GAS - CNO Gamma

* $(y_t / y_{t-1}) \sim \text{Gama}(\alpha_t, \lambda_t/\alpha_t)$; $\alpha_t > 0$, $\lambda_t > 0$
$y_t > 0$

$$E[y_t / y_{t-1}] = \lambda_t \quad e \quad V[y_t/y_{t-1}] = \lambda_t^2 / \alpha_t$$

* $$f(y_t / y_{t-1}; \alpha_t, \lambda_t) = \frac{1}{\Gamma(\alpha_t)} \frac{1}{(\alpha_t^{-1} \lambda_t)^{\alpha_t}} y_t^{\alpha_t - 1} \exp\left(\frac{-\alpha_t}{\lambda_t} y_t \right)$$

* $$\ln f(y_t / y_{t-1}; \alpha_t, \lambda_t) = -\ln \Gamma(\alpha_t) - \alpha_t \ln(1/\alpha_t) - \alpha_t \ln(\lambda_t) + (\alpha_t - 1) \ln y_t - \frac{\alpha_t}{\lambda_t} y_t$$

* $$\nabla^{\alpha} = \frac{\partial \ln f(y_t / y_{t-1})}{\partial \alpha_t} =$$
$$= \ln y_t - \frac{y_t}{\lambda_t} + \ln \alpha_t - \psi_1(\alpha_t) - \ln \lambda_t + 1$$
onde $\psi_1(\alpha_t) = \Gamma'(\alpha_t) / \Gamma(\alpha_t)$ $\Rightarrow$ função digamma

* $$\nabla^{\lambda} = \frac{\partial \ln f(y_t / y_{t-1})}{\partial \lambda_t} = \frac{\alpha_t}{\lambda_t} \cdot \left(\frac{y_t}{\lambda_t} - 1 \right)$$

obs: note que, como esperado, $E_{t-1}[\nabla^{\alpha}] = E_{t-1}[\nabla^{\lambda}] = 0$

* Com isso, temos

$$\nabla_t = \begin{pmatrix} \nabla_t^{\lambda} \\ \nabla_t^{\alpha} \end{pmatrix} = \begin{pmatrix} \ln(y_t) - y_t/\lambda_t + \ln(\alpha_t) - \psi_1(\alpha_t) - \ln(\lambda_t) + 1 \\ \alpha_t/\lambda_t \left(y_t/\lambda_t - 1 \right) \end{pmatrix}$$

* $I_{t|t-1} = \begin{bmatrix} I^{\alpha}_{t/t-1} & I^{\alpha,\lambda}_{t/t-1} \\ I^{\alpha/\lambda}_{t/t-1} & I^{\lambda}_{t/t-1} \end{bmatrix}$ com

$$I^{\alpha}_{t/t-1} = E_{t-1}\left[\nabla^{\alpha}_{t} \nabla'^{\alpha}_{t} \right] = E_{t-1}\left[\nabla^{\alpha}_{t} \right]^2 = -E_{t-1} \frac{\partial}{\partial \alpha_t}\left[\nabla^{\alpha}_{t} \right]$$

$$I^{\lambda}_{t/t-1} = E_{t-1}\left[\nabla^{\lambda}_{t} \nabla'^{\lambda}_{t} \right] = E_{t-1}\left[\nabla^{\lambda}_{t} \right]^2 = -E_{t-1} \frac{\partial}{\partial \lambda_t}\left[\nabla^{\lambda}_{t} \right]$$

$$I^{\alpha,\lambda}_{t/t-1} = E_{t-1}\left[\nabla^{\lambda}_{t} \nabla'^{\alpha}_{t} \right] = -E_{t-1} \frac{\partial}{\partial \alpha_t}\left[\nabla^{\lambda}_{t} \right] = -E_{t-1} \frac{\partial}{\partial \lambda_t}\left[\nabla^{\alpha}_{t} \right]$$

↳ Desenvolvendo os contas, chegamos em

$$I^{\alpha}_{t/t-1} = \psi_2(\alpha_t) - \frac{1}{\alpha_t} \quad \text{com} \quad \underbrace{\psi_2(\alpha_t)}_{\text{função Trigamma}} = \frac{\psi_1'(\alpha_t)}{\psi_1(\alpha_t)}$$

$$I^{\lambda}_{t/t-1} = \alpha_t / \lambda_t^2$$

$$I^{\alpha,\lambda}_{t/t-1} = 0$$

↳ Logo, $I_{t/t-1} = \begin{bmatrix} \psi_2(\alpha_t) - 1/\alpha_t & 0 \\ 0 & \alpha_t/\lambda_t^2 \end{bmatrix}$

* Até esse momento, utilizamos implicitamente a função de ligação identidade, $\alpha_t = f_{1t}$ e $\lambda_t = f_{2t}$.
No entanto, essa função não é adequada para garantir $\alpha_t > 0$ e $\lambda_t > 0$.
Usaremos a função de ligação logarítmica

$$\begin{cases} h_1(\alpha_T) = \tilde{f}_{1T} & \ln \alpha_T = \tilde{f}_{1T} \quad \text{ou} \quad \alpha_T = e^{\tilde{f}_{1T}}, \alpha_T = f_{1T} \\ h_2(d_T) = \tilde{f}_{2T} & \ln d_T = \tilde{f}_{2T} \quad \text{ou} \quad d_T = e^{\tilde{f}_{2T}}, d_T = f_{2T} \end{cases}$$

* Precisamos obter agora $\tilde{\nabla}_T$ e $\tilde{I}_{t/t-1}$

* Usaremos que $\tilde{\nabla}_T = (\dot{h}_t)^{-1} \nabla_T$

onde $\dot{h}_T = \begin{pmatrix} dh_1/df_{1T} & dh_1/df_{2T} \\ dh_2/df_{12} & dh_2/df_{2T} \end{pmatrix} = \begin{pmatrix} 1/\alpha_T & 0 \\ 0 & 1/d_T \end{pmatrix}$

$$\dot{h}_t' = \dot{h}_T \Rightarrow (\dot{h}_T')^{-1} = \begin{pmatrix} \alpha_T & 0 \\ 0 & d_T \end{pmatrix}$$

Logo

$$\tilde{\nabla}_T = \begin{pmatrix} \alpha_T & 0 \\ 0 & d_T \end{pmatrix} \begin{pmatrix} \ln(y_T) - Y_t/d_T + \ln(\alpha_T) - \psi_1(\alpha_T) - \ln(d_T) + 1 \\ \alpha_T/d_T \left(y_T/d_T - 1 \right) \end{pmatrix}$$

$$\begin{cases} \tilde{\nabla}_T^{\alpha} = \alpha_T \left(\ln(y_T) - Y_t/d_T + \ln(\alpha_T) - \psi_1(\alpha_T) - \ln(d_T) + 1 \right) \\ \tilde{\nabla}_T^{d} = \alpha_T \left(y_T/d_T - 1 \right) \end{cases}$$

De forma similar: $\tilde{I}_{t|t-1} = (\dot{h}_t')^{-1} I_{t/t-1} (\dot{h}_t')^{-1}$

$$\tilde{I}_{t/t-1} = \begin{pmatrix} \alpha_t & 0 \\ 0 & d_T \end{pmatrix} \begin{pmatrix} \psi_2(\alpha_T) - 1/\alpha_T & 0 \\ 0 & \alpha_T/d_T^2 \end{pmatrix} \begin{pmatrix} \alpha_T & 0 \\ 0 & d_T \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_T^2 \left(\psi_2(\alpha_T) - 1/\alpha_T \right) & 0 \\ 0 & \alpha_T \end{pmatrix}$$

* Seja um modelo GAS-CNO Gamma

$$p(y_t / f_t, F_t ; \psi) \sim \text{Gama}(\alpha, \lambda_t / \alpha)$$ → apenas λ_t varia no tempo

$$E[y_t / \alpha, \lambda_t] = \lambda_t \quad \text{e} \quad V[y_t / \alpha, \lambda_t] = \lambda_t^2 / \alpha$$

Parametrização $f_t = \ln \lambda_t$; $\underset{\sim}{\psi} = [\ln \alpha, \ln k_1, \ln k_2, \ln k_3, \eta, \phi]$