Polynomial Systems in Receptor Pharmacology

Gilles Gnacadja

AMGEN
Thousand Oaks, California, USA

Society for Industrial and Applied Mathematics

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Industry Days: A Southern California Perspective

Polynomial Systems in Receptor Pharmacology

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Pharmacology

Pharmacology: Study interactions between biological processes and therapeutic agents.

- **Pharmacokinetics**: *what the body does to the drug*.
- **Pharmacodynamics**: *what the drug does to the body*.
- ***"Pharmacostatics"*** at beginning of research pipeline: Study binding equilibrium *in vitro* of interactions between
  - receptors (biochemical recognition units),
  - ligands (pathogenic and therapeutic molecules),
  - other molecules.

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Current Practices – Illustrated with Trivial Receptor-Ligand Interaction

Chemistry

$$R + L \underset{k_{\text{off}}}{\overset{k_{\text{on}}}{\rightleftharpoons}} RL$$

Equilibrium state
Implicit equations
(Polynomial system)

$$[R]_{\text{Equil}} + [RL]_{\text{Equil}} = [R]_{\text{Total}}$$
$$[L]_{\text{Equil}} + [RL]_{\text{Equil}} = [L]_{\text{Total}}$$
$$[RL]_{\text{Equil}} = [R]_{\text{Equil}}[L]_{\text{Equil}}/K_{\text{d}}$$

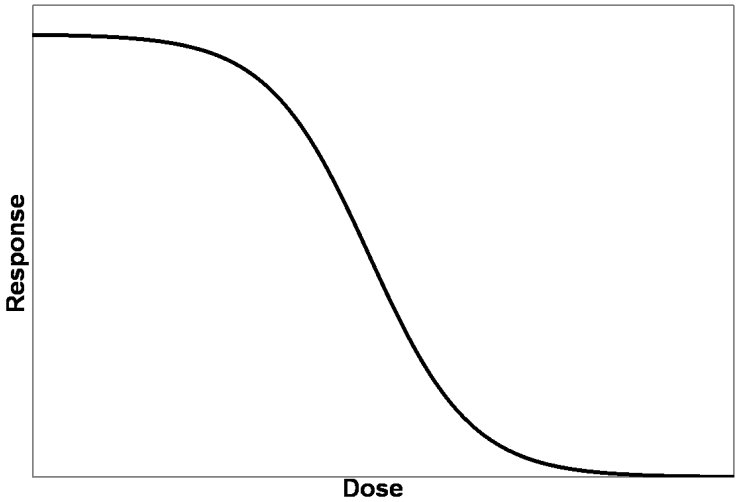
Equilibrium state
Explicit formula
(Forcibly derived)

$$[R]_{\text{Total}} \ll [L]_{\text{Total}}$$
$$[L]_{\text{Equil}} = [L]_{\text{Total}} \quad \text{(“No Ligand Depletion”)}$$
$$\frac{[RL]_{\text{Equil}}}{[R]_{\text{Total}}} = \frac{[L]_{\text{Total}}}{K_{\text{d}} + [L]_{\text{Total}}}$$

Issues with Forcibly Derived Formulas

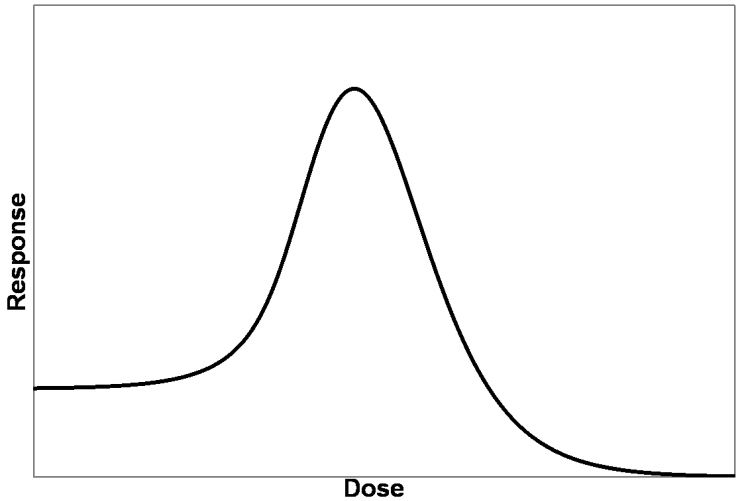
- $[R]_{\text{Total}} \ll [L]_{\text{Total}}$ is not mathematically sound.
  Operational wisdom: $[R]_{\text{Total}} < [L]_{\text{Total}}/10$.
- Not always applicable experimentally,
  e.g. in miniaturized assays.
- Need not be applicable *in vivo*.
- New formula derivation needed for every new interaction.
- Only possible with *"receptor-centric"* systems.
  (There is one receptor and all complex molecules contain it.)

One More Issue with Forcibly Derived Formulas



This shape is expected and is what forcibly derived closed-form formulas give.

Non-monotone responses do occur and cannot be simulated with forcibly derived closed-form formulas.



Excuses for Forcibly Derived Formulas

- Somewhat admissible:

  These formulas are simple and always give an answer. They are (often, not always) directionally correct.

- Not admissible:

  We have many sources of error, so why fix this one?

Polynomial Systems in Receptor Pharmacology

"No Ligand Depletion" Assumption Discarded for Competitive Antagonism

Miniaturized Receptor Binding Assays: Complications Arising from Ligand Depletion
Clare M. Scaramellini Carter, Juliet R. Leighton-Davies and Steven J. Charlton
J Biomol Screen 2007; 12; 255 originally published online Jan 26, 2007

$$\begin{array}{ccc} R & + L & \rightleftharpoons RL \\ + & & \\ A & & \\ \Downarrow\Uparrow & & \\ RA & & \end{array}$$

"No Ligand Depletion" Assumption Partially Discarded for Allosteric Modulation

0022-3565/08/3253-927–934$20.00
THE JOURNAL OF PHARMACOLOGY AND EXPERIMENTAL THERAPEUTICS

JPET 325:927–934, 2008

The Impact of Orthosteric Radioligand Depletion on the Quantification of Allosteric Modulator Interactions

Vimesh A. Avlani, David J. McLoughlin, Patrick M. Sexton, and Arthur Christopoulos

$$\begin{array}{ccc} R + L & \rightleftharpoons & RL \\ + & & + \\ A & & A \\ \Updownarrow & & \Updownarrow \\ RA + L & \rightleftharpoons & RLA \end{array}$$

with

$$[A]_{\text{Equil}} = [A]_{\text{Total}}$$

More Shots at the "No Ligand Depletion" Assumption

Biochimica et Biophysica Acta, 632 (1980) 464—469
© Elsevier/North-Holland Biomedical Press

COMPETITIVE BINDING STUDIES WITH MULTIPLE SITES EFFECTS ARISING FROM DEPLETION OF THE FREE RADIOLIGAND

J.W. WELLS, N.J.M. BIRDSALL, A.S.V. BURGEN and E.C. HULME

0026-895X/87/060603-07$02.00/0
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MOLECULAR PHARMACOLOGY, 31:603–609

Ligand Dissociation Constants From Competition Binding Assays: Errors Associated with Ligand Depletion

AVRAM GOLDSTEIN and RONALD W. BARRETT

Proc. Natl. Acad. Sci. USA
Vol. 84, pp. 6654–6658, October 1987
Biochemistry

An accurate method for determination of receptor–ligand and enzyme–inhibitor dissociation constants from displacement curves

AMNON HOROVITZ AND ALEXANDER LEVITZKI

ARCHIVES OF BIOCHEMISTRY AND BIOPHYSICS
Vol. 284, No. 1, January, pp. 26–29, 1991

A Simple Method for Calculating the Dissociation Constant of a Receptor (or Enzyme) · Unlabeled Ligand Complex from Radioligand Displacement Measurements

Robert L. Martin, Franco Renosto, and Irwin H. Segel

0026-895X/95/061197-07$3.00/0
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MOLECULAR PHARMACOLOGY, 47:1197–1203 (1995).

Interpretation of Binding Curves Obtained with High Receptor Concentrations: Practical Aid for Computer Analysis

STÉPHANE SWILLENS

Miniaturized Receptor Binding Assays: Complications Arising from Ligand Depletion
Clare M. Scaramellini Carter, Juliet R. Leighton-Davies and Steven J. Charlton
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0026-895X/08/7303-718–726$20.00
MOLECULAR PHARMACOLOGY
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Mol Pharmacol 73:718–726, 2008

Cell-Based and Biochemical Structure-Activity Analyses of Analogs of the Microtubule Stabilizer Dictyostatin

Brianne S. Raccor, Andreas Vogt, Rachel P. Sikorski, Charitha Madiraju, Raghavan Balachandran, Kia Montgomery, Youseung Shin, Yoshikazu Fukui, Won-Hyuk Jung, Dennis P. Curran, and Billy W. Day

Journal of Neuroscience Methods 188 (2010) 32–38

Concentration of receptor and ligand revisited in a modified receptor binding protocol for high-affinity radioligands: [^{3}H]Spiperone binding to D_2 and D_3 dopamine receptors

Juan Zhen, Tamara Antonio, Aloke K. Dutta, Maarten E.A. Reith

Polynomial Systems in Receptor Pharmacology

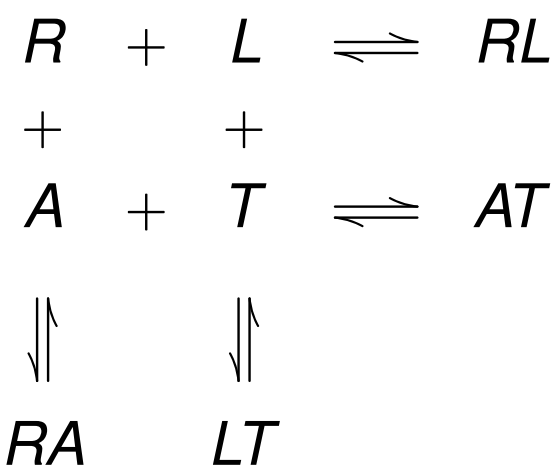
Goal

- Calculate binding equilibrium concentrations ...
- by *actually* solving the polynomial system of conservation and equilibrium conditions ...
- with the benefits of forcibly derived formulas, i.e. get an answer always and quickly.

Polynomial Systems in Receptor Pharmacology

The Receptor-Ligand-Antagonist-Trap System

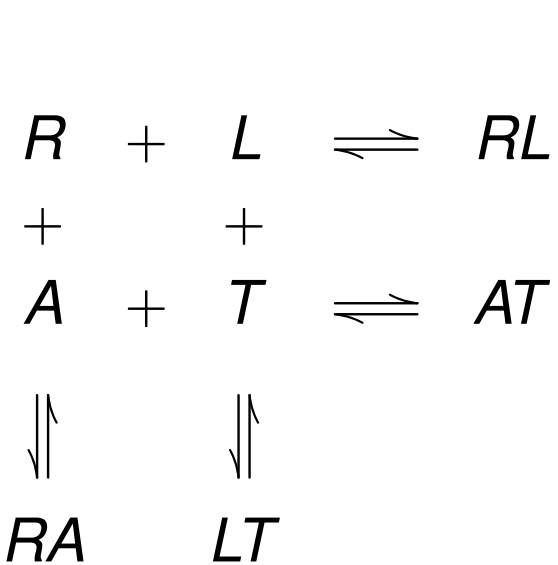
Polynomial System for Equilibrium State



$$
\begin{aligned}
[R] + K_{RL}[R][L] + K_{RA}[R][A] &= [R]_{\text{Total}} \\
[L] + K_{RL}[R][L] + K_{LT}[L][T] &= [L]_{\text{Total}} \\
[T] + K_{AT}[A][T] + K_{LT}[L][T] &= [T]_{\text{Total}} \\
[A] + K_{AT}[A][T] + K_{RA}[R][A] &= [A]_{\text{Total}}
\end{aligned}
$$

The Receptor-Ligand-Antagonist-Trap System

Polynomial System for Equilibrium State, Reformulated as a Fixed-Point Equation



$$[R] = \frac{[R]_{\text{Total}}}{1 + K_{RL}[L] + K_{RA}[A]}$$

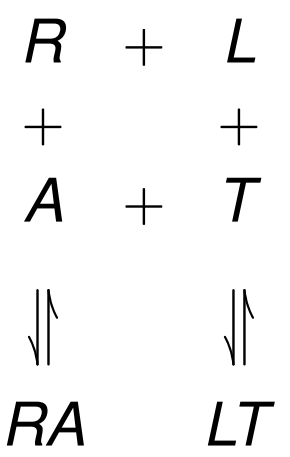
$$[L] = \frac{[L]_{\text{Total}}}{1 + K_{RL}[R] + K_{LT}[T]}$$

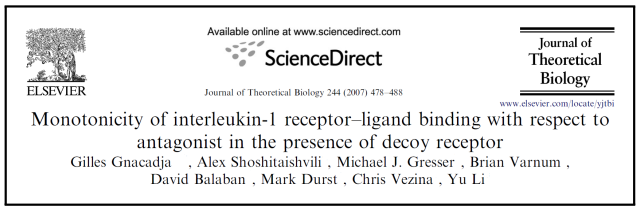
$$[T] = \frac{[T]_{\text{Total}}}{1 + K_{AT}[A] + K_{LT}[L]}$$

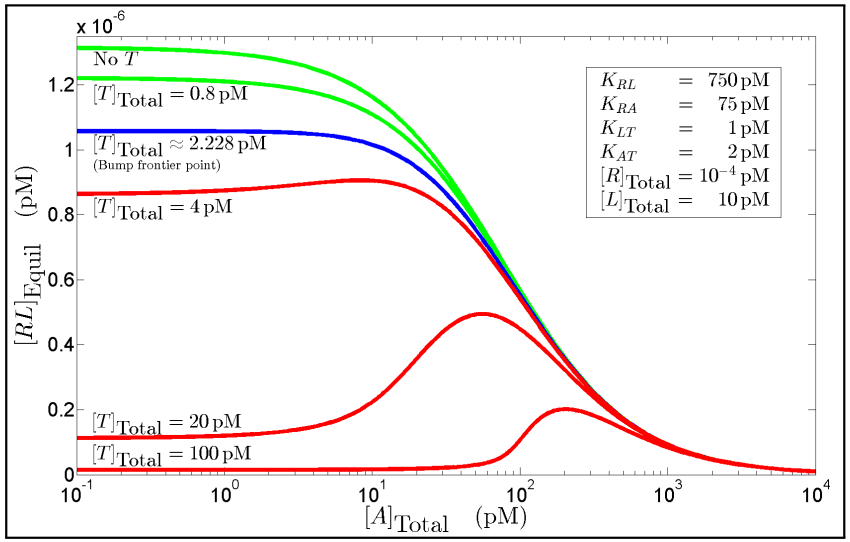
$$[A] = \frac{[A]_{\text{Total}}}{1 + K_{AT}[T] + K_{RA}[R]}$$

The Receptor-Ligand-Antagonist-Trap System

Polynomial System for Equilibrium State, Solved by Fixed-Point Iterations




Available online at www.sciencedirect.com
ScienceDirect
Journal of Theoretical Biology 244 (2007) 478–488
Journal of Theoretical Biology
www.elsevier.com/locate/yjtbi
ELSEVIER
Monotonicity of interleukin-1 receptor–ligand binding with respect to antagonist in the presence of decoy receptor
Gilles Gnacadja , Alex Shoshitaishvili , Michael J. Gresser , Brian Varnum , David Balaban , Mark Durst , Chris Vezina , Yu Li



- Fixed-point iterations converge, always and quickly. (Contraction w.r.t. Thompson metric; not L^1, L^2, L^∞, etc.)
- Experimentally observed motonicity features simulated and explained.
- Forcibly derived explicit formulas only produce always-increasing and always-decreasing curves.

Polynomial Systems in Receptor Pharmacology

The Polynomial System

$$x_i + \sum_{\alpha \in I} \alpha_i \, a_\alpha \, x^\alpha = b_i$$

$$i = 1, \ldots, n$$

Unknown $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geqslant 0}$

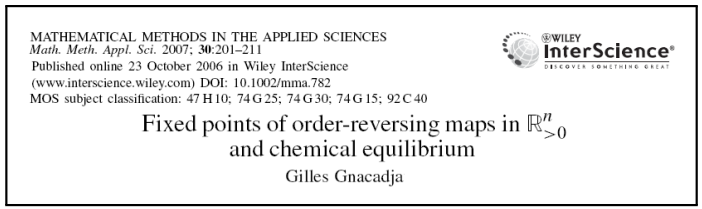
Given

$b_i \geqslant 0 \ , \ \forall \ i = 1, \ldots, n$

I finite $\subset \mathbb{Z}^n_{\geqslant 0} \backslash \{0_n, e_{n,1}, \ldots, e_{n,n}\}$

$a_\alpha \geqslant 0 \ , \ \forall \ \alpha = (\alpha_1, \ldots, \alpha_n) \in I$

$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$


MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2007; **30**:201–211
Published online 23 October 2006 in Wiley InterScience
(www.interscience.wiley.com) DOI: 10.1002/mma.782
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Fixed points of order-reversing maps in $\mathbb{R}^n_{>0}$ and chemical equilibrium

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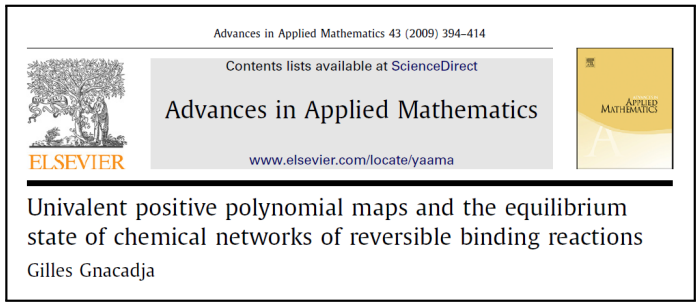
Advances in Applied Mathematics 43 (2009) 394–414
Contents lists available at ScienceDirect
Advances in Applied Mathematics
www.elsevier.com/locate/yaama

Univalent positive polynomial maps and the equilibrium state of chemical networks of reversible binding reactions

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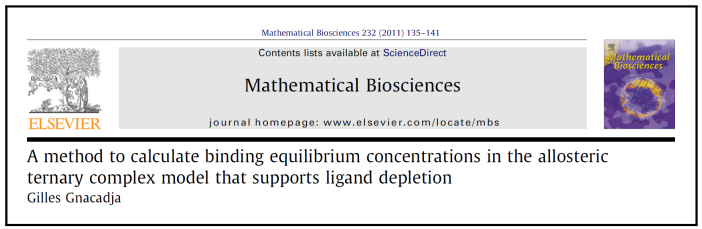
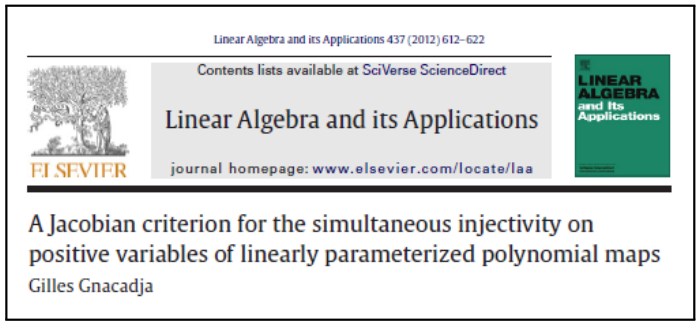

Mathematical Biosciences 232 (2011) 135–141
Contents lists available at ScienceDirect
Mathematical Biosciences
journal homepage: www.elsevier.com/locate/mbs

A method to calculate binding equilibrium concentrations in the allosteric ternary complex model that supports ligand depletion

Gilles Gnacadja


Linear Algebra and its Applications 437 (2012) 612–622
Contents lists available at SciVerse ScienceDirect
Linear Algebra and its Applications
journal homepage: www.elsevier.com/locate/laa

A Jacobian criterion for the simultaneous injectivity on positive variables of linearly parameterized polynomial maps

Gilles Gnacadja

Equivalent Fixed-Point Formulation

$$x_i = \frac{b_i}{1 + \sum_{\alpha \in I,\, \alpha_i > 0} \alpha_i \, a_\alpha \, x^{\alpha - e_{n,i}}} \qquad i = 1, \ldots, n$$

$$x = f(a, b, x)$$

$$f(a, b, \cdot) : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}$$ is smooth and order-reversing

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Solution Exists, is Unique, and is Infinitely Smooth

$$x_i + \sum_{\alpha \in I} \alpha_i \, a_\alpha \, x^\alpha = b_i \quad , \quad i = 1, \ldots, n$$

Existence — Brouwer Fixed Point Theorem

Uniqueness — Gale-Nikaidô Global Injectivity Theorem

GALE, D. and H. NIKAIDÔ
Math. Annalen 159, 81—93 (1965)
The Jacobian Matrix and Global Univalence of Mappings
By
DAVID GALE and HUKUKANE NIKAIDÔ in Providence and Osaka

Smoothness — Inverse Function Theorem

Polynomial Systems in Receptor Pharmacology

Solving by Fixed-Point Iterations

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Fixed points of order-reversing maps in $\mathbb{R}^n_{>0}$ and chemical equilibrium

Gilles Gnacadja

1. Fixed point iterations always converge if all complex molecules are (homo- or hetero-) dimers.
   - The map iterated is a contraction w.r.t. the Thompson metric (not L^1, L^2, L^∞, etc) and is order-reversing.
   - Iterates provide a descending sequence of boxes that converges to the solution point.
2. With k-mers with $k \geqslant 3$, the descending sequence of boxes converges, either to the solution point, or to a non-point box that contains it. ("Coupled fixed points")

First part partially rediscovered: M. G. A. van Dorp, F. Berger, E. Carlon
Computing Equilibrium Concentrations for Large Heterodimerization Networks
Physical Review E, Volume 84, Issue 3, September 2011

Solving by Fixed-Point Iterations

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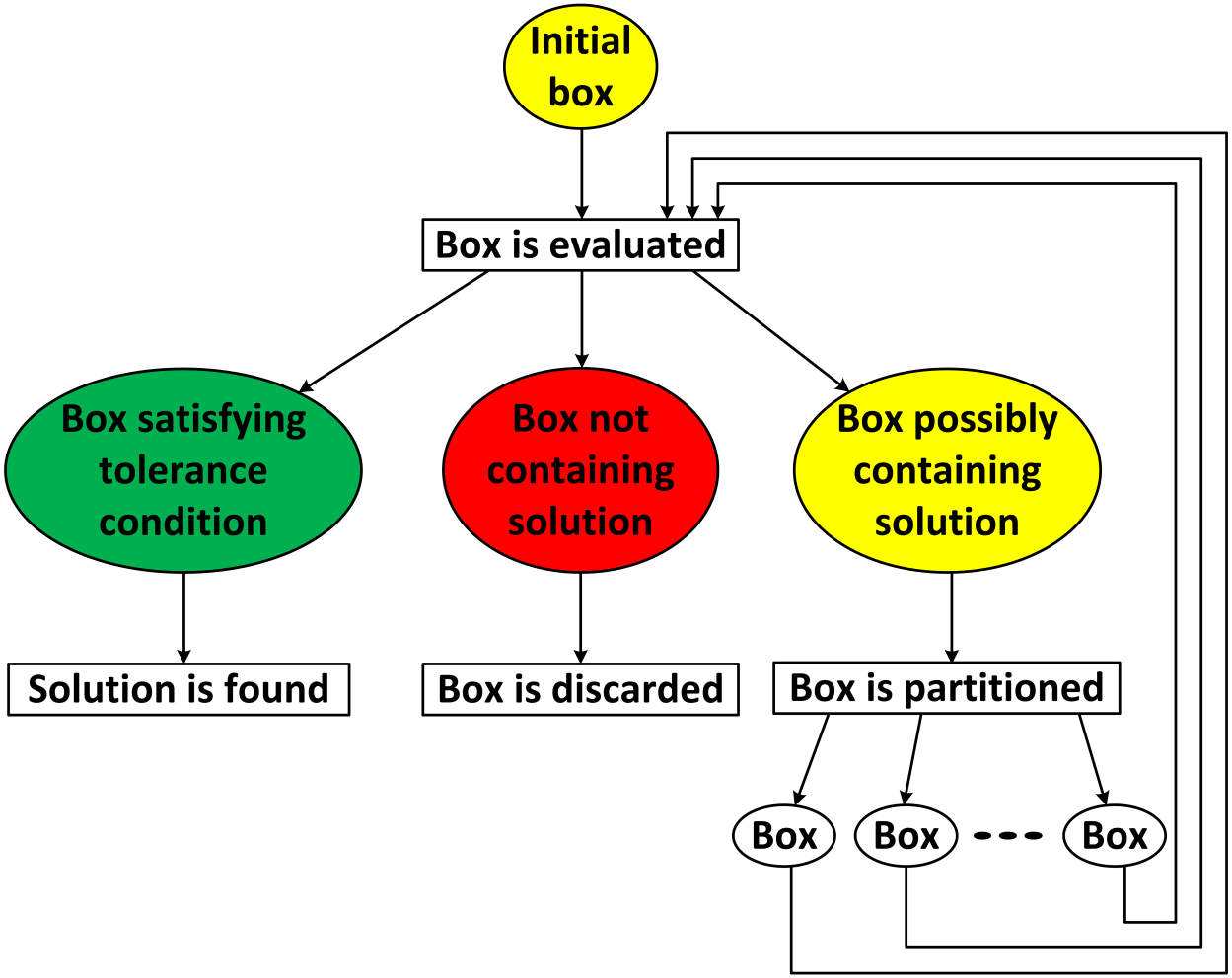
Polynomial Systems in Receptor Pharmacology

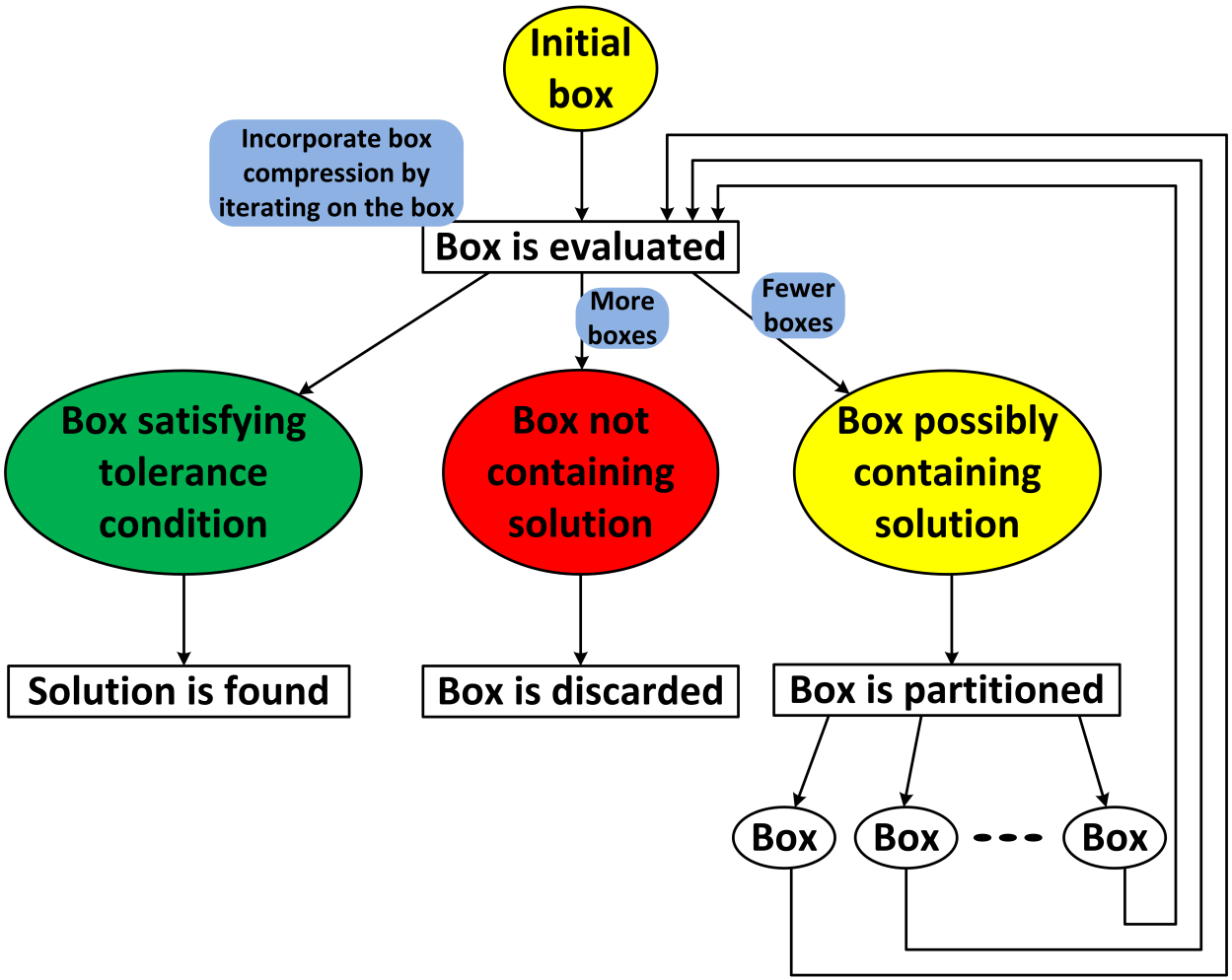
Solving by Cell Exclusion

If the descending sequence of boxes converges to a non-point box, divide the box and try the same on each subbox.
Do this again, and again...

Solving by Cell Exclusion



Strengthening Cell Discarding with Cell Compression



Summary

- Calculating binding equilibrium in “pharmacostatics” is about solving certain nice polynomial systems.
- Customary practice is to derive simple formulas at all cost.
  - Reduced accuracy
  - Only monotone response curves
- Algorithms exist to actually solve the polynomial systems with *a priori* assurance of success.
  - Exploit fixed-point formulation
  - Fixed-point iterations
  - Cell exclusion algorithm, strengthened by cell compression
  - Work in progress: stronger cell discarding conditions
- Enable scientists to study pharmacostatics of biochemical mechanisms that would otherwise be fudged or discarded.

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