INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
SEMESTER I 2023 − 24
MTL 104 (Linear Algebra and its Applications)
End-semester Examination

Date: 23/11/2023 Timing: 3:30 PM to 5:30 PM

Full Marks: 40

Question 1:

(a) Consider the set of real numbers $\mathbb{R}$ as a vector space over the field of rational numbers $\mathbb{Q}$. Show that the dimension of $\mathbb{R}$ is infinite. (3)

(b) Let V be a 6-dimensional vector space over the field $\mathbb{Z}_3$ (the field with three elements). Find the number of 1-dimensional subspaces of V. (3)

Question 2: Let V be any vector space over $\mathbb{C}$. Suppose that f and g are linear functionals on V such that the function h defined by $h(v) = f(v)g(v)$ is also a linear functional on V. Prove that either $f = 0$ or $g = 0$. (5)

Question 3: Suppose $R, T \in \mathcal{L}(\mathbb{R}^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbb{R}^3)$ such that $R = S^{-1}TS$. (5)

Question 4: Let $V = C([0,1])$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$. Let W be the subspace spanned by the linearly independent set $\{t, \sqrt{t}\}$.

(a) Find an orthonormal basis for W. (3)

(b) Let $h(t) = t^2$. Use the orthonormal basis obtained in (a) to obtain the "best" (closest) approximation of h in W. (3)

Question 5: Let V be the real inner product space consisting of the space of real-valued continuous functions on the interval $[-1, 1]$, with the inner product $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt$. Let W be the subspace of odd functions, i.e., functions satisfying $f(-t) = -f(t)$. Find the orthogonal complement of W. (6)

Question 6: Let V be a finite-dimensional inner product space. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint. (6)

Question 7: Let V be a finite-dimensional inner product space over a field F. Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in F$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \epsilon$. Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$. (6)

I do hereby undertake that as a student at IIT Delhi: (1) I will not give or receive aid in examinations, and (2) I will do my share and take an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honour Code.