Rethinking Teaching Abstract Algebra

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- I am unaware whether or not others have attempted an approach like this before.

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- A colleague offered the approach of introducing groups via a simplified study of Galois groups

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- Many students (at least in the US) do not see any of this material until late in their second year or their third year
- This does not serve students well whatsoever, in my opinion

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- Upon reaching a transition point, return to group and group-action definition of vector space

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- K is more than a group, it is a field. I.e. we have division, and our elements commute
- Proposal is to perturb this structure to reveal other algebraic structures

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- Until now we swept delicate notions of multiplicative inverses and division under the rug

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What becomes of K, and what becomes of V?

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Rings are kind of weird! We should provide examples.

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- the quaternions $\mathbb{H}$ with component-wise addition and a distributive multiplication form a non-commutative ring with identity
- polynomials with coefficients in a commutative ring R with identity form a commutative ring with identity, denoted $R[X]$, with component-wise addition and distributive multiplication
- $n \times n$ matrices with entries from an arbitrary ring form a non-commutative ring $M_n(R)$ for $n \geq 2$. If R has identity, so does $M_n(R)$

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Any real world execution of this may be met with (pleasant) surprise.

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And, as we more or less hinted at, if R is a field, M is a vector space.

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By now students have witnessed on some level a plethora of algebraic structures.

Thank you

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