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# How to hunt an invisible rabbit on a graph

Tatjana V. Abramovskaya [a], Fedor V. Fomin [b,c], Petr A. Golovach [b,c], Michał Pilipczuk [d]

[a] Department of Operations Research, St.-Petersburg State University, Universitetsky prospekt 28, Stary Peterhof, 198504, St-Petersburg, Russia

[b] Department of Informatics, University of Bergen, P.O. Box 7803, 5020 Bergen, Norway

[c] Steklov Institute of Mathematics at St. Petersburg, Russian Academy of Sciences, Russia

[d] Institute of Informatics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland

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ABSTRACT

We investigate Hunters & Rabbit game on graphs, where a set of hunters tries to catch an invisible rabbit that is forced to slide along an edge of a graph at every round. We show that the minimum number of hunters required to win on an $(n\times m)$-grid is $\lfloor \frac{\min\{n,m\}}{2} \rfloor + 1$. We also show that the extremal value of this number on $n$-vertex trees is between $\Omega(\log n/\log\log n)$ and $O(\log n)$.



## 1. Introduction

Our work originated from the following game puzzle. Hunter wants to shoot Rabbit who is hiding behind one of the three bushes growing in a row. Hunter does not see Rabbit, so he select one of the bushes and shoots at it. If Rabbit is behind the selected bush, then Hunter wins. Otherwise Rabbit, scared by the shot, jumps to one of the adjacent bushes. As Rabbit is infinitely fast, Hunter sees neither Rabbit's old nor new bush and has to select where to shoot again. Can Hunter always win in this game?

Of course, the answer is *yes*: Hunter has to shoot twice at the middle bush. If he misses the first time, it means that Rabbit was hiding either behind the leftmost or the rightmost bush. In both cases, the only adjacent bush where Rabbit can jump after the first shot is the middle one, thus the second shot at the middle bush finishes the game. A natural question is what happens if we have four bushes, and more generally, $n \geq 3$ bushes growing in a row? After a bit of thinking, the answer here is *yes* as

*E-mail addresses:* tanya.abramovskaya@gmail.com (T.V. Abramovskaya), fedor.fomin@uib.no (F.V. Fomin), petr.golovach@uib.no (P.A. Golovach), michal.pilipczuk@mimuw.edu.pl (M. Pilipczuk).

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well. This time Hunter wins by shooting consequently at the bushes $2, \ldots, n-1$ when $n$ is odd and at the bushes $2, \ldots, n$ when $n$ is even, and then repeating the same sequence of shots again.

In a slightly different situation, when bushes grow around a circle, say we have three bushes and Rabbit can jump from any of them to any of them, then Hunter cannot guarantee the success anymore. In this situation we need the second hunter and this brings us to the following setting. We consider Hunters & Rabbit game with two players, Hunter and Rabbit, playing on an undirected graph. Hunter player has a team of hunters who attempt to shoot the rabbit. At the beginning of the game, Rabbit player selects a vertex and occupies it. Then the players take turns starting with Hunter player. At every round of the game each of the hunters selects some vertex of the graph and the hunters shoot simultaneously at their respective aims. If the rabbit is not in a vertex that is hit by a shot, it jumps to an adjacent vertex. The rabbit is invisible to the hunters, but since we are interested in the guaranteed success of the hunters, we can assume that rabbit has a complete knowledge about all shots that the hunters plan. Hunter player wins if at some round of the game he succeeds to shoot the rabbit, and Rabbit player wins if the rabbit can avoid these situations forever. For a given graph $G$, we are interested in the minimum number of hunters sufficient to win in the Hunters & Rabbit game on $G$, for any strategy chosen by the rabbit player. We call this parameter the *hunting number* of a graph, and denote it by $h(G)$.

*Related work*. Britnell and Wildon studied the case with one hunter in [4]. They characterized the graphs for which one hunter (the prince in their terminology) can find the rabbit (the princess). This result was also independently discovered by Haslegrave [9], who not only characterized the graphs with hunting number one (in cat and mouse terminology) but also provided best possible capture times on such graphs. This problem is also mentioned as problem 6* in [6, p. 4] (as a problem of shooting shelters connected by tunnels) with a full solution given on pp. 52–54.

Hunters & Rabbit game is closely related to several pursuit-evasion and search games on graphs, see [7] for further references. In pursuit-evasion games a team of cops is trying to catch a robber located on the vertices of the graph. In cops–robbers terminology, Hunters & Rabbit is the Cops & Robber game, where the set of cops on helicopters (that is allowed to jump to any vertex) is trying to catch an invisible robber. The robber moves only to adjacent vertices and is forced to move every time the cops are in the air.

In particular, the classical Cops & Robbers games introduced independently by Winkler and Nowakowski [12] and by Quilliot [13] (see also the book by Bonato and Nowakowski [2] for the detailed introduction to the field), is the game where robber is visible, and the cops and robber move to adjacent vertices or remain on their present vertex. The variant of the game where the robber is invisible introduced by Tošić [14] and the variant where the cops use predefined paths as theirs search moves was introduced by Brass et al. [3]. Another related search game, node search, was introduced by Kirousis and Papadimitriou in [10,11]. Here cops can fly, that is move to any vertex they wish, the robber is invisible and very fast, that is can go to any vertex connected to his current location by a path containing no cops. Thus, Hunters & Rabbit can be seen as a variant of Tošić's game where cops have more power or as a variant of Kirousis–Papadimitriou's game, where the robber is more restricted. One more significant difference with mentioned games is that in most versions of Cops & Robbers games the robber is not forced to move at every round of the game, while in our setting the rabbit cannot stay at the same vertex for two consecutive rounds.

A randomized game called Hunter vs. Rabbit was considered by Adler et al. [1]; here, the hunter is allowed to move only along edges of the graph while there are no constraints on rabbit's moves.

*Our results and organization of the paper*. The remaining part of this paper is organized as follows. We give basic definitions and preliminary results in Section 2. We also show in this section that the hunting number of a graph does not exceed its pathwidth plus one and the bound is tight. In Section 3, we prove our first main result, namely that for an $(n \times m)$-grid $G$, it holds that $h(G) = \lfloor \frac{\min\{n,m\}}{2} \rfloor + 1$. This result is based on a new isoperimetric theorem that we find interesting on its own. In Section 4, we provide bounds on the hunting number of trees, which is the second contribution of the paper. We show that the hunting number of an $n$-vertex tree is always $O(\log n)$, but there are trees where it can be as large as $\Omega(\log n / \log \log n)$. We conclude with open problems in Section 5.

## 2. Basic definitions and preliminaries

We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$, the edge set is denoted by $E(G)$. For a set of vertices $S \subseteq V(G)$, $G[S]$ denotes the subgraph of $G$ induced by $S$, and by $G - S$ we denote the graph obtained from $G$ by the removal of all the vertices of $S$, that is the subgraph of $G$ induced by $V(G) \setminus S$. Let $G$ be an undirected graph. For a vertex $v$, we denote by $N_G(v)$ its *(open) neighborhood*, that is the set of vertices that are adjacent to $v$. The *closed neighborhood* of a vertex $v$ is $N_G[v] = N_G(v) \cup \{v\}$ and the *degree* of $v$ is denoted by $d_G(v) = |N_G(v)|$. For $S \subseteq V(G)$, we denote $N_G(S) = \cup_{v\in S} N_G(v) \setminus S$ and $\delta_G(S) = |N_G(S)|$. We omit the graph in the subscript whenever it is clear from the context. For positive integers $n$ and $m$, the $(n \times m)$-*grid* is the graph with the vertex set $\{(x, y)|1 \le x \le n, 1 \le y \le m\}$ where $x$ and $y$ are integers, such that two vertices $(x, y)$ and $(x', y')$ are adjacent if and only if $|x - x'| + |y - y'| = 1$. In other words, $(n \times m)$-grid is the Cartesian product of two paths with $n$ and $m$ vertices respectively. Throughout the paper $\log x$ denotes the logarithm of $x$ to base 2.

Consider the Hunters & Rabbit game on a graph $G$. Suppose that the Hunter player has $k$ hunters. A *hunters' strategy* is a (possible infinite) sequence $\mathcal{H} = (H_1, H_2, \dots)$ where $H_i \subseteq V(G)$ and $|H_i| \le k$ for $i \in \{1, 2, \dots\}$; the hunters shoot at each vertex of $H_i$ at the $i$th round of the game. Respectively, a *rabbit's strategy* is a sequence $\mathcal{R} = (r_0, r_1, \dots)$ of vertices of $G$ such that $r_i$ is adjacent to $r_{i-1}$ for $i \ge 1$; $r_0$ is an initial position of the rabbit, and it jumps from $r_{i-1}$ to $r_i$ after the $i$th shot of the hunters. For a set of vertices $S \subseteq V(G)$, a strategy $\mathcal{H}$ is a *winning hunters' strategy with respect to* $S$ if for any rabbit's strategy $\mathcal{R}$ such that $r_0 \in S$, there is $i \ge 1$ such that $r_{i-1} \in H_i$; $\mathcal{H}$ is a *winning hunters' strategy* if it is a winning hunters' strategy with respect to $V(G)$. Therefore, the hunting number $h(G)$ is the minimum $k$ such that there is a winning hunters' strategy for $k$ hunters. We also say that a rabbit's strategy $\mathcal{R}$ is a *winning rabbit's strategy against a hunters' strategy* $\mathcal{H}$ if $r_{i-1} \notin H_i$ for all $i \ge 1$.

As it is common for pursuit-evasion games with invisible fugitives, it is convenient to keep track of vertices that can or, respectively, cannot be occupied by the rabbit. Let $\mathcal{H} = (H_1, H_2, \dots)$ be a hunters' strategy. For $S \subseteq V(G)$, a vertex $v$ is *contaminated with respect to* $S$ after $i$th shot if there is a rabbit's strategy $\mathcal{R} = (r_0, r_1, \dots)$ such that $r_0 \in S$, $v = r_i$ and for any $j \in \{1, \dots, i\}$, $r_{j-1} \notin H_j$. Otherwise, we say that $v$ is *clear with respect to* $S$. If $S = V(G)$, then we simply say that $v$ is *contaminated* or *clear*. It is easy to see that if $X$ is a set of vertices contaminated at moment $i - 1$ with respect to some $S$, then the set of vertices contaminated at moment $i$ will be exactly $\Phi(X, H_i) = N(X \setminus H_i)$.

In our proofs we will be using the fact that we can always restrict our attention to finite strategies.

**Proposition 1.** *If $k$ hunters have a winning strategy on an $n$-vertex graph $G$ with respect to $S \subseteq V(G)$, then they have a winning strategy of length at most $2^n$.*

**Proof.** Consider the auxiliary arena graph, which is a directed graph $\mathcal{G}$ with the set of vertices $2^{V(G)}$ such that for any distinct $X, Y \subseteq V(G)$, $\mathcal{G}$ has the arc $(X, Y)$ if and only if there exists a set $H \subseteq V(G)$ of size at most $k$ such that $Y = \Phi(X, H)$. The graph $\mathcal{G}$ has $2^n$ vertices and at most $\binom{n}{k} \cdot 2^n$ arcs. It is easy to observe that $k$ hunters have a winning strategy on $G$ if and only if $\mathcal{G}$ has a directed walk that leads from $S$ and $\emptyset$: such paths correspond to Hunter's strategies, while the traversed vertices of $\mathcal{G}$ keep track of the set of contaminated vertices. Moreover, if such a walk exists, then there is also a directed simple path from $S$ to $\emptyset$, which corresponds to a winning strategy of length at most $2^n$. □

It is straightforward to observe that the hunting number is closed under taking subgraphs.

**Proposition 2.** *If $G_1$ is a subgraph of $G_2$, then $h(G_1) \le h(G_2)$.*

We also use the following property of Hunters & Rabbit on bipartite graphs.

**Lemma 1.** *Let $G$ be a bipartite graph and let $(V_1, V_2)$ be a bipartition of $V(G)$. Then $k$ hunters have a winning strategy on $G$ if and only if $k$ hunters have a winning strategy with respect to $V_1$.*

**Proof.** Clearly, if $k$ hunters have a winning strategy $\mathcal{H}$ on $G$, then $\mathcal{H}$ is a winning strategy with respect to $V_1$. Let $\mathcal{H}$ be a winning strategy on $G$ with respect to $V_1$. By Proposition 1, we can assume without loss of generality that $\mathcal{H} = (H_1, \dots, H_\ell)$ is finite. Moreover, we assume that $\ell$ is odd; otherwise, we just consider $\mathcal{H} = (H_1, \dots, H_\ell, H_\ell)$. Let $\mathcal{H}'$ be the strategy obtained by the concatenation of two copies of the sequence $\mathcal{H}$. We claim that $\mathcal{H}'$ is a winning strategy. To see it consider an arbitrary rabbit's

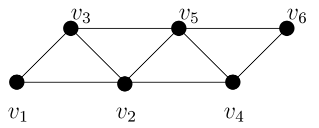


**Fig. 1.** An example of a graph $G$ with $h(G) = \mathbf{pw}(G) + 1 = 3$.

strategy $\mathcal{R} = (r_0, r_1, \ldots)$. If $r_0 \in V_1$, then there is $i \in \{1, \ldots, \ell\}$ such that $r_{i-1} \in H_i$ because $\mathcal{H}$ is a winning hunters' strategy with respect to $V_1$. Suppose then that $r_0 \in V_2$. If $r_{i-1} \notin H_i$ for $i \in \{1, \ldots, \ell\}$, then $r_\ell \in V_1$ because $\ell$ is odd. Then, there is $j \in \{1, \ldots, \ell\}$ such that $r_{\ell+j-1} \in H_j$. As in the rounds $\ell + 1, \ldots, 2\ell$ the hunters repeat $\mathcal{H}$, we have that the hunters shoot the rabbit in the $(\ell + j)$th round for some $j \in \{1, \ldots, \ell\}$. $\square$

We conclude the section by showing that the hunting number of a graph does not exceed its pathwidth plus one.

A *path decomposition* of a graph $G$ is a sequence $(X_1, \ldots, X_\ell)$ of subsets of $V(G)$ (called *bags*) such that

(i) $\bigcup_{i \in V(T)} X_i = V(G)$,
(ii) for each edge $xy \in E(G)$, $x, y \in X_i$ for some $i \in V(T)$, and
(iii) for each $x \in V(G)$, if $x \in X_i \cap X_j$ for some $1 \le i \le j \le \ell$, then $x \in X_k$ for all $k$ with $i \le k \le j$.

The *width* of a path decomposition $(X_1, \ldots, X_\ell)$ is $\max\{|X_i| \mid 1 \le i \le \ell\} - 1$. The *pathwidth* of a graph $G$ (denoted as $\mathbf{pw}(G)$) is the minimum width over all path decompositions of $G$.

**Proposition 3.** *For a graph $G$ it holds that $h(G) \le \mathbf{pw}(G) + 1$, and this bound is tight for graphs of pathwidth at least 2.*

**Proof.** Let $(X_1, \ldots, X_\ell)$ be a path decomposition of $G$ of width $k = \mathbf{pw}(G)$. We show that $\mathcal{H} = (X_1, \ldots, X_\ell)$ is a winning hunters' strategy for $k + 1$ hunters. To prove this, we show that all the vertices of $(\cup_{j=1}^{i} X_j) \setminus X_{i+1}$ are clear after the $i$th round, for all $i \in \{1, \ldots, \ell\}$; we assume here that $X_{\ell+1} = \emptyset$. It is straightforward to see that the claim holds for $i = 1$, because a vertex $v \in X_1$ can have a neighbor outside $X_1$ only if $v \in X_1 \cap X_2$, by the conditions (ii) and (iii) of the definition of a path decomposition. Assume that the claim holds for some $i \in \{1, \ldots, \ell\}$. If a vertex $v \in \cup_{j=1}^{i+1} X_j$ is contaminated after the $i + 1$st round, then $v$ has to be adjacent to a vertex $u$ that was contaminated after the $i$th round, and moreover $u \notin X_{i+1}$. By the inductive assumption we infer that $u \notin \cup_{j=1}^{i+1} X_j$, and hence by (ii) and (iii) of the definition of a path decomposition it follows that $v \in X_{i+1} \cap X_{i+2}$. Thus, no vertex of $(\cup_{j=1}^{i+1} X_j) \setminus X_{i+2}$ is contaminated after the $i + 1$st round, which proves the induction step. It remains to observe that after the $\ell$th round all the vertices of $G$ are clear. This means that the strategy $\mathcal{H}$ is winning.

Now we show tightness of the bound. First, we prove that the bound is tight for graphs of pathwidth 2.

Consider the graph $G$ shown in Fig. 1. It is straightforward to verify that $\mathbf{pw}(G) = 2$. We show that $h(G) \ge 3$. Consider an arbitrary hunters' strategy $\mathcal{H} = (H_1, H_2, \ldots)$ for 2 hunters. We prove that $\mathcal{H}$ cannot be a winning strategy by showing that for any $i \ge 1$, after the $i$th round the following invariant holds: at least 5 vertices of $G$ are contaminated, and the only vertices that can be clear are $v_1, v_3, v_4, v_6$. We shall denote this invariant by $(\diamondsuit)$.

Clearly, all the vertices are contaminated in the beginning, so $(\diamondsuit)$ holds before round 1. Suppose now that $(\diamondsuit)$ is satisfied before the $i$th round and we show that the same holds after the round. By symmetry and monotonicity under containment, it is sufficient to consider two cases.

**Case 1.** Vertex $v_1$ is clear and all other vertices are contaminated before the $i$th round. Notice that each of $v_2, v_4, v_5$ has at least 3 contaminated neighbors before the $i$th round. Hence, they are contaminated after the round as well. If $H_i = \{v_4, v_5\}$, then $v_1, \ldots, v_5$ are contaminated after the round and we have $(\diamondsuit)$. Assume then that $H_i \ne \{v_4, v_5\}$. Then $v_6$ is contaminated after the round. If $H_i = \{v_2, v_5\}$, then $v_1$ is contaminated after the round and we have $(\diamondsuit)$, because $v_1, v_2, v_4, v_5, v_6$ are contaminated after the round. If $H_i \ne \{v_2, v_5\}$, then $v_3$ gets contaminated and we have $(\diamondsuit)$, because $v_2, v_3, v_4, v_5, v_6$ are contaminated after the round.

**Case 2.** Vertex $v_3$ is clear and all other vertices are contaminated before the $i$th round. Notice that each of $v_2, v_3, v_4, v_5$ has at least 3 contaminated neighbors before the $i$th round. Hence, they are contaminated after the round. If $H_i = \{v_4, v_5\}$, then $v_1, \dots, v_5$ get contaminated and we have ($\diamondsuit$) after the round. If $H_i \neq \{v_4, v_5\}$, then $v_2, \dots, v_6$ get contaminated and we again have ($\diamondsuit$) after the round.

To show tightness of the bound for graphs of pathwidth $k \geq 2$, consider the graph $G'$ obtained from the graph $G$ shown in Fig. 1 as follows. We add a set $X$ of $k-2$ vertices and join them pairwise by edges to form a clique. Then every vertex of $X$ is joined by an edge with every vertex of $G$. It is straightforward to see that $\mathbf{pw}(G') = k$. We show that $h(G') = k + 1$. Let $\mathcal{H} = (H_1, H_2, \dots)$ be an arbitrary hunters' strategy for $k$ hunters. We prove that $\mathcal{H}$ is not a winning strategy by showing that for any $i \geq 1$, after the $i$th round the following invariant ($\diamondsuit\diamondsuit$) holds: the invariant ($\diamondsuit$) if fulfilled for the vertices of $G$ and the vertices of $X$ are contaminated.

As all the vertices are contaminated in the beginning, ($\diamondsuit\diamondsuit$) holds before round 1. Suppose that ($\diamondsuit\diamondsuit$) is satisfied before the $i$th round and we show that the same holds after the round.

If $X \subseteq H_i$, then at most 2 hunters shoot at the vertices of $G$ and, therefore, ($\diamondsuit$) holds for $G$ as it was shown above. Also in this case all the vertices of $X$ are contaminated after the $i$th round, because there is at least one contaminated before $i$th round vertex $u$ of $G$ such that $u \notin H_i$. We conclude that ($\diamondsuit\diamondsuit$) is fulfilled.

Suppose that $|X \setminus H_i| = 1$. Since the vertices of $X \setminus H_i$ are contaminated before the $i$th round, all the vertices of $G$ are contaminated after $i$th round. Since at most 3 hunters shoot at the vertices of $G$ and $G$ has at least 5 contaminated vertices before the $i$th round, the vertices of $X$ are contaminated after the round. Hence, ($\diamondsuit\diamondsuit$) holds.

Finally, assume that $|X \setminus H_i| \geq 2$ and consider distinct $x, y \in X \setminus H_i$. As $x$ is contaminated before the $i$-round, we have that all the vertices of $G$ are contaminated after $i$th round. It remains to observe that all the vertices of $X \setminus \{x\}$ are contaminated after the $i$th round, because $x$ is contaminated before the round, and $x$ is contaminated, because $y$ is contaminated before the $i$th round. We again have that ($\diamondsuit\diamondsuit$) holds. $\square$

We proved that the bound is tight for graphs of pathwidth at least 2. It can be noticed that if $\mathbf{pw}(G) = 1$, then $h(G) = 1$, because every component of $G$ is a caterpillar in this case, and as it was shown in [4], in this case $h(G) = 1$.

## 3. Hunting rabbit on a grid

In this section we compute the hunting number of an $(n \times m)$-grid. Throughout this section we assume that $n \leq m$. Recall that an $(n, m)$-grid has the vertex set $\{(x, y) \mid 1 \leq x \leq n, 1 \leq y \leq m\}$ and two vertices $(x, y)$ and $(x', y')$ are adjacent if and only if $|x - x'| + |y - y'| = 1$. For a vertex $(i, j)$, we say that $i$ is the *x-coordinate* and $j$ is the *y-coordinate* of $(i, j)$. Clearly, grids are bipartite graphs, and we assume in this section that $(V_1, V_2)$, where $V_1 = \{(x, y) \mid x+y \text{ is even}\}$ and $V_2 = \{(x, y) \mid x+y \text{ is odd}\}$, is the bipartition of the vertex set of a grid.

### 3.1. Isoperimetrical properties of grids

We need some isoperimetrical properties of subsets of $V_1$ for square grids. Let $G$ be an $(n \times n)$-grid. For $i \in \{2, \dots, 2n\}$, let $U_i = \{(x, y) \in V(G) \mid x + y = i\}$ (see Fig. 2) and

$$s(i) = |U_i| = \begin{cases} i - 1 & \text{if } i \leq n + 1, \\ 2n - i + 1 & \text{if } i > n + 1. \end{cases}$$

It is assumed that $U_i = \emptyset$ if $i \leq 1$ or $i > 2n$. We denote the vertices of $U_i$ by $u^i_1, \dots, u^i_{s(i)}$ and assume that they are ordered by the increase of their $y$-coordinate. For $i \in \{1 - n, \dots, n - 1\}$, $W_i = \{(x, y) \in V(G) \mid x - y = i\}$ (see Fig. 2) and $t(i) = |W_i| = n - |i|$; $W_i = \emptyset$ if $i \leq -n$ or $i \geq n$.

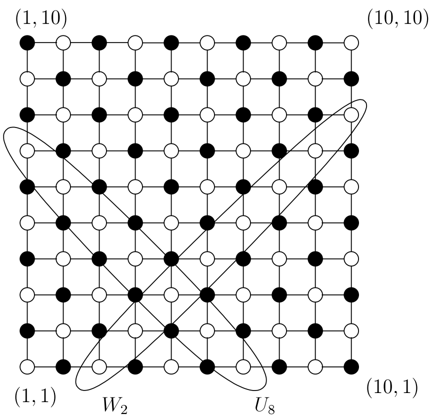


**Fig. 2.** Sets $U_8$ and $W_2$ in a $(10 \times 10)$-grid; the vertices of $V_1$ are depicted as white and the vertices of $V_2$ are black.

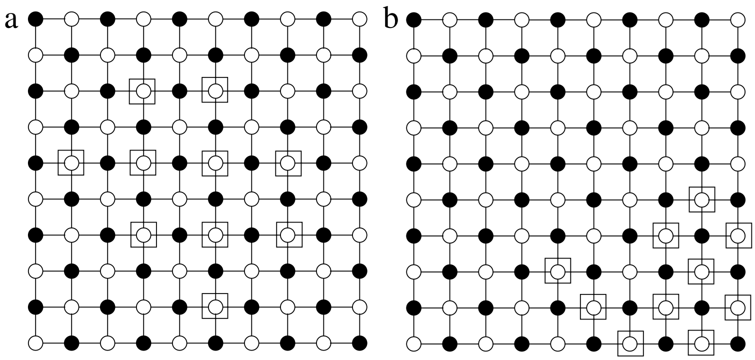


**Fig. 3.** An example of the down-right shifting; the vertices of $Q$ (in a) and $Q'$ (in b) are marked by square frames.

Let $W_i = \{w^i_1, \ldots, w^i_{t(i)}\}$ and assume that the vertices are ordered by the increase of their $y$-coordinate. Notice that

$$\begin{aligned} V_1 &= U_2 \cup U_4 \cup \cdots \cup U_{2n} \\ &= W_{2-2\lceil n/2\rceil} \cup \cdots \cup W_{-2} \cup W_0 \cup W_2 \cup \cdots \cup W_{2\lceil n/2\rceil -2} \end{aligned}$$

and

$$V_2 = U_3 \cup U_5 \cup \cdots \cup U_{2n-1} = W_{1-2\lfloor n/2\rfloor} \cup \cdots \cup W_{-1} \cup W_1 \cup \cdots \cup W_{2\lfloor n/2\rfloor -1}.$$

Let $Q \subseteq V_1$. We say that $Q'$ is obtained from $Q$ by the *down-right shifting* if it is constructed as follows: for each even integer $i \in \{2 \ldots 2n\}$, all the $r = |U_i \cap Q|$ vertices of $U_i \cap Q$ are replaced by $u^i_1, \ldots, u^i_r$ (see Fig. 3). Respectively, $Q'$ is obtained from $Q$ by the *down-left shifting* if for each even integer $i \in \{1-n, \ldots, n-1\}$, all the $r = |W_i \cap Q|$ vertices of $W_i \cap Q$ are replaced by $w^i_1, \ldots, w^i_r$.

**Lemma 2.** *If $Q'$ is obtained from $Q \subseteq V_1$ by the down-right (respectively, down-left) shifting, then $\delta(Q') \le \delta(Q)$.*

**Proof.** We prove the lemma for the down-right shifting. The proof for the down-left shifting uses symmetric arguments.

For an odd integer $i \in \{3, 5, \ldots, 2n-1\}$, let us define the following numbers:

(i) $c_i = |U_i \cap N(Q)|$, $c^-_i = |U_i \cap N(Q \cap U_{i-1})|$ and $c^+_i = |U_i \cap N(Q \cap U_{i+1})|$;

(ii) $d_i = |U_i \cap N(Q')|$, $d_i^- = |U_i \cap N(Q' \cap U_{i-1})|$ and $d_i^+ = |U_i \cap N(Q' \cap U_{i+1})|$.

By the construction of $Q'$ it is straightforward to verify that $c_i^- \geq d_i^-$ and $c_i^+ \geq d_i^+$. Since all elements of $Q$ that neighbor a vertex of $U_i$ reside either in $U_{i-1}$ or in $U_{i+1}$, we have that $c_i \geq \max\{c_i^-, c_i^+\}$. However, since vertices of $Q' \cap U_{i-1}$ are exactly the $|Q' \cap U_{i-1}|$ vertices of $U_{i-1}$ that have the smallest $y$-coordinate, and the same also holds for $Q' \cap U_{i+1}$, then it is easy to see that $d_i = \max\{d_i^-, d_i^+\}$. Hence, we obtain that

$$\begin{aligned}\delta(Q) &= \sum_{j=1}^{n-1} c_{2j+1} \geq \sum_{j=1}^{n-1} \max\{c_{2j+1}^-, c_{2j+1}^+\} \\ &\geq \sum_{j=1}^{n-1} \max\{d_{2j+1}^-, d_{2j+1}^+\} = \sum_{j=1}^{n-1} d_{2j+1} = \delta(Q'). \quad \square\end{aligned}$$

Thus, we already have two operations that preserve the cardinality of a set $Q$ while not increasing $\delta(Q)$: down-left and down-right shifting. We may now inspect sets $Q \subseteq V_1$ that are invariant with respect to both these operations, and it is easy to see that these are exactly sets conforming to the following definition. We say that $Q \subseteq V_1$ is a *pyramidal* set if for any $(x, y) \in Q$ such that $y \geq 2$, $(x-1, y-1) \in Q$ if $x \geq 2$ and $(x+1, y-1) \in Q$ if $x \leq n-1$.

For $i \in \{1, \dots, n\}$, let $R_i = \{(x, y) | 1 \leq x \leq n, y = i\}$, $X_i = R_i \cap V_1$ and $\overline{X}_i = R_i \cap V_2$. Let also

$$\ell(i) = |X_i| = \begin{cases} \lfloor n/2 \rfloor & \text{if } i \text{ is even,} \\ \lceil n/2 \rceil & \text{if } i \text{ is odd.} \end{cases}$$

Denote by $x_1^i, \dots, x_{\ell(i)}^i$ the vertices of $X_i$ and assume that they are ordered by the increase of their $x$-coordinate.

**Lemma 3.** *Suppose $Q \subseteq V_1$ is a pyramidal set. Then*

$$\delta(Q) = |Q| - |Q \cap X_n| + |N(Q \cap X_1) \cap \overline{X}_1|.$$

**Proof.** Take any $(x, y) \in N(Q)$ such that $y \geq 2$. As $(x, y) \in N(Q)$, then one of neighboring four vertices of $G$ belongs to $Q$, and due to $Q$ being pyramidal we have that $(x, y-1) \in Q$. Let us construct a matching $M$ between vertices of $N(Q)$ and vertices of $Q$ that matches every vertex $(x, y) \in N(Q)$ with $y \geq 2$ with vertex $(x, y-1) \in Q$. Then, on the side of $N(Q)$ the only unmatched vertices are the vertices of $N(Q) \cap R_1$, and from the fact that $Q$ is pyramidal it follows that these are exactly the vertices of $N(Q \cap X_1) \cap \overline{X}_1$. On the side of $Q$ the only unmatched vertices are the vertices of $Q \cap R_n = Q \cap X_n$. Thus, the claimed formula on $\delta(Q)$ follows. $\square$

We have already introduced shiftings along diagonals of the grid, so now we introduce shifting along the rows. Take any $Q \subseteq V_1$. We say that $Q'$ is obtained from $Q$ by the *left shifting* if it is constructed as follows: for each integer $i \in \{1 \dots n\}$, all the $r = |X_i \cap Q|$ vertices of $X_i \cap Q$ are replaced by $x_1^i, \dots, x_r^i$. See Fig. 4 for an example. Respectively, $Q'$ is obtained from $Q$ by the *right shifting* if for each integer $i \in \{1, n\}$, all the $r = |X_i \cap Q|$ vertices of $X_i \cap Q$ are replaced by $x_{\ell(i)-r+1}^i, \dots, x_{\ell(i)}^i$. A pyramidal set $Q \subseteq V_1$ is called *left-pyramidal* if for any $(x, y) \in Q$ with $x \geq 3$ we also have that $(x-2, y) \in Q$. Respectively, $Q$ is *right-pyramidal* if for any $(x, y) \in Q$ with $x \leq n-2$ we also have that $(x+2, y) \in Q$.

For a pyramidal set $Q \subseteq V_1$, let $i_1(Q) = 0$ if $(1, y) \notin Q$ for all $y \in \{1, \dots, n\}$ and $i_1(Q) = \max\{y \mid (1, y) \in Q\}$ otherwise. Similarly, let $i_2(Q) = 0$ if $(n, y) \notin Q$ for all $y \in \{1, \dots, n\}$ and $i_2(Q) = \max\{y \mid (n, y) \in Q\}$ otherwise.

**Lemma 4.** *Let $Q \subseteq V_1$ be a pyramidal set. If $i_1(Q) \geq i_2(Q)$ (as in Fig. 4), then $Q'$ obtained from $Q$ by the left shifting is left-pyramidal and satisfies $\delta(Q') \leq \delta(Q)$. Respectively, if $i_1(Q) \leq i_2(Q)$, then $Q'$ obtained from $Q$ by the right shifting is right-pyramidal and satisfies $\delta(Q') \leq \delta(Q)$.*

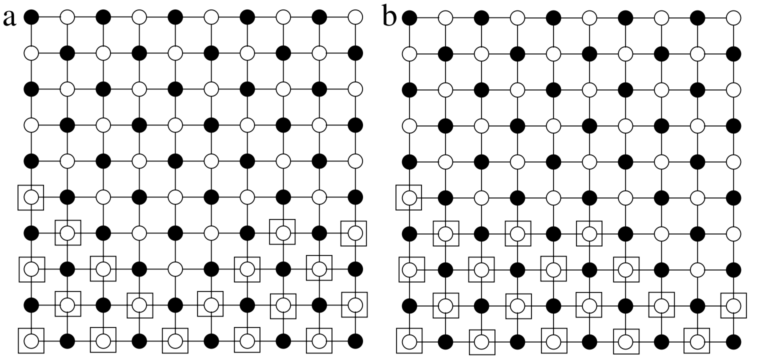


**Fig. 4.** An example of the left shifting, $i_1 = 5$ and $i_2 = 4$; the vertices of $Q$ (in a) and $Q'$ (in b) are marked by square frames.

**Proof.** We first prove that if $Q'$ is obtained by the left shifting, then $Q'$ is left-pyramidal, and if $Q'$ is obtained by the right shifting, then $Q'$ is right-pyramidal. It is straightforward to see that if $Q'$ is obtained by the left shifting, then $(x, y) \in Q'$ implies $(x-2, y) \in Q'$ whenever $x \geq 3$. Symmetrically, if $Q'$ is obtained by the right shifting, then $(x, y) \in Q'$ implies $(x+2, y) \in Q'$ whenever $x \leq n-2$. Hence, we only have to prove that $Q'$ is pyramidal, that is for any $(x, y) \in Q'$ such that $y \geq 2$, it holds that $(x-1, y-1) \in Q'$ provided $x \geq 2$ and $(x+1, y-1) \in Q'$ provided $x \leq n-1$. Let us fix some $(x, y) \in Q'$ with $y \geq 2$.

Assume first that $Q'$ is obtained from $Q$ by the left-shifting. If $X_y \cap Q = X_y$, that is $Q$ occupies the whole $X_y$, then because $Q$ is pyramidal, we also have $X_{y-1} \cap Q = X_{y-1}$. From the construction of $Q'$ it follows that $X_y \cap Q' = X_y$ and $X_{y-1} \cap Q' = X_{y-1}$, and the claimed condition holds for $(x, y)$ trivially. Suppose then that $X_y \setminus Q \neq \emptyset$. In such a situation it can be easily seen that $|Q \cap X_y| \leq |Q \cap X_{y-1}|$ because $Q$ is pyramidal, and hence also $|Q' \cap X_y| \leq |Q' \cap X_{y-1}|$. By the construction of $Q'$, we infer that $(x-1, y-1) \in Q'$ provided $x \geq 2$. The second property, that is $(x+1, y-1) \in Q'$ provided $x \leq n-1$, also follows if at least one of the following conditions holds: $|Q \cap X_y| < |Q \cap X_{y-1}|$ or $y$ is odd. Thus, the only remaining case is when $|Q \cap X_y| = |Q \cap X_{y-1}|$ and $y$ is even. Since $X_y \setminus Q \neq \emptyset$ and $Q$ is pyramidal, one can easily verify that the only situation when $|Q \cap X_y| = |Q \cap X_{y-1}|$ is the following: $n$ is even, $Q \cap X_y = \{x_r^y, \dots, x_{\ell(y)}^y\}$ for some $r \in \{2, \dots, \ell(y)\}$, where $x_{\ell(y)}^y = (n, y)$, and $Q \cap X_{y-1} = \{x_r^{y-1}, \dots, x_{\ell(y-1)}^{y-1}\}$. But then $(n, y) \in Q$ and $(1, y-1) \notin Q$ and we have that $i_1(Q) < y \leq i_2(Q)$. This is a contradiction with the assumption that $Q'$ is obtained by the left shifting.

The arguments for the case when $Q$ is obtained by the right shifting are exactly symmetric, and hence we omit the second check.

We are left with proving that $\delta(Q') \leq \delta(Q)$. Since we already know that both $Q'$ and $Q$ are pyramidal, from Lemma 3 we infer that it suffices to prove that $|N(Q \cap X_1) \cap \overline{X}_1| \geq |N(Q' \cap X_1) \cap \overline{X}_1|$. If $X_1 \subseteq Q$, then $Q' \cap X_1 = Q \cap X_1 = X_1$ and the condition holds trivially. Otherwise, if $X_1 \setminus Q \neq \emptyset$, it can be easily seen that $|N(Q \cap X_1) \cap \overline{X}_1| \geq |Q \cap X_1|$. On the other hand, by the construction of $Q'$ we have that $|N(Q' \cap X_1) \cap \overline{X}_1| = |Q' \cap X_1| = |Q \cap X_1|$ apart from the situation when $n$ is even and $Q'$ was obtained by the right shifting; In this case we have $|N(Q' \cap X_1) \cap \overline{X}_1| = |Q \cap X_1| + 1$, and this is the only situation left. Observe, however, that provided $n$ is even and $X_1 \setminus Q \neq \emptyset$, the only situation with $|N(Q \cap X_1) \cap \overline{X}_1| = |Q \cap X_1|$ is when $Q = \{x_1^1, x_2^1, \dots, x_r^1\}$ for some $r < \ell(1)$, and otherwise we are done. But then we would have that $(1, 1) \in Q$ and $(n-1, 1) \notin Q$, which, by $Q$ being pyramidal, implies that $i_1(Q) > 0 = i_2(Q)$. This is a contradiction with the fact that $Q'$ was obtained by the right shifting. $\square$

We say that a left-pyramidal set $Q \subseteq V_1$ has a *left spot at* $(i, j)$ if $(i, j) \in Q$ and $(i-1, j+1) \in V(G) \setminus Q$. Similarly, a right-pyramidal set $Q \subseteq V_1$ has a *right spot at* $(i, j)$ if $(i, j) \in Q$ and $(i+1, j+1) \in V(G) \setminus Q$. Obviously, all the left spots of a left pyramidal set have pairwise different $x$-coordinates, and the same also holds for right spots of right pyramidal sets.

The following two lemmas can be easily verified by a direct check using Lemma 3.

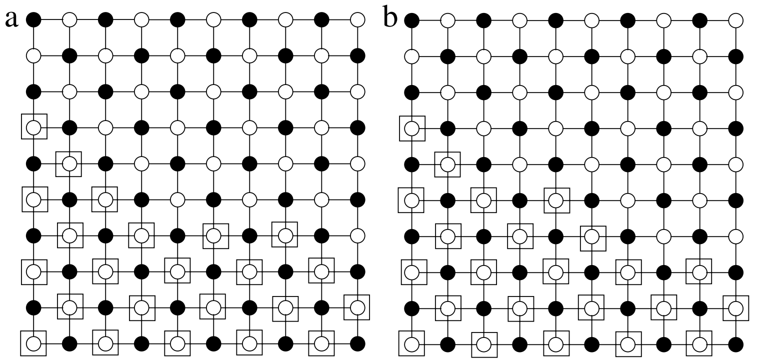

**Fig. 5.** Replacement of Lemma 5 for $(i_1, j_1) = (6, 4)$ and $(i_2, j_2) = (8, 4)$; the vertices of $Q$ (in a) and $Q'$ (in b) are marked by square frames.

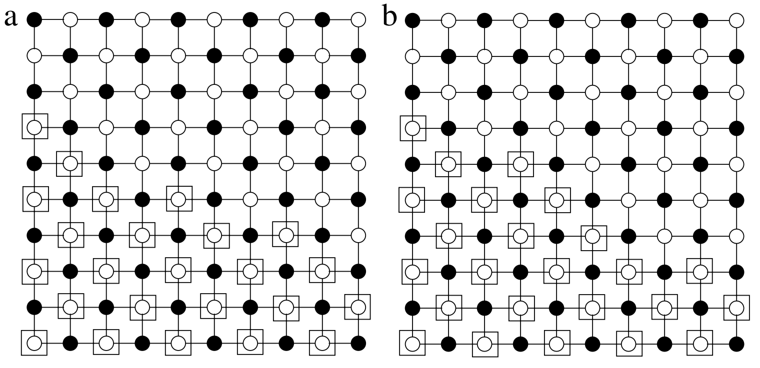

**Fig. 6.** Replacement of Lemma 5 for $(i_1, j_1) = (5, 5)$ and $(i_2, j_2) = (8, 4)$; the vertices of $Q$ (in a) and $Q'$ (in b) are marked by square frames.

**Lemma 5.** *Let $Q \subseteq V_1$ be a left-pyramidal set with two different left spots $(i_1, j_1)$ and $(i_2, j_2)$, such that $(i_1, j_1)$ and $(i_2, j_2)$ have the smallest and the largest x-coordinates among the left spots of $Q$, respectively. Construct $Q' = Q \setminus \{(i_2, j_2)\} \cup \{(i_1 - 1, j_1 + 1)\}$ (see* Figs. 5 and 6*). Then $Q'$ is also a left-pyramidal set and $\delta(Q') \leq \delta(Q)$.*

**Lemma 6.** *Let $Q \subseteq V_1$ be a right-pyramidal set with two different right spots $(i_1, j_1)$ and $(i_2, j_2)$, such that $(i_1, j_1)$ and $(i_2, j_2)$ have the largest and the smallest x-coordinates among the right spots of $Q$, respectively. Construct $Q' = Q \setminus \{(i_2, j_2)\} \cup \{(i_1 + 1, j_1 + 1)\}$. Then $Q'$ is also a right-pyramidal set and $\delta(Q') \leq \delta(Q)$.*

Note that the transformations of Lemmas 5 and 6 can be applied as long as the set $Q$ in question has at least two left (resp. right) spots.

Recall that $V_1 = U_2 \cup U_4 \cup \cdots \cup U_{2n}$. We define the ordering $v_1, \ldots, v_{\lceil n^2/2 \rceil}$ of the vertices of $V_1$ as follows: the sequence enumerates consequently the vertices of $U_2, U_4, \ldots, U_{2n}$ and the vertices of each $U_i$ are listed in the order $u_1^i, \ldots, u_{s(i)}^i$. For a positive integer $p$, let $Z_p = \{v_1, \ldots, v_p\}$. Recall also that $V_1 = W_{2-2\lceil n/2 \rceil} \cup \cdots \cup W_{-2} \cup W_0 \cup W_2 \cup \cdots \cup W_{2\lceil n/2 \rceil - 2}$. Respectively, we define another ordering $v'_1, \ldots, v'_{\lceil n^2/2 \rceil}$ of the vertices of $V_1$: the sequence enumerates consequently the vertices of $W_{2\lceil n/2 \rceil - 2}, W_{2\lceil n/2 \rceil - 4}, \ldots, W_{2-2\lceil n/2 \rceil}$ and the vertices of each $W_i$ are listed in the order $w_1^i, \ldots, w_{t(i)}^i$. For a positive integer $p$, we define $Z'_p = \{v'_1, \ldots, v'_p\}$.

**Theorem 1.** *Let $G$ be an $(n \times n)$-grid. Then for any $Q \subseteq V_1$, it holds that $\delta(Q) \geq \min(\delta(Z_p), \delta(Z'_p))$, where $p = |Q|$.*

**Proof.** Let $p \in \{1, 2, \ldots, \lceil \frac{n^2}{2} \rceil\}$ and let $\delta^* = \min\{\delta(Q) \mid Q \subseteq V_1, |Q| = p\}$.

First, we show that there is a left-pyramidal or a right-pyramidal set $Q \subseteq V_1$ of size $p$ with $\delta(Q) = \delta^*$. Let $Q$ be any set of size $p$ with $\delta(Q) = \delta^*$. Assume that $Q$ is chosen in such a way that the sum of $y$-coordinates of the vertices of $Q$ is minimum. Then $Q$ is pyramidal because otherwise we could apply the down-right or down-left shifting and obtain a set $Q'$ that would have smaller sum of $y$-coordinates of its vertices, and for which it would hold that $\delta(Q') \leq \delta(Q)$ by Lemma 2. Suppose now that $Q$ is neither left- nor right-pyramidal. If $i_1(Q) \geq i_2(Q)$, then let $Q'$ be the set obtained from $Q$ by the left shifting. By Lemma 4, we have that $\delta(Q') \leq \delta(Q)$ and $Q'$ is left-pyramidal. On the other hand, if $i_1(Q) < i_2(Q)$, then the set $Q'$ obtained from $Q$ by the right shifting is right-pyramidal and satisfies $\delta(Q') \leq \delta(Q)$ by Lemma 4. Since $\delta(Q) = \delta^*$ is minimum possible, in both cases we conclude that $\delta(Q') = \delta^*$.

Suppose now that there is a left pyramidal set $Q$ of size $p$ with $\delta(Q) = \delta^*$. Among all such sets we select $Q$ for which the sum of $x$-coordinates of its vertices is minimum. Then $Q$ has at most one spot, since otherwise using Lemma 5 we could construct a left-pyramidal set $Q'$ with $\delta(Q') \leq \delta(Q)$ and a smaller sum of $x$-coordinates of vertices. It remains to notice that if a left-pyramidal set of size $p$ has at most one spot, then in fact $Q = Z_p$.

The case when there is a right-pyramidal set $Q$ of size $p$ with $\delta(Q) = \delta^*$ is symmetric. Among all such sets we select $Q$ that maximizes the sum of $x$-coordinates of its vertices, and using Lemma 6 we argue that then $Q = Z'_p$.

Summarizing, in the first case we have that $\delta^* = \delta(Z_p)$ and in the second we have that $\delta^* = \delta(Z'_p)$, so we infer that $\delta^* = \min(\delta(Z_p), \delta(Z'_p))$. □

Using Theorem 1, it is possible to obtain an explicit expression for the tight lower bound for $\delta(Q)$, but such an expression is rather ugly. In particular, it can be noticed that there are cases when the bound is given by $\delta(Z_p)$ and cases when $\delta(Z'_p)$ is minimum. Consider, e.g., the $(6 \times 6)$-grid. Then $6 = \delta(Z_4) < \delta(Z'_4) = 7$ and $15 = \delta(Z_{12}) > \delta(Z'_{12}) = 14$. To compute the hunting number of a grid, we need the bound for one special case.

**Corollary 1.** *Let $G$ be an $(n \times n)$-grid, where $n \geq 4$ and $n$ is even. Then for any $Q \subseteq V_1$ with $|Q| = \frac{n^2}{4} - \frac{n}{2}$, it holds that $\delta(Q) \geq \frac{n^2}{4}$.*

**Proof.** Let $p = \frac{n^2}{4} - \frac{n}{2}$. Observe that $Z_p$ contains all the vertices of $U_2, \ldots, U_{n-2}$ and $\frac{n}{2} - 1$ vertices of $U_n$. Then $N(Z_p)$ contains all the vertices of $U_3, \ldots, U_{n-1}$ and $\frac{n}{2}$ vertices of $U_{n+1}$. Because $s(3) + \cdots + s(n-1) = 2 + \cdots + (n-2) = \frac{n^2}{4} - \frac{n}{2}$, we have that $\delta(Z_p) = \frac{n^2}{4}$. For the set $Z'_p$, we have that $Z'_p = W_{n-2} \cup \cdots \cup W_2$ and $N(Z'_p) = W_{n-1} \cup \cdots \cup W_1$. Hence, it follows that $\delta(Z'_p) = \frac{n^2}{4}$. By Theorem 1, $\delta(Q) \geq \min(\delta(Z_p), \delta(Z'_p)) = \frac{n^2}{4}$. □

### 3.2. The hunting number of a grid

Now we are ready to compute the hunting number of a grid.

**Theorem 2.** *Let $G$ be an $(n \times m)$-grid. Then $h(G) = \lfloor \frac{\min\{n,m\}}{2} \rfloor + 1$.*

**Proof.** Recall that we assume that $n \leq m$.

First, we prove that $h(G) \leq \lfloor \frac{n}{2} \rfloor + 1$. By Proposition 2, it is sufficient to show it for odd $n$. Therefore, we assume that $n$ is odd and, using Lemma 1, construct a winning strategy for $\frac{n-1}{2} + 1$ hunters with respect to $V_1$. Consecutively, for $i = 1, \ldots, m-1$, the hunter player makes the following sequence of shoots as it is shown in Fig. 7:

(i) shoot at $(i, 1), (i, 3), \ldots, (i, n)$,
(ii) for $j = 1, \ldots, (n+1)/2$, shoot at $(i+1, 1), (i+1, 3) \ldots, (i+1, 2j-1), (i, 2j), \ldots, (i, n-1)$ and then at $(i+1, 2), \ldots, (i+1, 2j), (i, 2j+1), \ldots, (i, n)$.

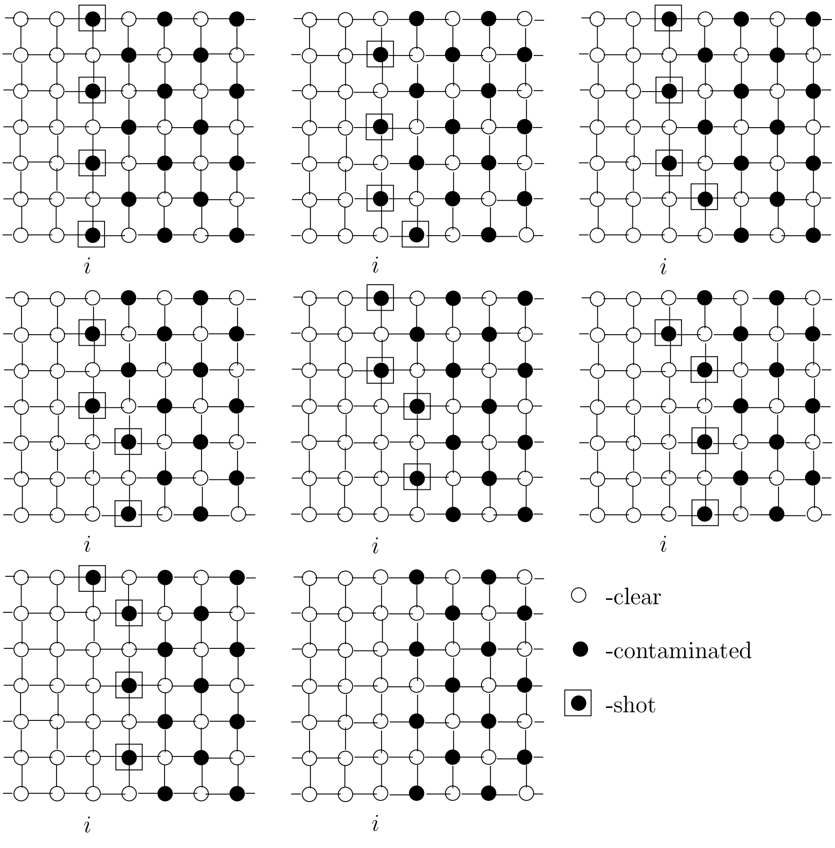


**Fig. 7.** The series of shots for each $i \in \{1, \dots, m-1\}$ for $n = 7$.

Finally, the hunter player shoots at $(m, 1), (m, 3), \dots, (m, n)$. It is straightforward to verify (see Fig. 7) that the following claim holds for each $i \in \{1, \dots, m\}$: after the $i$th series of rounds,

(i) the vertices of $V_{2-i \bmod 2}$ are clear;
(ii) the vertices $(x, y)$ for $x < i$ are clear.

This immediately implies that we have a winning hunter's strategy.

Now we show that $h(G) \geq \lfloor \frac{n}{2} \rfloor + 1$. By Proposition 2, it is sufficient to show it for $(n \times n)$-grids for even $n$. If $n = 2$, then a direct check shows that $h(G) = 2$ and the claim holds. Suppose then that $n \geq 4$. We show that the Hunter player has no winning strategy for $\frac{n}{2}$ hunters.

For the sake of contradiction, suppose $\mathcal{H} = (H_1, H_2, \dots)$ is Hunter's strategy for $\frac{n}{2}$ hunters. We show inductively that for every $i \geq 1$, each of the sets $V_1$ and $V_2$ has at least $\frac{n^2}{4}$ contaminated vertices after the $i$th shot. Clearly, all vertices are contaminated before the first shoot. Assume that the claim holds before $i$th round. Set $V_1$ contains at least $\frac{n^2}{4}$ contaminated vertices before the $i$th shot; let us denote this set of contaminated vertices by $A_{i-1}$. As $|H_i| \leq \frac{n}{2}$, we have a set $Q = A_{i-1} \setminus H_i$ of at least $\frac{n^2}{4} - \frac{n}{2}$ vertices that were contaminated before the $i$th shot and were not shot during the $i$th round. By applying Corollary 1 to any subset of $Q$ of size exactly $\frac{n^2}{4} - \frac{n}{2}$, we infer that $\delta(Q) \geq \frac{n^2}{4}$ and hence at least $\frac{n^2}{4}$ vertices of $V_2$ are contaminated after the $i$th shot. To show the symmetric claim for $V_2$, we can use exactly the same arguments, because we can apply Corollary 1 also to subsets of $V_2$; this follows from the assumption that $n$ is even and, therefore, $V_1$ and $V_2$ can be mapped to each other by an automorphism of $G$.

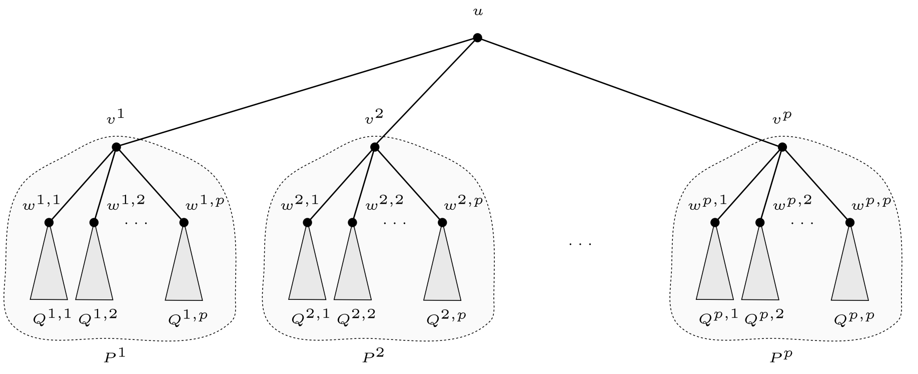


**Fig. 8.** Construction of tree $T_k$.

This is a contradiction with the assumption that $\mathcal{H}$ is a winning strategy for the Hunter player. As $\mathcal{H}$ was chosen arbitrarily, we have that $\frac{n}{2}$ hunters cannot hunt the rabbit. □

## 4. Hunting rabbit on a tree

In this section we provide upper and lower bounds on the hunting number of a tree.

It immediately follows from the results of Ellis, Sudborough and Turner [5] that any tree of pathwidth $t$ has at least $(5 \cdot 3^t - 1)/2$ vertices. Together with Proposition 3 it implies the following theorem.

**Theorem 3.** *For every $n$-vertex tree $T$, $h(T) \leq \log_3 \frac{2n+1}{5} + 1$.*

Next, we prove that the hunting number of an $n$-vertex tree can be as large as $\Omega(\log n/\log\log n)$. More precisely, we prove the following theorem.

**Theorem 4.** *For every positive integer $k$ there exists a tree $T_k$ such that $|V(T_k)| = 2^{O(k\log k)}$ and $h(T_k) \geq k$.*

The rest of this section is devoted to the proof of Theorem 4. The construction of the sequence of trees $(T_k)_{k=1,2,3,\dots}$ is inductive. We are going to think of each $T_i$ as of a rooted tree. For $T_1$ we take simply a path on three vertices with the middle vertex being the root. Let us define

$$p(k) = 2 \cdot ((4k-3)(k-1)+1).$$

To construct $T_k$ based on $T_{k-1}$, perform the following:

(i) Create the root $u$.
(ii) Add $p := p(k)$ children of $u$, denoted by $v^1, v^2, \dots, v^p$.
(iii) For every child $v^i$ of $u$, add $p$ subtrees $Q^{i,j}$ for $j = 1, 2, \dots, p$, all isomorphic to $T_{k-1}$ and with roots being children of $v^i$.

See Fig. 8 for an illustration. For $i = 1, 2, \dots, p$, by $P^i$ we denote the subtree of $T_k$ rooted at $v^i$. Furthermore, let $w^{i,j}$ be the root of subtree $Q^{i,j}$, for all $i, j \in \{1, 2 \dots, p\}$. By somehow abusing the notation, we will identify each subtree $P^i$ and $Q^{i,j}$ with its vertex set.

Observe now that we have recursive equation $|V(T_k)| = 1 + p(k) + p(k)^2 \cdot |V(T_{k-1})|$, from which it immediately follows that $|V(T_k)| = 2^{O(k\log k)}$. We are left with proving by induction that $h(T_k) \geq k$ for all positive integers $k$. For $k = 1$ we have $h(T_1) = 1$, so we proceed to the inductive step for $k \geq 2$. In the following, we denote $T = T_k$.

Let us fix the bipartition $(V_1, V_2)$ of $T$ such that $u \in V_1$ and $\{v^1, v^2, \dots, v^p\} \subseteq V_2$. We shall prove that $k-1$ hunters do not have a winning strategy on $T$ with respect to $V_1$, which by Lemma 1 is equivalent to the main claim. For the sake of contradiction, suppose that there is a winning strategy with respect to $V_1$ for $k-1$ hunters, and denote it by $\mathcal{H} = (H_1, H_2, H_3, \dots, H_m)$. Therefore, in the beginning of this strategy all the vertices of $V_1$ are contaminated, and at the end all the vertices of $T$ are clean. For $t = 0, 1, 2, \dots, m$, let $A_t$ be the set of contaminated vertices between hunters' shots $t$ and $t+1$. Thus, $A_0 = V_1, A_m = \emptyset, A_t \subseteq V_1$ for even $t$, and $A_t \subseteq V_2$ for odd $t$.

Let us fix a moment $t \in \{0, 1, \dots, m\}$, and consider subtree $Q^{i,j}$. We shall say that

(i) $Q^{i,j}$ is *contaminated at moment* $t$ if $Q^{i,j} \cap A_t \neq \emptyset$.
(ii) $Q^{i,j}$ is *full at moment* $t$ if $Q^{i,j} \cap A_t = Q^{i,j} \cap V_1$ provided that $t$ is even, and $Q^{i,j} \cap A_t = Q^{i,j} \cap V_2$ provided that $t$ is odd.
(iii) $Q^{i,j}$ is *well-contaminated at moment* $t$ if the following holds: supposing $t' \leq t$ is the latest moment not later than $t$ when $Q^{i,j}$ was full, then $|Q^{i,j} \cap H_{t''}| < k-1$ for all $t' < t'' \leq t$. In other words, since the last time $Q^{i,j}$ was full, it did not happen that *all* the available hunters were shooting at $Q^{i,j}$ in some round.

Observe that at moment $t = 0$ all the subtrees $Q^{i,j}$ are full, hence the definition of being well-contaminated is valid. Observe also that by the induction hypothesis and Lemma 1, each subtree $Q^{i,j}$ cannot be cleaned using less than $k-1$ hunters and beginning from any moment when it is full. This justifies the following claim.

**Claim 1.** *If a subtree $Q^{i,j}$ is well-contaminated at moment $t$, then it is also contaminated at moment $t$.*

Similarly as for $Q^{i,j}$, a subtree $P^i$ is *contaminated at moment* $t$ if $P^i \cap A_t \neq \emptyset$. Also, $P^i$ is *full at moment* $t$ if $P^i \cap A_t = P^i \cap V_1$ provided $t$ is even, and $P^i \cap A_t = P^i \cap V_2$ provided $t$ is odd.

We now prove a few auxiliary observations that will be used in the main proof.

**Claim 2.** *The following holds:*

(i) *Suppose that subtree $Q^{i,j}$ is contaminated at moment $t$ and for every $t'$ with $t < t' \leq t + (4k-6)$ we have that $H_{t'} \cap Q^{i,j} = \emptyset$. Then $Q^{i,j}$ is full at moment $t + (4k-6)$.*
(ii) *Suppose that subtree $P^i$ is contaminated at moment $t$ and for every $t'$ with $t < t' \leq t + (4k-4)$ we have that $H_{t'} \cap Q^{i,j} = \emptyset$. Then $P^i$ is full at moment $t + (4k-4)$.*

**Proof.** The claim follows immediately from the facts that the diameter of each $Q^{i,j}$ is equal to $4k-6$ and the diameter of each $P^i$ is equal to $4k-4$. This, in turn, follows from the observation that the diameter of $T_k$ is equal to $4k-2$, which can be proved via a straightforward induction. $\diamond$

For a moment $t$ ($0 \leq t \leq m$) and index $i$ ($1 \leq i \leq p$), let $n_i(t)$ be the number of subtrees $Q^{i,j}$, for $j = 1, 2, \dots, p$, that are well-contaminated at moment $t$. We shall say that

(i) $P^i$ is *lightly contaminated at moment* $t$ if $n_i(t) \leq p/2$;
(ii) $P^i$ is *heavily contaminated at moment* $t$ if $p/2 < n_i(t)$.

**Claim 3.** *Suppose $P^i$ is heavily contaminated at moment $t$, where $t$ is odd. Then $v^i \in A_t$.*

**Proof.** Suppose first that there was a moment $t_0$ with $t - (4k-5) \leq t_0 < t$, such that $v^i \in A_{t_0}$ but $v^i \notin H_{t_0+1}$; in particular $t_0$ is odd. Then it follows that $w^{i,j} \in A_{t_0+1}$ for every $j = 1, 2, \dots, p$, so in particular all the subtrees $Q^{i,j}$ became contaminated at moment $t_0+1$. Out of these subtrees, at most $(4k-5)(k-1)$ might contain some vertex of $H_{t'}$ for any $t'$ with $t_0 < t' \leq t$, which leaves at least one subtree $Q^{i,j}$ that did not contain any shots during all these moments. Since both $t_0$ and $t$ are odd and $Q^{i,j}$ consists of more than one vertex, we infer that $w^{i,j}$ remained contaminated at all the even moments between $t_0+1$ and $t-1$, so in particular $w^{i,j} \in A_{t-1}$. Since $w^{i,j} \notin H_t$, we have that $v^i \in A_t$.

Suppose now that no such moment $t_0$ exists. Recall that $P^i$ is heavily contaminated at moment $t$, that is the number of subtrees $Q^{i,j}$, for $j = 1, 2, \dots, p$, that are well-contaminated and, therefore, contaminated at moment $t$ is more than $p/2$. Since $t_0$ does not exist, we have that all the subtrees $Q^{i,j}$ that are contaminated at moment $t$, needed to be contaminated at all moments $t'$ with $\max\{t-(4k-5), 0\} \leq t' \leq t$: the only way a subtree $Q^{i,j}$ can become contaminated is by not shooting at $v^i$ when it is contaminated. At most $(4k-5)(k-1)$ of these subtrees might contain some vertex of $H_{t'}$ for any $t'$ with $\max\{0, t-(4k-5)\} < t' \leq t$, which leaves at least one subtree $Q^{i,j}$ that did not contain any shots during all these moments. By Claim 2, this subtree is full at moment $t-1$. Since $t$ is odd, this means that $w^{i,j} \in A_{t-1}$. As $w^{i,j} \notin H_t$, we again have that $v^i \in A_t$. $\diamond$

**Claim 4.** *Suppose $t$ is an odd moment, $v^i \in A_t$, and $v^i \notin H_{t+1}$. Then $P^i$ is heavily contaminated at moment $t + (4k - 5)$.*

**Proof.** By the assumption we have that $w^{i,j} \in A_{t+1}$ for every $j = 1, 2, \dots, p$. At most $(4k-6)(k-1)$ subtrees $Q^{i,j}$ can contain some vertex of $H_{t'}$ for $t + 1 < t' \leq t + (4k - 5)$. This leaves more than $p/2$ subtrees $Q^{i,j}$ that do not contain any shots during these moments. By Claim 2, all these subtrees are full at moment $t + (4k - 5)$, so in particular they are well-contaminated then. $\diamond$

Finally, we introduce a similar classification for the whole tree $T$ as for subtrees $P^i$. We shall say that

(i) $T$ is *lightly contaminated at moment $t$* if the number of heavily contaminated subtrees $P^i$ is at most $p/2$;
(ii) $T$ is *heavily contaminated at moment $t$* if the number of heavily contaminated subtrees $P^i$ is more than $p/2$.

The following two claims can be proved in exactly the same manner as Claims 3 and 4, with the modification that we use the second point of Claim 2 instead of the first one.

**Claim 5.** *Suppose $T$ is heavily contaminated at moment $t$, where $t$ is even. Then $u \in A_t$.*

**Claim 6.** *Suppose $t$ is an even moment, $u \in A_t$, and $u \notin H_{t+1}$. Then $T$ is heavily contaminated at moment $t + (4k - 3)$.*

We are finally ready to prove that $h(T_k) \geq k$ by exposing that the existence of the hunters' strategy $(H_1, H_2, H_3, \dots, H_m)$ leads to a contradiction. Let $t \in \{0, 1, \dots, m\}$ be the latest moment when $T$ was heavily contaminated. Since $T$ is heavily contaminated at moment 0, and lightly contaminated at moment $m$, we infer that such $t$ exists and satisfies $t < m$. By the maximality of $t$, we have that $T$ is lightly contaminated at moment $t + 1$. This means that some subtree $P^{i_0}$ ceased to be highly contaminated between moments $t$ and $t + 1$, which in turn means that some subtree $Q^{i_0,j_0}$ ceased to be well-contaminated between moments $t$ and $t + 1$. By the definition of being well-contaminated, subtree $Q^{i_0,j_0}$ could have ceased to be well-contaminated only if all the $k - 1$ hunters were shooting at it at moment $t + 1$, that is we have that $H_{t+1} \subseteq Q^{i_0,j_0}$. This implies in particular that $H_{t+1} \cap \{u, v^1, v^2, \dots, v^p\} = \emptyset$.

We now consider two cases depending on the parity of $t$.

Suppose first that $t$ is odd, so $A_t \subseteq V_2$. We have that more than $p/2$ subtrees $P^i$ are heavily contaminated at moment $t$, including the subtree $P^{i_0}$. By Claim 3, for each of these subtrees $P^i$ we have that $v^i \in A_t$. Since no vertex $v^i$ is contained in $H_{t+1}$, we can apply Claim 4 and infer that every subtree $P^i$ that was heavily contaminated at moment $t$ is again heavily contaminated at moment $t+(4k-5)$. Hence, at moment $t + (4k - 5)$ we again have more than $p/2$ heavily contaminated subtrees $P^i$, a contradiction with the maximality of $t$.

Suppose now that $t$ is even, so $A_t \subseteq V_1$. Since $T$ is heavily contaminated at moment $t$, by Claim 5 we have that $u \in A_t$. Since $u \notin H_{t+1}$, by Claim 6 we have that $T$ is again heavily contaminated at moment $t + (4k - 3)$, which contradicts the maximality of $t$.

We have obtained a contradiction in both of the cases, so this completes the inductive proof that $h(T_k) \geq k$. Thus Theorem 4 is proven.

## 5. Conclusions

In our paper we investigated the hunting number for grids and trees. In particular, we proved that the extremal value of the hunting number for $n$-vertex trees is between $\Omega(\log n/\log\log n)$ and $O(\log n)$. Very recently this result was improved by Gruslys and Méroueh in [8]. They showed that $h(T) \leq \lceil (1/2)\log n \rceil$ for an $n$-vertex tree $T$, and for any $\varepsilon > 0$ and any sufficiently large $n$, there is a tree $T$ of order $n$ such that $h(T) \geq (1/4 - \varepsilon)\log n$. Hence, they give asymptotically tight lower and upper bound for the maximum value of the hunting number of trees.

We conclude with a few open questions. We leave the algorithmic aspects of the problem completely untouched. For example, graphs with $h(G) = 1$ can be recognized in polynomial time due to characterization from [4]. However, we do not know if deciding whether $h(G) \leq 2$ can be done in polynomial time. While it is natural to assume that the problem is NP-hard or even PSPACE-hard, we do not have a proof confirming such an assumption. Also it would be interesting to see if the hunting number of a tree can be computed in polynomial time.

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