DEPARTMENT OF MATHEMATICS
MTL 122: Real and Complex Analysis

Mid Semester *Marks - 40*

All questions are compulsory. Assume everything done in class.

(1.) Show that the set of irrational numbers in $\mathbb{R}$ is not a F_σ set. [5 Marks]

(2.) Let (X_1, d_1) and (X_2, d_2) be compact metric spaces. Define a metric $D((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$ on $X_1 \times X_2$.
Show that the metric space $(X_1 \times X_2, D)$ is compact. [5 marks]

(3.) Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e, f\}, \{b, c, e, f\}, \{a, b, c, e, f\}, \{a, d, e, f\}, \{a, e, f\}, \{d, e, f\}, \{e, f\}\}$ be a topology defined on X.
Determine the interior, closure and boundary of the set $A = \{a, d\}$ and $B = \{b, e, f\}$ in $(X, \mathcal{T})$. [5 Marks]

(4.) Let (X, T_1) and (Y, T_2) be topological spaces and $f : X \to Y$. Prove that
i) If T_1 is discrete than f is continuous.
ii) If T_2 is indiscrete than f is continuous. [4 Marks]

(5.) Prove that a continuous image of a connected topological space is connected. [5 Marks]

(6.) Let $r = \frac{m}{n}$ be the unique representation for every rational (i.e. m and n be coprime). Define $f : \mathbb{R} \to \mathbb{R}$ as $f(i) = 0$ for all irrational i and $f(r) = \frac{1}{n}$ for all rational r. Show that f is continuous at all irrationals and discontinuous at all rationals.
Does there exist any function that is continuous at all rationals and discontinuous at all irrationals? Explain in brief. [6 Marks]

(7.) Let $g : [0, 1] \to \mathbb{R}$ be continuous with $g(1) = 0$. Show that the sequence $\{x^n g(x)\}$ converges uniformly to the zero function on $[0, 1]$. [5 Marks]

(8.) Prove that a sequence of equicontinuous functions converges uniformly in $C(X)$. [5 Marks]