Let **G=(V,T,P,S)** be a context free grammar.

Prove by induction on **n** where **n ≥ 1** that:

$\forall \mathbf{w} \in \mathbf{T}^*, \alpha \in (\mathbf{V} \cup \mathbf{T})^+, \alpha \Longrightarrow^n \mathbf{w} \longrightarrow \alpha \Longrightarrow_{lm}^n \mathbf{w}$

(From a high level point of view the claim says that if some terminal word **w** can be derived form $\alpha \in (V \cup T)^+$ then there exist a leftmost derivation to derive **w** from α)

Basis

If **n = 1** then we must show that $\forall \mathbf{w} \in \mathbf{T}^*, \alpha \in (\mathbf{V} \cup \mathbf{T})^+, \alpha \Longrightarrow^1 \mathbf{w} \longrightarrow \alpha \Longrightarrow_{lm}^1 \mathbf{w}$:

Let $w \in T^*$ and $\alpha \in (V \cup T)^+$ such that $\alpha \Longrightarrow^1 w$.

by definition of the $\Longrightarrow$ relation we get that:

$\exists \psi, \chi, \gamma \in (\mathbf{V} \cup \mathbf{T})^*, \mathbf{A} \in \mathbf{V}, \alpha = \psi \mathbf{A} \chi \wedge \mathbf{w} = \psi \gamma \chi \wedge (\mathbf{A} \longrightarrow \gamma) \in \mathbf{P}$

Now since $\mathbf{w} \in \mathbf{T}^*$ and since $\mathbf{w} = \psi \gamma \chi$ we get that $\psi \in \mathbf{T}^*$ and so:

$\exists \psi \in \mathbf{T}^*, \chi, \gamma \in (\mathbf{V} \cup \mathbf{T})^*, \mathbf{A} \in \mathbf{V}, \alpha = \psi \mathbf{A} \chi \wedge \mathbf{w} = \psi \gamma \chi \wedge (\mathbf{A} \longrightarrow \gamma) \in \mathbf{P}$

And we get by definition of $\Longrightarrow_{lm}$ relation that $\alpha \Longrightarrow_{lm}^1 \mathbf{w}$ as was to be shown.

Induction hypothesis

Suppose that for some **n = k ≥ 1** we have:
$\forall \mathbf{w} \in \mathbf{T}^*, \alpha \in (\mathbf{V} \cup \mathbf{T})^+, \alpha \Longrightarrow^k \mathbf{w} \longrightarrow \alpha \Longrightarrow_{lm}^k \mathbf{w}$

Induction step

We must prove that: $\forall \mathbf{w} \in \mathbf{T}^*, \alpha \in (\mathbf{V} \cup \mathbf{T})^+, \alpha \Longrightarrow^{k+1} \mathbf{w} \longrightarrow \alpha \Longrightarrow_{lm}^{k+1} \mathbf{w}$

Let $\mathbf{w} \in \mathbf{T}^*$ and let $\alpha \in (\mathbf{V} \cup \mathbf{T})^+$ such that $\alpha \Longrightarrow^{k+1} \mathbf{w}$

We get that $\exists \beta \in (\mathbf{V} \cup \mathbf{T})^*, \alpha \Longrightarrow \beta \Longrightarrow^k \mathbf{w}$

since $\alpha \Longrightarrow \beta$ and since **G** is a context-free grammar we get that it must be the case that $\beta \in (\mathbf{V} \cup \mathbf{T})^* \mathbf{V} (\mathbf{V} \cup \mathbf{T})^*$

Also we get by the induction hypothesis that $\beta \Longrightarrow_{lm}^k \mathbf{w}$.

Now because $\alpha \Longrightarrow \beta$ we get by definition of the $\Longrightarrow$ relation that:

$\exists \gamma, \delta, \zeta \in (\mathbf{V} \cup \mathbf{T})^*, \mathbf{A} \in \mathbf{V}, \alpha = \gamma \mathbf{A} \delta \wedge \beta = \gamma \zeta \delta \wedge (\mathbf{A} \longrightarrow \zeta) \in \mathbf{P}$

Now there are two possibilities $\gamma \in \mathbf{T}^* \vee \gamma \notin \mathbf{T}^*$

(1) If $\gamma \in \mathbf{T}^*$ we get:

$\exists \gamma \in \mathbf{T}^*, \delta, \zeta \in (\mathbf{V} \cup \mathbf{T})^*, \mathbf{A} \in \mathbf{V}, \alpha = \gamma \mathbf{A} \delta \wedge \beta = \gamma \zeta \delta \wedge (\mathbf{A} \longrightarrow \zeta) \in \mathbf{P}$

And by definition of the $\Longrightarrow_{lm}$ relation we get that $\alpha \Longrightarrow_{lm} \beta$ and since $\beta \Longrightarrow_{lm}^k \mathbf{w}$ we get that $\alpha \Longrightarrow_{lm}^{k+1} \mathbf{w}$ as was to be shown.

(2) If $\gamma \notin \mathbf{T}^*$ we get that it must be the case that $\gamma \in (\mathbf{V} \cup \mathbf{T})^* \mathbf{V} (\mathbf{V} \cup \mathbf{T})^*$ and so:

$\exists \gamma_1 \in \mathbf{T}^*, \mathbf{B} \in \mathbf{V}, \gamma_2 \in (\mathbf{V} \cup \mathbf{T})^*, \gamma = \gamma_1 \mathbf{B} \gamma_2$

And we get that $\alpha = \gamma_1 \mathbf{B} \gamma_2 \mathbf{A} \delta$ and $\beta = \gamma_1 \mathbf{B} \gamma_2 \zeta \delta$.

Since $\beta \Longrightarrow_{lm}^k \mathbf{w}$ and since $\mathbf{k \geq 1}$ we get that:

$\exists \theta \in (\mathbf{V} \cup \mathbf{T})^*, \beta \Longrightarrow_{lm} \theta \Longrightarrow_{lm}^{k-1} \mathbf{w}$

Now since $\beta \Longrightarrow_{lm} \theta$ we get that $\gamma_1 \mathbf{B} \gamma_2 \zeta \delta \Longrightarrow_{lm} \theta$

And since $\gamma_1 \in \mathbf{T}^*$ there must be some production rule in P that was used to (leftmost) derive θ from β of the form $\mathbf{B} \longrightarrow \eta$.

I.e.

$\exists \eta \in (\mathbf{V} \cup \mathbf{T})^*, (\mathbf{B} \longrightarrow \eta) \in \mathbf{P}$

By applying this rule we get $\gamma_1 \mathbf{B} \gamma_2 \zeta \delta \Longrightarrow_{lm} \gamma_1 \eta \gamma_2 \zeta \delta$ and so $\theta = \gamma_1 \eta \gamma_2 \zeta \delta$.

Now $\boldsymbol{\alpha = \gamma_1 B \gamma_2 A \delta \Longrightarrow_{lm} \gamma_1 \eta \gamma_2 A \delta \Longrightarrow \gamma_1 \eta \gamma_2 \zeta \delta = \theta}$.

Since $\theta \Longrightarrow_{\mathbf{lm}}^{\mathbf{k-1}} \mathbf{w}$ we get that in particular $\theta \Longrightarrow^{\mathbf{k-1}} \mathbf{w}$ and so

$\boldsymbol{\gamma_1 \eta \gamma_2 A \delta} \Longrightarrow^{\mathbf{k}} \mathbf{w}$

Now by the induction hypothesis we get that $\boldsymbol{\gamma_1 \eta \gamma_2 A \delta} \Longrightarrow_{\mathbf{lm}}^{\mathbf{k}} \mathbf{w}$

And we got that $\alpha = \gamma_1 B \gamma_2 A \delta \Longrightarrow_{\mathbf{lm}} \gamma_1 \boldsymbol{\eta} \gamma_2 A \delta \Longrightarrow_{\mathbf{lm}}^{\mathbf{k}} \mathbf{w}$

And so $\alpha \Longrightarrow_{\mathbf{lm}}^{\mathbf{k+1}} \mathbf{w}$ as was to be shown.

Q.E.D.