*] = L[r_2^*]$$

<u>Proof of *Theorem 1*</u>

Let $L_1, L_2 \in \mathcal{P}(\Sigma^*)$ be such that $L_1 \subseteq L_2$, we must show that $L_1^* \cdot L_2^* = L_2^*$:

Since $L_1 \subseteq L_2$ we get that $L_1^* \subseteq L_2^*$, and so $L_1^* \cdot L_2^* \subseteq L_2^* \cdot L_2^* = L_2^*$.
Now since $\{\varepsilon\} \subseteq L_1^*$ we get that $L_2^* = \{\varepsilon\} \cdot L_2^* \subseteq L_1^* \cdot L_2^*$.
By the two set inclusions we can conclude that $L_1^* \cdot L_2^* = L_2^*$ as was to be shown.

Q.E.D.

<u>Proof of *Theorem 2*</u>

Let $L_1, L_2 \in \mathcal{P}(\Sigma^*)$ be such that $L_1 \subseteq L_2$, we must show that $L_2^* \cdot L_1^* = L_2^*$:

Since $L_1 \subseteq L_2$ we get that $L_1^* \subseteq L_2^*$, and so $L_2^* \cdot L_1^* \subseteq L_2^* \cdot L_2^* = L_2^*$.
Now since $\{\varepsilon\} \subseteq L_1^*$ we get that $L_2^* = L_2^* \cdot \{\varepsilon\} \subseteq L_2^* \cdot L_1^*$.
By the two set inclusions we can conclude that $L_2^* \cdot L_1^* = L_2^*$ as was to be shown.

Q.E.D.

Proof of ***Corollary 1***

Let $r_1, r_2 \in R$ be such that $L[r_1] \subseteq L[r_2]$, we must show that $L[r_1^* r_2^*] = L[r_2^*]$:

Since $L[r_1] \subseteq L[r_2]$ we get by ***Theorem 1*** that $L[r_1]^* \cdot L[r_2]^* = L[r_2]^*$, and so $L[r_1^* r_2^*] = L[r_1^*] \cdot L[r_2^*] = L[r_1]^* \cdot L[r_2]^* = L[r_2]^* = L[r_2^*]$ as was to be shown.

Q.E.D.

Proof of ***Corollary 2***

Let $r_1, r_2 \in R$ be such that $L[r_1] \subseteq L[r_2]$, we must show that $L[r_2^* r_1^*] = L[r_2^*]$:

Since $L[r_1] \subseteq L[r_2]$ we get by ***Theorem 2*** that $L[r_2]^* \cdot L[r_1]^* = L[r_2]^*$, and so $L[r_2^* r_1^*] = L[r_2^*] \cdot L[r_1^*] = L[r_2]^* \cdot L[r_1]^* = L[r_2]^* = L[r_2^*]$ as was to be shown.

Q.E.D.
