INDIAN INSTITUTE OF TECHNOLOGY DELHI
DEPARTMENT OF MATHEMATICS
SEMESTER I 2023 − 24
MTL 104 (Linear Algebra and its Applications)
Minor

Date: 15/09/2023 Timing: 8 AM to 9:40 AM

Full Marks: 30

Question 1: (a) Find a basis and the dimension for each of the following vector spaces. Here $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers respectively. [Justification is not needed. There are no partial marks in each of the portions]

(i) The vector space $\mathbb{C}^2$ over the field $\mathbb{R}$. (1)

(ii) The vector space of complex symmetric (3×3) matrices (entries from $\mathbb{C}$) over the field $\mathbb{R}$. (1)

(iii) The vector space of real skew-symmetric (4×4) matrices (entries from $\mathbb{R}$) over the field $\mathbb{R}$. (1)

(b) Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$ (over the field $\mathbb{C}$)? Justify. (2)

(c) Prove that F^∞ is infinite-dimensional. (2)

Question 2: Let $\mathcal{P}_n(x)$ be the vector space (over $\mathbb{R}$) which consists of all the real polynomials of degrees at most n, $n \in \mathbb{N}$. Prove or disprove: there exists a basis $p_0(x), p_1(x), p_2(x), p_3(x)$ of $\mathcal{P}_3(x)$ such that none of the polynomials $p_0(x), p_1(x), p_2(x), p_3(x)$ has degree 2. Find a basis for the quotient space $\mathcal{P}_6(x)/\mathcal{P}_3(x)$. Justify your answer. $(3 + 2)$

Question 3: Suppose V is a finite-dimensional vector space. Let U, W be subspaces of V. Prove that $(U + W)^0 = U^0 \cap W^0$ and $(U \cap W)^0 = U^0 + W^0$. $(3 + 3)$

Question 4: Consider $\mathbb{R}^3$ as a vector space over the field $\mathbb{R}$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. Show that T has a one-dimensional invariant subspace. Will T have a two-dimensional invariant subspace? Justify your answer. $(2 + 4)$

Question 5: Suppose $T \in \mathcal{L}(\mathbb{R}^{10})$ is defined by $T(x_1, \ldots, x_{10}) = (x_1 + \cdots + x_{10}, \cdots, x_1 + \cdots + x_{10})$. Find all eigenvalues and eigenvectors of T. Discuss whether T is diagonalizable or not. $(4 + 2)$